 What can you do tomorrow??

You are now free to work out deB's formulas. ~~[illegible]~~

Reflection positivity. Suppose given a V ~~vector space~~ f.d. Hilbert space and a family $\phi(t) \in \mathcal{L}(V)$ for $t \geq 0$. such that $\phi(0) = 1$, $\phi(t+t') = \phi(t)\phi(t')$.

Free field theory, Gaussian

Think about real Gaussian stochastic process

$$Y \xrightarrow{\iota\varepsilon^* + A^*} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{\varepsilon} Y$$

Review yesterday's formulas.

$$\varepsilon = \tfrac{1}{2}(a+b) \qquad\qquad \iota\varepsilon + A = \iota a$$

$$A = \tfrac{\iota}{2}(a-b) \qquad\qquad \iota\varepsilon^* + A^* = \tfrac{\iota}{2}(a^* + b^*) - \tfrac{\iota}{2}(a^* - b^*)$$

$$= \iota b^*$$

$$(\iota\varepsilon^* + A^*)\varepsilon = \iota b^* \tfrac{1}{2}(a+b) = \tfrac{\iota}{2}(1 + b^* a)$$

$$(\iota\varepsilon^* + A^*)A = \iota b^* \tfrac{i}{2}(a-b) = \tfrac{1}{2}(1 - b^* a)$$

I think you should work in $Y \oplus Y$ as much as possible. Unitary picture $\|y_1\|^2 - \|y_2\|^2$

$$W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}, \quad W^0 = W \oplus \begin{matrix} \mathrm{Ker}(a^*) \\ \oplus \\ \mathrm{Ker}(b^*) \end{matrix}.$$

What is interesting? $\begin{pmatrix} 1 \\ z \end{pmatrix} Y$ is isotropic for the hermitian form when $|z| = 1$. $W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y$ is a line consisting of

$$\begin{aligned} y_1 &= ax + v^+ \\ y_2 &= bx + v^- \end{aligned} \qquad \text{where } z y_1 = y_2$$

$$S(z): V^+ \to V^-$$

Solutions of $(az - b)x = -zv^+ + v^- \implies \underbrace{S(z)}_{(1-bb^*)(1-zab^*)^{-1}} z v^+ = v^-$

 ~~[illegible]~~

Another description is ~~[illegible]~~ $W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| a^* y_1 = b^* y_2 \right\}$

So $W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y = \left\{ y \in Y \middle| (z^{-1} a^* - b^*) y = 0 \right\}$.

$= \mathrm{Ker}\left\{ (az - b)^* : Y \to X \right\}$ for $|z| = 1$.

Is there a simple way to see that

$$y \longmapsto \tilde{y}(z) = \left(e^-, (1 - z ab^*)^{-1} y \right)$$

gives an isometric embedding $Y \longrightarrow L^2(S^1)$.

Go back to $W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ isotropic for $\left(\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)$

i.e. $(\varepsilon x', Ax) = (Ax', \varepsilon x)$, $W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (y_1, Ax) = (y_2, \varepsilon x) \ \forall x \right\}$

$$W^0 \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y = \left\{ \begin{pmatrix} y \\ \lambda y \end{pmatrix} \middle| (y, Ax) = (\lambda y, \varepsilon x) \ \forall x \right\}$$

$$\Updownarrow \quad (y, (\bar{\lambda}\varepsilon - A)x) = 0 \quad \forall x$$

$$\Updownarrow \quad (\lambda \varepsilon^* - A^*) y = 0 .$$

In the J-matrix picture you have $u^\lambda = \begin{pmatrix} u_1^\lambda \\ \vdots \\ u_{n+1}^\lambda \end{pmatrix}$ defined

$$\boxed{(\lambda I_{n+1} - \tilde{A}) u^\lambda = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n+1} u_{n+2}^\lambda \end{pmatrix}} .$$

Recall

$u_1^\lambda = 1$

$$\tilde{A} = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & \ddots & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_n \\ & & & a_n & b_{n+1} \end{pmatrix}$$

$a_1 u_2^\lambda = (\lambda - b_1) u_1^\lambda$

$a_2 u_3^\lambda = (\lambda - b_2) u_2^\lambda - a_1 u_1^\lambda$

$a_n u_{n+1}^\lambda = (\lambda - b_n) u_n^\lambda - a_{n-1} u_{n-1}^\lambda$

$a_{n+1} u_{n+2}^\lambda = (\lambda - b_{n+1}) u_{n+1}^\lambda - a_n u_n^\lambda$

61 You want to derive a spectral representation for ~~these~~ element of y. Review simplest version.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "e_{n+1}"] \\ X \arrow[r, "(\lambda\varepsilon - A)"] & Y \arrow[d, "\varepsilon^*"] \\ & X \end{tikzcd}
\qquad
\begin{tikzcd} Y \arrow[r, "\begin{pmatrix}\varepsilon^* \\ e^*_{n+1}\end{pmatrix}"] & X \oplus \mathbb{C} \arrow[r, "(\lambda\varepsilon - A \;\; e_{n+1})"] & Y \end{tikzcd}$$

$$y = (\lambda\varepsilon - A \quad e_{n+1}) \begin{pmatrix}\varepsilon^* \\ e^*_{n+1}\end{pmatrix} (\lambda - \varepsilon^* A)^{-1} y$$

$$\tilde{y}(\lambda) = e^*_{n+1}(\lambda - A_\varepsilon)^{-1} y$$

$\tilde{y}(\lambda)$ is a ~~real~~ rational function with n simple poles the eigenvalues of $\varepsilon^* A$

$$(\lambda\varepsilon - A)x_n + \tilde{y}(\lambda)e_{n+1} = y$$

Go back to $u^\lambda = \begin{pmatrix} u_1^\lambda \\ \vdots \\ u^\lambda_{n+1}\end{pmatrix}$ $\qquad (\lambda - A_{n+1})u^\lambda = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n+1}u^\lambda_{n+2}\end{pmatrix}$

$$\begin{aligned}(\lambda - \bar{\mu})(u^\mu, u^\lambda) &= (u^\mu, \lambda u^\lambda) - (\mu u^\mu, u^\lambda) \\ &= (u^\mu, (\lambda - \tilde{A})u^\lambda) - ((\mu - \tilde{A})u^\mu, u^\lambda) \\ &= u^{\bar{\mu}}_{n+1}\, a_{n+1} u^\lambda_{n+2} - a_{n+1} u^{\bar{\mu}}_{n+2} u^\lambda_{n+1} \\ &= a_{n+1}\begin{vmatrix} u^{\bar{\mu}}_{n+1} & u^\lambda_{n+1} \\ u^{\bar{\mu}}_{n+2} & u^\lambda_{n+2}\end{vmatrix}\end{aligned}$$

$$(x, \lambda\varepsilon^*_\varepsilon u^\lambda) \qquad (x, A^* u^\lambda)$$
$$\parallel \qquad\qquad \parallel$$
$$\begin{pmatrix} u^\lambda \\ \lambda u^\lambda \end{pmatrix} \in W^0 \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix}\mathbb{C} \qquad (\varepsilon x, \lambda u^\lambda) \overset{?}{=} (Ax, u^\lambda)$$

$$\left(\begin{pmatrix} u^\mu \\ \mu u^\mu \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} u^\lambda \\ \lambda u^\lambda \end{pmatrix}\right) = (\lambda - \bar{\mu})(u^\mu, u^\lambda) = a_{n+1}\begin{vmatrix} u^{\bar{\mu}}_{n+1} & u^\lambda_{n+1} \\ u^{\bar{\mu}}_{n+2} & u^\lambda_{n+2}\end{vmatrix}$$

~~[illegible]~~

basic problem: Given an operator A on Y, cyclic vector v_0

$\xi \longmapsto v_0^*(\lambda - A)^{-1}\xi$ ~~to embed Y into~~ transforms Y to rational functions. Injective

$$(\lambda - A)^{-1}\xi = (\lambda^{-1} + \lambda^{-2}A + \lambda^{-3}A^2 + \cdots)\xi$$

so

$$\oint \frac{d\lambda}{2\pi i} \frac{f(\lambda)}{(\lambda - A)}\xi = f(A)\xi$$

$$\int \frac{d\lambda}{2\pi i} f(\lambda)\, v_0^*(\lambda - A)^{-1}\xi = v_0^* f(A)\xi$$

Go back to $W = \begin{pmatrix}\varepsilon \\ A\end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y\end{matrix}$ $\quad W^0 \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y = \mathbb{C}\begin{pmatrix}u^\lambda \\ \lambda u^\lambda\end{pmatrix}$

$$0 = \left(\begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y \\ \lambda y\end{pmatrix}\right) = (\varepsilon x, \lambda y) - (Ax, y)$$
$$= (x, (\lambda\varepsilon^* - A^*)y) = 0$$

$$y \in \operatorname{Ker}(\lambda\varepsilon^* - A^*)$$

suppose you pick a line L in W^0/W $\ni$ $p_1 L \notin \varepsilon X$. This gives an extension of the partial operator. Example $L = \begin{pmatrix} u^{\lambda_0} \\ \lambda_0 u^{\lambda_0}\end{pmatrix}$. What is the spectrum of the resulting operator? ~~You seem to have a line~~ Should be related to the ~~resulting~~ λ such that $\begin{pmatrix} u^\lambda \\ \lambda u^\lambda\end{pmatrix}$ maps to L in W^0/W.

Have $W = \begin{pmatrix}\varepsilon \\ A\end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y\end{matrix}$ $\quad W^0 \cap \begin{pmatrix}1 \\ \lambda\end{pmatrix} Y = \mathbb{C}\begin{pmatrix}u^\lambda \\ \lambda u^\lambda\end{pmatrix}$

$\mathbb{C}u^\lambda = \operatorname{Ker}(\lambda\varepsilon^* - A^*)$. A line L in W^0/W, ~~[illegible]~~ ~~such that~~ "independent of ε" corresponds to an extension of $A\varepsilon^{-1}$ to an operator on Y which has eigenvalues, so you should get a pencil of positive divisors of degree $n+1$. ~~Another description~~ You have to distinguish between the λ $\mathbb{P}^1$ and $\mathbb{P}^1(W^0/W)$. There is a natural map $\lambda \longmapsto \operatorname{Ker}(\lambda\varepsilon^* - A^*) \xrightarrow{\sim} W^0 \cap \binom{1}{\lambda}Y \hookrightarrow W^0/W$ which sends real axis to isotropic lines in W^0/W

degree?

Philosophy: A line in W^0/W is a kind of boundary condition to be added in order to obtain a well defined operator $\tilde{A}$ having a spectrum, resolvent, etc. When W is enlarged to the graph Γ of this operator then the resolvent is linked to the way $\Gamma_{\tilde{A}}$ and $\binom{1}{\lambda}Y$ intersect

$$\binom{1}{\tilde{A}}Y \cap \binom{1}{\lambda}Y = \left\{ \binom{y}{\lambda y} \,\middle|\, \lambda y = \tilde{A}y \right\}$$

To invert $\begin{pmatrix} 1 & 1 \\ \lambda & \tilde{A} \end{pmatrix}$:

$$\begin{tikzcd}
 & Y \arrow[d, "\binom{1}{\lambda}"] \arrow[dr, "-\tilde{A}+\lambda"] & \\
Y \arrow[r, "\binom{1}{\tilde{A}}"] \arrow[dr, "\lambda - \tilde{A}"'] & Y \oplus Y \arrow[r, "(-\tilde{A}\ 1)"] \arrow[d, "(+\lambda\ -1)"] & Y \\
 & Y &
\end{tikzcd}$$

$$\begin{pmatrix} 1 & 1 \\ \tilde{A} & \lambda \end{pmatrix}^{-1} = \begin{pmatrix} \lambda & -1 \\ -\tilde{A} & 1 \end{pmatrix} \frac{1}{\lambda - \tilde{A}}$$

What are the natural questions?? You really are missing the appropriate viewpoint. First question is how to describe the spectrum. For each λ you have this line $W^0 \cap \binom{1}{\lambda}Y$ mapping to W^0/W and the line $L \hookrightarrow W^0/W$. So the spectrum is described

 as ~~those~~ consisting λ s.t. this is not transverse i.e.
if ~~[illegible]~~ $W'/W = L$, then $W^0 \cap \begin{pmatrix}1\\ \frac{1}{\varepsilon}\end{pmatrix} Y \to W^0/W$ vanishes. So we have a dual section of the sub line bundle

$$0 \longrightarrow \underset{\text{degree } -n}{\mathrm{Ker}(\lambda\varepsilon^* - A^*)} \longrightarrow \overset{n+1}{Y} \xrightarrow{\lambda\varepsilon^* - A^*} \overset{n}{X} \longrightarrow 0$$

So the spectrum is a divisor of degree n:

approach $W = \begin{pmatrix}\varepsilon\\ A\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ lines in W^0/W
except for the line $\begin{pmatrix}0\\ e_{n+1}\end{pmatrix}\mathbb{C}$ $\qquad \mathbb{C}e_{n+1} = (\varepsilon X)^\perp = \mathrm{Ker}(\varepsilon^*)$
are the same as the ~~graphs of~~ extensions of $\begin{pmatrix}\varepsilon\\ A\end{pmatrix} X$ to $\begin{pmatrix}1\\ \tilde{A}\end{pmatrix} X$
where $\tilde{A} : Y \longrightarrow Y \ni \quad (\varepsilon x', \tilde{A} y) = (A x', y) \quad \forall x' \in X,\ y \in Y$
equiv. $\varepsilon^* \tilde{A} = A^*$, $\tilde{A}\varepsilon = A$. In terms of J-matrix

$$\tilde{A} = \begin{bmatrix} b_1 & a_1 & & & 0 \\ a_1 & \ddots & \ddots & & \\ & \ddots & & a_{n-1} & \\ 0 & & a_{n-1} & b_n & a_n \\ & & & a_n & * \end{bmatrix} \qquad * \text{ arbitrary}$$

So extensions are described by $b_{n+1} \in \mathbb{C}$, $\tilde{A}$ hermitian $\iff b_{n+1}$ real.

The interesting point is to represent lines in W^0/W in the form $W^0 \cap \begin{pmatrix}1\\ \mu\end{pmatrix} Y \simeq \mathrm{Ker}(\mu\varepsilon^* - A^*)$.
What this means is you will choose b_{n+1} to be the coefficient arising from $\begin{pmatrix}u^\mu\\ \mu u^\mu\end{pmatrix}$. This means $\tilde{A} u^\mu = \mu u^\mu$

$$e^*_{n+1}(\tilde{A} u^\mu) = a_n u^\mu_n + b_{n+1} u^\mu_{n+1}$$

and this is to be μu^μ_n : $a_n u^\mu_n + (b_{n+1} - \mu) u^\mu_{n+1} = 0$

65 So we fix the boundary condition so that u^i is an eigen~~function~~ vector

$\tilde{A}u^i = i\,u^i$.

Now that we have this $\tilde{A}$ which is nearly hermitian you will get a spectral representation once you know $\tilde{A} - \tilde{A}^*$ which is essentially the imaginary part of $b_{n+1}\, i - \frac{a_n u_n^i}{u_{n+1}^i}$.

Back to refl positivity. Try to understand the simple harmonic oscillator.

$\langle 0 \,|\, x(t_1)\, x(t_2) \cdots x(t_n) \,|\, 0 \rangle$ somehow time ordered.

$x(t_k) = e^{-\frac{i}{\hbar}Ht}\, x\, e^{+\frac{i}{\hbar}Ht}$

Basic example: Forced simple harmonic oscillator
$m\ddot{x} + kx = F(t) \in C_0^\infty(\mathbb{R})$

Review: $W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\quad W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y = \left\{ \begin{pmatrix} y \\ \lambda y \end{pmatrix} \middle| \begin{matrix} (\lambda\varepsilon^* - A^*)y \\ = 0 \end{matrix} \right\}$

W°/W is 2 diml. $\quad W^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (y_1, Ax) = (y_2, \varepsilon x) \;\; \forall x \right\}$

e.g. $y_1 = \varepsilon x$, $y_2 = Ax$

i.e. $A^* y_1 = \varepsilon^* y_2$

Suppose e_{n+1} is a unit vector gen. $\mathrm{Ker}(\varepsilon^*) = (\varepsilon X)^\perp$. ~~Take~~

$W^\circ \supset W + \begin{pmatrix} \mathrm{Ker}\, A^* \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathrm{Ker}\, \varepsilon^* \end{pmatrix}$

Can you have $\begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} \in \begin{matrix} \mathrm{Ker}\, A^* \\ \oplus \\ \mathrm{Ker}\, \varepsilon^* \end{matrix}$ i.e. $\varepsilon^* A x = 0$ & $A^* \varepsilon x = 0$. Yes.

Any line L in W°/W corresp to a $(n+1)$-subsp. V of $\begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ ~~corres. to~~ cont W, & ~~e_{n+1}~~

Look at $p_1: V \longrightarrow Y$ either $p_1 V \subset \varepsilon X$ whence ~~contains~~ ~~Ker~~ $\mathrm{Ker}(p_1|V)$ is a line in $\begin{matrix} 0 \\ \oplus \\ Y \end{matrix}$ cont in W° ie. $(\varepsilon X)^\perp$.

or $p_1 : V \xrightarrow{\sim} Y$ and then V is graph of $\tilde{A} : Y \to Y$

must have $\Gamma_{\tilde{A}} \subset W^\circ$ i.e. $(y, Ax) = (\tilde{A}y, \varepsilon x)$ $\forall x, y$

i.e. $A = \tilde{A}^* \varepsilon$ and $W \subset \Gamma_{\tilde{A}}$ i.e.

$\tilde{A}\varepsilon = A$. $\boxed{A^* = \varepsilon^* \tilde{A}}$ Everything

about $\tilde{A}$ is determined by $e_{n+1}^* \tilde{A} e_{n+1} = b_{n+1} \in \mathbb{C}$

But you want the de Branges picture, which is based on a specific choice for $\tilde{A}$, namely he uses the line $\operatorname{Img}\{W^\circ \cap \binom{1}{\mu} Y \longrightarrow W^\circ/W\}$ where $\mu = i$.

so $\Gamma_{\tilde{A}} = W \oplus \mathbb{C}\binom{u^i}{iu^i}$ $\tilde{A} u^i = i u^i$

which means $\det(\lambda - \tilde{A})$ has the factor $\lambda - i$.

Now u^μ killed $\mu\varepsilon^* - A^*$ in general.

~~u^i is general~~ You want isometric embedding

I think I want to use $e_{n+1}^* (\lambda - \tilde{A})^{-1} y$ for the isometric embedding. Need to know about $\tilde{A} - \tilde{A}^*$.

$\varepsilon^*(\tilde{A} - \tilde{A}^*) = A^* - (\tilde{A}\varepsilon)^* = A^* - A^* = 0$, also

$(\tilde{A} - \tilde{A}^*)\varepsilon = A - A = 0$.

Can you relate $\tilde{A}$ to the stuff before

$$\begin{array}{ccc} X & \xrightarrow{\lambda\varepsilon - A} & Y \\ & & \downarrow{\scriptstyle i\varepsilon^* + A^*} \\ & & X \end{array}$$

$\varepsilon = \frac{1}{2}(a+b)$ $\qquad i\varepsilon - A = ib$

$A = \frac{i}{2}(a-b)$ $\qquad -i\varepsilon^* - A^* = -ib^*$

$\qquad i\varepsilon^* + A^* = ib^*$

$(i\varepsilon^* + A^*)\varepsilon = ib^* \frac{1}{2}(a+b) = \frac{i}{2}(1 + b^*a)$

$(i\varepsilon^* + A^*)A = ib^* \frac{i}{2}(a-b) = \frac{1}{2}(1 - b^*a)$

$-i(i\varepsilon^* + A^*)\varepsilon = \frac{1}{2}(1 + b^*a)$

$\varepsilon^*\varepsilon + A^*A = 1$

67 Start with $d\mu$ on $\mathbb{R}$ prob. measure, where scalar product on $\mathbb{C}[\lambda]$. Restrict to $Y = F_{n+1}$ polys of degree $\leq n$. Get $\binom{\mathcal{E}}{A} F_n$. ~~[illegible]~~ Reproducing kernel? What can you say about point evaluation.

$$y(\alpha) = (e_\alpha, y) \quad \text{in } Y = F_{n+1}.$$

$$(e_\alpha, (A-\alpha)x) = 0 \quad \forall x$$

so $\quad ((A^* - \bar{\alpha}) e_\alpha, x) = 0 \quad \forall x \in F_n \quad$ so

Confused.
$$Y = F_{n+1} = \mathbb{C}1 + \mathbb{C}\lambda + \cdots + \mathbb{C}\lambda^n$$
$$X = F_n = \mathbb{C}1 + \quad \mathbb{C}\lambda^{n-1}$$

$$y(\alpha) = (e_\alpha, y) \quad \text{defines } e_\alpha \in Y \text{ for } \alpha \in \mathbb{C}.$$

Then $\quad (e_\alpha, (\lambda - \alpha)x) = 0 \quad$ so $\quad e_\alpha \perp (\alpha\mathcal{E} - A)X$

so $\quad (\bar{\alpha}\mathcal{E}^* - A^*) e_\alpha = 0$

~~By the measure Take~~ Let $p_1, \ldots, p_{n+1}$ be the orthogonal polys. ~~[illegible]~~ Then

$$e_\alpha = \sum_i \overline{p_i(\alpha)}\, p_i \qquad e(\alpha, \lambda)$$

so that
$$(e_\alpha, y) = \sum_i p_i(\alpha) (p_i, y)$$
$$(e_\alpha, p_j) = \sum_i p_i(\alpha)\, \delta_{ij} = p_j(\alpha).$$

$$\int \bar{e}(\alpha, \lambda)\, y \, d\mu(\lambda) = y(\alpha)$$

$$\int e(x'', x')\, d\mu(x') \int e(x', x)\, y(x)\, d\mu(x) = \int \overbrace{y(x')}^{e(x'', x')}$$

$$\int d\mu(x')\, e(x'', x')\, e(x', x) = e(x'', x)$$

Review. $W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset W^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (y_1, Ax) = (y_2, \varepsilon x) \quad \forall x \right\}$

~~[illegible]~~ Consider a line V/W in W°/W where $W \subset V \subset W^\circ$. Assuming $V \cap \underbrace{\mathrm{Ker}(p_1 : W^\circ \to Y)}_{\begin{pmatrix} 0 \\ \mathrm{Ker}\,\varepsilon^* \end{pmatrix}} = 0$ then $p_1 : V \xrightarrow{\sim} Y$ so $V = \begin{pmatrix} 1 \\ \tilde{A} \end{pmatrix} \cancel{X}$

where $\tilde{A}\varepsilon = A$ and $(y, Ax) = (\tilde{A}y, \varepsilon x) \quad \forall x, y$

equiv. $\varepsilon^* \tilde{A} = A^*$. ~~[illegible]~~ $\therefore \tilde{A}^* \varepsilon = A$, so

$(\tilde{A}^* - \tilde{A})\varepsilon = 0 \implies \varepsilon^*(\tilde{A}^* - \tilde{A}) = 0.$ So $\tilde{A}^* - \tilde{A}^*$

$= e_{n+1} b_{n+1} e^*_{n+1}$ $\quad e_{n+1}$ unit v. gen. $\mathrm{Ker}(\varepsilon^*)$.

$$W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y = \left\{ \begin{pmatrix} y \\ \lambda y \end{pmatrix} \middle| (y, Ax) = (\lambda y, \varepsilon x) \quad \forall x \right\}.$$

i.e. $(\lambda \varepsilon^* - A^*) y = 0$

$$W^\circ \cap \begin{pmatrix} 1 \\ \mu \end{pmatrix} Y = \begin{pmatrix} 1 \\ \mu \end{pmatrix} \mathrm{Ker}(\mu\varepsilon^* - A^*) = \begin{pmatrix} u^\mu \\ \mu u^\mu \end{pmatrix}$$

What would you like to do? Couple to a transmission line. Look at Hardy space

$H = L^2(\mathbb{R}, \frac{d\omega}{2\pi}) = H^+ \oplus H^-$ You would like to make on $H^- \oplus Y \oplus H^+$ ~~[illegible]~~ a self adjoint op. ~~[illegible]~~ combining $A\varepsilon^{-1}$ with ~~[illegible]~~ mult by ω on $H^\pm$. Work with subspaces of $\begin{matrix} H^- & & Y & & H^+ \\ \oplus & + & \oplus & + & \oplus \\ H^- & & Y & & H^+ \end{matrix}$. What happens with H^+

~~[illegible]~~ $W^+ = \begin{pmatrix} 1 \\ \omega \end{pmatrix} D^+_\omega \subset \begin{pmatrix} H^+ \\ \oplus \\ H^+ \end{pmatrix}$

$W^+ = \left\{ \begin{pmatrix} f \\ \omega f \end{pmatrix} \middle| \int (1+\omega^2)|f|^2 \frac{d\omega}{2\pi} < \infty \right\}$ Keep on trying.
$f \in H^+$

$\varepsilon = \frac{a+b}{2}$

$(W^+)^\circ = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \middle| (f_1, \omega f) = (f_2, f) \quad \forall f \in D^+_\omega \right\}$ $\quad A = \frac{i(a-b)}{2}$

$W^+ = \begin{pmatrix} 1 \\ \omega \end{pmatrix} D^+_\omega \subset \begin{pmatrix} H^+ \\ H^+ \end{pmatrix} \xrightarrow{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}} \begin{pmatrix} H^+ \\ \oplus \\ H^+ \end{pmatrix}$ $\quad \begin{pmatrix} \varepsilon \\ A \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 \\ \omega \end{pmatrix} = \cancel{\begin{pmatrix} 1-i\omega \\ 1+i\omega \end{pmatrix}}\ \cancel{[illegible]} \begin{pmatrix} 1-i\omega \\ 1+i\omega \end{pmatrix}$ $\quad \begin{pmatrix} a \\ b \end{pmatrix} = i\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} \varepsilon \\ A \end{pmatrix}$

$$\begin{cases} a = \varepsilon - iA & 1-i\omega \\ b = \varepsilon + iA & 1+i\omega \end{cases}$$

$$\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}\begin{pmatrix} 1 \\ \omega \end{pmatrix} D_\omega = \begin{pmatrix} 1-i\omega \\ 1+i\omega \end{pmatrix} D_\omega$$

codim 1

$(1-i\omega)D_\omega = H^+$ since $1-i\omega = 0$ when $\omega = -i$

$(1+i\omega)D_\omega = (\omega - i)D_\omega$

kernel of evaluation at $\omega = i$. So $(W^+)^\oplus$ has

a

Review. Take $\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ and $\begin{pmatrix} 1 \\ \omega \end{pmatrix} D_\omega \subset \begin{matrix} H^+ \\ \oplus \\ H^+ \end{matrix}$

$$\begin{pmatrix} 1 \\ \omega \end{pmatrix} D_\omega = \begin{pmatrix} (1+\omega^2)^{-1/2} \\ \omega(1+\omega^2)^{-1/2} \end{pmatrix} H^+ \subset \begin{matrix} H^+ \\ \oplus \\ H^+ \end{matrix}$$

$W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad W^0 = W \oplus \begin{matrix} \mathrm{Ker}(a^*) \\ \oplus \\ \mathrm{Ker}(b^*) \end{matrix}$

do harmonic oscillator

$x'' + x = F(t)$ has solution

$$x = \int_{-\infty}^{t} G(t-t')F(t')\,dt' \qquad (\partial_t^2 + 1)G(t) = \delta(t)$$

vanishing as $t \to -\infty$

$$G(t) = \begin{cases} \sin t & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

and general solution

$$x = \mathrm{Re}(Ae^{-it}) + \int_{-\infty}^{t} \sin(t-t')F(t')\,dt'$$

~~which is~~ $\cancel{\int_{-\infty}^{\infty} \sin(t-t')F(t')dt'}$ What happens as $t \to \infty$

$$\int_{-\infty}^{t} \sin(t-t')F(t')\,dt' = \int_{-\infty}^{t} \mathrm{Re}\big(-ie^{i\overbrace{(t-t')}^{u}}\big)F(t')\,dt'$$

$$= \int_0^\infty \mathrm{Re}(-ie^{iu})F(t-u)\,du \qquad ?$$

70 for $t \gg 0$ dryer stop 11:00, then 80 min

$$x(t) = \int_{-\infty}^{\infty} \mathrm{Re}\left(-i e^{i(t-t')}\right) F(t')\,dt'$$

$$= \mathrm{Re}\; -i e^{it} \int_{-\infty}^{\infty} e^{-it'} F(t')\,dt'$$

$H = \omega a^* a$ $\qquad H = \frac{1}{2m}\left(p^2 + \frac{1}{2}\omega^2 q^2\right)$ $\qquad [p,q] = \left[\frac{\hbar}{i}\partial_x, x\right]$

$(\omega q - ip)(\omega q + ip)$ $\qquad = \frac{\hbar}{i}$

$[q,p] = i\hbar$

$$\cancel{[\omega q - ip, \omega q + ip]} = \omega\hbar i^2 - i\omega\frac{\hbar}{i} = -2\omega\hbar$$

$$[a, a^*] = \left[\frac{\omega q + ip}{\sqrt{2\hbar\omega}}, \frac{\omega q - ip}{\sqrt{2\hbar\omega}}\right] = 1$$

$$\omega\frac{(\omega q - ip)(\omega q + ip)}{2\hbar\omega}$$

$$\dot q = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot p = -\frac{\partial H}{\partial q} = -kq$$

$$m\ddot q + kq = 0 \qquad \frac{k}{m} = \omega^2$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2$$

$$= \hbar\omega \underbrace{\left(\frac{-ip}{\sqrt{2m\omega\hbar}} + \sqrt{\frac{m\omega}{2\hbar}}q\right)}_{a^*} \overbrace{\left(\frac{ip}{\sqrt{2m\omega\hbar}} + \sqrt{\frac{m\omega}{2\hbar}}q\right)}^{a} = \hbar\omega\left(H - \frac{1}{2}\right)$$

not important

$$\left[\frac{ip}{\sqrt{2m\omega\hbar}}, \sqrt{\frac{m\omega}{2\hbar}}q\right] = \frac{1}{2\hbar}\hbar = \frac{1}{2}$$

suppose $H = \omega a^* a$ $\qquad q = a + a^*$

$$\langle 0 | q e^{-itH} q | 0\rangle = \langle 0 | a e^{-it\omega a^* a} a^* | 0\rangle$$

$$= \langle 0 | a e^{-it\omega} a^* | 0\rangle = e^{-it\omega}$$

$t = -i\tau$ $\qquad e^{-i(-i\tau)H} = e^{-\tau H}$

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Review: $W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad L^2(\mathbb{R}, \frac{d\omega}{2\pi}) = H^- \oplus H^+$ Hardy spaces

To Understand $\begin{pmatrix} 1 \\ \omega \end{pmatrix} D_\omega \subset \begin{matrix} H^+ \\ \oplus \\ H^+ \end{matrix}$

$$\varepsilon = \tfrac{1}{2}(a+b)$$
$$A = \tfrac{i}{2}(a-b)$$
$$\varepsilon - iA = a$$
$$\varepsilon + iA = b$$

$$\begin{tikzcd} D_\omega \arrow[r, "\begin{pmatrix} 1 \\ \omega \end{pmatrix}"] \arrow[rr, bend right] & \begin{matrix} H^+ \\ \oplus \\ H^+ \end{matrix} \arrow[r, "\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}"] & \begin{matrix} H^+ \\ \oplus \\ H^+ \end{matrix} \end{tikzcd}$$

$a = 1 - i\omega$ $\qquad ba^{-1} = \dfrac{1+i\omega}{1-i\omega}$

$b = 1 + i\omega$

Idea is that $1+i\omega$ vanishes at $\omega = i$ so that $(1+i\omega)D_\omega$ should have codim 1.

You need $L^2(S^1)$ $\qquad L^2(\mathbb{R})$

z $\qquad \dfrac{1+i\omega}{1-i\omega} = \dfrac{-\omega+i}{\omega+i}$

1 $\qquad \displaystyle\int_{-\infty}^{\infty} \left| \frac{\sqrt{2}}{\omega+i} \right|^2 \frac{d\omega}{2\pi} = \int \frac{2}{1+\omega^2} \frac{d\omega}{2\pi}$

$$= \frac{2}{2\pi} \arctan \omega \Big]_{-\infty}^{\infty} = \frac{\pi}{\pi} = 1$$

Go back to

$H = \quad H^- \oplus Y \oplus H^+$

Let's return to $H = \cdots \oplus u^{-1}V^- \oplus aX \oplus V^+ \oplus uV^+ \oplus \cdots$

$\qquad\qquad\qquad\qquad \cdots \oplus u^{-1}V^- \oplus V^- \oplus bX \oplus uV^+ \oplus \cdots$

and try for the hermitian analogues. So what to do next?? Is there a simple way to describe H?

Suppose you try to generalize $g^* u^n g = (g^* u g)^n$ for $n \geq 0$ to the continuous case.

~~del~~ Look for H with $u^t = e^{ith}$ h herm.

$j^* u^t j = (j^* u j)^t$ for $t \geq 0$. Meaning?

~~Suppose~~ Maybe it works.

$\int$?

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discrete case:

$$j^* u^n j = \overset{\gamma}{\overbrace{(j^* u j)}}^n \qquad n \geq 0$$

$$= (\gamma^*)^{-n} \qquad n \leq 0.$$

form

$$\sum_{n \in \mathbb{Z}} z^{-n} j^* u^n j = \sum_{n \geq 0} (z^{-1}\gamma)^n + \sum_{n > 0} (z\gamma^*)^n$$

$$= \frac{1}{1 - z^{-1}\gamma} + \frac{z\gamma^*}{1 - z\gamma^*}$$

$$= \frac{1}{1 - z^{-1}\gamma} \Big(\underbrace{1 - z\gamma^* + (1 - z^{-1}\gamma) z\gamma^*}_{1 - \gamma\gamma^*} \Big) \frac{1}{1 - z\gamma^*}$$

analogue is

$$\int_{-\infty}^{\infty} e^{-i\omega t} j^* u^t j \, dt \qquad \begin{cases} e^{t\beta} & t \geq 0 \\ e^{-t\beta^*} & t \leq 0 \end{cases}$$

$$\int_0^\infty e^{-i\omega t} e^{t\beta} dt + \int_{-\infty}^0 e^{-i\omega t} e^{-t\beta^*} dt$$

$$\left[\frac{e^{(-i\omega + \beta)t}}{-i\omega + \beta} \right]_0^\infty + \left[\frac{e^{-(i\omega + \beta^*)t}}{-(i\omega + \beta^*)} \right]_{-\infty}^0$$

$$\frac{1}{i\omega - \beta} - \frac{1}{i\omega + \beta^*} = \frac{1}{i} \left(\frac{1}{\omega + i\beta} - \frac{1}{\omega - i\beta^*} \right)$$

$$= \frac{1}{\omega - i\beta^*} \underbrace{\left(\frac{\omega - i\beta^* - (\omega + i\beta)}{i} \right)}_{-(\beta + \beta^*)} \frac{1}{\omega + i\beta} = \frac{1}{\omega - \alpha^*} \left(-i(\alpha - \alpha^*) \right) \frac{1}{\omega - \alpha}$$

Put $\beta = i\alpha$

73 Can you use this somehow. The idea is to produce ~~B~~ α nearly hermitian operator directly from $\begin{pmatrix} \varepsilon \\ A \end{pmatrix}$

We expect to find a self adjoint operator H e^{itH} such that $y^* e^{itH} y = e^{it\alpha}$

so $y^* \dfrac{1}{\omega - H} y = \dfrac{1}{\omega - \alpha}$ $\qquad \mathrm{Im}(\alpha) \geq 0$

So what?

$W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad W^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (y_1, Ax) = (y_2, \varepsilon x) \ \forall x \right\}$

$W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y = \left\{ \begin{pmatrix} y \\ \lambda y \end{pmatrix} \middle| y \in \mathrm{Ker}(\lambda \varepsilon^* - A^*) \right\}$.

partial unitary picture

$W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad W^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (y_1, ax) = (y_2, bx) \ \forall x \right\}$

i.e. $a^* y_1 = b^* y_2$

given $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W^\circ$ let $x = a^* y_1$. Then

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} a a^* y_1 \\ b a^* y_2 \end{pmatrix} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$ $\qquad$ where $a^* y_1' = 0$, $b^* y_2' = 0$.

$\therefore \quad W^\circ = W \oplus \begin{matrix} \mathrm{Ker}\, a^* \\ \oplus \\ \mathrm{Ker}\, b^* \end{matrix}$ $\qquad L_z = W^\circ \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y = \left\{ \begin{pmatrix} y \\ zy \end{pmatrix} \middle| a^* y = z b^* y \right\}$

$y \in \mathrm{Ker}(a^* - z b^*)$

$$\begin{tikzcd}
0 \arrow[r] & L_z \arrow[r] \arrow[d] & Y \arrow[r, "a^* - z b^*"] \arrow[d, "\binom{1}{z}"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W^\circ \arrow[r] \arrow[d] & Y \oplus Y \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & Y \arrow[r, equal] & Y & &
\end{tikzcd}$$

 Review again. Consider $W = \binom{a}{b} X \subset \begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix}$ isotropic $\|ax\|^2 = \|bx\|^2$ for $\|y_1\|^2 - \|y_2\|^2$. Find $W^\circ = W \oplus \begin{smallmatrix} \mathrm{Ker}(a^*) \\ \oplus \\ \mathrm{Ker}(b^*) \end{smallmatrix}$. ~~What is the unitary~~ Now pick a line in W°/W. The one you take is $\binom{0}{\mathrm{Ker}\, b^*}$. Note that any line in W°/W corresponds to a V, $W \subset V \subset W^\circ$. ~~Is V the graph of~~ When is V a graph. Look at $p_1 : V \to Y$

Wait. You know that $W^\circ \ominus W = \begin{smallmatrix} \mathrm{Ker}\, a^* \\ \oplus \\ \mathrm{Ker}\, b^* \end{smallmatrix}$, so $V \ominus W$ is a line $\subset \begin{smallmatrix} \mathrm{Ker}\, a^* \\ \oplus \\ \mathrm{Ker}\, b^* \end{smallmatrix}$. ~~So the import~~

~~You know that~~

Let's review the situation in the partial unitary $O(n)$ case. You have Y a $n+1$ dim Hilb space, an n dim v.S. X and maps $a, b : X \to Y$ $\ni$ 1) $az - b : X \hookrightarrow Y$ all $z \in \mathbb{C} \cup \infty$ 2) $\|ax\| = \|bx\|$ all x. $\therefore$ get $\| \ \|$ on $X \ni$ $a^*a = b^*b = 1$.

~~[illegible]~~ Form

$$H_{nt} = \underline{aX \oplus V^+} \oplus uV^+ \oplus$$
$$\overline{V^- \oplus bX} \oplus$$

$$y = aa^*y + \pi^+ y$$
$$uy = ba^*y + u\pi^+ y$$
$$= aa^*ba^*y + \pi^+ ba^* y + u\pi^+ y$$
$$u^2 y = aa^*(ba^*)^2 y + \pi^+ (ba^*)^2 y + u\pi^+(ba^*) y + u^2 \pi^+ y$$

$$y = u^{-N}\Big\{ aa^*(ba^*)^N y + u^{N} \pi^+ (ba^*)^n y +$$

so move into functions

$$y \rightsquigarrow \pi^+ y + u^{-1}\pi^+(ba^*) y + u^{-2}\pi^+(ba^*)^2 y + \cdots$$
$$\pi^+ (1 - z^{-1} ba^*)^{-1} y$$

$$Y \xrightarrow{\begin{pmatrix} z^{-1}a^* \\ e^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V^+ \end{matrix} \xrightarrow{(az-b \quad e)} Y$$

solution of $(az-b)x + ev^+ = y$ is $\begin{pmatrix} z^{-1}a^* \\ e^* \end{pmatrix}(1 - z^{-1}ba^*)^{-1} y$

$$(az-b)z^{-1}a^* + ee^* = 1 - z^{-1}ba^*$$

How do you use, organize, these ideas?

~~You want to deform~~

You have

$$Y \xrightarrow{\begin{pmatrix} -b^* \\ e^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V^- \end{matrix} \xrightarrow{(az-b \quad e)} Y$$

How to organize? You might work with $\begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

$$W = \begin{pmatrix} a \\ b \end{pmatrix} X, \quad \begin{pmatrix} 1 \\ z \end{pmatrix} Y, \quad W^0 = W \oplus \begin{pmatrix} V^+ \\ \oplus \\ V^- \end{pmatrix}$$

Your previous success is based upon the splitting $Y = \begin{matrix} X \\ \oplus \\ V^+ \end{matrix}$ or $Y = \begin{matrix} X \\ \oplus \\ V^- \end{matrix}$. You somehow use the

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Start again. You have $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ ~~$a^*a = b^*b = 1$~~

$\|ax\|^2 = \|bx\|^2 \ \forall x \iff W = \begin{pmatrix} a \\ b \end{pmatrix} X$ isot. from $\|y_1\|^2 - \|y_2\|^2$.

$W^0 = W \oplus \begin{matrix} \mathrm{Ker}(a^*) \\ \oplus \\ \mathrm{Ker}(b^*) \end{matrix}$. The basic spectral representation $Y \xrightarrow{\begin{pmatrix} +b^* \\ e^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V^- \end{matrix} \xrightarrow{(-az+b \quad e_-)} Y$

arises from the splitting $Y \xrightarrow{\begin{pmatrix} b^* \\ e^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V^- \end{matrix} \xrightarrow{(b, e)} Y$. Leads to solution

of $(az-b)x = -y + \tilde{y}(z)e$ is $\begin{pmatrix} x \\ \tilde{y}(z) \end{pmatrix} = \begin{pmatrix} b^* \\ e^* \end{pmatrix}(1 - zab^*)^{-1} y$

But then you ~~know that~~ can prove that $\int |\tilde{y}(z)|^2 \frac{d\theta}{2\pi} = \|y\|^2$

It's this residue trick you need to understand better. YES. How might you prove it. First you might take 2 elements $y, y' \in Y$ and somehow understand

$$\left(y', (1-\bar{z}^{-1} b a^*)^{-1} e \, e^*(1-zab^*)^{-1} y \right) \qquad ee^* = 1 - bb^*$$

trick

$$\frac{1}{1-z^{-1}\gamma} + \frac{z\gamma^*}{1-z\gamma^*} = \frac{1}{1-z^{-1}\gamma}\left((1-z^{-1}\gamma) z\gamma^* + 1 - z\gamma^*\right)\frac{1}{1-z\gamma^*}$$

$$\frac{1}{1-z^{-1}\gamma}(1-\gamma\gamma^*)\frac{1}{1-z\gamma^*} = \frac{A}{1-z^{-1}\gamma} +$$

Invariant approach - see next 2 pages.
$\mathbb{P}^1 = \mathbb{P}T$, where T is 2-dim'l equipped with pseudoscalar product, Y is a Hilbert space, $T \otimes Y$ has $\otimes$product pseudoscalar product, canonical sequence

$$0 \to \mathcal{O}(-1) \otimes Y \to \mathcal{O} \otimes T \otimes Y \to \mathcal{O}(1) \otimes Y \to 0$$

L^2 sections of $\mathcal{O}(-1)$ over the real $\mathbb{P}^1$ should form a Hilbert space in an intrinsic way, hence also L^2 sections of $\mathcal{O}(-1) \otimes Y$. So what else happens? You now want to proceed to spectral representation. You need to choose a line in W^0/W and conjugate (or adjoint) line. Corresponds to choose V, $W < V < W^0$ and its annihilator V^0. Interested in $V \neq V^0$

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Invariant approach. T 2 dimensional space with hermitian form of signature $1,-1$. Y Hilbert space. $T\otimes Y$ is Krein space. Have basic exact sequence

$$0 \longrightarrow \mathcal{O}(-1)\otimes Y \xrightarrow{\mathcal{O}\otimes} T\otimes Y \longrightarrow \mathcal{O}(1)\otimes Y \longrightarrow 0$$

Take W isotropic in $T\otimes Y$ for the pseudo scalar product, get W^0/W. Where is the K-mod?

Example: $T = \mathbb{C}^2 \quad |z_1|^2 - |z_2|^2$

$T\otimes Y = \underline{Y\oplus Y} \supset W^0 \quad \|y_1\|^2 - \|y_2\|^2$

$$0 \longrightarrow \begin{pmatrix}1\\z\end{pmatrix} Y \hookrightarrow \begin{matrix}Y\\ \oplus \\ Y\end{matrix} \xrightarrow{(z\ \ -1)} Y \longrightarrow 0$$

So get $\mathcal{O}\otimes W^0 \longrightarrow \mathcal{O}(1)\otimes Y$ which should be OKAY provided $\mathcal{O}\otimes W^0$ and $\mathcal{O}(-1)\otimes Y$ intersect transversally, i.e. $W^0 + \begin{pmatrix}1\\z\end{pmatrix} Y = \begin{matrix}Y\\ \oplus \\ Y\end{matrix}$. How is this related to $W \cap \begin{pmatrix}1\\z\end{pmatrix} Y = 0$? There should be some relation ~~between~~

$W^0 = \left\{ \begin{pmatrix}y_1\\y_2\end{pmatrix} \middle| \ a^*y_1 = b^*y_2 \right\}$

$W^0 \cap \begin{pmatrix}1\\z\end{pmatrix} Y = \left\{ \begin{pmatrix}y\\zy\end{pmatrix} \middle| \ a^*y = b^*(zy) \right\} \simeq \mathrm{Ker}\,(a^* - \bar{z} b^*)$.

$= \left((a - \bar{z} b)X\right)^\perp$

$\therefore$ ~~If no bound states~~

$$W \cap \begin{pmatrix}1\\z\end{pmatrix} Y = \left\{ \begin{pmatrix}ax\\bx\end{pmatrix} \middle| \ bx = zax \right\} \simeq \mathrm{Ker}\,(az - b)$$

annihilator relationship

$$\left(W^0 + \begin{pmatrix}1\\z\end{pmatrix} Y \right)^0 = W \cap \begin{pmatrix}1\\ \bar{z}^{-1}\end{pmatrix} Y$$

So how do you proceed at this point?

We know that W^0/W has induced pseudoscalar product. Suppose ~~it has dim~~ we have $\mathcal{O}(n)$ case

Picking a line

$$\left(W^\circ \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y\right)^\circ = W + \begin{pmatrix} 1 \\ \bar z^{-1} \end{pmatrix} Y$$

$$\begin{tikzcd}
 & & Y \arrow[d, "\begin{pmatrix} 1 \\ \bar z^{-1} \end{pmatrix}"] & \\
0 \arrow[r] & W \arrow[r, "\begin{pmatrix} a \\ b \end{pmatrix}"] & Y \oplus Y \arrow[r] \arrow[d, "\begin{pmatrix} \bar z^{-1} & -1 \end{pmatrix}"] & {} \\
 & & Y &
\end{tikzcd}$$

? not imp. ~~So not~~

What do you need to understand? You ~~whd~~ choose V, $W \subset V \subset W^\circ$ $L = V/W$ line in $\begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$. There is a conjugate line L°

$$\left(\begin{pmatrix} 1 \\ F \end{pmatrix} Y\right)^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \ (y_1, y) = \underset{\displaystyle (F^* y_2, y)}{\overset{}{(y_2, Fy)}} \quad \forall y \right\}$$

$$= \begin{pmatrix} F^* \\ 1 \end{pmatrix} Y$$

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In the intrinsic picture you have chosen V, $W < V < W^\circ$. This should yield a spectrum namely ~~those points~~ the complement of those points of $\omega \in \mathbb{P}^1 T$ such that W and $\mathcal{O}(-1) \otimes_\omega Y$ are complementary. ~~Somehow~~ Somehow you want to ~~[illegible]~~ embed Y in L^2 sections on the real circle. If you take the standard ~~=~~ picture where $V/W = \operatorname{Ker} a^*$ or $\operatorname{Ker} b^*$ you know what to do. So one way to proceed would be to try to arrange this by choosing the ~~polarization~~ coordinates. ~~Choose that~~

Suppose given $\overset{n}{W} \subset \overset{2}{T} \otimes \overset{n+1}{Y}$ isotropic, and $\overset{n+1}{V}$, $\overset{n}{W} < \overset{n+1}{V} < \overset{n+2}{W^0}$. Choose a polar. of $T = \begin{pmatrix}\mathbb{C}\\ \oplus \\ \mathbb{C}\end{pmatrix}$ ps sc pr $|z_1|^2 - |z_2|^2$

$W = \begin{pmatrix}a\\ b\end{pmatrix}X$, $\|ax\|^2 = \|bx\|^2$ $\qquad W^0 = W \oplus \begin{matrix}\mathrm{Ker}\,a^*\\ \oplus\\ \mathrm{Ker}\,b^*\end{matrix}$ $\qquad V$

intersects to give a line $L \subset \begin{matrix}\mathrm{Ker}\,a^*\\ \oplus\\ \mathrm{Ker}\,b^*\end{matrix}$. Actually we can restrict the hermitian form to V, it vanishes on W and has a sign >0 or <0 on $V - W$. In fact ~~there's a map to $\mathbb{R}$~~ V/W is a complex line with scalar product.

You have $V \oplus l_\omega \otimes Y \xrightarrow{\sim} T \otimes Y$

for ω not in the spectrum and then your quotient line V/W, so what equations are you solving?

$$\begin{pmatrix}1\\ \gamma\end{pmatrix}Y + \begin{pmatrix}1\\ z\end{pmatrix}Y \xrightarrow{\sim} \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$$

So it seems that I get an element of V/W for any triple (z, y_1, y_2) $\quad z \notin \operatorname{spec}\gamma$

$$(z \ \ -1) : \begin{pmatrix}1\\ \gamma\end{pmatrix}Y \xrightarrow{\sim} Y$$

$$z - \gamma : Y \xrightarrow{\sim} Y$$

so given y

$$\begin{array}{ccc}\mathcal{O}\otimes V & \longrightarrow & \mathcal{O}(1)\otimes Y\\ \downarrow & & \\ \mathcal{O}\otimes(V/W) & & \end{array}$$

 Review. You have $W \subset V \subset W^\circ \subset T \otimes Y$

and $0 \longrightarrow \mathcal{O}(-1) \otimes Y \longrightarrow \mathcal{O} \otimes T \otimes Y \longrightarrow \mathcal{O}(1) \otimes 1 \longrightarrow 0$

The hermitian form on $T \otimes Y$ ~~restricts to~~ restricts to 0 on W so you get a 1-diml quotient V/W with pos. def. herm. form. Spectral transform, namely go from Y to functions on the real $\mathbb{P}^1 \subset \mathbb{P}(T)$. Spectrum = where $\begin{pmatrix}1\\z\end{pmatrix} Y$ and V are not complementary. off spectrum. Assume $V = \begin{pmatrix}1\\ \gamma\end{pmatrix} Y$

$$\begin{pmatrix}1\\ \gamma\end{pmatrix} Y \oplus \begin{pmatrix}1\\ z\end{pmatrix} Y = \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} z & -1\end{pmatrix}\begin{pmatrix}1\\ \gamma\end{pmatrix} = z - \gamma : Y \longrightarrow Y \quad \text{is an isom.}$$

Anyway

$$\begin{pmatrix}1 & 1\\ \gamma & z\end{pmatrix}^{-1} = \begin{pmatrix} z & -1\\ -\gamma & 1\end{pmatrix}(z-\gamma)^{-1}.$$

At the moment you have a map from $\begin{pmatrix}y_1\\ y_2\end{pmatrix} \in \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$

to

$$\begin{pmatrix}1\\ \gamma\end{pmatrix} \underbrace{\begin{pmatrix} z & -1\end{pmatrix}(z-\gamma)^{-1}\begin{pmatrix}y_1\\ y_2\end{pmatrix}}_{z(z-\gamma)^{-1}y_1 - (z-\gamma)^{-1}y_2 \;=\; (z-\gamma)^{-1}(zy_1 - y_2)}$$

This is the element of V which is to be projected onto V/W.

It's worth looking for a proof that

$y \longmapsto \pi (z-\gamma)^{-1} y = \tilde{y}(z)$ is unitary embedding

$$\tilde{y}(z)^* y(z) = y^*(z^{-1} - \gamma^*)^{-1} \pi^* \pi (z-\gamma)^{-1} y$$

$$(1-z\gamma^*)^{-1} + z^{-1}\gamma(1-z^{-1}\gamma)^{-1}$$
$$= (1-z\gamma^*)^{-1}\big((1-z\gamma^*)z^{-1}\gamma + (1-z^{-1}\gamma)\big)(1-z^{-1}\gamma)^{-1}$$
$$= (1-z\gamma^*)^{-1}(1-\gamma^*\gamma)(1-z^{-1}\gamma)^{-1}$$

Is there an intrinsic way to do this.

$$\begin{tikzcd} & V \arrow[dl, "\pi"'] \arrow[dr, "s"] \arrow[r, phantom, "\subset"] & T\otimes Y \arrow[d] \arrow[r, phantom, "\supset"] & l_\omega\otimes Y \\ V/W & & T/l_\omega\otimes Y & \end{tikzcd}$$

So you get an ~~embedding~~ a map $Y \xrightarrow{\nu} (T/l_\omega)^*\otimes V \to (T/l_\omega)^*\otimes V/W$

You want this for l_ω "real"

~~Idea~~ Roughly $\tilde{y}(z) = \pi(z-\gamma)^{-1}y = \pi z^{-1}(1-z^{-1}\gamma)^{-1}y$ is analytic ~~on $|z|$~~ on and outside $|z|=1$.

I guess what's intriguing is inner product between different $\pi(z-\gamma)^{-1}y$. In the ~~skew~~ partial hermitean case what's interesting is the pairing between u^λ and u^μ. This seems to involve W and W^0.

In the J-matrix case you have u^λ entire in λ and a formula $(u^\mu, u^\lambda) = \frac{1}{\bar\mu-\lambda}(\quad)$ hermitian form ~~skew form~~ applied to bdry values.

which will vanish when $\lambda=\bar\mu$ because the lines L_λ $L_{\bar\lambda}$ are orth.

So this leads us to $\boxed{\text{ignore } V}$ and concentrate on the family of $L_\lambda \subset W^0/W$

$L_\omega = W^0 \cap l_\omega\otimes Y$. Given two: ω, ω' what can you say? $L^0_\omega = W + l_{\bar\omega}\otimes Y$

$$0 \to L_\omega \to \underset{n+2}{W^0} \to \underset{n+1}{(T/l_\omega)\otimes Y} \to 0$$

looks like $L_\omega = \mathcal{O}(-n-1)$.

So you have two of these lines $L_\omega, L_{\omega'}$. Conv

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bring this discussion to an end.

$$W \subset T \otimes Y \supset l_z \otimes Y$$

$$W^\circ \cap (l_z \otimes Y) = L_z \text{ maps inj into } W^\circ/W$$

main ideas? From the data T, Y, W you seem to get the 2 dim'l W°/W (Krein space) and this subline bundle $\mathcal{L}$ of $\mathcal{O} \otimes W^\circ/W$ over $\mathbb{P}T$ with certain adjointness properties. Let's describe this as well as we can.

$$W = \begin{pmatrix} a \\ b \end{pmatrix} X \qquad W^\circ = W \oplus \begin{matrix} \mathrm{Ker}\, a^* \\ \oplus \\ \mathrm{Ker}\, b^* \end{matrix} \qquad \text{You need}$$

$$W^\circ \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \qquad \begin{pmatrix} y \\ zy \end{pmatrix} = \begin{pmatrix} ax \\ bx \end{pmatrix} + \begin{pmatrix} v^+ \\ v^- \end{pmatrix}$$

i.e. $z(ax + v^+) = bx + v^-$

$$(az - b)x = -zv^+ + v^-$$

So the image of L_z in $W^\circ/W = \begin{matrix} \mathrm{Ker}(a^*) \\ \oplus \\ \mathrm{Ker}(b^*) \end{matrix}$

Consists of $\begin{pmatrix} v^+ \\ v^- \end{pmatrix}$ such that $-zv^+ + v^- \in (az-b)X$. Means all $\begin{pmatrix} v^+ \\ zS(z)v^+ \end{pmatrix}$ $v^- \in V^-$.

$S(z)(zv^+) = v^-$

Notice that the degree of $zS(z)$ is $n+1$.

Next go back to $W \subset T \otimes Y \supset l_z \otimes Y$

$L_z = W^\circ \cap (l_z \otimes Y) \hookrightarrow W^\circ/W$. Pencil of hyperplane sections of degree $n+1$.

83 Intrinsically you have $\mathcal{L} \simeq \mathcal{O}(n-1)$ embedded in $\mathcal{O} \otimes (W^0/W)$. If you take quotient lines of W^0/W (or lines) then you get ~~sections~~ maps $\mathcal{L} \longrightarrow \mathcal{O}$ i.e. divisors ≥ 0 of degree $n+1$. ~~For these various interesting points~~ metric possibilities. W^0/W has hermitian form so $\mathbb{P}(W^0/W)$ has a real projective line. Now the herm. form on W^0/W pulls back to the one which is restriction of given herm. form on $T \otimes Y$. But $\begin{pmatrix} y \\ zy \end{pmatrix} \rightsquigarrow (1-|z|^2)\|y\|^2$, so $z \mapsto L_z$ from $\mathbb{P}(T)$ to $\mathbb{P}(W^0/W)$ preserves real $\mathbb{P}T$ and two disks.

Let's start now with $W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ equipped with $\left(\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = (y_1', y_2) - (y_2', y_1)$

W isotropic means $(\varepsilon x', Ax) = (Ax', \varepsilon x)$

i.e. $A^* \varepsilon = \varepsilon^* A$ (wrt of scalar prod on X).

Intrinsic version T 2diml Krein, Y $n+1$ diml Hilb, $T \otimes Y$ then $2n+2$ diml Krein, W n diml isot. in $T \otimes Y$, W^0/W ~~then~~ 2 diml Krein, this is the port or terminals. So $\omega \in \mathbb{P}T$, l_ω corresp line in T, assume $W \cap (l_\omega \otimes Y) = 0$ $\forall \omega$ (no bound states), $l_\omega^0 = l_{\bar\omega}$, $\bar\omega$ = reflection of ω through the real $\mathbb{P}^1$ given by the null lines for the Krein form, so $W \cap (l_\omega \otimes Y) = 0 \iff W^0 + l_{\bar\omega} \otimes Y = T \otimes Y$ $\therefore$ as ω varies $L_\omega = W^0 \cap (l_\omega \otimes Y)$ is a line subbundle of W^0, $0 \to L_\omega \to \underset{n+2}{W^0} \to \underset{n+1}{T/l_\omega \otimes Y} \to 0$, so $\{L_\omega\} \simeq \mathcal{O}(-n-1)$. Also $\omega \mapsto \operatorname{Img}\{L_\omega \hookrightarrow W^0/W\}$ gives an alg. map $\mathbb{P}T \xrightarrow{Z} \mathbb{P}(W^0/W)$, covered by a line bundle $\mathcal{L} \longrightarrow \mathcal{O}(-1)$ ~~such that Krein form on W~~ compatible with

84 Krein forms since the Krein form on W^0 descends to W^0/W! Z is the response function. It preserves the null circles and the $+, -$ disks, has degree $n+1$.

To get spectral rep for elements of Y choose $W \subset \overset{n+1}{V} \subset W^0$ so that V/W is a ~~pos~~ neg line, then V^0/W is a ~~nega~~posi-tive line. Get spectrum of $\omega \ni V \cap (l_\omega \otimes Y) \neq 0$ off the spectrum get $V \oplus l_\omega \otimes Y = 0$, this true for ω pos. or real since Krein form on V is <0 off W and on $l_\omega \otimes Y$ is ≥ 0. So spect in LHP. Off spectrum we have ~~$V \xrightarrow{\sim} T/l_\omega \otimes Y$ and $V \to V/W$, so we get $\mathcal{O}(-1) \otimes Y \xrightarrow{\sim} \mathcal{O} \otimes V \to \mathcal{O} \otimes V/W$,~~

Spectral repn. V/W neg line in W^0/W, V^0/W corresp. pos. line. Claim $V \cap l_\omega \otimes Y = 0$ for $\mathrm{Im}(\omega) \geq 0$ because the Krein form on $V-W$ is ≤ 0 and ≥ 0 on $l_\omega \otimes Y$. Thus $V \xrightarrow{\sim} T/l_\omega \otimes Y$ in closed UHP.

What's important, what do I want to emphasize? The Krein space $T \otimes H$

What is the response of a trans line? Here H is infinite diml. A transm. line is the direct sum of a shift and its adjoint. ~~Look at an For You~~

Suppose Y Hilbert space with s such that $s^*s = 1$ and $\mathrm{Ker}(s^*)$ one diml. OK. $W = \binom{1}{s} Y \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$, $*$

$W^0 = \binom{s^*}{1} Y$? $\quad \left\{ \binom{y_1}{y_2} \,\middle|\, (y_1, y) = (y_2, sy) \ \forall y \right\} = \left\{ \binom{s^* y_2}{y_2} \right\}$

$(y_2, sy) = (s^* y_2, y)$

Then $W^0 = W \oplus \begin{pmatrix} 0 \\ \oplus \\ \mathrm{Ker}\, s^* \end{pmatrix}$ and $\overline{W} \cap \binom{1}{z} Y$

$\binom{y}{sy} = \binom{y}{zy} \qquad (z - s) y = 0 \implies (z s^* - 1) y = 0$

$\implies y = 0$ for $|z| < 1$.

In infinite dims need to be careful.

~~W~~ Spectrum

$$(1-z^{-1}S)y = 0 \Rightarrow y=0 \text{ for } \left|\tfrac{1}{z}\right|<1$$

You are confused. You probably should review what happens when $W=$? For a partial unitary $\begin{pmatrix}a\\b\end{pmatrix}X \subset \begin{matrix}Y\\\oplus\\Y\end{matrix}$ there is a complete picture for $|z|\neq 1$, namely two spectral representations associated to the contractions ba^* and ab^*

$$W^0 = W \oplus \begin{matrix}\operatorname{Ker} a^*\\ \oplus \\ \operatorname{Ker} b^*\end{matrix} \quad \cap \begin{pmatrix}1\\z\end{pmatrix}Y \quad \ni \begin{pmatrix}ax\\bx\end{pmatrix} + \begin{pmatrix}v^+\\v^-\end{pmatrix} = \begin{pmatrix}y\\zy\end{pmatrix}$$

$$z(ax+v^+) = (bx+v^-)$$

$$(az-b)x = -zv^+ + v^-$$

$$|z|<1: \qquad v^- = (1-\cancel{ba^*})(1-zab^*)^{-1}zv^+$$

$$|z|>1: \qquad zv^+ = (1-aa^*)(1-z^{-1}ba^*)^{-1}v^-$$

~~...~~ Suppose $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1\\S\end{pmatrix}$ with $S^*S=1$.

$$ab^* = S^* \qquad ba^* = S$$

$$W^0 = \begin{pmatrix}1\\S\end{pmatrix}Y + \begin{matrix}0\\\oplus\\\operatorname{Ker}(S^*)\end{matrix}$$

In general you find the response is a map $V^+ \to V^-$ for $|z|<1$. and a map $V^- \to V^+$ for $|z|>1$.

so in the shift case the response line is $\begin{pmatrix}0\\0\end{pmatrix}\mathbb{C}$ for $|z|<1$.

and $\begin{pmatrix}0\\1\end{pmatrix}\mathbb{C}$ for $|z|>1$.

You need to make sense of transm line ~~theory~~ First

 Trans. line is direct sum of in and out

out: $V^+ \oplus uV^+ \oplus u^2 V \oplus \cdots$

in $V^- \oplus u^{-1}V^- \oplus u^{-2}V^- \oplus \cdots$

Intrinsic picture of either is a $W \subset T \otimes Y$ such that W^0/W is one dimensional, sign of herm. form on W^0/W gives in or out type. To find out polarize T: $W = \binom{a}{b} X \subset \begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix} \supset \binom{1}{z} Y = \ell_z \otimes Y$.

W isotropic means $a^*a = b^*b = 1_X$. Find $W^0 = W \oplus \begin{smallmatrix} \operatorname{Ker} a^* \\ \oplus \\ \operatorname{Ker} b^* \end{smallmatrix}$, so if W^0/W dim 1 either $\operatorname{Ker} a^*$ or $\operatorname{Ker} b^*$ is $\mathbb{C}$ & other is zero.

Assume sign $= -$ on W^0/W, i.e. $W^0/W = \begin{smallmatrix} 0 \\ \oplus \\ \operatorname{Ker} b^* \end{smallmatrix}$. Then a isom, so $W = \binom{1}{\gamma} Y \subset \begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix}$ where $\gamma^*\gamma = 1$ and $\operatorname{Ker}\gamma^*$ dim 1. $W^0 = \binom{\gamma^*}{1} Y$. Ask now about response. When is $W^0 \oplus \binom{1}{z} Y = \begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix}$?

iff $Y \xrightarrow[\sim]{\binom{\gamma^*}{1}} W^0 \subset \begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix} \xrightarrow{(z\ \ -1)} Y$ is an isom, i.e. $1 - z\gamma^*$ is invertible. True for $|z| < 1$, get spectral embedding

$$Y \xrightarrow{\binom{\gamma^*}{1}(1-z\gamma^*)^{-1}} W^0 \longrightarrow W^0/W = \operatorname{Ker}(\gamma^*)$$

$(-\gamma \ \ 1)$

that is $\quad y \longmapsto (1-\gamma\gamma^*)(1-z\gamma^*)^{-1} y$.

Problem: When you discussed response you first asked for $W \cap \binom{1}{z} Y = 0$ and then formed varying line L_z in W^0 whose image in W^0/W gives the response function. Spectrum not discussed until V chosen. Here there are only two choices for V, namely $V = W^0$ or $V = W$.

Properties of $1 - z\gamma^*$ for $|z| > 1$? Does it have kernel $a_0 \neq 0$

$$\gamma^*(a_0, a_1, \dots) = (a_1, a_2, \dots) \overset{?}{=} z^{-1}(a_0, a_1, \dots)$$

$$a_1 = z^{-1}a_0,\ a_2 = z^{-1}a_1, \dots \qquad a_n = z^{-n}a_0$$

and this sequence is in ℓ^2. So spectrum for $V = W^0$ is closed disk $|z| \geq 1$.

87 Now take $V = W = \binom{1}{g} Y \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

$$Y \xrightarrow{\binom{1}{g}} V \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \xrightarrow{(+z \ \ -1)} Y \qquad (z \ \ -1)\begin{pmatrix} 1 \\ g \end{pmatrix} = z - g$$

$(z-g)^{-1} = z^{-1}(1-z^{-1}g)^{-1} = \sum_{n \geq 0} z^{-n-1} g^n$ is invertible for $|z|>1$.

~~But you don't~~ but there is no line to project $\binom{1}{g}(z-g)^{-1}y$ into.

Summarize. Considering $W = \binom{1}{g} Y \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$, $W^0 = \binom{g^*}{1} Y$ where $g^* g = 1$, $\operatorname{Ker} g^*$ dim 1. First study the response, i.e. the intersection $L_z = W^0 \cap \binom{1}{z} Y$.

$$L_z = \begin{pmatrix} g^* \\ 1 \end{pmatrix} \operatorname{Ker}(1 - z g^* : Y \to Y).$$

Case 1. $|z| > 1$. In this case L_z has dim 1 for all $|z|>1$ including ∞, and the line L_z projects onto W^0/W.

Case 2, $|z| < 1$ In this case $L_z = 0$

Response function for a transmission line. $\begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ is the ~~Hilbert space~~ polarized Kreen space

$$\begin{matrix} \cdots \oplus u^{-1}V^- \oplus V^- \oplus V^+ \oplus uV^+ & \qquad \|y_1\|^2 \\ \searrow \quad \searrow \quad \searrow \quad \searrow & \\ \cdots \oplus u^{-1}V^- \oplus V^- \oplus V^+ \oplus uV^+ & \qquad -\|y_2\|^2 \end{matrix}$$

W is the graph of the arrows so that $(aX)^\perp = V^-$ up $\binom{V^-}{0}$ and $(bX)^\perp = V^+$ down $\binom{0}{V^+}$ (observe signs are wrong)

$\therefore W^0/W = \begin{pmatrix} V^- \\ \oplus \\ V^+ \end{pmatrix}$ $\qquad L_z = \begin{pmatrix} y \\ zy \end{pmatrix} \in W^0$. Suppose $|z|<1$

start with $\xi \in V^+$, then you get

$$\begin{pmatrix} z^{-1}\xi + z^{-2}u\xi + z^{-3}u^2\xi + \cdots \\ \\ \xi + z^{-1}u\xi + z^{-2}u^2\xi + \cdots \end{pmatrix} \in L_z$$

provided $|z|>1$. And a similar element starting from

$\eta \in V^-$

$$\begin{pmatrix} \cdots + z u^{-1}\eta + \eta \\ \\ \cdots + z^2 u^{-1}\eta + z\eta \end{pmatrix} \in L_z \quad \text{provided } |z|<1.$$

These only possibilities for L_z $|z| \neq 1$. Image of former is $0 \oplus V^+$ in W^0/W and image of the latter is $V^- \oplus 0$. So the response function is constant on each of the disks. We have

$$Z_z = \begin{cases} 0 \oplus V^+ & \text{for } |z|>1 \\ V^- \oplus 0 & \text{for } |z|<1. \end{cases}$$

Next couple a transmission line to a 1-port of type $O(n)$. You do this by means of an isomorphism between the terminals. Actually you take the direct sum of the Y-Hilbert spaces and the direct sum of the W's with a sign change on the Krein form. Then you need a maximal isot subspace of $W_1^0/W_1 \oplus W_2^0/W_2$), so there should be degenerate couplings. The dimensions are funny. NO Krein isos are $U(2,2)$ dim 4, Lagrangian subspaces descr. by unitaries $U(2)$ dim 4.

Review the situation. The problem is to understand coupling a partial unitary to a transm. line. The result is a unitary operator, only thing we can ask is the spectral measure arising from a convenient cyclic vector

89 Review. When you couple a 1-port to a transmission line you obtain a Hilbert space and unitary operator. ~~[illegible]~~ There are two obvious cyclic vectors and a less obvious one. ~~from the d.b.~~ The obvious ones are V^+, V^- associated ~~cyclic~~ measure is $\frac{d\theta}{2\pi}$. These are related by $S(z)$, factoring $S = p/q$ leads to less obvious one. zeroes of q are outside S^1. q is the deBranges function.

Question ~~whether the de Branges [illegible] theory~~ how this coupling can be understood in terms of the response functions. Given two 1-ports if you couple them the spectrum is given by appropriate difference of the response functions

LC circuit. Before considered $C' \oplus C$, as symplectic

$$E = L(-i\omega)I \qquad\qquad I = C(-i\omega)E$$

You want to bring in power

$$P = EI \qquad\qquad \int EI\,dt = \int L\dot{I}I\,dt$$

~~For a leaf~~

$$= \frac{1}{2}LI^2 + \text{const}$$

$$\int EI\,dt = \int EC\dot{E}\,dt = \frac{1}{2}CE^2$$

You would like to take 2 diml space of $\begin{pmatrix} E \\ I \end{pmatrix}$, allow E, I to be complex, define hermitian form signature $1,-1$.

There has to be an intelligent way to handle this, somehow. You want to fit the situation into a $T \otimes Y$ somehow.

Consider $E = L\partial_t I$ $\qquad E = L(-i\omega)I$

power is? You have ^{T} equipped with hermitian form. You would like to associate a 2 diml space to each edge.

Have basic 2 plane of $\begin{pmatrix}E\\I\end{pmatrix}$ and for ω the line $\begin{pmatrix}L(-i\omega)\\1\end{pmatrix}$, hermitian form

$$\left(\begin{pmatrix}E_1\\I_1\end{pmatrix}, \begin{pmatrix}0 & i\\-i & 0\end{pmatrix}\begin{pmatrix}E_2\\I_2\end{pmatrix}\right) = i(E_1I_2 - I_1E_2)$$

$$\begin{pmatrix}E_1\\I_1\end{pmatrix}\begin{pmatrix}E_2\\I_2\end{pmatrix}$$

<u>Maybe you need Kähler stuff</u>

What you do in the real Lagrangian case for an LC circuit. Basic space is $C^1 \oplus C_1$ with hyperbolic skew form. This is the sum of hyperbolic planes for each edge. Skew form is

$$\left\langle \begin{pmatrix}E_1\\I_1\end{pmatrix}, \begin{pmatrix}E_2\\I_2\end{pmatrix}\right\rangle = E_1I_2 - I_1E_2$$

and any line is of course isotropic. Now for frequency s you want the line $\begin{pmatrix}Ls\\1\end{pmatrix}\mathbb{R}$

What's the relation between the skew form and the hermitian form?

skew form $\begin{pmatrix}x_1'\\x_2'\end{pmatrix}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$= x_1'x_2 - x_2'x_1$ $\quad$ extend to $\mathbb{C}^2$ sesq.

$$\frac{1}{i}\begin{pmatrix}\bar z_1'\\ \bar z_2'\end{pmatrix}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} = \frac{\bar z_1' z_2 - \bar z_2' z_1}{i}$$ ~~skew~~ herm. form

if $z' = z$ $\quad \dfrac{\bar z_1 z_2 - \bar z_2 z_1}{i} = 2\,\mathrm{Im}(\bar z_1 z_2)$ $\quad$ seems to be type (1,1)

91 Try harder. First point is that a symplectic space when complexified is naturally a Krein space. Why. Equivalence between hermitian and skew herm. forms. Take $\mathbb{R}^2$ and a skew form
$$\omega\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$$
$$= a\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \qquad a \text{ real.}$$

extend sesquilinear to the complexification
$$\omega\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = a\begin{vmatrix} \bar{x}_1 & y_1 \\ \bar{x}_2 & y_2 \end{vmatrix}$$
$$= a\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \text{skew herm.}$$

if you mult. by i then $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ +i & -i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ +i & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

~~So you want to define~~ The problem is to fit LC circuits into your abstract framework. T_Ω should be specified. $\mathbb{R}^2$ with volume $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$, complexified becomes $\mathbb{C}^2$ with $(\pm i)\begin{vmatrix} \bar{x}_1 & y_1 \\ \bar{x}_2 & y_2 \end{vmatrix}$

you want $\frac{1}{i}\begin{vmatrix} 1 & 1 \\ \bar{\omega} & \omega \end{vmatrix} = \frac{\omega - \bar{\omega}}{i} > 0$ for $\omega \in UHP$.

In the case of an inductance, you have $\begin{pmatrix} E \\ I \end{pmatrix} \in \mathbb{R}^2$ with $\begin{vmatrix} E_1 & E_2 \\ I_1 & I_2 \end{vmatrix}$, hence herm. form $\frac{1}{i}\begin{vmatrix} \bar{E}_1 & E_2 \\ \bar{I}_1 & I_2 \end{vmatrix}$ on $\mathbb{C}^2$.

$$L_\omega = \begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix}\mathbb{C} \qquad \begin{vmatrix} L(+i\bar{\omega}) & L(-i\omega) \\ 1 & 1 \end{vmatrix}$$
$$= L\, i(\bar{\omega} + \omega) =$$

92 capacitance

$$L_\omega = \begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix} \mathbb{C} \qquad \begin{vmatrix} 1 & 1 \\ C(i\bar\omega) & C(-i\omega) \end{vmatrix} = C(-i\omega - i\bar\omega) = -Ci(\omega+\bar\omega)$$

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try the line

$$L_s = \begin{pmatrix} Ls \\ 1 \end{pmatrix} \mathbb{C} \qquad \begin{vmatrix} L\bar s & Ls \\ 1 & 1 \end{vmatrix} = L(\bar s - s)$$

$$L_s = \begin{pmatrix} 1 \\ Cs \end{pmatrix} \mathbb{C} \qquad \begin{vmatrix} 1 & 1 \\ C\bar s & Cs \end{vmatrix} = C(s-\bar s)$$

This is a pain. How do I proceed to organize this?

Concentrate on what you have on $C^1 \oplus C_1$

Pairing between factors. Important is for each s a subspace $N_s \subset \begin{matrix} C^1 \\ \oplus \\ C_1 \end{matrix}$. ~~There is an L-part~~

This is the direct sum of L, C situations

$$N_s^{ind} = \begin{pmatrix} Ls \\ 1 \end{pmatrix} C_1^{ind} \qquad N_s^{cap} = \begin{pmatrix} 1 \\ Cs \end{pmatrix} C^{1,cap}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = i \begin{vmatrix} \bar x_1 & y_1 \\ \bar x_2 & y_2 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 \\ \bar\omega & \omega \end{vmatrix} = \omega - \bar\omega$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \bar x_1 y_2 + \bar x_2 y_1 \qquad 1s + \bar s 1 = s + \bar s$$

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -i & \\ & i \end{pmatrix}.$$ ~~Now you have to divide~~

Now you understand T. Y is 1 dim'l

It seems that I need another ingredient

T is fixed, say $T = \mathbb{C}^2$ with $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$l_s = \binom{1}{s}\mathbb{C} \subset T$ $\qquad l_s^\circ = \left\{ \binom{y_1}{y_2} \,\middle|\, \binom{1}{s}^* \begin{pmatrix}0&1\\1&0\end{pmatrix}\binom{y_1}{y_2} = 0 \right\}$

$y_2 + \bar{s} y_1 = 0$

$y_2 = -\bar{s} y_1$

$\therefore l_s^\circ = l_{-\bar{s}}$ So what do you get ?? So what next? There is a difficulty here. It seems that all we get in the case $\dim(Y) = 1$ is ? What do you need or want?

LC circuit to what extent is it a harmonic oscillator – it should be except for $\omega = 0, \infty$. Have discuss modes of the homogeneous system. Have real space $C^1 \oplus C_1$ symplectic structure given by natural pairing $C^1 \times C_1 \to \mathbb{R}$. (determined up to sign) $\forall s$ have impedance subspace $\Gamma_s \subset C^1 \oplus C_1$ subspaces $\binom{Ls}{1} C^1_{ind} \oplus \binom{1}{Cs} C^1_{cap}$ L, C positive def. quadratic forms, so Γ_s is Lagrangian. Another Lag. subsp is $W = \delta C^0 \oplus Z_1$. ~~Free modes~~ Spectrum consists of $s \ni W$ not transverse to Γ_s. But you need s complex. So basically you need to complexify

Try again. Basic phase space is $\binom{\{E_\sigma\}}{\{I_\sigma\}} = N$, σ runs over the edges. $\Gamma_s =$ subspace $\ni \frac{E_\sigma}{I_\sigma} = Ls$ ind, $= \frac{1}{Cs}$ cap.

so $\{\Gamma_s\}$ subvector bundle of $\mathcal{O} \otimes N$

Analyze hermitian forms. equivalent on a cx v.s. V

hermitian bilinear form $H(x,y)$

real symm $S(x,y)$ on underlying real v.s. $\ni S(ix,iy) = S(x,y)$

real skew-symm $A(x,y)$ ——— $A(ix,iy) = A(x,y)$

real quadratic form $Q(x)$ ——— $Q(ix) = Q(x)$

$H(x,y) = S(x,y) + i A(x,y)$ real + imag parts.

$H(x,y) = \overline{H(y,x)} \iff S$ symm, A skew symm.

$$S(x, iy) + iA(x, iy) = i\big(S(x,y) + iA(x,y)\big)$$

$$\therefore \quad A(x, iy) = S(x,y)$$

$$A(y, ix) = -A(ix, y) = -A(i^2x, iy) = A(x, iy)$$

A skew $\Longleftrightarrow$ S symm.

$$Q(x) = \underbrace{S(x,x)}_{= H(x,x) =} = A(x, ix)$$

Suppose V is the complexification of V_r, Choose basis: $V = \mathbb{C}^n$, $V_r = \mathbb{R}^n$. A hermitian bilin. $H(x,y)$ Same as herm. matrix which splits into a real symm matrix + i times skew symm. matrix. Point is that $H(x,y)$ is determined by sesquilinearity to $x, y \in V_r$ and then $H(x,y) = \underbrace{S(x,y)}_{\text{real symm}} + i\underbrace{A(x,y)}_{\text{real skew symm.}}$

$$\cancel{H(x_1 + ix_2, y_1 + iy_2) = \big(H(x_1, y_1) + H(x_2, y_2)\big) + i\big(H(x_1, y_2) - H(x_2, y_1)\big)}$$

$$S = 0 \Longleftrightarrow H(x,x) = S(x,x) = 0 \quad \forall x \in V_r$$

Thus equivalence between skew symm. forms on V_r and herm forms on V_c such that V_r is isotropic.

Back to LC circuit. You have $C^1 \oplus C_1$ = phase space real with a symplectic form (up to $\pm$). Complexification then has natural Krein form.

Notice that any real subspace isotropic wrt A is autom. isotropic wrt H since $S(x,x) = 0$ for $x \in V_r$.

95 ~~[illegible]~~ You have to look at $V_{\mathbb{R}} = \left\{ \begin{pmatrix} E \\ I \end{pmatrix} \in \mathbb{R}^2 \right\}$ Skew form is

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} = E_1 I_2 - I_1 E_2 = \begin{vmatrix} E_1 & E_2 \\ I_1 & I_2 \end{vmatrix}$$

The corresponding hermitian form should be

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix}^* \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} = i \begin{vmatrix} \bar{E}_1 & E_2 \\ \bar{I}_1 & I_2 \end{vmatrix}$$

when $\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} E_j \\ I_j \end{pmatrix}$ $j = 1, 2$.

$$i(\bar{E} I - \bar{I} E)$$
$$= 2 \operatorname{Im}(\bar{I} E)$$

So what next? Impedance $\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} Ls \\ 1 \end{pmatrix} I$

$$i \begin{vmatrix} \overline{Ls}\,\bar{I} & Ls I \\ \bar{I} & I \end{vmatrix} = i L (\bar{s} - s) |I|^2 = 2 \operatorname{Im}(s) L |I|^2$$

$$i \begin{vmatrix} \bar{E} & E \\ C\bar{s}\bar{E} & Cs E \end{vmatrix} = i C |E|^2 (s - \bar{s}) = C |E|^2 (-2 \operatorname{Im} s)$$

E, I

Think real

Phase space 4 diml before the symplectic reduction.

$$\boxed{E_L = L \dot{I}_L \qquad C \dot{E}_C = I_C}$$

$$\boxed{I_L = I_C \quad \text{and} \quad E_L = -E_C}$$

96 Try to understand what you can ~~need to consider~~ show the eigenvalues are purely imaginary. The argument involves real spaces and quadratic forms. It involves Siegel UHP Symmetric complex matrices with positive real part. ~~Consider all~~ Consider Symmetric

Have Lagrangian subspace

$$\begin{matrix} \delta C^0 & \subset & C^1 \\ \oplus & & \\ Z_1 & \subset & C_1 \end{matrix}$$

$$\begin{tikzcd}
0 \arrow[r] & \delta C^0 \arrow[r] & C^1 \arrow[r] \arrow[d, "N_s"] & C^1/\delta C^0 \arrow[r] & 0 \\
0 & C_1/Z_1 \arrow[l] & C_1 \arrow[l] & Z_1 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

when is $W = \delta C^0 \oplus Z_1$ transv. to Γ_{N_s}

$$\begin{matrix} \delta C^0 \\ \oplus \\ Z_1 \end{matrix} \cap \begin{pmatrix} 1 \\ N_s \end{pmatrix} C^1 \ni \quad w \qquad \begin{matrix} w \\ \\ N_s w \end{matrix}$$

The intersection is $\{ w \in \delta C^0 \mid N_s w \in Z_1 \}$. What argument to give that this can't happen unless $Re(s) = 0$. The argument is by self-pairing. You take. In this situation you have for $x \in C^1$ say $x = (x_C, x_L)$ that

diagonal matrices

$$\begin{pmatrix} x_L \\ x_C \end{pmatrix}^* N_s \begin{pmatrix} x_L \\ x_C \end{pmatrix} = x_L^* \frac{1}{Ls} x_L + x_C^* Cs \, x_C$$

97 ~~Ex~~ Example. Let V be a complex vector space, let $V^\dagger$ = anti dual = dual with opposite complex structure, have pairing $V^\dagger \underset{\mathbb{R}}{\otimes} V \longrightarrow \mathbb{C}$ which is sesquilinear. ~~~~

~~~~ $\langle t \,|\, v \rangle$ $\quad t \in V^\dagger,\ v \in V$. ~~~~

~~is clear how make hermitian symmetric~~. So on ~~~~ $V^\dagger \oplus V$ you have a ~~herm~~. form. sesquilinear form, namely $\begin{pmatrix} t_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ v_2 \end{pmatrix} \longmapsto \langle t_1, v_2 \rangle$ which you can symmetrize in herm.

$$H\left(\begin{pmatrix} t_1 \\ v_1 \end{pmatrix}; \begin{pmatrix} t_2 \\ v_2 \end{pmatrix}\right) = \langle t_1, v_2 \rangle + \underbrace{\overline{\langle t_2, v_1 \rangle}}_{\substack{\text{linear in } t_2 \\ \text{anti linear in } v_1}}$$

~~Let~~ Given V, W $\mathbb{C}$-vector spaces, ~~then~~ Let $F(v, w)$ be sesquilinear: ~~~~ linear in w anti-linear in v, e.g. $V = \mathbb{C}^m$, $W = \mathbb{C}^n$, α $m \times n$ matrix $F(v, w) = v^* \alpha w$. ~~~~ Get another sesq. form $G(w, v) = \overline{F(v, w)} = w^* \alpha^* v$ and then ~~~~

$$H\left(\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}\right) = F(v_1, w_2) + G(w_1, v_2)$$

$$= v_1^* \alpha w_2 + w_1^* \alpha^* v_1 = \begin{pmatrix} v_1^* & w_1^* \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ w_2 \end{pmatrix}$$

~~So there is a natural analogue of~~ ~~If you pick a Hilbert space structure, then~~

So it seems that there is a ~~~~ Kreinian analogue of symplectic. ~~~~

~~~~ Back to LC circuit. Begin with space of $\begin{pmatrix} E \\ I \end{pmatrix} \in \mathbb{C}^2$ and the Hermitian bilinear form

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} = \bar{E}_1 I_2 + \bar{I}_1 E_2 = 2\mathrm{Re}(\bar{E}_1 I_1) \quad \text{if } \begin{matrix} E_2 = E_1 \\ I_2 = I_1 \end{matrix}$$

$$\begin{pmatrix} Ls \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Ls \\ 1 \end{pmatrix} = L\bar{s} + Ls = L(s+\bar{s}) = \overset{L}{\cancel{2L}}(2\mathrm{Im}\, s)$$

$$\begin{pmatrix} 1 \\ Cs \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ Cs \end{pmatrix} = (1 \quad C\bar{s}) \begin{pmatrix} Cs \\ 1 \end{pmatrix} = C(2\mathrm{Im}\, s)$$

£1 = 9.35 FF

OKAY let's check. You have the above standard hermitian form on $C^L \oplus C_1$ namely $E\bar{I} + \bar{I}E = 2\,\mathrm{Re}(E\bar{I})$, pairing $\begin{pmatrix} E_1 \\ I_1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}$.

Next have subspace $Z_1^{\mathbb{E}} \subset C_1$ which is isotropic, and the annihilator is $\delta C^0 \oplus C_1$, so there should be no problem with $\delta C^0 \oplus Z_1$ being maximal isotropic.

Summarize. ~~[illegible]~~ You made some progress toward linking LC circuits to the invariant version of p. unitaries

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Consider LC network. Up to now you have studied the "configuration space" viewpoint, namely, you pick say the voltage space C^L as config. space. You ~~[illegible]~~ have a quadratic (hermitian) form on C^L depending on s?

Let's get this straight. ~~[illegible]~~ Start with a real situation and then complexify. Real ~~[illegible]~~ situation is a vector space D, the dual space D^*, a quadratic form Q_s on D depending on s a subspace Z of D spectrum ~~is the~~ those s such that Q_s is nondeg.

LC network. You are used to a "configuration space version.

99 LC network "configuration space" version

You have a real voltage function space C^1 and dual current function space C_1, ~~the~~ the impedance of the edges yields a map $N_s : C^1 \longrightarrow C_1$ for real s direct sum of types graph $N_s = \begin{pmatrix} Ls \\ 1 \end{pmatrix} C_1$ or $\begin{pmatrix} 1 \\ Cs \end{pmatrix} C^1$

Your configuration space is a real space V together with a quadratic form $Q_s(v)$ which is the direct sum $V_L \oplus V_C$ $\quad Q_s = (Ls)^{-1} \oplus Cs$.

Check: $(E, I) = (E, Q_s E) = s^{-1}(E_L, L^{-1}E_L) + s(E_C, CE_C)$

So you have a real vector space V split into $V_L \oplus V_C$ and ~~...~~ $Q_s = \underline{s^{-1}(E_L, L^{-1}E_L) \oplus s(E_C, CE_C)}$

for pos. def. quad forms L^{-1}, C $\quad s^{-1}L^{-1}(E) + s\,C(E)$

on V_L and V_C resp. Next we have a subquotient, of V - restrict to conservative equivalently node potentials / const voltage functions and divide by ~~voltages functions~~ node potentials supported on the ext. vertices. Look at the induced quadratic form ~~Q~~ on subquotient.

Real Quadratic form version.

Complex hermitian version should be: Exactly the same, namely V is a complex vector space split into $V_L \oplus V_C$ with hermitian pos. def forms ~~...~~ L^{-1} and C on V_L, V_C resp. Get a modified version

You want to organize, merge two themes:

analysis of partial unitaries - here you encounter isotropic subspaces in a Krein space. In this theory ~~...~~ there is a Hilbert space Y around LC networks. ~~...~~ Somehow this is adapted a symplectic or Krein viewpoint.

You need to double - hermitian forms become isotropic subspaces.

~~Ref the following~~ Complexify an LC network.

Originally E, I are real functions of t.

~~Consider~~ You need to double! You have C^1 and C_1 spaces of complex edge voltage functions and edge current functions. These are anti-dual in a ~~natural~~ preferred way because of the "power" $\bar{I}E$ for each edge.

Given $E(t)$, $I(t)$ real with compact support, then $\int_{-\infty}^{\infty} E(t)\,I(t)\,dt$ = power into the edge

$$= \int_{-\infty}^{\infty} E(\omega)\,I(-\omega)\,\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \overline{I(\omega)}\,E(\omega)\,\frac{d\omega}{2\pi}$$

I guess my point is that complex "phase space" is $C^1 \oplus C_1$ and it carries a natural hermitian form, hyperbolic type; also skew-hermitian multiplying by i.

The impedance of each edge yields a subbundle $N_s \subset C^1 \oplus C_1$ direct sum of either $\begin{pmatrix} Ls \\ 1 \end{pmatrix}\mathbb{C}$ or $\begin{pmatrix} 1 \\ Cs \end{pmatrix}\mathbb{C}$

for $\mathrm{Re}(s) = 0$ this subspace should be isotropic ~~Many of N_s is~~. The quotient bundle $s \mapsto C^1 \oplus C_1 / N_s$ is holom. + pure of type $\mathcal{O}(1)$, so we can identify $C^1 \oplus C_1$ with the space of holom. section. We ~~split $C^1 \oplus C_1$ into~~ should get a canonical isomorphism

$T \otimes Y \xrightarrow{\sim} C^1 \oplus C_1$ OKAY

~~Take a~~ polarized Hilbert space $\mathcal{U}_+ \oplus \mathcal{U}_-$

~~First point~~ Maybe all that's involved is ~~changes~~ to take Y to be a ~~polarized~~ Krein space and then $T \otimes Y$ should have the tensor product

101. You need some improvement

Begin with a complex vector space Ω form direct sum $D = \Omega \oplus \Omega^\dagger$ where $\Omega^\dagger$ is the anti-dual, so we have a sesquilinear pairing $\Omega^\dagger \otimes_{\mathbb{R}} \Omega \xrightarrow{\langle\, ,\, \rangle} \mathbb{C}$. Define skew hermitian form on D by

$$H\left(\begin{pmatrix} x \\ \lambda \end{pmatrix}, \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}\right) = -\overline{\langle \lambda_1, x \rangle} + \langle \lambda, x_1 \rangle$$

Consider a graph $\begin{pmatrix} 1 \\ T \end{pmatrix}\Omega \subset D$. When is this isotropic?

$$H\left(\begin{pmatrix} x \\ Tx \end{pmatrix}, \begin{pmatrix} x_1 \\ Tx_1 \end{pmatrix}\right) = -\overline{\langle Tx_1, x \rangle} + \langle Tx, x_1 \rangle = 0$$

means? $\quad T : \Omega \to \Omega^\dagger$

$T^\dagger : \Omega^{\dagger\dagger} \to \Omega^\dagger$ defined by

$$\langle T^\dagger x, x' \rangle = \overline{\langle Tx', x \rangle}$$

means T is hermitian, i.e. $\langle Tx, x' \rangle$ herm. symmetric in x, x'.

Ω complex v.s. $\Omega^\dagger$ anti-dual, a map $T : \Omega \to \Omega^\dagger$ is equivalent to a sesquilinear form $H(x, x') = \langle Tx, x' \rangle$

$T : \Omega \to \Omega^\dagger$ same as ~~an anti~~ $T : \Omega \to \Omega^*$ $\quad T(cx) = \bar{c}T(x)$

$T^\dagger : \underset{\Omega}{\overset{\Omega^{\dagger\dagger}}{\shortparallel}} \to \Omega^\dagger \qquad \langle T^\dagger x, x' \rangle \qquad Tx'$

Assume Ω is a Hilb space so that one has a canonical isom $\Omega \xrightarrow{\sim} \Omega^\dagger$. Then $\Omega \oplus \Omega^\dagger = \Omega \oplus \Omega$ equipped with the skew herm. form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. One has $\left(\begin{pmatrix} 1 \\ T \end{pmatrix}\Omega\right)^\circ = \begin{pmatrix} 1 \\ T^* \end{pmatrix}\Omega$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ Tx \end{pmatrix}}_{\begin{pmatrix} Tx \\ -x \end{pmatrix}} = y_1^* Tx - y_2^* x = \left((T^* y_1)^* - y_2^*\right) x = 0 \ \forall x$$

$$\Rightarrow y_2 = T^* y_1$$

~~Now Hilbert~~ Suppose now Ω is a polarized Hilbert space $\Omega = \Omega^+ \oplus \Omega^-$ and we equip it with the hermitian operator $\begin{pmatrix} \omega & 0 \\ 0 & -\omega^{-1} \end{pmatrix} = \omega \pi_+ - \omega^{-1} \pi_-$

Then for each $\omega \in \mathbb{P}^1$ you have a subspace of Ω, namely the graph of this operator which is isotropic wrt the canonical skew-herm. form when ω is real. The point to make perhaps is that you get a holom subbundle over $\mathbb{P}^1$ of $\mathcal{O} \otimes (\Omega \oplus \Omega)$.

Know that this holom. subbundle is pure of type $\mathcal{O}(-1)$. Things you know:

$$0 \longrightarrow \Gamma_\omega \longrightarrow \mathcal{O} \otimes \begin{matrix}\Omega \\ \oplus \\ \Omega\end{matrix} \longrightarrow Q_\omega \longrightarrow 0$$

$\Omega^{\oplus 2}$ canon. isom. to $\Gamma_{hol}(\mathbb{P}^1, Q)$.

$\Omega^{\oplus 2}$

~~Take Hilbert space H~~

Review. Start with a Hilbert space X, form double $\begin{matrix}X \\ \oplus \\ X\end{matrix}$ with skew-hermitian form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$

If $T : X \longrightarrow X$ is linear then

$$\left(\begin{pmatrix} 1 \\ T \end{pmatrix} X \right)^0 = \left\{ \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \;\middle|\; \begin{pmatrix} x \\ Tx \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = 0 \right.$$

$$x^* x_2' = (Tx)^* x_1' \qquad \forall x$$

$$x_2' = T^* x_1'$$

$$= \begin{pmatrix} 1 \\ T^* \end{pmatrix} X$$

So that $\begin{pmatrix} 1 \\ T \end{pmatrix} X$ is isotropic $\iff T = T^*$.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cancel{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

103 Consider the double $\begin{matrix} X \\ \oplus \\ X \end{matrix}$. Then $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ~~is an~~ ~~autom~~ preserves the skew-hermitian form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \cancel{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^* \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2 \end{pmatrix}$$

and it carries $\begin{pmatrix} 1 \\ T \end{pmatrix} X$ into $\begin{pmatrix} T \\ -1 \end{pmatrix} X$.

So it seems that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \begin{matrix} X \\ \oplus \\ X \end{matrix} \longrightarrow \begin{matrix} X \\ \oplus \\ X \end{matrix}$$

is an autom of the skew-herm. form and it carries $\begin{pmatrix} 1 \\ \omega \end{pmatrix} X$ into $\begin{pmatrix} \omega \\ -1 \end{pmatrix} X = \begin{pmatrix} 1 \\ -\omega^{-1} \end{pmatrix} X$. This tells me how to treat an LC circuit in the framework of $T \otimes Y$ where $T = \begin{matrix} \mathbb{C} \\ \oplus \\ \mathbb{C} \end{matrix}$ standard skew herm. f. $l_\omega = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \mathbb{C}$ and Y is a Hilbert space ~~of ...~~

$\begin{pmatrix} E \\ I \end{pmatrix} \in \mathbb{C}^2$ skew-herm. form is $\begin{pmatrix} E_1 \\ I_1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}$

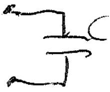
Impedance line is $\begin{pmatrix} 1 \\ C\omega \end{pmatrix} \mathbb{C}$ $= \begin{vmatrix} \bar{E}_1 & E_2 \\ \bar{I}_1 & I_2 \end{vmatrix}$

Still not clear, ~~...~~ if I restrict to real frequencies. What's the problem Maybe you should use time evolution

---

$$\begin{pmatrix} 1 \\ s \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix} = \begin{pmatrix} 1 & \bar{s} \end{pmatrix} \begin{pmatrix} s \\ 1 \end{pmatrix} = s + \bar{s}$$

In the end you get $H\left(\begin{pmatrix} 1 \\ s \end{pmatrix} \otimes y\right) = 2\mathrm{Re}(s) \|y\|^2$

Consider , behavior described by

$E(t)$, $I(t)$ satisfying $I(t) = C\dot{E}(t)$. Power is $E(t)I(t) = CE\dot{E} = \partial_t\left(\frac{1}{2}CE^2\right)$ so $\int_a^b EI\,dt = \frac{1}{2}CE^2\Big]_a^b$ so the energy going in between times $a + b$. Perhaps you should think of $E(t), I(t) = 0$ for $t \ll 0$ and of exponential growth at $t \to +\infty$, which is appropriate for LT. Frequency analysis. $E(t) = \mathrm{Re}\left(E(\omega)e^{-i\omega t}\right)$ also for I where $E(\omega), I(\omega)$ are complex amplitudes satisfying $I(\omega) = C(-i\omega)E(\omega)$. Power generally is $\int E(t)I(t)\,dt = \int_{-\infty}^{\infty} E(\omega)I(-\omega)\frac{d\omega}{2\pi}$

$$= \int_0^\infty \frac{d\omega}{\pi} \frac{E(\omega)I(-\omega) + E(-\omega)I(\omega)}{2} \quad \mathrm{Re}\left(\overline{E(\omega)}\,I(\omega)\right).$$

For $I(\omega) = C(-i\omega)E(\omega)$ $\quad \mathrm{Re}\left(\overline{E(\omega)}\,I(\omega)\right) = \mathrm{Re}(-i\omega)\,C|E(\omega)|^2 = 0$ for ω real corresp to $\int_{-\infty}^{\infty} EI(t)\,dt = 0$ if E, I have comp. support.

So the picture is the following. An edge yields a 2 dim'l complex space of $\begin{pmatrix} E \\ I \end{pmatrix}$ equipped with a hermitian form $\mathrm{Re}(\bar{E}I) = \frac{1}{2}(\bar{E}I + \bar{I}E)$

$$= \begin{pmatrix} E \\ I \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix}.$$

At frequency ω, $\begin{pmatrix} E \\ I \end{pmatrix}$ is restricted to lie in the line $\begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix}\mathbb{C}$ which is isotropic for this hermitian form. In general $\begin{pmatrix} 1 \\ T \end{pmatrix}\mathbb{C}^n$ is isotropic for $\begin{pmatrix} E \\ I \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix}$

$E, I \in \mathbb{C}^n \iff$

$$\begin{pmatrix} 1 \\ T \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ T \end{pmatrix} = \begin{pmatrix} 1 & T^* \end{pmatrix} \begin{pmatrix} T \\ 1 \end{pmatrix} = T + T^*$$

vanishes i.e. T skew symmetric. ~~This skew symm. feature is part~~

$$\begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix}\mathbb{C}^n = \begin{pmatrix} 1 \\ \frac{-1}{L(i\omega)} \end{pmatrix}$$

~~What you should~~ Picture: for ~~an~~ a C edge you assoc. a 2diml space of $\begin{pmatrix}E\\I\end{pmatrix}$ with herm. form $\begin{pmatrix}E\\I\end{pmatrix}^*\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}E\\I\end{pmatrix}$ and ~~the~~ subspace $\begin{pmatrix}1\\C(-i\omega)\end{pmatrix}\mathbb{C}$ depending on frequency ω which is isot for $\omega\in\mathbb{R}$. For an L edge the same except the line is $\begin{pmatrix}L(-i\omega)\\1\end{pmatrix}\mathbb{C}$.

~~On~~ $g^*\begin{pmatrix}0&1\\-1&0\end{pmatrix}g = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$ i.e. $g^{-1} = \begin{pmatrix}0&-1\\+1&0\end{pmatrix}g^*\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

$$= \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\bar a&\bar c\\\bar b&\bar d\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}-\bar c&\bar a\\-\bar d&\bar b\end{pmatrix} = \begin{pmatrix}\bar d&-\bar b\\-\bar c&\bar a\end{pmatrix}$$

$= \frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$. If $ad-bc=1$, this means $g\in SL_2(\mathbb{R})$ ~~So the~~ Note $\Rightarrow |\det g|^2 = 1$.

so the group of such g contains $SL_2(\mathbb{R})$ and $c\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $|c|=1$. So if you are interested in 2 diml V with herm. form of ~~type~~ sign $1,-1$, then any two are isom. & auto gp is $SL_2(\mathbb{R})\mathbb{T} < GL_2(\mathbb{C})$.

~~What~~ Our structure

What I need to do is to go directly from the family of $\begin{pmatrix}E\\I\end{pmatrix}\in\mathbb{C}^2$, $\begin{pmatrix}E\\I\end{pmatrix}^*\begin{pmatrix}0&1\\+1&0\end{pmatrix}\begin{pmatrix}E\\I\end{pmatrix}$, $\begin{pmatrix}1\\C(-i\omega)\end{pmatrix}\mathbb{C}$ or $\begin{pmatrix}L(-i\omega)\\1\end{pmatrix}\mathbb{C}$ to a Hilbert space Y, the Krein space $T\otimes Y$ and family $l_\omega\otimes Y$ where $T=\mathbb{C}^2$, $l_\omega = \begin{pmatrix}1\\-i\omega\end{pmatrix}$, $\begin{pmatrix}x_1\\x_2\end{pmatrix}^*\begin{pmatrix}0&1\\+1&0\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$. You need an isom $\mathbb{C}^2\ni\begin{pmatrix}x_1\\x_2\end{pmatrix}\longmapsto$

~~Take Hilbert~~ You compare $T=\mathbb{C}^2$, ~~with~~ $l_\omega = \begin{pmatrix}1\\-i\omega\end{pmatrix}\mathbb{C}$ and $H\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = \begin{pmatrix}x_1\\x_2\end{pmatrix}^*\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \bar x_1x_2+\bar x_2x_1 = 2\mathrm{Re}(\bar x_1x_2)$

106 to $P = \mathbb{C}^2$, $\mathcal{S}_\omega = \begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix}\mathbb{C}$, $H\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = 2\mathrm{Re}(\bar{x}_1 x_2)$

or $= \begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix}\mathbb{C}$

Consider $T \xrightarrow{\begin{pmatrix} c^{-1/2} & 0 \\ 0 & c^{1/2} \end{pmatrix}} P$ in the C-case

$$\begin{pmatrix} c^{-1/2} & 0 \\ 0 & c^{1/2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c^{-1/2} & 0 \\ 0 & c^{1/2} \end{pmatrix}$$

$$\begin{pmatrix} c^{-1/2} & 0 \\ 0 & c^{1/2} \end{pmatrix}\begin{pmatrix} 1 \\ s \end{pmatrix}\mathbb{C} = \begin{pmatrix} c^{-1/2} \\ c^{1/2}s \end{pmatrix}\mathbb{C} = \begin{pmatrix} 1 \\ cs \end{pmatrix}\mathbb{C}$$

$$= \begin{pmatrix} 0 & c^{-1/2} \\ c^{1/2} & 0 \end{pmatrix}\begin{pmatrix} c^{-1/2} & 0 \\ 0 & c^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$T \xrightarrow{\begin{pmatrix} 0 & iL^{1/2} \\ iL^{-1/2} & 0 \end{pmatrix}} P$

$$\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}\mathbb{C} \mapsto \begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix}\mathbb{C} \qquad \begin{pmatrix} 0 & L \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix} = \begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}^*\begin{pmatrix} 0 & 1 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix} = \begin{pmatrix} 1 & +i\bar{\omega} \end{pmatrix}\begin{pmatrix} -i\omega & 1 \end{pmatrix}$$
$$= i(\bar{\omega} - \omega) = 2\,\mathrm{Im}(\omega)$$

$$\begin{pmatrix} -i\omega \\ 1 \end{pmatrix}^*\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -i\omega \\ 1 \end{pmatrix} = \begin{pmatrix} i\bar{\omega} & 1 \end{pmatrix}\begin{pmatrix} 1 & -i\omega \end{pmatrix}$$
$$= i\bar{\omega} - i\omega = 2\,\mathrm{Im}(\omega)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix} = \begin{pmatrix} -i\omega \\ 1 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{scalar}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix}^*\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix} = C(-i\omega) + \overline{C(-i\omega)}$$
$$= Ci(\bar{\omega} - \omega) = (2\,\mathrm{Im}\,\omega)C$$

$$\begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix}^*\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix} = Li\bar{\omega} - Li\omega$$
$$= (2\,\mathrm{Im}\,\omega)L$$

107 Review. An LC circuit has 2 description configuration space: ① polarized Hilbert space $\Omega = \Omega^+ \oplus \Omega^-$ plus a subquotient F_2/F_1. ~~Ba~~

~~[illegible]~~ You

Start again. Begin again. Concrete model LC network is a graph with C, L edges. Each edge has "phase space", ~~[illegible]~~ states $\binom{E}{I} \in \mathbb{C}^2$, hermitian form (power) $\begin{pmatrix} E \\ I \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix} = 2\,\mathrm{Re}(\bar{E} I)$

For an C-edge there is a line $l_\omega = \begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix} \mathbb{C}$ $\quad C > 0$

For an L- ——————— $l_\omega = \begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix} \mathbb{C}$ $\quad L > 0$

~~[illegible]~~ for $\omega \in S^2 = \mathbb{C} \cup \{\infty\}$ which is isotropic for ω real. (for this herm. form graphs ~~are~~ $\binom{1}{T}\mathbb{C}$ isotropic iff $T^* = -T$).

$\therefore$ Each edge gives a 2-dim complex phase space equipped with herm. form type $(1,-1)$ and the family l_ω of lines.

$$\begin{pmatrix} 0 & iL^{1/2} \\ iL^{-1/2} & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & iL^{1/2} \\ iL^{-1/2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -iL^{-1/2} \\ -iL^{1/2} & 0 \end{pmatrix} \begin{pmatrix} iL^{-1/2} & 0 \\ 0 & iL^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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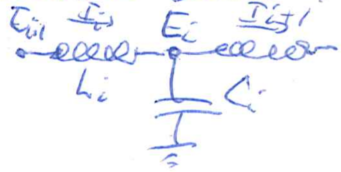
Organize your thoughts about connecting an LC network to a transmission line.

First transmission line equation



$$\begin{cases} E_{i-1} - E_i = L_i\,\partial_t I_i \\ I_i - I_{i+1} = C_i\,\partial_t E_i \end{cases}$$

$$\partial_x E + \ell\,\partial_t I = 0$$
$$\rho\,\partial_x I + \partial_t E = 0$$

assume speed $\frac{1}{\sqrt{\ell\gamma}} = 1$

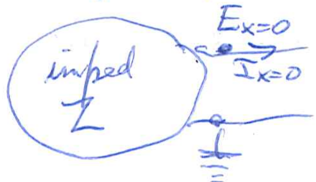
$\ell = \rho \quad \gamma = \rho^{-1}$

$$(\partial_x + \partial_t)(E + \rho I) = 0$$
$$(\partial_x - \partial_t)(E - \rho I) = 0$$

outgoing

~~outgoing~~ incoming

$$E + \rho I = A e^{-s(x-t)}$$
$$E - \rho I = B e^{+s(x+t)}$$



$$\frac{E_{x=0}}{I_{x=0}} = -Z$$

$$\begin{pmatrix} 1 & \rho \\ 1 & -\rho \end{pmatrix}\begin{pmatrix} E_{x=0} \\ I_{x=0} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{st}$$

$$\frac{-Z+\rho}{-Z-\rho} = \frac{A}{B}$$

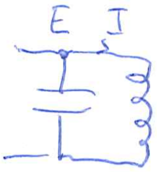
$$\boxed{\frac{A}{B} = \frac{-\rho + Z}{\rho + Z}}$$

typical Z is Ls $\quad \frac{1}{Cs}$

$$S = \frac{A}{B} = \frac{Ls - 1}{Ls + 1}$$

or $$\frac{\frac{1}{Cs} - 1}{\frac{1}{Cs} + 1} = \frac{1 - Cs}{1 + Cs}$$

I should take the case



$$\frac{E}{I} = \frac{1}{\frac{1}{Ls} + Cs} = \frac{Ls}{LCs^2 + 1}$$

$$S = \begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix}\left(\frac{Ls}{LCs^2+1}\right) = \frac{-LCs^2 + Ls - 1}{LCs^2 + Ls + 1}$$

$$s = \frac{-L \pm \sqrt{L^2 - 4LC}}{2LC}$$ neg. real part

~~What you need now. Take a coherent~~

What you need to do now is to decide how intrinsic coupling to a transmission line is. You have a picture of an LC network – subquotient of a polarized Hilbert space, namely the space of 1-cochains ~~equipped with inner~~ ~~product~~ split into C + L types with the inner product $C|E|^2$ resp $L^{-1}|E|^2$. The modified form $sC|E|^2$ resp $s^{-1}L^{-1}|E|^2$ induces a hermitian (for s real) form on the subquotient. skew-herm. (for $s \in i\mathbb{R}$). So you have a ck line with hermitian form. For an actual circuit the line has a basis – voltage at the external node. So the hermitian form is $Z_s|E|^2$. ~~When you couple to a transm~~

$$\begin{array}{ccc} C^0 & \xrightarrow{\delta} & C^1 \\ \downarrow{\scriptstyle \omega_n} & & \\ \mathbb{C} & & \end{array}$$

What can ~~you~~ do intrinsically. You have a complex line J and a sesquilinear form hermitian for real s Can form $J \oplus J^{\dagger}$. First do real case. You have a real line J and a quadratic form on it Can form $J \oplus J^{*}$ symplectic + graph of quad form is isotropic. ~~We take here~~ Complex case graph of a sesqui form $J \longrightarrow J^{\dagger}$. For a general subquotient J of a polar. Hilb. space you get a sesquilinear form $Z_s(j_1, j_2)$ which is hermitian for s real (herm. means $Z_s(j,j) \in \mathbb{R}$) and skew-herm. for $s \in i\mathbb{R}$ (skew-herm. means $i \cdot$ herm).

If $J = \mathbb{C}^n$ then $Z_s(j_1, j_2) = (j_1, Z_s j_2)$

110 ~~Observat~~ Missing point. A transmission line with unit speed has an impedance which identifies voltage + current spaces

$$\partial_x E + \rho\,\partial_t I = 0 \qquad \text{YES.} \quad (\partial_x + \partial_t)(E+\rho I) = 0$$

$$\rho\,\partial_x I + \partial_t E = 0 \qquad (\partial_x - \partial_t)(E - \rho I)$$

Solutions of frequency ω are

$$E + \rho I = A e^{-s(x-t)}$$
$$E - \rho I = B e^{s(x+t)}$$

so you get
$$\begin{pmatrix} E+\rho I \\ E-\rho I \end{pmatrix}_{x=0} = \begin{pmatrix} 1 & \rho \\ 1 & -\rho \end{pmatrix}\begin{pmatrix} E \\ I \end{pmatrix}_{x=0} = \begin{pmatrix} A \\ B \end{pmatrix} e^{st}$$

so
$$\frac{A}{B} = \begin{pmatrix} 1 & \rho \\ 1 & -\rho \end{pmatrix}(-Z) = \frac{-Z+\rho}{-Z-\rho} = \frac{Z-\rho}{Z+\rho}$$

Lesson seems to be that the ρ:

Structure of a 1-port, complex 2 diml space equipped with a hermitian form of ~~type~~ signature $1, -1$, also a line l_ω depending on $\mathcal{O}(1)\otimes\Gamma(\mathcal{O}(n))$ — in finite case $\omega \mapsto l_\omega$ is a rational maps from ω sphere to $\mathbb{P}^1$. ~~In Nature~~

Trans. line. $\mathcal{O}\otimes\Gamma(\mathcal{O}(n))$

$$\begin{tikzcd}
 & \mathcal{O}\otimes\Gamma(\mathcal{O}(n)) \arrow[d] & & \\
\mathcal{O}(-n) \arrow[ur, hook, "can"] \arrow[r, hook] & \mathcal{O}\otimes\mathbb{C}^2 \arrow[r, two heads] \arrow[dr, hook] & \mathcal{O}(n) & \text{Given} \\
 & & \mathcal{O}\otimes\Gamma(\mathcal{O}(n)) \arrow[u, "can"] & \\
 & \mathcal{O}(-1)\otimes\Gamma(\mathcal{O}(n-1)) \arrow[ur, hook] & &
\end{tikzcd}$$

Seems strange but something might work. Go backwards, You have $0 \to \mathcal{O}(-1)\otimes Y \to \mathcal{O}\otimes T\otimes Y \to \mathcal{O}(1)\otimes Y \to 0$, and W isotropic in $T\otimes Y$

Go back to symplectic case. T 2diml symplectic, Y n diml quadratic, $T\otimes Y$ symp. W isotropic in $T\otimes Y$. Assume $W \cap \mathcal{O}(-1)_\omega\otimes Y = 0$ $\forall\omega$, then

111 $\quad W^0 + \mathcal{O}(-1)_\omega \otimes Y = 0 \quad \forall \omega$ so get $W^0 \cap \mathcal{O}(-1)_\omega \otimes Y \hookrightarrow W^0/W$

nice intersection

$$\mathcal{O}\otimes W^0 \longrightarrow$$

Use $l_\omega^{\subset T}$ for $\mathcal{O}(-1)_\omega$, and

$$E_\omega = W^0 \cap (l_\omega \otimes Y)$$

$$\begin{tikzcd}
E_\omega \arrow[r, hook] \arrow[dr, hook] & \mathcal{O}\otimes \overset{n+1}{W^0} \arrow[r] \arrow[d] \arrow[dr] & \mathcal{O}\otimes T\otimes Y \arrow[d] \\
& \mathcal{O}\otimes(W^0/W) \arrow[d, two heads] & \mathcal{O}(1)\otimes \overset{n}{Y} \\
& Q_\omega = E_\omega^* &
\end{tikzcd}$$

Suppose Y n diml $\quad W$ $n-1$ diml

W^0 $n+1$ diml $\quad E_\omega \simeq \mathcal{O}(-n)$. Can you reverse the prosess, namely start from W^0/W 2diml symplectic and $\mathcal{O}\otimes \overset{n-1}{W}$

$$\begin{tikzcd}
& \mathcal{O}\otimes \overset{n-1}{W} \arrow[d, hook] & \\
\mathcal{E} \arrow[r, hook] \arrow[d, equal] & \mathcal{O}\otimes \overset{n+1}{\Gamma(\mathcal{O}(n))} \arrow[r] \arrow[d] & \mathcal{O}(1)\otimes \overset{n}{Y} \arrow[d] \\
\mathcal{E} \arrow[r, hook] & \mathcal{O}\otimes(W^0/W) \arrow[r, two heads] & \mathcal{E}^*
\end{tikzcd}$$

Try to reverse the symplectic version.

T 2diml symp. $\quad Y$ ndl nondeg quadratic $\quad T\otimes Y$ symp.

W isotropic in $T\otimes Y$, assume $\mathcal{O}\otimes W$ transversal to $\mathcal{O}(-1)\otimes Y$ over $\mathbb{P}_1 = \mathbb{P}_1 T$, i.e. $W \cap l_\omega \otimes Y = 0 \quad \forall\omega$

Then $W^0 + l_\omega \otimes Y = T\otimes Y \quad \forall \omega$ so get vector bundle

112 $\mathcal{E}$ $E_\omega = W^\circ \cap (\ell_\omega \otimes Y)$. $\mathcal{E}$ should be Lagrangian inside $\mathcal{O}\otimes W^\circ/W$.

~~$0 \to \mathcal{O}(-1)\otimes Y \to \mathcal{O}\otimes T \otimes Y \to \mathcal{O}(1)\otimes Y \to 0$~~

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O}\otimes W^\circ \arrow[r] \arrow[d] & \mathcal{O}(1)\otimes Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(-1)\otimes Y \arrow[r] \arrow[d] & \mathcal{O}\otimes T\otimes Y \arrow[r] \arrow[d] & \mathcal{O}(1)\otimes Y \arrow[r] & 0 \\
 & \mathcal{O}\otimes(T\otimes Y/W^\circ) \arrow[r, equal] \arrow[d] & \mathcal{O}\otimes \underbrace{(T\otimes Y/W^\circ)}_{W^*} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

$n-1 = \dim W$
$n = \dim Y$
$\deg \mathcal{E} = -n$
$\operatorname{rank} \mathcal{E} = 1$

You want to reverse this process. So what do we have? How to proceed?. To Start with
$\mathcal{E} \hookrightarrow \mathcal{O}\otimes W^\circ/W \to \mathcal{E}^*$ $\quad$ $\mathcal{E}$ Lagrangian

Problem: Classify Lagrangian subbundles $\mathcal{L}$ of $\mathcal{O}\otimes V$ over $\mathbb{P}_1$ where V is a symplectic vector space. First case $\dim V = 2$. Then $\mathcal{L} \cong \mathcal{O}(n)$ for some $n \geq 0$. So we have a line bundle $\mathcal{L}^*$ with 2 independent sections.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r, hook] \arrow[d] & \mathcal{O}\otimes V \arrow[r] \arrow[dr] & \mathcal{L}^* \arrow[r] & 0 \\
 & \mathcal{O}\otimes H^0(\mathcal{L}^*)^* \arrow[ur] & & \mathcal{O}\otimes H^0(\mathcal{L}^*) \arrow[u] &
\end{tikzcd}$$

Do we get a quadratic function on $H^0(\mathcal{L}^*)^*$? deg.

$$\begin{tikzcd}
0 \arrow[d] & 0 \\
\mathcal{L} \arrow[d] & \mathcal{L}^* \arrow[u] \\
\mathcal{O}\otimes H^0(\mathcal{L}^*)^* \arrow[d] & \mathcal{O}\otimes H^0(\mathcal{L}^*) \arrow[u] \\
\mathcal{O}(1)\otimes H^0(-)^* \arrow[d] & \mathcal{O}(-1)\otimes H^0(\mathcal{L}^*(-1)) \arrow[u] \\
0 & 0 \arrow[u]
\end{tikzcd}$$

Somehow you can fit together the canon. res. of $\mathcal{O}(n)$ and its dual.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-1)\otimes S_{n-1} \arrow[r] & \mathcal{O}\otimes S_n \arrow[r] & \mathcal{O}(n) \arrow[r] & 0 \\
 & & \mathcal{O}\otimes V \arrow[u] & \mathcal{O}\otimes V \arrow[u] & \\
0 & \mathcal{O}(1)\otimes S^*_{n-1} \arrow[l] & \mathcal{O}\otimes S^*_n \arrow[l] \arrow[u] & \mathcal{O}(-n) \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

This seems fairly clear. **NO.** You probably need to use two disjoint divisors of degree n.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[d, hook] & \mathcal{O}\otimes W^\circ \arrow[r, two heads] \arrow[d, hook] & \mathcal{O}(1)\otimes Y \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{O}(-1)\otimes Y \arrow[r] \arrow[d] & \mathcal{O}\otimes T\otimes Y \arrow[r] \arrow[d] & \mathcal{O}(1)\otimes Y \arrow[r] & 0 \\
 & \mathcal{O}\otimes W^* \arrow[r, phantom, "="] & \mathcal{O}\otimes (W^*) & &
\end{tikzcd}$$

Maybe you should begin with $\mathcal{L} \hookrightarrow \mathcal{O}\otimes \overset{V}{(W^\circ/W)}$ and construct W°. But you observe that $\mathcal{L} \hookrightarrow \mathcal{O}\otimes W^\circ \twoheadrightarrow \mathcal{O}(1)\otimes Y$ must be the canonical resolution of $\mathcal{L}$, and then $\Gamma(\mathcal{O}\otimes W^\circ) \to \Gamma(\mathcal{O}(1)\otimes Y)$ will be the

$$\begin{matrix} \Gamma(\mathcal{O}\otimes W^\circ) & \to & \Gamma(\mathcal{O}(1)\otimes Y) \\ \parallel & & \parallel \\ W^\circ & \subset & T\otimes Y \end{matrix}$$

corresp K-modules. Now use V symplectic and $\mathcal{L}$ Lagrangian to get

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{O}\otimes V \arrow[r] \arrow[d] & \mathcal{L}^* \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
0 \arrow[r] & \mathcal{O}(-1)\otimes Y^* \arrow[r] \arrow[d] & \mathcal{O}\otimes (W^\circ)^* \arrow[r] \arrow[d] & \mathcal{L}^* \arrow[r] & 0 \\
 & \mathcal{O}\otimes (W^\circ)^* \arrow[r, phantom, "="] & \mathcal{O}\otimes (W^\circ)^* & &
\end{tikzcd}$$

$W^\circ \subset T\otimes Y$

Thus get canon. isom. of the K-modules.

~~Start~~ What happens is that $W^0 \subset T \otimes Y$ is the K-module for $\mathcal{L}$ and $Y^* \to T \otimes (W^0)^*$ is the K-module for $\mathcal{L}(1)$. Other ways

$$0 \to \mathcal{O}(-1) \otimes Y^* \to \mathcal{O} \otimes (W^0)^* \to \mathcal{L}^* \to 0$$

$$(W^0)^* = H^0(\mathcal{L}^*) \qquad Y^* = H^0(\mathcal{L}^*(-1))$$

$$0 \to \mathcal{L} \to \mathcal{O} \otimes W^0 \to \mathcal{O}(1) \otimes Y \to 0$$

$$Y = H^1(\mathcal{L}(-1)) \qquad H^1(\mathcal{L}(-2)) = W^0$$

natural duality, but

$$0 \to \mathcal{L}(-1) \to \mathcal{O}(-1) \otimes V \to \mathcal{L}^*(-1) \to 0$$

gives $\underset{\underset{Y^*}{\parallel}}{H^0(\mathcal{L}^*(-1))} \xrightarrow{\sim} \underset{\underset{Y}{\parallel}}{H^0(\mathcal{L}(-1))}$.

Basic data — symplectic space V and Lagrangian subbundle $\mathcal{L}$ of $\mathcal{O} \otimes V$ over $\mathbb{P}_1 T$. $\underline{H^0(\mathcal{L}) = 0}$

Wrong direction. Start with nondeg. quadratic form on Y, T 2dim'l symplectic, $T \otimes Y$ then symplectic, $W \subset T \otimes Y$ isotropic, assume $W \cap (l_\omega \otimes Y) = 0$ all $\omega \in \mathbb{P}$, whence $W^0 + l_\omega \otimes Y = T \otimes Y$ $\forall \omega$

whence get $\mathcal{L}_\omega = W^0 \cap l_\omega \otimes Y \hookrightarrow W^0/W$

$$0 \to \mathcal{L} \to \mathcal{O} \otimes W^0 \to \mathcal{O}(1) \otimes Y \to 0$$

must be canonical resolution so $Y \xrightarrow{\sim} H^1(\mathcal{L}(-1))$ $\quad H^1(\mathcal{L}(-2)) \xrightarrow{\sim} W^0$

$$0 \to \mathcal{O}(-1) \otimes Y^* \to \mathcal{O} \otimes (W^0)^* \to \mathcal{L}^* \to 0$$

$$(W^0)^* \xrightarrow{\sim} H^0(\mathcal{L}^*) \qquad H^0(\mathcal{L}^*(-1)) \xrightarrow{\sim} Y^*$$

$$0 \to \mathcal{L} \to \mathcal{O} \otimes V \to \mathcal{L}^* \to 0$$

$$\underbrace{H^0(\mathcal{L}^*(-1))}_{Y^*} \xrightarrow{\sim} \underbrace{H^0(\mathcal{L}(-1))}_{Y}$$

115 So ~~that~~ now that this is clear you ~~[illegible]~~ want to work in the real line. ~~Let us consider~~ Details of the alg. situation. Use coord z.

$T = \mathbb{C}^2 \quad \ell_z = \begin{pmatrix}1\\z\end{pmatrix}\mathbb{C}. \quad T\otimes Y = \begin{matrix}Y\\\oplus\\Y\end{matrix}$ You need an isom $Y^* \simeq Y$, non degenerate symm pairing

Naturally $\begin{matrix}Y\\\oplus\\Y^*\end{matrix}$ is symplectic and those Lag. subspaces which are graphs $\begin{pmatrix}1\\F\end{pmatrix}Y$ ~~to~~ have form $\begin{pmatrix}1\\F\end{pmatrix}Y \quad F = F^*$

$F: Y \to Y^*$ symmetric, But to make sense of $\begin{pmatrix}1\\z\end{pmatrix}$ you need a fixed q.f. So $T\otimes Y \quad \ell_z$ have a standard form. ~~[illegible]~~

Real case $\quad T = \mathbb{R}^2 \quad$ skew form $\begin{pmatrix}x_1\\x_2\end{pmatrix}^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}x_1'\\x_2'\end{pmatrix}$

Y ~~[illegible]~~ Euclidean space $\quad = x_1x_2' - x_2x_1' = \begin{vmatrix}x_1 & x_1'\\x_2 & x_2'\end{vmatrix}$

$T\otimes Y = \begin{matrix}Y\\\oplus\\Y\end{matrix}$ with $\begin{pmatrix}y_1\\y_2\end{pmatrix}^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}y_1'\\y_2'\end{pmatrix} = y_1^ty_2' - y_2^ty_2'$

$\Gamma_\alpha = \begin{pmatrix}1\\\alpha\end{pmatrix}Y$ is isotropic means $\begin{pmatrix}1\\\alpha\end{pmatrix}^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1\\\alpha\end{pmatrix} = \alpha - \alpha^t = 0$

In fact $\Gamma_\alpha^0 = \left\{\begin{pmatrix}y_1\\y_2\end{pmatrix} \,\middle|\, \underbrace{\begin{pmatrix}y\\\alpha y\end{pmatrix}^t\begin{pmatrix}&1\\-1&\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}}_{y^ty_2 - y^t\alpha^ty_1} = 0\right\} = \Gamma_{\alpha^t}$

$\therefore y_2 = \alpha^ty_1$

So now consider ~~[illegible]~~ W isotropic in $\begin{matrix}Y\\\oplus\\Y\end{matrix}$

$W = \begin{pmatrix}\varepsilon\\A\end{pmatrix}X$. If you want W is subspace of $\begin{matrix}Y\\\oplus\\Y\end{matrix}$

$\varepsilon = pr_1|W, \; A = pr_2|W \qquad W^0 = \left\{\begin{pmatrix}y_1\\y_2\end{pmatrix} \,\middle|\, \begin{pmatrix}\varepsilon x\\Ax\end{pmatrix}^t\begin{pmatrix}&1\\-1&\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} = 0\right\}$

$(\varepsilon x)^ty_2 = (Ax)^ty_1$ or $\varepsilon^ty_2 = A^ty_1 \qquad \varepsilon^tA = A^t\varepsilon$

t denotes $*$ ~~with~~ wrt same scalar prod on ~~X~~

$W \cap \begin{pmatrix}1\\\lambda\end{pmatrix}Y = \left\{\begin{pmatrix}\varepsilon\\A\end{pmatrix}x \,\middle|\, \lambda\varepsilon x = Ax\right\}$ ~~[illegible]~~

You need to complexify – choose orthonormal basis for Y and X, then have solution of $(\lambda\varepsilon - A)x = 0 \quad x \in X_c$.

$$\lambda x = Ax$$
$$(x, \lambda x) = (x, Ax) \quad \bar\lambda \|x\|^2$$
$$\lambda\|x\|^2 \quad (Ax, x) = (\lambda x, x)$$

$$0 = (\varepsilon x, (\lambda\varepsilon - A)x) = \lambda\|\varepsilon x\|^2 - (\varepsilon x, Ax)$$

$$0 = ((\bar\lambda\varepsilon - A)x, \varepsilon x) = \bar\lambda\|\varepsilon x\|^2 - (Ax, \varepsilon x)$$

$\therefore \lambda = \bar\lambda$. $\boxed{\text{Point } (\varepsilon x, Ax) \in \mathbb{R}}$

So make assumption that no bound states, this is something you test ~~over~~ in the real setting. In particular you want ε inj $(\lambda = \infty)$ A inj $(\lambda = 0)$.

More review $T = \mathbb{R}^2$ skew form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{vmatrix} x_1 & x_1' \\ x_2 & x_2' \end{vmatrix}$

Y Euclidean space, $T \otimes Y = \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$, ~~with~~ skew form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = y_1^* y_2' - y_2^* y_1'$$

$$\Gamma_\alpha = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} Y \qquad \Gamma_\alpha^0 = \begin{pmatrix} 1 \\ \alpha^t \end{pmatrix} Y \qquad (\varepsilon x)^* y_2 = (Ax)^* y_1$$

W isot. in $\begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \qquad W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \varepsilon^* y_2 = A^* y_1 \right\}$

$$W_c \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y_c = \left\{ \begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} \middle| (\lambda\varepsilon - A)x = 0 \right\}$$

$\boxed{\begin{matrix} \varepsilon^* A = A^* \varepsilon \\ \text{if } W \subset W^0 \end{matrix}}$

$$0 = (\varepsilon x, (\lambda\varepsilon - A)x) = \lambda\|\varepsilon x\|^2 - (\varepsilon x, Ax) \in \mathbb{R} \quad \therefore \lambda \in \mathbb{R}.$$

~~$0 = ((\bar\lambda\varepsilon - A)x, \varepsilon x) = \bar\lambda\|\varepsilon x\|^2 - (Ax, \varepsilon x)$~~

Continue & calculate W^0, ε inj so can arrange $\varepsilon^*\varepsilon = 1$. $Y = \varepsilon X \oplus \operatorname{Ker}(\varepsilon^*)$. ~~Res.~~ $\quad \varepsilon^* y_2 = A^* y_1$

$$\begin{matrix} Y \\ \oplus \\ Y \end{matrix} \supseteq \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \oplus \begin{pmatrix} 0 \\ \oplus \\ \operatorname{Ker}\varepsilon^* \end{pmatrix}$$

Assume $\varepsilon^* y_2 = A^* y_1$

~~$(\varepsilon x, y_1) = (Ax, y_2)$~~

$(\varepsilon x, y_2) = (Ax, y_1)$

117 Given y_1 $\quad x \longmapsto (Ax, y_1)$ can be represented as $(\varepsilon x, y_2)$ uniquely with $y_2 \in \varepsilon X$.

Define a map $Y \xrightarrow{\theta} X$ by requiring

$$(Ax, y) = (\varepsilon x, \varepsilon\theta y) = (x, \theta y) \quad \therefore \theta = A^*$$

W^0 seems to consist of $\begin{pmatrix} \varepsilon \\ A \end{pmatrix}$?,

Start with $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W^0$ i.e. $\varepsilon^* y_2 = A^* y_1$

or $(\varepsilon x, y_2) = (Ax, y_1) \quad \forall x$ $\quad$ But

$$(Ax, y_1) = (x, A^* y_1)$$
$$= (\varepsilon x, \varepsilon A^* y_1)$$

$\cancel{(\varepsilon x, \varepsilon\varepsilon^* y_2 + (1-\varepsilon\varepsilon^*) y_2) = (x, \varepsilon^* y_2)}$

Claim $\cancel{y_1}$ so $y_2 \equiv \varepsilon A^* y_1$ ~~mod~~ $(\operatorname{Ker} \varepsilon^*)$.

$$W^0 = \begin{pmatrix} 1 \\ \varepsilon A^* \end{pmatrix} Y + \begin{pmatrix} 0 \\ \oplus \\ \operatorname{Ker} \varepsilon^* \end{pmatrix}$$

<u>Proof</u>: Given $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W^0$ i.e. $(\varepsilon x, y_2) = (Ax, y_1) \quad \forall x$.

Then $(\varepsilon x, y_2) = (x, A^* y_1) = (\varepsilon x, \varepsilon A^* y_1) \implies y_2 - \varepsilon A^* y_1 \in \operatorname{Ker} \varepsilon^*$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ \varepsilon A^* y_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ y_2 - \varepsilon A^* y_1 \end{pmatrix}$$

εX

$\varepsilon A^* \varepsilon x$

$= \varepsilon\varepsilon^* A x = Ax$

Note $(\varepsilon A^*)^* = A\varepsilon^*$

$$\begin{bmatrix} B & \gamma \\ \gamma^* & \end{bmatrix}$$

$\pi = 1 - \varepsilon\varepsilon^*$.

$$A\varepsilon^* = \underbrace{\varepsilon\varepsilon^* A \varepsilon^*}_{\varepsilon A^* \varepsilon\varepsilon^*} + \pi A \varepsilon^*$$

$$\varepsilon A^* = \varepsilon A^* \varepsilon\varepsilon^* + \varepsilon A^* \pi$$

So it should be possible to uniquely extend the partial ~~symm~~ symm. op. $\begin{pmatrix}\varepsilon\\ A\end{pmatrix}$ to a symm. operator $\tilde{A}$ on Y $\ni$ $\pi(\tilde{A})\pi = 0$. This gives a kind of canonical extension. ~~[illegible]!!!~~ ~~In the end you seem to get~~ W°/W is symplectic and you have ~~constructed a~~ found a canonical Lagrangian subspace. In fact we have $W^\circ/W \simeq \begin{matrix}\operatorname{Ker}\varepsilon^*\\ \oplus\\ \operatorname{Ker}\varepsilon^*\end{matrix}$

What is the answer? You have

$$W = \begin{pmatrix}\varepsilon\\ A\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix} \quad \text{and } W \subset \begin{pmatrix}1\\ \tilde{A}\end{pmatrix} Y \subset W^\circ$$

$\varepsilon\varepsilon^* A \varepsilon^*$

$$\tilde{A} = A\varepsilon^* + \varepsilon A^* - \varepsilon A^* \varepsilon\varepsilon^*$$

$$= A\varepsilon^* + \varepsilon A^* \pi = \pi A \varepsilon^* + \varepsilon A^*$$

Now we have ~~a simple problem~~ to find $W^\circ \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y$,

$$W^\circ = \left\{ \begin{pmatrix}y_1\\ y_2\end{pmatrix} \,\middle|\, A^* y_1 = \varepsilon^* y_2 \right\} \qquad W^\circ \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y \simeq \begin{pmatrix}1\\ \lambda\end{pmatrix} \operatorname{Ker}(\lambda\varepsilon^* - A^*)$$

You guess that the response function should have a simple form, like what you found for an LC network. ~~The basic problem here~~ The idea here is that the response is ~~[illegible]~~ Lagrangian subbundle $\mathcal{L} \subset \mathcal{O} \otimes (W^\circ/W)$ and since $W^\circ/W = \begin{pmatrix}\operatorname{Ker}\varepsilon^*\\ \operatorname{Ker}\varepsilon^*\end{pmatrix}$, $\mathcal{L}_\omega$ should be the graph of a ~~[illegible]~~ symmetric operator on $\operatorname{Ker}(\varepsilon^*)$.

Idea resolvent of $\left(\begin{array}{c|c} \varepsilon^* A = A^* \varepsilon & \varepsilon A^* \pi \\ \hline \pi A \varepsilon^* & 0 \end{array}\right)$. Something like this

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda - B & -\gamma \\ -\gamma^* & \lambda \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$(d - ca^{-1}b)^{-1} = \left(\lambda - \gamma^* \frac{1}{\lambda - B}\gamma\right)^{-1}$$

You are trying for [redacted] something like [redacted] inducing a quadratic form on a subspace or quotient space. ~~So two~~

So you run into a [redacted] familiar situation namely you take the resolvent and ~~project into something~~ compress into a generating subspace. The answer is very easy. The problem is to fit it into something [redacted].

Now what. $\varepsilon^*\varepsilon = 1$. ~~EA~~ Can write

$$Y = \varepsilon X \oplus \operatorname{Ker} \varepsilon^* \qquad \gamma =$$

$$\tilde{A} = \left[\begin{array}{c|c} \varepsilon\varepsilon^* A \varepsilon^* & \varepsilon A^* \pi \\ \hline \pi A \varepsilon^* & 0 \end{array}\right]$$

You need to organize all this stuff. ~~What~~ How. ~~Go back~~ To construct $\mathcal{L}_\omega = W^0 \cap \binom{1}{\omega} Y$

$$-\delta \log \det \begin{pmatrix} \lambda - \alpha & -\gamma \\ -\gamma^* & \lambda - \beta \end{pmatrix} = \operatorname{tr} \begin{pmatrix} \lambda - \alpha & -\gamma \\ -\gamma^* & \lambda - \beta \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & +\delta\beta \end{pmatrix}$$

$$= \operatorname{tr} \frac{1}{\lambda - \beta - \gamma^* \frac{1}{\lambda - \alpha}\gamma} \delta\beta$$

Note that is hermitian for λ real.

120 [illegible] ~~What's going on?~~

~~Given $\varepsilon^*_{y_1} = 0$, then A y_1~~

You want to calculate W^0 where $W = \begin{pmatrix} \varepsilon x \\ A x \end{pmatrix}$

Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W^0$, i.e. $A^* y_1 = \varepsilon^* y_2$ remove $\begin{pmatrix} \varepsilon\varepsilon^* y_1 \\ A\varepsilon^* y_1 \end{pmatrix}$ from $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ to assume $\varepsilon^* y_1 = 0$. Now we have $\forall x$ $(y_1, Ax) = (x_1, x)$ for ~~some~~ a unique x_1

$\underset{(y_2, \varepsilon x)}{\overset{\parallel}{}}$

i.e. A $(A^* y_1, x) = (x_1, x)$.

so it seems that if $\varepsilon^* y_1 = 0$ then $\begin{matrix} y_1 \\ \varepsilon A^* y_1 \end{matrix}$

Wait given $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W^0$ i.e. $A^* y_1 = \varepsilon^* y_2$ then

$\begin{pmatrix} y_1 \\ \varepsilon A^* y_1 \end{pmatrix}$ satisfies $A^* y_1 = \varepsilon^*(\varepsilon A^* y_1)$, so $\Gamma_{\varepsilon A^*} \subset W^0$

A $\cancel{\varepsilon A}$

also $(\Gamma_{\varepsilon A^*})^0 = \Gamma_{A\varepsilon^*}$?

The point is that [illegible] $\begin{pmatrix} 1 \\ \varepsilon A^* \end{pmatrix} Y \subset W^0$

because $\varepsilon A^*(\varepsilon x) = \varepsilon\varepsilon^* A x$ NO

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} y_1 \\ \varepsilon A^* y_1 \end{pmatrix}}_{} + \begin{pmatrix} 0 \\ y_2 - \varepsilon A^* y_1 \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} \varepsilon\varepsilon^* y_1 \\ \varepsilon A^* \varepsilon\varepsilon^* y_1 \end{pmatrix} + \begin{pmatrix} y_1 - \varepsilon\varepsilon^* y_1 \\ \\ \end{pmatrix}$$

$\varepsilon\varepsilon^* A \varepsilon y_1$

Given y_1 $\exists! x_1$ $\ni (y_1, Ax) = (\varepsilon x_1, \varepsilon x)$ $\forall x$

i.e. $x_1 = A^* y_1$

$\therefore \begin{pmatrix} y_1 \\ \varepsilon A^* y_1 \end{pmatrix} \in W^0$

1

$A\varepsilon^*$

$\begin{matrix} y_1 \\ A\varepsilon^* y_1 \end{matrix} \in W^0$ $\quad A^* y_1 \overset{?}{=} \varepsilon^* A \varepsilon^* y_1$

$\parallel$

$A^* \varepsilon\varepsilon^* y_1$

NO