Rn 390 – 675

In much other paper dated '02.

Paper 458 dated Jane 6, '98

455 – back of sheet dated Dec, '98

390 Review: You have a partial unitary operator $Y = aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX$

$\xi_\pm$ ~~unit vectors~~. From this you get $S(z)$ analytic for $|z|<1$ $|S(z)|<1$ ~~st~~ such that

$$(za-b)x = -\xi_+ + S(z)\xi_-$$

in the functional rep this says

$$(z-\zeta)x(\zeta) = -S(\zeta) + S(z)$$

close to pt evaluator $\dfrac{1-\overline{S(w)}S(z)}{1-\bar{w}z}$?

So what are you going to do?

Take an example. First $X=0$. ξ_+, ξ_- are two unit vector in Y and $\exists\ t \in S^1 \ni \xi_+ = t\xi_-$ and then $S(z)=t$. Now let $c\xi_+ = h\xi_-$ with $|h|<1$. Eigenvalue equation $h = zS(z)$ in this case is $h = zt$ or $z = t^{-1}h$.

Next: shift Let $X = \mathbb{C}1$ $\begin{matrix} a=1 \\ b=\zeta \end{matrix}$ $Y = \mathbb{C}1 \oplus \mathbb{C}\zeta$

ξ_+ unit vector in $V_+ = \mathbb{C}\zeta$

ξ_- ———— $V_- = \mathbb{C}1$ take $\xi_- = 1$.

suppose $\xi_+ = t\zeta$ $|t| = 1$.

$$(z-\zeta)x = -t\zeta + S(z) \qquad S(z) = tz$$

~~eigenvector~~ $h = zS(z)$ $h = tz^2$ $z^2 = ht^{-1}$.

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391 Continue ~~[illegible]~~: Suppose we have ~~[illegible]~~ a partial unitary arising from ~~moments~~ a measure. $X = \underbrace{\mathbb{C}1 + \cdots + \mathbb{C}\zeta^{n-1}} = P_{n-1}$, $Y = P_n$

$n = 1$. $\quad X = \underbrace{\mathbb{C}1}_{p_0} \oplus \mathbb{C}\underbrace{(\zeta + h_1)}_{p_1}$ $\qquad (1, \zeta + h_1) = 0 \qquad h_1 = -(1, \zeta)$

$$(\zeta + h_1, \zeta + h_1) = (\zeta, \zeta + h_1) = 1 - |h_1|^2$$

$$Y = \mathbb{C}\underbrace{(1 + \bar{h}_1\zeta)}_{g_1} \oplus \mathbb{C}\zeta$$

$$\xi_- = \frac{1 + \bar{h}_1\zeta}{1 - |h_1|^2} \qquad \xi_+ = \frac{\zeta + h_1}{1 - |h_1|^2}$$

$$(\zeta - z)x = -\xi_+ + S(z)\xi_-$$

$$S(z) = \frac{\xi_+}{\xi_-}(z) = \frac{z + h_1}{1 + \bar{h}_1 z}$$

$$h = zS(z) = z\frac{z + h_1}{1 + \bar{h}_1 z}$$

$$h\frac{g_1}{1 - |h_1|^2} = z\frac{p_1}{1 - |h_1|^2} \qquad \cancel{zp_1}$$

eigenvalue equation $\qquad zp_1 - hg_1 = 0 \qquad$ Should this h be h_2?

measure had moments $\quad \mu_0 = 1, \ \mu_1 = \bar{h}_1$

$(X, c) \qquad \dim(X) = 1. \qquad H = L^2(S^1, d\mu)$

$$d\mu = \underbrace{\left(\sum_{n \geq 0} \zeta^{-n}c^n + \sum_{n \geq 1} \zeta\bar{c}^n\right)}\frac{d\theta}{2\pi}$$

$$\frac{1}{1 - c\zeta^{-1}} + \frac{\bar{c}\zeta}{1 - \bar{c}\zeta} = \frac{1 - |c|^2}{|1 - c\zeta^{-1}|^2}$$

$$L^2(S^1, d\mu) \longleftrightarrow L^2(S^1)$$

$$\|f\|^2_{d\mu} = \int |f|^2 \frac{1-|c|^2}{|1-\bar{c}\zeta|^2} \frac{d\theta}{2\pi}$$

$$\frac{f(1-|c|^2)^{1/2}}{1-c\zeta^{-1}} \longleftarrow\!\shortmid\; f \;\longmapsto \frac{f(1-|c|^2)^{1/2}}{1-\bar{c}\zeta}$$

$$\frac{1-\bar{c}\zeta}{(1-|c|^2)^{1/2}} f \longleftarrow\!\shortmid\; 1$$

$$1 \longmapsto \frac{1-c\zeta^{-1}}{(1-|c|^2)^{1/2}}$$

so the scattering this way is

$$1 \longmapsto \frac{1-c\zeta^{-1}}{(\quad)^{1/2}} \longmapsto \frac{1-c\zeta^{-1}}{1-\bar{c}\zeta}$$

Look more carefully at $L^2(S^1, d\mu)$ $\qquad \int \zeta^n d\mu = c^n \quad n \geq 0$

$$\frac{\zeta^{-1}-\bar{c}}{(1-|c|^2)^{1/2}},\ 1,\ \frac{\zeta-c}{(1-|c|^2)^{1/2}},\ \zeta\frac{\zeta-c}{(1-|c|^2)^{1/2}},\ \ldots \quad \text{orth basis.}$$

$$\frac{1-\bar{c}\zeta}{(\quad)^{1/2}} = \xi_- \qquad \xi_+$$

$$(\zeta-c, \zeta-c) = (\zeta, \zeta-c) = 1-(\zeta,1)c = 1-\bar{c}c$$

$$(1, \zeta^n(\zeta-c)) = c^{n+1} - c^n c = 0 \qquad n \geq 0.$$

$$S(\zeta)\xi_- = \xi_+ \qquad S(\zeta) = \frac{\zeta-c}{1-\bar{c}\zeta}$$

So what have you found.

Take unitary S, $Y = H^+ \ominus zSH^+$, $X = H^+ \ominus SH^+$

~~[illegible]~~ Y

$$\begin{array}{ccc} H^+ & \xrightarrow{aX} & SH^+ \\ \mathbb{C}1 = V^- \big\downarrow & & \big\downarrow V^+ = \mathbb{C}S \\ zH^+ & \xrightarrow{bX} & zSH^+ \end{array}$$

Then you complete this partial unitary to a unitary $u = ba^* + \xi_- h \xi_+^*$ or maybe a contraction. Take $h = 1$ first. Eigenvalues for u should be roots of $1 - zS(z) = 0$, equivalently, poles of the Pick function $i\dfrac{1+zS}{1-zS}$ which is real on the boundary. So ~~it [illegible]~~ you need to identify the (H, u, ξ) assoc. to the C.T. $i\dfrac{1+zS}{1-zS}$ of zS with Y, u and some cyclic vector. Is there a natural cyclic vector.

Start the other way with $L^2(S^1, d\mu)$, $u =$ mult by ~~[illegible]~~ $\zeta = e^{i\theta}$, $\xi = 1$

$$f(z) = \int i \underbrace{\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}}_{\frac{2}{1-z\zeta^{-1}} - 1} d\mu$$

$$\left(\frac{1}{1-\bar{z}\zeta}, \frac{1}{1-\bar{w}\zeta}\right) = \int \frac{1}{1-z\zeta^{-1}} \frac{1}{1-\bar{w}\zeta} d\mu$$

$$\frac{1}{1-z\zeta^{-1}} + \frac{1}{1-\bar{w}\zeta} - 1 \qquad \left(= \sum_{n\geq 0} z^n \zeta^{-n} + \sum_{n>0} \bar{w}^n \zeta^n\right)$$

$$= \frac{1}{1-z\zeta^{-1}} + \frac{\bar{w}\zeta}{1-\bar{w}\zeta} = \frac{1-z\bar{w}}{(1-z\zeta^{-1})(1-\bar{w}\zeta)}$$

$$\left(\frac{1}{1-\bar{z}\zeta}, \frac{1}{1-\bar{w}\zeta}\right) = \frac{1}{1-z\bar{w}} \frac{1}{2i} \int i \left(\frac{2}{1-z\zeta^{-1}} - 1 + \frac{2}{1-\bar{w}\zeta} - 1\right) d\mu$$

$$= \frac{1}{2i} \frac{f(z) - \overline{f(w)}}{1-z\bar{w}}$$

The elements $\dfrac{1}{1-\bar{z}\zeta}$ lie in $H^2(S^1, d\mu)$ & probably span

394 Assume nice form for $d\mu$, say

$$d\mu = |E|^2 \frac{d\theta}{2\pi} \quad \text{with } E \text{ analytic in the disk}$$

Szego theory. If ρ is smooth >0 on S^1, then

$$\log \rho = \sum a_n \zeta^n \qquad a_n = \int e^{-in\theta} \log\rho \, \frac{d\theta}{2\pi}$$

$$\bar{a}_n = a_{-n}$$

$$= \underbrace{\frac{a_0}{2} + \sum_{n\geq 1} a_n \zeta^n}_{g(\zeta)} + \underbrace{\frac{a_0}{2} + \sum_{n\geq 1} \bar{a}_n \zeta^{-n}}_{\overline{g(\zeta)}} \qquad \therefore \rho = E\bar{E}, \quad E = e^g$$

So when $d\mu/\rho$ is nice then you should have an isom with $L^2(S^1)$.

$$L^2\left(S^1, |E|^2 \frac{d\theta}{2\pi}\right) \xleftarrow{\sim} L^2\left(S^1, \frac{d\theta}{2\pi}\right)$$

$$\frac{f}{E} \longleftarrow\!\shortmid \; f$$

$$\frac{1}{1-\bar{z}\zeta} \longmapsto \frac{E(\zeta)}{1-\bar{z}\zeta}$$

$$\frac{1}{2i} \frac{f(z) - \overline{f(w)}}{1 - z\bar{w}} = \int \frac{1}{1-z\zeta^{-1}} \frac{1}{1-\bar{w}\zeta} |E(\zeta)|^2 \frac{d\theta}{2\pi}$$

$$= \int \frac{\overline{E(\zeta)}}{1-z\zeta^{-1}} \frac{E(\zeta)}{1-\bar{w}\zeta} \frac{d\theta}{2\pi} \qquad \frac{d\zeta}{2\pi i \zeta}$$

analytic outside S^1 — analytic inside S^1

~~shrink cont~~

$$= \int \frac{\overline{E(\bar{\zeta}^{-1})}}{\zeta - z} \frac{E(\zeta)}{1-\bar{w}\zeta} \frac{d\zeta}{2\pi i} \qquad \overline{E(\bar{\zeta}^{-1})} \text{ not analytic in disk.}$$

$$= \frac{\overline{E(\bar{z}^{-1})}\, E(z)}{1 - \bar{w} z}$$

395 Better is:

$$f(z) = \int i\left(\frac{2}{1-z\xi^{-1}} - 1\right) \overline{E(\bar{\xi}^{-1})}\, E(\xi) \frac{d\xi}{2\pi i \xi}$$

~~What is~~ I am trying to go from S to a measure. Recall the equivalence you want to prove.

Pick function up to additive real const.
positive harmonic function
measure
cyclic unitary rep of $\mathbb{Z}$
scattering function $S(z)$ ~~How about going~~

~~Need to get f.d. case~~

periodic cases:
periodic Pick function on UHP descends to a Pick function on D

Review + write up Pick function stuff, including periodic ones.

Start in D. $f(z)$ analytic for $|z| < 1+\varepsilon$

$$f(z) = \sum_{n\geq 0} a_n z^n \qquad a_n = \int \frac{f(\xi)}{(\xi)^{n+1}} \frac{d\xi}{2\pi i}$$

$$= \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$$

$$2i \operatorname{Im} f(z) = \sum a_n z^n - \sum \bar{a}_n \bar{z}^n$$

$$a_n = \int_0^{2\pi} 2i \operatorname{Im} f(e^{i\theta})\, e^{-in\theta} \frac{d\theta}{2\pi} \qquad n \geq 1$$

$$a_0 - \bar{a}_0 = \underline{\hspace{6cm}} \qquad n = 0.$$

$$f(z) = \sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} z^n \int_0^{2\pi} (2i \operatorname{Im}(f))\, e^{-in\theta} \frac{d\theta}{2\pi} + \bar{a}_0$$

measure $d\mu$ on S^1 is a positive linear functional on continuous functions.

$f(1) = 0$ $\qquad f \geq 0$

$|z-1|^2$

$$\rho(z) = \sum_{\text{finite}} a_n z^n \geq 0 \qquad \overline{a_n} = a_{-n}$$

Assume $\rho(1) = 0$. Is $\rho(z)$ divisible by $|z-1|^2$?

$$\underline{\left|z^{1/2} - z^{-1/2}\right|^2} = |z-1|^2 = (\bar{z}-1)(z-1) = 1 - z - z^{-1} + 1$$

Consider $\rho(z) = \sum_{|n| \leq N} a_n z^n \qquad \rho(z) \geq 0$ for $|z| = 1$.

fdl thm of alg. says
$$\rho(z) = c z^m \prod_{j=1}^{k} (z - \alpha_j) \qquad \forall \alpha_j \neq 0.$$
$$\overset{\shortparallel}{\overline{\rho(\bar{z}^{-1})}} = \bar{c} z^{-m} \prod_{j=1}^{k} (z^{-1} - \bar{\alpha}_j)$$

$$(-\bar{\alpha}_j) z^{-1} (\bar{\alpha}_j^{-1} + z)$$

$$\rho(z) = c z^m \prod_{j=1}^{k} (z - \alpha_j) = \overline{\rho(\bar{z}^{-1})} = \bar{c} z^{-m} \prod_{j=1}^{k} \overbrace{(z^{-1} - \bar{\alpha}_j)}$$
$$= \bar{c} \prod_{j=1}^{k} (-\bar{\alpha}_j) \, z^{-m-k} \prod_{j=1}^{k} (z - \bar{\alpha}_j^{-1})$$

$c = \bar{c} \prod_{j=1}^{k} (-\bar{\alpha}_j)$, $\quad m = -m + k$, $\quad (\alpha_j) = (\bar{\alpha}_j^{-1})$ up to a perm.

$k = 2m$. Remove from ρ root pairs over S^1.

Divide ρ by $(z-\alpha)(z^{-1} - \bar{\alpha})$ where $|\alpha| < 1$ say decrease m by 1 k by 2, can suppose then that $|\alpha_j| = 1$ $j = 1, \dots, k$. Then you argue that α_1 must be a double root.

Better proof. $\rho(z) = \sum_{|n| \le N} a_n z^n \geq 0$ for $|z| = 1$.

$\rho(z)$ is rational, poles $\{0, \infty\}$. Let $\rho(\alpha) = 0$. $(\alpha \neq 0, \infty)$

$\overline{\rho(\bar{z}^{-1})} = \rho(z) \Rightarrow \rho(\bar{\alpha}^{-1}) = 0$. If $\alpha \neq \bar{\alpha}^{-1}$ i.e. $\alpha \notin S^1$, then $\rho_1(z)$, $\underbrace{\frac{\rho(z)}{z^{-1}(z-\alpha)(z-\bar{\alpha}^{-1})}}_{\text{lower degree.}} = \frac{\rho(z)}{|z-\alpha|^2}$ for $|z| = 1$.

If $\alpha = \bar{\alpha}^{-1}$, then look at $\rho_1(z) = \frac{\rho(z)}{(z-\alpha)(z^{-1}-\bar{\alpha})}$ rational poles $\subset \{0, \alpha = \bar{\alpha}^{-1}, \infty)$ if ρ double zero at α no problem. If ρ simple zero at α then $\rho_1(z)$ has a pole. Plot $\rho(e^{i\theta})$

$e^{i\theta} = \alpha$

Claim $\rho(z) = \left| \prod_{\substack{\text{roots } \alpha \text{ of} \\ \rho \ni |\alpha| \le 1}} (z - \alpha_i) \right|^2$

What sort of digression are you making?

Basically you want the equiv. of certain things.

1) $d\mu = \underbrace{\sum_{n \in \mathbb{Z}} \mu_n \zeta^{-n} \frac{d\theta}{2\pi}}_{\text{dist in general}}$ is a measure

2) matrix $\mu_{j,k} = \mu_{k-j}$ is ≥ 0 and not zero.

3) $\sum_{n \geq 0} \mu_n \bar{z}^n + \sum_{n > 0} \bar{\mu}_n z^n > 0$ for $|z| < 1$.

Real point is maybe to be found inside C^* theory.

 ~~Think carefully~~ ~~Think out carefully~~

Think algebraically, begin with moments $\{\mu_n\}$, initially an arbitrary sequence, ~~[illegible]~~

~~[illegible]~~ Equivalently a linear fcl on $\mathbb{C}[z, z^{-1}]$. Impose positivity condition, how to be formulated? In the most algebraic version you require (μ_n) to be a pos. def. fn. on $\mathbb{Z}$. However you might want to consider the moments through degree N. Inverse system? There is ~~an [illegible]~~ an idea here, namely the Schur coeffs. $h_1, h_2, \ldots, h_n$ are equivalent to $\mu_1, \ldots, \mu_N$. This is the formal theory.

Things to do maybe? maybe work out equivalence between following ~~[illegible]~~ notions of positivity for $\mu = \{\mu_n\}$

$$\mu_{kl} = \mu_{l-k} \quad \text{positive} \ \text{def.}$$

$$\sum_{n\geq 0} \mu_n \bar{z}^n + \sum_{n\geq 1} \bar{\mu}_n z^n > 0$$

This is not probably very important

$$\left(\frac{1}{1-\bar{z}\,\zeta}, \frac{1}{1-\bar{w}\,\zeta^{-1}} \right) = \int \frac{1}{1-\bar{z}\,\zeta} \frac{1}{1-\bar{w}\,\zeta^{-1}} d\mu \quad \text{etc.}$$

~~Focus on Schur exp.~~ To go from one to other.

$d\mu$ better $\mu_0, \ldots, \mu_{n_0}$ ~~[illegible]~~

399 Do something to sort out the confusion.

Consider a partial unitary of type $O(n)$:

$Y = aX \oplus V_+ = V_- \oplus bX$ $\quad a^*a = b^*b = 1.$ $\quad \dim(X) = n$, $\dim(Y) = n+1$

no bound states, choose unit vectors $\xi_\pm \in V_\pm$,

$S(z)$ corresponding scattering function

$$S(z) = \left(\xi_-, \frac{1}{1 - z a b^*}\xi_+\right).$$ You know that

$$y \longmapsto \left(\xi_-, \frac{1}{1-zab^*}y\right) = \xi_-^* \frac{1}{1-zab^*} y = \tilde{y}(z)$$

is an isometric embedding of Y ~~onto~~ $H^+ \ominus zSH^+ = H^+ \cap zSH^-$. $\quad \xi_- \longmapsto 1$, $\xi_+ \longmapsto S$.

So you can take concrete picture $Y = H^+ \cap zSH^-$, $X = H^+ \cap SH^-$, $\xi_- = 1$, $\xi_+ = S$. Also writing

$$S = \frac{p}{q} = \frac{e^{i\phi}\prod_1^n (z - a_j)}{e^{-i\phi}\prod_{j=1}^n (1 - z\bar{a}_j)} \qquad q(z) = z^n \overline{p(\bar{z}^{-1})}.$$ Then

$$Y \xrightarrow{\sim} H^+ \cap zSH^- \xrightarrow[\sim]{\cdot q} qH^+ \cap zpH^- = H^+ \cap z^{n+1}H^- = P_n$$

$$y \longmapsto \xi_-^* \frac{1}{1-zab^*} y$$

$$Y = \frac{1}{q} P_n \subset H^+$$

$$X = \frac{1}{q} P_{n-1} \qquad \frac{1}{q}$$

So you divide by q to get the filtrations

standard situation: $d\mu = \frac{1}{|q|^2}\frac{d\theta}{2\pi}$ scattering is $\frac{p}{q}$ where $p = \bar{q}$ roughly. Let's get the details straight.

~~The S degree n Blashke product~~

Suppose $S(z) : S^1 \longrightarrow \mathbb{T}$ smooth and $|z| < 1$. real analytic. Then S has a degree n.

400 $\dim H^+ \cap SH^- = ?$

Write $S = \frac{p}{q}\, e^{g}/e^{g_\#}$ — Blashke prod. deg n. Need more detail $S = S_B$? Wait. If S cont. on $|z| \leq 1$ analytic in interior and $|S(z)| = 1$ when $|z| = 1$, then S must be a Blashke product, because $\overline{S(\bar{z}^{-1})} = S(z)$ extends S to a meron. function

~~Go where next? Go back Hanzikle~~.

Where to? Fix attention to S of degree n.

Consider $d\mu$ measure on S^1 support $n+1$ pts.

Try to fix the moments $\mu_0, \dots, \mu_{n-1}$

look at $\int i \underbrace{\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}}_{1+2\sum_{n\geq 1} z^n \zeta^{-n}} d\mu = f(z)$

~~$f(z) \neq \frac{f(0)+f(0)}{2}$~~

Why do I find this hard? ~~Difficult~~

Maybe you should study the case of an S where $h_n = 0$ for $n >> 0$. $q_n = q_{n+1} = \cdots$

$p_{n+k} = z^k p_n$ $k \geq 0$. You should have a simple ~~formula relating~~ relation between $d\mu$

Important case: $\prod (1 - |h_n|^2) > 0$.

leads to $q_\infty \in 1 + zH^2(d\mu)$ Albores

and sim. $H^2(d\mu) = H^+$

$\frac{q_\infty}{\|q_\infty\|} \longleftarrow 1$

$\int |f|^2 d\mu = \int \left|\frac{f}{q_\infty}\right|^2 \|q_\infty\|^2 \frac{d\theta}{2\pi}$; $d\mu = \frac{\|q_\infty\|^2}{|q_\infty|^2} \frac{d\theta}{2\pi}$

401 So you get ~~a~~ a formula for the measure. Why is this interesting? You seem to have an interesting limit as $n \to \infty$. You need a principle

Conclusion: Again you can start with a finite Blashke product: S_m of degree m, and then you put $S_n = z^{n-m} S_m$. At the same time q_n doesn't change.

Let $S(z)$ be rational function of degree n mapping D to D. Then you have the Schur expansion

$$S_0 = \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & h_{n+1} \\ \bar{h}_{n+1} & 1 \end{pmatrix} (0)$$

with $|h_1|, \dots, |h_n| < 1$ and $|h_{n+1}| \leq 1$.

Take $n = 1$. $S_0 = \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} (z S_1)$ $\qquad S_1 = h_2$

OKAY $\qquad S = \dfrac{p_{\prime\prime}}{q_{1}}$

Your problem is what end to ~~example~~ examine. Begin with either S or μ. Do S first. No because you're confused by non unitary S.

~~Think again of the diff~~

Begin with μ, form $L^2(S^1, d\mu)$ construct

$p_n \in (z^n + P_{n-1}) \cap P_{n-1}^{\perp}$ $\qquad q_n \in (1 + zP_{n-1}) \cap zP_{n-1}^{\perp}$

$p_n^* \in (z^{-n} + \underbrace{P_{n-1}^*}_{z^{-n+1}P_{n-1}}) \cap (P_{n-1}^*)^{\perp}$ $\qquad z^n p_n^* \in (1 + zP_{n-1}) \cap \{zP_{n-1}\}^{\perp}$

$p_{n+1} - zp_n \in P_n \cap (zP_{n-1})^{\perp}$ $\qquad p_{n+1} = zp_n + h_{n+1} q_n$

$\quad \| \quad h_{n+1} q_n$ $\qquad q_{n+1} = \bar{h}_{n+1} z p_n + q_n$

$p_n - h_n q_{n-1} = zp_{n-1}$ $\qquad q_n - \bar{h}_n z p_{n-1} = q_{n-1}$

$\|p_n\|^2 + |h_n|^2 \|q_{n-1}\|^2 = \|p_{n-1}\|^2$ $\qquad (q_n, p_n) - h_n (zp_{n-1}, p_n) = 0$

$-h_n \|q_{n-1}\|^2 = (q_{n-1}, zp_{n-1})$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$zp_n + h_n q_{n-1}$

$$(q_n, p_n) = (\cancel{q_{n-1}} + \bar h_n z p_{n-1}, p_n)$$

$$(q_n, p_n) = (q_n, \cancel{z p_{n-1}} + h_n q_{n-1})$$

$$= (q_{n-1} + h_n z p_{n-1}, h_n q_{n-1})$$

$$\begin{tikzcd} P_n \arrow[r, hook, "\tilde p_{n+1}"] & P_{n+1} \\ zP_{n-1} \arrow[r, hook, "z\tilde p_n"'] \arrow[u, hook, "\tilde q_n"] & zP_n \arrow[u, hook, "\tilde q_{n+1}"'] \end{tikzcd}$$

$$\begin{pmatrix} \tilde p_{n+1} \\ \tilde q_n \end{pmatrix} = \underbrace{\begin{pmatrix} k_{n+1} & h_{n+1} \\ -\bar h_{n+1} & k_{n+1} \end{pmatrix}}_{\in U_2}\begin{pmatrix} z\tilde p_n \\ \tilde q_{n+1} \end{pmatrix}$$

$a, d > 0.$ $a = d.$

$k_{n+1}^2 = 1 - |h_{n+1}|^2$

$$(\tilde q_{n+1}, \tilde p_{n+1}) = h_{n+1}$$

$$(\tilde q_n, z\tilde p_n) = (-\bar h_{n+1} z\tilde p_n + \cancel{k_{n+1}\tilde q_{n+1}}, z\tilde p_n) = -h_{n+1}$$

What to do? You want to start with $L^2(S^1, d\mu)$ and work out ~~the~~ orth poly theory. Point to follow: other measures whose first k moments agree with those of $d\mu$. ~~Then How to proceed~~. Suppose $\mu_0 = 1$, ~~and given μ~~ suppose given $\mu_1, \dots, \mu_n$ yielding $P_{n-1} \xrightarrow{z} P_n$, $p_0 = 1, p_1, \dots, p_n$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & h_1 \\ \bar h_1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$p_n = z p_{n-1} + h_n q_{n-1}$$

$$q_n = \bar h_n z p_{n-1} + q_{n-1}$$

$$\cancel{(q_n, p_n) = (q_n, z p_{n-1} + h_n q_{n-1})}$$

$$\cancel{z p_{n-1} + h_n q_{n-1}}$$

$$\cancel{= h_n}$$

$$(q_n, p_n) = h_n (q_n, q_{n-1})$$

$$= h_n (q_n, q_n - \bar h_n z p_{n-1}) = h_n \|q_n\|^2$$

403 Where next? You started with $L^2(S^1, d\mu)$ formed the partial unitary $P_{n-1} \overset{\subset}{\underset{z}{\longrightarrow}} P_n = Y$ using the moments $\mu_0, \dots, \mu_n$. ~~Using~~
The partial unitary ~~dilates to an cut~~ has a "canonical" or "standard" dilation, so there should be an n-th approx measure, de Branges type.

$$Y \hookrightarrow H^+ \cap zSH^- \xrightarrow{g_n} H^+ \cap z^{n+1}H^- = P_{n-1}$$

$$y \longmapsto \left\{ \frac{1}{1 - zab^*} y \right\}^*_-$$

So the claim is that $\frac{1}{|g_n|^2}\frac{d\theta}{2\pi}$ is the nth order approx to $d\mu$.

You seem to have all the ingredients, but the picture is not completely clear. You to be able to go directly between S's and measures as suggested by the Schur expansion. At the moment we have

$d\mu \longmapsto (\mu_0, \dots, \mu_n) \equiv$ scalar product making $P_{n-1} \hookrightarrow P_n$ a partial un.

$$\downarrow$$

$$\begin{pmatrix} p_n \\ g_n \end{pmatrix} \longleftrightarrow S_n = \frac{p_n}{g_n}$$

$$\begin{pmatrix} p_n \\ g_n \end{pmatrix} \longleftrightarrow \frac{1}{|g_n|^2}\frac{d\theta}{2\pi}$$

So what's taking place? The ~~measure~~ measure

measure $\longmapsto$ Schur sequence $\longmapsto S$

Given $d\mu$ you construct orth poly sequence getting the h_n. In turn the h_n sequence yield $S_n = \frac{p_n}{g_n}$ and a sequence of measures $d\mu_{\leq n}$. You want simple examples. What should these be?

404 You want simple examples, but it's not clear what they should be. ~~Pert?~~

A finite support $d\mu$ on S^1, say $n+1$ pt support, get partial unitary $P_{n-1} \hookrightarrow P_n$ with a unitary boundary condition. Count dims, assume prob. measure $n+1+n = 2n+1$ real dims. The partial unitary amounts to $h_1, \dots, h_n$ $\underline{2n \text{ dim}}$ + unit bdry cond. $|h_{n+1}|=1$, again $2n+1$ real dims. Moments depend only on the p.u.,

Review what you know about the good situation.

$$Y = aX \oplus V_+ = V_- \oplus bX \qquad V_\pm = \mathbb{C}\xi_\pm \quad \text{unit v.}$$

eigenv. eqn.
$$(az-b)x = -v_+ + v_-$$

solution anal. for $|z|<1$
$$x = (1-zb^*a)^{-1}b^*v_+ = b^*(1-zab^*)^{-1}v_+$$
$$v_- = (1-bb^*)(1-zab^*)^{-1}v_+$$

refinement
$$(az-b)x = -y + \tilde{y}(z)\xi_-$$
$$x = b^*(1-zab^*)^{-1}y$$
$$\tilde{y}(z) = \xi_-^*(1-zab^*)^{-1}y.$$

$$y = \xi_- \qquad \underbrace{\tilde{\xi}_-(z)} = \xi_-^*(1-zab^*)^{-1}\xi_- = \xi_-^*\xi_- = 1$$

$$y = \xi_+ \qquad \tilde{\xi}_+(z) = \xi_-^*(1-z^{-1}ab^*)^{-1}\xi_+ = S(z).$$

$$zax + v_+ = bx + v_-$$
$$|z|^2\|x\|^2 + \|v_+\|^2 = \|x\|^2 + \|v_-\|^2$$
$$(1-|z|^2)\|x\|^2 = \|v_+\|^2 - \|v_-\|^2$$

so if x is defined up to $|z|=1$.

Unitary embedding $\qquad \tilde{y}(z) = \sum_{n\ge0} z^n \xi_-^* c^{*n} y$

$$\|\tilde{y}\|^2 = \sum_{n\ge0} \xi_-^* c^{*n} y \cdot \xi_-^* c^{*n} y \qquad \text{etc.}$$

405 ~~tentatively~~ Got $Y \hookrightarrow H^+ \cap zS(z) H^- \xrightarrow{\cdot q_n} q_n H^+ \cap z p_n H^-$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^+ \cap z^{n+1} H^- = P_n$

so $Y \simeq P_n \frac{1}{q_n}$ $\qquad X \simeq P_{n-1} \frac{1}{q}$

get vector $\frac{1}{q_n}$ such that $z^j \frac{1}{q_n}$ $0 \le j \le n$ is basis for Y.

Where are you? Go back to $d\mu$. $\int d\mu = 1$.

Form. $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$ $\qquad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\|p_n\|^2 = \prod_{j=1}^{n} (1 - |h_j|^2)$$

$p_1 = z p_0 + h_1 q_0$

$\|p_1\|^2 + |h_1|^2 \|p_0\|^2 = 1.$

$p_n = (z^n + P_{n-1}) \cap P_{n-1}^\perp$ $\qquad \tilde{p}_n = \xi_+$ $\qquad S_n = \frac{p_n}{q_n}$

$q_n = (1 + z P_{n-1}) \cap z P_{n-1}^\perp$ $\qquad \tilde{q}_n = \xi_-$.

$X = P_{n-1} \quad Y = P_n$ inside $L^2(S^1, d\mu)$

Starting with $d\mu$ you get a sequence of partial unitaries, in fact a sequence of polys. q_n $n \ge 0$

$$\left(P_{n-1}, \| \ \|^2_{d\mu}\right) \xrightarrow[q_n]{\|q_n\| = \frac{1}{q_n}} \quad H^+$$

$$|f(z)|^2 \frac{1}{|\bar{h}_n z p_{n-1} + q_{n-1}|^2} \frac{d\theta}{2\pi}$$

$$f(z)\overline{f(z)} \frac{q_{n-1} z^{-n+1} p_{n-1}}{(\bar{h}_n z p_{n-1} + q_{n-1})(h_n z^{-n} q_{n-1} + z^{-n+1} p_{n-1})} \frac{dz}{2\pi i z} \frac{1}{|q_{n-1}|^2}$$

Basically you need to be able to do this step. Maybe it involves quasi-determinants

Go back to $Y = aX \oplus V_+^* = bX \oplus V_-^*$ $\quad V_\pm = \mathbb{C}\xi_\pm$

$$c = ba^* + \xi_- h \xi_+^*$$

$$c^* = ab^* + \xi_+ \bar{h} \xi_-^*$$

$$\xi_-^* \frac{1}{1-zc^*} = \xi_-^* \frac{1}{1-zab^* - z\Delta c^*} = \xi_-^*\left(G_0 + G_0 z\Delta c^* G_0 + \cdots\right)$$

$$z\xi_+\bar{h}\xi_-^*$$

$$\xi_-^* G_0 + \underbrace{(\xi_-^* G_0 \xi_+ z)\bar{h}}_{S(z)z\bar{h}}\xi_-^* G_0 + \cdots$$

$$\xi_-^* \frac{1}{1-zc^*} y = \frac{1}{1-S(z)z\bar{h}} \xi_-^* G_0 y$$

$$1 - S(z)z\bar{h} = 1 - \frac{p}{q} z\bar{h} = \frac{q - \bar{h}zp}{q}$$

so you have

$$\xi_-^* \frac{1}{1-zc^*} y = \frac{q}{q-\bar{h}zp}\,\xi_-^* \frac{1}{1-zab^*} y \qquad \text{OKAY}$$

$$(1-cc^*)^{1/2} = (\overbrace{1-bb^*}^{\xi_-\xi_-^*} - \xi_-|h|^2\xi_-^*)^{1/2} = (1-|h|^2)^{1/2}$$

$$(1-|h|^2)^{1/2}\xi_-^* \frac{1}{1-zc^*} y = \underbrace{\frac{q(1-|h|^2)^{1/2}}{q-\bar{h}zp}}_{\tilde{q}_n/q_{n+1}}\,\xi_-^* \frac{1}{1-zab^*} y$$

it works, but it is pretty hard to make clear.

Problems: 1) When $q_\infty\ \exists$ i.e. $h_n \in \ell^2$

then does $q_\infty(z) = \det(1-zc^*)$ in some Hilb-Schm. sense?

~~also check~~ ??

Given (H, u, ξ) take $\xi_+ = \xi$, $\xi_- = u(\xi)$

$H = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$. You now have a partial unitary; what's S ??

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What's S???

$$u = ba^* + \xi_- \xi_+^*$$

$$(za-b)x = -y + \tilde{y}(z)\xi_-$$

$$x = (1-zb^*a)^{-1}b^*y = b^*(1-zab^*)^{-1}y$$

$$\tilde{y}(z) = \xi_-^*(1-zab^*)^{-1}y$$

$$c = ba^* + \xi_- t \xi_+^*$$

$$c^* = ab^* + \xi_+ \bar{t} \xi_-^*$$

$$c^* - \xi_+ \bar{t} \xi_-^* = ab^*$$

$$S(z) = \xi_-^*(1 - zc^* + z\xi_+ \bar{t} \xi_-^*)^{-1}\xi_- \quad ?$$

take $0 \le t \le 1$. You want to perturb the unitary

$$c_t = u - \xi_- t \xi_+^*$$

$$c_t = ba^* + \xi_-(1-t)\xi_+^*$$

$$c_t^* = u^{-1} - \xi_+ t \xi_-^*$$

You might find it better to interchange u and u^{-1}

Start again. You have (H, u, ξ) cyclic unit rep of $\mathbb{Z}$. You can form partial unitaries, maybe 2,

$$c_t = u(1 - t\xi\xi^*) = u \text{ if } t=0$$

$$= u(1-\xi\xi^*) \quad t=1$$

$$= u \cdot \text{projection onto } \xi^\perp$$

$$= u\, aa^* \quad \text{if } Ker(a^*) = \mathbb{C}\xi$$

$$u\xi\xi^* = \xi_-\xi_+^*$$

$$\frac{1}{1-zc_t^*}$$

rank 1 projector

$$c = u -$$

408 So basically you take perturbation $u + tue$, e rank 1 projection.

$e = \xi \xi^*$

$ue = u(\xi)\xi^*$

$$\frac{1}{1 - z(u+tue)^*} = \frac{1}{1 - zu^{-1} - ztue}$$

$$= G + GztueG + \cdots$$

In order to calculate you probably need ξ^* on the front.

You want to perturb $\frac{1}{1-zu^*}$

$$\frac{1}{1-zu^* - z\Delta u^*} = \frac{1}{1-zu^*} + \frac{1}{1-zu^*} z\Delta u^* \frac{1}{1-zu^*} + \cdots$$

$$\Delta u^* = \xi \eta^*$$

$$\eta^* \frac{1}{1-zu^* - z\Delta u^*} = \eta^* \frac{1}{1-zu} + \left(\eta^* \frac{1}{1-zu} \xi z\right) \eta^* \frac{1}{1-zu}$$

$$= \frac{1}{1 - \left(\eta^* \frac{1}{1-zu}\xi\right) z} \eta^* \frac{1}{1-zu}$$

It should be simple.

$$c = ba^* + \xi_- h \xi_+^*$$

$$c^* = ab^* + \xi_+ \bar{h} \xi_-^*$$

$$cc^* = bb^* + \xi_- |h|^2 \xi_+^*$$

$$1 - cc^* = 1 - bb^* - \xi_- |h|^2 \xi_+^*$$

$$= \xi_- (1 - |h|^2) \xi_-^*$$

$$\xi_-^* \frac{1}{1 - zc^*} = \xi_-^* \frac{1}{1 - zab^* - z\xi_+ \bar{h} \xi_-^*} = \cancel{\xi_-^* \frac{1}{1-zc^*} z \xi_+ \bar{h} \xi_-^* z}$$

$$= \xi_-^* \frac{1}{1-zab^*} + \left(\xi_-^* \frac{1}{1-zab^*} \xi_+ \bar{h}\right) \xi_-^* \frac{1}{1-zab^*} + \left(\xi_-^* \frac{1}{1-zab} \xi_+ z \bar{h}\right)\left(\xi_-^* \frac{1}{1-zab^*} \xi_+ z\right) \xi_-^* \frac{1}{1-zab^*}$$

$$= \frac{1}{1 - \xi_-^* \frac{1}{1-zab^*} \xi_+ z \bar{h}} \xi_-^* \frac{1}{1-zab^*}$$

$$S(z) = \xi_-^* \frac{1}{1-zab^*} \xi_+$$

$$\xi_-^* \frac{1}{1-zc^*} = \frac{1}{1-S(z)zh} \xi_-^* \frac{1}{1-zab^*}$$

With luck you might be able to interchange c, c_0

$c_0 = ba^*$ $\quad c_0^* = ab^*$

$$c = c_0 + \underbrace{\xi_+ \bar{h} \xi_-^*}_{\Delta c}$$

$\xi_+ \bar{h} \xi_-^*$

$$\xi_-^* \frac{1}{1-zc^*} = \xi_-^* \frac{1}{1-zc_0^* - z\Delta c^*} = \xi_-^* G_0 + \xi_-^* G_0 z \Delta c^* G_0 + \xi_-^* G_0 z \Delta c^* G_0 z \Delta c^* G_0 + \cdots$$

$$\xi_-^* \frac{1}{1-zc^*} = \frac{1}{1-\xi_-^* \frac{1}{1-zc_0^*} \xi_+ z\bar{h}} \xi_-^* \frac{1}{1-zc_0^*}$$

$$\xi_-^* \frac{1}{1-zc_0^*} = \xi_-^* \frac{1}{1-zc^* + z\Delta c^*}$$

$$= \xi_-^* G + \xi_-^* G(-z\Delta c^*) G + \cdots$$

$$= \frac{1}{1+\xi_-^* \left(\frac{1}{1-zc^*}\right) \xi_+ z\bar{h}} \xi_-^* \frac{1}{1-zc^*}$$

How is this supposed to help?

It should be easy! $\qquad Y = L^2(S^1, d\mu)$

$aX \oplus V^+$ $\qquad u(\xi_+) = \xi_-$

$V^- \oplus bX$ $\qquad \xi_+ = S$

~~Review~~ Review ~~what~~ partial unitary + boundary condition

$Y = aX \oplus V^+ = V^- \oplus bX$

$y = ax_1 + v_+ = v_- + bx$

$u(y) = bx_1 + u(v_+) = zv_- + zbx$

amounts to $x_1 = zX$ $\quad u(v_+) = zv_-$ and $(za-b)x = -v_+ + v_-$

$S(z)v_+ = v_-$

410 This should be simple, but you want ~~to~~ relate it to the Pick function. You should be very close,

$$W=\begin{pmatrix}a\\b\end{pmatrix}X \subset W^\circ=\begin{pmatrix}a\\b\end{pmatrix}X\oplus\begin{pmatrix}V^+\\ \oplus\\ V^-\end{pmatrix} \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix} \supset \begin{pmatrix}1\\z\end{pmatrix}Y$$

$$V=\begin{pmatrix}1\\c\end{pmatrix}Y$$

$$ca=b \qquad a^*=b^*c$$
$$c^*b=a \qquad c^*c\,a=c^*b=a$$
$$cc^*b=b$$

$$Y \xrightarrow{\begin{pmatrix}1\\c\end{pmatrix}} \begin{matrix}Y\\ \oplus\\ Y\end{matrix} \xrightarrow{(z\ \ -1)} Y$$

so $$y \longmapsto \begin{pmatrix}1\\c\end{pmatrix}(z-c)^{-1}y \longmapsto \begin{pmatrix}\pi_+\\ \pi_- c\end{pmatrix}\frac{1}{z-c}\,y \in \begin{matrix}V_+^*\\ \oplus\\ V_-^*\end{matrix}$$

$c=ba^*$
$$\pi_+\frac{1}{z-c}\,y=(1-aa^*)\frac{1}{z-ba^*}\,y$$

$$\pi_- c=(1-bb^*)c=(1-bb^*)ba^*=0.$$

In general
$$c=ba^*+\pi_- h\,\pi_+ \qquad \pi_- c=\pi_- h\,\pi_+$$

$$\begin{pmatrix}\pi_+\\ \pi_- c\end{pmatrix}\frac{1}{z-c}\,y = \begin{matrix}(1-aa^*)\dfrac{1}{z-c}\,y\\ h(1-bb^*)\dfrac{1}{z-c}\,y\end{matrix}$$

Roughly what's happening is that you are computing a map from Y to the intrinsic ~~Hilbert space~~ Hardy space of holomorphic sections of $\mathcal{O}(-1)$ over the disk.

~~Same thing~~

How to handle a measure? What you know. In finite dimensions you get eigenvector expansion from residues of the resolvent. Contour integral of the resolvent

$$\eta=\frac{1}{2\pi i}\oint\frac{1}{z-u}\,\eta\,dz$$

$$=\sum_{|\zeta|=1}\underbrace{\xi_\zeta\,\xi_\zeta^*}_{\pi_\zeta}\,\eta$$

4 11 I think that you might want to the $r \uparrow 1$ limit. Replace u, u^{-1} by $ru; ru^{-1}$

Let's proceed in a straightforward fashion

Suppose given $Y = aX \oplus V_+ = V_- \oplus bX$ $\quad V_\pm = \mathbb{C}\xi_\pm$

$|\xi_\pm| = 1.$ $\quad u = ba^* + \xi_- \xi_+^*$

Eigenvector equation for u: $\quad (z-u)\eta = 0$

$\eta = ax_1 + v_+ = v_- + bx$ $\quad u(\eta) = bx_1 + u(v_+)$

$\quad z\eta = zbx + zv_-$

$\therefore x_1 = zx.$ $\quad \therefore$ $\boxed{\begin{aligned}&(1)\ (za-b)x = -v_+ + v_- \\ &(2)\quad zv_- = uv_+\end{aligned}}$ $\leftarrow$ this is a translation of $(z-u)\eta = 0$

~~I solve~~ Solving (1) yields if $v_+ = \xi_+$, then

$v_- = \xi_- S(z)$ $\quad$ in general $\quad \boxed{(3)\ v_- = \xi_- S(z)\xi_+^* v_+}$

where $S(z) = \xi_-^*(1 - zab^*)^{-1}\xi_+$. Combine (3), (2)

to get $\quad z\xi_- S(z)\xi_+^* v_+ = uv_+ = \xi_- \xi_+^* v_+$

$\boxed{zS(z) = 1}$ gives eigenvalues for u.

You want to go further, say a spectral representation for the elements of Y. You want to associate to each $y \in Y$ a transform $\tilde{y}(z)$. Resolvent of u?

$(z-u)\eta = y$ $\quad$ Inhomogeneous equation

$\eta = ax_1 + v_+ = bx + v_-$

$(z-u)\eta = zbx + zv_- - bx_1 - \overbrace{u(v_+)}^{\xi_-\xi_+^* v_+} \overset{?}{=} y$ $\quad zv_- - u(v_+) = (1-bb^*)y$

~~impose condition $zx = x_1$~~ $\Rightarrow zX - X_1 = b^* y$

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$$a(zx - b^*y) + v_+ = bx + v_-$$

$$zv_- - \xi_-\xi_+^* v_+ = (1-bb^*)y = \xi_-\xi_-^* y$$

$$(az-b)x = ab^*y - v_+ + v_-$$

$$-(b^*az-1)x = -b^*ab^*y + b^*v_+$$

$$x = \frac{1}{1-zb^*a}(b^*v_+ - b^*ab^*y)$$

$$\boxed{x = b^*\frac{1}{1-zab^*}(v_+ - ab^*y)}$$

$$z\xi_-^* v_- - \xi_+^* v_+ = \xi_-^* y$$

$Y = aX \oplus V_+ = bX \oplus V_-$ $\qquad V_\pm = \mathbb{C}\xi_\pm, \quad \|\xi_\pm\| = 1.$

$$c = ba^* + \xi_- h \xi_+^*$$
$$c^* = ab^* + \xi_+ \bar{h} \xi_-^*$$

$$V = \begin{pmatrix}1\\c\end{pmatrix} Y = \begin{pmatrix}a\\b\end{pmatrix}X \oplus \begin{pmatrix}1\\c\end{pmatrix}V^+ \hookrightarrow \begin{matrix}Y\\ \oplus\\ Y\end{matrix} \xrightarrow{(z\ \ -1)} Y$$

$$\begin{pmatrix}\xi_+\\ h\xi_-\end{pmatrix}\mathbb{C}$$

$$\begin{pmatrix}1\\c\end{pmatrix}\frac{1}{z-c}y \longleftarrow\!\!\shortmid\ y$$

$$\downarrow$$

$$\begin{pmatrix}\xi_+\xi_+^*\\ h\xi_-\xi_+^*\end{pmatrix}\frac{1}{z-c}y$$

$$\parallel$$

$$\begin{pmatrix}\xi_+\\ h\xi_-\end{pmatrix}\xi_+^*\frac{1}{z-c}y$$

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413 Note the pseudo norm squared of $\begin{pmatrix}\xi_+ \\ h\xi_-\end{pmatrix}\xi_+^*(z-c)^{-1}y$ is $(1-|h|^2)|\xi_+^*(z-c)^{-1}y|^2$, which checks with $(1-c^*c)^{1/2}(z-c)^{-1}y$ since

$$c^*c = aa^* + \xi_+|h|^2\xi_+^*$$
$$(1-c^*c)^{1/2} = \xi_+(1-|h|^2)^{1/2}\xi_+^*$$

So you should have an isometric embedding of y into H^-. Now you need to understand what happens as $|h| \to 1$. Basically $\xi_+^*(z-c)^{-1}y$ is a rational function of z analytic for $|z|>1$ and vanishing at $z=\infty$. (Assume $\dim Y<\infty$)

Eigenvector $\quad \eta = ax_1 + v_+ = bx + v_-$

$$(z-u)\eta = zbx + zv_- - bx_1 - \xi_- h\xi_+^* v_+ = y$$

$$\Longrightarrow \begin{cases} zx - x_1 = b^*y \\ zv_- - \xi_- h\xi_+^* v_+ = (1-bb^*)y \end{cases}$$

$$\xi_+^*\eta = \xi_+^* v_+$$

$$a(zx - b^*y) + v^+ = bx + v_-$$

$$(az-b)x = ab^*y - v_+ + v_-$$

$$\cancel{(z-a^*b)x = b^*y - b^*v + \cdots = b^*(y+v_-)}$$

$$(z-a^*b)x = b^*y + a^*v_-$$

$$x = (z-a^*b)^{-1}b^*y + (z-a^*b)^{-1}a^*v_-$$

414 Review: You have

$$Y = aX \oplus V_+ = bX \oplus V_- \qquad V_\pm = \mathbb{C}\xi_\pm ,\ \|\xi_\pm\| = 1$$

$$c = ba^* + \xi_- h \xi_+^* \qquad |h| \le 1$$

$$V = \begin{pmatrix}1\\c\end{pmatrix} Y = \begin{pmatrix}a\\b\end{pmatrix} X + \begin{pmatrix}\xi_+\\h\xi_-\end{pmatrix}\mathbb{C} \xrightarrow[\sim]{(z\ \ -1)} Y$$

$$\begin{pmatrix}1\\c\end{pmatrix}(z-c)^{-1} y \longleftarrow y$$

$$\downarrow$$

$$\begin{pmatrix}\xi_+\\h\xi_-\end{pmatrix} \xi_+^*(z-c)^{-1} y$$

rational function of z
~~analytic outside~~
poles inside D vanishes at ∞

You want somehow to understand $h \uparrow 1$.

$$(z-c)\eta = z(ax + v_+) - (bx + \xi_- h \xi_+^* v_+) = y$$

$$(za - b)x + (z\xi_+ - \xi_- h)\xi_+^* v_+ = y$$

Use ~~geometric~~ perturb. exp.

$$c = c_0 + \Delta c = ba^* + \xi_- h \xi_+^*$$

$$\xi_+^* \frac{1}{z-c} = \xi_+^* \frac{1}{z-c_0} + \xi_+^* \frac{1}{z-c_0}\underbrace{\Delta c}_{\xi_- h \xi_+^*}\frac{1}{z-c_0} +$$

$$= \underbrace{\left(1 + Sh + (Sh)^2 + \cdots\right)}_{\frac{1}{1-Sh}} \xi_+^* \frac{1}{z-c_0}$$

$$S = \xi_+^* \frac{1}{z - ba^*} \xi_-$$

415 At this point you need an example.

$$\xi_+^* \frac{1}{1-\bar{z}c} \xi_- = \xi_+^* \frac{z}{z-c} \xi_- = z\, S(z^{-1})$$

Example. $n=0$ $X=0$, $Y=\mathbb{C}$ $\xi_+ = \xi_-$

$c = h.$

$$S_-(z) = \xi_+^* \frac{1}{z-c} \xi_- = \frac{1}{z-h}$$

$$y \longmapsto \xi_+^* \frac{1}{z-c} y = \frac{y}{z-h} \qquad y \in \mathbb{C}.$$

$$\int \frac{|y|^2}{|\zeta-h|^2} \frac{d\theta}{2\pi} = \int \frac{|y|^2}{|1-h\zeta^{-1}|^2} \frac{d\theta}{2\pi} = \frac{|y|^2}{1-|h|^2}$$

$$\int \frac{1-|z|^2}{|1-z\zeta^{-1}|^2} \frac{d\theta}{2\pi} = 1$$

$$\int \frac{1}{(\zeta-z)(1-\bar{z}\zeta)} \frac{d\zeta}{2\pi i} = \frac{1}{1-\bar{z}z}$$

$n=1.$

Review. Start with an S say Blaschke prod.

$Y = H^+ \cap S(z) z H^-$ $\qquad X = H^+ \cap S H^-$

Given $d\mu$ you can form $\frac{1}{1-\bar{z}\zeta}$ inside $H^2(d\mu)$

$$\int \frac{1}{1-\bar{z}\zeta} \frac{1-z\bar{w}}{1-\bar{w}\zeta} d\mu = \int \frac{1}{1-z\zeta^{-1}} (1-\bar{w}\zeta + \bar{w}\zeta - z\bar{w}) \frac{1}{1-\bar{w}\zeta}$$

$$= \int \left(\frac{1}{1-z\zeta^{-1}} + \frac{\bar{w}\zeta}{1-\bar{w}\zeta} \right) d\mu = \int \left(\frac{1-\frac{1}{2}}{1-z\zeta^{-1}} + \frac{1}{1-\bar{w}\zeta} - \frac{1}{2} \right)$$

$$= \frac{1}{2i} \int i \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} + i \frac{1+\bar{w}\zeta}{1-\bar{w}\zeta} \, d\mu = \frac{f(z) - \overline{f(w)}}{2i}$$

416 Fastest way to see this

$$(1-z\bar{w})\int \frac{1}{1-\bar{z}\zeta}\frac{1}{1-\bar{w}\zeta}\,d\mu = (1-z\bar{w})\underbrace{\sum_{\substack{n\geq 0\\ p\geq 0}} \bar{z}^{n}\bar{w}^{p}\int \zeta^{-n+p}\,d\mu}_{\sum_{n>0} z^{n}\zeta^{-n}+1+\sum_{p>0}\bar{w}^{p}\zeta^{p}}$$

Discuss the problem. You have a p.u. and need to understand divisors. Go over the problem. There are two things to relate. First is a measure $d\mu$ on the circle. Foundation.

~~Given H, u, ξ and a vector $\xi \neq 0$ ...~~

Given (H,u,ξ), equiv. $d\mu$ on S^1, equiv. Pick fcn, equiv pos. harm. fcn. (H,u,ξ) equiv to a p.u. with unitary bdry condition, so there is a whole circle of related measures which might be interesting. Anyway

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Discussion. ~~You need to do~~ The problem: To work out the equivalence between

1) (H,u,ξ) cyclic unit rep of $\mathbb{Z}$
2) $d\mu$
3) positive harmonic fn. on D
4) Pick function $F(z)$ on D mod add. real const.
5) partial unitary $Y = aX \oplus V_+ = V_- \oplus bX$, dim $V_\pm = 1$ having no bound states together with unit. boundary condition $V_+ \xrightarrow{\sim} V_-$
6) $S(z)$ anal. function on D bdd by 1
7) Schur sequence $h_1, \ldots$ either inf. in D or finite with last in S^1 earlier ones in D.

Main technical difficulty is to relate Pick function F to S.

It's likely that difficulty arises because you are not comfortable with non unitary S.

417 Today you must seriously tackle non unitary S

Start with $L^2(S^1, d\mu)$, where $d\mu = \rho \frac{d\theta}{2\pi}$

Szegö ~~situation~~. Choose $H^2(S^1, d\mu) \ominus z H^2(S^1, d\mu) \ni g$

~~so~~ $\|g\| = 1$, $|g|^2 = 1$ assuming

$\int \rho \frac{d\theta}{2\pi} = 1$. Let's go the other way. Given $\rho > 0$

smooth enough, form $\log \rho = \sum c_n z^n$, put

$g(z) = \frac{c_0}{2} + \sum_{n \geq 0} c_n z^n$ so $\log \rho = g(z) + \overline{g(z)}$

$\rho = |e^g|^2$ $\rho \frac{d\theta}{2\pi}$

Start with $\rho(z) > 0$ suff smooth.

$\log(\rho) = \sum_{n \in \mathbb{Z}} c_n z^n = g(z) + \overline{g(z)}$ $g(z) = \frac{c_0}{2} + \sum_{n \geq 1} c_n z^n$

$\rho = |e^g|^2$ set $g = e^{-g}$ $\frac{1}{|g|^2} = \rho$.

Then you get $\rho \frac{d\theta}{2\pi} = \left|\frac{1}{g}\right|^2 \frac{d\theta}{2\pi}$ $\int |g|^2 \frac{1}{|g|^2} \frac{d\theta}{2\pi} = 1$

$$L^2(S^1) \xrightarrow{\sim} L^2\left(S^1, \rho \frac{d\theta}{2\pi}\right) \xleftarrow{\sim} L^2(S^1)$$

$\frac{f}{g}$ f

$f \longmapsto \frac{f\bar{g}}{\cdots}$ leads to a scattering situation with $S = \frac{\bar{g}}{g} = e^{g - \bar{g}} = e^{2i\,\mathrm{Im}(g)}$

But this S is not analytic

You now want to calculate S for the partial unitary you get by removing 1 from the domain of u. You also want the Pick function associated to $d\mu = \frac{1}{|g|^2} \frac{d\theta}{2\pi}$.

418 ~~So? What??~~

$$\int_0^{2\pi} \frac{1}{1 - z\zeta^{-1}} e^{g(\zeta)} e^{\overline{g(\zeta)}} \frac{d\theta}{2\pi}$$

when might you be able to evaluate this? $e^{g(z)} = \frac{1}{g(z)}$ is analytic for $|z| \leq 1$. You would like $e^{\overline{g(\zeta)}} = e^{\overline{g(\bar\zeta^{-1})}}$ to extend to an analytic function $\frac{1}{\overline{g(\bar z^{-1})}}$ inside the disk. What is going to happen is that you get singularities. ~~Aim for~~ Look for simple examples. ~~You expect~~ We can choose $e^g = \frac{1}{g}$ simplest appear to be to take $e^g = \frac{1}{g} = z - \bar a^{-1}$ $|a| < 1$.

then $\frac{1}{\overline{g(\bar z^{-1})}} = z^{-1} - a$

$$\frac{1}{g(z)} = z - \alpha \qquad |\alpha| > 1 \qquad \frac{1}{\overline{g(\bar z^{-1})}} = z^{-1} - \bar\alpha$$

sing at $z = 0$

$$\int \frac{1}{\zeta - z} (\zeta - \alpha)(\zeta^{-1} - \bar\alpha) \frac{d\zeta}{2\pi i}$$

$$= \int \frac{(\zeta - \alpha)(1 - \bar\alpha\zeta)}{\zeta(\zeta - z)} \frac{d\zeta}{2\pi i} = \mathrm{Res}_0 + \mathrm{Res}_z$$

$z - \alpha - \bar\alpha z^2 + |\alpha|^2 z$

$$= \frac{\alpha}{-z} + \frac{(z-\alpha)(1-\bar\alpha z)}{z} = \frac{\alpha + (z-\alpha)(1-\bar\alpha z)}{z}$$

$$= \frac{z - \bar\alpha z^2 + |\alpha|^2 z}{z} = 1 - \bar\alpha z + |\alpha|^2$$

419 $\displaystyle\int i \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} (\zeta-\alpha)(\zeta^{-1}-\bar\alpha) \frac{d\zeta}{2\pi i \zeta}$

$$= \int i 2 \frac{(\zeta-\alpha)(\zeta^{-1}-\bar\alpha)}{1-z\zeta^{-1} \quad \zeta} \frac{d\zeta}{2\pi i} - \int i \frac{(\zeta-\alpha)(\zeta^{-1}-\bar\alpha)}{\zeta} \frac{d\zeta}{2\pi i}$$

$$= \int 2i \frac{(\zeta-\alpha)(1-\bar\alpha\zeta)}{(\zeta-z)\,\zeta} \frac{d\zeta}{2\pi i} - \int i \frac{\overbrace{(\zeta-\alpha)(1-\bar\alpha\zeta)}^{\zeta-\alpha+|\alpha|^2\zeta-\alpha\zeta^2}}{\zeta^2} \frac{d\zeta}{2\pi i}$$

$$= 2i \underbrace{\left\{ \frac{\cancel{+\alpha}}{\cancel{+z}} + \frac{\overbrace{(z-\alpha)(1-\bar\alpha z)}^{z-\cancel{\alpha}-\bar\alpha z^2+|\alpha|^2 z}}{z} \right\}}_{1+|\alpha|^2-\bar\alpha z} - i\left\{ 1+|\alpha|^2 \right\}$$

$$= i\left(1+|\alpha|^2-\bar\alpha z\right)$$

$$e^g = \frac{1}{g} = \frac{1}{z-\alpha}$$

$$\int i \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} \frac{1}{(\zeta-\alpha)(\zeta^{-1}-\bar\alpha)} \frac{d\zeta}{2\pi i\zeta}$$

$$= \int i \frac{(\zeta+z)}{(\zeta-z)(\zeta-\alpha)(1-\bar\alpha\zeta)} \frac{d\zeta}{2\pi i} = \mathrm{Res}_z + \mathrm{Res}_{\frac{1}{\bar\alpha}}$$

$$= i\left\{\frac{2z}{(z-\alpha)(1-\bar\alpha z)}\right\} + i\left\{\underbrace{\frac{\left(\frac{1}{\bar\alpha}+z\right)}{\left(\frac{1}{\bar\alpha}-z\right)\left(\frac{1}{\bar\alpha}-\alpha\right)(-\bar\alpha)}}_{\frac{(1+\bar\alpha z)}{(1-\bar\alpha z)(|\alpha|^2-1)}}\right\}$$

$$= i \frac{1}{(z-\alpha)(1-\bar\alpha z)} \Big\{ \underbrace{2z + \frac{(z-\alpha)(1+\bar\alpha z)}{|\alpha|^2-1}}_{\frac{(z+\alpha)(-1+\bar\alpha z)}{|\alpha|^2-1}} \Big\} = i\frac{z+\alpha}{z-\alpha}\frac{-1}{|\alpha|^2-1}$$

$$= i\frac{\alpha+z}{\alpha-z}\frac{1}{|\alpha|^2-1}$$

420 Let S be a Blaschke product of degree n.

* First discuss what you can do.

Suppose given $h_1, h_2, \dots, h_n, \dots$ in D with $h_n = 0$ for $n >> 0$.

I have run into: Question: Let $S : D \to D$ be rational. Does the Schur expansion of S have to be finite.

$$\frac{2z+2-2}{4-\cancel{z}-1} = \frac{2z}{3-z}$$

$$S(z) = \frac{z+1}{2} \qquad S(0) = \frac{1}{2} \qquad \frac{\frac{z+1}{2} - \frac{1}{2}}{1 - \frac{1}{2}\left(\frac{z+1}{2}\right)} = \frac{z}{\frac{3}{2} - \frac{z}{2}}$$

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} S = \frac{S(z) - \frac{1}{2}}{1 - \frac{1}{2} S(z)} = z S_1(z) \qquad \| \qquad \frac{2z}{3-z}$$

mistake

$$S_1(z) = \frac{1}{3-z} \qquad \frac{\frac{1}{3-z} - \frac{1}{3}}{1 - \frac{1}{3}\frac{1}{3-z}} = \frac{3-(3-z)}{9-3z-1} = \frac{z}{8-3z}$$

$$S_2(z) = \frac{1}{8-3z} \qquad \frac{\frac{1}{8-3z} - \frac{1}{8}}{1 - \frac{1}{8}\frac{1}{8-3z}} = \frac{8-(8-3z)}{64-24z-1} = \frac{3z}{63-24z}$$

$$S_3(z) = \frac{1}{21-8z} \qquad \frac{\frac{1}{21-8z} - \frac{1}{21}}{1 - \frac{1}{21}\frac{1}{21-8z}} = \frac{21-(21-8z)}{441-168z-1} = \frac{8z}{440-168z} \qquad \|$$

$$S_4(z) = \frac{1}{55-21z} \qquad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,$$

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$$S_0(z) = \frac{z+1}{2}$$

$$\frac{\frac{z+1}{2} - \frac{1}{2}}{1 - \frac{1}{2}\frac{z+1}{2}} = \frac{2(z+1)-2}{4-(z+1)} = \frac{2z}{3-z}$$

$$S_1(z) = \frac{2}{3-z} \qquad \frac{\frac{2}{3-z} - \frac{2}{3}}{1 - \frac{2}{3}\frac{2}{3-z}} = \frac{6-2(3-z)}{9-3z-4} = \frac{2z}{5-3z}$$

$$S_2(z) = \frac{2}{5-3z} \qquad \frac{\frac{2}{5-3z} - \frac{2}{5}}{1 - \frac{2}{5}\frac{2}{5-3z}} = \frac{10-2(5-3z)}{25-15z-4} = \frac{6z}{21-15z}$$

Conclude $\quad p_n = zp_{n-1} + h_n q_{n-1} \qquad S(z) = \frac{p_n}{q_n} \qquad S(0) = h_n$

$\quad q_n = \bar{h}_n z p_{n-1} + q_{n-1}$

$$\frac{S(z) - h_n}{1 - \bar{h}_n S(z)} = \frac{p_n - h_n q_n}{q_n - \bar{h}_n p_n} = \frac{zp_{n-1} - |h_n|^2 z p_{n-1}}{(1-|h_n|^2)q_{n-1}}$$

$q_n = \bar{h}_n z p_{n-1} + q_{n-1}$

$p_n = zp_{n-1} + h_n q_{n-1}$

$\underbrace{q_n - \bar{h}_n z p_{n-1}}_{} - |h_n|^2 q_{n-1}$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$S_n = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} (z S_{n-1})$$

focus on what you know best. Start with $d\mu$ construct p_n, q_n, h_n, $S_n = \frac{p_n}{q_n}$. Fix level n. Then $Y_n = P_n$ inside $L^2(S^1, d\mu)$ has the inner product $\int \frac{|f|^2}{|q_n|^2} \frac{d\theta}{2\pi}$. You did not succeed in proving this directly yet.

Given $\quad Y = aX \oplus V_+ = bX \oplus V_-, \quad V_\pm = \mathbb{C}\xi_\pm, \quad \|\xi_\pm\| = 1$

get isom. embedding $\quad Y \hookrightarrow H^+$

$c^*c = (ba^*)^*(ba^*) = aa^*$ $\qquad y \longmapsto \xi_-^* \frac{1}{1-zab^*} y$

get $\quad Y \xrightarrow{\sim} H^+ \cap S(z) z H^-$

422 What should be the approach.

Posiblity 1. Take S Blashke product of degree n. Construct a measure on S^1 associated to S.

Given $d\mu$ construct h_n, p_n, q_n $\qquad$ h_k for $k \leq n$ depends only on μ_k for $k \leq n$.

$S = \frac{p_n}{q_n}$. $\quad P_n \hookrightarrow H^2$? $\quad$ My approach amounts to looking ~~at~~ $P_{n-1} \xrightarrow[z]{} P_n$ with $d\mu$-scalar product as a partial unitary, with $\xi_+ = \tilde{p}_n$, $\xi_- = \tilde{q}_n$ then using the embedding from scattering theory.

You ought to do this several times.

$Y = aX \oplus V_+ = bX \oplus V_-$ $\qquad V_\pm = \mathbb{C}\xi_\pm \quad \|\xi_\pm\| = 1.$

idea for gluing $L^2 \otimes V_-$, $L^2 \otimes V_+$ together via an S should you use a Krein space? This might naturally give $\|x\|^2 - \|Sx\|^2$.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y \\ cy \end{pmatrix} = y_1^* y - y_2^* c^* y = (y_1 - c^* y_2)^* y$$

$$\therefore \left(\begin{pmatrix} 1 \\ c \end{pmatrix} Y \right)^0 = \begin{pmatrix} c^* \\ 1 \end{pmatrix} Y$$

Try to fit in Szego stuff. $\qquad \mu_0 = \int 1 \, d\mu$

$d\mu$, $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$ $\qquad p_0 = q_0 = 1 \qquad \tilde{p}_0 = \frac{1}{\mu_0}$

You have a model for p.u. of type $\Theta(n)$.

Given $Y = aX \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus bX$ $\qquad \|\xi_+\| = \|\xi_-\| = 1$

you get map $y \longmapsto \xi(z-$

$(za - b)x = -y + \hat{y}(z)\xi_-$

$(1 - zb^*a)x = b^*y \qquad (1-bb^*)(1-ab^*)^{-1}y$

423 $V = \begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} 0 \\ V_- \end{pmatrix} \hookrightarrow \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \longrightarrow Y$

Wait. $c = ba^* + \xi_- h \xi_+^*$

$c^* = ab^* + \xi_+ \bar{h} \xi_-^*$

$\begin{pmatrix} +z^{-1} \\ 1 \end{pmatrix} Y$ killed by $(1 \ \ -z^{-1})$

$\parallel$

$\begin{pmatrix} 1 \\ z \end{pmatrix} Y$

$$\begin{pmatrix} 1 \\ c \end{pmatrix} Y = \begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} \xi_+ \\ h\xi_- \end{pmatrix} \mathbb{C}$$

$$\begin{pmatrix} c^* \\ 1 \end{pmatrix} Y = \begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} \bar{h}\xi_+ \\ \xi_- \end{pmatrix} \mathbb{C}$$

$\begin{pmatrix} 1 \\ z \end{pmatrix} Y^0 = \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} Y = \begin{pmatrix} 1 \\ \bar{z}^{-1} \end{pmatrix} Y = \begin{pmatrix} 1 \\ z \end{pmatrix} Y$ when $|z| = 1$

$$\begin{pmatrix} c^* \\ 1 \end{pmatrix} Y \hookrightarrow \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \xrightarrow{(-z \ \ 1)} Y$$

$\xi_-^* (1 - zc^*)^{-1} y$ is the embedding

check $c^* = ab^*$ OK

Notice that you took annihilator of $\begin{pmatrix} 1 \\ z \end{pmatrix} Y$ to get $\begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} Y$ which for $|z| = 1$ is $\begin{pmatrix} z^{-1} \\ 1 \end{pmatrix} Y = \begin{pmatrix} 1 \\ z \end{pmatrix} Y$.

Back to Szego stuff. Suppose $d\mu$ such that $h_k = 0$ for $k > n$, i.e. $\frac{p_k}{q_k} = \frac{z^{k-n} p_n}{q_n}$.

Then $\overline{P_\infty} \hookrightarrow H^2$ prob an isom.

$f \longmapsto \frac{f}{q_n}$

$H^2(S^1, d\mu)$

~~Suppose~~

Missing idea? perhaps is determinant interpretation of p_n and q_n. Might lead to quasi-dets

424 Begin with $d\mu$ construct h_k, p_k, g_k ~~$k \geq ...$~~

suppose $h_k = 0$ for $k > n$, whence $p_k = z^{k-n} p_n$

$g_k = g_n$ for $k \geq n$. Then you get sim. emb.

$$\overline{P_\infty} \text{ in } L^2(S^1, d\mu) \hookrightarrow H^+$$

probably an isom

$$f \longmapsto \frac{f}{g_n}$$

You should have

$$P_k \text{ in } L^2(S^1, d\mu) \xrightarrow{\sim} H^+ \cap \overbrace{\frac{p_k}{g_n} z H^-}^{\frac{z^{k+1}H^-}{g_n}}$$

which should lead to $\overline{P}_\infty \xrightarrow{\sim} H^+$. This should give point evaluators for $H^2_+(S^1, d\mu)$. namely

$$\frac{\overline{g(w)}\, g(z)}{1 - \overline{w} z} \longleftarrow \frac{1}{1-\overline{w} z}$$

because

$$\int \overline{\frac{\overline{g(w)}\, g(z)}{1-\overline{w} z}}\, f \frac{1}{|g(z)|^2} \frac{d\theta}{2\pi}$$

$$= \int \frac{g(w)}{z - w} f(z) \frac{1}{g(z)} \frac{dz}{2\pi i}$$

$$= g(w) f(w) \frac{1}{g(w)} = f(w).$$

what about

$$\int \overline{\frac{1}{1-\overline{z}\zeta}} \frac{1}{1-\overline{w}\zeta} \frac{1}{|g(\zeta)|^2} \frac{d\theta}{2\pi}$$

Can you describe the positive harmonic function ~~extens~~ yielding $\frac{1}{|g(\zeta)|^2}$ on the boundary

425 $\quad -\log \frac{1}{|g(\zeta)|^2} = +\log(g(\zeta)) + \overline{\log(g(\zeta))}$

$$= 2\,\mathrm{Re}\log g(\zeta).$$

so

~~basically you have~~

$$g(\zeta) \rightsquigarrow g(z) \qquad \overline{g(\zeta)} = \overline{g(\bar\zeta^{-1})} \rightsquigarrow \overline{g(\bar z^{-1})}$$

so you find that $\frac{1}{|g(\zeta)|^2}$ on S^1 extends

$$g(z) = 1 - \bar h z$$

then $\quad g(z)\,\overline{g(\bar z^{-1})} = (1-\bar h z)(1-h z^{-1})$ $\quad$ no good at $z=0$.

$$L^2(S^1, d\mu) \supset P_n = P_{n-1} \oplus \mathbb{C}\underbrace{\tilde p_n}_{\xi_+} = P_{n-1} \oplus \mathbb{C}\underbrace{\tilde q_n}_{\xi_-}$$

$$(z-u)x = -\xi_+ + S(z)\xi_-$$

interpret as polys to get:

$$(z-\zeta)\,x(\zeta) = -\tilde p_n(\zeta) + S(z)\tilde q_n(\zeta)$$

$$\Rightarrow S(z) = \frac{p_n(z)}{q_n(z)}$$

But now take $\quad y \in P_n$

$$(z-u)x = -y + \hat y(z)\xi_-$$

$$(z-\zeta)x(\zeta) = -y(\zeta) + \hat y(z)\tilde q_n(\zeta) \qquad \therefore \hat y(z) = \frac{y(z)}{\tilde q_n(z)}$$

This is all clear, but how do you see that the embedding is isometric?

$$\frac{p_n(z)q_n(\zeta) - q_n(z)p_n(\zeta)}{z-\zeta}$$

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$$p_n(z)\,q_n(\zeta) - q_n(z)\,p_n(\zeta)$$

$$= \begin{vmatrix} p_n(z) & p_n(\zeta) \\ q_n(z) & q_n(\zeta) \end{vmatrix} = \begin{vmatrix} z p_{n-1}(z) + h_n q_{n-1}(z) & p_n(\zeta) \\ \bar{h}_n z p_{n-1}(z) + q_{n-1}(z) & q_n(\zeta) \end{vmatrix}$$

$$\overset{?}{=} \begin{vmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{vmatrix} \cdot \begin{vmatrix} z p_{n-1}(z) & \zeta p_{n-1}(\zeta) \\ q_{n-1}(z) & q_{n-1}(\zeta) \end{vmatrix}$$

You have to get over this hurdle.

First examine $\quad Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$

$$(za-b)\,x = -y + \hat{y}(z)\,\xi_-$$

$$(1 - z b^* a)\,x = b^* y \qquad x = b^*(1 - z a b^*)^{-1} y$$

$$\hat{y}(z) = \xi_-^*(1 - z \underbrace{ab^*}_{c^*})^{-1} y \qquad c = ba^*$$

$$\int |\hat{y}(z)|^2 \frac{d\theta}{2\pi} = \int \Big(y, \frac{1}{z - c} \underbrace{\xi_- \xi_-^* \frac{1}{1 - zc^*}}_{1 - bb^* = 1 - cc^*} y\Big) \frac{dz}{2\pi i}$$

intuitively use residue calc.

$$\int |\hat{y}(z)|^2 \frac{d\theta}{2\pi} = \int \Big| \underbrace{\xi_-^* \frac{1}{1 - zc^*}}_{\sum z^n \xi_-^* (c^*)^n y} y \Big|^2 \frac{d\theta}{2\pi} \qquad \|y\|^2 - \lim_n \|c^{*n} y\|^2$$

$$\parallel$$

$$= \sum_{n \geq 0} |\xi_-^* (c^*)^n y|^2 = \sum_{n \geq 0} \|c^{*n} y\|^2 - \|c^{*n+1} y\|^2$$

~~So what seems important~~ This is the scattering viewpoint.

427 How can I use this profitably? You can continue with the scattering viewpoint to establish that that $Y \xrightarrow{\sim} H^+ \cap SzH^-$. $S(z) = \xi_-^* \frac{1}{1-zC^*} \xi_+$ and we get $\int |S|^2 \frac{d\theta}{2\pi} = 1$ when $(C^*)^n \xi_+ \to 0$. On the other hand $|S(z)| < 1$ should be for $|z| < 1$.?

$$y = ax_1 + v_+ = bx + v_-$$

$$(z-u)y = zbx + zv_- - bx_1 - u(v_+)$$

$$x_1 = zX.$$

$$zaX + v_+ = bx + v_- \qquad (za-b)x = -v_+ + v_-$$

$$|z|^2 \|x\|^2 + \|v_+\|^2 = \|x\|^2 + \|v_-\|^2$$

$$\|v_+\|^2 - \|v_-\|^2 = (1-|z|^2)\|x\|^2$$

If $v_+ = \xi_+$, then $v_- = S(z)\xi_-$ so $1 - |S(z)|^2 = (1-|z|^2)\|x\|^2 \geq 0.$

So you learn that S is an inner fn.

~~Also~~ All this appears very simple, i.e. the scattering viewpoint yields a lot.

<u>Try to organize it.</u>

Basic idea seems to be this. Given $Y = aX \oplus V^+ = bX \oplus V^-$ then you dilate ~~to get another~~

$$\cdots \oplus u^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{\substack{\| \\ V^- \oplus bX}} \oplus uV^+ \oplus u^2V^+ \oplus \cdots$$

$$L^2(S^1, V^-) \longleftrightarrow H \longleftrightarrow L^2(S^1, V^+)$$

428. Review constructions: Given X, c contraction, get dilation $H, u, \varepsilon : X \longrightarrow H$, $\varepsilon^* u^n \varepsilon = \begin{cases} c^n & n \geq 0 \\ c^{*-n} & n \leq 0 \end{cases}$

H = Completion of $\mathbb{C}[u,u^{-1}] \otimes X$ with

$$(u^k x, u^l x') = (x, \varepsilon^* u^{l-k} \varepsilon\, x')$$

decomposition of H:

$$\cdots \oplus u^{-1} V_- \oplus \varepsilon X \oplus V_+ \oplus u V_+ \oplus \cdots$$

V_+ = completion of X wrt $\|x\|^2 - \|cx\|^2$

V_- = ——————————— $\|x\|^2 - \|c^* x\|^2$

$$\|x_0 + u x_1\|^2 = \|x_0\|^2 + (x_0, c x_1) + (c x_1, x_0) + \|x_1\|^2$$

$$= \|x_0 + c x_1\|^2 + \|x_1\|^2 - \|c x_1\|^2$$

$$\|(u-c) x_1\|^2 = \|x_1\|^2 - \|c x_1\|^2$$

Embeddings $L^2(S^1, V_\pm) \longrightarrow H$ and their adjoints.

$$\|u^{-1} x_1 + x_2\|^2 = \|x_1\|^2 + (c^* x_1, x_2) + (x_2, c^* x_1) + \|x_2\|^2$$

$$= \|c^* x_1 + x_2\|^2 + \|x_1\|^2 - \|c^* x_2\|^2$$

$$\|x_1 - u c^* x_1\|^2 = \|x_1\|^2 - \|c^* x_1\|^2$$

$$V^+ \xrightarrow{\sim} \overline{(u-c)X} \qquad V^- \longrightarrow \overline{(1-uc^*)X}$$

$$\pi_+ x \longmapsto (u-c)x \qquad \pi_- x \longmapsto 1 - c^* x$$

$$(u^n \pi_- x_n, x) = (u^n (1 - uc^*) x_n, x)$$

$$= 0 \qquad n < 0$$

$$= (x_n, (1 - u^{-1} c) c^n x) \qquad n \geq 0$$

$$= (x_n, (1 - c^* c) c^n x) \qquad ''$$

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$$\Big(\sum_n{}' u^n \pi_- x_n , x\Big) = \sum_{n\geq 0} \big(\pi_- x_n , \pi_- c^n x\big)$$

$$= \int \Big(\sum_n{}' \pi_- x_n z^n , \pi_- \sum_{n\geq 0}{}' z^{-n} c^n x\Big) \frac{d\theta}{2\pi}$$

adjoint of $\sum z^n \pi_- x_n \longmapsto \sum u^n \pi_- x_n$ is

$$\pi_- \frac{1}{1-z^{-1}c} x \longleftarrow\!\!| \; x$$

$H \quad \oplus\, u^{-1} V_-^{\#} \oplus X \oplus V_+^{\#} \oplus$

$\pi_+ x = (u-c)x$
$\quad = (u\varepsilon - \varepsilon c)x$

$$\pi_- x = (1-uc^*)x$$
$$= (\varepsilon - u\varepsilon c^*)x$$

$$\Big(\sum_{n\in\mathbb{Z}} u^n \pi_- x_n , \varepsilon x\Big) = \sum_{n\geq 0} \big(u^n \pi_- x_n , \varepsilon x\big)$$

$$= \sum_{n\geq 0} \big(u^n (\varepsilon - u\varepsilon c^*) x_n , \varepsilon x\big)$$

$$= \sum_{n\geq 0} \big((c^n - c^{n+1} c^*) x_n , x$$

$$= \sum_{n\geq 0} \big(x_n , (1-cc^*) c^{*n} x\big)$$

$$= \sum_{n\geq 0} \big(\pi_- x_n , \pi_- c^{*n} x\big)$$

$$= \Big(\sum_{n\in\mathbb{Z}} u^n \pi_- x_n , \underbrace{\sum_{n\geq 0}{}' u^n \pi_- c^{*n} x}_{\pi_- \frac{1}{1-zc^*} x}\Big)$$

430 $\quad H \xrightarrow{a} L^2(S^1, V_+^{\oplus})$, $\quad H \xrightarrow{b} L^2(S^1, V_-^{\oplus})$

$$\|ax+by\|^2 = \|x+a^*by\|^2 + \|y\|^2 - \|a^*by\|^2$$
$$= \|b^*ax+y\|^2 + \|x\|^2 - \|b^*ax\|^2$$

$$(b^*ax, b^*ax) = (ax, bb^*ax)$$
$$\|x\|^2 - \|b^*ax\|^2 = (ax, (1-bb^*)ax)$$

~~So you start with~~

Given $X, c.$ get $H, u, \varepsilon : X \to H$ $\qquad \varepsilon^* u^n \varepsilon = \begin{cases} c^n & n \geq 0 \\ c^{*-n} & n \leq 0 \end{cases}$

decomposition $\oplus\, u^{-1}(V_-) \oplus X \oplus V_+^{\oplus} \oplus u V_+^{\oplus} \oplus$

$$V_+ = \overline{(u-c)X}, \qquad \|(u-c)x\|^2 = \|x\|^2 - \|cx\|^2$$
$$\pi_+ x = (u-c)x$$
$$V_- = \overline{(1-uc^*)X} \qquad \|(1-uc^*)x\|^2 = \|x\|^2 - \|c^*x\|^2$$
$$\pi_- x = (1-uc^*)x.$$

$$\left(\sum_{n\in\mathbb{Z}} u^n \pi_+ x_n, x\right) = \sum_{n<0} \left(u^n(u-c)x_n, x\right)$$

$(u^{n+1}x_n, x) - \underbrace{(u^n c x_n, x)}_{c x_n,\, c^{-n} x}$

$$= \sum_{n<0} (x_n, c^{-n-1}x) - (cx_n, c^{-n}x)$$

$(x_{-n}, c^{n-1}x) - (cx_{-n}, c^n x)$

$(x_{-n}, (1-c^*c)c^{n-1}x)$.

$$\left(\sum_{n\in\mathbb{Z}} u^{-n}\pi_+ x_{-n}, x\right) = \sum_{n>0} \left((u-c)x_{-n}, u^n x\right)$$

$$= \sum_{n>0} (\pi_+ x_{-n}, \pi_+ c^{n-1}x) = \sum_{n<0} (\pi_+ x_n, \pi_+ c^{-n-1}x)$$

$$= \left(\sum_{n<0} u^n \pi_+ x_n, \underbrace{\sum_{n<0} u^n \pi_+ c^{-n-1} x}_{\sum_{n\geq 1} z^{-n}\pi_+ c^{n-1}x}\right)$$

$\pi_+ \dfrac{z^{-1}}{1-z^{-1}c} x = \pi_+ \dfrac{1}{z-c} x$

 Again ~~What happens~~

$$\left(\sum_{n\in\mathbb{Z}} u^n \pi_+ x_n , \varepsilon x\right) = \sum_{n\in\mathbb{Z}} \left(u^n(ux_n - cx_n), x\right)$$

$$= \sum_{n\leq -1} (x_n, u^{-n-1}x) - (cx_n, u^{-n}x)$$ zero for $n \geq 0$

$$= \sum_{n\leq -1} (x_n, c^{-n-1}x) - (x_n, cc^{-n-1}x)$$

$$= \sum_{n\leq -1} \left(\pi_+ x_n , \pi_+ c^{-n-1} x\right)$$

$$= \left(\sum_n u^n \pi_+ x_n , \underbrace{\sum_{n\leq -1} u^n \pi_+ c^{-n-1} x}\right)$$

$$\sum_{n\geq 0} z^{-n-1} \pi_+ c^n x \longleftarrow \sum_{n\geq 0} u^{-n-1} \pi_+ c^n x$$

$$\parallel$$

$$\pi_+ \frac{z^{-1}}{1-z^{-1}c} x = \pi_+ \frac{1}{z-c} x$$

So what do you learn??

$$u^{-1}V_-^* \oplus V_-^* \oplus X \oplus V_+^* \oplus uV_+^* \oplus$$

$$\pi_- x = (u^{-1} - c^*)x \qquad \pi_+ x = (u-c)x$$

$$\left(\sum_{n\in\mathbb{Z}} u^n \pi_+ x_n , x\right) = \sum_{n\leq -1} \left((u-c)x_n, u^{-n}x\right)$$

$$= \sum_{n\geq 1} \left((u-c)x_{-n}, u^n x\right) = \sum_{n\geq +1} (x_{-n}, c^{n-1}x) - (cx_{-n}, c^n x)$$

$$= \sum_{n\geq +1} \left(\pi_+ x_{-n}, \pi_+ c^{n-1} x\right) = \left(\sum_n u^{-n} \pi_+ x_{-n} , \sum_{n\geq +1} u^{-n} \pi_+ c^{n-1} x\right)$$

 ~~Nothing Most~~ No improvement.

Summary:

$$\begin{tikzcd} & H \arrow[dl] \arrow[dr] & \\ L^2(S^1, V_-) & X \arrow[l] \arrow[r] & L^2(S^1, V_+) \end{tikzcd}$$

$$\pi_- \frac{1}{1 - z c^*} x \longleftarrow\!\mid x \longmapsto \pi_+ \frac{1}{z - c} x$$

Where to head? Puzzle: why do $\frac{1}{|q_{n-1}|^2} \frac{d\theta}{2\pi}$ and $\frac{1}{|q_n|^2} \frac{d\theta}{2\pi}$ have the same moments in degrees $< n$.

$$\int z^j \frac{1}{|\tilde{q}_{n-1}|^2} \frac{d\theta}{2\pi} \overset{?}{=} \int z^j \frac{1}{|\tilde{q}_n|^2} \frac{d\theta}{2\pi} \qquad j < n.$$

Why it's true? Go back to $Y = aX + \mathbb{C}\xi_+ = bX + \mathbb{C}\xi_-$
$Y = P_n$, $X = P_{n-1}$, $\xi_+ = \tilde{p}_n^*$, $\xi_- = \tilde{q}_n$ and I have

$$\left(\begin{array}{l} Y \xrightarrow{\sim} H^+ \cap S(z) z H^- \\ X \xrightarrow{\sim} H^+ \cap S(z) H^- \end{array} \right.$$

$$y \longmapsto \xi_-^* \frac{1}{1 - zab^*} y \qquad \begin{array}{l} \xi_- \longmapsto 1 \\ \xi_+ \longmapsto S(z). \end{array}$$

So what is happening? True understanding might come from comparing unitary embeddings.

Begin with $(X, a^* b)$ and analyze the scattering ~~dia~~ embedding

$$x \mapsto \pi_- \frac{1}{1 - z b^* a} x$$

$$\begin{array}{cc} X & V^+ \\ V^- & aX \end{array}$$

$$\begin{tikzcd} H^+ \arrow[r, "X"] & SH^+ \\ zH^+ \arrow[u, hook, "\overset{\perp}{\mathbb{C}\xi_-}"] \arrow[r, "aX"'] & SzH^+ \arrow[u, hook, "S\,\mathbb{C}\xi_+"'] \end{tikzcd}$$

$$\xi_-^* \frac{1}{1 - zab^*} y \qquad \xi_-^* \frac{1}{1 - zab^*} ax = \xi_-^* a \frac{1}{1 - z b^* a} x$$

433 So basically what you need to analyze is a boundary condition. Roughly you start with (X, c) construct ~~[illegible]~~ an isometric embedding of X into H^+ in fact an isom of X with H^+/SH^+. It should be simple and straightforward - you make unitary assumption which guarantee that the dilation of (X, c) is $L^2(S^1)$. There are two ways to proceed. Try to handle $ba^* + \xi_- h \xi_+^*$ on Y. This you mostly did. Or stick with $X \xrightarrow{\sim} H^+/SH^+$ and look for ξ_+, ξ_- in X.

Review latter. Point on $H^+/SH^+ = H^+ \cap SH^-$ you have contraction ~~op~~ operator, hence partial unitary $aX' \ni x \in H^+ \cap SH^- \ni zx \perp SH^+$

$zx \in SH^-$

$aX' = H^+ \cap z^{-1}SH^-$. ~~[illegible]~~ $\xi_+ \in X$ and $(\xi_+, x) = 0$

$\Rightarrow z^{-1}x$ ~~[illegible]~~ $\in H^+$ $\quad \xi_+ = 1$ iff $S(0) = 0$.

$$\frac{S(z) - S(0)}{z} \in H^+ \qquad S^{-1}\frac{S(z)-S(0)}{z} \in \overset{H^+\,?}{\cancel{[illegible]}}$$

$$\left(S^{-1}\frac{S(z)-S(0)}{z}, \overset{H^+}{f} \right) = \left(\frac{S(z)-S(0)}{z}, Sf \right)$$

$$= \left(S - S(0), Szf \right) = \underset{\substack{\parallel \\ 0}}{(1, zf)} - S(0)\underset{\substack{\parallel \\ 0}}{(1, Szf)}$$

$$1 - \overline{S(0)}\,S(z) \in H^+ \cap SH^-$$

$$\left(1 - \overline{S(0)}\,S(z), Sf\right) = S(0)f(0) - S(0)(1, f) = 0$$

$$\underbrace{\left(1-\overline{S(0)}\,S(z),\ zf\right)}_{\perp\, SH^+} = -S(0)\Big(\mathcal{S},\ \underbrace{zf}_{\in X}\Big) = 0.$$

So it seems that $\begin{cases} \xi_- = 1-\overline{S(0)}\,S(z) \\ \xi_+ = \dfrac{S(z)-S(0)}{z}. \end{cases}$

---

simple example. $X = \mathbb{C}$ $\quad c = 0$. Then $\varepsilon^* u^n \varepsilon = 0 \quad n \neq 0$

so that $H = L^2(S^1) = \cdots \oplus u^{-1}V^- \oplus \cancel{X} \oplus V^+ \oplus \cdots \qquad X = \mathbb{C}\varepsilon$

$V_+^+ = \overline{(u-c)X} = \mathbb{C}u\varepsilon \qquad V_- = (1-uc^*)X = \mathbb{C}\varepsilon = X.$

$\xi_+ = u\varepsilon \quad , \quad \xi_- = \varepsilon$. ~~Anyway~~ Review calculation

$$H = \cdots \oplus u^{-1}V_- \oplus X \oplus V_+ \oplus uV_+ \oplus \cdots$$

better
$$H = \cdots \oplus u^{-1}V_- \oplus X \oplus V_+ \oplus uV_+ \oplus \cdots$$
$$\cdots \oplus u^{-1}V_- \oplus V_- \oplus uX \oplus uV_+ \oplus \cdots$$

$$\|x_0 + ux_1\|^2 = \|x_0\|^2 + (x_0, cx_1) + (cx_1, x_0) + \|cx_1\|^2$$
$$+ \|x_1\|^2 - \|cx_1\|^2$$

$$\|x_0 + ux_1\|^2 = \|x_0 + cx_1\|^2 + \|x_1\|^2 - \|cx_1\|^2$$
$$= \|c^*x_0 + x_1\|^2 + \|x_0\|^2 - \|c^*x_0\|^2$$

$$\|(u-c)x_1\|^2 = \|x_1\|^2 - \|cx_1\|^2$$

$$\|(1-uc^*)x_0\|^2 = \|x_0\|^2 - \|c^*x_0\|^2 \qquad \varepsilon, \pi_\pm : X \longrightarrow H$$

$$\pi_+ x = (u-c)x, \quad \pi_- x = (1-u c^*)x, \quad V_\pm = \overline{\pi_\pm X}$$

$$\ell^2 \otimes V_+ \hookrightarrow X \qquad \Big(\sum_{n\in\mathbb{Z}} u^n \pi_+(x_n),\ \varepsilon x\Big)$$

$$= \sum_{n\geq 0} \left(u^{-n}(u-c)x_{-n},\ \varepsilon x\right) = \sum_{n\geq 1} (x_{-n},\ u^{n-1}\varepsilon x) - (cx_{-n},\ u^n \varepsilon x)$$

$$= \sum_{n\geq 1} \underbrace{(x_{-n},\ c^{n-1}\varepsilon x) - (cx_{-n},\ cc^{n-1}\varepsilon x)}_{(\pi_+ x_{-n},\ \pi_+ c^{n-1}x)} \qquad \pi_+ \frac{z^{-1}}{1-z^{-1}c}x$$

$$(\pi_+ x_{-n},\ \pi_+ c^{n-1}x) = \Big(\sum_n u^n \pi_+(x_{-n}),\ \sum_{n\geq 1} u^{-n}\pi_+ c^{n-1}x\Big)$$

435 Simple example $X = \mathbb{C}\varepsilon$ $\quad c = 0$

If X finite dim, ~~[illegible]~~ and no bound states, then S is unitary.

Simple example. In general take $L^2(S^1; d\mu)$ or (Y, u, ξ) and consider the partial unitary obtained by removing ξ from the domain of u.

$aX = \xi^{\perp}$ $\qquad bX = (u\xi)^{\perp}$. Simplest example is $\frac{d\mu}{2\pi}$, so you have the following picture of Y

$$\begin{array}{lccccc}
Y & , & \zeta^{-1}\xi \;, & \xi \;, & \zeta\xi \;, & \zeta^2\xi \\
aX & , & \zeta^{-1}\xi \;, & 0 \;, & \zeta\xi \;, & \\
bX & & \zeta^{-1}\xi & \xi & 0 &
\end{array}$$

better

$$Y = \bigoplus_n \mathbb{C}\zeta^n \underset{b = \zeta}{\overset{a = \text{inc.}}{\leftleftarrows}} X = \bigoplus_{n \neq 0} \mathbb{C}\zeta^n$$

Contraction $c = ba^*$ kills ζ^0 otherwise is mult by ζ

$$\zeta^{-2} \to \zeta^{-1} \to \zeta^{0} \to \circ \qquad \zeta^{1} \to \zeta^{2} \to$$

change so as to kill V_-

436 Simplest inf diml example. Take $X = L^2(S^1)$ orth basis ζ^n

let $c\zeta^n = \zeta^{n+1}$ for $n \neq -1$

$\qquad\quad 0$ for $n = -1$.

Thus (X,c) is the dir. sum of shift ζ on H^2 and its adjoint on H^2_-. Obvious dilation

$$
\begin{tikzcd}
 & & u\varepsilon\zeta^0 \arrow[r] & u^2\varepsilon\zeta^1 \arrow[r] & {} \\
{} \arrow[r] & \varepsilon\zeta^{-1} \arrow[r] & \varepsilon\zeta^0 \arrow[u] & \varepsilon\zeta^1 \arrow[r] & \varepsilon\zeta^2 \arrow[r] & {} \\
{} \arrow[r] & u^{-2}(\varepsilon\zeta^1) \arrow[r] & u^{-1}(\varepsilon\zeta^1) \arrow[ur] & &
\end{tikzcd}
$$

$$\xi_+ = (u\varepsilon - \varepsilon c)\zeta^{-1}$$

$$
\begin{tikzcd}
 & & & u(\varepsilon\zeta^{-1}) \arrow[r] & u^2(\varepsilon\zeta^{-1}) \arrow[r] & {} \\
\varepsilon X \quad {} \arrow[r] & \varepsilon\zeta^{-2} \arrow[r] & \varepsilon\zeta^{-1} \arrow[ur] & \varepsilon\zeta^0 \arrow[r] & \varepsilon\zeta^1 \arrow[r] & {} \\
{} \arrow[r] & u^{-1}(\varepsilon\zeta^0) \arrow[r] & u^{-1}(\varepsilon\zeta^0) \arrow[ur] & & &
\end{tikzcd}
$$

$$\xi_- = (\varepsilon - u\varepsilon c^*)\zeta^0$$

In this case $L^2(S^1, V_+)$ and $L^2(S^1, V_-)$ are $\perp$

so $S = 0$.

$\qquad\qquad\qquad\qquad c =$ mult by h

Another example. $X = \mathbb{C}$ $\quad |h| < 1$.

$\varepsilon = \varepsilon 1 \qquad \xi_+ = \dfrac{(u-h)\varepsilon}{\sqrt{1-|h|^2}} \qquad \dfrac{(1-u\bar{h})\varepsilon}{\sqrt{1-|h|^2}} = \xi_-$

What happens in this. So you get a basis for H $\quad u^n\xi_- \quad n \le 0 \qquad u^n\xi_+ \quad n \ge 0$?

$$\mathbb{C}\varepsilon + \mathbb{C}u\varepsilon \qquad (1-|h|^2)\,\xi_-^*\xi_+ = \big((1-u\bar{h})\varepsilon, (u-h)\varepsilon\big)$$

$$-\big(\bar{h}, \varepsilon^*(1-u^{-1}h)\varepsilon\big)$$

$$-\big(\bar{h}, h - \bar{h}h\big)$$

Example. $X = \mathbb{C}$ c = mult by h $|h| < 1$.

$$\pi_+ 1 = (u-h)\varepsilon 1 \qquad ((u-h)\varepsilon 1, (u-h)\varepsilon 1)$$

$$\pi_- 1 = (\varepsilon - u\varepsilon c^*) 1 \qquad = (\varepsilon 1, (1-u^{-1}h)\varepsilon 1) = 1-|h|^2$$

$$= (1-u\bar{h})\varepsilon 1$$

$$\mathbb{C}\xi_+ \qquad \xi_+ = \frac{(u-h)\varepsilon 1}{\sqrt{1-|h|^2}}$$

$$Y = \overline{\varepsilon X + u\varepsilon X} = \varepsilon X \oplus V_+$$

$$= u\varepsilon X \oplus V_- \quad \mathbb{C}\xi_- \qquad \xi_- = \frac{(1-u\bar{h})\varepsilon 1}{\sqrt{1-|h|^2}}$$

$$S = \frac{u-h}{1-u\bar{h}}$$

dilation has basis $u^n \xi_-$ $n \leq 0$ and $u^n \xi_+$ $n \geq 0$.

$$(\xi_+, \xi_-) = \frac{1}{1-|h|^2} \underbrace{((u-h)\varepsilon 1, (1-u\bar{h})\varepsilon 1)}_{(-h, (1-|h|^2))} = -h$$

You know that $H = \overline{\sum u^n \varepsilon X}$ is $L^2(S^1, d\mu)$ where $d\mu = \rho \frac{d\theta}{2\pi}$ $\rho = \sum \mu_n \zeta^{-n}$ $\mu_n = \begin{cases} h^n & n \geq 0 \\ \bar{h}^{-n} & n \leq 0 \end{cases}$

$$\rho = \frac{1}{1-h\zeta^{-1}} + \frac{\bar{h}\zeta}{1-\bar{h}\zeta} = \frac{1-|h|^2}{|1-h\zeta^{-1}|^2}$$

~~Also you~~ What is your aim?. You want to understand better the process of going from $d\mu$ to (h_n) to S. In the case $d\mu = \frac{d\theta}{2\pi}$ the sequence (h_n) is 0 and S is zero. You now want another example.

~~Of course there are two possi~~ Be careful since you have two S functions assoc. to $d\mu$. The first, ~~als~~ already mentioned, ~~above~~ is obtained by the othog poly sequence + number h_n. The other is the ~~[illegible]~~ S assoc. to the partial unitary obtained by removing 1 from domain of ζ in $L^2(S^1, d\mu)$

438 Consider a finite (h_n) sequence situation; meaning? $\mathcal{O}(n)$ type partial unitary

There's a clean statement about partial unitaries of $\mathcal{O}(n)$ type, namely, classified by divisors degree n in D, or monic polys of degree n with all roots in D.

Given $Y = aX \oplus V^+ = bX \oplus V^-$ a partial unitary of type $\mathcal{O}(n)$, dilate contraction ab^* to get a scattering situation

$$
\begin{array}{ccccc}
L^2(S^1, V_-) & & H & & L^2(S^1, V_+) \\
\pi_- \frac{1}{1-zc^*} x & \longleftarrow & \varepsilon x & \longmapsto & \pi_+ \frac{1}{z-c} x \\
\pi_- x & \longmapsto & (1-u\varepsilon c^*)x & \longmapsto & \pi_+ \frac{1}{z-c}(1-zc^*)x \\
\pi_- \frac{1}{1-zc^*}(z-c)x & \longleftarrow & (u\varepsilon-\varepsilon c)x & \longleftarrow & \pi_+ x
\end{array}
$$

Take $Y = aX \oplus V^+ = bX \oplus V^-$ of $\mathcal{O}(n)$ type same as a herm. scalar prod. on P_n such mult by z is unitary and $\|1\| = 1$. Then you get p_k, q_k, h_k for $k \le n$. properties. $q_n = \det(1 - zab^*)$

How to organize? Go over things.

c contraction on X $\quad H, u, \varepsilon : X \to H \quad \varepsilon^* u^n \varepsilon = c^n \quad n \geq 0$

$\pi_+(x) = (u\varepsilon - \varepsilon c)x \qquad (\varepsilon x', u^j(u\varepsilon - \varepsilon c)x) = 0 \quad j \geq 0$

$\pi_-(x) = (1 - u\varepsilon c^*)x \qquad ((u\varepsilon - \varepsilon c)x', u^j(u\varepsilon - \varepsilon c)x) = 0 \quad j > 0$

$$\left(u^n \pi_+ x', \varepsilon x\right) = \left(\pi_+ x', u^{-n}\varepsilon x\right)$$

$$= \left(u^n(u\varepsilon - \varepsilon c)x', \varepsilon x\right) = 0 \qquad n \geq 0$$

$$= \left((u\varepsilon - \varepsilon c)x', u^{-n}\varepsilon x\right) \qquad n \leq -1$$

$$= \left(\varepsilon x', u^{-n-1}\varepsilon x\right) - \left(\varepsilon c x', u^{-n}\varepsilon x\right)$$

$$= \left(x', c^{-n-1}x\right) - \left(cx', c^{-n}x\right)$$

$$= \left(\pi_+ x', \pi_+(c^{-n-1}x)\right) \qquad -n \geq 1$$

$$= \Big(z^n \pi_+ x', \underbrace{\sum_{k\geq 1} z^{-k}\pi_+(c^{k-1}x)}\Big)$$

ENUF $\qquad \pi_+ \dfrac{z^{-1}}{1-z^{-1}c}\, x = \pi_+ \dfrac{1}{z-c}\, x$

Suppose you start with $d\mu$, form $L^2(S^1, d\mu)$ and

$$P_0 \subset P_1 \subset P_2 \subset \cdots \subset L^2(S^1, d\mu)$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ h_n & 1 \end{pmatrix}\begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \text{starting from } p_0 = q_0 = 1.$$

Recall $p_n \in (z^n + P_{n-1}) \cap P_{n-1}^{\perp} \qquad q_n \in (1 + zP_{n-1}) \cap zP_{n-1}^{\perp}$

Fix an n ~~look~~ at $Y = P_n \qquad X = P_{n-1} \underset{b=z}{\overset{a=1}{\rightrightarrows}} P_n$

See if you can get this all set up.

How? You need ~~both~~ $X\ Y\ Z$? You're missing some important things

440 Basic idea should involve $X \underset{b}{\overset{a}{\rightrightarrows}} Y$

$$P_{n-1} \underset{b=za}{\overset{a=in}{\rightrightarrows}} P_n \underset{b_1}{\overset{a_1}{\rightrightarrows}} P_{n+1}$$

a^*b $\quad a_1^*b_1$

ba^* $\quad b_1a_1^*$

When we view ba^{-1} as a partial unitary on P_n ~~…~~ we calculate its δ using ba^*

Go back to $V = \binom{1}{c} Y \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \xrightarrow{(z \ -1)} \check{l}_z \otimes Y$

$$a_1^*b_1 f = a_1^* z f$$

$b_1 a = a_1 b$ $\quad = zf \quad$ if $f \in P_{n-1}$

$$a_1^* \underbrace{b_1 a}_{a_1 b} = ba^*a \quad ?$$

$\qquad b \qquad\qquad b$

$p_{n+1} \perp P_n = a_1 P_n$

$g_n \in P_n$

$$a_1^* b_1 p_n = a_1^* z p_n = a_1^*(p_{n+1} - h_{n+1} g_n)$$

$$= -h_{n+1} a_1^* g_n = -h_{n+1} g_n$$

Thus the contraction $a_1^* b_1$ is $ba^* - h_{n+1} \tilde{g}_n \tilde{p}_n^*$

$- \xi_{n-} h_{n+1} \xi_{n+}^*$

uX

$X \cap uX \qquad X + uX$

X

$u^{-1}X \cap X$

$u^{-1}X$

$$\begin{matrix} & p_n & \\ P_{n-1} & \xrightarrow{a} & P_n \\ g_n \ b \downarrow & & \downarrow b_1 \ g_{n+1} \\ P_n & \xrightarrow{a_1} & P_{n+1} \\ & p_{n+1} & \end{matrix}$$

Problem: Maybe compare

$(za - b) x_{n-1}$

$X \underset{b}{\overset{a}{\rightrightarrows}} Y \qquad c = a^*b \qquad a^*a = b^*b = 1.$

$X \xrightarrow{\pi_\pm} V_\pm$ defd. by $\|\pi_+ x\|^2 = \|x\|^2 - \|cx\|^2$

$\|\pi_- x\|^2 = \|x\| - \|c^*x\|^2 \qquad \pi_\pm X = V_\pm$

441

$$x \longmapsto \pi_+ \frac{1}{z-c} x \overset{\text{means}}{=} \sum_{n\geq 0} z^{-n-1} \pi_+ c^n x \in H^2_-(S^1, V_+)$$

$$\sum_{n\geq 0} \|\pi_+ c^n x\|^2 = \|x\|^2 - \lim_{n\to\infty} \|c^n x\|^2$$

$$x \longmapsto \pi_- \frac{1}{1-zc^*} x = \sum_{n\geq 0} z^n \pi_- c^{*n} x \in H^2_+(S^1, V_-)$$

$$\sum_{n\geq 0} \|\pi_- c^{*n} x\|^2 = \|x\|^2 - \lim_{n\to\infty} \|c^{*n} x\|^2$$

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projects. Review recovering X from scattering data.

$$\ell^2 \otimes V_+ \xrightarrow{\varepsilon_+} H \xleftarrow{\varepsilon_-} \ell^2 \otimes V_-$$

$$\|\varepsilon_+ f_+ + \varepsilon_- f_-\|^2$$
$$= \|f_+ + \varepsilon_+^* \varepsilon_- f_-\|^2 + (f_-, (1-\varepsilon_-^* \varepsilon_+ \varepsilon_+^* \varepsilon_-) f_-)$$
$$= \|f_- + \varepsilon_-^* \varepsilon_+ f_+\|^2 + (f_+, \underbrace{(1-\varepsilon_+^* \varepsilon_- \varepsilon_-^* \varepsilon_+)}_{\varepsilon_+^*(1-\varepsilon_-\varepsilon_-^*)\varepsilon_+} f_+)$$

Somehow you end up with $\begin{pmatrix} (1-s^*s)^{1/2} & -s^* \\ s & (1-ss^*)^{-1/2} \end{pmatrix}$

$$\|ax_0 + bx_1\|^2 = \|x_0 + \overbrace{a^* b}^{c} x_1\|^2 + (x_1, (1-c^*c)x_1)$$
$$= \|x_1 + c^* x_0\|^2 + (x_0, (1-cc^*)x_0)$$

$aX + V_+$

$bX + V_-$

$$(\quad) \quad bX \oplus V_- = aX + bX = aX \oplus V_+$$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \longmapsto \begin{pmatrix} x_0 + cx_1 \\ \sqrt{1-c^*c}\, x_1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & \sqrt{1-c^*c} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \longmapsto \begin{pmatrix} \sqrt{1-cc^*}\, x_0 \\ c^* x_0 + x_1 \end{pmatrix} = \begin{pmatrix} \sqrt{1-cc^*} & 0 \\ c^* & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

442 Composite unitary.

$$\cancel{\begin{pmatrix}\sqrt{1-cc^*} & 0\\ c^* & 1\end{pmatrix}\begin{pmatrix}1 & -c(1-c^*c)^{-1/2}\\ 0 & (1-c^*c)^{-1/2}\end{pmatrix} = \begin{pmatrix}(1-cc^*)^{1/2} & -c\\ c^* & (1-c^*c)^{-1/2}\end{pmatrix}}$$

~~Check~~ $$\cancel{\begin{pmatrix}(1-cc^*)^{1/2} & -c\\ c^* & (1-c^*c)^{-1/2}\end{pmatrix}\begin{pmatrix}(1-cc^*)^{1/2} & c\\ -c^* & (1-c^*c)^{-1/2}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}$$

Let's try again, what?, $c = ba^* + \xi_- h \xi_+^*$ $\|\xi_\pm\|=1$

perturbation analysis $Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$

basic scattering map $\sqrt{1-|h|^2}\,\xi_-^* \dfrac{1}{1-zc^*}$ $1-cc^* = \xi_-(1-|h|^2)\xi_-^*$

$$\xi_-^*\left(\frac{1}{1-zc^*}\right) = \xi_-^*\frac{1}{1-zc_0^*} + \left(\xi_-^*\frac{1}{1-zc_0^*}\xi_+\right)z\bar{h}\,\xi_-^*\frac{1}{z-c_0^*} + \cdots$$

$$= \frac{1}{1-S(z)z\bar{h}}\,\xi_-^*\frac{1}{1-zc_0^*}$$

Repeat this. The ~~sem~~ embedding $y \longmapsto \sqrt{1-|h|^2}\,\xi_-^*\dfrac{1}{1-zc^*}y$ from Y to H^2, associated to the perturbed contraction $c = ba^* + \xi_- h\xi_+^*$, can be written

$$y \longmapsto \frac{\sqrt{1-|h|^2}}{1-S(z)z\bar{h}}\,\xi_-^*\frac{1}{1-zc_0^*}y$$

See how this looks when $S(z) = \dfrac{p_0}{q_0}$ $\qquad p_1 = zp_0 - \bar{h}q_0$

$\qquad q_1 = -hzp_0 + q_0$

$$\underbrace{\boxed{\frac{\sqrt{1-|h|^2}}{1-\frac{p_0}{q_0}z\bar{h}}\,\frac{1}{q_0}}}_{\dfrac{\sqrt{1-|h|^2}}{q_1}} q_0\,\xi_-^*\frac{1}{1-zc_0^*}y$$

$$\frac{\sqrt{1-|h|^2}}{q_1}\,q_1\,\xi_-^*\frac{1}{1-zc_0^*}y =$$

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$$Y \xrightarrow{\xi_-^* \frac{1}{1-zc_0^*}} H^2 \xrightarrow{g_0} H^2\left(\frac{d\theta}{|g_0|^2 2\pi}\right)$$

$$Y \xrightarrow{\sqrt{1-|h|^2}\ \xi_-^* \frac{1}{1-zc^*}} H^2 \xrightarrow{g_1} H^2\left(\frac{d\theta}{|g_1|^2 2\pi}\right)$$

$$g_1 \overbrace{\frac{\sqrt{1-|h|^2}}{1-\frac{p_0}{g_0}zh} \xi_-^* \frac{1}{1-zc_0^*}} = \sqrt{1-|h|^2}\ g_0\, \xi_-^* \frac{1}{1-zc_0^*}$$

But actually ~~there's an interesting point~~ there should be a quasi-determinant way to see this. Take orth. poly. viewpt. Suppose $Y = P_n$. You start with $\xi_-^* \frac{1}{1-zc_0^*} = (1-bb^*)\frac{1}{1-zab^*}$, contraction $c_0 = ba^*$, mult by $g_0 = "g_{n-1}" = \det(1-zc_0^*)$, cofactor matrix = More precisely

$$\det(1-zc_0^*)\, \xi_-^* \frac{1}{1-zc_0^*} = \xi_-^* \frac{\det(1-zc_0^*)}{1-zc_0^*}$$

is a row of the cofactor matrix for $1-zc_0^*$

You seem to have an identity, namely

$$\left(g_0\, \xi_-^* \frac{1}{1-zc_0^*} = \xi_-^* \frac{\det(1-zc_0^*)}{1-zc_0^*}\right) \text{ times } \sqrt{1-|h|^2}$$

is equal to

$$g_1\, \xi_-^* \frac{1}{1-zc^*} = \xi_-^* \frac{\det(1-zc_1^*)}{1-zc_1^*}$$

444 First a review

$$c = ba^* + \xi_- h \xi_+^*$$
$$c^* = ab^* + \xi_+ \bar h \xi_-^*$$
$$(1-cc^*)^{1/2} = \xi_-(1-|h|^2)^{1/2}\xi_-^*$$

Isometric embedding of Y assoc. to c is

$$\xi_-^*(1-cc^*)^{1/2}\frac{1}{1-zc^*} = \sqrt{1-|h|^2}\;\xi_-^*\frac{1}{1-z(c_0+\Delta c)^*}$$

$$c_0^* = ab^* \qquad \Delta c = \xi_+ \bar h \xi_-^*$$

$$= \sqrt{1-|h|^2}\left\{\xi_-^*\frac{1}{1-zc_0^*} + \underbrace{\xi_-^*\frac{1}{1-zc_0^*}\xi_+}_{S(z)z\bar h} z\bar h\,\xi_-^*\frac{1}{1-zc_0^*} + \cdots\right\}$$

$$= \frac{(1-|h|^2)^{1/2}}{1-S(z)z\bar h}\,\xi_-^*\frac{1}{1-zc_0^*}. \quad \text{Thus}$$

$$\boxed{\underbrace{\xi_-^*(1-cc^*)^{1/2}\frac{1}{1-zc^*}}_{\substack{\text{isom emb. } Y \hookrightarrow H^+ \\ \text{assoc. to } c}} = \underbrace{\frac{(1-|h|^2)^{1/2}}{1-S(z)z\bar h}\,\xi_-^*\frac{1}{1-zc_0^*}}_{\substack{\text{isom emb. } Y \hookrightarrow H^+ \\ \text{assoc. to } c_0 = ba^*.}}}$$

Suppose now $Y = P_n$, $X = P_{n-1}$, $a = \mathrm{inc}$, $b = za$

We know $V^- = \mathbb{C} q_n$, $V^+ = \mathbb{C} p_n$ and natural choices for are $\xi_+ =$ [redacted] $\tilde p_n$, [redacted] $\xi_- = \tilde q_n$ $\qquad S(z) = \frac{\tilde p}{\tilde q}$

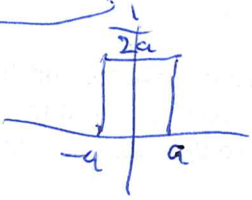
$$1 - S(z)z\bar h = \frac{q_n - \bar h z p_n}{q_n} = \frac{q_{n+1}}{q_n} \quad \text{if } h_{n+1} = -\bar h$$

$$= \frac{\tilde q_{n+1}}{\tilde q_n}(1-|h|^2)^{1/2} \qquad \|q_{n+1}\| = (1-|h|^2)^{1/2}\|q_n\|$$

Thus
$$\boxed{\tilde q_{n+1}\,\xi_-^*(1-cc^*)^{1/2}\frac{1}{1-zc^*} = \tilde q_n\,\xi_-^*\frac{1}{1-zc_0^*}}$$

445 Digression on getting ~~started~~ - you would like to understand Carleman's inequality of Hormander's, about ~~growth of~~ compact support smooth functions. Idea to take convolution of step functions

$$\underbrace{\int_{-a}^{a} e^{i\xi x}\,\frac{dx}{2a}} = \frac{e^{i\xi a}-e^{-i\xi a}}{i\xi\, 2a} = \frac{\sin a\xi}{a\xi}$$

F.T. of 

$a_n > 0$

$$\prod_{n=1}^{\infty} \frac{\sin a_n\xi}{a_n\xi} = \prod_{n=1}^{\infty}\left(1-\frac{a_n^2\xi^2}{3!}+\frac{a_n^4\xi^4}{5!}-\dots\right)$$

This infinite product converges for a_n square summable but you want $\sum a_n < \infty$ in order to have a function of compact support. The actual function being a convolution of ≥ 0 fns. is ≥ 0. The F.T. $\prod_{1}^{\infty} \frac{\sin a_n\xi}{a_n\xi}$ should be entire with all zeros real, growth $e^{\left(\sum a_n\right)|\mathrm{Im}\,\xi|}$

Return to our perturbation calculation

$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$ $\qquad \|\xi_\pm\| = 1$.

$c_0 = ba^*$ $\qquad c = ba^* + \xi_- h \xi_+^*$ $\qquad |h| < 1$

Look inside $\begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ Krein space $T \otimes Y$, $\mathcal{O}(-1) \hookrightarrow \mathcal{O}\otimes T \twoheadrightarrow \mathcal{O}(1)$

Does it help?

$$\begin{pmatrix}1\\ c_0\end{pmatrix} Y = \begin{pmatrix}a\\ b\end{pmatrix} X \oplus \begin{pmatrix}\xi_+\\ 0\end{pmatrix}\mathbb{C}$$

$$\begin{pmatrix}1\\ c\end{pmatrix} Y = \begin{pmatrix}a\\ b\end{pmatrix} X \oplus \begin{pmatrix}\xi_+\\ h\xi_-\end{pmatrix}\mathbb{C}$$

$$\left(\begin{pmatrix}1\\ c\end{pmatrix} Y\right)^{0} = \left\{ \begin{pmatrix}y_1\\ y_2\end{pmatrix} \;\middle|\; (y_1, y) = (y_2, cy) \quad \forall y \in Y \right\}$$

equiv $\quad y_1 = c^* y_2$

446 $\left(\begin{pmatrix}1\\c\end{pmatrix}Y\right)^{\circ} = \begin{pmatrix}c^*\\1\end{pmatrix}Y$

$$\begin{pmatrix}c_0^*\\1\end{pmatrix}Y = \begin{pmatrix}a\\b\end{pmatrix}X \oplus \begin{pmatrix}0\\\xi_-\end{pmatrix}\mathbb{C} \qquad c_0^* = ab^*$$

$$\begin{pmatrix}c^*\\1\end{pmatrix}Y = \begin{pmatrix}a\\b\end{pmatrix}X \oplus \begin{pmatrix}\bar h\xi_+\\\xi_-\end{pmatrix}\mathbb{C} \qquad c^* = ab^* + \xi_+\bar h\xi_-^*$$

$$\mathcal{O}(-1) \longrightarrow T \xrightarrow{(-z\ \ 1)} \mathcal{O}(1)$$
$$\mathcal{O}(-1) = \mathbb{C}\begin{pmatrix}1\\z\end{pmatrix}$$

$$\begin{pmatrix}c^*\\1\end{pmatrix}Y \hookrightarrow \begin{matrix}Y\\\oplus\\Y\end{matrix} \xrightarrow{(-z\ \ 1)} Y$$

$$\begin{pmatrix}c^*\\1\end{pmatrix}y \longmapsto (1-zc^*)y$$

$$\underbrace{\begin{pmatrix}c^*\\1\end{pmatrix}(1-zc^*)^{-1}y} \longleftrightarrow y$$

project this into $\left(\begin{pmatrix}a\\b\end{pmatrix}X\right)^{\circ} \Big/ \begin{pmatrix}a\\b\end{pmatrix}X$

to get $\begin{pmatrix}\bar h\xi_+\\\xi_-\end{pmatrix}\xi_-^*(1-zc^*)^{-1}y$

which has $\|\ \ \|^2 = (1-|h|^2)\left|\xi_-^*(1-zc^*)^{-1}y\right|^2$

Actually is there a direct way to see that the intrinsic map yields an isometric embedding?

~~Question: Given $L^2(S^1, d\mu)$~~

Questions. 1) Finite support measure arising from $|h|=1$.

2) Constructing the operator with given seq. (h_n), can you do it better, in a simpler way, using $T\otimes Y$.

3) Removing 1 from domain of u on $L^2(S^1, d\mu)$, calculate S.

447 $\quad u_{tt} = y^2(u_{xx} + u_{yy}) \qquad u = e^{-i\omega t} e^{i\xi x} v(y)$

$$-\frac{\omega^2}{y^2} v = \cancel{y}(-\xi^2 + \partial_y^2) v$$

$$\left(\frac{\omega^2}{y^2} - \xi^2 + \partial_y^2\right) v = 0$$

better $\quad \omega^2 v = (-y^2\partial_y^2 + \xi^2 y^2) v$

$$(y\partial_y)^2 = y^2\partial_y^2 + y\partial_y \qquad y\partial_y y \partial_y v = y^2\partial_y^2 v + y\partial_y v$$

$$y^2\partial_y^2 = (y\partial_y)^2 - (y\partial_y) \qquad y^{-1/2}(y^2\partial_y^2) y^{1/2} = (y\partial_y + \tfrac{1}{2})^2 - (y\partial_y + \tfrac{1}{2})$$

$$= \left(y\frac{\partial}{\partial y}\right)^2 + \cancel{y\partial_y} + \frac{1}{4} - \frac{1}{2}$$

$$= (y\partial_y)^2 - \frac{1}{4}$$

$$\omega^2 y^{-1/2} v = \left(-y^2 \underbrace{\left(\partial_y + \tfrac{1}{2y}\right)\left(\partial_y + \tfrac{1}{2y}\right)}_{\partial_y^2 + \frac{1}{2y}\partial_y - \frac{1}{2y^2} + \frac{1}{2y}\partial_y + \frac{1}{4y^2}} + \xi^2 y^2\right)(y^{-1/2} v)$$

$$\omega^2 y^{-1/2} v = \Big(\underbrace{-y^2\partial_y^2 - y\partial_y}_{-(y\partial_y)^2} + \frac{1}{4} + \xi^2 y^2\Big)(y^{-1/2} v)$$

$$\left(\omega^2 - \frac{1}{4}\right) \underbrace{y^{-1/2} v}_{w} = \underbrace{\left(-(y\partial_y)^2 + \xi^2 y^2\right)}(\underbrace{y^{-1/2} v}_{w}) \qquad \ln y =$$

equivalent to $\quad -\partial_x^2 + \xi^2 e^{2x} \quad$ on the real line

I am interested in eigenfunctions decaying as $x \to +\infty$

Maybe you want $\quad e^{i\xi x_{old}} \quad$ to be anti-periodic

$e^{i\xi} = -1 \qquad \boxed{\xi = \pi}$