448 Question: Suppose you then go ahead and find the eigenfunction for Δ which is anti periodic under $x \mapsto x+1$ and decays ~~as~~ $y \to \infty$. Then you would like to ~~[illegible]~~ make autom. by summing over $\mathbb{Q}$.

I think there is a difficulty with all this, namely, ~~[illegible]~~ requiring anti-periodicity under $x \mapsto x+1$ is not related to the kind of spinor representation where the center $\{\pm 1\}$ of $SL_2(\mathbb{R})$ acts non-trivially

Questions you might consider: 1) relation of Bessel K function to $v(y)$ discussed above 2) what's known about bound states for the automorphic wave equation.
3) Link between $L^2(S^1)$ + Δ on D.

Think about introducing $z^{1/2}$. ~~[illegible]~~ This maybe yields a $\mathbb{Z}/2$ grading. Consider

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Where this might ~~[illegible]~~ be motivated: Periodic de B functions, example:

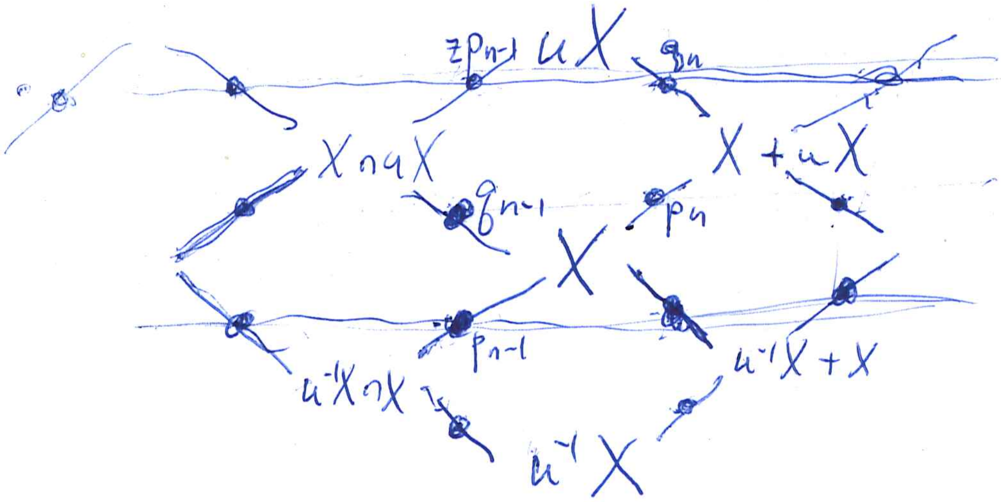
$$S(z) = \frac{z-\alpha}{1-\bar{\alpha} z} \approx \frac{z^{1/2} - \alpha z^{-1/2}}{z^{-1/2} - \bar{\alpha} z^{1/2}} = \frac{e^{i\pi\lambda} - \alpha e^{-i\pi\lambda}}{e^{-i\pi\lambda} - \bar{\alpha} e^{i\pi\lambda}}$$

$$E(\lambda) = e^{i\pi\lambda} - \alpha e^{-i\pi\lambda} \quad \text{entire} \quad \text{roots: } e^{2\pi i\lambda} = \alpha \in D$$
$$\Rightarrow \lambda \in UHP.$$

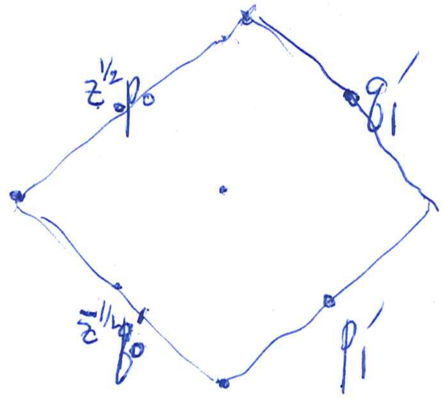
$$\frac{E(\lambda)}{\overline{E(\bar{\lambda})}} = \frac{e^{i\pi\lambda} - \alpha e^{-i\pi\lambda}}{e^{-i\pi\lambda} - \bar{\alpha} e^{i\pi\lambda}} = \frac{e^{2\pi i\lambda} - \alpha}{1 - \bar{\alpha} e^{2\pi i\lambda}} = \begin{pmatrix} 1 & -\alpha \\ -\bar{\alpha} & 1 \end{pmatrix} (e^{2\pi i\lambda})$$
$\in D$ for $\lambda \in UHP$

The question is whether you get a simpler ~~version~~ construction of the Hilbert space ~~[illegible]~~ for (h_n).

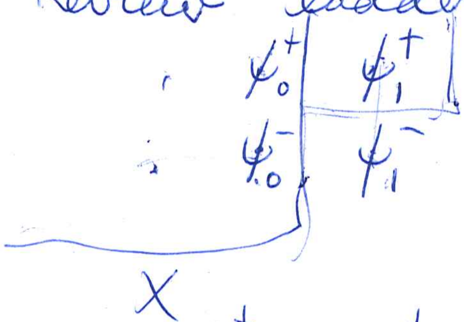
449 Go back to

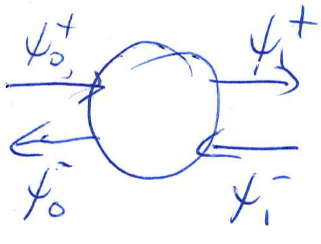


$$\begin{pmatrix} p_1' \\ q_1' \end{pmatrix} = \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} \begin{pmatrix} z^{1/2} p_0 \\ \bar{z}^{-1/2} p_0' \end{pmatrix}$$

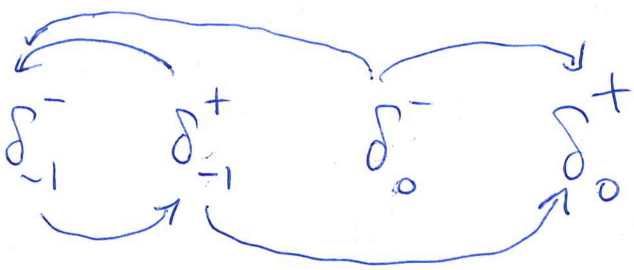


Review ladder





$$\begin{tikzcd}
\delta_{-1}^+ \arrow[r] & \delta_0^+ \arrow[r] \arrow[d, shift left] & \delta_1^+ \\
\delta_{-1}^- \arrow[u] & \delta_0^- \arrow[l] \arrow[u, shift left] & \delta_1^- \arrow[l] \arrow[u]
\end{tikzcd}$$



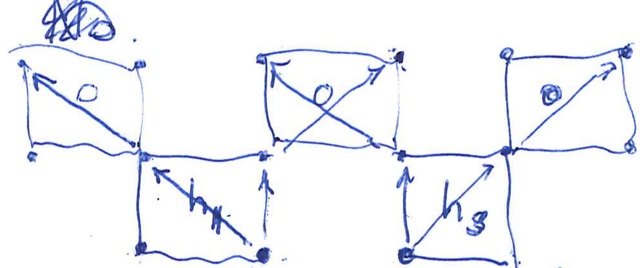
450 Put into words what intrigues you.

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

this reminds you of replacing $S(z)$ by $S(z)z$ so that V_+, V_- become $\perp$.
there is some significance of this to periodic deB functions, you saw that $z - a = e^{2\pi i\lambda} - a$ $= e^{i\pi\lambda}\left(e^{i\pi\lambda} - ae^{-i\pi\lambda}\right) = z^{1/2}\left(z^{1/2} - az^{-1/2}\right)$ factorization into deB functions,

Note that $E(\lambda) = e^{i\pi\lambda} - ae^{-i\pi\lambda}$ is anti-periodic: $E(\lambda+1) = -E(\lambda)$. So what? You feel there another Hilbert space around, a double covering of some sort.

Review construction of unitary op corresp to a sequence (h_n). Basic idea is to have 2 diml unitaries.



$\otimes^{c}$

~~What do you need?~~ Sample

Review what you know about S. Assume $S(z)$ inner-analytic for $|z|<1$ bdd by 1 $\lim_{|z|\to 1} |S(z)| = 1$ a.e. frame as closed subspace of H^2 stable under z. Put

$$Y = H^2 \ominus zSH^2 = H^2 \cap zSH^2_-$$

$$\begin{tikzcd} & \xi = 1 \quad H^2 \quad X = aX \arrow[dl, dash] \arrow[dr, dash] & \\ zH^2 \arrow[dr, dash] & & SH^2 \arrow[dl, dash] \\ bX = zX & S(z)zH^2 & S = \xi_+ \end{tikzcd}$$

~~shift~~

451 Maybe you problem is the $\mathbb{Z}$ action on H^2. Go back over intrinsic H^2.

$$d\left(\frac{az+b}{cz+d}\right) = \frac{(\cancel{c}z+d)\,a\,dz - (\cancel{a}z+b)\,c\,dz}{(cz+d)^2} = \frac{dz}{(cz+d)^2}$$

$$\cancel{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* dz}$$

$$z = e^{i\theta}$$

$$\frac{dz}{2\pi i z} = \frac{d\theta}{2\pi}$$

$$\frac{1}{2\pi i}\, d\log\left(\frac{az+b}{cz+d}\right) = \frac{1}{2\pi i}\,\frac{\dfrac{dz}{(cz+d)^2}}{\dfrac{az+b}{cz+d}} = \frac{1}{2\pi i}\,\frac{dz}{\underbrace{(az+b)}_{\bar d z+\bar c}\underbrace{(cz+d)}_{\bar b z+\bar a}}$$

$$= \frac{1}{2\pi i}\quad \frac{dz}{z}\,\frac{1}{|cz+d|^2}$$

$$\therefore\quad \frac{1}{2\pi i}\, d\log\left(\frac{\bar d z+\bar c}{cz+d}\right) = \frac{1}{2\pi i}\, d\log z\, \frac{1}{|cz+d|^2}$$

$$f(\theta)\left(\frac{d\theta}{2\pi}\right)^{1/2} \qquad \left(\frac{1}{2\pi i}\, d\log\left(\frac{\bar d z+\bar c}{cz+d}\right)\right)^{1/2} = \left(\frac{d\theta}{2\pi}\,\frac{1}{|cz+d|^2}\right)^{1/2}$$

$$= \frac{1}{|cz+d|}\left(\frac{d\theta}{2\pi}\right)^{1/2}$$

$\therefore$ $f(e^{i\theta})\left(\dfrac{d\theta}{2\pi}\right)^{1/2}$ transforms to $f\left(\dfrac{ae^{i\theta}+b}{ce^{i\theta}+d}\right)\dfrac{1}{|ce^{i\theta}+d|}\left(\dfrac{d\theta}{2\pi}\right)^{1/2}$

~~AO you want~~ You defined an action

$$\begin{pmatrix} \bar d & \bar b \\ c & d \end{pmatrix} : f(z)\left(\frac{dz}{2\pi i z}\right)^{1/2} \rightsquigarrow \frac{1}{|cz+d|}\, f\left(\frac{\bar d z+\bar c}{cz+d}\right)\left(\frac{dz}{2\pi i z}\right)^{1/2}$$

452. Alternative approach. Make $SU(1,1)$ act on $L^2(S^1)$ preserving $\| \|^2$.

$$\int |f(z)|^2 \frac{d\theta}{2\pi} \overset{?}{=} \int \left| f\left(\frac{az+b}{cz+d}\right)\right|^2 \frac{1}{|cz+d|^2} \frac{d\theta}{2\pi}$$

so $\begin{pmatrix} \bar{d} & \bar{b} \\ c & d \end{pmatrix}$: $f(z) \rightsquigarrow f\left(\frac{az+b}{cz+d}\right)\frac{1}{cz+d}$

$$\int |f(x)|^2 \frac{dx}{\pi} \overset{?}{=} \int \left| f\left(\frac{ax+b}{cx+d}\right)\right|^2 \frac{1}{|cx+d|^2} \frac{dx}{\pi}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$: $f(x) \rightsquigarrow \frac{1}{cx+d} f\left(\frac{ax+b}{cx+d}\right)$

Let $f(x) \longmapsto \sum e^{2\pi i y(\cancel{x}+n)} f(\cancel{x+n})$

~~You take~~

Today Jan 5, 98 you want some progress, a better understanding of periodic case. First recall

$$z = \frac{1-s}{1+s} = \frac{1-(-i\lambda)}{1+(-i\lambda)} = \frac{1+i\lambda}{1-i\lambda} \qquad \lambda = i\,\frac{1-z}{1+z}$$

~~It will take some thought.~~

First point. periodic (inner functions / Pick functions) on UHP ~~same~~ are the same as inner functions / Pick functions on D, via $f(\lambda) = g(e^{2\pi i\lambda})$. ~~Also true~~ ~~for all functions~~ If $S(\lambda)$ analytic on $\mathbb{R}$, then corresp. $S(z)$ analytic for $|z| \le 1$ $|S(z)| = 1$ so $S(z)$ finite Blashke product

4.53 Today spend time on periodic case to gain insight into $z^{1/2}$ stuff. Use de B theory about $L^2(\mathbb{R})$ to understand orthogonal polys on the circle. You specify a finite Blashke product $S(z)$, say $\frac{z-\alpha}{1-\bar{\alpha} z}$ with $|\alpha|<1$. Then get $X = H^2_{S^1}/SH^2_{S^1}$, $Y = H^2_{S^1}/zS^2H^2_{S^1}$ very simple situation. Now lift $S(z)$ to $S(e^{2\pi i \lambda})$ and you get $H^2_{\mathbb{R}}/SH^2_{\mathbb{R}}$.

Concentrate to see if there is anything here. You are hunting for the right Hilbert space. ~~Look for~~

Given (X, c) look for $(H, u^{1/2}, \varepsilon)$, $\varepsilon : X \hookrightarrow H$ such that

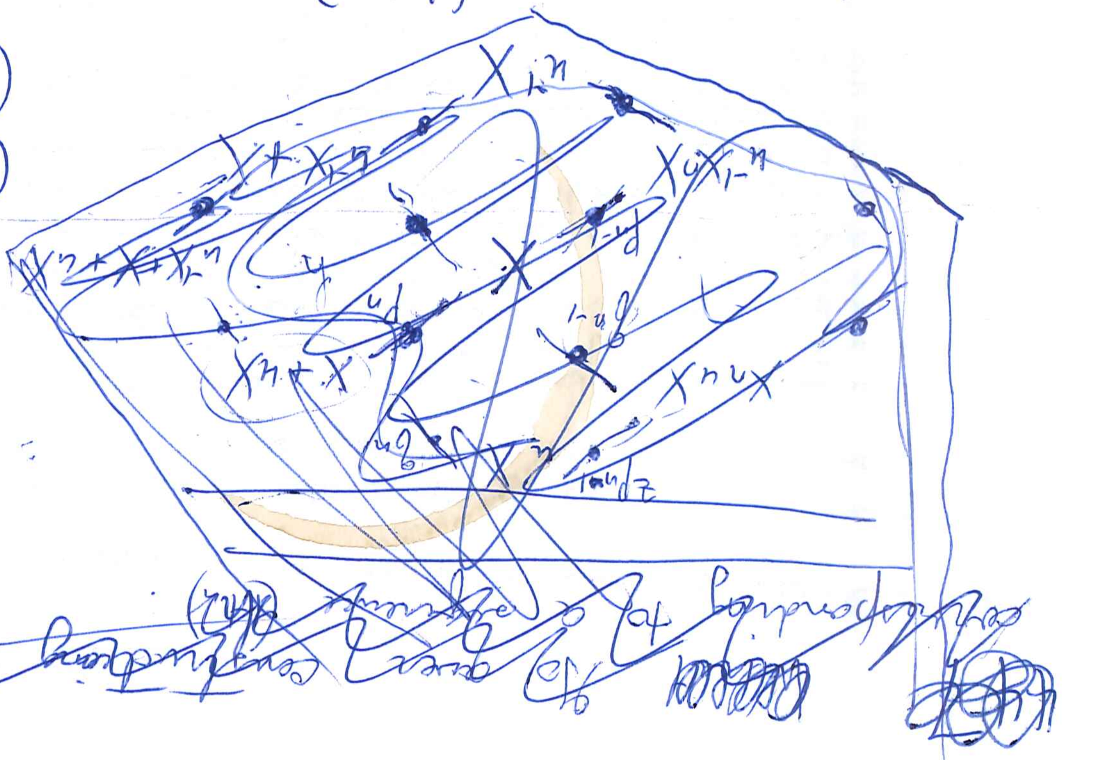
$$\varepsilon^* u^{n/2} \varepsilon = \begin{cases} c^{n/2} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Go back to $(H, u, \varepsilon : \mathbb{C} \to H) \to \varepsilon^* u^n \varepsilon = h^n$.

Then get $H : \cdots \oplus \mathbb{C}u^{-1}\xi_- \oplus \mathbb{C}\varepsilon \oplus \mathbb{C}\xi_+ \oplus \mathbb{C}u\xi_+ \oplus \cdots$ orth basis.

$$\xi_+ = \frac{(u-h)\varepsilon}{(1-|h|^2)^{1/2}} \qquad \xi_- = \frac{(1-\bar{h}u)\varepsilon}{(1-|h|^2)^{1/2}} \qquad (\xi_-, \xi_+) = -h$$

$$((1-\bar{h}u)\varepsilon, (u-h)\varepsilon)$$
$$= ((1-\bar{h}u)\varepsilon, \varepsilon)(-h)$$
$$= (1-|h|^2)(-h)$$



454 $\quad \varepsilon^* u^{1/2} \varepsilon = 0$

IDEA: $e^{\varepsilon\emptyset + t\emptyset^2}$, version of the moment theory for?

Repeat. Begin with $X = \mathbb{C}$ and the contraction $C\varepsilon = h\varepsilon$ with $|h| < 1$. Get dilation $(H, u, \varepsilon : \mathbb{C} \to H)$ $\quad \varepsilon^* u^n \varepsilon = c^n \quad n \geq 0$.
You ask whether there's a natural way to adjoin $u^{1/2}$, i.e. form $H + u^{1/2} H$. This should be an orth direct sum, so that $u^{1/2} \mapsto -u^{1/2}$ (Galois) is unitary.

$\mathbb{C} \xrightarrow{e^{2\pi i\lambda}} \mathbb{C} - \{0\}$ $\quad$ Take an $S(z)$ simplest $S(z) = z$

Pull back to $S = e^{2\pi i\lambda}$, form $H^2_{\mathbb{R}} / S H^2_{\mathbb{R}} = X$.
~~Somehow you have~~ You have $L^2(\mathbb{R}) \supset H^2_{\mathbb{R}}$ acted on by translation. So $L^2(\mathbb{R})$ has comm. unitaries $f(\lambda) \mapsto f(\lambda+1)$ and $f(\lambda) \mapsto e^{2\pi i\lambda} f(\lambda)$

Maybe focus upon $\mathbb{C} \xrightarrow{e^{\pi i\lambda}} \mathbb{C}^* \xrightarrow{z^2} \mathbb{C}^*$ and compare $H^+ / S(z^2) H^+ \qquad H^+ / S(z) H^+$

Start with an $S(z)$ with Schur expansion
$S(z) = \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & h_2 \\ \bar{h}_2 & 1 \end{pmatrix}$, better suppose $S(z)$ finite Blashke product, simple example $S(z) = \frac{z-a}{1-\bar{a}z}$
Write? You're confused: Just take
$S(z) = \frac{p_n(z)}{q_n(z)} = c \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}$. Maybe better to take
$S_n = \frac{p_n}{q_n}$ where $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots$

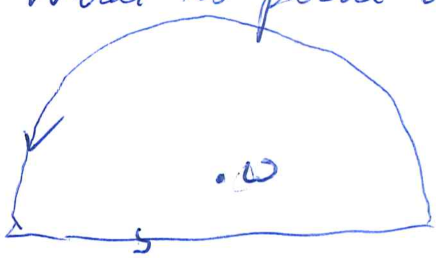
But then

455 $z^2 - \alpha = (z-\sqrt{\alpha})(z+\sqrt{\alpha})$ so

if $p_n(z) = \prod_{j=1}^{n}(z-\alpha_j)$, then $p_n(z^2)$ has the same form of twice the degree. How do I study $S(z^2)$?

$$S(z) = \begin{pmatrix} 1 & h_1 \\ \bar h_1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

$z = e^{2\pi i \lambda}$. Object: periodic inner function $S(\lambda)$ on UHP analytic on $\mathbb{R}$. Assoc. to $S(\lambda)$ is a Hilb space $X = H^+ \cap SH^-$. What is point eval? for H^+.



$$\left(\oint + \int_{-\infty}^{\infty}\right) \frac{-i}{\lambda-\omega} f(\lambda)\, \frac{d\lambda}{\pi}$$

$$f(\omega) = \oint \frac{1}{\lambda-\omega} f(\lambda)\frac{d\lambda}{2\pi i} = \left(\int_{-\infty}^{\infty} + \int_{\frown}\right) \underbrace{\frac{1}{2i(\lambda-\omega)}}_{\frac{i}{2(\lambda-\bar\omega)}} f(\lambda)\,\frac{d\lambda}{\pi}$$

~~eval~~ $ev_\omega = \dfrac{1}{2i(\bar\omega-\lambda)}$

$$\|ev_\omega\|^2 = \frac{1}{2i(\bar\omega-\omega)} = \frac{1}{4\,\mathrm{Im}(\omega)} \qquad \|ev_\omega\| = \frac{1}{2\sqrt{\mathrm{Im}(\omega)}}$$

$$|f(\omega)| \le \frac{1}{2\sqrt{\mathrm{Im}\,\omega}}\|f\|$$

alg. way. $e_{\bar\omega,\lambda} = \dfrac{1}{2i(\bar\omega-\lambda)}$

$$(e_{\bar\omega}, e_{\bar\omega'}) = \frac{1}{2i(\bar\omega'-\omega)}$$

so $e_{\bar\omega,\lambda} = \dfrac{1}{2i(\bar\omega-\lambda)}$ pt. eval for H^+.

456. to given $S(\lambda)$ inner, $f \in SH^+$

$$(k_\omega, f) = f(\omega)$$ to find k_ω: $\ni S^{-1}k_\omega \in H^+$

$$(k_\omega, Sg) = S(\omega) g(\omega) \qquad \forall g \in H^+$$

$$(\underbrace{k_\omega S^{-1}}_{\in H^+}, g) = S(\omega) g(\omega) \qquad \forall g \in H^+$$

$$\therefore \frac{k_\omega S^{-1}}{\overline{S(\omega)}} = \frac{1}{2i(\bar\omega - \lambda)}$$

$$\therefore k_\omega = \frac{1}{2i(\bar\omega-\lambda)} \overline{S(\omega)}\, S(\lambda)$$

$$k_\omega = \frac{\overline{S(\omega)}\, S(\lambda)}{2i(\bar\omega - \lambda)}$$ pt. eval. for SH^+

ev. for $H^+ \ominus SH^+$ should be $\dfrac{1 - \overline{S(\omega)}\, S(\lambda)}{2i(\bar\omega - \lambda)}$

simple case $S(\lambda) = e^{2\pi i \lambda}$

$$\frac{1 - e^{2\pi i(\lambda - \bar\omega)}}{2i(\bar\omega - \lambda)}$$

$$S(\lambda) = \frac{e^{\pi i \lambda} - a e^{-\pi i \lambda}}{e^{-i\pi\lambda} - \bar a e^{+i\pi\lambda}}$$

Consider $d\mu$ on S^1 $\qquad F(z) = \int i \underbrace{\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}}_{\frac{2}{1-z\zeta^{-1}} - 1} d\mu(\zeta)$

How do you relate F to the Schur expansion of S.

457 Anyway $\mu_n = \int \xi^n \, d\mu$ $\quad L^2(S^1, d\mu) \supseteq H^2(S^1, d\mu) \ni \dfrac{1}{1-\bar{z}\xi}$

$$(ev_{\bar{w}}, f)_{d\mu} = f(\bar{w})$$

$$\int \overline{\frac{1}{1-\bar{w}\xi}} \, \frac{1}{1-\bar{z}\xi} \, d\mu$$

$$= \sum_{m\geq 0} \sum_{n \geq 0} w^m \bar{z}^n \underbrace{\int \xi^{-m+n} \, d\mu}_{\mu(n-m)}$$

$$= \sum_{k \in \mathbb{Z}} \mu(k) \sum_{\substack{n-m=k \\ n,m\geq 0}} w^m \bar{z}^n$$

$k<0$: $\displaystyle\sum_{n\geq 0} w^{n-k} \bar{z}^n = \frac{w^{-k}}{1-w\bar{z}}$

$k \geq 0$: $\displaystyle\sum_{m \geq 0} w^m \bar{z}^{m+k} = \frac{\bar{z}^k}{1-w\bar{z}}$

$$\therefore (1-w\bar{z}) \int \overline{\frac{1}{1-\bar{w}\xi}} \, \frac{1}{1-\bar{z}\xi} = \frac{1}{1-\bar{z}\xi} + \frac{w\xi^{-1}}{1-w\xi^{-1}}$$

Go over deformation

Idea. Hilbert space + unitary + projection
$\qquad\quad H \qquad\qquad\qquad u \qquad\qquad \xi\xi^*$

what can you generate? reps of a group
$\mathbb{Z} * (\mathbb{Z}/2)$. (n point)

Program. Finite support measure on S^1 yields orth poly sequence, ~~yields~~ n. point prob measure on S^1 should be same as partial unitary of type $O(n)$ + unitary bdry conditions.

458 Jan 6, 98 You have problems with taking a prob measure on S^1, equiv., cyclic unitary rep of $\mathbb{Z}$, and removing the cyclic vector to obtain a partial unitary. The problem becomes evident, arises, when you ~~~~ try to relate the Pick function assoc. to the measure to something more familiar like the Schur sequences.

Try this. Begin with $S(z)$ Blashke prod of degree n – equiv. to a divisor of degree n in D & scalar of modulus 1. Focus on what arises from an n-point prob. measure $S(z) = \frac{p_n(z)}{q_n(z)}$

$$p_n(z) = \det(z-c), \quad q_n(z) = \det(1-zc^*)$$
$$= \prod_{j=1}^{n}(z-a_j) \qquad = \prod_{j=1}^{n}(1-z\bar{a}_j)$$

& Repeat: begin with $S(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-\bar{a}_j z}$ & form ~~~~ $X = H^+/p_n H^+$, better you form partial unitary corresp to S. You get a

Situation: $\quad p = p_n(z) = \prod_{j=1}^{n}(z-a_j) \qquad S = \frac{p}{q}$

Situation. Given S degree n Bl. product you have to do the deformation from $h=0$ to h, ~~~~ $|h|=1$. ~~~~ This is a critical obstruction point, namely how to go ~~~~ from a smooth >0 measure to ~~~~ discrete one. How to proceed? Answer. Examples perturbation method. Stages. $c = ba^* + \xi_- h \xi_+^*$

459 Another viewpt would be to focus on contractions of a given type. Consider the family of outgoing subspaces of H^+ of codim n. These have form pH^+ where $p(z) = z^n + \text{lower}$ ~~monic~~ has all roots in D. Known you get family of

$$\begin{tikzcd} & H^+ & \\ zH^+ \arrow[ur] & & \\ & & pH^+ \arrow[uul] \\ & zpH^+ \arrow[uul] \arrow[ur] & \end{tikzcd}$$

varying holom. in p.

Fill in details. Given S of degree n, write $S = \frac{p}{q}$, whence

$$H^+ \cap SH^- \xrightarrow{q} qH^+ \cap pH^- = H^+ \cap z^n H^- = P_{n-1}$$

so you get deB measure $\frac{1}{|q|^2}\frac{d\theta}{2\pi}$ const

Try harder.

Start on the other side. Suppose given an $n+1$ pt support measure $d\mu$. ~~better … $d\mu$~~

$P_n \xrightarrow{\sim} L^2(S^1, d\mu)$. Construct orth system.

$p_j \in (z^j + P_{j-1}) \cap (P_{j-1}^\perp)$ $\quad 0 \le j \le n$

$q_j \in (1 + zP_{j-1}) \cap zP_{j-1}^\perp$ $\quad$ "

$p_j = zp_{j-1} + h_j q_{j-1}$

$q_j = \bar{h}_j z p_{j-1} + q_{j-1}$ $\quad 0 \le j \le n$

also have for $j = n+1$ $\quad p_{n+1} = q_{n+1} = 0$

$0 = \bar{h}_{n+1} z p_n + q_n$

$\underbrace{1 - \bar{h} z S_n(z) = 0}_{\text{spectrum}}$

460 There seems to be an error of judgment about whether contractions correspond to S's. The problem arises from ~~conditions~~ different types of S's.

S inner i.e. $S(z)$ analytic bdd by 1 for $|z|<1$ and $|S(z)|=1$ a.e. for $|z|=1$. Same as a unit. op commuting with z $\ni$ $SH^+ \subset H^+$, up to a scalar $|1|$ is same as outgoing subspace. Gives a contraction:

$X = H^+ \cap SH^-$, c = mult by z compressed.

leads to first nbd. $Y = H^+ \cap zSH^- = \underset{S}{aX} \oplus V^+ = \underset{1}{V} \oplus bX$

There is a definite unitary operator on Y extending ba^{-1} in fact a circles worth

So it seems clear that

S modulo π factors = outgoing subspace = (X,c) ?... = partial unitary Y

$S \longmapsto SH^+ \longmapsto X = H^+ \cap SH^-$ (c induced by z) $\longmapsto$ $Y = H^+ \cap zSH^-$

If we actually ask for S on the nose, this gives an isom. $V^- \xrightarrow{S} V^+$, whence you get a unitary

But next you want a measure, this perhaps involves something new. Basic idea.

$d\mu$ $\quad Y = P_n \xrightarrow{\sim} L^2(S^1, d\mu)$

$n+1$

No take S of degree n. Then you have a unitary op. on $Y = H^+ \cap zSH^-$

$= aX \oplus \underset{\xi^+}{\mathbb{C}S} = \underset{\xi^-}{\mathbb{C}1} \oplus bX$ $\qquad$ ba^{-1} on aX, S^{-1} on $\mathbb{C}S$.

Obstructed still ~~completely~~

I seemed to have learned that an inner S determined a unitary operator on $Y = H^+ \cap zSH^-$ extending the partial unitary $ba^{-1}: X \longrightarrow zX$.
To find what you want to. You can start with $d\mu$ · $n+1$ pt support and try to recover $d\mu$

~~Basic idea~~

Consider $\overbrace{d\mu}^{\text{prob. ms.}}$ $n+1$ pt support.

$$Y = P_{\leq n} \xrightarrow{\sim} L^2(S^1, d\mu)$$

$$aX = P_{n-1}, \qquad bX = zP_{n-1}$$

orth. polys. $p_0 = 1, \dots, p_n = \xi_+$ up to norm.
$q_0 = 1, \dots, q_n = \xi_-$

$$S_n = \frac{p_n}{q_n} = \frac{\xi^+}{\xi^-}$$

What are you trying to do? To go ~~back~~ between an inner fn S of degree n and a prob measure with $n+1$ pt support. Above gives $d\mu \rightsquigarrow \left(\frac{p_j}{q_j}\right) \rightsquigarrow S = \frac{p_n}{q_n}$.
To go in other direction you have at present the following recipe. Given S form $Y = H^+ \cap zSH^-$ $X =$ etc. get unitary operator with spectrum given by $1 - zS(z) = 0$ and get cyclic vector from the K-module $X \overset{a}{\underset{b}{\rightrightarrows}} Y$
~~What~~ You ~~guess something~~ guess; $\frac{1+zS(z)}{1-zS(z)}$ should be the Pick function belonging to the measure?

~~Consider something Is there noone who can handle~~

Anyway what's probably involved is the interaction between the resolvents on both sides. Classical approach, example, formula for Green's fn. with jump at critical point.

Go over stuff again — especially $c = ha^* + \xi_- h \xi_+^*$

462 Wait: You seem to have learned that to go from a measure to an S is trickier than expected. Compare approaches. Given $d\mu$ $\overset{\text{card}}{\text{supp}} = n+1$ you ~~work~~ start with

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Given S of degree n you get a unitary op on $Y = H^+ \ominus zSH^+$ and a cyclic V.

Recall $\xi_- = 1$

$$\begin{array}{ccccc} & & H^+ & & \\ & zH^+ & & \overset{aX}{\searrow} & \\ & \overset{bX}{\searrow} & & & SH^+ \\ & & zSH^+ & \overset{S = \xi_+}{} & \end{array}$$

$Y = aX \oplus \mathbb{C}S = bX \oplus \mathbb{C}1$

$u = z + S^{-1}$

$u = ba^* + 1S^*$

Preview GNS idea. $\rho : A \underset{\text{linear}}{\longrightarrow} B$ $\quad \rho(1) = 1$.

M A-mod
N B-mod

$N \underset{i}{\overset{j}{\rightleftarrows}} M$ $\qquad jain = \rho(a)n$

$\Gamma = A \oplus A \otimes B \otimes A$

$a_1 \otimes b \otimes a_2 \longmapsto a_1 i b j a_2$ on M

When $B = RA$, then $\exists!$ $RA \longrightarrow \mathcal{L}(M)$ OK

$\hat{\rho} a \longmapsto jai$

so $\Gamma = A * \mathbb{C}F$

You need a better understanding of the transition from contraction to unitary ~~transit Make a~~

~~You want to take a look at the~~

Recap situation at the end of yesterday. Discussing

$d\mu \longleftrightarrow S$ $\quad d\mu$ $n+1$ pt supp $\quad S$ degree n.

Given $d\mu$ get partial unitary of type $O(n)$ with

463 unitary boundary condition

Given $d\mu$, $|\mathrm{supp}| = n+1$, get p_j q_j $0 \le j \le n$

$z p_n = h q_n$ $|h| = 1$. Why should $h = 1$?

Other direction. $Y = H^+ \cap zS H^- = aX \oplus \mathbb{C}S = \mathbb{C}1 \oplus bX$

~~What else is there~~

$u\, aX = zX$

$u(S) = 1$

$u = ba^* + \xi_- \xi_+^*$

$$\underbrace{ax_1 + v^+}_{} = bx + v^-$$

$$\downarrow u \qquad \qquad \downarrow z$$

$bx_1 + u(v^+) = zbx + zv^-$ $\Rightarrow$ $x_1 = zx$

$u(v^+) = zv^-$

$v^+ = S(z)v_-$

~~$zv^- = S(z)u(v^-)$~~

$$\boxed{\begin{aligned}(za-b)x &= -v^+ + v^- \\ u(v^+) &= zv^-\end{aligned}}$$

Take $v_- = 1$, then $v_+ = S$??

Use ζ for the variable pt on S^1.

$u = ba^* + \xi_- \xi_+^*$ Assume $y \ne 0 \ni u(y) = z(y)$.

z a number.

$y = ax_1 + c_1\xi_+ = c_2\xi_- + bx$

$u(y) = bx_1 + c_1\xi_-$ $zy = zc_2\xi_- + zbx$

$\therefore (x_1 = zx)$. $c_1 = zc_2$

$a(zx) + zc_2\xi_+ = bx + c_2\xi_-$

$(za-b)x = -zc_2\xi_+ + c_2\xi_-$ take $c_2 = 1$.

$(za-b)x = -z\xi_+ + \xi_-$

~~$(\zeta - \zeta)x = \ldots - z S(\zeta) + \ldots$~~

464 Try again. Given $S(z)$ Blashke product of degree n. Form $X = H^+ \cap SH^-$, $Y = H^+ \cap zSH^-$
$\xi_- = 1 \in Y$, $\xi_+ = S1 \in Y$. Then $Y = aX \oplus \mathbb{C}\xi_+$
$= bX \oplus \mathbb{C}\xi_-$. ~~$(a-b)x = y + y(\xi)\xi_-$~~ Define
$u = ba^* + \xi_-\xi_+^*$ on Y, u is unitary. Let
$(u-\zeta)y = 0$, $y \neq 0$. $y = ax_1 + c_1\xi_+ = bx + c\xi_-$

$$0 = (u-\zeta)(y) = bx_1 + c_1\xi_- - \zeta bx - \zeta c\xi_- \qquad \begin{array}{l} x_1 = \zeta x \\ c_1 = \zeta c \end{array}$$

$$a\zeta x + \zeta c\xi_+ = bx + c\xi_-$$

$$(a\zeta - b)x = (-\zeta\xi_+ + \xi_-)c \qquad \text{say } c = 1.$$

$$(a\zeta - b)x = -\zeta\xi_+ + \xi_-$$

interpret as functions ~~...~~ inside H^+

$$(\zeta - z)x_\zeta(z) = -\zeta S(z) + 1$$

you have problems with almost every ζ
ζ is fixed, let $z \longrightarrow \zeta$ get ζ sat $1 - \zeta S(\zeta) = 0$

Now continue ~~...~~ Write $S = \frac{p_n}{\tilde{p}_n}$?

I think I see the mistake. YES!! The Blashke products arising as $\frac{p_n}{\tilde{p}_n} = \frac{z^n + \text{lower}}{1 + z\,\text{lower}}$ have ~~...~~ special phase, $\prod \left(\frac{z-a_j}{1-\bar{a}_j z}\right)$! Yes!!

You still don't understand the trick. But you are beginning to understand the difficulties.

RECAP. There's a problem going between $d\mu$ and S. First point is that ~~a Blashke pro~~ an S

of degree n does determine a unitary of degree n and a cyclic vector unique up to S^1 factor; hence a measure of $n+1$ pt supp. Namely to S you assoc. a partial unitary of type $\mathcal{O}(n)$ with unit bdry condition

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RECAP. Given $S = e^{i\phi}\prod_{j=1}^{n} \frac{z-a_j}{1-\bar{a}_j z}$ $\quad |a_j|<1.$

Form. $$Y = H^+ \cap zSH^- \simeq H^+ / z\prod_{j=1}^{n}(z-a_j)H^+$$

get $$Y = aX \oplus \mathbb{C}S = zaX \oplus \mathbb{C}1$$

You get a cyclic vector, how $\quad S = e^{i\phi}\frac{p_n}{q_n}$

$$Y = H^+ \cap zSH^- \xrightarrow{\cdot q_n} H^+ \cap z^{n+1}H^- = P_n$$

P_n equipped with $\quad \|f\|^2 = \int \left|\frac{f}{\tilde{q}_n}\right|^2 \frac{d\theta}{2\pi}$

So the cyclic vector is $\frac{1}{\tilde{q}_n}$. Note this does not involve the phase $e^{i\phi}$. You need a better handle on this. Maybe the problems arise from direction - in the finite measure case you start at h_1 end, + with S you start at h_{n+1}.

Problem. Suppose given $d\mu$ equiv. H, u, ξ so you have u and $\xi\xi^*$. Thus you are given a unitary and a projection of rank 1. You have a program already in this situation, namely take $\xi_+ = \xi$ and $\xi_- = u(\xi)$. $\quad c =$

466 Consider u, ξ. Form $c = ba^* + \xi_- h \xi_+^*$

$\xi_+ = \xi$ $\quad \xi_- = u(\xi_+)$. Basically $1 = aa^* + \xi_+\xi_+^*$

so $u = ba^* + u(\xi_+)\xi_+^*$.

$$\delta \log \det(z - c) = tr \frac{1}{z-c}(-\delta c) = -tr \frac{1}{z-c}\delta c$$

$$-\delta \log \det (z-c) = tr\left(\frac{1}{z-c}\delta c\right) \qquad \delta c = \xi_- \delta h \xi_+^*$$

$$= tr \frac{1}{z-c_0} + \frac{1}{z-c_0}\Delta c$$

$$c = ba^* + \xi_- h \xi_+^* \qquad -\delta \log \det(z-c) = tr\left(\frac{1}{z-c}\delta c\right)$$

$$= tr\left(\frac{1}{z-c}\xi_- \delta h\, \xi_+^*\right) = \left(\xi_+^* \frac{1}{z-c}\xi_-\right)\delta h$$

$$\xi_+^* \frac{1}{z-c} = \xi_+^* \frac{1}{z-c_0} + \xi_+^* \frac{1}{z-c_0}\xi_- h\, \xi_+^* \frac{1}{z-c_0} + \cdots$$

$$\xi_+^* \frac{1}{z-c}\xi_- = S(z^{-1}) + S(z^{-1})\, h\, S(z^{-1})$$

$$= \frac{S(z^{-1})}{1 - S(z^{-1})h}$$

$$-\delta \log \det(z-c) = \frac{S(z^{-1})\,\delta h}{1 - S(z^{-1})h} = -\delta \log (1 - S(z^{-1})h)$$

$$\boxed{\det(z-c) = (1 - S(z^{-1})h)\det(z-c_0)}$$

$$c^* = ab^* + \xi_+ \bar{h}\, \xi_-^*$$

$$\xi_-^* \frac{1}{1-zc^*} = \xi_-^* \frac{1}{1-zc_0^*}$$

$$+ \xi_-^* \frac{1}{1-zc_0^*} z\xi_+ \bar{h}\, \xi_-^* \frac{1}{1-zc_0^*} = \frac{1}{1 - \xi_-^*(1-zc_0^*)^{-1}\xi_+ z\bar{h}}\, \xi_-^* \frac{1}{1-zc_0^*}$$

$$\delta \log \det(1 - zc^*) = \operatorname{tr} \frac{+1}{1 - zc^*}(-z\xi_+ \delta\bar{h}\xi_-^*)$$

$$-\delta \log\det(1-zc^*) = \underbrace{\left(\xi_-^* \frac{1}{1-zc^*}\xi_+\right)}_{S(z)} z\delta\bar{h} = -\delta\log\left(1 - S(z)z\bar{h}\right)$$

$$\boxed{\det(1-zc^*) = \left(1 - S(z)z\bar{h}\right)\det(1-zc_0^*)}$$

Need a good viewpoint: A prob. measure is equivalent to seq $\mu_1, \mu_2, \dots \mu_n$. The collection of moments through order n

Proceed formally. measure = sequence of moments satisfying pos. condition Can say two measures agree to order n when $\mu_j = \mu_j'$ $j \le n$

What you want to do is find the S belonging to ba^*. You need to work ~~backwa~~ the pert. theory backwards.

You want somehow to go from $u = ba^* + \xi_- \xi_+^*$ to ba^*. You have $\frac{1}{z-u}$ and $\xi_+^* \frac{1}{z-u}\xi_-$.

Maybe you know something about $\xi_+^* \frac{1}{z-u}\xi_-$ since $\xi_- = u\xi_+$. $\xi_+^* \frac{1}{z-u} u\xi_+ = \xi_+ \frac{-1}{1-zu^{-1}}\xi_+$

$$= -\sum_{n=0}^{\infty} z^n \mu_{-n}$$

Wait: ~~this is essentially the Pick fn. of $d\mu$.~~ ~~$\det(1-zc^*) = (1-S(z)z\bar{h})\det(1-zc_0^*)$~~

$$\xi_+^* \frac{1}{z-u} u\xi_+ = \int \frac{1}{z-\zeta}\zeta\, d\mu = -\int \frac{1}{1-z\zeta^{-1}}\, d\mu$$

Anyway ~~without~~ you have to write up stuff

$$c = ba^* + \xi_- h \xi_+^*$$

$$u = u(aa^* + \xi_+ \xi_+^*) = ba^* + \xi_- \xi_+^*$$

How are $\frac{1}{z-u}$ $\frac{1}{z-c_0}$ related? What's important here are the functions $\underbrace{\xi_+^* \frac{1}{z-u} \xi_-}_{S_1(z^{-1})}$, $\underbrace{\xi_+^* \frac{1}{z-c_0} \xi_-}_{S_0(z^{-1})}$ because the perturbation is $\xi_- \xi_+^*$.

In general $S_h(z^{-1}) = \xi_+^* \frac{1}{z-c_h} \xi_-$. Formulas are

$$\xi_+^* \frac{1}{z-c_h} = \xi_+^* \left(\frac{1}{z-c_0} + \frac{1}{z-c_0} \xi_- h \xi_+^* \frac{1}{z-c_0} + \cdots \right.$$

$$\xi_+^* G_h = \xi_+^* \big(G_0 + G_0 (\xi_- h \xi_+^*) G_0 + G_0 (\quad) G_0 (\quad) G_0 +$$

$$= \xi_+^* G_0 + (\xi_+^* G_0 \xi_- h) \xi_+^* G_0 + (\xi_+^* G_0 \xi_- h)^2 \xi_+^* G_0 + \cdots$$

$$\boxed{\xi_+^* G_h = \frac{1}{1 - S_0(z^{-1}) h} \xi_+^* G_0} \qquad \boxed{S_h(z^{-1}) = \frac{1}{1 - S_0(z^{-1}) h} S_0(z^{-1})}$$

$$\xi_+^* G_0 = \frac{1}{z - c_h + \xi_- h \xi_+^*} = \xi_+^* G_h + \xi_+^* G_h (-\xi_- h \xi_+^*) G_h + \cdots$$

$$\boxed{\xi_+^* G_0 = \frac{1}{1 + S_h(z^{-1}) h} \xi_+^* G_h} \qquad \boxed{S_0(z^{-1}) = \frac{1}{1 + S_h(z^{-1}) h} S_h(z^{-1})}$$

$$\boxed{1 + S_h(z^{-1}) h =} \; 1 + \frac{S_0(z^{-1}) h}{1 - S_0(z^{-1}) h} = \boxed{\frac{1}{1 - S_0(z^{-1}) h}}$$ So what?

$$c_h = ba^* + \xi_- h \xi_+^*$$

$$-\delta \log \det (z - c_h) = \mathrm{tr} \frac{1}{z-c_h} \xi_- \delta h \xi_+^* = \xi_+^* \frac{1}{z-c_h} \xi_- \delta h$$

$$= S_h(z^{-1}) \delta h = \frac{1}{1 - S_0(z^{-1}) h} S_h(z^{-1}) \delta h$$

$$= -\delta \log (1 - S_0(z^{-1}) h)$$

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$$\boxed{\begin{aligned} \det(z-c_h) &= \left(1-S_0(z^{-1})h\right)\det(z-c_0) \\ \hline \det(z-c_0) &= \frac{1}{1-S_0(z^{-1})h}\det(z-c_h) \\ &= \left(1+S_h(z^{-1})h\right)\det(z-c_h) \end{aligned}}$$

$c_h = ba^* + \xi_- h \xi_+^*$

$c_h^* = ab^* + \xi_+ \bar{h} \xi_-^*$

Other side $\boxed{\xi_-^* \frac{1}{1-zc_h^*}\xi_+ = S_h(z)}$

$$\frac{1}{1-zc_h^*} = \frac{1}{1-zc_0^* - z\xi_+ \bar{h}\xi_-^*}$$

$$\xi_-^*\frac{1}{1-zc_h^*} = \xi_-^*\frac{1}{1-zc_0^*} + \xi_-^*\frac{1}{1-zc_0^*}(\xi_+ z\bar{h}\xi_-^*)\frac{1}{1-zc_0^*} + \xi_-^*\frac{1}{1-zc_0^*}(\xi_+^* z\bar{h}\xi_-)\frac{1}{1-zc_0^*}(\xi_+^* z\bar{h}\xi_-)\frac{1}{1-zc_0^*}$$

$$= \frac{1}{1-\xi_-^*\frac{1}{1-zc_0^*}\xi_+ z\bar{h}}\,\xi_-^*\frac{1}{1-zc_0^*}$$

$$\boxed{\xi_-^* G_h = \frac{1}{1-S_0(z)z\bar{h}}\xi_-^* G_0} \qquad \boxed{S_h(z) = \frac{1}{1-S_0(z)z\bar{h}}S_0(z)}$$

$$\boxed{1+S_h(z)z\bar{h}} = 1+\frac{1}{1-S_0(z)z\bar{h}}S_0(z)z\bar{h} = \boxed{\frac{1}{1-S_0(z)z\bar{h}}}$$

$$-\delta\log\det(1-zc_h^*) = \operatorname{tr}\frac{1}{1-zc_h^*}z\xi_+\delta\bar{h}\xi_-^* = S_h(z)\,z\,\delta\bar{h}$$

$$\cancel{= -\delta\log(1+S_h(z)z} = \frac{1}{1-S_0(z)z\bar{h}}S_0(z)z\delta\bar{h}$$

$$= -\delta\log\left(1-S_0(z)z\bar{h}\right)$$

$$\boxed{\begin{aligned} \det(1-zc_h^*) &= \left(1-S_0(z)z\bar{h}\right)\det(1-zc_0^*) \\ \hline \det(1-zc_0^*) &= \left(1+S_h(z)z\bar{h}\right)\det(1-zc_h^*) \end{aligned}}$$

Now take $h=1$ whence $c_h = u$. Do you get a relation between the moments and S_0?

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$$S_1(z) = \xi_-^* \frac{1}{1-zu^*} \xi_+ \qquad \xi_- = u\xi_+$$
$$\xi_-^* = \xi_+^* u^*$$
$$= \xi_+^* \frac{u^*}{1-zu^*} \xi_+$$

equiv. to the ~~involves~~ moments of the measure.

~~$\det(1 - z S_h$~~

$$1 + S_1(z) z = \frac{1}{1 - S_0(z) z}$$

$$1 + S_1(z) z = \xi_+^* \left(1 + \frac{zu^*}{1-zu^*}\right) \xi_+ = \xi_+^* \frac{1}{1-zu^*} \xi_+$$

$$\therefore \quad \xi_+^* \frac{1}{1-zu^*} \xi_+ = \frac{1}{1 - S_0(z) z} \qquad \frac{2}{1-zu^*} - 1 = \frac{1+zu^*}{1-zu^*}$$

$$\xi_+^* \, i \frac{1+zu^*}{1-zu^*} \xi_+ = i \frac{1 + S_0(z) z}{1 - S_0(z) z}$$

$$\xi_- h \xi_+^* \qquad z \xi_+ \bar{h} \xi_-^*$$

Recap (H, u, ξ) $\quad |\xi| = 1 \quad$ ξ cyclic.

$$u = ba^* + \xi_- \xi_+^* \qquad \xi_+ = \xi \qquad \xi_- = u(\xi).$$
$$u^* = ab^* + \xi_+ \xi_-^*$$

$$S_1(z) = \xi_-^* \frac{1}{1-zu^*} \xi_+ = \xi_-^* \left(\frac{1}{1-zc_0} + \frac{1}{1-zc_0} z \xi_+ \xi_-^* \frac{1}{1-zc_0} + \cdots \right) \xi_+$$
$$= S_0(z) + S_0(z) z S_0(z) + \cdots = \frac{S_0(z)}{1 - S_0(z) z}$$

$$S_1(z) = (u\xi)^* \frac{1}{1-zu^*} \xi = \xi^* \frac{u^*}{1-zu^*} \xi$$

$$1 + z S_1(z) = \xi^* \left\{ 1 + \frac{zu^*}{1-zu^*} \right\} \xi = \xi^* \frac{1}{1-zu^*} \xi$$
$$\parallel$$
$$1 + z \frac{S_0}{1 - S_0 z} = \frac{1}{1 - S_0(z) z}$$

$$\therefore \quad \frac{1}{1 - S_0(z) z} = \xi^* \frac{1}{1-zu^*} \xi$$

471 You want to put together the general S.

c contraction on X, to dilate: $\varepsilon : X \to Y \circlearrowleft u$, $\varepsilon^*\varepsilon = 1$

$\varepsilon^* u \varepsilon = c$ $\quad \|\varepsilon x_0 + u\varepsilon x_1\|^2 = \|x_0 + cx_1\|^2 + (x_1, (1-c^*c)x_1)$

$= \|c^*x_0 + x_1\|^2 + (x_0, (1-cc^*)x_0)$.

$X \xrightarrow{\varepsilon} H \circlearrowleft u$ $\quad \varepsilon^* u^n \varepsilon = c^n$

$u^{-n} = c^{*-n}$

$\pi_*(x) = u\varepsilon x - \varepsilon c x$

$\|\pi_+(x)\|^2 = (u\varepsilon x - \varepsilon c x, u\varepsilon x - \varepsilon c x)$

$= \|x\|^2 - \|cx\|^2$.

H: $\quad \oplus u^{-1}V_- \oplus X \oplus V_+^* \oplus uV_+ \oplus \cdots$

$n \geq 1$

$(\pi_-(x), u^n \pi_+ x') = ((\varepsilon - u\varepsilon c^*)x, u^n(u\varepsilon - \varepsilon c)x')$

$= (x, (\varepsilon^* - c\varepsilon^* u^{-1}) u^n (u\varepsilon - \varepsilon c)x')$

$$\left.\begin{array}{l} \varepsilon^* u^{n+1}\varepsilon - c\varepsilon^* u^n \varepsilon \\ - \varepsilon^* u^n \varepsilon c + c\varepsilon^* u^{n-1}\varepsilon c \end{array}\right\} = 0 \text{ for } n \geq 1$$

for $n=0$ get $-c + cc^*c$

$n = -1$ $\quad \begin{cases} 1 - cc^* \\ -c^*c + cc^{*2}c \end{cases}$

$(\varepsilon x, \sum_{n\in\mathbb{Z}} u^n \pi_* x_n) = (\varepsilon x, \sum_{n>0} u^{-n}(u\varepsilon - \varepsilon c)x_{-n})$

$= (x, \sum_{n>0} (c^{*n-1} - c^{*n}c) x_{-n}$ $\qquad \dfrac{z^{-1}}{1-z^{-1}c} = \dfrac{1}{z-c}$

$= \sum_{n\geq 1} ((c^{n-1} - c^*c^n)x, x_{-n}) = (\pi_+ \sum_{n\geq 1} z^{-n} c^{n-1} x, \sum z^{-n} \pi_+ x_{-n})$

472.

$$\pi_- x = \varepsilon x - u\varepsilon(c^* x)$$

$$(\pi_- x,\ u^n \pi_- x') = \left(\varepsilon x - u\varepsilon c^* x,\ u^n(\varepsilon x - u\varepsilon c^* x)\right)$$

$$\pi_-^* u^n \pi_- = (\varepsilon^* - c\varepsilon^* u^{-1})\, u^n (\varepsilon - u\varepsilon c^*) \qquad n \geq 1$$

$$= \underbrace{\varepsilon^* u^n \varepsilon}_{c^n} - c\underbrace{\varepsilon^* u^{n-1}\varepsilon}_{c^{n-1}} - \underbrace{\varepsilon^* u^{n+1}\varepsilon}_{c^{n+1}} c^* + c\underbrace{\varepsilon^* u^n \varepsilon}_{c^n} c^*$$

$$\varepsilon^* u^n \pi_+ = \varepsilon^* u^n (u\varepsilon - \varepsilon c) = \underbrace{\varepsilon^*(u^{n+1})\varepsilon}_{c^{n+1}} - \underbrace{\varepsilon^* u^n \varepsilon}_{c^n} c = 0.$$

$$\pi_+^* u^n \pi_+ = (\varepsilon^* u^{-1} - c^*\varepsilon^*)\, u^n (u\varepsilon - \varepsilon c) \qquad n \geq 1$$

$$= 0.$$

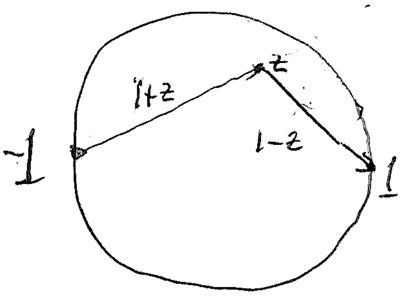
Prop to the effect $\exists$ decomp.

You have today connected Pick functions ~~measures~~ with scattering functions in the disk cases. Prepare to move to UHP. You need to understand analogous theory functions.

$$z = \frac{1-s}{1+s} \qquad s = \frac{1-z}{1+z} \qquad \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|z| < 1 \longleftrightarrow \operatorname{Re}(s) > 0$$

$$s = -i\lambda$$



$$z = \frac{1+i\lambda}{1-i\lambda} \qquad \lambda = i\,\frac{1-z}{1+z}$$

deg 1 rational fns.

$$\frac{\zeta + z}{\zeta - z} = \frac{\zeta + \frac{1+i\lambda}{1-i\lambda}}{\zeta - \frac{1+i\lambda}{1-i\lambda}} = \frac{\zeta(1-i\lambda) + 1 + i\lambda}{\zeta(1-i\lambda) - 1 - i\lambda}$$

$$= \frac{(-i\zeta + i)\lambda + (\zeta + 1)}{(-i\zeta - i)\lambda + (\zeta - 1)} =$$

473 $\zeta = e^{i\theta} = \frac{1+ix}{1-ix}$ $\qquad z = \frac{1+i\lambda}{1-i\lambda}$

$$\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} = \frac{\zeta+z}{\zeta-z} = \frac{\frac{1+ix}{1-ix}+\frac{1+i\lambda}{1-i\lambda}}{\frac{1+ix}{1-ix}-\frac{1+i\lambda}{1-i\lambda}} = \frac{(1+ix)(1-i\lambda)+(1-ix)(1+i\lambda)}{(1+ix)(1-i\lambda)-(1-ix)(1+i\lambda)}$$

$$= \frac{1+x\lambda}{i(x-\lambda)}$$

$$\frac{d\theta}{2\pi} = \frac{d\zeta}{2\pi i\zeta} = \frac{1}{2\pi i}\left(\frac{i}{1+ix}+\frac{i}{1-ix}\right)dx = \frac{1}{1+x^2}\frac{dx}{\pi}$$

$$i\,\frac{1+ze^{-i\theta}}{1-ze^{i\theta}}\,\frac{d\theta}{2\pi} = \frac{1+x\lambda}{x-\lambda}\,\frac{1}{1+x^2}\,\frac{dx}{\pi}$$

$$= \frac{1+x^2+x(\lambda-x)}{x-\lambda}\,\frac{1}{1+x^2}\,\frac{dx}{\pi}$$

$$= \left(\frac{1}{x-\lambda} - \frac{x}{1+x^2}\right)\frac{dx}{\pi}$$

What do you need next? You want eventually to ~~use~~ understand periodic de B functions. theory.

Formulate in term of $S(\lambda) = \frac{E^\#(\lambda)}{E(\lambda)}$ e.g. $E(\lambda) = e^{-i\pi\lambda}$. Partial unitary – what to do about Z?

$$\gamma^t = \varepsilon^* u^t \varepsilon \qquad t \geq 0$$

Dilate to get (H, u^t, ε), some pos. def. function on $\mathbb{R}$

$$\rho(t) = \begin{cases}\gamma^t & t \geq 0\\ (\gamma^*)^{-t} & t \leq 0\end{cases}$$

maybe you want $e^{\gamma t}$

You probably want γ around $\gamma - \gamma^* \geq 0$.

$$\pi_-(x) = \lim_{t \downarrow 0} \frac{(U^t \varepsilon - \varepsilon C^t)}{t} x$$

$$\|e^{-\gamma t} y\|^2 - \|e^{-\gamma(t+\delta t)} y\|^2 = \|\bigcirc e^{-\gamma t} y\|^2$$

$-(\gamma+\gamma^*)$

UHP version, a contraction c becomes $e^{-t\gamma}$

$e^{it\tau}$ $\quad e^{itH}$ $\quad U^t = e^{itA}$ $\quad e^{itC}$

In any case set this up.

$$\varepsilon^* e^{itA} \varepsilon = e^{itC} \qquad t \geq 0$$

Get this moving.

$$U^t \varepsilon x(t)$$

You start with $U^t = e^{iAt}$ 1-par un. gp. on H

$\varepsilon : X \hookrightarrow H$ assume $\varepsilon^* e^{iAt} \varepsilon = e^{iCt}$ $t \geq 0$.

To reconstruct H from (X, C).

$$\left(e^{iAt_1} \varepsilon x_1, e^{iAt_2} \varepsilon x_2\right) = (x_1, \underbrace{\varepsilon^* e^{iA(t_2-t_1)} \varepsilon}\, x_2)$$

$$e^{iC(t_2-t_1)} \qquad t_2 \geq t_1$$
$$e^{-iC(t_1-t_2)} \qquad t_2 \leq t_1$$

You want $\pi_+^* : X \to V_+$ some completion of X.

Use $e^{-\gamma t}$ $\qquad \varepsilon^* e^{iAt} \varepsilon = e^{-\gamma t}$ $\qquad t \geq 0$

You want

$$\|x\|^2 = \|x\|^2 - \lim_{t\to\infty} \|e^{-\gamma t} x\|^2 = \lim_{t\to\infty} \left[-\|e^{-\gamma t} x\|^2\right]_0^t$$

$$= \int_0^\infty dt \left(-\frac{d}{dt}\right) \|e^{-\gamma t} x\|^2 \quad \cancel{= \int_0^\infty dt \left(-\frac{d}{dt}\right)(x, e}$$

$$\left(-\frac{d}{dt}\right)\left(e^{-\gamma t} x, e^{-\gamma t} x\right) = \left(\gamma e^{-\gamma t} x, e^{-\gamma t}\right) + \left(e^{-\gamma t} x, \gamma e^{-\gamma t} x\right)$$

$$= \left(e^{-\gamma t} x, (\gamma^* + \gamma) e^{-\gamma t} x\right)$$

475 So $\|\pi_+ x\|^2 = (x, (\gamma^* + \gamma)x) = -\frac{d}{dt}\|e^{-\gamma t}x\|^2\Big]_{t=0}$

How to use this? Scattering picture.

H should be

$$L^2(\mathbb{R}_{<0}, V_-) \oplus X \oplus L^2(\mathbb{R}_{>0}, V_+)$$

but you will have to work to define $U^t = e^{iAt}$

This continuous setting should be harder, but you should learn how to work with it. Actually the scattering might be easy, and the reconstruction should be handled with functions of the frequency parameter λ or ω.

$$\|e^{-\gamma t}x\|^2 = (e^{-\gamma t}x, e^{-\gamma t}x)$$

$$\frac{d}{dt}\|e^{-\gamma t}x\|^2 = -(\gamma e^{-\gamma t}x, e^{-\gamma t}x) - (e^{-\gamma t}x, \gamma e^{-\gamma t}x)$$

$$-\frac{d}{dt}\|e^{-\gamma t}x\|^2 = \Big(e^{-\gamma t}x, \underbrace{(\gamma + \gamma^*)}_{\geq 0} e^{-\gamma t}x\Big) \overset{\text{def}}{=} \|\pi_+(e^{-\gamma t}x)\|^2$$

$$\Big[-\|e^{-\gamma t}x\|^2\Big]_0^\infty = \int_0^\infty \|\pi_+(e^{-\gamma t}x)\|^2\,dt$$

$$\|x\|^2 - \lim_{t\to\infty}\|e^{-\gamma t}x\|^2 \qquad\qquad \|t \mapsto \pi_+(e^{-\gamma t}x)\|^2 \text{ in } L^2(\mathbb{R}_{>0}, V_+)$$

$$H^2(\mathbb{R}, V_+)$$

$$\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{-\gamma t}e^{-st}\,dt = \frac{1}{s+\gamma}$$

$$\lambda \mapsto \pi_+\left(\frac{1}{-i\lambda+\gamma}\right)x = \pi_+\frac{i}{\lambda+i\gamma}x$$

 Other direction

$$\|x\|^2 = \lim_{t\to+\infty} \|e^{-\gamma^* t}x\|^2 = \int_0^\infty \|\pi_+(e^{-\gamma^* t}x)\|^2\,dt$$

$$= \|t \mapsto \pi_+(e^{-\gamma^* t}x)\|^2 \text{ in } L^2(\mathbb{R}_{>0}, V_+)$$

If things are symmetrical then you change γ to γ^* to get $\pi_+ \frac{i}{\lambda+i\gamma^*}x$

Now γ is almost skew hermitian,

Mixed up. $\gamma+\gamma^*$ rank 1 and >0.

spectrum of γ in RHP.

so that $e^{-\gamma t}$ decays as $t\to+\infty$

spectrum of γ^* also in RHP

so that $e^{-\gamma^* t}$ decays as $t\to+\infty$.

OK. Let $\|\pi x\|^2 = (x, (\gamma+\gamma^*)x) = 2\,\mathrm{Re}(x,\gamma x)$

$$\left\|\pi \frac{i}{\lambda+i\gamma}x\right\|^2 = \int_{-\infty}^{\infty}\left(\frac{i}{\lambda+i\gamma}x,_{\gamma+\gamma^*}\frac{i}{\lambda+i\gamma}x\right)\frac{d\lambda}{2\pi}$$

$$= \int_{-\infty}^{\infty}\left(x, \frac{1}{\lambda-i\gamma^*}(\gamma+\gamma^*)\underbrace{\frac{1}{\lambda+i\gamma}}_{\text{anal in UHP}}x\right)\frac{d\lambda}{2\pi}$$

if res. calc. works then

$$= 2\pi i\left(x, (\gamma+\gamma^*)\frac{1}{i\gamma^*+i\gamma}x\right)\frac{1}{2\pi} = \|x\|^2$$

$$\left\|\pi\frac{i}{\lambda+i\gamma^*}x\right\|^2 = \int_{-\infty}^{\infty}\left(x, \frac{1}{\lambda-i\gamma}(\gamma+\gamma^*)\frac{1}{\lambda+i\gamma^*}x\right)\frac{d\lambda}{2\pi}$$

477 assume $\dim(\mathcal{X})=1$. $\gamma \in$ RHP $\mathcal{X}=\mathbb{C}\xi$

$\xi = 1$.

$$\frac{i}{\lambda+i\gamma} \quad \text{outgoing state}$$

$$\frac{i}{-\lambda+i\gamma^*} \quad \text{incoming ~~state~~}$$

$$S(\lambda) = \frac{\lambda+i\gamma^*}{\lambda+i\gamma}$$ wrong. The guess

is that you must replace λ by $-\lambda$ in the incoming picture $\therefore S(\lambda) = \dfrac{-\lambda+i\gamma^*}{\lambda+i\gamma}$

pole at $-i\gamma$ $\in$ Lower HP

zero at $i\gamma^* \in$ UHP

Guess at the moment is $\quad \mathcal{E}_+^* \mathcal{E}x = \pi \dfrac{i}{\lambda - i\gamma} x$

$$\mathcal{E}_-^* \mathcal{E}x = \pi \frac{i}{-\lambda+i\gamma^*} x$$

Somehow you have to construct the UHP analogues

γ spectrum $\subset$ RHP , $\gamma+\gamma^*$ rank 1 ≥ 0.

$$\Big(\underbrace{\frac{i}{\omega-\bar{\lambda}}}_{e(\bar{\lambda},\omega)}, f\Big) = \int_{-\infty}^{\infty} \frac{-i}{\omega-\lambda} f(\omega)\, \frac{d\omega}{2\pi} = f(\lambda)$$

$$(e(\bar\lambda,\omega), f) = \int_{-\infty}^{\infty} \overline{\frac{i}{\omega-\bar\lambda}}\, f(\omega)\, \frac{d\omega}{2\pi} = \int \frac{1}{\omega-\lambda} f(\omega)\, \frac{d\omega}{2\pi i} = f(\lambda).$$

$$(e_{\bar\lambda}, e_{\bar\mu}) = e_{\bar\mu,\lambda} = \frac{i}{\lambda-\bar\mu}$$

$$\|e_{\bar\lambda}\|^2 = \frac{i}{\lambda-\bar\lambda} = \frac{i}{2i\,\mathrm{Im}\lambda} = \frac{1}{2\,\mathrm{Im}\lambda}$$

Cauchy Schwartz $\quad |f(\lambda)| = |(e_{\bar\lambda}, f)| \leq \dfrac{1}{(2\,\mathrm{Im}\lambda)^{1/2}} \|f\|$.

478 [redacted] $S(\lambda)$ anal in UHP bdd by 1 and $|S(\lambda)| = 1$ a.e. on $\mathbb{R}$. Consider $X = H^+ \cap SH^-$

Your philosophy is that S should correspond to a partial unitary (without bound states) equivalently a partial hermitian.

$$\zeta = e^{i\theta} = \frac{1+i\omega}{1-i\omega} \qquad \left(\frac{1+i\omega}{\sqrt{1+\omega^2}}\right)^2 = \frac{(1+i\omega)^2}{(1+i\omega)(1-i\omega)} = \frac{1+i\omega}{1-i\omega}$$

$$e^{i\theta/2} = \frac{1+i\omega}{\sqrt{1+\omega^2}}$$

Good examples? $H^+/(\lambda-i)H^+$ $\quad S = \frac{\lambda-i}{\lambda+i}$

$H^+/e^{2\pi i\lambda}H^+$ $\quad S = e^{2\pi i\lambda} = \frac{e^{\pi i\lambda}}{e^{-\pi i\lambda}}$

Take an $S(\lambda)$ e.g. $e^{2\pi i\lambda}$ form $X = H^+/e^{2\pi i\lambda}H^+$ on which you have $e^{it\lambda}$ $t \geq 0$, as well as $z = \frac{1+i\lambda}{1-i\lambda}$. Wait.

Discuss the philosophy. You have $L^2 = H^+ \oplus H^-$ canonically associated to a (linear) disk D in $\mathbb{CP}^1$. Given an S analytic in D with radial limits $|\ | = 1$ a.e. get $X = H^+ \ominus SH^+ = H^+ \cap SH^-$. Functions on ∂D give mult ops on L^2. ~~Interested~~ in $e^{it\omega} = U^t$ 1-param. unitary gp. Also $\frac{1+i\omega}{1-i\omega}$.

What do we have on X? Contraction, 1-param. semi-group of contractions. Basically you should focus on what is needed to reconstruct L^2.

$$e_{\bar\lambda,\omega} = \frac{i}{\omega-\bar\lambda} \quad : \quad \int_{-\infty}^{\infty} \frac{-i}{\omega-\lambda} f(\omega)\frac{d\omega}{2\pi} = f(\lambda).$$

point eval. for SH^+ is $\overline{S(\lambda)}\,\frac{i}{\omega-\lambda}\,S(\omega)$

since $g = Sf \Rightarrow \left(\overline{S(\lambda)}\, e_{\bar\lambda} S,\ Sf\right) = S(\lambda)\left(e_{\bar\lambda}, f\right) = S(\lambda)f(\lambda) = g(\lambda)$

point eval. for $H^+ \ominus SH^+$ is different

$$e_{\bar\lambda} - \overline{S(\lambda)}\, S e_{\bar\lambda}$$

since $e_{\bar\lambda} - \overline{S(\lambda)}\, S e_{\bar\lambda} \in H^+ \cap (SH^+)^\perp$

$$\left(e_{\bar\lambda} - \overline{S(\lambda)}\, S e_{\bar\lambda},\ Sg\right) = (Sg)(\lambda) - S(\lambda)g(\lambda) = 0$$

So for $S(\lambda) = e^{2\pi i\lambda}$ get

$$i\,\frac{1-e^{-2\pi i\bar\lambda}\,e^{2\pi i\omega}}{\omega-\bar\lambda}$$

Look: you need to reconstruct H. Either have the contraction $C = \varepsilon^*\frac{1+i\omega}{1-i\omega}\varepsilon$ or the semigrp. $\varepsilon^* e^{it\omega}\varepsilon$ which should be $e^{-\gamma t}$ where $-\gamma = \varepsilon^* i\omega\varepsilon$ in a suitable sense.

So how to do the scattering?

Take $X = H^+ \cap SH^-$ semigroup $\varepsilon^* e^{it\omega}\varepsilon$ $t \geq 0$. where $\varepsilon : X \hookrightarrow H^+$. I think it should be true that $\varepsilon^* e^{it\omega}\varepsilon = e^{-\gamma t}$, $-\gamma = \varepsilon^* i\omega\varepsilon$. This is unbounded so you have to be careful about its meaning. You guess of course that the graph of $-\gamma$ corresponds to the contraction $\varepsilon^*\frac{1+i\omega}{1-i\omega}\varepsilon$

480 Learn how to do this. You need to do this carefully. The basic Hilbert space is $L^2(\mathbb{R}, \frac{d\omega}{2\pi})$ which is isom. via F.T. to $L^2(\mathbb{R}, dt)$

$$\phi(t) = \int e^{-i\omega t} f(\omega) \frac{d\omega}{2\pi} \qquad f(\omega) = \int e^{+i\omega t} \phi(t)\, dt$$

I will try to work inside $H^2(\mathbb{R}, \frac{d\omega}{2\pi})$ which consists of analytic function $f(\lambda)$ on the UHP should be L^2 on $\operatorname{Im} \lambda = $ const.

$\lambda = +is$

$e^{i\lambda t} = e^{-st}$

$$f(\lambda) = \int_0^\infty e^{i\lambda t} \phi(t)\, dt$$

$$f(\omega + ia) = \int_0^\infty e^{i\omega t} e^{-at} \phi(t)\, dt$$

$$|f(\lambda)| \leq \int_0^\infty \underbrace{|e^{i\lambda t}|^2}_{e^{-2\operatorname{Im}(\lambda)t}} dt \int_0^\infty |\phi(t)|^2 dt$$

$$\frac{1}{2\operatorname{Im}(\lambda)} \|\phi\|^2$$

$$e_{\bar\lambda} = \frac{i}{\omega - \bar\lambda} \qquad \|e_{\bar\lambda}\|^2 = \frac{i}{\lambda - \bar\lambda} = \frac{1}{2\operatorname{Im}(\lambda)}$$

$f(\omega) \longmapsto f(\omega + ia)$ corresp. to $\phi(t) \longmapsto e^{-at}\phi(t)$

obviously a 1-param. family of contraction operators $\lim_{a \downarrow 0}$ increases to identity

converges in L^2 norm by dominated convergence

$$\|\phi\|^2 - \|e^{-at}\phi\|^2$$

Return to the problem at hand.

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$$f(\omega) = \int e^{i\omega t_1}\phi(t_1)\,dt_1 \qquad t+t_1=t_2$$

$$e^{i\omega t}f(\omega) = \int e^{i\omega(t+t_1)}\phi(t_1)\,dt_1 = \int e^{i\omega t_2}\phi(t_2-t)\,dt_2$$

~~Suppose~~ You want $\|f\|^2 - \|\varepsilon^* e^{i\omega t}f\|^2$

$$= (f, f) - (e^{i\omega t}f, \varepsilon\varepsilon^* e^{i\omega t} f)$$

$$= \Big(f, \underbrace{(1 - e^{-i\omega t}\varepsilon\varepsilon^* e^{i\omega t})f}_{e^{-i\omega t}(1-\varepsilon\varepsilon^*)e^{i\omega t}f}\Big)$$

Look at on the t side.

$$\phi(\tau) \longmapsto \phi(\tau - t) \longmapsto \chi_{[0,1]}\phi(\tau-t)$$

$$f(\omega) = \int_0^1 e^{i\omega t}\underbrace{\phi(t)}\,dt$$

ϕ support in $[0,1]$

$$e^{i\omega\tau}f(\omega) = \int_\tau^{1+\tau} e^{i\omega t}\phi(t-\tau)\,dt$$

$$\varepsilon^* e^{i\omega\tau}f(\omega) = \int_\tau^1 e^{i\omega t}\phi(t-\tau)\,dt$$

$$\frac{d}{d\tau}\varepsilon^* e^{i\omega\tau}f(\omega) =$$

$$f(\omega) = \int_0^1 e^{i\omega u}\phi(u)\,du$$

$$e^{i\omega t}f(\omega) = \int_0^1 e^{i\omega(t+u)}\phi(u)\,du$$

$$\varepsilon^* e^{i\omega t}f(\omega) = \int_0^{1-t} e^{i\omega(t+u)}\phi(u)\,du$$

$$\frac{d}{dt}\varepsilon^* e^{i\omega t}f = i\omega\int_0^{1-t} e^{i\omega(t+u)}\phi(u)\,du - e^{i\omega}\phi(1-t)$$

$$\frac{d}{dt}\varepsilon^* e^{i\omega t}f\Big|_{t=0} = i\omega f - e^{i\omega}\underbrace{\phi(1)}_{\int e^{-i\omega'}f(\omega')\frac{d\omega'}{2\pi}}$$

Iterating a parametrix idea, recall the construction of heat kernels where you take a path with the correct tangent vector and you iterate to obtain the semi group. Central limit theorem

Review situation, actual the example you are considering, continuous shift.

General case: Given an inner function $S(\lambda)$ on the UHP, you have $\varepsilon : X = H^+ \cap SH^- \hookrightarrow L^2(\mathbb{R}, \frac{d\lambda}{2\pi})$ and then $\varepsilon^* e^{it\lambda} \varepsilon$ should be a 1-parameter semi group of contractions.

$$\varepsilon^* e^{i\lambda t_1} \varepsilon \, \varepsilon^* e^{i\lambda t_2} \varepsilon \overset{?}{=} \varepsilon^* e^{i\lambda (t_1 + t_2)} \varepsilon$$

should be true because $e^{i\lambda t} H^+ \subset H^+$ and $X = H^+ \cap SH^- \xleftarrow[\varepsilon^*]{\sim} H^+/SH^+$, take Take

$$\begin{array}{ccccc} SH^+ & \xhookrightarrow{S} & H^+ & \xrightarrow{\varepsilon^*} & X \\ \downarrow{\scriptstyle e^{it\lambda}} & & \downarrow{\scriptstyle e^{it\lambda}} & & \downarrow \\ SH^+ & \xhookrightarrow{S} & H^+ & \xrightarrow{\varepsilon^*} & X \end{array}$$

$$e^{i\lambda t}(\operatorname{Ker} \varepsilon^*) \subset (\operatorname{Ker} \varepsilon^*)$$

$$\Longrightarrow \quad \varepsilon^* e^{i\lambda t}(1 - \varepsilon^* \varepsilon) = 0$$

You want the infinitesimal generator of this semi group, should be an unbdd operator in general, so you aim for the graph.

$$\varepsilon^* e^{i\lambda t} \varepsilon = e^{i\gamma t}$$

$\operatorname{spec}(\gamma) \subset$ UHP

$\frac{1}{2i}(\gamma - \gamma^*) \geq 0$.

$$\frac{1 + i\lambda t}{1 - i\lambda t} \qquad \frac{1 + i\lambda t}{\sqrt{1 + \lambda^2 t^2}}$$

Maybe can understand graph. Perhaps you should see if $\varepsilon^* \lambda \varepsilon = \gamma$ in some sense.

$$\varepsilon^* \frac{1 + i\lambda}{1 - i\lambda} \varepsilon = \frac{1 + i\gamma}{1 - i\gamma} \ ?$$

 Situation: $S(\lambda)$ inner fn on UHP

$X = H^+ \cap SH^-$ ~~...~~

$X \xhookrightarrow[\varepsilon]{} L^2(\mathbb{R}, \frac{d\omega}{2\pi})$ $\qquad X \underset{\varepsilon}{\overset{\varepsilon^*}{\rightleftarrows}} H^+$

$$\begin{array}{ccccccccc} 0 & \to & H^+ & \xrightarrow{S} & H^+ & \xrightarrow{\varepsilon^*} & X & \to & 0 \\ & & \downarrow e^{i\lambda t} & & \downarrow e^{i\lambda t} & & \downarrow & & \\ 0 & \to & H^+ & \xrightarrow{S} & H^+ & \xrightarrow{\varepsilon^*} & X & \to & 0 \end{array}$$

know that $\varepsilon^* e^{i\lambda t}\varepsilon$ is a 1-par. semi-gp of contractions on X

What's the program? Recall yesterday's ideas, namely start with $e^{i\gamma t}$ on X, and construct scattering reps.

$$\left[\|e^{i\gamma t}x\|^2\right]_{t=\infty}^{t=0} = \int_0^\infty dt\left(-\frac{d}{dt}\right)\left(x, (e^{+i\gamma t})^* e^{i\gamma t}x\right)$$

$$\| \qquad\qquad = \int_0^\infty dt\left(x, (e^{i\gamma t})^*\left((+i\gamma)^* + (i\gamma)\right)e^{i\gamma t}x\right)(-1)$$

$$\|x\|^2 - \lim_{t\to+\infty}\|e^{i\gamma t}x\|^2$$

$$= \int_0^\infty dt\left(e^{i\gamma t}x, \underbrace{(-i\gamma + i\gamma^*)}_{\frac{\gamma-\gamma^*}{i}}e^{i\gamma t}x\right) \qquad \geq 0$$

Viewpoint: You have a concrete semigp $e^{i\gamma t}$ defined as $\varepsilon^* e^{i\lambda t}\varepsilon$. You want to construct the quadratic form $\left(x, \frac{\gamma-\gamma^*}{i}x\right)$, really you need the Hilbert space ~~on~~ for which this is the scalar product, should be the completion of the domain, probably the ~~...~~ Krein form restricted to the graph.

We have to review Krein form + ~~...~~ partial herm. ops.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1^* y_1 - y_2^* y_2$$

$$\begin{pmatrix} 1 \\ c \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ c \end{pmatrix} = 1 - c^* c$$

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$$\begin{pmatrix}1\\ \gamma\end{pmatrix}^* \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = y_1 - \gamma^* y_2$$

$$\left(\begin{pmatrix}1\\ \gamma\end{pmatrix}Y\right)^\circ = \begin{pmatrix}\gamma^*\\ 1\end{pmatrix}Y \qquad \left(\begin{pmatrix}1\\ c\end{pmatrix}Y\right)^\circ = \begin{pmatrix}c^*\\ 1\end{pmatrix}Y$$

C.T. $\lambda = i\dfrac{1-z}{1+z}$ UHP $\leftrightarrow$ D

$$\boxed{\begin{pmatrix}c & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}c^*\\ 1\end{pmatrix} = cc^* - 1}$$

$$z = \frac{1+i\lambda}{1-i\lambda} = \begin{pmatrix}i & 1\\ -i & 1\end{pmatrix}(\lambda)$$

$$\frac{1}{2}\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}i & 1\\ -i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}\begin{pmatrix}i & 1\\ i & -1\end{pmatrix} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\begin{pmatrix}1\\ \gamma\end{pmatrix}^*\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\ \gamma\end{pmatrix} = \begin{pmatrix}1 & \gamma^*\end{pmatrix}\begin{pmatrix}-i\gamma\\ i\end{pmatrix} = -i\gamma + i\gamma^* = \frac{\gamma-\gamma^*}{i}$$

$$\begin{pmatrix}1\\ \gamma\end{pmatrix}^*\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}1 & \gamma^*\end{pmatrix}\begin{pmatrix}-iy_2\\ +iy_1\end{pmatrix} = i(-y_2 + \gamma^* y_1)$$

$$\left(\begin{pmatrix}1\\ \gamma\end{pmatrix}Y\right)^\circ =$$

$$0 = \begin{pmatrix}1\\ \gamma\end{pmatrix}^*\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}1 & \gamma^*\end{pmatrix}\begin{pmatrix}-y_2\\ y_1\end{pmatrix} = -y_2 + \gamma^* y_1$$

$$\therefore \left(\begin{pmatrix}1\\ \gamma\end{pmatrix}Y\right)^\circ = \begin{pmatrix}1\\ \gamma^*\end{pmatrix}Y$$

what's next? ~~[illegible]~~ You want to consider dissipative γ. Keep X f.d. Then no problem with

485 [redacted] Review scattering for (X, γ)

$\dim(X) < \infty$ to simplify $\frac{\gamma-\gamma^*}{i} \geq 0$.

$$\|x\|^2 - \lim_{t\to\infty} \|e^{i\gamma t}x\|^2 = -\int_0^\infty dt \frac{d}{dt}\left(e^{i\gamma t}x, e^{i\gamma t}x\right)$$

$$= \int_0^\infty dt\,(-1)\left(e^{i\gamma t}x, \left((i\gamma)^* + (i\gamma)\right) e^{i\gamma t}x\right)$$

$$= \int_0^\infty dt \left(e^{i\gamma t}x, \frac{\gamma-\gamma^*}{i} e^{i\gamma t}x\right) = \int_0^\infty dt \left\|\pi(e^{i\gamma t}x)\right\|^2$$

where $\pi : X \longrightarrow V$ is completion wrt $\|\pi x\|^2 = \left(x, \frac{\gamma-\gamma^*}{i}x\right)$

[redacted] Assume $\operatorname{spec}(\gamma) \subset$ UHP. $\Longrightarrow \|e^{i\gamma t}x\| \to 0 \quad \forall x$.

get isom. embed $x \longmapsto \pi(e^{i\gamma t}x)$, $X \hookrightarrow L^2(\mathbb{R}_{\geq 0}, V)$

$$\int_0^\infty dt\, e^{i\omega t}\pi(e^{i\gamma t}x) = \pi\left(\frac{i}{\omega+\gamma}x\right) \qquad H^2(\,;V)$$

[redacted] similarly get isom embed using γ^*.

[redacted]

$$\|x\|^2 - \lim_{t\to-\infty}\|e^{+i\gamma^* t}x\|^2 = \int_{-\infty}^0 dt\left(e^{i\gamma^* t}x, \underbrace{\left((i\gamma^*)^* + (i\gamma^*)\right)}_{\frac{\gamma-\gamma^*}{i}} e^{i\gamma^* t}x\right)$$

$$= \int_{-\infty}^0 dt \left\|\pi(e^{i\gamma^* t}x)\right\|^2$$

$\operatorname{spec}(i\gamma^*) \subset$ RHP $\Rightarrow e^{i\gamma^* t}x \to 0 \quad t \to -\infty$

so get isom embed. $x \mapsto \pi(e^{i\gamma^* t}x)$, $X \hookrightarrow L^2(\mathbb{R}_{\leq 0}, V)$

$$\int_{-\infty}^0 dt\, e^{i\omega t}\pi(e^{i\gamma^* t}x) = \pi\left(\frac{1}{i(\omega+\gamma^*)}x\right)$$

Can you calculate the scattering, find X inside?
You should check what you have

So far you have $X \hookrightarrow L^2(\mathbb{R}_{\geq 0}, V)$

$$x \longmapsto \int_0^\infty dt\ \nu(e^{i\gamma t}x)$$

digress to the unitary case. Given (X, c), set V_+ = comp. of X for $\nu_+(x) = \|x\|^2 - \|cx\|^2$. Then get

$$x \longmapsto \nu_+ z$$

Get signs straight. $\quad u^n =$ mult by z^n

$u^t =$ mult by $e^{i\omega t}$

$$x \mapsto \sum_{n\geq 0} \|(1-c^*c)^{1/2} c^n x\|^2 = \|\nu_+ \sum_{n\geq 0} z^{-n} c^n x\|^2$$

$$= \|\nu_+ \frac{1}{1-z^{-1}c} x\|^2$$

$\pi_+ \frac{1}{1-z^{-1}c} x$ analytic outside S^1

$\pi_- \frac{1}{1-zc^*} x$ —— inside S^1

Continuous case. Maybe you will have to change notation.

(X, c)

$$\|\pi_+ x\|^2 = \|x\|^2 - \|cx\|^2$$

$$\|\pi_- x\|^2 = \|x\|^2 - \|c^* x\|^2$$

$$x \longmapsto \sum_{n\geq 0} \|\pi_+ c^n x\|^2 = \|x\|^2 - \lim_{n\to\infty} \|c^n x\|^2$$

$$\sum_{n\geq 0} \|\pi_- c^{*n} x\|^2 = \|x\|^2 - \lim_{n\to\infty} \|c^{*n} x\|^2$$

$$\ell^2 \otimes V_+ \qquad\qquad \ell^2 \otimes V_-$$

go back to your $\varepsilon : X \hookrightarrow H$ $\qquad \varepsilon^* u^n \varepsilon = c^n \quad \forall n.$

If you write

$$\cdots \oplus u^{-1} V_- \oplus \varepsilon X \oplus V_+ \oplus u V_+ \oplus \cdots$$

when scattering is perf., then get $X \hookrightarrow \ell^2_{\geq 0} \otimes V_-$

487 cont. case $(X, e^{i\gamma t})$.

what is H to be? $\int e^{i\gamma t} \mathcal{E}(x(t))\, dt$

$e^{i\gamma}$ So you should write down H in the discrete case $\sum_{n\in\mathbb{Z}} u^n \,\mathcal{E} x_n$

$$\left\| \int_0^\infty U^t \,\mathcal{E}(x_t)\,dt \right\|^2$$

$$= \left\| \int_0^\infty e^{i\gamma t} x_t \,dt \right\|^2 + \quad ?$$

---

maybe go back to

$$\left\| \sum_{n\geq 0} u^n x_n \right\|^2 = \|x_0\|^2 + \left(x_0, \sum_{n\geq 1} u^n x_n\right) + \left(\sum_{n\geq 1} u^n x_n, x_0\right)$$
$$+ \left\| \sum_{n\geq 0} u^n x_{n+1} \right\|^2$$

$$\left\| \sum_{n\geq 0} c^n x_n \right\|^2 = \|x_0\|^2 + \left(x_0, \sum_{n\geq 1} c^n x_n\right) + \left(\sum_{n\geq 1} c^n x_n, x_0\right)$$
$$+ \left\| c \sum_{n\geq 0} c^n x_{n+1} \right\|^2$$

$$\left\| \sum_{n\geq 0} u^n x_n \right\|^2 - \left\| \sum_{n\geq 0} u^n x_{n+1} \right\|^2 = \left\| \sum_{n\geq 0} c^n x_n \right\|^2 - \left\| \sum_{n\geq 0} c^{n+1} x_{n+1} \right\|^2$$

$$\left\| \sum_{n\geq 0} u^n x_{n+1} \right\|^2 - \left\| \sum_{n\geq 0} u^n x_{n+2} \right\|^2 = \left\| \sum_{n\geq 0} c^n x_{n+1} \right\|^2 - \left\| \sum_{n\geq 0} c^{n+1} x_{n+2} \right\|^2$$

$$\left\| \sum_{n\geq 0} u^n x_n \right\|^2 - \lim_{\overrightarrow{k}} \left\| \sum_{n\geq 0} u^n x_{n+k} \right\|^2 = \left\| \sum_{n\geq 0} c^n x_n \right\|^2 + \left\| \Pi_+ \left(\sum c^n x_{n+1}\right) \right\|^2$$
$$+ \left\| \Pi_+ \sum_{n\geq 0} c^n x_{n+2} \right\|^2 + \cdots$$

488 Given $(X, e^{i\gamma t})$ what is the dilation

Assume $\dim(X) < \infty$ $\qquad \frac{\gamma - \gamma^*}{i} \geq 0$.

$$\|x\|^2 - \lim_{t\to+\infty} \|e^{i\gamma t} x\|^2 = \int_0^\infty dt \, \frac{d}{dt} \|e^{i\gamma t} x\|^2 (-1)$$

$$= \int_0^\infty dt \, \left(e^{i\gamma t} x, \left((i\gamma)^* + (i\gamma)\right) e^{i\gamma t} x\right)(-1)$$

$$= \int_0^\infty dt \, \left(e^{i\gamma t} x, \left(\frac{\gamma - \gamma^*}{i}\right) e^{i\gamma t} x\right)$$

$$= \int_0^\infty dt \, \|\nu(e^{i\gamma t} x)\|^2 \qquad \|\nu(x)\|^2 = \left(x, \frac{\gamma - \gamma^*}{i} x\right)$$

$$= \Big\| \underbrace{\int_0^\infty e^{i\omega t} \, \nu(e^{i\gamma t}) \, dt}_{\nu\left(\frac{-1}{i\omega + i\gamma} x\right)} \Big\|^2$$

Check. $$\left\| \nu \frac{i}{\omega + \gamma} x \right\|^2 = \int \frac{d\omega}{2\pi} \left(\frac{i}{\omega + \gamma} x, \left(\frac{\gamma - \gamma^*}{i}\right) \frac{i}{\omega + \gamma} x \right)$$

$$= \int \frac{d\omega}{2\pi} \left(x, \frac{-i}{\omega + \gamma^*} \left(\frac{\gamma - \gamma^*}{i}\right) \frac{i}{\omega + \gamma} x \right)$$

pole at $\omega = -\gamma^*$ $\qquad$ anal in UHP

Ignore transform - use

$$x \longmapsto \nu(e^{i\gamma t} x), \quad t \geq 0 \qquad \varepsilon^* u^t \varepsilon x = e^{i\gamma t} x \quad t > 0$$

Now $\varepsilon^* u^t \varepsilon x = e^{+i\gamma^* t} x \qquad t < 0 \qquad \varepsilon^* u^{-s} \varepsilon = e^{(i\gamma)^* s}$

$e^{+i\gamma^* t}$

$$\|x\|^2 - \lim_{t\to-\infty} \|e^{i\gamma^* t} x\|^2 = \int_{-\infty}^0 dt \, \frac{d}{dt} \left(e^{i\gamma^* t} x, e^{i\gamma^* t} x\right)$$

$$= \int_{-\infty}^0 dt \left(e^{i\gamma^* t} x, \left(\overbrace{(i\gamma^*)^* + (i\gamma^*)}^{\frac{\gamma - \gamma^*}{i}}\right) e^{i\gamma^* t} x\right)$$

489 $$\|x\|^2 - \lim_{t\to-\infty} \|e^{i\gamma^* t}x\|^2 = \int_{-\infty}^{0} dt\, \|\nu(e^{i\gamma^* t}x)\|^2$$

so we have the transform

$$x \longmapsto \nu(e^{i\gamma^* t}x), \quad t\leq 0.$$

Anyway see if you can connect X to $L^2(\mathbb{R}_{>0}, V)$

You take direct sum but then you need to define u^t on pairs $(\varepsilon x, f(t'))$. $u^t(\varepsilon x, 0) = (e^{i\gamma t}x, ?)$

$u^t \varepsilon x = \varepsilon^* u^t \varepsilon x$, something ~~like x~~ with the right size $\|x\|^2 - \|e^{i\gamma t}x\|^2 = \int_t^0 dt \frac{d}{dt}(e^{i\gamma t}x, e^{i\gamma t}x)$

$$= \int_0^t dt \Big(e^{i\gamma t}x, \underbrace{\big((i\gamma)^* + i\gamma\big)}_{\frac{\gamma - \gamma^*}{i}} e^{i\gamma t}x\Big)(-1)$$

so clear you want $\nu(e^{i\gamma t'}x)$, $0\leq t' \leq t$

Guess: $u^\tau(\varepsilon x, f(t)) = \big(\varepsilon e^{i\gamma \tau}x,\ \nu(e^{i\gamma t}x)_{0<t<\tau} + f(t-\tau)_{t<\tau}\big)$

~~$u^t(\varepsilon x + f(t'))$~~

$$u^t\big(\varepsilon x + f(t')_{0<t'}\big) = \varepsilon e^{i\gamma t}x + \nu e^{i\gamma t'}x_{\,0<t'<t} + f(t-t')_{t'<t<\infty}$$

What you need next is ~~$\varepsilon^* u^{-t}$~~

$$\big(x, \varepsilon^* u^{-t} f(t')_{0<t'}\big) = \big(u^t\varepsilon x, f(t')_{0<t'}\big)$$
$$= \big(\nu e^{i\gamma t'}x_{\,0<t'<t},\ f(t')_{0<t'}\big)$$
$$= \int_0^t dt' \big(\nu e^{i\gamma t'}x, f(t')\big) = \Big(x, \int_0^t dt'\, e^{-i\gamma^* t'}\nu^* f(t')\Big)$$

490 What is ν? $\nu : X \longrightarrow V \ni \|\nu x\|^2 = (x, \frac{\gamma-\gamma^*}{i} x)$. Concretely identify: $V = \overline{\left(\frac{\gamma-\gamma^*}{i}\right)^{1/2} X}$

and $\nu x = \left(\frac{\gamma-\gamma^*}{i}\right)^{1/2} x$ $\quad (\nu x', \nu x) = (x', (\ )x)$

$\therefore$ ~~...~~. Put $\rho = \left(\frac{\gamma-\gamma^*}{i}\right)^{1/2}$ $\quad \rho \geq 0$ rank 1 say.

$\rho = \xi k^2 \xi^*$ $\quad \nu x = k \xi^* x$

$$\|\nu x\|^2 = (k\xi^* x, k\xi^* x) = k^2(x, \xi\xi^* x)$$
$$= (x, \rho x)$$

$\therefore \nu x = k\xi^* x$ and $\nu^* = k\xi$

Repeat: $\frac{\gamma-\gamma^*}{i} = \xi k^2 \xi^*$ $\quad k > 0,\ |\xi| = 1$

Let $V = \mathbb{C}$ let $\nu : X \longrightarrow V$ be $\nu x = k\xi^* x$

Then $(\nu x, \nu x) = |k\xi^* x|^2 = k^2 x^* \xi \xi^* x = x^* \frac{\gamma-\gamma^*}{i} x$

Thus $\nu x = k\xi^* x$ i.e. $\nu = (k\xi)^*$

whence $\nu^* = k\xi$

$$\int_0^t dt' \, e^{-i\gamma^* t'} k\xi f(t')$$

You have to begin again with $X, \gamma \ni$

$\frac{\gamma-\gamma^*}{i} = \xi k^2 \xi^*$ $\quad |\xi| = 1,\ k > 0.$

$$u^t \varepsilon x = e^{i\gamma t} x \oplus (k\xi^* e^{i\gamma t'} x)_{0<t'<t}$$

$$(x, \varepsilon^* u^{-t} (f(t'))_{0<t'}) = (u^t \varepsilon x, (f(t'))_{0<t'})$$
$$= ((k\xi^* e^{i\gamma t'} x)_{0<t'<t}, f(t')_{0<t'})$$

$$=$$

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$$L^2(\mathbb{R}_{<0}, V) \oplus X \oplus L^2(\mathbb{R}_{>0}, \underset{\underset{\mathbb{C}}{\shortparallel}}{V})$$

$$\varepsilon^* u^t \varepsilon x = e^{i\gamma t} \qquad t \geq 0.$$

$$\|x\|^2 - \|e^{i\gamma t} x\|^2 = \int_0^t dt' \frac{d}{dt}(e^{i\gamma t'}x, e^{i\gamma t'}x)(-1)$$

$$= \int_0^t dt' (e^{i\gamma t'}x, \underbrace{((i\gamma)^* + i\gamma)(-1)}_{\frac{\gamma - \gamma^*}{i} = \xi k^2 \xi^*} e^{i\gamma t'}x)$$

$k^2 > 0$
$|\xi| = 1$

$$= \int_0^t dt' \, \|k \xi^* e^{i\gamma t'} x\|^2$$

$$u^t \varepsilon x = \varepsilon(e^{i\gamma t} x) \oplus (k\xi^* e^{i\gamma t'} x)$$

$0 < t' < t$

$k\xi^* e^{i\gamma t'} x \, \chi_{(0,t)}(t')$

certainly isometric.

Now take $f \in L^2(\mathbb{R}_{>0})$ & calc.

$$\varepsilon^* u^{-t} f$$

$$(x, \varepsilon^* u^{-t} f) = (u^t \varepsilon x, f) = \int_0^t \overline{k\xi^* e^{i\gamma t'} x} \, f(t') \, dt'$$

$$= \int_0^t x^* e^{-i\gamma^* t'} \xi \, k \, f(t') \, dt'$$

$$\varepsilon^* u^{-t} f = \int_0^t (e^{-i\gamma^* t'} \xi) \, k f(t') dt'$$

This is a start but what you want is to take $g \in L^2(\mathbb{R}_{<0})$

492 Given $(X, e^{i\gamma t})$ to construct $H, u^t, \varepsilon : X \to H$
$\varepsilon^* u^t \varepsilon = e^{i\gamma t}$ for $t > 0$. Idea

$$\|x\|^2 - \|e^{i\gamma t}x\|^2 = \int_0^t dt' \underbrace{\left(-\frac{d}{dt'}\right)(e^{i\gamma t'}x, e^{i\gamma t'}x)}_{-1\,(e^{i\gamma t'}x,\,((i\gamma)^* + i\gamma)e^{i\gamma t'}x)}$$

$$= \int_0^t dt' \left(e^{i\gamma t'}x, \overbrace{\frac{\gamma - \gamma^*}{i}}^{\nu^*\nu} e^{i\gamma t'}x\right)$$

$$= \int_0^t dt' \, \|\nu e^{i\gamma t'}x\|^2$$

here $\nu : X \to V$ $\ni$ $\nu^*\nu = \frac{\gamma - \gamma^*}{i}$ and $\overline{\nu X} = V$.

~~scattering becomes Next define~~

$$u^t \varepsilon x = \varepsilon e^{i\gamma t} x + (1 - \varepsilon\varepsilon^*) u^t \varepsilon x$$

Let's do the discrete case first.

Given (X, c) $\qquad H, u, \varepsilon : X \to H$ $\qquad \varepsilon^* u^n \varepsilon = c^n$, $\varepsilon^* u^{-n} \varepsilon = c^{*n}$ $\quad n \geq 0$

$\pi_+ x = u\varepsilon x - \varepsilon c x \qquad V_+ = \overline{\pi_+ V} \subset H.$

$\varepsilon^* \pi_+ x = 0 \qquad (\pi_+ x, \pi_+ x) = (u\varepsilon x, u\varepsilon x - \varepsilon c x)$
$= \|x\|^2 - \|cx\|^2$

~~$n \geq 1 \quad (\pi_+ x', u^n \pi_+ x) = (x', (\varepsilon^* u^{-1} - c^* \varepsilon^*) u^n (u\varepsilon - \varepsilon c) x)$~~

$$(\pi_+ x', u^n \pi_+ x) = ?$$

$$\pi_+^* u^n \pi_+ = (u\varepsilon - \varepsilon c)^* u^n (u\varepsilon - \varepsilon c)$$
$$= (\varepsilon^* u^{-1} - c^* \varepsilon^*)(u^{n+1}\varepsilon - u^n \varepsilon c)$$
$$= \varepsilon^* u^n \varepsilon - \varepsilon^* u^{n-1} \varepsilon c - c^* \varepsilon^* u^{n+1} \varepsilon + c^* \varepsilon^* u^n \varepsilon c$$
$$n \geq 1 \qquad = c^n - c^{n-1} c - c^* c^{n+1} + c^* c^n c = 0$$
$$n = 0 \qquad = 1 - c^* c - \cancel{c^* c} + \cancel{c^* c}$$

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$$\boxed{\begin{array}{lll} \varepsilon^* u^n \pi_+ &= 0 & n \geq 0 \\ \pi_+^* u^n \pi_+ &= 0 & n \geq 1 \\ &= 1 - c^* c & n = 0. \end{array}}$$

$$\varepsilon^* u^n (u\varepsilon - \varepsilon c) = \varepsilon^* u^{n+1}\varepsilon - (\varepsilon^* u^n \varepsilon) c = c^{n+1} - c^n \varepsilon = 0.$$

$$\pi_+^* u^n \pi_+ = (u\varepsilon - \varepsilon c)^* u^n \pi_+ \qquad n \geq 0$$

$$= (\varepsilon^* u^{-1} - c^* \varepsilon^*) u^n \pi_+ = \varepsilon^* u^{n-1} \pi_+ - c^* \cancel{\varepsilon^* u^n \pi_+}$$

$$= 0 \qquad n \geq 1$$

$$= \varepsilon^* u^{-1} (u\varepsilon - \varepsilon c) = \varepsilon^* u^{-1} u \varepsilon - \varepsilon^* u^{-1} \varepsilon c$$

$$= 1 - c^* c.$$

$$\pi_- = u^{-1}\varepsilon - \varepsilon c^* \qquad V_- = \overline{\pi_- X}$$

$$\boxed{\begin{array}{lll} \varepsilon^* u^{-n} \pi_- &= 0 & n \geq 0 \\ \pi_-^* u^{-n} \pi_- &= 0 & n \geq 1 \\ &= 1 - c c^* & n = 0. \end{array}}$$

$$\pi_-^* u^n \pi_+ = (u^{-1}\varepsilon - \cancel{\varepsilon c^*})^* u^n \pi_+ \qquad n \geq 0$$

$$= 0$$

$$H = \oplus\, u^{-1} V^- \oplus V^- \oplus \varepsilon X \oplus V^+ \oplus u V^+ \oplus \cdots \qquad \text{YES.}$$

orthogonal direct sum

$$\Big\| \sum_{n \geq 0} u^n x_n \Big\|^2 = \Big\| x_0 + \sum_{n \geq 1} u^n x_n \Big\|^2 = \|x_0\|^2 + \Big(x_0, \sum_{n \geq 1} u^n x_n\Big) + \Big(\sum_{n \geq 1} u^n x_n, x_0\Big) + \Big\|\sum_{n \geq 1} u^n x_n\Big\|^2$$

$$\Big\| \sum_{n \geq 0} c^n x_n \Big\|^2 =$$

$$\Big\| \sum_{n \geq 0} u^n x_n \Big\|^2 - \Big\| \sum c^n x_n \Big\|^2 = \sum$$

$$\Big\| \sum_{n \geq 1} c^n x_n$$

494 $$\Big\|\sum_{n\geq 0} u^n x_n\Big\|^2 - \Big\|\sum_{n\geq 0} u^n x_{n+1}\Big\|^2 = \Big\|\sum_{n\geq 0} c^n x_n\Big\|^2 - \Big\|c\sum_{n\geq 0} c^n x_{n+1}\Big\|^2$$

$$\Big\|\sum_{n\geq 0} u^n x_{n+1}\Big\|^2 - \Big\|\sum_{n\geq 0} u^n x_{n+2}\Big\|^2 = \Big\|\sum_{n\geq 0} c^n x_{n+1}\Big\|^2 - \Big\|c\sum_{n\geq 0} c^n x_{n+2}\Big\|^2$$

$$\Big\|\sum_{n\geq 0} u^n x_n\Big\|^2 - \lim_{k\to\infty}\Big\|\sum_{n\geq 0} u^n x_{n+k}\Big\|^2 = \Big\|\sum_{n\geq 0} c^n x_n\Big\|^2 + \sum_{k=1}^{\infty}\Big\|(1-c^*c)^{1/2}\sum_{n\geq 0} c^n x_{n+k}\Big\|^2$$

In the continuous case you need to do something not so obvious.

$$\varepsilon^*(\varepsilon x_0 + u\varepsilon x_1) = x_0 + cx_1 = 0 \qquad x_0 = -cx_1$$

$$\varepsilon x_0 + u\varepsilon x_1 = -\varepsilon c x_1 + u\varepsilon x_1 = (u\varepsilon - \varepsilon c)x_1 .$$

$$u^t\varepsilon x = \underbrace{\varepsilon\varepsilon^* u^t \varepsilon x}_{\varepsilon e^{i\gamma t}x} + (1-\varepsilon\varepsilon^*)u^t\varepsilon x \qquad \varepsilon.$$

To take a limit as $t \downarrow 0$. $\qquad u^t\varepsilon - \varepsilon e^{i\gamma t}$

$$\|(u^t\varepsilon - \varepsilon e^{i\gamma t})x\|^2 =$$

$$\|u^t\varepsilon x\|^2 - (u^t\varepsilon x, \varepsilon e^{i\gamma t}x) - (\varepsilon e^{i\gamma t}x, u^t\varepsilon x) + \|\varepsilon e^{i\gamma t}x\|^2$$

$$\|x\|^2 - \|e^{i\gamma t}x\|^2 \cancel{- \|e^{i\gamma t}x\|^2} \cancel{+ \|e^{i\gamma t}x\|^2}$$

$$= \cancel{\|x\|^2} - (x + i\gamma t x, x + i\gamma t x)$$

$$= -(x, i\gamma t x) - (i\gamma t x, x)$$

$$= \left(x, \frac{\gamma - \gamma^*}{i}x\right)t$$

49F ~~[illegible]~~ example. $X = \mathbb{C}$ $\quad x = 1 \quad \gamma \in UHP$

$$x^* u^t x = \begin{cases} e^{i\gamma t} & t \geq 0 \\ e^{i\gamma^* t} & t \leq 0 \end{cases} = \int e^{i\omega t} \rho(\omega) \frac{d\omega}{2\pi}$$

$$\rho(\omega) = \int_{-\infty}^{0} e^{-i\omega t} e^{i\gamma^* t}\, dt + \int_0^\infty e^{-i\omega t} e^{i\gamma t}\, dt$$

$$= \frac{1}{-i\omega + i\gamma^*} + \frac{1}{i\omega - i\gamma} = \frac{i}{\omega - \gamma^*} + \frac{-i}{\omega - \gamma}$$

$$= \frac{(-i)(\gamma - \gamma^*)}{(\omega - \gamma^*)(\omega - \gamma)} = \frac{k^2}{|\omega - \gamma|^2} \qquad k^2 = \frac{\gamma - \gamma^*}{i} = 2\,\mathrm{Im}\,\gamma$$

$$L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) \qquad L^2\left(\mathbb{R}_\omega, \rho \frac{d\omega}{2\pi}\right) \qquad L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right)$$

$$\frac{ik}{\omega - \bar\gamma} \longleftarrow x = 1 \longrightarrow \frac{ik}{\omega - \gamma}$$

$$\frac{ik}{\omega - \bar\gamma} = k\, e_{\bar\gamma}$$

$$\int_{-\infty}^{\infty} \overline{\left(\frac{i}{\omega - \bar\gamma}\right)} f(\omega) \frac{d\omega}{2\pi} = \int_{\circlearrowleft \gamma} \frac{1}{\omega - \gamma} f(\omega) \frac{d\omega}{2\pi i} = f(\gamma).$$

$$\left\| \frac{i}{\omega - \gamma} \right\|^2 = \frac{i}{\gamma - \bar\gamma} = \frac{1}{2\,\mathrm{Im}\,\gamma} = \frac{1}{k^2}$$

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so in this example $H = L^2(\mathbb{R})$ $\quad x = \varepsilon 1 = \dfrac{ik}{\omega - \bar\gamma}$

$$x^* u^t x = \int \overline{\frac{ik}{\omega - \bar\gamma}}\, e^{it\omega} \frac{ik}{\omega - \bar\gamma} \frac{d\omega}{2\pi}$$

$$= \int \frac{k^2}{\omega - \gamma} \frac{e^{i\omega t}}{\omega - \bar\gamma} \frac{d\omega}{2\pi} = \frac{k^2 e^{i\gamma t}}{\gamma - \bar\gamma}\, i = e^{i\gamma t}$$

496 Given X, $e^{i\gamma t}$ to define H. Maybe define scattering

$$\|x\|^2 - \|e^{i\gamma t}x\|^2 = -\int_0^t dt' \frac{d}{dt'}\|e^{i\gamma t'}x\|^2$$

$$= \int_0^t dt' \left(e^{i\gamma t'}x, \underbrace{-\left((i\gamma)^* + i\gamma\right)}_{\frac{\gamma-\gamma^*}{i} = \nu^*\nu} e^{i\gamma t'}x\right) = \int_0^t dt' \|\nu e^{i\gamma t'}x\|^2$$

Assume $\|e^{i\gamma t}x\| \to 0$ as $t \to +\infty$ $\forall x$, get

$$\|x\|^2 = \int_0^\infty dt' \|\nu e^{i\gamma t'}x\|^2 = \int \frac{d\omega}{2\pi} \left\|\nu \frac{-i}{\omega-\gamma} x\right\|^2$$

$$\int_0^\infty e^{-i\omega t'} \nu e^{i\gamma t'} dt' = \frac{+1}{i\omega - i\gamma}$$ so

$L^2(\mathbb{R},$

$$u^t \varepsilon x = \varepsilon e^{i\gamma t}x + \int_0^t dt' \left(u^{t-t'}\nu(e^{i\gamma t'}x)\right)$$

meaning

discrete case

$$x = aa^*x + \pi_+ x$$
$$ux = ba^*x + u\pi_+ x$$
$$u^2x = ba^*ba^*x + u\pi_+ ba^*x + u^2\pi_+ x$$
$$= aa^*(ba^*)^2x + \pi_+(ba^*)^2x + u\pi_+(ba^*)x + u^2\pi_+ x$$
$$u^n x = aa^*(ba^*)^n x + \pi_+(ba^*)^n x + u\pi_+(ba^*)^{n-1}x + \dots + u^n\pi_+ X$$

$$\boxed{u^t\varepsilon x = \varepsilon e^{i\gamma t}x + \int_0^t dt' \, u^{t'}\left(\nu e^{i\gamma(t-t')}x\right)}$$

Your problem is now to make sense of $\int dt\, u^t \nu(x_t) = \int u^t \nu(dx_t)$ in H

497 H Hilbert space, u^t 1-param. unit group $\varepsilon : X \to H$ $\varepsilon^* u^t \varepsilon = \begin{cases} e^{i\gamma t} & t \geq 0 \\ e^{i\gamma^* t} & t \leq 0 \end{cases}$

Perhaps you are describing some sort of stochastic integral.

$$u^t \varepsilon x = \varepsilon e^{i\gamma t} x + \int_0^t dt' \, u^{t-t'} \left(\nu e^{i\gamma(t')} x \right)$$

$$\|x\|^2 = \|e^{i\gamma t} x\|^2 + \int_0^t dt' \|\nu e^{i\gamma(t')} x\|^2$$

$$-\frac{d}{dt} \|e^{i\gamma t} x\|^2 = \Big(e^{i\gamma t} x, \underbrace{(-(i\gamma)^* - (i\gamma))}_{\frac{\gamma - \gamma^*}{i} = \nu^* \nu} e^{i\gamma t} x\Big)$$

Simple example. Assume $\frac{\gamma - \gamma^*}{i}$ rank 1

so $\frac{\gamma - \gamma^*}{i} = \xi k^2 \xi^*$ in X. $|\xi| = 1$.

You need to make sense of $\int_0^\infty dt' \, u^{-t'} \nu (e^{\gamma t'} x)$

Example: $X = \mathbb{C}$ $\xi = x = 1$.

$$\int_0^\infty dt' \, e^{-i\omega t'} \nu (e^{\gamma t} x) = \nu \left(\frac{1}{i\omega - i\gamma} x \right)$$

$H = L^2(\mathbb{R}, \rho \frac{d\omega}{2\pi})$ $\rho = \frac{k^2}{|\omega - \gamma|^2}$

$\varepsilon x = 1$ function

$u^t = e^{i\omega t}$

~~Start~~ the example again $X = \mathbb{C}$, $\gamma \in UHP$

seek H, u^t, $\varepsilon : \mathbb{C} \to H$ so $\varepsilon \in H$ $\ni$ $\varepsilon^* u^t \varepsilon = \begin{cases} e^{i\gamma t} & t \geq 0 \\ e^{i\gamma^* t} & t \leq 0 \end{cases}$

$\varepsilon^* u^t \varepsilon = \int e^{i\omega t} \rho(\omega) \frac{d\omega}{2\pi}$ $\rho(\omega) = \int e^{-i\omega t} \varepsilon^* u^t \varepsilon \, dt$

$$= \int_{-\infty}^0 e^{-i\omega t} e^{i\gamma^* t} dt + \int_0^\infty e^{-i\omega t} e^{i\gamma t} dt = \frac{1}{-i\omega + i\gamma^*} + \frac{1}{i\omega - i\gamma} = \frac{i}{\omega - \gamma^*} + \frac{-i}{\omega - \gamma}$$

498 $\rho(\omega) = \dfrac{-i\gamma + i\gamma^*}{(\omega-\gamma^*)(\omega-\gamma)} = \dfrac{\gamma-\gamma^*}{i} \dfrac{1}{|\omega-\gamma|^2}$ so this is H

$\varepsilon = 1 \in L^2(\mathbb{R}, \rho \frac{d\omega}{2\pi})$ $u^t = e^{i\omega t}$

$u^t\varepsilon - \varepsilon e^{i\gamma t} = e^{i\omega t} - e^{i\gamma t}$

Wait you have the model $L^2(\mathbb{R}, \rho\frac{d\omega}{2\pi})$ for H with $\varepsilon = 1$ and $u^t = e^{i\omega t}$. But there are other models for H, u^t, ε

$$H = L^2(\mathbb{R}, \frac{d\omega}{2\pi}),\ u^t = e^{i\omega t},\ \varepsilon = \frac{ik}{\omega - \bar\gamma}$$

Chk: $\varepsilon^* u^t \varepsilon = \int \overline{\dfrac{ik}{\omega-\bar\gamma}}\, e^{i\omega t} \dfrac{ik}{\omega-\bar\gamma} \dfrac{d\omega}{2\pi} = e^{i\gamma t} \underbrace{\dfrac{ik^2}{\gamma-\bar\gamma}}_{=1}$

H^2 should be the subspace spanned by $e^{i\omega t}\varepsilon$

orthogonal complement $H^2 \ominus \mathbb{C}\varepsilon = \underbrace{\left(\dfrac{\omega-\gamma}{\omega-\bar\gamma}\right)}_{S} H^2$ functions vanishing at γ.

Now look at $\left(\dfrac{\omega-\gamma}{\omega-\bar\gamma}\right)H^2$ and $u^t = e^{i\omega t}$.

Get shift. You know that $H^2 \ominus \mathbb{C}\overset{\varepsilon X}{\overset{\shortparallel}{\varepsilon}} = SH^2$ which has the "basis $Se^{i\omega t}$, $t \geq 0$; "essentially" orthonormal.

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continue with example $X = \mathbb{C},\ \overset{x=1}{\vee},\ \mathrm{Im}(\gamma) > 0$.

$\varepsilon^* u^t \varepsilon = e^{i\gamma t}$ Get $H = L^2(\mathbb{R}, \rho\frac{d\omega}{2\pi})$ $\rho = \dfrac{k^2}{|\omega-\gamma|^2}$

$k^2 = \dfrac{\gamma-\gamma^*}{i} = 2\,\mathrm{Im}\,\gamma$ $\varepsilon x = 1 \in H$

$$L^2(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}) \qquad L^2(\mathbb{R}, \frac{d\omega}{2\pi})$$

want $H^2_+(\mathbb{R}, \frac{d\omega}{2\pi})$ ~~consist of~~ like $H^2_+(S^1, \frac{d\theta}{2\pi}) \ni \sum_{n\geq 0} a_n z^n$

rep. kernel $\dfrac{1}{1-\bar w z}$ $H^2_+(\mathbb{R}, \frac{d\omega}{2\pi}) \ni \int_0^\infty e^{i\omega t}\phi(t)\,dt$

~~Ep.~~ rep. kern $\dfrac{1}{\omega-\bar\lambda}$ $\int \overline{\dfrac{i}{\omega-\bar\lambda}} f(\omega)\dfrac{d\omega}{2\pi} = f(\lambda)$

You want to decompose H:

$$H = H^2_- \oplus \varepsilon X \oplus H^2_+$$

you want to somehow ~~...~~ get the first element of H^2_+ which is $1 = \int_0^\infty e^{i\omega t}\,\delta(t)\,dt$, ~~so~~ this is not in H but when multiplied by elements of H^2_+ ~~...~~ gives elements of H. Let's use one of the pictures

$$\begin{array}{ccccc}
L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) & \xrightarrow{\ \sim\ } & L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}\right) & \xleftarrow{\ \sim\ } & L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) \\
\frac{ik}{\omega-\gamma} & \longleftarrow & 1 = \varepsilon x & \longmapsto & \frac{ik}{\omega-\bar{\gamma}} \\
\int e^{i\omega t}\frac{\omega-\bar\gamma}{\omega-\gamma}\phi(t)\,dt & \longleftarrow & \int e^{i\omega t}\frac{\omega-\bar\gamma}{ki}\phi(t)\,dt & \longleftarrow & \int e^{i\omega t}\phi(t)\,dt \\
\frac{\omega-\bar\gamma}{\omega-\gamma}f(\omega) & & \frac{\omega-\bar\gamma}{ki}f(\omega) & & f(\omega)
\end{array}$$

NO you want

$$L^2\left(\mathbb{R}, \frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi}\right)$$

$$\|$$

$$\overbrace{H^2_-\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) \oplus \varepsilon X \oplus H^2_+\left(\mathbb{R}, \frac{d\omega}{2\pi}\right)}$$

$$\begin{array}{ccc}
\| & & \| \\
L^2(\mathbb{R}_{<0}, dt) & & L^2(\mathbb{R}_{>0}, dt) \\
 & & \uparrow \\
 & & \text{gen. by } \delta(t)
\end{array}$$

Try $\quad f(\omega) = \int_0^\infty e^{i\omega t}\phi(t)\,dt \longmapsto \underbrace{\frac{\omega-\bar\gamma}{ik}f(\omega)}_{\text{is } \perp \text{ to } \varepsilon x = 1.}$

$$\int \frac{\omega-\bar\gamma}{ik}f(\omega)\frac{k^2}{|\omega-\gamma|^2}\frac{d\omega}{2\pi} = \int f(\omega)\frac{-ik}{\omega-\gamma}\frac{d\omega}{2\pi}.$$

~~4~~ 700 Try $f(\omega) = \int_0^\infty e^{i\omega t}\phi(t)\,dt \longmapsto \frac{|\omega-\gamma|^2}{k^2} f(\omega)$?

You need an isom. embedding $L^2(\mathbb{R}_{>0}, dt) \longleftrightarrow L^2(\mathbb{R}, \rho\frac{d\omega}{2\pi})$

$$f(\omega) \longmapsto \frac{\omega-\gamma}{ik} f(\omega) \mapsto \int \underbrace{\left|\frac{\omega-\gamma}{ik} f(\omega)\right|^2 \frac{k^2}{|\omega-\gamma|^2}}_{|f|^2} \frac{d\omega}{2\pi}$$

$$\int \bar{1}\ \frac{\omega-\gamma}{ik} f(\omega) \frac{k^2}{|\omega-\gamma|^2} \frac{d\omega}{2\pi} = \int f(\omega) \frac{-ik}{\omega-\bar{\gamma}} \frac{d\omega}{2\pi}$$

$$\int \bar{1}\ \frac{\omega-\bar{\gamma}}{ik} f(\omega) \ \text{——} \ = \int f(\omega) \frac{-ik}{\omega-\gamma} \frac{d\omega}{2\pi} = kf(\gamma)$$

Try again. $|c|<1$. discrete case first $\varepsilon^* u^n \varepsilon = c^n \quad n \geq 0.$

$$\varepsilon^* u^n \varepsilon = \begin{cases} c^n & n \geq 0 \\ \bar{c}^{-n} & n \leq 0 \end{cases} = \int z^n \rho \frac{d\theta}{2\pi}$$

$$\rho = \sum z^{-n} \begin{cases} c^n & n \geq 0 \\ \bar{c}^{-n} & n < 0 \end{cases} = \sum_{n\geq 0} (z^{-1}c)^n + \sum_{n<0} z^{+n}\bar{c}^{+n}$$

$$= \frac{1}{1-z^{-1}c} + \frac{z\bar{c}}{1-z\bar{c}} = \frac{1-|c|^2}{|1-z\bar{c}|^2}$$

Decompose $H = L^2(S^1, \rho\frac{d\theta}{2\pi})$: $\cdots \oplus u^{-1}V_{-} \oplus V_{-} \oplus X \oplus V_{+} \oplus uV_{+} \oplus \cdots$

$$u\varepsilon - \varepsilon c = (u-c)\varepsilon$$

~~$(u-c)\varepsilon,\ u^n(u-c)\varepsilon$~~ $n \geq 0$

$$\varepsilon^* u^n (u-c)\varepsilon = c^{n+1} - c^n c = 0.$$

$$(u\varepsilon - \varepsilon c)^* u^n (u\varepsilon - \varepsilon c) = \varepsilon^* u^{n-1}(u\varepsilon - \varepsilon c)$$

$$= c^n - (\varepsilon^* u^{n-1}\varepsilon)c = \begin{cases} 0 & n \geq 1 \\ 1-c^*c. & n=0 \end{cases}$$