501 Cont version

$$\varepsilon^* u^t \varepsilon = \underbrace{\begin{cases} e^{i\gamma t} & t \geq 0 \\ e^{i\gamma^* t} & t \leq 0 \end{cases}} = \int e^{i\omega t} \rho(\omega) \frac{d\omega}{2\pi}$$

$$\rho(\omega) = \int e^{-i\omega t} \qquad dt = \int_{-\infty}^{0} e^{-i\omega t} e^{i\gamma^* t}\, dt + \int_0^\infty e^{-i\omega t} e^{i\gamma t}\, dt$$

$$= \frac{1}{-i\omega + i\gamma^*} + \frac{1}{i\omega - i\gamma} = \frac{i}{\omega-\gamma^*} + \frac{-i}{\omega-\gamma} = \frac{(-i)(\gamma-\gamma^*)}{|\omega-\gamma|^2}$$

So $H = L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}\right)$ $\qquad \varepsilon x = 1.$

$$\varepsilon^*\left(u^t\varepsilon - \varepsilon e^{i\gamma t}\right) = 0 \qquad \text{for } t \geq 0.$$

$$\left(u^t\varepsilon - \cancel{\varepsilon e^{i\gamma t}}\right)^*\left(u^t\varepsilon - \varepsilon e^{i\gamma t}\right) = \varepsilon^* u^{-t}\left(u^t\varepsilon - \varepsilon e^{i\gamma t}\right)$$

$$= 1 - \underbrace{\varepsilon^* u^{-t}\varepsilon}_{e^{-i\gamma^* t}}\, e^{i\gamma t} = 1 - (e^{i\gamma t})^* e^{i\gamma t}$$

$$= 1 - (1+i\gamma t)^*(1+i\gamma t) + O(t^2)$$

$$= 1 - (1 - i\gamma^* t + i\gamma t) + O(t^2)$$

$$= -i(\gamma-\gamma^*)t + O(t^2).$$

$$\left(u^{t'}\varepsilon - \cancel{\varepsilon e^{i\gamma t'}}\right)^*\left(u^t\varepsilon - \varepsilon e^{i\gamma t}\right) = \varepsilon^* u^{-t'+t}\varepsilon - \varepsilon^* u^{-t'}\varepsilon\, e^{i\gamma t}$$

$$= e^{i\gamma(t-t')} - e^{-i\gamma^* t'} e^{i\gamma t}$$

Puzzle $\qquad \left(u^t\varepsilon - \varepsilon e^{i\gamma t}\right)^*\left(u^t\varepsilon - \varepsilon e^{i\gamma t}\right) = k^2 t + O(t^2).$

$\varepsilon^*\left(u^t\varepsilon - \varepsilon e^{i\gamma t}\right)$ $\qquad \int \left(e^{i\omega t} - e^{i\gamma t}\right) \frac{k^2}{(\omega-\gamma)(\omega-\bar{\gamma})}\frac{d\omega}{2\pi}$

$= 0$ by residue calcul.

503 $\quad V_{\pm}$ $\quad$ The idea is to look at $(u^t_\varepsilon - \varepsilon e^{i\gamma t})x$ as $t \downarrow 0$. You should get the completion of X wrt the scalar product

$$\left((u^t_\varepsilon - \varepsilon e^{i\gamma t})x', (u^t_\varepsilon - \varepsilon e^{i\gamma t})x\right)$$

$$= (x', x) - (e^{i\gamma t}x', e^{i\gamma t}x)$$

$$= \left(x', \left(\frac{\gamma - \gamma^*}{i}\right)x\right)t$$

So clearly $V_{\pm}$ is the completion of X wrt $\|x\|^2 = \left(x, \left(\frac{\gamma-\gamma^*}{i}\right)x\right)$. Question: Is there a canonical isomorphism $V \oplus \varepsilon X = \varepsilon X \oplus V$?

$$\|\varepsilon x_0 + u^t{}_\varepsilon x_1\|^2 = \|x_0 + e^{i\gamma t}x_1\|^2 + \|(u^t{}_\varepsilon - \varepsilon e^{i\gamma t})x_1\|^2$$

At this point you ~~know that~~ have lots of things in the space $\simeq L^2(\mathbb{R}_{>0}, dt)$ you seek.

$$\int f(\omega)\frac{k^2}{(\omega-\gamma)(\omega-\bar{\gamma})}\frac{d\omega}{2\pi} = f(\gamma)$$

for $f \in H^2(\mathbb{R}, \rho\frac{d\omega}{2\pi})$

Review $X = \mathbb{C}$, $e^{i\gamma t}$ $\gamma \in UHP$

$$\varepsilon^* u^t \varepsilon = \underbrace{\begin{cases} e^{i\gamma t} & t \geq 0 \\ e^{i\gamma^* t} & t \leq 0 \end{cases}} = \int e^{i\omega t}\rho(\omega)\frac{d\omega}{2\pi}$$

$$\rho(\omega) = \int e^{-i\omega t}\, dt = \left(\frac{k^2}{|\omega-\gamma|^2}\right) \qquad k^2 = \frac{\gamma-\gamma^*}{i} = 2\operatorname{Im}\gamma$$

$H = L^2(\mathbb{R}, \rho\frac{d\omega}{2\pi})$, $u^t = e^{i\omega t}$, $\varepsilon = 1$

Look at outgoing subspaces of H, stable $e^{i\omega t}$ $t \geq 0$
Obvious one gen. by ε call it W

$$\begin{tikzcd}
L^2(\mathbb{R}, \rho\frac{d\omega}{2\pi}) & \in L^2(\mathbb{R}, \frac{d\omega}{2\pi}) \\
\cup & \cup \\
W \ni 1 = \varepsilon \arrow[r, leftrightarrow] & \frac{ik}{\omega-\bar{\gamma}} \in H^2_+(\quad) \\
\cup & \\
W \cap \varepsilon^\perp \arrow[r, leftrightarrow] & \frac{\omega-\gamma}{\omega-\bar{\gamma}} H^2_+
\end{tikzcd}$$

So ~~[illegible]~~ you get decomposition

$$H = W^\perp \oplus W = \underset{\underset{H^2_-}{\uparrow S}}{W^\perp} \oplus \underbrace{\mathbb{C}\varepsilon \oplus \underset{\underset{SH_+}{\uparrow}}{W \cap \varepsilon^\perp}}_{H_+}$$

get complete description corresp to codim 1 outgoing subspaces.

You need the analogy of $\boxed{V_\pm}$ How

503 Repeat. $X = \mathbb{C}$, $e^{i\gamma t}$ $\gamma \in UHP$.

H, u^t, $\varepsilon : 1 \to \varepsilon$, $\varepsilon^* u^t \varepsilon = \begin{cases} e^{i\gamma t} & t \geq 0 \\ e^{i\gamma^* t} & t \leq 0 \end{cases}$

$= \int e^{i\omega t} \rho(\omega) \frac{d\omega}{2\pi}$ $\rho(\omega) = \frac{k^2}{|\omega-\gamma|^2}$ $k^2 = \frac{\gamma-\gamma^*}{i}$

$\varepsilon = 1 \in L^2(\mathbb{R}, \rho \frac{d\omega}{2\pi})$

Problem: to embed $L^2(R_{\geq 0},$ ~~dt~~$, V)$ into H.

V = completion of X wrt. $\|\nu x\|^2 = \left(x, \frac{\gamma-\gamma^*}{i} x\right)$ $\therefore V = C$
and $\nu : X \to V$ is k.

So $H = L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2} \frac{d\omega}{2\pi}\right)$ $\varepsilon = 1$ $u^t \varepsilon = e^{i\omega t}$

$$\begin{tikzcd}
H \\
L^2(\mathbb{R}, \frac{d\omega}{2\pi}) \arrow[u, shift right=2, "\frac{\omega-\bar\gamma}{ki}"] \arrow[u, shift left=2, "\frac{\omega-\gamma}{ki}"']
\end{tikzcd}$$

Use the ω picture to split H.

Better might be

$$\begin{tikzcd}
H \arrow[d, shift right=2, "\frac{ki}{\omega-\bar\gamma}"'] \arrow[d, shift left=2, "\frac{ki}{\omega-\gamma}"] \\
L^2(\mathbb{R}, \frac{d\omega}{2\pi})
\end{tikzcd}$$

Focus on

$$\begin{tikzcd}
L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2} \frac{d\omega}{2\pi}\right) \arrow[r] & L^2(\mathbb{R}, \frac{d\omega}{2\pi}) \\
1 \arrow[r, mapsto] & \frac{ki}{\omega-\bar\gamma} \in H^2_+ \\
e^{it\omega} 1 \arrow[r, mapsto] & e^{it\omega} \frac{ki}{\omega-\bar\gamma} \in H^2_+
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}\frac{ki}{\omega-\gamma} \oplus \left(\frac{\omega-\bar\gamma}{\omega-\gamma}\right) H^2_- \arrow[d, equal] & & L^2(\rho \frac{d\omega}{2\pi}) & & & \\
H^2_- \arrow[r, "\sim"] & W_- \arrow[ur, hook] & & W_+ \arrow[ul, hook'] & H^2_+ \arrow[l, "\sim"'] & = \mathbb{C}\frac{ki}{\omega-\gamma} \oplus \left(\frac{\omega-\gamma}{\omega-\bar\gamma}\right) H^2_+ \\
\frac{ki}{\omega-\gamma} & \varepsilon \arrow[u, phantom, sloped, "\in"] & X \arrow[ul, hook'] \arrow[ur, hook] & \varepsilon \arrow[u, phantom, sloped, "\in"] & \frac{ki}{\omega-\bar\gamma} \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

Apparently we can say that

$$L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}\right) = \frac{\omega-\bar\gamma}{ki} H^2_- \oplus \mathbb{C}\overset{1}{\overset{\shortparallel}{\varepsilon}} \oplus \frac{\omega-\gamma}{ki} H^2_+$$

$$\int \overline{\frac{\omega-\bar\gamma}{ki} f_-}\; \frac{\cancel{\omega-\gamma}}{\cancel{ki}} f_+ \; \frac{k^2}{(\omega-\bar\gamma)(\cancel{\omega-\gamma})}\; \frac{d\omega}{2\pi}$$

$$= \int \frac{\omega-\gamma}{\omega-\bar\gamma}\, \underline{\bar f_-}\, \underline{f_+}\, \frac{d\omega}{2\pi} = 0$$

$$\int \cancel{\varepsilon}\; \frac{\cancel{\omega-\gamma}}{ki} f_+ \; \frac{k^2}{(\cancel{\omega-\gamma})(\omega-\bar\gamma)}\; \frac{d\omega}{2\pi} = 0$$

$$\begin{tikzcd}
L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}\right) \arrow[d, "\frac{ki}{\omega-\bar\gamma}"] & = & \frac{\omega-\bar\gamma}{ki} H^2_- \arrow[d, "\simeq"] & \oplus & \mathbb{C}\varepsilon \arrow[d, "\simeq"] & \oplus & \frac{\omega-\gamma}{ki} H^2_+ \arrow[d, "\simeq"] \\
L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) & & H^2_- & \oplus & \mathbb{C}\frac{ki}{\omega-\bar\gamma} & \oplus & \left(\frac{\omega-\gamma}{\omega-\bar\gamma}\right) H^2_+
\end{tikzcd}$$

You've been calculating using the ω picture. But what you want ~~to do~~ is the t picture. A first question. ~~$e^{i\omega t}\varepsilon = [illegible]\, e^{i\omega t}1$~~ Write $u^t\varepsilon$ as $\varepsilon(?) + \frac{\omega-\gamma}{ki} f_+(\omega)$

$$e^{i\omega t} = \varepsilon a + \frac{\omega-\gamma}{ki} f_+(\omega). \qquad a = e^{i\gamma t}$$

$$f_+(\omega, t) = \frac{e^{i\omega t} - e^{i\gamma t}}{\omega-\gamma}\, ki$$

$$\phi_+(t,t) = \int e^{+i\omega t'}\left(\frac{e^{i\omega t} - e^{i\gamma t}}{\omega-\gamma}\right) ki\, \frac{d\omega}{2\pi}$$

505 $$L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}\right) = \frac{\omega-\bar\gamma}{ki} H^2_- \oplus \mathbb{C}1 \oplus \frac{\omega-\gamma}{ki} H^2_+$$

t fixed $$u^t 1 = e^{i\omega t} = c(t) + \frac{\omega-\gamma}{ki} f_+(\omega,t)$$

$$\Longrightarrow \quad c(t) = e^{i\gamma t}, \quad f_+(\omega,t) = \frac{e^{i\omega t}-e^{i\gamma t}}{\omega-\gamma} ki$$

$f_+(\cdot,t)\in H^2_+$ hence $$f_+(\omega,t) = \int_0^\infty e^{i\omega t'}\phi_+(t',t)\,dt'$$

$$\phi_+(t',t) = \int \frac{d\omega}{2\pi} e^{-i\omega t'} \frac{e^{i\omega t}-e^{i\gamma t}}{\omega-\gamma} ki$$

$$= \int \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t')}}{\omega-\gamma} ki - \underbrace{\int \frac{d\omega}{2\pi} \frac{\overset{H^-}{e^{-i\omega t'}} e^{i\gamma t}}{\omega-\gamma} ki}$$

0 if $t'>t$ $\qquad$ 0 for $0<t'$

$-k e^{i\gamma(t-t')}$ $t'<t$ $\qquad$ $+ke^{i\gamma(t-t')}$ $t'<0$

seems that $$\underset{t>0}{\phi_+(t',t)} = \begin{cases} 0 & t'>t \text{ or } t'<0 \\ -ke^{i\gamma(t-t')} & 0<t'<t \end{cases}$$

$$\frac{\omega e^{i\omega t}-\gamma e^{i\gamma t}}{\omega-\gamma} = \frac{(\omega-\gamma)e^{i\omega t} - \gamma(e^{i\gamma t}-e^{i\omega t})}{\omega-\gamma}$$

check. $$f_+(\omega,t) = ki\frac{e^{i\omega t}-e^{i\gamma t}}{\omega-\gamma}$$

$$\phi_+(t',t) = \int\frac{d\omega}{2\pi}\left(e^{-i\omega t'} ki \frac{e^{i\omega t}-e^{i\gamma t}}{\omega-\gamma}\right)$$

if $t'>t$ this in H_- so get 0

if $t'<0$ this is in H_+ so get 0 ~~except for the pole at $\omega>\gamma$~~

if $0<t'<t$, then get $\int\frac{d\omega}{2\pi} ki \frac{e^{i\omega(t-t')}}{\omega-\gamma} = -ke^{i\gamma(t-t')}$

506 You are still missing something which might be pretty simple.

$$H \qquad \| u^t \varepsilon x - \varepsilon e^{i\gamma t} x \|^2 = \|x\|^2 - \|e^{i\gamma t} x\|^2$$

$$\|x\|^2 - (\varepsilon^* u^t \varepsilon x, e^{i\gamma t} x) \qquad = \int_0^t dt' \left(-\frac{d}{dt'}\right) \|e^{i\gamma t'} x\|^2$$

My hope is that in

$$= \int_0^t dt' \underbrace{\left(e^{i\gamma t'} x, \frac{\gamma - \gamma^*}{i} e^{i\gamma t'} x\right)}_{\|\nu(e^{i\gamma t'} x)\|^2}.$$

You can directly define

$L(\mathbb{R}_{\geq 0}, dt', V) \hookrightarrow H$. You understand $\nu : X \to V$

Missing idea. Fix Δt, $u = u^{\Delta t}$, $c = e^{i\gamma \Delta t}$ then have $H, u, \varepsilon^* u^n \varepsilon = c^n \quad n \geq 0$ so that you have the dilation of the contraction c. Whence

$$\left(u^a \varepsilon - \varepsilon e^{i\gamma a}\right)^* u^t \left(u^b \varepsilon - \varepsilon e^{i\gamma b}\right)$$

$$\varepsilon^* \left(u^{t+b} \varepsilon - u^t \varepsilon e^{i\gamma b}\right) = c^{t+b} - c^t c^b = 0$$

$$\varepsilon^* \left(u^{-a+t+b} \varepsilon - u^{-a+t} \varepsilon c^b\right) = c^{-a+t+b} - \underbrace{c^{-a+t}}_{} c^b$$

provided $-a+t \geq 0$

It clearly works, but you need the details. ~~Shit~~

Example: $\gamma \in UHP \quad X = \mathbb{C}$.

$$H = L^2\left(\mathbb{R}, \frac{k^2}{|\omega - \gamma|^2} \frac{d\omega}{2\pi}\right) \qquad u^t = e^{i\omega t}, \quad \varepsilon = 1$$

You want to embed $L^2(\mathbb{R}_{\geq 0}, dt') \hookrightarrow H$ step fns.

$$\chi_{(a,b)}(t') \longmapsto u^a(u^{b-a}\varepsilon - \varepsilon c^{b-a}) = u^b \varepsilon - u^a \varepsilon c^{b-a}$$

$$\chi_{(0,b)}(t') \longmapsto u^b \varepsilon - \varepsilon c^b \qquad u^a \frac{u^r \varepsilon - \varepsilon c^r}{r}$$

$$\delta_a(t') \longmapsto \lim_{r \to 0} \frac{u^{a+r}\varepsilon - u^a \varepsilon c^r}{r} \qquad \frac{e^{i\omega r} - e^{i\gamma r}}{r} = i(\omega - \gamma)$$

507 Basically what happens is that H splits into 3 parts. εX , $\sum_{t>0} (u^t\varepsilon - \varepsilon c^t) X$, $\sum_{t<0} (u^{t}\varepsilon - \varepsilon c^{t}) X$

$$u^a(u^t\varepsilon - \varepsilon c^t) = (u^{a+t}\varepsilon - u^a\varepsilon c^t) - a$$

$$u^{a+t}\varepsilon - \varepsilon c^{a+t} = u^a\varepsilon c^t - \varepsilon c^{a+t}$$

$$= (u^a\varepsilon - \varepsilon c^a)c^t$$

$$(u^t\varepsilon - \varepsilon c^t)x \overset{?}{\subseteq} (u^{t+a}\varepsilon - \varepsilon c^{t+a})X \ ?$$

$$u^2\varepsilon - \varepsilon c^2$$

$$y = aa^*y + \pi_+ y$$

$$uy = ba^*x + u\pi_+ x$$

$$= aa^* ba^* x + \pi_+ ba^* x + u\pi_* x$$

$$u^2 y = (ba^*)^2 x + u\pi_+ ba^* x + u^2\pi_+ x$$

~~$ux = \varepsilon\varepsilon^* ux +$~~

$$u\varepsilon x = \underbrace{\varepsilon\varepsilon^* u\varepsilon x}_{\varepsilon c x} + \underbrace{(1-\varepsilon^*\varepsilon)u\varepsilon x}_{\nu_+ x}$$

$$u^2\varepsilon x = u\varepsilon(cx) + u\nu_+ x$$

$$= \varepsilon c^2 x + \nu_+ cx + u\nu_+ x$$

$$u^3\varepsilon x = \varepsilon c^3 x + \nu_+ c^2 x + u\nu_+ cx + u^2\nu_+ x$$

$$u^n\varepsilon x - \varepsilon c^n x = \sum_{j=0}^{n-1} u^j \nu_+ c^{n-1-j} x$$

$$= \sum_{j=0}^{n-1} u^{n-1-j}\nu_+ c^j x$$

508 (X, c) $\quad H = L^2(S^1, d\mu)$

$$d\mu = \left(\sum_{n\geq 0} z^{-n} c^n + \sum_{n>0} z^n c^{*n}\right)\frac{d\theta}{2\pi} \qquad \frac{1}{1-z^{-1}c} + \frac{zc^*}{1-zc^*} = \frac{1}{1-zc^*}\left(1 - c^*c\right)\frac{1}{1-z^{-1}c}$$

$$\Big\| \sum_{\text{finite}} u^n \varepsilon x_n \Big\|^2 = \sum_{m,n} \left(x_m, \varepsilon^* u^{-m} u^n \varepsilon x_n\right)$$

$$= \int \left(\sum z^m x_n\right)^* d\mu \left(\sum z^n x_n\right)$$

$$= \int \left(\frac{1}{1-z^{-1}c}\sum z^m x_m\right)^* (1-c^*c)\frac{1}{1-z^{-1}c}\sum z^n x_n \frac{d\theta}{2\pi}$$

$$\sum_{n\geq 0} \bar{z}^n c^n + \sum_{n>0} z^n c^{*n} = \frac{1}{1-\bar{z}c} + \frac{zc^*}{1-zc^*}$$

$$= \frac{1}{1-zc^*}\left(1 - zc^* + zc^*(1-\bar{z}c)\right)\frac{1}{1-\bar{z}c}$$

$$= \frac{1}{1-zc^*}\left(1 - c^*|z|^2 c\right)\frac{1}{1-\bar{z}c}$$

~~Standard vector!! [illegible]!!~~

partial hermitian operator. $\quad$ $$\begin{tikzcd} X \arrow[r, hook, shift left, "\eta"] \arrow[r, shift right] & Y \end{tikzcd}$$

formulas. $$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}^* \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1'^* y_1 - y_2'^* y_2$$

$$\begin{pmatrix} 1 \\ m \end{pmatrix}^* \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1 - m^* y_2 = 0$$

$$\therefore \left(\begin{pmatrix} 1 \\ m \end{pmatrix} Y\right)^\circ = \begin{pmatrix} m^* \\ 1 \end{pmatrix} Y$$

509 what to do? $\lambda = i\frac{1-z}{1+z}$ $z = \frac{1+i\lambda}{1-i\lambda} = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\lambda$

~~$\frac{1}{2}\begin{pmatrix} -1 & +1 \\ -i & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} i & 1 \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$~~

$$\frac{1}{2}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

So you consider $\left(\begin{pmatrix} a \\ b \end{pmatrix} X\right)^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{a^* y_1 - b^* y_2} = 0 \right\}$

$\begin{pmatrix} a \\ b \end{pmatrix} X \subset \left(\begin{pmatrix} a \\ b \end{pmatrix} X\right)^0$ means $\boxed{a^* a = b^* b.}$

$$\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} X = \begin{pmatrix} -ia + ib \\ a + b \end{pmatrix} X$$

~~$\eta =$~~ $-ia + ib$

~~$\eta^* = ia^* - ib^*$~~ ~~$\eta^*\eta = (ia^* - ib^*)(-ia + ib)$~~

~~$= (a^* - b^*)(a - b)$~~

~~$= a^* a + b^* b - a^* b - b^* a$~~

$$az - b = a\frac{1+i\lambda}{1-i\lambda} - b = \frac{a + ia\lambda - b + ib\lambda}{1-i\lambda} = \frac{(a-b) + (ia+ib)\lambda}{1-i\lambda}$$

$$= \frac{i}{1-i\lambda}\left((-ia+ib) + (a+b)\lambda\right)$$

$\lambda\eta - \alpha$

$\eta = a+b$

$\alpha = i(a-b)$

$az - b$

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$$\left(\begin{pmatrix}\eta\\ \alpha\end{pmatrix} X\right)^{\circ} = \left\{ \begin{pmatrix}y_1\\ y_2\end{pmatrix} \;\middle|\; \underbrace{\begin{pmatrix}\eta\\ \alpha\end{pmatrix}^{*}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}}_{\eta^* y_2 = \alpha^* y_1} = 0 \right\}$$

$$\left(\begin{pmatrix}\eta\\ \alpha\end{pmatrix} X\right)^{\circ} \supset \begin{pmatrix}\eta\\ \alpha\end{pmatrix} X \iff \boxed{\eta^*\alpha = \alpha^*\eta}$$

$$\eta^*\alpha = (a^* + b^*)\, i(a-b) = i\left[a^*a + b^*a - a^*b - b^*b\right]$$

$$\alpha^*\eta = -i(a^* - b^*)(a+b) = -i\left[a^*a + a^*b - b^*a - b^*b\right]$$

$$\eta^*\alpha = \alpha^*\eta \iff a^*a = b^*b$$

Begin with a review of the unitary case. $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\quad a^*a = 1$, $b^*b = 1$.

Given

$$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

$$c_h = \underbrace{ba^*}_{c_0} + \xi_- h \xi_+^* \qquad |h| \le 1.$$

You should start with u unitary, ξ cyclic, $\xi_+ = \xi$, $u(\xi_+) = \xi_-$ $\quad u = c_1$. To relate the spectral measure assoc. to u, ξ. $\quad \xi^* u^n \xi = \int \zeta^n d\mu$.

to the scattering function $\quad S_0(z) = \xi_-^* \dfrac{1}{1 - z c_0^*} \xi_+$ $\qquad c_0 = ba^*$, $c_0^* = ab^*$

$$S_h(z) = \xi_-^* \frac{1}{1 - z c_h^*} \xi_+$$

$$S_1(z) = (u\xi)^* \frac{1}{1 - z u^{-1}} \xi = \xi^* \frac{u^{-1}}{1 - z u^{-1}} \xi$$

$$1 + z S_1(z) = \xi^* \frac{1}{1 - z u^{-1}} \xi$$

$$\frac{1}{1 - z S_0(z)}$$

$$\therefore \frac{1}{1 - z S_0(z)} = \xi^* \frac{1}{1 - z u^{-1}} \xi$$

$$c_h^* = ab^* + \xi_+ \bar{h} \xi_-^*$$

$$\xi_-^* \frac{1}{1 - z c_h^*} = \xi_-^* \frac{1}{1 - z c_0^*} + \xi_-^* \frac{1}{1 - z c_0^*} z \xi_+ \bar{h} \xi_-^* \frac{1}{1 - z c_0^*}$$

$$S_h(z) = \frac{1}{1 - S_0(z) z\bar{h}} S_0(z)$$

$$1 + z\bar{h} S_h(z) = 1 + \frac{z\bar{h} S_0(z)}{1 - S_0(z) z\bar{h}} = \frac{1}{1 - S_0(z) z\bar{h}}$$

511 Work out scattering both discrete + cont.

(X, c) given. Let $(H, u, \varepsilon : X \to H)$ be dilation structure

$$H \quad \cdots \oplus uV_- \oplus \varepsilon X \oplus V_+ \oplus uV_+ \oplus \cdots$$

$V_+ = \overline{(u\varepsilon - \varepsilon c)X} \longleftarrow$ ~~X completed~~ $\nu_-(x) = (u^{-1}\varepsilon - \varepsilon c^*)x$

define $\nu_\pm : X \longrightarrow H$ by $\nu_+(x) = (u\varepsilon - \varepsilon c)x$

set $V_\pm = \overline{\nu_\pm X}$, then $X \xrightarrow{\nu_\pm} V_+$ identifies

$V_\pm$ both completion of X under $\|\nu_+ x\|^2 = \|x\|^2 - \|cx\|^2$

$$\|\nu_- x\|^2 = \|x\|^2 - \|c^* x\|^2$$

$$\varepsilon^* u^n \nu_+(x) = \varepsilon^* u^n (u\varepsilon - \varepsilon c)x$$

$$= (c^{n+1} - c^n c)x = 0 \qquad n \geq 0.$$

$$\nu_+^* u^n \nu_+ = (u\varepsilon - \cancel{\varepsilon c})^* u^n (u\varepsilon - \varepsilon c)$$

$$= \varepsilon^* u^{n-1}(u\varepsilon - \varepsilon c) = 0 \qquad n \geq 1.$$

$\therefore$ εX, $u^n V_+$ for $n \geq 0$ are orthogonal.

similarly εX, $u^{-n} V_-$ ———————

$$(u^{-p}\nu_-)^*(u^q \nu_+) = \underline{\nu_-^* u^{\overset{n}{\boxed{p+q}}} \nu_+} \qquad n \geq 0.$$

$$= (u^{-1}\varepsilon - \cancel{\varepsilon c^*})^* u^n (u\varepsilon - \varepsilon c)$$

$$= \varepsilon^* u^{n+1}(u\varepsilon - \varepsilon c) = 0.$$

~~$u^{-1}\nu_+ = u^{-1}(u\varepsilon - \varepsilon c) = \varepsilon - u^{-1}\varepsilon c$~~

$$\begin{tikzcd}
\bigoplus_{n\in\mathbb{Z}} u^n V_- \arrow[r, hook, "j_-"'] & H \arrow[l, bend right, "j_-^*"'] \arrow[r, bend left, "j_+^*"] & \overset{(2)}{\bigoplus_{n\in\mathbb{Z}}} u^n V_+^{*} \arrow[l, hook', "j_+"]
\end{tikzcd}$$

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$$\left(j_+^* \varepsilon x, \sum u^n (x_n)\right) = \left(\varepsilon x, \sum u^n \nu_+(x_n)\right)$$

$$\left(\varepsilon x, u^{-m} \nu_+(x'_{-m})\right) = \left[u^m \varepsilon x, \underline{(u\varepsilon - \varepsilon c)} x'_{-m}\right) = 0 \qquad m \leq 0$$

$$= \left((u\varepsilon - \varepsilon c)^* u^m \varepsilon x, x'_{-m}\right)$$

$$= \left((\varepsilon^* u^{-1} - c^* \varepsilon^*) u^m \varepsilon x, x'_{-m}\right) \qquad n \geq 1$$

$$= \left(c^{m-1} x - c^* c^m x, x'_{-m}\right)$$

$$= \left(\nu c^{m-1} x, \nu x'_{-m}\right)$$

$$= \left(u^{-m} \nu c^{m-1} x, u^{-m} x'_{-m}\right)$$

$$= \left(\sum_{n \geq 1} u^{-n} \nu c^{n-1} x, u^{-m} x'_{-m}\right)$$

$$j_+^*(\varepsilon x) = \sum_{n \geq 0} u^{-n-1} \nu(c^n x)$$

$$u(\varepsilon x) = \varepsilon c x + \nu_+ x$$

$$u^2(\varepsilon x) = u \varepsilon c x + u \nu_+ x$$

$$= \varepsilon c^2 x + \nu_+ c x + u \nu_+ x$$

$$u^3(\varepsilon x) = \varepsilon c^3 x + \nu_+ c^2 x + u \nu_+ c x + u^2 \nu_+ x$$

$$\varepsilon x = u^{-3} \varepsilon c^3 x + u^{-3} \nu_+ c^2 x + u^{-2} \nu_+ c x + u^{-1} \nu_+ x$$

$$\varepsilon x \longmapsto \sum_{n \geq 0} u^{-n-1} \nu_+ c^n x \rightsquigarrow \nu_+ \left\{ \sum_{n \geq 0} z^{-n-1} c^n x \right\}$$

$$\nu_+ \frac{z^{-1}}{1 - z^{-1} c} x = \nu_+ \frac{1}{z - c} x$$

other direction

$$u^{-1}\varepsilon x = \nu_- x + \varepsilon c^* x$$

$$u^{-2}\varepsilon x = u^{-1}\nu_- x + \nu_- c^* x + \varepsilon c^{*2} x$$

$$u^{-3}\varepsilon x = u^{-2}\nu_- x + u^{-1}\nu_- c^* x + \nu_- c^{*3} x + \varepsilon c^{*4} x$$

$$\varepsilon x \rightsquigarrow \sum_{n\geq 0} u^{n+1}\nu_- c^{*n} x \rightsquigarrow \nu \sum_{n\geq 0} z^{n+1} c^{*n}_{\ x} = \nu_- \frac{1}{z^{-1} - c^*} x$$

$$\left(\varepsilon x,\ u^n \nu_*(x_n)\right) = \left(x, \varepsilon^* u^n (u^{-1}\varepsilon - \varepsilon c^*) x_n\right) \qquad n \geq 1$$

$$= \left(x, (c^{n-1} - c^n c^*) x_n\right)$$

$$= \left(c^{*n-1} x, (1 - cc^*) x_n\right)$$

$$= \left(\nu_- c^{*n-1} x,\ \nu_- x_n\right)$$

$$= \left(\sum_{k\geq 1} u^k \nu_- c^{*k-1} x,\ u^n \nu_- x_n\right)$$

$$\varepsilon x \rightsquigarrow \sum_{k\geq 1} u^k \nu_- c^{*k-1} x$$

Continuous case - $u^a(u^t\varepsilon - \varepsilon c^t) + (u^a\varepsilon - \varepsilon c^a)c^t$

$$= u^{a+t}\varepsilon - \varepsilon c^{a+t}$$

$$H = L^2(\mathbb{R}_{\leq 0}, V) \oplus \varepsilon X \oplus L^2(\mathbb{R}_{\geq 0}, V)$$ ~~answer again~~

$$\boxed{u^t \varepsilon x - \varepsilon c^t x = \int_0^t u^{t'} \nu(c^{t-t'} x)\, dt'}$$

$$\varepsilon x - u^{-t}\varepsilon c^t x = \int_0^t u^{-t+t'} \nu(c^{t-t'} x)\, dt'$$

$$= \int_0^t u^{-t''} \nu(c^{t''} x)\, dt''$$

Now put in $c^t = e^{i\gamma t}$ and $u^t = e^{i\omega t}$

$$\int_0^\infty e^{-i\omega t}\,\nu(e^{i\gamma t}x)\,dt' = \nu\left(\frac{1}{i(\omega-\gamma)}x\right) \quad ..$$

So you need to describe this ν map $\nu : X \to H$, possibly enlarged in the Sobolev sense. Yes Although $\nu x \notin H$ $(\nu x)^*$ is defined on some nice subspace of H. When $X = \mathbb{C}$. take $x = 1$. Then you have

$$\varepsilon 1 = \int_0^\infty e^{-i\omega t}\,\nu(c^t x)\,dt = \int_0^\infty e^{-i\omega t} e^{i\gamma t}\,\nu(1)\,dt$$

$$= \frac{1}{i(\omega-\gamma)}\,\nu(1)$$ It's likely that ?

Example. $H = L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}\right)$ $\varepsilon = 1$

$$\wr|$$

$L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right)$ $\varepsilon = \frac{ki}{\omega-\bar\gamma}$

the image of $L^2(\mathbb{R}_{>0}, dt)$ is SH^2_+ $S = \frac{\omega-\gamma}{\omega-\bar\gamma}$

You have $\|u^t\varepsilon x - \varepsilon c^t x\|^2 = (u^t\varepsilon x - \cancel{\varepsilon c^t x}, u^t\varepsilon x - \varepsilon c^t x)$ $t \geq 0$

$$= \|x\|^2 - \|\underbrace{c^t x}_{e^{i\gamma t}x}\|^2 = (x, \underbrace{(-i)(\gamma-\gamma^*)}_{\nu^*\nu}x)t + O(t^2)$$

Hope that $\lim_{t\downarrow 0} \frac{u^t\varepsilon x - \varepsilon c^t x}{\sqrt{t}} = \nu x$ exists in an ~~some~~ appropriate Sobolev space

and that $u^t\varepsilon x - \varepsilon c^t x = \int_0^t u^{t'}\nu(c^{t-t'}x)\,dt'$ holds.

try applying $\int_0^\infty e^{-i\omega t} .. dt$. get

$$\frac{1}{i(\omega-H)}\varepsilon x - \varepsilon\frac{1}{i(\omega-\gamma)}x = \frac{1}{i(\omega-H)}\,\nu\left(\frac{1}{i(\omega-\gamma)}x\right)$$

515 $\varepsilon^* \frac{1}{\omega - H}\varepsilon - \frac{1}{\omega-\gamma} = $ ✓

Use $\overset{\omega}{}$ model $\lim_{t\downarrow 0}\frac{u^t\varepsilon - \varepsilon c^t}{\sqrt{t}} = \lim_{t\downarrow 0}\frac{e^{i\omega t}-e^{i\gamma t}}{\sqrt{t}}$

$$u^t\varepsilon x - \varepsilon c^t x = \frac{e^{i\omega t}-e^{i\gamma t}}{\omega-\gamma}ki$$

Go over yesterday model

$$\begin{tikzcd}
L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}\right) \arrow[d, "\wr"', "\cdot\frac{ki}{\omega-\bar\gamma}"] & = & \frac{\omega-\bar\gamma}{ki}H^2_- \arrow[d, "\wr"] & \oplus \ \mathbb{C}1 \arrow[d] & \oplus \ \frac{\omega-\gamma}{ki}H^2_+ \arrow[d] \\
L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) & = & H^2_- & \oplus \ \mathbb{C}\frac{ki}{\omega-\bar\gamma} & \oplus \ \left(\frac{\omega-\gamma}{\omega-\bar\gamma}\right)H^2_+
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{matrix}u^t\varepsilon\\ -\varepsilon e^{i\gamma t}\end{matrix} = \left(e^{i\omega t} - e^{i\gamma t}\right) = \frac{\omega-\gamma}{ki}\cdot\frac{e^{i\omega t}-e^{i\gamma t}}{\omega-\gamma}ki \arrow[d] \\
\frac{\cancel{\omega-\gamma}}{\omega-\bar\gamma}\,\frac{e^{i\omega t}-e^{i\gamma t}}{\cancel{\omega-\gamma}}ki
\end{tikzcd}$$

So $u^t\varepsilon - \varepsilon c^t$ is rep. by $\boxed{\frac{e^{i\omega t}-e^{i\gamma t}}{\omega-\bar\gamma}ki} \in H^+$

But $u^t\varepsilon - \varepsilon c^t = \int_0^t u^{t'} c^{t-t'}\nu\, dt'$

$$\frac{d}{dt}\left(\frac{e^{i\omega t}-e^{i\gamma t}}{\omega-\bar\gamma}ki\right)\Bigg|_{t=0} = \frac{i\omega e^{i\omega t}-i\gamma e^{i\gamma t}}{\omega-\bar\gamma}ki\Bigg|_{t=0} = \frac{i(\omega-\gamma)}{\omega-\bar\gamma}ki$$

agrees with $\frac{d}{dt}\left(e^{i\omega t}\varepsilon - \varepsilon e^{i\gamma t}\right)\Big|_{t=0} = i(\omega-\gamma)\varepsilon$

It's now clear that $\frac{d}{dt}(a^t\varepsilon - \varepsilon e^{i\gamma t})\Big|_t =$

$i(\omega-\gamma)\varepsilon = i\frac{\omega-\gamma}{\omega-\gamma}k_i$, not in SH_+^2

but is sort of the "first orthonormal" basis elt., the others ~~being~~ given by $e^{i\omega t}$ acting on this.

You now want to understand continuous case, also partial hermitian operators.

$$az-b = a\frac{1+i\lambda}{1-i\lambda} - b = \frac{a+i\lambda a - b + i\lambda b}{1-i\lambda} = \frac{a-b+(ia+ib)\lambda}{1-i\lambda}$$

$$= \frac{i(a-b)-(a+b)\lambda}{1-i\lambda} = \frac{(a+b)\lambda - i(a-b)}{-(1-i\lambda)}$$

$\varepsilon = a+b$

$\alpha = i(a-b)$

$$\varepsilon^*\alpha = (a^*+b^*)(ia-ib) = i(1-a^*b+b^*a-1)$$

$$\alpha^*\varepsilon = (-i)(a^*b-b^*a)$$

$$\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix}X = \begin{pmatrix}1&1\\ i&-i\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}X \quad \therefore\ \varepsilon^*\alpha = \alpha^*\varepsilon \iff a^*a = b^*b.$$

$$\frac{1}{2}\begin{pmatrix}1&1\\ i&-i\end{pmatrix}\begin{pmatrix}1&0\\ 0&-1\end{pmatrix}\begin{pmatrix}1&-i\\ 1&i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1&1\\ i&-i\end{pmatrix}\begin{pmatrix}1&-i\\ -1&-i\end{pmatrix} = \begin{pmatrix}0&-i\\ i&0\end{pmatrix}$$

$$\left(\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix}X\right)^0 = \left\{\begin{pmatrix}y_1\\ y_2\end{pmatrix} \,\middle|\, \overbrace{\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix}^*\begin{pmatrix}0&-1\\ 1&0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}}^{-\varepsilon^*y_2+\alpha^*y_1} = 0\right\}$$

$\cup$?

$$\alpha^*y_1 = \varepsilon^*y_2$$

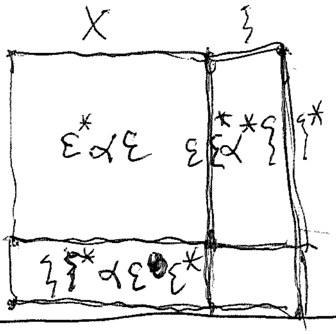
$\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix}X$

$$\alpha^*\varepsilon = \varepsilon^*\alpha \quad ✓$$

517 Calculate $\left(\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix}X\right)^{\circ} = \left\{\begin{pmatrix}y_1\\ y_2\end{pmatrix} \middle| \alpha^* y_1 = \varepsilon^* y_2\right\}$.

Equip X with scalar prod + $\varepsilon^*\varepsilon = 1$. Assume of type $O(n)$, let ξ be a unit vector $\in \mathrm{Ker}(\varepsilon^*)$

$\therefore\ Y = \varepsilon X \oplus \mathbb{C}\xi \qquad 1 = \varepsilon\varepsilon^* + \xi\xi^*$



$$\left(\begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix}X\right)^{\circ} = \begin{pmatrix}\varepsilon\\ \alpha\end{pmatrix}X + \begin{pmatrix}\xi\\ \varepsilon\alpha^*\xi\end{pmatrix}\mathbb{C} +$$

Try to do things invariantly. First a contraction of c on Y same as a graph subspace $\begin{pmatrix}1\\ c\end{pmatrix}Y$ on which Krein form is ≥ 0

$$\frac{1}{2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix} = i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}1-i\gamma\\ 1+i\gamma\end{pmatrix}^*\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1-i\gamma\\ 1+i\gamma\end{pmatrix} = \begin{pmatrix}1\\ \gamma\end{pmatrix}^* i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ \gamma\end{pmatrix}$$

$$= i(-\gamma + \gamma^*) = \frac{\gamma - \gamma^*}{i}$$

$$\frac{1-i\gamma}{1+i\gamma} \text{ contraction} \iff \frac{\gamma-\gamma^*}{i} \geq 0.$$

So what do you want to do?

Invariant approach to scattering. Look from c viewpt.

Given (Y, c) form $H = \oplus\, \overline{\nu_- Y} \oplus jY \oplus \overline{\nu_+ Y} \oplus u\overline{\nu_+ Y} \oplus \cdots$

$\nu_+ y = uj - jc$

$\nu_- y = u^{-1}j - jc^*$

$$\cdots \oplus V_- \oplus jY \oplus V_+ \oplus uV_+ \oplus \cdots$$

$$ujy = jcy + \nu_+ y$$
$$u^2jy = jc^2y + \nu_+ cy + u\nu_+ y$$
$$u^3jy = jc^3y + \nu_+ c^2y + u\nu_+ cy + u^2\nu_+ y$$

$$u^{n+1} j y = j c^{n+1} y + \sum_{i=0}^{n} u^{n-i} \nu_+ c^{i} y$$

$$\nu_+ \frac{z^{-1}}{1-z^{-1}c} y$$

$$\nu_+ \sum z^{-i-1} c^{i} y$$

$$jy = ~~u^{-n-1} j c^{n+1} y~~ + \sum_{i\geq 0} u^{-1-i} \nu_+ c^{i} y \rightsquigarrow \nu_+ \left(\frac{1}{z-c} y\right)$$

other direction

$$\nu_-\left(\frac{1}{z^{-1}-c^*} y\right)$$

So have

$L^2(S^1, V_+)$ $\quad zH^2_+(S^1, V_+) \oplus H^2_+(S^1, V_+)$

$H \quad zH^2_-(S^1, V_-) \oplus jY \oplus H^2_+(S^1, V_+)$

$$\oplus u^{-3}V_+ \oplus u^{-2}V_+ \oplus u^{-1}V_+ \oplus V_+ \oplus uV_+ \oplus$$

$$\cong \qquad \cong \qquad \cong \qquad \cong$$

$$H: \oplus u^{-1}V_- \oplus V_- \oplus jY \oplus V_+ \oplus uV_+ \oplus$$

$$\cong \qquad \cong \qquad \cong$$

$$\cdots \oplus u^{-1}V_- \oplus V_- \oplus uV_- \oplus u^2V_-$$

$$\frac{uV_- \oplus u^2V_- \oplus \cdots}{S(V_+ + uV_+ + \cdots)}$$

519 I'm still working on the continuous case. There's a philosophy I need to ~~f~~ find, better, I need to develop the invariant viewpoint.

Discuss discrete scattering picture.

(Y, c) $\quad H, u, j \quad$ $j^*u^nj = c^n \quad n \geq 0$, $\quad (c^*)^{-n} \quad n \leq 0$

$\nu_+ \cancel{y} = (uj - jc)\cancel{y}$ $\qquad V_\pm = \overline{\nu_\pm X}$

$\nu_- \cancel{y} = (u^{-1}j - jc^*)\cancel{y}$

$H \overset{\oplus}{=} u^{-1}V_- \oplus V_- \oplus j\cancel{Y} \oplus V_+ \oplus uV_+ \oplus \cdots$ $\qquad uj = jc + \nu_+$

outgoing rep.

$ujx = jcx + \nu_+x$

$u^2jx = \underbrace{uj}_{jc+\nu_+}cx + u\nu_+x$

$= jc^2x + \nu_+cx + u\nu_+x$

$u^3jx = jc^3x + \nu_+c^2x + u\nu_+cx + u^2\nu_+x$

$u^njx = jc^nx + \sum_{i=0}^{n-1} u^{i}\nu_+c^{n-1-i}x$

$jx = u^{-n}jc^nx + \sum_{i=0}^{n-1} u^{\overset{n-1-i}{-n+i}}\nu_+c^{n-1-i}x$

$\sum_{i=0}^{n-1} u^{-i-1}\nu_+c^ix$

$u^{n+1}jx = jc^{n+1}x + \nu_+c^nx + u\nu_+c^{n-1}x + \cdots + u^n\nu_+x$

$jx = u^{-n-1}jc^nx + \underbrace{u^{-n-1}\nu_+c^nx + u^{-n}\nu_+c^{n-1}x + \cdots + u^{-1}\nu_+x}$

$$\boxed{jx = u^{-n-1}jc^nx + \sum_{k=0}^{n} u^{-k-1}\nu_+c^kx}$$

$\|x\|^2 = \|c^{n+1}x\|^2 + \sum_{k=0}^{n}\|\nu_+c^kx\|^2$ $\qquad \|c^nx\| \to 0 \quad \forall x$

$\Rightarrow \quad j : \cancel{X} \underset{\text{isom.}}{\hookrightarrow} \overset{(2)}{\bigoplus_{k\geq 1}} u^{-k}V_-$

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$$\begin{array}{llllll}
\ell^2(\mathbb{Z}, V_+): & u^{-2}V_+ & u^{-1}V_+ \mid & V_+ & uV_+ & \\
 & \| & & \| & \| & \| \\
H: & u^{-1}V_- \quad V_- & \cancel{jY} & V_+ & uV_+ & \\
 & \| \qquad \| & & \| & & \\
\ell^2(\mathbb{Z}, V_-) & u^{-1}V_- \oplus V_- & \oplus\, uV_- & u^2V_+ & u^3V_+ &
\end{array}$$

Continuous case $\quad \overset{H \longrightarrow}{(Y, e^{i\gamma t})} \quad (-i)(\gamma - \gamma^*) \geq 0.$

Look first at the discrete fin. diml case to get the picture. Begin with (Y, c), construct dilation $E = E_+ + E_-$ $\quad E_+ \cap E_- = jY$

$$\begin{tikzcd}
 & E & \\
\cdots \oplus V_- \oplus jY = E_- \arrow[ur, hook] \arrow[dr, hook] & & E_+ = jY \oplus V_+ \oplus uV_+ \oplus \cdots \arrow[ul, hook'] \\
 & Y \arrow[ur, hook] &
\end{tikzcd}$$

View in terms of a vector bundle on $\mathbb{P}^1$

What is the invariant viewpoint?? oriented circle in $\mathbb{P}^1$ gives rise to an <u>intrinsic</u> <u>Hilbert space</u> with polarization.

Let $|c| < 1$. $\quad w = \dfrac{\bar{d}z + \bar{c}}{cz + d}$ $\qquad ds^2 = g(dz, dz)$

$$|w|^2 = \frac{(dz+\bar{c})(\bar{c}\bar{z}+\bar{d})}{|cz+d|^2} \qquad \boxed{dw = \frac{dz}{(cz+d)^2}} \qquad \begin{aligned} &= \rho(|z|^2)\,|dz|^2 \\ &= \rho(|w|^2)\,|dw|^2 \end{aligned}$$

$$\rho(|z|^2)\,|dz|^2 = \rho \qquad\qquad \frac{\rho(|w|^2)}{\rho(|z|^2)} =$$

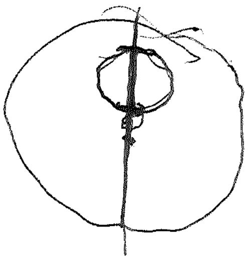
$$w = \frac{\bar{d}z + \bar{c}}{cz+d} \qquad |d|^2 - |c|^2 = 1.$$

$$1 - |w|^2 = \frac{|cz+d|^2 - |\bar{d}z+\bar{c}|^2}{|cz+dz|^2}$$

$$\begin{array}{l} |c|^2|z|^2 + \cancel{\bar{c}d}\bar{z} + \cancel{c\bar{d}}z + |d|^2 \\ - |d|^2|z|^2 - \bar{d}\cancel{z}c - \cancel{dz}\bar{c} - |c|^2 \end{array} = \frac{1}{(|d|^2-|c|^2)(1-|z|^2)}$$

$$1 - |w|^2 = \frac{1-|z|^2}{|cz+d|^2} \qquad \|dw\| = \frac{\|dz\|}{|cz+d|^2}$$

$$\frac{\|dw\|}{1-|w|^2} = \frac{\|dz\|}{1-|z|^2}$$



NE dist $d(z, z+dz) = \dfrac{|dz|}{1-|z|^2}$

$$d(rz, rz+rdz) = \frac{r|dz|}{1-r^2|z|^2} \leq \frac{r|dz|}{1-|z|^2} = r\,d(z, z+dz)$$

$$V_- \quad jX \quad V_+ \quad uV_+$$

$$u^{-1}(uj - jc)x = (j - u^{-1}jc)x$$

$$= -\underbrace{(u^{-1}j - jc^*)x}_{V_-} - \underbrace{jc^*cx}_{jX}$$

$$L^2(S^1, V_-) \xrightarrow{j_-} E \xleftarrow{j_+} L^2(S^1, V_+)$$

$$\boxed{\cancel{u^{-1}\nu_+ x} = -\cancel{\nu_- cx} - jc^*cx}$$

assume $\overline{\mathrm{Img}(j_-) + \mathrm{Img}(j_+)} = E$

equivalently $\mathrm{Ker}(j_-^*) \cap \mathrm{Ker}(j_+^*) = 0$, enough this to be true on X. Then you get $S = j_-^* j_+ : V_+ \to \bigoplus_{n\geq 1} u^n V_-$

$$\nu_+(x) = (uj - jc)(x) \rightsquigarrow \underbrace{z\,\nu_- \frac{1}{z^{-1}-c^*}x - \nu_- \frac{1}{z^{-1}-c^*}cx}$$

$$\nu_- \frac{z}{1-zc^*}(z-c)x = \nu_- \frac{1}{z^{-1}-c^*}(z-c)x$$

$$\boxed{S(z)\nu_+ x = \nu_- \frac{1}{z^{-1}-c^*}(z-c)x}$$

would $\dfrac{1}{1-zc^*}(z-c)$ be better.

$$u^{-1}\nu_+ x = u^{-1}(uj - jc)x = jx - u^{-1}jcx$$

$$\nu_- cx = (u^{-1}j - jc^*)cx$$

$$u^{-1}\nu_+ x + \nu_- cx = jx - jc^*cx = j(1-c^*c)x$$

$$\boxed{u^{-1}\nu_+ x = -\nu_- cx + j(1-c^*c)x}$$

$$\begin{array}{cc} \|u^{-1}\nu_+ x\|^2 & \|\quad\| + \|\quad\| \\ \shortparallel & \shortparallel \\ \|x\|^2 - \|cx\|^2 & \|cx\|^2 - \|c^*cx\|^2 + \|(1-c^*c)x\|^2 \end{array}$$

$$c^*c - c^*\cancel{c}c^*c + 1 - 2c^*c + \cancel{(c^*c)^2}$$

$$S^* = j_+^* j_- : V_- : \bigoplus_{n\geq 1} u^{-n}V_+$$

$$\nu_- x = (u^{-1}j - jc^*)x \rightsquigarrow z^{-1}\nu_+ \frac{1}{z-c}x - \nu_+\frac{1}{z-c}c^*x$$

$$= \nu_+ \frac{1}{z-c}(z^{-1} - c^*)x$$

$$\boxed{\underbrace{S^*(z^{-1})}\nu_- x = \nu_+ \frac{1}{z-c}(z^{-1}-c^*)x}$$

indicates $S^* = j_+^* j_-$ in the functions an S^1 picture is analytic in outer disk

Now

$$L_+ \xrightarrow{j_+} E \xleftarrow{j_-} L_-$$

$$\|j_+\xi + j_-\eta\|^2 = \|\xi + S^*\eta\|^2 + \big(\eta, (1-SS^*)\eta\big)^2$$
$$= \|S\xi + \eta\|^2 + \big(\xi, (1-S^*S)\xi\big).$$

So picture

$$L^+ \qquad \underline{u^{-2}V_+ \oplus u^{-1}V_+} \oplus V_+ \oplus uV_+ \oplus$$

$$\cap \qquad\qquad \| \qquad \| \qquad \|$$

$$E \quad \overline{\cdots \oplus u^{-1}V_- \oplus V_- \oplus} \; \underline{jX \oplus V_+ \oplus uV_+ \oplus \cdots}$$

$$\| \quad \| \quad \| \qquad\qquad \cup$$

$$L^- \quad \cdots \oplus u^{-1}V_- \oplus V_- \oplus \overline{uV_- \oplus u^2V_+ \oplus \cdots}$$

What do you wish to accomplish? You know you can recover X where $\operatorname{Ker} j_+^* j \cap \operatorname{Ker} j_-^* j = 0$.

~~You need to define~~

You ultimate want the Schur expansion

$$T \otimes Y \supseteq \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \supset \underbrace{\begin{pmatrix} 1 \\ c \end{pmatrix} Y}_{V} \supset \underbrace{\begin{pmatrix} a \\ b \end{pmatrix} X}_{W} \subset \underbrace{\begin{pmatrix} c^* \\ 1 \end{pmatrix} Y}_{V^\circ}$$

$$0 = \begin{pmatrix} 1 \\ c \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1 - c^* y_2 \qquad ca = b \qquad c^*b = a$$

$$b^*c = a^*$$

Somehow you want to get $\begin{pmatrix} 1 \\ ba^* \end{pmatrix} X \subset \begin{matrix} X \\ \oplus \\ X \end{matrix}$

---

both + understand the UHP case especially perturbation, determinant

Now it's time to understand dB theory. Start with $Y, \gamma \ni \frac{\gamma - \gamma^*}{i}$ rank $1 > 0$. Begin with $\dim Y < \infty$.

then get $\quad Y \xrightarrow{\nu} H^+ \cap SH^- \qquad y \mapsto \nu \frac{1}{\omega - \gamma^*} y = \hat{y}(\omega)$

Check

$$\int |\hat{y}(\omega)|^2 \frac{d\omega}{2\pi} = \int \left(\nu \frac{1}{\omega - \gamma^*} y\right)^* \left(\nu \frac{1}{\omega - \gamma^*} y\right) \frac{d\omega}{2\pi} \qquad \frac{\gamma - \gamma^*}{i}$$

$$= \int y^* \frac{1}{\omega - \gamma} \boxed{\nu^* \nu} \frac{1}{\omega - \gamma^*} y \frac{d\omega}{2\pi} = \oint y^* \overset{2\pi i}{\cancel{\frac{\gamma - \gamma^*}{i}}} \frac{1}{\cancel{\omega - \gamma^*}} y \frac{d\omega}{2\pi}$$

524 Now factor $S(\omega)$ into ~~[illegible]~~ $\frac{\overline{E(\bar{\omega})}}{E(\omega)} = E^{\#}(\omega)$ where

$$E(\omega) = \prod_1^n (\omega - \bar{a}_i) \qquad a_i \in UHP.$$

Then $$Y \xrightarrow{\sim} H^+ \cap \frac{E^{\#}}{E} H^- \xrightarrow{\cdot E} EH^+ \cap E^{\#} H^-$$

$$(\omega + i) H^+ \text{ contains } 1 = (\omega+i)\frac{1}{\omega+i}$$

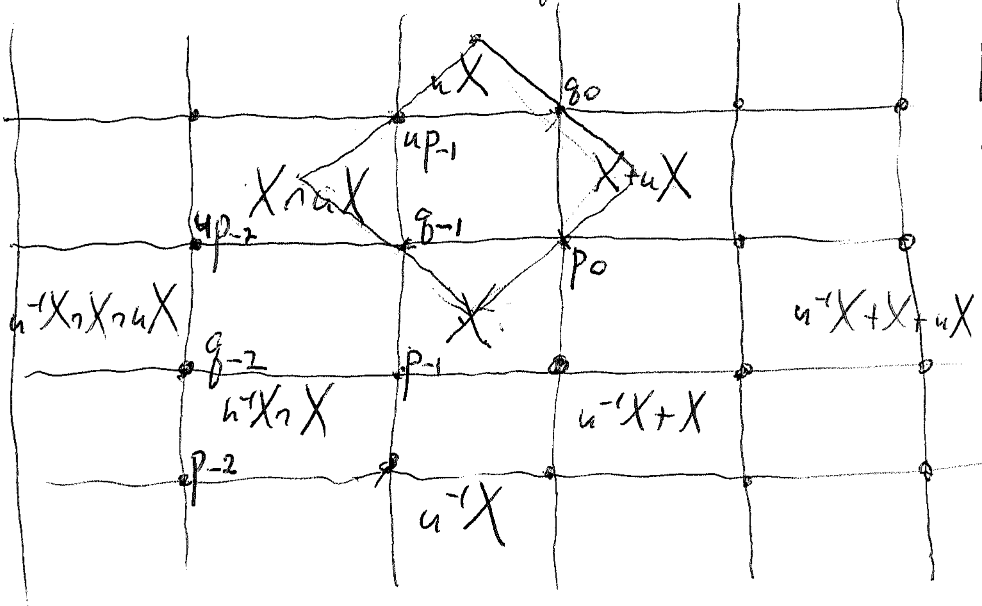
~~Probable~~ Probably true that $EH^+ \cap E^{\#} H^- =$ polys in λ of degree $< n$.

The other point is that $E(\omega) = \det(\omega - \gamma^*)$.

There's another idea arising, namely that the embedding is some kind of ~~[illegible]~~ quasi-det. ~~[illegible]~~ ~~[illegible]~~

Now ~~you~~ can ask about de B's ~~[illegible]~~ filtration. Clearly you have ~~[illegible]~~ the ~~[illegible]~~spaces $P_0 \subset P_1 \subset \cdots \subset P_{n-1}$ of polys. How does this arise from your space Y.

~~[illegible]~~ Study a contraction c such that $1 - c^*c$, $1 - cc^*$ have rank 1. Construct $S(z)$, you want the operator ~~se~~ picture of the Schur expansion. ~~You want the operator picture~~ Use E, u to form the bfiltration.



$$p_0 = up_{-1} + t_0 q_{-1}$$
$$q_0 = \bar{t}_0 u p_{-1} + q_{-1}$$

There might be a better way to proceed, namely build from the terminals, as in orthogonal polys. You know this should be exhaustive $\Longleftrightarrow$ no states with 0 scattering data.

$E: \quad \oplus V_- \oplus jX \oplus V_+ \oplus uV_+$ $\qquad \xi_+, \xi_-$ unit vector sp. V_+, uV_-

outgoing space $L_+ = V_+ \oplus uV_+ \oplus \cdots = \bigoplus_{n\geq 0} \mathbb{C}u^n\xi_+$

incoming $L_- = \oplus u^{-1}V_- \oplus V_- = \bigoplus_{n\leq 0} \mathbb{C}u^n\xi_-$. Try to fit this into the mould you know. Here you work in

$$Y = X \oplus \underbrace{V_+}_{\mathbb{C}\xi_+} = \underbrace{uV_-}_{\xi_-} \oplus uX$$

Look – the program is clear, you are looking at a partial unitary completed by a $\overset{\text{dissipative}}{\text{boundary}}$ condition, there should be a p_n, q_n ~~the~~ system independent of this bdry condition

Case you know: $E = Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$, $c_h = ba^* + \xi_- h \xi_+^*$

$u = ba^* + \xi_-\xi_+^*$. Let $p_0 = \xi_+$, $q_0 = \xi_-$. You might be making the same error as before, the wrong end. So far have $X \overset{p_0}{\subset} Y \overset{q_0}{\supset} uX$

Let's organize exactly what we want. There's a general process for going from $(h_n)_{n\in\mathbb{Z}}$ to a unitary. This construction is like the Schur expansion – you write it down.

You want to show it corresponds to the scatt. situation. Directional problem.

As n increases you get further down into X. You begin with X, c. Suppose you start with.

[redacted]

$$\begin{array}{cccccc}
\ell^2\otimes V_+ & & \cdots & u^{-1}V_+ & V_+ & uV_+ \\
 & & f & & \| & \| \\
E & & u^{-1}V_- & X & V_+ & uV_+ \\
\| & & \| & & g & \\
\ell^2\otimes V_- & & u^{-1}V_- & V_- & uV_- & u^2V_-
\end{array}$$

what's the best approach. I think this might be the orthogonal complement of X. Perhaps mimic orthog polys.

Start with $Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$

Partial unitary. Will try to get deeper into X.

[redacted] Use model $Y = H^+/zS(z)H^+ = H^+ \cap zSH^-$

$aX = X = H^+ \cap SH^-$ $\quad \xi_+ = S$

$bX = zX = zH^+ \cap zSH^-$ $\quad \xi_- = 1$.

$h = S(0) = (1, S)$ $\quad S - h \in bX$ $\quad 1 - \bar{h}S \in Y$

$1 - \bar{h}S \in aX$? $\quad (1-\bar{h}S, S) = (1,S) - h = 0$

$p_0 = S$ $\quad p_1 = p_0 - h q_0$

$q_0 = 1$ $\quad q_1 = q_0 - \bar{h} p_0$

Consider (X, c). $\quad aX = \operatorname{Ker}(1-c^*c)$ $\quad a = \text{inc } X \hookrightarrow Y$

$bX = \operatorname{Ker}(1-cc^*)$ $\quad b = ca$

Get $c = ba^* \oplus \xi_- h \xi_+^* : aX \oplus \mathbb{C}\xi_+ \longrightarrow bX \oplus \mathbb{C}\xi_-$

~~Important~~. Your idea is to put $p_0 = \xi_+$, $q_0 = \xi_-$

then $h = q_0^* p_0 = \xi_-^* \xi_+$ $\quad q_0^*(p_0 - hq_0) = h - h = 0$

527 ∴ you find $p_0 - hq_0 \in (\mathbb{C}\xi_+)^\perp = bX$

get $\exists!\, p_1 \ni \quad cp_1 = p_0 - hq_0$

Again consider (Y, c) put $X = \operatorname{Ker}(1 - c^*c)$ $\quad \underset{\text{inc.}}{a : X \to Y}$

$Y = aX \oplus \mathbb{C}\underbrace{\xi_+}_{\text{unit v.}} = bX \oplus \mathbb{C}\underbrace{\xi_-}_{\text{unit v.}}$ $\quad bX = caX = \operatorname{Ker}(1 - cc^*)$

$c = ba^* + \xi_- h \xi_+^*$

$c\xi_+ = \xi_- h$

Do this inside~~s~~ $\quad E = \oplus \underbrace{V_-} \oplus \mathcal{J} \oplus \underbrace{V_+} \oplus \cdots$

$v_-\xi_- = (u^{-1}\mathcal{J} - \mathcal{J}c^*)\xi_- \qquad v_+\xi_+ = (u\mathcal{J} - \mathcal{J}c)\xi_+$

$uv_-\xi_- = (\mathcal{J} - u\mathcal{J}c^*)\xi_-$ ~~to its easy~~

But you know $Y = aX + \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX.$

$c\xi_+ = h\xi_- \qquad c^*\xi_- = \bar{h}\xi_+$

$\boxed{v_+\xi_+ = u\mathcal{J}\xi_+ - h\mathcal{J}\xi_-}$

$v_-\xi_- = u^{-1}\mathcal{J}\xi_- - \bar{h}\mathcal{J}\xi_+$

$\boxed{uv_-\xi_- = \mathcal{J}\xi_- - \bar{h}u\mathcal{J}\xi_+}$

Inside Y you have

There are non-trivial calculations. Consider

$\underbrace{Y = aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX}$ $\qquad c = ba^* + \xi_- h \xi_+^*$

$\qquad c^* = ab^* + \xi_+ \bar{h} \xi_-^*$

any partial unitary, so that the inner product $\xi_-^*\xi_+$ can be arbitrary.

Analysis of ~~the~~ a boundary condition

$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$, $c = ba^* + \xi_- h \xi_+^*$

$Z = \overline{jY + ujY} = jY \oplus \overline{\nu_+ Y} = u\overline{\nu_- Y} \oplus ujY$

$c\xi_+ = h\xi_-$

$\nu_+ y = (uj - jc)y \qquad j^*\nu_+ = 0$

$(\nu_+ y, \nu_+ y) = (ujy, (uj - jc)y) = \|y\|^2 - \|cy\|^2$

$c = ba^* + \xi_- h \xi_+^* \qquad 1 - c^*c = \cancel{aa^*} + \xi_+(1 - |h|^2)\xi_+^*$

$c^* = ab^* + \xi_+ \bar{h} \xi_-^* \qquad \therefore\ \nu_+(aX + \xi_+ m) = \nu_+ \xi_+ m$

so $Z = jY \oplus \mathbb{C}\nu_+\xi_+ = \mathbb{C}u\nu_-\xi_- \oplus ujY$

$= jaX \oplus \mathbb{C}\xi_+ \oplus \mathbb{C}\nu_+\xi_+ = jbX \oplus \mathbb{C}\xi_- \oplus \mathbb{C}\nu_-\xi_-$

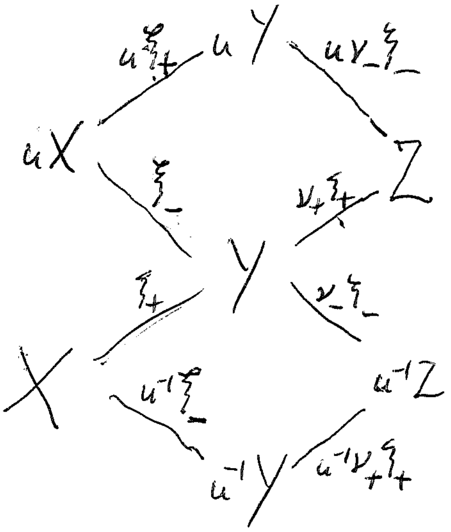
What aims? Describe s

Begin with $Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$ $\quad \|\xi_\pm\| = 1 \quad a^*a = b^*b = 1_X$

$c = ba^* + \xi_- h \xi_+^*$ where $|h| < 1$.

handle by eigenvector equation? ~~[illegible]~~

orthogonal polys. Work in E $\quad j : Y \hookrightarrow E$



$(\xi_-, u\xi_+) = (\xi_-, c\xi_+)$
$= (\xi_-, h\xi_-) = h$

$\nu_+\xi_+ = au\xi_+ + bu\nu_-\xi_-$

$\xi_- = -\bar{b}u\xi_+ + au\nu_-\xi_-$

$$\boxed{au\nu_-\xi_- = \xi_- + \bar{b}u\xi_+}$$

$a + \frac{|b|^2}{a} = \frac{1}{a}$

$\nu_+\xi_+ = au\xi_+ + b\left(\frac{1}{a}\xi_- + \frac{\bar{b}}{a}u\xi_+\right)$

$\boxed{a\nu_+\xi_+ = b\xi_- + u\xi_+} \implies 0 = b + (\xi_-, u\xi_+) = b + h$

You seek the operator picture of the Schur expansion. ~~[illegible] You have~~

You start with X, γ let $Y = gX \oplus V_+ = V_- \oplus ugX$.

$\therefore Y = \overline{gX + ugX}$ $a = g|X$, $b = ug|X$.

$a^*b = g^*ug = c$. Assume $O(n)$ type.

Then by structure for such Kronecker modules get $X = P_{n-1}$ $Y = P_n$ $a = $ incl of P_{n-1}, $b = z\cdot : P_{n-1} \to P_n$

and $V_+ = \mathbb{C}\xi_+$ $V_- = \mathbb{C}\xi_-$

Now here $\xi_+ = \tilde{p}_n$ $\xi_- = \tilde{q}_n$ you don't require p_n to be monic, nor q_n to have constant term 1, but you just visualize the recursion relations. So you ought to be able to find $\tilde{p}_{n-1}, \tilde{q}_{n-1}$ such that

$$\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z\tilde{p}_{n-1} \\ \tilde{q}_{n-1} \end{pmatrix}$$

and to continue.

So start with $Y = aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX$ where $\|\xi_\pm\| = 1$. First invariant is $\xi_-^*\xi_+ = (\xi_-, \xi_+)$. If this vanishes then $\xi_+ \in bX$, and $\xi_- \in aX$. Set $X' = X \cap$

$$\underbrace{X \oplus V_+}_{} = \overbrace{V_- \oplus uX}^{}$$

If $V_- \perp V_+$ then $V_+ \subset uX$ and you want $(u^{-1}V_+)^\perp \subset X$

You want y such that $y \perp V_+$ so $y \in X$ and then also $uy \perp \xi_+$ ~~[illegible]~~ i.e. $uy \in X$ $\therefore y \in X \cap u^{-1}X$.

530 So you have $X' = u^{-1}X \cap X \perp u^{-1}\xi_+, \xi_+$

Then

$$\begin{tikzcd}
 & \overset{=P_{n-1}}{uX} \arrow[dr, hook, "\xi''\,1"] & \\
\underset{=P_{n-2}}{X \cap uX} \arrow[ur, hook, "z^n = \xi_+"] \arrow[dr, "\xi_-"', "1''"] & & Y = P_n \; ? \\
 & \underset{P_{n-1}}{X} \arrow[ur, hook, "\xi_+ = z^n"'] &
\end{tikzcd}$$

so how do I get this to work? You have

$$Y = X \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus uX$$

form $W = X \cap uX = \{\xi_+, \xi_-\}^\perp$

$$~~X \cap uX~~ + \mathbb{C}\xi_+ + \mathbb{C}\xi_- = Y$$

$$\begin{tikzcd}
 & & uP_{n-1} = uX \arrow[dr, "q_n"] & \\
 & uP_{n-2} = X \cap uX \arrow[ur, hook, "up_{n-1}"] \arrow[dr, "q_{n-1}"] & & P_n = Y \\
uP_{n-3} = u^{-1}X \cap X \cap uX \arrow[ur, hook, "up_{n-2}"] \arrow[dr, "q_{n-2}"] & & P_{n-1} = X \arrow[ur, hook, "p_n"] & \\
 & P_{n-2} = u^{-1}X \cap X \arrow[ur, hook, "p_{n-1}"] \arrow[dr, "u^{-1}q_{n-1}"] & & \\
 & & u^{-1}P_{n-1} = u^{-1}X &
\end{tikzcd}$$

$$\begin{tikzcd}
 & uX \arrow[dr, dash, "\xi_-"] & \\
uW = X \cap uX \arrow[ur, dash, "u\xi'_+"] \arrow[dr, dash, "\xi'_-"'] & & Y \\
 & X \arrow[ur, dash, "\xi_+"'] & \\
W = u^{-1}X \cap X \arrow[ur, dash, "\xi'_+"] \arrow[dr, dash] & & \\
 & u^{-1}X &
\end{tikzcd}$$

$$\xi'_- = \xi_- - \xi_+ \overbrace{\xi_+^* \xi_-}^{\alpha}$$

$$1 = \|\xi_-\|^2 = \|\xi'_-\|^2 + |\alpha|^2$$

$$\xi'_- = \frac{1}{\sqrt{1 - |\xi_+^* \xi_-|^2}} \left(\xi_- - \xi_+ \xi_+^* \xi_- \right)$$

$$u\xi'_+ = \frac{1}{\sqrt{1 - |\xi_-^* \xi_+|^2}} \left(\xi_+ - \xi_- \underbrace{\xi_-^* \xi_+} \right)$$

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$$\begin{pmatrix} u\xi'_+ \\ \xi'_- \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & -\overset{h}{\boxed{\xi_-^*\xi_+}} \\ -\underset{\bar{h}}{\boxed{\xi_+^*\xi_-}} & 1 \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} u\xi'_+ \\ \xi'_- \end{pmatrix} \qquad h = \xi_-^*\xi_+$$

Repeat this. Start with a partial unitary

$Y = aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX$ Let $h = (\xi_-, \xi_+) = \xi_-^*\xi_+$

$W = aX \cap bX = Y \cap \{\xi_+, \xi\}^\perp$

$$\begin{tikzcd}
 & bX \arrow[dr, "\xi_-"] & \\
W \arrow[ur, "u\xi'_+"] \arrow[dr, "\xi'_-"'] & & Y \\
 & aX \arrow[ur, "\xi_+"'] & \\
u^{-1}W \arrow[ur, "\xi'_+"'] & &
\end{tikzcd}$$

$$\frac{1}{\sqrt{1-|h|^2}}(\xi_+ - \xi_- h) = u\xi'_+$$

$$\frac{1}{\sqrt{1-|h|^2}}(\xi_- - \xi_+ \bar{h}) = \xi'_-$$

$$\frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} u\xi'_+ \\ \xi'_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} u\xi'_+ \\ \xi'_- \end{pmatrix}$$

This works simply.

Let's begin again with (X, c) form $Y = \overline{jX + ujX}$

$= \underbrace{jX}_{aX} \oplus V_+ = V_- \oplus \underbrace{ujX}_{bX}$, choose $\xi_\pm$.

then $a^*b = j^*uj = c$.

$$1 - c^*c = 1 - b^*aa^*b = b^*\xi_+\xi_+^*b$$
$$1 - cc^* = 1 - a^*bb^*a = a^*\xi_-\xi_-^*a$$

$b^*\xi_+ \sim u\xi'_+ \qquad a^*\xi_- \sim \xi'_-$

532 What are you going to find out from the scattering.

$$\begin{array}{ll}
L^2(S^1,V_+): & \qquad\quad \underline{u^{-2}V_+ \oplus u^{-1}V_+ \oplus}\ V_+ \oplus uV_+ \\
 & \qquad\qquad\quad \cap \qquad\qquad\quad \| \qquad \| \\
E: & \overline{\oplus u^{-2}V_- \oplus u^{-1}V_- \oplus jX}\ \underline{\oplus\ V_+ \oplus uV_+} \\
 & \quad \| \qquad\qquad \| \qquad\qquad\qquad \cup \\
L^2(S^1,V_-): & \oplus u^{-2}V_- \oplus u^{-1}V_- \oplus \overline{V_- \oplus uV_- \oplus u^2V_-}
\end{array}$$

So what can you actually say? Ultimately you get a repn. by pairs $\binom{f}{g}$ ~~[illegible]~~ $f \in L^2(S^1,V_+)$ and $g \in L^2(S^1,V_-)$, probably

$$\|j_+f + j_-g\|^2 = \|f + \underbrace{j_+^*j_-}_{S^*}g\|^2 + \|g\|^2 - \|S^*g\|^2$$

$$= \|\underbrace{j_-^*j_+}_{S}f + g\|^2 + \|f\|^2 - \|Sf\|^2$$

$$E = H^2_-(S^1,V_-) \oplus jX \oplus H^2_+(S^1,V_+)$$

other approach $\qquad X \xrightarrow{(j_+^*j,\ j_-^*j)} H^2_-(S^1,V_+) \oplus H^2_+(S^1,V_-)$

$$j_+^*jx = \nu_+\left(\frac{1}{z-c}x\right)$$

~~[illegible]~~

$$j_-^*jx = \nu_-\left(\frac{1}{1-zc^*}x\right)$$

Discuss: ~~[illegible] the simplest thing to do~~

~~[illegible]~~ Problems of S its properties & reconstruction

You start with X, c ~~Problem of reconstruc~~

Eigenvalue equation.

533 So what comes next? You have to put the whole picture together. The direction to go is to work inside X and to ignore the dilation E as much as possible.

$$(\cancel{Y}, c) \qquad \cancel{X} = \begin{tikzcd} \mathrm{Ker}(1-c^*c) \arrow[r, shift left, "c"] & \mathrm{Ker}(1-cc^*) \arrow[l, shift left, "c^*"] \end{tikzcd} \quad \text{unitary}$$

$$c(1-c^*c) = (1-cc^*)c \qquad \begin{tikzcd} \mathrm{Im}(1-c^*c) \arrow[r, shift left] & \mathrm{Im}(1-cc^*) \arrow[l, shift left] \end{tikzcd}$$

~~You are working out~~

Viewpoint: Given (X, c) have $\nu_+\left(\frac{1}{z-c}x\right)$, $\nu_-\left(\frac{1}{1-zc^*}x\right)$ in $H^2_-(S^1, V_+)$ and $H^2_+(S^1, V_-)$

$$\nu_+\left(\frac{1}{z-c}x\right) = \sum_{n\geq 0} z^{-1-n}\nu_+(c^n x)$$

You probably want to go the other way

assumption =

$$X \xleftarrow{\quad} H^2_+(S^1, V_-) \times \overset{z^{-1}}{H^2_-}(S^1, V_+)$$

and you ought to be able to ~~handle~~ describe the kernel using $S(z): V_+ \to V_-$

$$\begin{tikzcd} H^2_+(S^1, V_-) + H^2(S^1, V_+) \subset L^2(S^1, V_-) + L^2(S^1, V_+) \subset E \arrow[d, "j^*"] \\ X \end{tikzcd}$$

---

what's the map. $X \leftarrow H^2_+(S^1, V_-) = V_- \oplus zV_- \oplus z^2V_- \oplus \cdots$

$z^n\xi_-$ goes to $j^*u^n\boxed{\xi_-}$? you need to put V_- inside of X

Basically you have to get back to working inside X. Replace X by Y. You have (Y, c), get description

$$c = ba^* + \xi_- h \xi_+^*$$

$$\xi_+ : V_+ \xrightarrow{\sim} \mathrm{Ker}(a^*)$$

$$\xi_- : V_- \xrightarrow{\sim} \mathrm{Ker}(b^*)$$

$$\|hx\| < \|x\|$$

Go over $Y = aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX$

$$c_h = ba^* + \xi_- h \xi_+^*$$
$$c_h^* = ab^* + \xi_+^* \bar{h} \xi_-^*$$

$$\xi_-^*\left(\frac{1}{1-zc_h^*} y\right) \qquad \xi_+^*\left(\frac{1}{z-c_h} y\right)$$
$$\| \qquad\qquad\qquad \|$$
$$f_-^* f y \qquad\qquad f_+^* f y$$

---

need relation between contractions and partial unitaries.

$c_0 = ba^*$ $\quad c_0^* = ab^*$

Do perturbation stuff. $Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$

$a^*a = b^*b = 1.$

$$c_h = ba^* + \xi_- h \xi_+^* \qquad 1 - c_h^* c_h = \xi_+ (1-h^*h)\xi_+^*$$
$$c_h^* = ab^* + \xi_+ h^* \xi_-^* \qquad 1 - c_h c_h^* = \xi_-(1-hh^*)\xi_-^*$$

embedding really the incoming rep. $\xi_-^*\left(\frac{1}{1-zc_h^*}\right)$

$$= \xi_-^* \frac{1}{1-zab^* - z\xi_+ h^* \xi_-^*} = \xi_-^* \frac{1}{1-zc_0^*} + \left(\xi_-^* \frac{1}{1-zc_0^*} z\xi_+ h^*\right)\xi_-^* \frac{1}{1-zc_0^*} + \cdot$$

$$= \frac{1}{1 - zS_0(z)h^*} \; \xi_-^* \frac{1}{1-zc_0^*}$$

$$S_h = \frac{1}{1-zS_0 h^*} S_0 \qquad 1 + zS_h h^* = \frac{1}{1-zS_0h^*} zS_0 h^* + 1$$

$$\boxed{1 + zS_h h^* = \frac{1}{1-zS_0 h^*}}$$

$$-\delta \log \det(1 - zc_h^*) = \operatorname{tr} \frac{1}{1-zc_h^*}(+z\xi_+ \delta h^* \xi_-^*)$$
$$= \operatorname{tr}\left(\xi_-^* \frac{1}{1-zc_h^*} \xi_+ \, z\delta h^*\right) = S_h z \delta h^*$$

$$-\delta \log(1 - S_0 z h^*) = \frac{1}{1-S_0 z h^*} S_0 z \delta h^* = S_h z \delta h^*$$

$$\frac{\det(1-zc_h^*)}{\det(1-zc_0^*)} = 1 - S_0 z h^*$$

$$A\frac{1}{A-B} = 1 + B\frac{1}{A-B}$$
$$A = A - B + B$$

$$(za-b)x = -y + \hat{y}(z)\xi_-$$

$$\Gamma = \begin{pmatrix}a\\ b\end{pmatrix}X + \begin{pmatrix}0\\ \xi_-\end{pmatrix}\mathbb{C} \hookrightarrow \begin{matrix}Y\\ \oplus\\ Y\end{matrix} \xrightarrow{(z\ \ -1)} Y$$

$$\begin{pmatrix}a\\ b\end{pmatrix}x + \begin{pmatrix}0\\ \xi_-\end{pmatrix}(-t) \longmapsto (za-b)x + t\xi_- = y$$

Compare $c_0 = ba^*$ with $c_h = ba^* + \xi_- h \xi_+^*$

$c = ba^*$
$c^* = ab^*$

$$\begin{pmatrix}a\\ b\end{pmatrix}X \Big| \ \Gamma_{c_h} = \begin{pmatrix}a\\ b\end{pmatrix}X + \begin{pmatrix}\xi_+\\ h\xi_-\end{pmatrix}\mathbb{C} \ ?$$

Work in $Y \oplus Y$. You want $\xi_-^* \frac{1}{1-zc^*} y$. You want to consider

$$\begin{pmatrix}c^*\\ 1\end{pmatrix}Y = \begin{pmatrix}a\\ b\end{pmatrix}X + \begin{pmatrix}h^*\xi_+\\ \xi_-\end{pmatrix}t \xrightarrow{(z\ \ -1)} Y$$

$$z\overset{c_h^* y_1}{\left(ax + \xi_+ h^* t\right)} - \overset{y_1}{\left(bx + \xi_- t\right)} = \lambda^{-1} y$$

$c = ba^*$
$c^* = ab^*$

solve this for $\begin{pmatrix}a\\ b\end{pmatrix}x + \begin{pmatrix}\xi_+ h^*\\ \xi_-\end{pmatrix}t = \begin{pmatrix}c_h^* y_1\\ y_1\end{pmatrix}$

then and apply ξ_-^* to y_1

What's going on? You are calculating $\xi_-^*\left(\frac{1}{1-zc^*}y\right)$

But to get the ism. embedding you need ν_- not ξ_-^*, and $\nu_-^*\nu_- = 1 - cc^* = \underbrace{1 - ba^*ab^*}_{\xi_-\xi_-^*} - \xi_- h h^* \xi_-^*$

$= \xi_- (1-hh^*)\xi_-^*$ $\qquad \nu_- = (1-hh^*)^{1/2}\xi_-^*$

This raises the analyticity question again.

536 What can you hope to do? Derive the perturbation formulas without using the geometric series.

$$\frac{1}{1-zc_h^*} = \frac{1}{1-zc_0^*} \overset{\frac{1}{1-zc_0^*}+}{\xi_+} z h^* \xi^* \frac{1}{1-zc_h^*}$$

$$\xi^*_- \frac{1}{1-zc_h^*} = \overset{\xi^*_- \frac{1}{1-zc_0^*}+}{\xi^*_-} \frac{1}{1-zc_0^*} \xi_+ z h^* \, \xi^*_- \frac{1}{1-zc_h^*}$$

$$R = R_0 + S_0 z h^* R \qquad \therefore R = \frac{1}{1-S_0 z h^*} R_0$$

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A-B}$$

How to handle this stuff with $1-h^*h$ etc.

Example. $\|c\| < 1$ $\qquad \rho_n = \begin{cases} c^n & n \geq 0 \\ c^{*-n} & n \leq 0 \end{cases}$

$$\int z^n \rho \frac{d\theta}{2\pi} = \rho_n \qquad \rho = \sum_{n\geq 0} z^{-n} c^n + \sum_{n \geq 1} z^n c^{*n}$$

$$j^* u^n j = \int z^n \rho \frac{d\theta}{2\pi}$$

$$\rho = \frac{1}{1-zc^*}(1-c^*c)\frac{1}{1-z^{-1}c} = \frac{1}{1-z^{-1}c}(1-cc^*)\frac{1}{1-zc^*}$$

$$\begin{tikzcd} L^2(S^1, \frac{d\theta}{2\pi}; V_-) & E = L^2(S^1, \rho\frac{d\theta}{2\pi}, X) \arrow[l] \arrow[r, "\sim"] & L^2(S^1, \frac{d\theta}{2\pi}; V_+) \\ \nu_-\left(\frac{1}{1-zc^*} f\right) & f(z) \arrow[l, mapsto] \arrow[r, mapsto] & \nu_+\left(\frac{1}{1-\underset{\substack{\uparrow \\ \text{or } z^{-1}c}}{z^{-1}c}} f\right) \end{tikzcd}$$

Let's handle the case where $V_\pm$ are not just lines.

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$c = ba^* + \xi_- h \xi_+^*$ here $\xi_+ \xi_+^* = 1 - aa^*$

$c^* = ab^* + \xi_+ h^* \xi_-^*$ $\xi_- \xi_-^* = 1 - bb^*$

say $\xi_+ : \mathbb{C}^d \xrightarrow[\sim]{} \operatorname{Ker} a^*$ etc. Now how to handle this? What is $\nu_\pm : X \longrightarrow V_\pm$? $\|\nu_+ y\|^2 = \|y\|^2 - \|cy\|^2$

$\nu_+^* \nu_+ = \xi_+ (1 - h^* h) \xi_+^*$ so can take $V_+ = \mathbb{C}^d$

and $\nu_+ = (1 - h^* h)^{1/2} \xi_+^*$ $\nu_- = (1 - hh^*)^{1/2} \xi_-^*$

Then the incoming repn is

$$y \longmapsto \nu_- \frac{1}{1 - zc_h^*} y = (1 - hh^*)^{1/2} \xi_-^* \frac{1}{1 - zc_h^*} y$$

$$(1-hh^*)^{1/2} \xi_-^* \frac{1}{1 - zc_h^*} = {}_\wedge^{(1-hh^*)^{1/2}} \xi_-^* \frac{1}{1 - zc_0^*} + {}_\wedge^{(1-hh^*)^{1/2}} \xi_-^* \frac{1}{1 - zc_0^*} \left(-z\xi_+ h^* \xi_-^*\right) \frac{1}{1 - zc_h^*} \quad ?$$

$$S_0 = \xi_-^* \frac{1}{1 - zc_0^*} \xi_+$$

$$\xi_-^* \frac{1}{1 - zc_h^*} = \frac{1}{1 - S_0 z h^*} \xi_-^* \frac{1}{1 - zc_0^*}$$

$$\boxed{(1-hh^*)^{1/2} \xi_-^* \frac{1}{1 - zc_h^*} = (1-hh^*)^{1/2} \frac{1}{1 - S_0 z h^*} \xi_-^* \frac{1}{1 - zc_0^*}}$$

What is the scattering $S_h(z)$? Something like

$$(1 - c_h c_h^*)^{1/2} \frac{1}{1 - zc_h^*} (z - c_h) (1 - c_h^* c_h)^{1/2} \quad ?$$

$$L^2(S^1, \rho \tfrac{d\theta}{2\pi}, X) \longrightarrow L^2(S^1, \tfrac{d\theta}{2\pi}, X)$$

$$(1 - c^* c)^{1/2} \frac{1}{z - c} f \longleftarrow\!\!| \; f \longmapsto (1 - cc^*)^{1/2} \frac{1}{1 - zc^*} f$$

$$g \longmapsto (1 - cc^*)^{1/2} \frac{1}{1 - zc^*} (z - c) (1 - c^* c)^{-1/2} g$$

538 Scattering for a contraction. This time take (X, c), form $Y = \overline{aX + bX}$ where

$$\|ax_0 + bx_1\|^2 = \|x_0\|^2 + (x_0, a^*bx_1) + (a^*bx_1, x_0) + \|x_1\|^2$$

$$= \|x_0 + a^*bx_1\|^2 + \underbrace{(x_1, \underbrace{(1 - b^*a\,a^*b)}_{b^*(1-aa^*)b}x_1)}_{\|bx_1\|^2 - \|a^*bx_1\|^2}$$

$$a^*b = g^*ug = c$$

You might as well work inside E

$$Y = \overline{gX + ugX} = \overline{aX + bX}.$$

$$= gX \oplus V_+ = V_- \oplus ugX$$

$$\boxed{c = a^*b = g^*ug}$$

The scattering operator is

$$S(z) = (1-bb^*)(1-z\overset{?}{ab^*})^{-1}(1-aa^*) : V^+ \to V^-$$

How do I straighten this out?

Eigenvector equation

$$\eta = \begin{array}{l} + u^{-1}v_-^1 + \underbrace{ax_1 + v_+}_{} \\ \qquad\qquad\quad \| \\ \qquad\qquad \overline{v_- + bx} + uv_+^1 + u^2v_+^2 \end{array}$$

$$u\eta = z\eta$$

$$uv_+^1 = uv_+$$

$$uu^{-1}v_-^1 = zv_-$$

$$\frac{u}{b}ax_1 = zbx$$

$$\boxed{v_+^1 = z^{-1}v_+}$$

$$\boxed{v_-^1 = zv_-}$$

$$\boxed{x_1 = zx}$$

$$zax + v_+ = v_- + bx$$

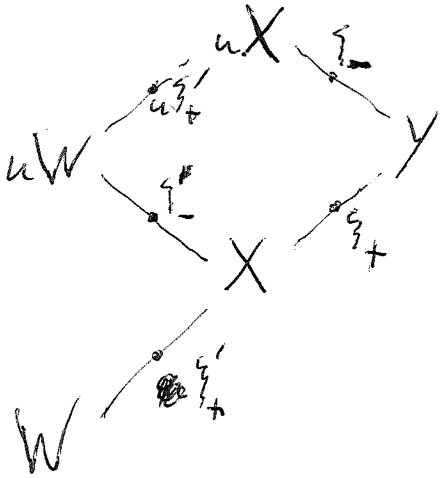
$$\boxed{(za - b)x = -v_+ + v_-}$$

$$(1 - zb^*a)x = b^*v_+ \qquad x = b^*(1 - zab^*)^{-1}v_+$$

$$v_- = \left(1 - \cancel{zab^*} + (\cancel{z}a - b)\,b^*\right)(1 - zab^*)^{-1}v_+$$

539 You have to keep straight c on X and the contraction c_0 on Y. $c = j^*uj = a^*b$ $c_0 = ba^*$. Two ways to proceed. up or down

Try going down. $uW = aX \cap bX = X \cap uX$



$$\xi_+ - \xi_-\overbrace{\xi_-^*\xi_+}^{h} = u\xi_+'(1-h^*h)^{1/2}$$

$$\xi_- - \xi_+\xi_+^*\xi_- = \underbrace{\xi_-'(1-hh^*)^{1/2}}_{\tilde{\xi}_-'}$$

$$\tilde{\xi}_-'^*\tilde{\xi}_-' = (\xi_-^* - \cancel{\xi_-^*\xi_+\xi_+^*})(\xi_- - \xi_+\xi_+^*\xi_-)$$

$$= \xi_-^*\xi_- - \xi_-^*\xi_+\xi_+^*\xi_-$$

$$1 - h\,h^*$$

$$\begin{pmatrix}\xi_+ & \xi_-\end{pmatrix}\begin{pmatrix}1 & -h^* \\ -h & 1\end{pmatrix} = \begin{pmatrix}u_*\xi_+' & \xi_-'\end{pmatrix}\begin{pmatrix}(1-h^*h)^{1/2} & 0 \\ 0 & (1-hh^*)^{1/2}\end{pmatrix}$$

$$\begin{pmatrix}1 & -h^* \\ -h & 1\end{pmatrix}\begin{pmatrix}(1-h^*h)^{-1/2} & \\ & (1-hh^*)^{-1/2}\end{pmatrix} = \begin{pmatrix}(1-h^*h)^{-1/2} & -h^*(1-hh^*)^{-1/2} \\ -h(1-h^*h)^{-1/2} & (1-hh^*)^{-1/2}\end{pmatrix}$$

So this works OKAY. How do you propose to understand the scattering? Ultimately you want to obtain the Schur expansion.

Idea: Go back to the eigenvector picture.

Try going up.

$$\begin{tikzcd}
 & & uY \arrow[dr, dash, "\xi'_-"] & \\
 & uX \arrow[ur, dash, "u\xi_+"] \arrow[dr, dash, "\xi_-"] & & Z \\
uW \arrow[ur, dash] \arrow[dr, dash] & & Y \arrow[ur, dash, "\xi'_+"'] & \\
 & X \arrow[ur, dash, "\xi_+"'] & & \\
W \arrow[ur, dash] & & &
\end{tikzcd}$$

On each of W, X, Y, Z you have a contraction induced by u, namely, $x \longmapsto ux \xmapsto[\text{on } X]{\text{proj}} j_X^* ux$

$$\begin{tikzcd}
 & & uX+u^2X \\
 & uX \arrow[ur, hook] \arrow[dr, hook, "b"] & \\
X \cap uX \arrow[ur, hook] \arrow[dr, hook] & & X+uX \\
 & X \arrow[ur, hook, "a"'] \arrow[dr, hook] & \\
u^{-1}X \cap X \arrow[ur, hook] & & u^{-1}X+X \\
 & u^{-1}X \arrow[ur, hook] &
\end{tikzcd}$$

~~[illegible]~~

Start with (X, c) define $aW = \mathrm{Ker}(1-c^*c)$

$bW = \mathrm{Ker}(1-cc^*)$, $ba^{-1} = c : aW \longrightarrow bW$

ξ_+ unit vector spanning $\mathrm{Ker}(a^* : X \longrightarrow W)$

ξ_- ________ $\mathrm{Ker}(b^* : \cancel{X} \longrightarrow W)$

$c = ba^* + \xi_- h \xi_+^*$ on X. $c_0 = ba^*$

541 You want the scattering associated to (X,c), need $\nu_+ : X \longrightarrow \nu_+ X$ completion with respect to $\|x\|^2 - \|cx\|^2 = (x, (1-c^*c)x)$

$$c^* = ab^* + \xi_+ h^* \xi_-^* \qquad c = ba^* + \xi_- h \xi_+^*$$

$$c^*c = aa^* + \xi_+ h^*h \xi_+^* \qquad 1-c^*c = \xi_+ (1-h^*h) \xi_+^*$$

$$cc^* = bb^* + \xi_- h h^* \xi_-^* \qquad 1-cc^* = \xi_-(1-hh^*)\xi_-^*$$

You can identify $\nu_+ : X \longrightarrow \overline{\nu_+ X}$ with $(1-h^*h)^{1/2}\xi_+^* : X \longrightarrow \overline{(1-h^*h)X}$, similarly $\nu_- : X \longrightarrow \overline{\nu_- X}$ close w $(1-hh^*)^{1/2}\xi_-^*$. Then the reps are $\nu_+ \frac{1}{z-c}x$

$$\left\|\nu_+ \frac{1}{z-c}x\right\|^2 = \sum_{n\geq 0} \underbrace{\left\|\cancel{z^{-1-n}}\,\nu_+ c^n x\right\|^2}_{\|c^n x - c^{n+1}x\|^2} \qquad \nu_- \frac{1}{1-zc^*}x$$

incoming rep. $x \longmapsto (1-hh^*)^{1/2}\xi_-^* \frac{1}{1-z(c_0^* + \xi_+ h^* \xi_-^*)}x$

$$= (1-hh^*)^{1/2}\xi_-^*\left\{\frac{1}{1-zc_0^*} + \frac{1}{1-zc_0^*}(z\xi_+h^*\xi_-^*)\frac{1}{1-zc^*}\right\}$$

~~$(1-hh^*)^{1/2}$~~ ξ^*

$$\underbrace{\xi_-^*\frac{1}{1-zc_h^*}} = \xi_-^*\frac{1}{1-zc_0^*} + \underbrace{\xi_-^*\frac{1}{1-zc_0^*}\xi_+ zh^*}\xi_-^*\frac{1}{1-zc_h^*}$$

$$\boxed{(1-S_0 z h^*)\xi_-^*\frac{1}{1-zc_h^*} = \xi_-^*\frac{1}{1-zc_0^*}}$$

$$(1-hh^*)^{1/2}\xi_-^*\frac{1}{1-zc_h^*} = (1-hh^*)^{1/2}\frac{1}{1-S_0zh^*}\,\xi_-^*\frac{1}{1-zc_0^*}$$

Is there some way to check this? Like whether the operator $(1-hh^*)^{1/2}\frac{1}{1-S_0zh^*}$ is unitary

$$\frac{1}{1-S_0 z h^*} = \frac{1}{1-\frac{p_0}{q_0} z h^*} = \frac{q_0}{q_0 - h_0 z p_0}$$

This does change the denominator. ~~There's too much happening.~~ You need to organize this stuff. Try new idea: Instead of embedding X into $H^2_+(S^1, V_-)$, try handling ~~[illegible]~~ X as a quotient of $H^2_+(S^1, V_-)$

$$\begin{array}{cccccccc} \cdots\oplus & uV^- & \oplus & jX & \oplus & & & \\ & \| & & & & & & \\ \cdots\oplus & uV^- & \oplus & \overbrace{V_- \oplus uV_- \oplus \cdots} & & & & \end{array}$$

so you look at $j^*: V_- \to X$.

See if this helps. You want to work in X. You ought to see what can

$$\left(\left(jX, \sum_{k=0}^{\infty} u^k \nu_- x_k\right) = \left(jx, j_- \sum z^k \nu_- x_k\right)\right)$$

$$= \left(jx, \sum_0^{\infty} u^k (j - u c^*) x_k\right)$$

$$= \sum_0^{\infty} \left(x, c^k(1-cc^*) x_k\right)$$

$$= \sum_0^{\infty} \left(c^{*k}x, (1-cc^*) x_k\right)$$

$$= \sum_0^{\infty} \left(\nu_- c^{*k}x, \nu_- x_k\right) = \left(\sum_{k\geq 0} z^k \nu_- c^{*k} x, \sum_{k=0}^{\infty} z^k \nu_- x_k\right)$$

$$j_-^* jx = \sum_{k\geq 0} z^k \nu_-(c^{*k}x) = \nu_- \frac{1}{1-zc^*} x$$

543 New idea: Given (X, c)

set $\nu_- : X \longrightarrow \overline{\nu_- X} = V_-$ completion $\|\nu_- x\|^2 = \|x\|^2 - \|c^* x\|^2$
$= (x, (1-cc^*)x)$

define $H^2_+(S^1, V_-) \longrightarrow X$

$$\begin{matrix} \cup \\ \bigoplus_{k\geq 0} z^k \nu_- x_k \longmapsto \sum_{k\geq 0} c^k \overbrace{(1-cc^*)}^{\nu_-^* \nu_-} x_k \end{matrix}$$

$$\sum \|x_k\|^2 - \|c^* x_k\|^2 < \infty$$

$$\sum_{k\geq 0}' (x_k, (1-cc^*)x_k) < \infty .$$

idea: $x \longmapsto \nu_-\left(\frac{1}{1-zc^*} x\right) = \sum_{k\geq 0} z^k \nu_-(c^{*k} x)$

$j_-^* j : X \longrightarrow H^2_+(S^1, V_-)$

$$\|j_-^* j x\|^2 = \sum_{k\geq 0} \|c^{*k} x\|^2 - \|c^{*k+1} x\|^2 = \|x\|^2 - \lim_{k\to\infty} \|c^{*k} x\|^2$$

compute adjoint to be $\sum' z^k \nu_- x_k \longmapsto \underbrace{\sum c^k (1-cc^*) x_k}_{c^k (1-cc^*)^{1/2} (1-cc^*)^{\frac{1}{2}} x_k}$

Is it true that $\|c^{*k} x\| \longrightarrow 0 \ \forall x \iff \sum_{n\geq 0} j^* u^n V_- = X$.

$$j^* u^n \nu_- X = j^* u^n (j - u j c^*) X = (c^n - c^{n+1} c^*) X$$

You're probably being stupid. Go back to

$$\begin{matrix} & u^{-1}V_- \oplus & jX \oplus V_+ & \oplus \, uV_+ \\ & & & \cup \\ \| & \| & & \\ & u^{-1}V_- \oplus & V_- \oplus uV_- & \oplus \, u^2 V_- \end{matrix}$$

You can project: $E \xrightarrow{j_-^*} L^2(S^1, V_-)$, $j_-^* j =$ the projection of jX onto $L^2(S^1, V_-)$ or $H^2_+(S^1, V_-)$. The proj. has same norm $\iff jX \subset H^2_+(S^1, V_-)$.

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$$X \overset{j}{\hookrightarrow} (jX \oplus V_+ \oplus \cdots) \overset{j_+}{\hookleftarrow} H^2_+(S^1, V_-)$$

look at map $j^* j_+$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "f^*"] \end{tikzcd}$$

$$\overline{f^*Y} = X \iff \Big(\underset{\underset{(fx, y)}{\|}}{(x, f^*Y)} = 0 \Rightarrow x = 0 \Big) \iff f \text{ inj}$$

$$(fx, y) = 0$$

Thus the adjoint $j^* j_+$ of $j_+^* j$ is injective $\iff$ $j^* j_+$ has dense image. i.e.

$$\overline{\sum c^k (1 - cc^*) X} = X.$$

Somehow you are trying to understand if $V_\pm$ generate X. Lets try to straighten this out.

Suppose you start with $Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$

Actually,

$$\cdots \oplus u^{-1}V_- \oplus \underbrace{X \oplus V_+} \oplus \cdots$$

$$\overbrace{V_- \oplus uX} \oplus \cdots$$

You want to know if ξ_+, ξ_- generate E under u, u^{-1} action

For Monday's lecture discuss $|S| = 1$

545 Given (X, c), have dilation E

form uX

$uW = X \cap uX$ $\qquad X + uX = Y$

X

$W = u^{-1}X \cap X$ $\qquad u^{-1}X + X = u^{-1}Y$

$u^{-1}X$

$\xi_+ : V_+ \xrightarrow{\sim} Y \ominus X$

$\xi_- : V_- \xrightarrow{\sim} Y \ominus uX$

Analyze the square

$$\begin{tikzcd}
 & & uX \arrow[dr, squiggly, dash, "\xi_-"] & \\
 & uW \arrow[ur, dash, "u\xi'_+"] \arrow[dr, dash, "\xi'_-"'] & & Y \\
 & & X \arrow[ur, dash, "\xi_+"'] & \\
 & W \arrow[ur, dash, "\xi'_+"'] & &
\end{tikzcd}$$

~~cont~~ $h = \xi_-^*\xi_+$

$$\xi_+ - \xi_-\overbrace{(\xi_-^*\xi_+)}^{h} = u\xi'_+k$$

$$(\xi_+ - \cancel{\xi_- h})^*(\xi_+ - \xi_- h) = (u\xi'_+k)^*(u\xi'_+k)$$

$$1 - \underbrace{\xi_+^*\xi_-}_{h^*}h \qquad \overset{\|}{k^*k}$$

$$\therefore k = (1 - h^*h)^{1/2}$$

$$\xi_- - \xi_+\overbrace{\xi_+^*\xi_-}^{h^*} = \xi'_-\ell$$

$$(\xi_- - \cancel{\xi_+ h^*})^*(\xi_- - \xi_+h^*) = (\xi'_-\ell)^*(\xi'_-\ell) = \ell^*\ell$$

$$1 - hh^* \qquad \ell = (1 - hh^*)^{1/2}$$

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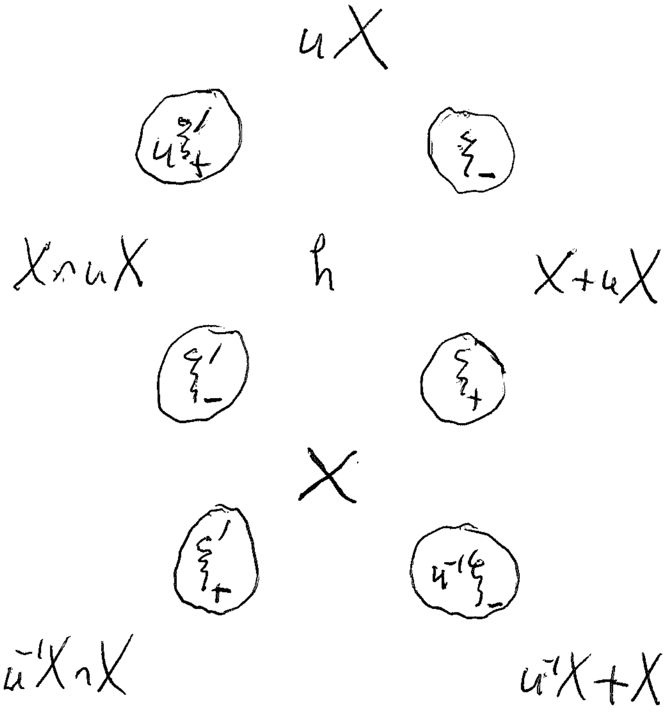


$$\xi_+ - \xi_- \overbrace{\xi_-^* \xi_+}^{h} = u\xi_+'(1-h^*h)^{1/2}$$

$$\xi_- - \xi_+ \underbrace{\xi_+^* \xi_-}_{h^*} = \xi_-'(1-hh^*)^{1/2}$$

$$\begin{pmatrix} \xi_+ & \xi_- \end{pmatrix} \begin{pmatrix} 1 & -h^* \\ -h & 1 \end{pmatrix} = \begin{pmatrix} u\xi_+' & \xi_-' \end{pmatrix} \begin{pmatrix} (1-h^*h)^{1/2} & 0 \\ 0 & (1-hh^*)^{1/2} \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ & \xi_- \end{pmatrix} \begin{pmatrix} (1-h^*h)^{-1/2} & -h^*(1-hh^*)^{-1/2} \\ -h(1-h^*h)^{-1/2} & (1-hh^*)^{-1/2} \end{pmatrix} = \begin{pmatrix} u\xi_+' & \xi_-' \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \xi_+ & \xi_- \end{pmatrix} = \begin{pmatrix} u\xi_+' & \xi_-' \end{pmatrix} \begin{pmatrix} (1-h^*h)^{-1/2} & h^*(1-hh^*)^{-1/2} \\ h(1-h^*h)^{-1/2} & (1-hh^*)^{-1/2} \end{pmatrix}}$$

$$-h(1-h^*h)^{-1} + (1-hh^*)^{-1/2}h(1-h^*h)^{-1/2}$$

$1-$

$$(1-h^*h)^{-1} - \underbrace{h^*(1-hh^*)^{-1/2}h}_{h^*h(1-h^*h)^{-1}}(1-h^*h)^{-1/2}$$

547 Given (X,c) put $W = \mathrm{Ker}(1-c^*c)$,
let $a : W \longrightarrow$ be inc., $b = ca : W \longrightarrow X$
note $b = c : \mathrm{Ker}(1-c^*c) \xrightarrow[\text{unitary}]{\sim} \mathrm{Ker}(1-cc^*)$. $V_+ = . -$

$c = ba^* + \xi_- h \xi_+^*$ where $\|h\| < 1$

Yesterday you were making progress toward understanding the scattering of a contraction.

Given (X,c) put $W = \mathrm{Ker}(1-c^*c)$, $a : W \to X$ the inc,
$b : X \longrightarrow$? Observe

$$\begin{aligned} X &= \mathrm{Ker}(1-c^*c) \oplus \overline{\mathrm{Im}(1-c^*c)} \\ & \qquad c\downarrow \uparrow c^* \\ &= \mathrm{Ker}(1-cc^*) \oplus \overline{\mathrm{Im}(1-cc^*)} \end{aligned}$$

$W = \mathrm{Ker}(1-c^*c) \xhookrightarrow{a} X$ inclusion, $b : W \longrightarrow X$ induced by c

$ca = b \qquad c^*b = c^*ca = a.$

$$\begin{tikzcd} & X \arrow[d, shift left, "c"] \\ W \arrow[ur, "a"] \arrow[r, "b"'] & X \arrow[u, shift left, "c^*"] \end{tikzcd}$$

What viewpoint to take?

$$\begin{pmatrix} a \\ b \end{pmatrix} W \subset \begin{pmatrix} 1 \\ c \end{pmatrix} X, \begin{pmatrix} c^* \\ 1 \end{pmatrix} X$$

$$\left(\begin{pmatrix} a \\ b \end{pmatrix} W \right)^0 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \, a^*x_1 = b^*x_2 \right\}$$

contains $\begin{pmatrix} 1 \\ c \end{pmatrix} X \iff a^* = b^*c$ i.e. $\boxed{a = c^*b}$

548 $Y = \overline{jX + ujX} = jX \oplus \overline{\nu_+ X} = u\overline{\nu_- X} \oplus ujX$

$\nu_+ : X \to \overline{\nu_+ X}$ completion wrt $\|\nu_+ x\|^2 = (x, (1-c^*c)x)$

$\nu_+ x = ujx - jcx$ $\qquad \nu_+^*\nu_+ = (j^*u^{-1} - c^*j^*)(uj - jc) = 1 - c^*c$

$\nu_- : X \to \overline{\nu_- X}$ comp wrt $\|\nu_- x\|^2 = (x, (1-cc^*)x)$

$u\nu_- x = u(u^{-1}j - jc^*)x = (j - ujc^*)(x)$.

$X = aW \oplus \overline{(1-c^*c)X} = bW \oplus \overline{(1-cc^*)X}$

$1 = aa^* + \xi_+^* \xi_+ = bb^* + \xi_- \xi_-^*$

$c = ba^* + \xi_- h \xi_+^* \qquad c^* = ab^* + \xi_+ h^* \xi_-^*$

$1 - c^*c = 1 - (aa^*) - \xi_+ h^*h \xi_+^* = \cancel{aa^*} + \xi_+^*\xi_+ - \cancel{aa^*} - \xi_+ h^*h \xi_+$

$1 - c^*c = \xi_+ (1 - h^*h) \xi_+^*$

$1 - cc^* = \xi_- (1 - hh^*) \xi_-^*$

$X = aW \oplus \xi_+\mathbb{C} = bW \oplus \xi_-\mathbb{C} \qquad c = ba^* + \xi_- h \xi_+^*$

Next have $Y = \overline{jX + ujX} = X + uX$

$= X \oplus \eta_+\mathbb{C} = uX \oplus \eta_-\mathbb{C}$

~~$(u-g)^*(u-c) = (u^{-1} c^*)(u-c$~~

$\eta_+\mathbb{C} = \overline{\nu_+ X} = \overline{(uj - jc)X} \qquad (uj - jc)^*(uj - jc) = 1 - c^*c$

$$\boxed{\begin{aligned} \eta_+ &= (uj\xi_+ - jh\xi_-)(1 - |h|^2)^{-1/2} \\ \eta_- &= \overline{\nu_- X} = (j\xi_- - u\bar{h}\xi_+)(1 - |h|^2)^{-1/2} \end{aligned}}$$

$(j - ujc^*)^*(j - ujc^*)\xi_- = (1 - cc^*)\xi_- = \xi_-(1 - |h|^2)$

ujX

$\bigcirc\, uj\xi_*$ $\quad$ $\bigcirc\, \eta_-$

ujW $\quad$ Y

$\bigcirc\, j\xi_-$ $\quad$ jX $\quad$ $\bigcirc\, \eta_+$

jW

$u^{-1}jX$

You want to apply $j_-^* : E \to L^2(S^1)$

this sends $\eta_- \longmapsto 1$

$\eta_+ \longmapsto S$

$$\begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} \begin{pmatrix} uj\xi_+ \\ j\xi_- \end{pmatrix}$$

$$\begin{pmatrix} uj\xi_+ \\ j\xi_- \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & h \\ +\bar{h} & 1 \end{pmatrix} \begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix}$$

$$\begin{pmatrix} z\tilde{\xi}_+ \\ \tilde{\xi}_- \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} S \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} S \\ 1 \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} \begin{pmatrix} z\tilde{\xi}_+ \\ \tilde{\xi}_- \end{pmatrix}$$

$\times \frac{1}{1-|h|^2}$

$$\eta_-^* \eta_+ = (j\xi_- - u\bar{h}j\xi_+)^* (uj\xi_+ - jh\xi_-)$$

$$= -\xi_+^* j^* h (j\xi_+ - u^{-1}jh\xi_-)$$

$$= -(h - \underbrace{\xi_+^* h c^* h \xi_-}_{h^2\bar{h}}) = h(1-|h|^2)$$

$$x \longmapsto \gamma_-\left(\frac{1}{1-zc^*}x\right), \quad \nu_+\left(\frac{1}{z-c}x\right)$$

but what is the norm on this ~~couple~~ pair

suppose given $S : L^2(S^1, V_-) \longrightarrow L^2(S^1, V_+)$. to what?

$$\|jx + ky\|^2 = \|x + (j^*k)y\|^2 + \|y\|^2 - \|\overset{j^*k}{\cancel{k}}y\|^2$$

$$= \|k^*jx + y\|^2 + \|x\|^2 - \|k^*jx\|^2$$

$$\begin{tikzcd}
& X \arrow[d, "j"] & \\
L^2(S^1, V_-) \arrow[r, hook, "j_+"] & E & L^2(S^1, V_+) \arrow[l, hook', "j_-"']
\end{tikzcd}$$

* * *

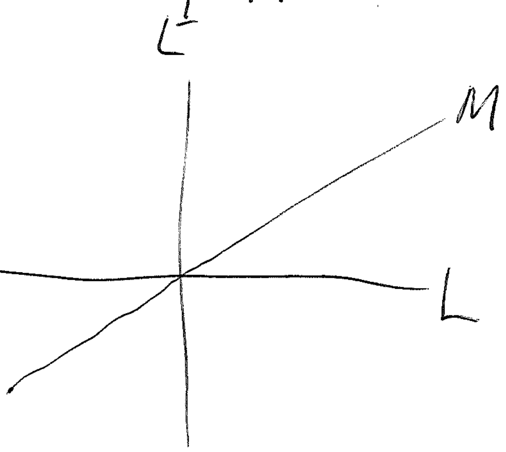
Set up the general Grassmannian ~~situation~~. Diff notation.

$$\begin{tikzcd}
L \arrow[dr, hook, "a"] & \\
& E \\
M \arrow[ur, hook, "b"'] &
\end{tikzcd}
\qquad
\begin{aligned} a^*a &= 1 \\ b^*b &= 1. \end{aligned}$$

Can you find the projection onto

$\overline{aL+bM} \subset E$, $\overline{aL+bM}^{\perp} = \mathrm{Ker}(a^*) \cap \mathrm{Ker}(b^*)$.

$$\begin{tikzcd}
& M \arrow[d, "b"] & \\
L \arrow[r, "a"] & E \arrow[r, "a^*"] \arrow[d, "b^*"] & L^{\perp} \\
& M^{\perp} &
\end{tikzcd}$$



551 Concentrate

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[dr, "c=b^*a"'] & E \arrow[d, "b^*"] \\ & W \end{tikzcd}$$

You want I think to construct the orthogonal projection onto $\overline{aV+bW} \subset E$. $\overline{aV+bW}^{\perp} = \mathrm{Ker}(a^*) \cap \mathrm{Ker}(b^*)$, so you need to use ~~[illegible]~~ something like

$$E \xrightarrow{\begin{pmatrix} a^* \\ b^* \end{pmatrix}} \begin{matrix} V \\ \oplus \\ W \end{matrix} \xrightarrow{?} \begin{matrix} V \\ \oplus \\ W \end{matrix} \xrightarrow{(a \ \ b)} E$$

$$(a \ \ b)\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} a^* \\ b^* \end{pmatrix}(a \ \ b) \overset{?}{=} (a \ \ b)$$

$$\begin{pmatrix} 1 & c^* \\ c & 1 \end{pmatrix}\underbrace{\begin{pmatrix} & \\ & \end{pmatrix}}\begin{pmatrix} 1 & c^* \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & c^* \\ c & 1 \end{pmatrix}$$

So ⎯⎯ should be ~~[illegible]~~ the inverse of $\begin{pmatrix} 1 & c^* \\ c & 1 \end{pmatrix}$ when this is invertible.

$(1-c^*c)^{-1} - c^*c(1-c^*c)^{-1}$

$$\begin{pmatrix} 1 & c^* \\ c & 1 \end{pmatrix}^{-1} \approx \begin{pmatrix} 1 & -c^* \\ -c & 1 \end{pmatrix} \Big/ \frac{1}{1-cc^*}$$

$c = b^*a$

$cc^* = b^*aa^*b$

$c^*c = a^*bb^*a$

$$\begin{pmatrix} 1 & c^* \\ c & 1 \end{pmatrix}\begin{pmatrix} (1-c^*c)^{-1} & -c^*(1-cc^*)^{-1} \\ -c(1-c^*c)^{-1} & (1-cc^*)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & c^* \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & -c^* \\ -c & 1 \end{pmatrix} = \begin{pmatrix} 1-c^*c & 0 \\ 0 & 1-cc^* \end{pmatrix}$$

But instead of this matrix stuff you should use what you know: ~~[illegible]~~ $\overline{aV+bW} \simeq aV \oplus (\quad)^{1/2}W$

552 Try again!!!!!! $c = b^*a$

$$\|\underbrace{av + bw}_{\xi}\|^2 = \|\underbrace{v + \overset{c^*}{\widehat{a^*b}}\,w}_{a^*\xi}\|^2 + (w, (1-cc^*)w)$$

$$= \|\underbrace{b^*a}_{c}v + w\| + (v, (1-c^*c)v)$$

$$\xi \longmapsto \begin{cases} a^*\xi \\ \sqrt{1-cc^*}\, b^*\xi \end{cases}$$

Given ξ take $a^*\xi$ to get aV component. Remove from ξ to get $\xi - aa^*\xi$ and you want to remove from this bw, you want

$$\xi - aa^*\xi - \overset{(b-ac^*)}{b}w \in \operatorname{Ker} a^* \cap \operatorname{Ker} b^*$$

$$\boxed{c = b^*a,\ c^* = a^*b}$$

$\therefore\ a^*bw = 0.$

Stupid. You know $\overline{aV+bW} = aV \oplus \overline{(b-ac^*)W}$ where $(b-ac^*)^*(b-ac^*) = (b^* - \cancel{ca^*})(b-ac^*) = 1-cc^*.$

$$a^*\left(\xi - aa^*\xi - (b-ac^*)w\right) = 0$$

$$b^*\xi - ca^*\xi - (1-cc^*)w = 0 \quad ?$$

Again. $V \xrightarrow{a} E \xleftarrow{b} W$ $\quad a^*a = 1 \quad b^*b = 1$

$b^*a = c \quad a^*b = c^*$

$$\overline{aV+bW} = aV \oplus \overline{(b-ac^*)W} \qquad \underbrace{(b-\cancel{ac^*})^*(b-ac^*)}_{=1-cc^*}$$

$\xi = av + (b-ac^*)w$ + something killed by a^*, b^*.

$v = a^*\xi \qquad (1-cc^*)w = (b-ac^*)^*\xi = b^*\xi - ca^*\xi$

$$w = (1-cc^*)^{-1}(b^*\xi - ca^*\xi)$$

553 Start again. $X \xrightarrow{a} E \xleftarrow{b} X'$ $\quad Y = \overline{aX + bX'}$

$$\|ax + bx'\|^2 = \|x + cx'\|^2 + \|(1-c^*c)^{1/2}x'\|^2 \qquad c = a^*b$$
$$= \|c^*x + x'\|^2 + \|(1-cc^*)^{1/2}x\|^2 \qquad c^* = b^*a$$

$$\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & c^* \end{pmatrix}\begin{pmatrix} 1 \\ c \end{pmatrix} + \begin{pmatrix} 0 & (1-c^*c)^{1/2} \end{pmatrix}\begin{pmatrix} 0 \\ (1-c^*c)^{1/2} \end{pmatrix}$$

Given $\xi \in E$ find its projection onto Y.

$$\xi = ax + bx' + (\text{elt killed by } a^*, b^*).$$

$$\xi = ax + (b-ac)x' + \underline{\qquad\qquad}$$

$$\boxed{a^*\xi = x}$$

$$(b^* - c^*a^*)\xi = (b^* - \cancel{c^*a^*})(b-ac)x' = (1-c^*c)x'.$$

$$\therefore \boxed{x' = (1-c^*c)^{-1}(b^* - c^*a^*)\xi}$$

$$x = a^*\xi \qquad\qquad Y = aX \oplus \overline{V_+}$$
$$x \qquad\qquad \simeq X \oplus \overline{(1-c^*c)^{1/2}X'}$$

$$ax + (b-ca)x' \longleftarrow\!\shortmid\ x + (1-c^*c)^{1/2}x'$$
$$\xi \longmapsto a^*\xi + (1-c^*c)^{-1/2}(b-ca)^*\xi$$

Go back to scattering

$$\begin{array}{lcccccc}
L^2(V_+) & \cdots \oplus z^{-1}V_+ & \oplus & V_+ & \oplus & zV_+ & \\
\uparrow j_+ & \cap & & \| & & \| & \| \\
E & \oplus\ u^{-1}V_- \oplus jX & \oplus & V'_+ & \oplus & uV_+ & \oplus \cdots \\
\uparrow j_- \quad \| & \| & & & \cup & & \\
L^2(V_-) & \oplus\ z^{-1}V_- & \oplus & V_- & \oplus & zV_- & \oplus \cdots
\end{array}$$

$$j_+^* j x = \nu_+\left(\frac{1}{z-c}x\right) = \sum_{n\geq 0} z^{-1-n}\nu_+ c^n x \quad \in H^2_-(S^1, V_+)$$

$$\|j_+^* j x\|^2 = \|x\|^2 - \lim_{n\to\infty}\|c^n x\|^2$$

554 Decide whether you can characterize when $jX \subset j_+ L^2(V_+) + j_- L^2(V_-)$. You need to calculate the norm of $j_+ j_+^* j x + j_- j_-^* j x$. The main point should be

$$\left(\underbrace{j_+ j_+^* j x}, \underbrace{j_- j_-^* j x'}\right)$$

$$\left(\sum_{n \geq 0} u^{-1-n} \nu_+ c^n x \,,\ \sum_{k \geq 0} u^k \nu_- c^{*k} x' \right)$$

$$= \sum_{n \geq 0, k \geq 0} \left(\underbrace{\nu_+}_{uj - jc} c^n x \,,\ u^{1+n+k} \underbrace{\nu_-}_{j - ujc^*} c^{*k} x' \right)$$

table of inner products.

$$j^* u^n \nu_+ = \begin{cases} 0 & n \geq 0 \\ (c^*)^{-n-1}(1 - c^* c) & n \leq -1 \end{cases}$$

$$j^* u^n \nu_- = \begin{cases} c^n (1 - cc^*) & n \geq 0 \\ 0 & n \leq -1 \end{cases}$$

$$\nu_+^* u^n \nu_- = \begin{cases} (1 - c^* c)\, c^{n-1} (1 - c c^*) & n \geq 1 \\ -c^* + c^* c c^* & n = 0 \\ 0 & n \leq -1 \end{cases}$$

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$$(j_+ j_+^* j x, j_- j_-^* j x') = \Big(\sum_{n\geq 0} u^{-1-n} \nu_+ c^n x, \sum_{k\geq 0} u^k \nu_+ c^{*k} x'\Big)$$

$$= \sum_{n,k\geq 0} (c^n x, \nu_+^* u^{1+n+k} \nu_- c^{*k} x')$$

$$= \sum_{n,k\geq 0} (c^n x, (1-c^*c) c^{n+k} (1-cc^*) c^{*k} x')$$

$$= \sum_{n,k\geq 0} \big(c^{*n}(1-c^*c)c^n x, c^k(1-cc^*)c^{*k} x'\big)$$

$$= \Big(\sum_{n\geq 0} c^{*n}(1-c^*c) c^n x, \sum_{k\geq 0} c^k(1-cc^*)c^{*k} x'\Big)$$

~~$x - \lim c^{*n} c^n x$~~

$$= \lim_{\substack{n\to\infty \\ k\to\infty}} (x - c^{*n}c^n x, x' - c^k c^{*k} x')$$

$$\|j_+ j_+^* j x\|^2 = \lim_{n\to\infty} \|x\|^2 - \|c^n x\|^2$$

$$\|j_- j_-^* j x'\|^2 = \lim_{n\to\infty} \|x'\|^2 - \|c^{*n} x'\|^2$$

$$(j_+ j_+^* j x, j_- j_-^* j x') = \lim_{\substack{n\to\infty \\ k\to\infty}} (x - c^{*n}c^n x, x' - c^k c^{*k} x')$$

$$(j_- j_-^* j x', j_+ j_+^* j x) = \lim_{\substack{n\to\infty \\ k\to\infty}} (x' - c^k c^{*k} x', x - c^n c^{*n} x)$$

Review logic. You want to find a criterion that $jX \subset \overline{j_+ L^2(S^1, V_+) + j_- L^2(S^1, V_-)}$. The idea will be project jX onto this closure and calculate the norm.

$$E \supset \overline{aV + bW}$$

$$\parallel$$

$$aV \oplus \overline{(b-ac)W}$$

$a^*b = c$

$b^*a = c^*$

$c^*c = b^*aa^*b$

$1-c^*c = b^*(1-aa^*)b$

$a^*(b-ac)=0$ $\quad (b-ac)^*(b-ac) = 1-c^*c$

So let $\xi \in E$, write $\xi = av + (b-ac)w +$ something in Ker a^*, ~~b^*~~ $(b-ac)^*$

$v = a^*\xi$ $\quad (1-c^*c)w = b^*\xi$ NO

~~$\|av + (b-ac)w\|^2 = \|v\|^2 + (w, (1-c^*c)w)$~~

~~$= \|a^*\xi\|^2 + (b^*\xi, (1-c^*c)^{-1}b^*\xi)$~~

~~$= \|a^*\xi\|^2 + ((1-c^*c)^{-1/2}b^*\xi, (1-c^*c)^{-1/2}b^*\xi)$~~

~~$1 = (1-c^*c)^{-1/2}b^*(1-aa^*)b(1-c^*c)^{-1/2}$~~

~~$\& (1-c^*c)^{-1/2} \& 1 +$~~

Repeat $\quad E \supset \overline{aV + bW}$

$$\parallel$$

$$aV \oplus \overline{(b-ac)W}$$

$a^*b = c$, $b^*a = c^*$

$a^*a = 1$

$a^*(b-ac) = 0$

$(b-ac)^*(b-ac) = (1-c^*c)$

$$\xi = av + (b-ac)w$$

$a^*\xi = v$, $\quad (b-ac)^*\xi = (1-c^*c)w$

$$b^*\xi - c^*a^*\xi = (1-c^*c)w$$

$$\|av + (b-ac)w\|^2 = \|v\|^2 + (w, (1-c^*c)w)$$

$$= \|a^*\xi\|^2 + \| (1-c^*c)^{-1}(b^*\xi - c^*a^*\xi), b^*\xi - c^*a^*\xi)$$

I know $\|a^*\xi\|^2$, $\|b^*\xi\|^2$ $\quad (a^*\xi, b^*\xi)$

557 Suppose $V_\pm$ 1 diml. generated by unit vectors ξ_+, ξ_- resp. All you is the matrix, really pos def matrix function on $\mathbb{Z}$, in order to get $\overline{L^2(S^1,V_+)+L^2(S^1,V_-)}$.

$$\begin{pmatrix} \boxed{\xi_+^* u^n \xi_+} & \xi_+^* u^n \xi_-\ S^* \\ \xi_-^* u^n \xi_+\ S & \boxed{\xi_-^* u^n \xi_-} \end{pmatrix}$$

This should be simple. Basically you have some inner product on pairs.

$$\begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \|f\|^2 + (g, Sf) + (f, S^*g) + \|g\|^2$$

Draw picture

$$\underbrace{jX \oplus V_+}_{\parallel} \oplus uV_+ \oplus \cdots$$

$$\cdots \oplus u^{-1}V_- \oplus \overline{V_- \oplus ujX}$$

Form u, u^{-1} inv. subspace generated by V_+, V_-

Given by a measure on S^1 values in M_2

~~mea~~ $\rho \frac{d\theta}{2\pi}$ $\qquad \rho = \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix}$ $\qquad 1 - |S|^2$

What's the relation to jX? You have

$$X \overset{j}{\hookrightarrow} E \hookleftarrow L^2(S^1, V_+ \oplus V_-).$$

Basically you have a to understand an element of $\overline{aV + bW}$ where $V \overset{a}{\hookrightarrow} E \overset{b}{\hookleftarrow} W$ $\qquad a^*a = b^*b = 1 \qquad a^*b = c$, $b^*a = c^*$

Know $$\overline{aV+bW} = aV \oplus \overline{\underbrace{(b - ac)}_{(1-aa^*)b}W} = bW \oplus \overline{\underbrace{(a - bc^*)}_{(1-bb^*)a}V}$$

558 Given $\xi \in E$, write $\xi = av + (1-aa^*)bw + \operatorname{Ker} a^*, b^*$

$$a^*\xi = v, \quad \underbrace{b^*(1-aa^*)\xi}_{\underset{\sim}{b}^* - \underset{\sim}{c}^*\underset{\sim}{a}^*} = \underbrace{b^*(1-aa^*)b}_{1-c^*c}\, w$$

Take $\xi = jx$, then $\underset{\sim}{a}^*\xi = j_+^* j\xi = \sum_{n\geq 0} z^{-1-n}\xi_+^* c^n x$

$$\underset{\sim}{b}^*\xi = j_-^* j\xi = \sum_{n\geq 0} u^n \xi_-^*(c^{*n}x)$$

$$\underset{\sim}{b}^* - \underset{\sim}{c}^*\underset{\sim}{a}^* = \sum_{n\geq 0} z^n \xi_-^*(c^{*n}x) - \underbrace{S(z)}_{\xi_-^*\frac{1}{1-zc^*}\xi_+} \sum z^{-1-n}\xi_+^*(c^n x)$$

$$\boxed{\begin{array}{c} \xi_+^* \frac{1}{z-c} x \\ \xi_-^* \frac{1}{1-zc^*} x - \xi_-^* \frac{1}{1-zc^*} \xi_+ \xi_+^* \frac{1}{z-c} x \end{array}}$$

This pair should be the projection of jx onto $j_+L^2(S^1) + j_-L^2(S^1)$

A better method, Can I compute the norm of the projection in terms of $a^*\xi$, $b^*\xi$

Work inside $\overline{aV + bW} = aV \oplus \overline{\underbrace{(b-ac)}_{(1-aa^*)b} W}$

$$= bW + \overline{\underbrace{(a-bc^*)}_{(1-bb^*)a} V}$$

?

~~$\|ax + bx'\|^2 = \|x\|^2 + ($~~

$$\begin{tikzcd} Y \arrow[r, "\begin{pmatrix} a^* \\ b^* \end{pmatrix}"] & V \oplus W \arrow[r, "(a \ b)"] & Y \end{tikzcd}$$

$$(a \ \ b)\begin{pmatrix} a^* \\ b^* \end{pmatrix} = \Big(aa^*$$

$$\begin{tikzcd}
V \oplus W \arrow[r, "\begin{pmatrix} a & b \end{pmatrix}"] \arrow[rr, bend right] & E \arrow[r, "\begin{pmatrix} a^* \\ b^* \end{pmatrix}"] & V \oplus W
\end{tikzcd}$$

$\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}$ invertible

$\begin{pmatrix} a^* \\ b^* \end{pmatrix}\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}$

assume invertible

$c = a^* b$, $c^* = b^* a$

You want the norm in V calculated in $V \oplus W$. so you want ~~something~~ same herm. matrix on $V \oplus W$ i.e. involving c, c^* which when ~~combined~~ composed with $\begin{pmatrix} a^* \\ b^* \end{pmatrix}$ yields the norm you want.

$$\|av + bw\|^2 = \|v\|^2 + (v, cw) + (w, c^*v) + \|w\|^2$$

$$\|$$

$$\begin{pmatrix} v^* & w^* \end{pmatrix}\begin{pmatrix} a^* \\ b^* \end{pmatrix}\begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v^* & w^* \end{pmatrix}\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}$$

Assume

Important relation

$$\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}^{-1}\begin{pmatrix} a^* \\ b^* \end{pmatrix}\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a^* \\ b^* \end{pmatrix}^{-1}$$

Let $\xi = \begin{pmatrix} a^* \\ b^* \end{pmatrix}^{-1}\begin{pmatrix} v \\ w \end{pmatrix}$. $\|\xi\|^2 = \left\|\begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}^{-1}\begin{pmatrix} v \\ w \end{pmatrix}\right\|^2$

$$= \begin{pmatrix} v \\ w \end{pmatrix}^*\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}^{*\,-1}\begin{pmatrix} a & b \end{pmatrix}^*\begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}^{-1}\begin{pmatrix} v \\ w \end{pmatrix}$$

$$= \begin{pmatrix} v \\ w \end{pmatrix}^*\begin{pmatrix} 1 & c \\ c^* & 1 \end{pmatrix}^{-1}\begin{pmatrix} v \\ w \end{pmatrix}.$$

Interpret this $v = \xi_+^* \frac{1}{z-c} x$, $w = \xi_-^* \frac{1}{1-zc^*} x$

$a = j_+$ $b = j_-$ $\underset{\underset{c^*}{\|}}{b^*a} = j_-^* j_+$

$$\begin{pmatrix} v \\ w \end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix}^{-1} \begin{pmatrix} v \\ w \end{pmatrix}$$

$$S = \xi_-^* \frac{1}{1-zc^*} \xi_+$$

$$= \frac{1}{1-|S|^2} \begin{pmatrix} x^* \frac{1}{z^{-1}-c^*} \xi_+ \\ x^* \frac{1}{1-z^{-1}c} \xi_- \end{pmatrix} \begin{pmatrix} 1 & -S^* \\ -S & 1 \end{pmatrix} \begin{pmatrix} \xi_+^* \frac{1}{z-c} x \\ \xi_-^* \frac{1}{1-zc^*} x \end{pmatrix}$$

Try again. In E you have ξ_+, ξ_- unit vectors

$\Rightarrow$ $\xi_\pm^* u^n \xi_\pm = \delta_n$ $\xi_-^* u^n \xi_+ = S_n$?

$$\begin{tikzcd} L^2(S^1) \arrow[r, "j_+"] & E & L^2(S^1) \arrow[l, "j_-"'] \\ z^n \arrow[r, mapsto] & u^n \xi_+ & \\ & u^n \xi_- & z^n \arrow[l, mapsto] \end{tikzcd}$$

$j_-^* j_+$

You have to connect $\xi_\pm \in V_\pm$ with X, c

$\nu_+ x = (uj - jc)(x)$ $\nu_*^* \nu_* = (j^* u^{-1} - \cancel{c^*} j^*)(uj - jc) \cancel{x}$
$= 1 - c^* c$

Go through the rest.
It should be enough to choose $\xi_\pm \in V_\pm = \overline{\nu_\pm X}$ of norm $\underline{1}$. Keep on trying.

$$\overset{aX}{\overset{\|}{\boxed{jX}}} \oplus \mathbb{C}\xi_+$$
$$\|$$
$$\mathbb{C}\xi_- \oplus \underset{bX}{\boxed{ujX}}$$

You want $S(z) = \xi_-^* \frac{1}{1-zb^*a} \xi_+$
$S(z)^* = \xi_+^* \frac{1}{1-\bar{z}a^*b} \xi_-$

You need to sort this out.

$u(jx) = j(cx) +$

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$$u(ax) = bx = aa^*bx + \xi_+\xi_+^*bx$$

$$u^2(ax) = ba^*bx + u\xi_+\xi_+^*bx$$

$$= a(a^*b)^2x + \xi_+\xi_+^*ba^*bx + u\xi_+\xi_+^*bx$$

$$u^3(ax) = a(a^*b)^3x + \xi_+\xi_+^*b(a^*b)^2x + u\xi_+\xi_+^*ba^*bx + u^2\xi_+\xi_+^*bx$$

$$ax = u^{-3}a(a^*b)^3x + u^{-3}\pi_+(ba^*)^2bx + u^{-2}\xi_+\xi_+^*ba^*bx + u^{-1}\xi_+\xi_+^*bx$$

$$\text{scattering} = \sum_{n\geq 0} u^{-1-n}\xi_+\xi_+^*(ba^*)^nbx$$

$$\mapsto \underbrace{\xi_+\xi_+^*}_{1-aa^*}\frac{1}{z-ba^*}bx$$

problem here is $\xi_+\xi_+^*b = \nu_+$

$$\xi_+\xi_+^*uj\frac{1}{z-\underset{j^*uj}{a^*b}}x$$

Go back to (X,c) assume $V_\pm$ 1-diml. choose $\xi_\pm$ of norm 1 in $V_\pm$, but what is $\nu_\pm : X \longrightarrow V_\pm$.

$a\cancel{X}\oplus V_+$

$V_-\oplus bX$

$$y = aa^*y + \pi_+y$$

$$uy = aa^*ba^*y + \pi_+ba^*y + u\pi_+y$$

$$u^2y = aa^*(ba^*)^2y + \pi_+(ba^*)^2y + u\pi_+ba^*y + u^2\pi_+y$$

$$ujx = jcx + \nu_+x$$

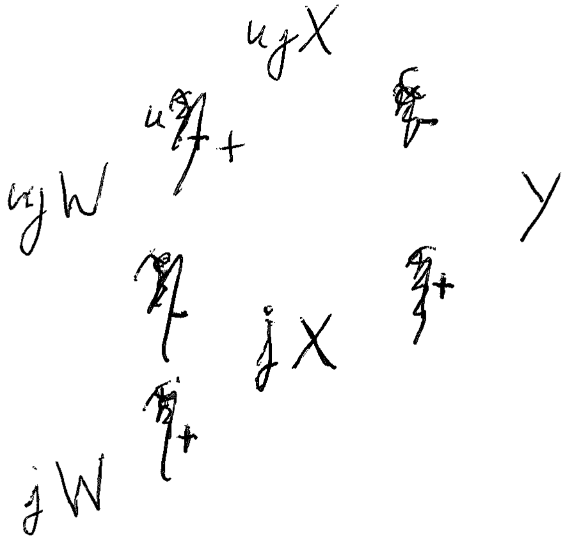
$$u^2jx = jc^2x + \nu_xcx + u\nu_+x$$

$$\cdots\oplus u^{-1}V_-\oplus \boxed{\begin{array}{c} jX\oplus V_+ \\ \| \\ V_-\oplus ujX \end{array}} \oplus uV_+ \oplus u^2V_+\oplus\cdots$$

What you want to do is

Big calculation. $X = aW\oplus\xi_+\mathbb{C} = bW\oplus\xi_-\mathbb{C}$

$$c = ba^* + \xi_-h\xi_+^*$$



~~Consider~~ You really want to use only

$$Y = \overline{jX + ujX} = aX \oplus \xi_+ \mathbb{C} = \xi_- \mathbb{C} \oplus bX$$

where $\underset{\sim}{\xi_+}$ is a unit vector generating $V_\pm$

$$\begin{tikzcd} aX \arrow[dr, "ba^*"] & \xi_+ \mathbb{C} \\ \xi_- \mathbb{C} & bX \end{tikzcd}$$

Suppose you specify the scale

What you should do is to ~~[illegible]~~ construct and study what arises from the scattering data

$\xi_+^* u^n \xi_+$	$\xi_+^* u^n \xi_-$
$\xi_-^* u^n \xi_+$	$\xi_-^* u^n \xi_-$

$S_n = \xi_-^* u^n \xi_+$

zero for $n > 0$

so $S(z) = \sum z^{-n} S_n$

$= \sum_{n \geq 0} z^n S_{-n}$

$$S(z) = \xi_-^* \frac{1}{1 - zu^{-1}} \xi_+$$

$$c_y = ba^* \qquad c_y^* = ab^*$$

$$1 - c_y c_y^* = 1 - bb^* \qquad 1 - c_y^* c_y = 1 - aa^*$$

563 Try restricting attention to a c satisfying

$$c = cc^*c \Rightarrow c^* = c^*cc^* \qquad (c^*c)^2 = c^*c$$

$$1 - c^*c = \xi_+\xi_+^* \qquad 1 - cc^* = \xi_-\xi_-^*$$

$$c(1-c^*c) = c\xi_+\xi_+^* \Rightarrow c\xi_+ = 0 \quad \text{sim.} \quad c^*\xi_- = 0$$

$$\left[\begin{array}{l} j_+^* j x = \xi_+^* \dfrac{1}{z-c}x \\ j_-^* j x = \xi_-^* \dfrac{1}{1-zc^*}x \end{array}\right.$$

~~$V_+ \oplus zV_+$~~

~~$\oplus z^{-1}V_- \oplus jX \oplus V_+ \oplus zV_+ \oplus$~~

~~$V_- \oplus u\,jX$~~

~~$\oplus z^{-1}V_- \oplus V_- \oplus zV_-$~~

$$S = \xi_-^* \frac{1}{1-zc^*}\xi_+ \qquad S^* = \xi_+^* \frac{1}{1-z^{-1}c}\xi_-$$

$$SS^* = \xi_-^* \frac{1}{1-zc^*} \underbrace{\xi_+\xi_+^*}_{(1-c^*c)} \frac{z}{z-c}\xi_-$$

$$= \xi_-^* \frac{1}{1-zc^*}\left(z - c + (1-zc^*)c\right)\frac{1}{z-c}\xi_-$$

$$= \xi_-^*\left(\frac{1}{1-zc^*} + c\frac{1}{z-c}\right)\xi_-$$

---

setup up the pos. def $\overset{\text{matrix}}{}$ function assoc. to

$$\varepsilon = (\xi_+^* \;\; \xi_-^*) : \mathbb{C}^2 \longrightarrow E$$

$$\varepsilon^* u^n \varepsilon = \begin{pmatrix} \xi_+^{**} \\ \xi_-^* \end{pmatrix} u^n (\xi_+ \;\; \xi_-)$$

56 4/7 Question. Take $S(z) = h$ $|h| \le 1$

What's the corresponding Hilbert space picture?

~~[illegible]~~ Don't think about X instead use S to construct an $H, u, \xi_\pm$.

Focus on H, u gener. by $\xi_\pm$ glued via S.

The point you missed was that a constant $S(z) = h_0$ has an infinite Schur expansion if $|h_0| < 1$ name

$$S(z) = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \quad \text{etc.}$$

So the Schur expansion stops only with an h_n of modulus one

---

operator situation corresponding to a Schur expansion

Go over the ideas main point is the way unitary ops are constructed. For my lecture I need the formula for the scattering ~~in terms of~~ operator from a partial unitary. Apply to $Y = \rho X \oplus V_+ = \eta X \oplus V_-$

There's a (small?) problem going from the shift sort of definition of S to a function of z. You want the

Idea: For a unitary operator you know $\frac{1}{z - u}$ exists.

Get things done. You want now to exploit the idea that the resolvent of u is defined ~~in~~ each of the disks, and that this should ~~leads~~ lead to a nice ~~[illegible]~~ picture. For the first time you see your old ~~[illegible]~~ picture involving the graph of the propagator emerging.

Now how do you exploit this knowledge?

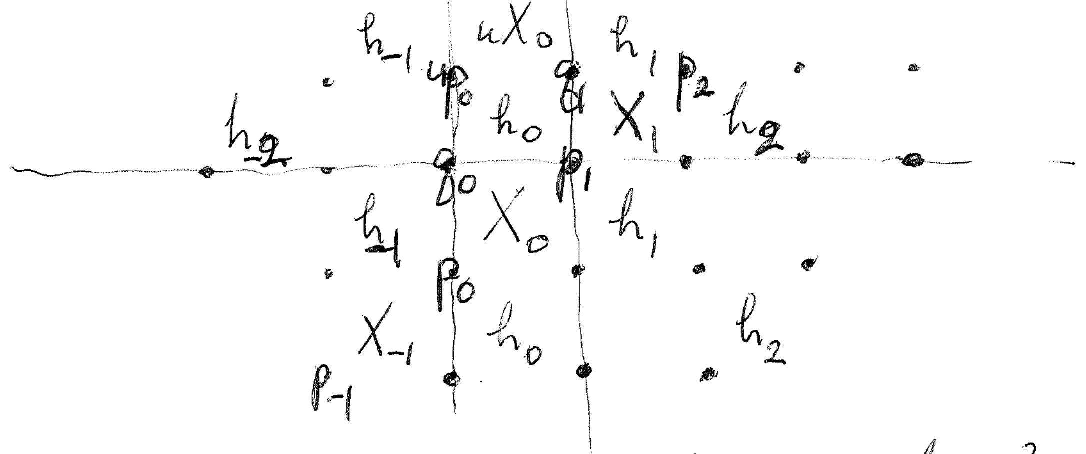
56 ~~F~~ This will be a measure on the circle

$$\varepsilon^* u^n \varepsilon = \int z^n \rho \frac{d\theta}{2\pi} \qquad \rho(z) \geq 0 \text{ in } M_2\mathbb{C}$$

$$\rho = \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \qquad S = \xi_-^* \frac{1}{1-zC^*} \xi_+$$

For the rest of this afternoon I want to improve the notation ~~construction~~ construction of the unitary operator ~~with~~ corresponding to ~~of~~ a sequence of Schur parameters $(h_n)_{n\in\mathbb{Z}}$

The construction is easy enough, namely, your ~~[illegible]~~ Hilbert space has ~~[illegible]~~ a sequence of ~~[illegible]~~ orthonormal bases. Picture as before, namely, an orthonormal bases for each row in $\mathbb{Z}\times\mathbb{Z}$.



How might you study this? First idea consists ~~in [illegible] forming something~~ of passing to a spectral repn, so that ~~[illegible]~~

~~[illegible]~~ Idea: Go back to Green's function idea with ~~[illegible]~~ boundary conditions on either sides. Exploit the fact that $\frac{1}{z-u}$ defined for $|z|<1$ and $|z^{-1}|<1$. Basically then given $x \in X$ you consider $\frac{1}{z-u}x$ or $\frac{1}{1-zu^{-1}}x$

566 Spend today preparing for the next two days lectures. Possible topics – relation between contractions and partial unitaries, – eigenvector equation, probably to be replaced by Green's function an X. – double array picture of a unitary.

Consider then a [redacted] partial unitary

$$Y = aX \oplus \underset{\Sigma_+\mathbb{C}}{\underset{\|}{V_+}} = bX \oplus \underset{\Sigma_-\mathbb{C}}{\underset{\|}{V_-}} .$$ Dilate as usual

New viewpoint is ~~the~~ ~~You have the propagator~~ ~~there is little little~~ the boundary condition idea, resolvent $\frac{1}{z-u}x$ has boundary values given, essentially by the eigenvalue equation.

So [illegible] formulate this. Start with ~~the eigen~~

$j: X \hookrightarrow E \circlearrowleft^{u}$ Now look at $\frac{1}{z-u}x$

This should be defined for $|z| \neq 1$. For $|z|>1$ by the series $\frac{1}{z-u}x = \frac{1}{z}\frac{1}{1-\frac{u}{z}}x = \sum_{n\geq 0} z^{-n-1}u^n x$

and for $|z|<1$ by the series

$$\frac{1}{z-u}x = \frac{1}{zu^{-1}-1}u^{-1}x = -\sum_{n\geq 0} z^n u^{-n-1}x$$

), and as usual things are linked by the analytic cont.

But now observe that if you apply j^* you get ~~back~~ something involving c or c^*.

$$j^*\frac{1}{z-u}jx\cancel{\phi} = \sum_{n\geq 0} z^{-n-1}c^n x \qquad |\cancel{z}|>1.$$

$$j^*\frac{1}{z-u}jx = -\sum_{n\geq 0} z^n c^{*\,n+1}x \qquad |z|<1.$$

567 So now what can you hope for??
understand scattering)

$$j^* \frac{1}{1-z^{-1}u} j x = j^* \sum_{n\geq 0} z^{-n} u^n j x = \frac{1}{1-z^{-1}c} x \qquad |z|>1$$

$$j^* \frac{1}{1-z^{-1}u} j x = j^* \frac{1}{z(u^{-1}-z^{-1})} z u^{-1} j x = - j^* \frac{1}{1-zu^{-1}} z u^{-1} j x$$

$$= -\sum_{n\geq 1} z^n j^* u^{-n} j x = -\sum_{n\geq 1} z^n c^{*n} x \qquad |z|<1$$

$$= \frac{-zc^*}{1-zc^*} x$$

$$\frac{1}{1-z^{-1}c} + \frac{zc^*}{1-zc^*} = \frac{1}{1-z^{-1}c}\Big(\underbrace{1-zc^* + (1-z^{-1}c)zc^*}_{1-cc^*}\Big)\frac{1}{1-zc^*}$$

So there's still a lot to understand here.

Basically the same things keep coming up.

Question: $H^2(\mathbb{R})$ with $H^2(S^1)$

Idea: You have the Poisson summation link between $L^2(\mathbb{R})$ and $L^2(S^1)$. What about $H^2(\mathbb{R})$ and $H^2(S^1)$? This should be straightforward.

$\mathbb{Z}$ action $\lambda \longmapsto \lambda+1$ on the UHP

$$f(\lambda) = \int_0^\infty e^{i\lambda x}\phi(x)\,dx \qquad f(\lambda+1) \longleftrightarrow e^{ix}\phi(x)$$

$$\sum_{n\in\mathbb{Z}} f(\lambda+n) = \int_0^\infty \sum_{n\in\mathbb{Z}} e^{i(\lambda+n)x}\phi(x)\,dx$$

$$= \int_0^\infty \boxed{\sum_{n\in\mathbb{Z}} e^{inx}} e^{i\lambda x}\phi(x)\,dx$$

Go over Poisson summation formula for Schwartz functions.

You have

$$\phi(x) = \int \frac{d\xi}{2\pi} e^{i\xi x} f(\xi)$$

$$f(\xi) = \int dx\, e^{-i\xi x} \phi(x).$$

Idea is to make smooth functions a T^2 acts on $\mathcal{S}(\mathbb{R})$ translation $\phi(x) \longmapsto \phi(x+1)$, mult by $e^{2\pi i x}$

That comes later. First you need the basic argument

$$\phi(x) \longmapsto \sum_{n\in\mathbb{Z}} \phi(x+n) \qquad \text{periodic}$$

so it has a F.S.

$$\sum_{n\in\mathbb{Z}} \phi(x+n) = \sum_{k\in\mathbb{Z}} e^{2\pi i kx} \int_0^1 \sum_{n\in\mathbb{Z}} \phi(y+n) \boxed{e^{-2\pi i ky}} dy$$

$$e^{-2\pi i k(y+n)}$$

$$= \sum_{k\in\mathbb{Z}} e^{2\pi i kx} f(k)$$

$$\sum_{n\in\mathbb{Z}} e^{2\pi i t(x+n)} \phi(x+n) = \sum_{k\in\mathbb{Z}} e^{2\pi i kx} \int_0^1 \sum_{n\in\mathbb{Z}} e^{2\pi i t(y+n)} \phi(y+n) e^{-2\pi i k \overset{y+n}{y}} dy$$

$$= \sum_{k\in\mathbb{Z}} e^{2\pi i kx} \underbrace{\int_{-\infty}^{\infty} e^{2\pi i t y} \phi(y) e^{-2\pi i ky} dy}_{f(k-t)}$$

So what can we do?

Go to $\lambda \in$ UHP. You have translation $\lambda \longmapsto \lambda + t \quad t\in\mathbb{R}$. [redacted] You want

Given $f(\lambda)$ form $\sum_{n\in\mathbb{Z}} f(\lambda+n)$ [illegible] $e^{2\pi i t(\lambda+n)} = g(\lambda)$

Then

$$g(\lambda) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\lambda} \int_0^1 g(\lambda') e^{-2\pi i \lambda' k} d\lambda'$$

$$= \sum_{k\in\mathbb{Z}} e^{2\pi i k\lambda} \int_0^1 \sum f(\lambda'+n) e^{2\pi i t(\lambda'+n)} e^{-2\pi i \overset{\lambda'+n}{\lambda'} k} d\lambda'$$

$$= \sum_{k\in\mathbb{Z}} e^{2\pi i k\lambda} \hat{f}(k-t)$$

So what can we do now? ~~[illegible]~~

[illegible] Rough idea. L

[illegible] Consider $H^2(\mathbb{R}, \frac{d\lambda}{2\pi})$ and the unitary operator $f(\lambda) \longmapsto f(\lambda+1)$. [illegible] On the F.T. level this [illegible] should be mult by e^{it} on $L^2(\mathbb{R}_{\geq 0}, \frac{dt}{2\pi})$

$$f(\lambda) = \int_0^\infty e^{+it\lambda} \phi(t) dt \qquad z^n = (e^{i\lambda})^n$$

$$\phi(t) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-it\lambda} f(\lambda)$$

So you have a unitary operator mult by $e^{2\pi i t}$ on $L^2(\mathbb{R}_{>0}, dt)$. ~~[illegible]~~ So what happens? You want perhaps to use the usual smooth Poisson formula.

Ultimately you need to understand how to compare these Hilbert spaces, namely $H^2(\mathbb{R})$ with $H^2(S^1)$. What happens roughly is that you have a kind of bundle, or family of Hilbert spaces. Geometrically if you look at $L^2(\mathbb{R}, dt)$ with mult op $e^{2\pi i t}$ then you get ~~a direct is an~~ a family of Hilbert spaces over the circle, the fibre at $z \in S^1$ being $\ell^2(\lambda + \mathbb{Z})$ where $e^{2\pi i(\lambda+\mathbb{Z})} = z$.

I need to work out tomorrow's lecture.

$$Y = aX \oplus \xi_+\mathbb{C} = bX \oplus \xi_-\mathbb{C}$$

$$E \quad \cdots u^{-2}\xi_-\mathbb{C} \oplus u^{-1}\xi_-\mathbb{C} \oplus \overbrace{aX \oplus \xi_+\mathbb{C}}^{} \overset{\shortparallel}{} \underbrace{\xi_-\mathbb{C} \oplus bX}_{} \oplus u\xi_+\mathbb{C} \oplus u^2\xi_+\mathbb{C} \oplus \cdots$$

What do I do? What can you say about

$$j^* \frac{1}{z-u} j x = \sum_{n \geq 0} z^{-n-1} c^n x \qquad \text{for } |z| > 1.$$

$$\shortparallel$$

$$\frac{1}{z-c} x$$

$$= -\sum_{n<0} z^{-n-1} c^{*-n} x = -\sum_{n \geq 1} z^{n-1} c^{*n} x = -\frac{c^*}{1-zc^*} x$$

$$\frac{1}{z-c} + \frac{c^*}{1-zc^*} = \frac{1}{z-c} \underbrace{\left(1 - \cancel{zc^*} + (\cancel{z}-c)c^*\right)}_{(1-cc^*)} \frac{1}{1-zc^*}$$

$$= \frac{1}{1-zc^*} \underbrace{\left(1 - \cancel{zc^*} + c^*(\cancel{z}-c)\right)}_{1-c^*c} \frac{1}{z-c}$$