First begin with Cauchy problem with line $t=0$, where you have space of $\psi(x)$, inner prod $\int \psi^* \psi \, dx$ and time evolution operator e^{tX} $\quad \frac{1}{i}X = \begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix}$. Idea is that you have expansion into eigenfns. of X:

$$\phi(x) = \int_\omega \psi_\omega(x) \langle \psi_\omega | \phi \rangle$$

Hopefully $IH(\phi) = \int_\omega IH(\psi_\omega) \, |\langle \psi_\omega | \phi \rangle|^2$ or some variant thereof.

You need the eigenfn. expansion.

$$\phi(x) = \int \frac{dk}{2\pi} e^{ikx} \hat{\phi}(k) \qquad \hat{\phi}(k) = \int dx \, e^{-ikx} \phi(x)$$

$$e^{tX}\phi(x) = \int \frac{dk}{2\pi} e^{ikx} \, e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}} \hat{\phi}(k)$$

$$\begin{vmatrix} k-\omega & 1 \\ 1 & -k-\omega \end{vmatrix} = -k^2+\omega^2-1 = 0 \qquad \omega = \pm(k^2+1)^{1/2}$$

$$(k-\omega)\psi^1 + \psi^2 = 0 \qquad (\omega-k)\psi^1 = \psi^2$$
$$\psi^1 - (k+\omega)\psi^2 = 0 \qquad (\omega+k)\psi^2 = \psi^1$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\omega-k & \omega-k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\omega-k & \omega-k \end{pmatrix}\begin{pmatrix} -\omega & 0 \\ 0 & \omega \end{pmatrix}$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\omega-k & \omega-k \end{pmatrix}\begin{pmatrix} -\omega & 0 \\ 0 & \omega \end{pmatrix}\begin{pmatrix} \omega-k & -1 \\ \omega+k & 1 \end{pmatrix}\frac{1}{2\omega}$$

$$\therefore \; e^{tX}\phi(x) = \int \frac{dk}{2\pi} e^{ikx} \left\{ \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} e^{-it\omega} (\omega-k \;\; -1) \hat{\phi}(k) + \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} e^{i\omega t} (\omega+k \;\; 1) \hat{\phi}(k) \right\} \frac{1}{2\omega}$$

two comp. projectors

$$\begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} \frac{1}{2\omega} \begin{pmatrix} \omega-k & -1 \end{pmatrix} = \frac{1}{2\omega} \begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ \omega-k \end{pmatrix} \frac{1}{2\omega} \begin{pmatrix} \omega+k & 1 \end{pmatrix} = \frac{1}{2\omega} \begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix}$$

So you know the expansion of $\phi(x)$ into eigenfunctions for X. There is a pos. energy

$$e^{tX}\phi(x) = \int \frac{dk}{2\pi} \left\{ \frac{e^{i(kx-\omega t)}}{2\omega} \begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix} + \frac{e^{i(kx+\omega t)}}{2\omega} \begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix} \right\} \hat{\phi}(k)$$

Formula
$$e^{tX}\phi(x) = \int \frac{dk}{2\pi} \left\{ e^{i(kx-\omega t)} \underbrace{\begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix} \frac{1}{2\omega}} + e^{i(kx+\omega t)} \underbrace{\begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix} \frac{1}{2\omega}} \right\} \hat{\phi}(k)$$

these annihilate each other + add to identity hence they are complementary projectors.

At this point you see that functions $\phi(x)$ split into components indexed by $(k, \pm\omega)$. Your aim is to calculate IH for suitable $\phi(x)$, smooth compact support?

Recall your attitude yesterday, where you found on E_ω for $\omega\in\mathbb{R}$ a hermitian form by Wronskian + conjugation methods. Repeat this. Looking at

DE. $$\omega\psi_\omega = \frac{1}{i}X\psi_\omega = \begin{pmatrix}\frac{1}{i}\partial_x & h\\ \bar h & -\frac{1}{i}\partial_x\end{pmatrix} \qquad \frac{1}{i}\partial_x\psi_\omega = \begin{pmatrix}\omega & -h\\ \bar h & -\omega\end{pmatrix}\psi_\omega$$

$$-W(\sigma\psi_\omega,\psi_\omega) = -\begin{vmatrix}\bar\psi^2_\omega & \psi^1_\omega\\ \bar\psi^1_\omega & \psi^2_\omega\end{vmatrix} = |\psi^1_\omega|^2 - |\psi^2_\omega|^2$$

This is also $\psi^*_\omega\varepsilon\psi_\omega$ and it ~~satisfies~~ is ind of ~~$\partial_x\psi^*_\omega$~~ x for $\omega\in\mathbb{R}$. In fact if $\psi(x,t)=e^{i\omega t}\psi_\omega(x)$

Then $\partial_t(\psi^*\psi) = \partial_x(\psi^*\varepsilon\psi)$ $\qquad e^{-i\bar\omega t}e^{i\omega t} = e^{i(\omega-\bar\omega)t}$

$$\Vert$$

$$\boxed{i(\omega-\bar\omega)\psi^*_\omega\psi_\omega = \partial_x(\psi^*_\omega\varepsilon\psi_\omega)}$$

~~Another~~ What is E_ω? It should consist of

$$\cancel{e^{ikx}\begin{pmatrix}\omega+k & 1\\ 1 & \omega+k\end{pmatrix}\frac{1}{2\omega}\hat\phi(k) + e^{-ikx}\begin{pmatrix}\omega+k & -1\\ -1 & \omega-k\end{pmatrix}\frac{1}{2\omega}\hat\phi(-k)}$$

$$\boxed{e^{ikx}\begin{pmatrix}\omega+k & 1\\ 1 & \omega-k\end{pmatrix}\frac{1}{2\omega}\hat\phi(k) + e^{-ikx}\begin{pmatrix}\omega-k & +1\\ +1 & \omega+k\end{pmatrix}\frac{1}{2\omega}\hat\phi(-k)}$$

$$\begin{pmatrix}\frac{1}{i}\partial_x & 1\\ 1 & -\frac{1}{i}\partial_x\end{pmatrix} e^{-ikx}\begin{pmatrix}+1\\ \omega+k\end{pmatrix} = e^{-ikx}\begin{pmatrix}-k & 1\\ 1 & +k\end{pmatrix}\begin{pmatrix}+1\\ \omega+k\end{pmatrix}$$

$$= e^{-ikx}\begin{pmatrix}\omega\\ \omega^2+k\omega\end{pmatrix} = \omega e^{-ikx}\begin{pmatrix}1\\ \omega+k\end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix} e^{ikx}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix} = e^{ikx}\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix}$$

$$= e^{ikx}\begin{pmatrix} \omega \\ \underset{\omega(\omega-k)}{1-k\omega+k^2} \end{pmatrix} = \omega\, e^{ikx}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix}.$$

Simpler to say

$$\psi_\omega = e^{ikx}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix} c^1 + e^{-ikx}\begin{pmatrix} 1 \\ \omega+k \end{pmatrix} c^2$$

$$= \begin{pmatrix} 1 & 1 \\ \omega-k & \omega+k \end{pmatrix}\begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix}\begin{pmatrix} c^1 \\ c^2 \end{pmatrix}$$

$$\psi_\omega^* \varepsilon \psi_\omega = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^* \begin{pmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{pmatrix}\begin{pmatrix} 1 & \omega-k \\ 1 & \omega+k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -(\omega-k) & -(\omega+k) \end{pmatrix}$$

$$\times \begin{pmatrix} e^{ikx} & \\ & e^{-ikx} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$- \left| \begin{pmatrix} e^{ikx} \\ e^{ikx}(\omega-k) \end{pmatrix}^c \quad \begin{pmatrix} e^{ikx} \\ e^{ikx}(\omega-k) \end{pmatrix} \right|$$

$$= - \begin{vmatrix} e^{-ikx}(\omega-k) & e^{ikx} \\ e^{-ikx} & e^{ikx}(\omega-k) \end{vmatrix} = -(\omega-k)^2 + 1$$

$$= -\omega^2 + 2k\omega - k^2 + 1$$

$$\left(1-(\omega-k)^2\right)\frac{(\omega+k)}{\omega+k} = \frac{\omega+k-(\omega-k)}{\omega+k} = \frac{2k}{\omega+k}$$

$$\left(1-(\omega+k)^2\right)\frac{\omega-k}{\omega-k} = \frac{\omega-k-(\omega+k)}{\omega-k} = \frac{-2k}{\omega-k}$$

There seems to be a mistake ~~No~~ it OK 915

Consider $i\omega\psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi$

Look for soln with $\psi = e^{ikx}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

$$\omega\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

$$(k-\omega)v^1 + v^2 = 0$$
$$(-k-\omega)v^2 + v^1 = 0$$

$$(\omega-k)v^1 = v^2$$
$$(\omega+k)v^2 = v^1$$

$$\omega^2 - k^2 = 1.$$

Ass. $|\omega| > 1$

let $k = (\omega^2-1)^{1/2}$

$$\boxed{\psi_\omega(x) = e^{ikx}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix}c^1 + e^{-ikx}\begin{pmatrix} 1 \\ \omega+k \end{pmatrix}c^2}$$

$$\psi_\omega(x) = \begin{pmatrix} e^{ikx}c^1 + e^{-ikx}c^2 \\ e^{ikx}(\omega-k)c^1 + e^{-ikx}(\omega+k)c^2 \end{pmatrix}$$

$$|\psi^1_\omega|^2 - |\psi^2_\omega|^2 = \left|e^{ikx}c^1 + e^{-ikx}c^2\right|^2$$
$$- \left|e^{ikx}(\omega-k)c^1 + e^{-ikx}(\omega+k)c^2\right|^2$$
$$= |c^1|^2 + \overline{c^1}e^{-2ikx}c^2 + \overline{c^2}e^{2ikx}c^1 + |c^2|^2$$
$$-(\omega-k)^2|c^1|^2 - \overline{c^1}(\omega-k)(\omega+k)e^{-2ikx}c^2$$
$$-\overline{c^2}(\omega+k)(\omega-k)e^{2ikx}c^1 - (\omega+k)^2|c^2|^2$$
$$= |c^1|^2\Big(\underbrace{1-(\omega-k)^2}_{2k(\omega-k)}\Big) + |c^2|^2\Big(\underbrace{1-(\omega+k)^2}_{-2k(\omega+k)}\Big)$$

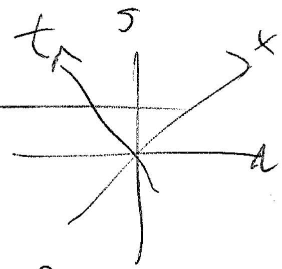
ω-Eigenspace for $\begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix}$ consists of

$$\psi_\omega(x) = \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} e^{ikx} c_1 + \begin{pmatrix} 1 \\ \omega+k \end{pmatrix} e^{-ikx} c_2$$

$$= \begin{pmatrix} 1 & 1 \\ \omega-k & \omega+k \end{pmatrix} \begin{pmatrix} e^{ikx} c_1 \\ e^{-ikx} c_2 \end{pmatrix}$$

Then
$$\psi_\omega^* \varepsilon \psi_\omega = \begin{pmatrix} e^{ikx} c_1 \\ e^{-ikx} c_2 \end{pmatrix}^* \underbrace{\begin{pmatrix} 1 & \omega-k \\ 1 & \omega+k \end{pmatrix} \overset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\wedge} \begin{pmatrix} 1 & 1 \\ \omega-k & \omega+k \end{pmatrix}}_{\begin{pmatrix} 1-(\omega-k)^2 & 0 \\ 0 & 1-(\omega+k)^2 \end{pmatrix}} \begin{pmatrix} e^{ikx} c_1 \\ e^{-ikx} c_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^* \begin{pmatrix} 1-(\omega-k)^2 & 0 \\ 0 & 1-(\omega+k)^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$



Start again
$$-\partial_r \psi^1 = i h \psi^2$$
$$\partial_s \psi^2 = i \bar{h} \psi^1$$

$$-\partial_t + \partial_x = \partial_r$$
$$\partial_t + \partial_x = \partial_s$$

$$\partial_s \overline{\psi^2} = -i h \overline{\psi^1}$$

$$\partial_t = \frac{1}{2}(-\partial_r + \partial_s)$$
$$\partial_x = \frac{1}{2}(\partial_r + \partial_s)$$

$$\partial_t(\psi^*\psi) = \partial_x(\psi^* \varepsilon \psi)$$

$$(-\partial_r + \partial_s)(\psi^*\psi) = (\partial_r + \partial_s)(\psi^*\varepsilon\psi)$$

$t = -r+s$
$x = r+s$

Better is $(\psi^*\psi)\,dx + (\psi^*\varepsilon\psi)\,dt$

$$(\psi^*\psi)(dr+ds) + (\psi^*\varepsilon\psi)(-dr+ds)$$

$$= (\psi^*\psi - \psi^*\varepsilon\psi)\,dr + (\psi^*\psi + \psi^*\varepsilon\psi)\,ds$$

$$= (+2)\, \psi_2^*\psi_2\, dr + 2\psi_1^*\psi_1\, ds$$

$$-\partial_s(\psi_2^*\psi_2) + \partial_r(\psi_1^*\psi_1) = 0$$

Check

$$\partial_s(\psi_2^*\psi_2) = \psi_2^*(i\bar{h}\psi_1) + (i\bar{h}\psi_1)^*\psi_2$$
$$= i\bar{h}\,\psi_2^*\psi_1 - ih\,\psi_1^*\psi_2$$
$$\partial_r(\psi_1^*\psi_1) = \psi_1^*(-ih\psi_2) + (-ih\psi_2)^*\psi_1$$
$$= -ih\,\psi_1^*\psi_2 + i\bar{h}\,\psi_2^*\psi_1$$

$$\boxed{\partial_r|\psi_1|^2 = \partial_s|\psi_2|^2} \qquad |\psi_1|^2 ds + |\psi_2|^2 dr$$

$$\psi = \int e^{i(r\rho - s\rho^{-1})}\begin{pmatrix}1\\-\rho\end{pmatrix} f(\rho)\,\frac{d\rho}{2\pi}$$

$$\psi_1 = \int e^{i(r\rho - s\rho^{-1})} f(\rho)\,\frac{d\rho}{2\pi} = \int e^{-is\rho^{-1}}\left(e^{ir\rho}f(\rho)\right)\frac{d\rho}{2\pi}$$

$$\int |\psi_1(r,s)|^2 ds = \qquad \text{is} \qquad \textit{Easier in the present form}$$

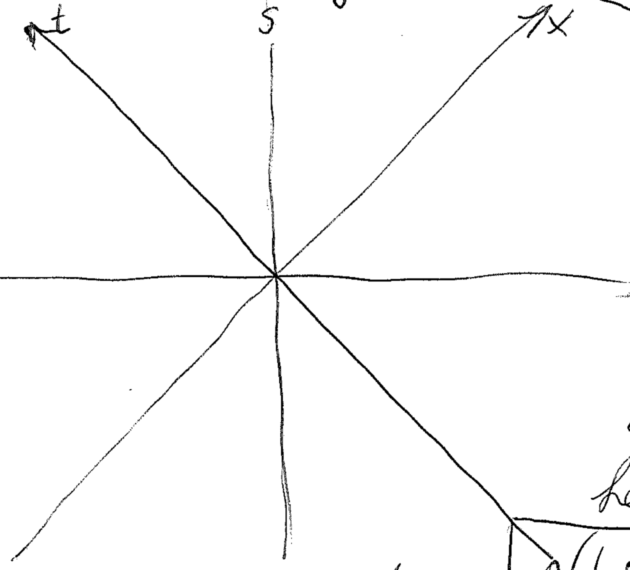
$$\psi_2(r,s) = \int e^{ir\rho}\left(e^{-is\rho^{-1}}(-\rho)f(\rho)\right)\frac{d\rho}{2\pi}$$

$$\int|\psi_2(r,s)|^2 dr = \int \left|e^{-is\rho^{-1}}(-\rho)f(\rho)\right|^2 \frac{d\rho}{2\pi}$$
$$= \int \rho^2|f(\rho)|^2\,\frac{d\rho}{2\pi} \qquad \text{is ind. of } r$$

$$\psi_1(r,s) = \int e^{i(r\xi^{-1})}\, e^{-is\xi} f(\xi^{-1})(+\xi^{-2})\,\frac{d\xi}{2\pi}$$

$$\int|\psi_1(r,s)|^2 ds = \int\left|e^{i(r\xi^{-1})}f(\xi^{-1})\frac{1}{\xi^2}\right|^2\frac{d\xi}{2\pi}$$
$$= \int |f(\xi^{-1})|^2\frac{1}{\xi^4}\frac{d\xi}{2\pi} = \int|f(\rho)|^2\rho^4\frac{d\rho}{\rho^2\,2\pi} = \int|\rho f(\rho)|^2\frac{d\rho}{2\pi}$$

$$\psi(r,s) = \int e^{i(r\rho - s\rho^{-1})} \begin{pmatrix}1\\-\rho\end{pmatrix} \tilde{f}(\rho)\, \frac{d\rho}{\rho\cdot 2\pi}$$



need better understanding of the continuous grid space you have the analog of increasing + decreasing staircases, non characteristic curves. Each curve gives a hermitian form on grid space by integrating

$$\boxed{2\left(|\psi_1|^2 ds + |\psi_2|^2 dr\right) = \psi^*\psi\, dx + \psi^*\varepsilon\psi\, dt}$$

$x = r+s$
$t = -r+s$

Get the fmla straight.

$$\psi^*\psi\, d(r+s) + (\psi^*\varepsilon\psi)\, d(-r+s)$$

$$= \psi^*(1-\varepsilon)\psi\, dr + \psi^*(1+\varepsilon)\psi\, ds$$

$$= 2\left(|\psi_2|^2 dr + |\psi_1|^2 ds\right)$$

$$\begin{pmatrix}1 & \\ & 1\end{pmatrix} \quad \begin{pmatrix}1 & \\ & -1\end{pmatrix}$$

Something to straighten out is the harmonic oscillator picture. At the moment you have a state space with time evolution operator and hopefully two hermitian forms. I want the indefinite form to arise from a symplectic form + conjugation in the standard way.

Go back to $\partial_t\psi = \begin{pmatrix}\partial_x & ih\\ i\bar{h} & -\partial_x\end{pmatrix}\psi = X\psi$ defining the state space with time evolution. Eigenvalue eqn

$$\omega\psi = \begin{pmatrix}\frac{1}{i}\partial_x & h\\ \bar{h} & -\frac{1}{i}\partial_x\end{pmatrix}\psi \quad \text{or} \quad \frac{1}{i}\partial_x\psi = \begin{pmatrix}\omega & -h\\ \bar{h} & -\omega\end{pmatrix}\psi$$

Conjugation operation $\psi^\sigma = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}^\sigma = \begin{pmatrix}\bar{\psi}^2\\ \bar{\psi}^1\end{pmatrix}$

${}^{\sigma}(\omega\psi) = \bar{\omega}\,{}^{\sigma}\psi \qquad {}^{\sigma}\begin{pmatrix} \frac{1}{i}\partial_x & h \\ \tilde{h} & -\frac{1}{i}\partial_x \end{pmatrix} = \begin{pmatrix} \frac{1}{i}\partial_x & h \\ \bar{h} & -\frac{1}{i}\partial_x \end{pmatrix}$

Conjugation induces $\sigma : E_\omega \xrightarrow{\sim} E_{\bar{\omega}}$. Suppose

$\partial_t \psi = X\psi \qquad \psi(x,t) = \int_\omega e^{i\omega t}\,\hat{\psi}(x,\omega)\,\frac{d\omega}{2\pi}$

$({}^{\sigma}\psi)(x,t) = \int e^{-i\bar{\omega} t}\,({}^{\sigma}\hat{\psi})(x,\omega)\,\frac{d\omega}{2\pi}$

?

$$\partial_t\psi = X\psi \Longrightarrow \partial_t({}^{\sigma}\psi) = {}^{\sigma}X\,{}^{\sigma}\psi$$

$${}^{\sigma}X = {}^{\sigma}\begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix} = \begin{pmatrix} -\partial_x & -ih \\ -i\bar{h} & \partial_x \end{pmatrix} = -X$$

~~So if $\psi(x,t)$ satisfies $\partial_t\psi = X\psi$,~~
~~then $({}^{\sigma}\psi)(x,t)$ satisfies $\partial_t({}^{\sigma}\psi) = -X\,{}^{\sigma}\psi$~~
~~which means $\phi = {}^{\sigma}\psi(x,-t)$ satisfies $\partial_t\phi$?~~
~~$\psi(x,t) \longmapsto \sigma(\psi(x,t))$~~

If $\psi = \psi(x,t)$ sat. $\partial_t\psi = X\psi$, then

$\sigma\psi : (x,t) \longmapsto \sigma(\psi(x,t))$ sat $\partial_t(\sigma\psi) = -X(\sigma\psi)$, so

${}^{\sigma}\psi : (x,t) \longmapsto \sigma(\psi(x,-t))$ " $\partial_t\,{}^{\sigma}\psi = X\,{}^{\sigma}\psi$, ~~and~~

this agrees with

$$\psi(x,t) = \int e^{i\omega t}\,\hat{\psi}(x,\omega)\,\frac{d\omega}{2\pi}$$

$$\boxed{{}^{\sigma}\psi(x,t) = (\sigma\psi)(x,-t)} = \int e^{+i\omega t}\,\sigma(\hat{\psi}(x,\omega))\,\frac{d\omega}{2\pi} \Longrightarrow \boxed{\widehat{{}^{\sigma}\psi}(x,\omega) = \sigma\hat{\psi}(x,\omega)}$$

Now you have conjugation defined on both pictures: $\left({}^{\sigma}\psi\right)(x,t) = \sigma\left(\psi(x,-t)\right)$, $\left({}^{\sigma}\hat{\psi}\right)(x,\omega) = \sigma\left(\hat{\psi}(x,\omega)\right)$

$\hat{\psi}(x,\omega)$ is a section of the ~~[illegible]~~ eigenfunction bundle $\coprod_{\omega\in\mathbb{R}} E_\omega$. So now I ~~[illegible]~~ have time reflection through the x-axis. ~~These are~~ This is a natural feature of the IH picture

~~[illegible]~~

$$\partial_t\psi = \begin{pmatrix}\partial_x & ih\\ i\bar{h} & -\partial_x\end{pmatrix}\psi(x,t) \qquad \omega\hat{\psi} = \begin{pmatrix}\frac{1}{i}\partial_x & h\\ \bar{h} & -\frac{1}{i}\partial_x\end{pmatrix}\hat{\psi}(x,\omega)$$

$$\psi(x,t) = \int e^{i\omega t}\hat{\psi}(x,\omega)\frac{d\omega}{2\pi} \qquad \hat{\psi}(x,\omega) = \int e^{-i\omega t}\psi(x,t)\,dt$$

$$\int \psi^*\varepsilon\psi\,dt = \int\left(\int e^{+i\omega t}\hat{\psi}(x,\omega)^*\frac{d\omega}{2\pi}\right)^*\varepsilon\psi(x,t)\,dt$$

$$= \int \hat{\psi}(x,\omega)^*\int e^{-i\omega t}\varepsilon\psi(x,t)\,dt\,\frac{d\omega}{2\pi}$$

$$= \int \underbrace{\hat{\psi}(x,\omega)^*\varepsilon\hat{\psi}(x,\omega)}\frac{d\omega}{2\pi}$$

$$\underbrace{\hat{\psi}^*\varepsilon\hat{\psi} = |\hat{\psi}^1|^2 - |\hat{\psi}^2|^2}_{\text{independent of } x}$$

$$\omega\hat{\psi} = \left(\frac{1}{i}\varepsilon\partial_x + A\right)\hat{\psi}$$

$$\partial_x\left(\hat{\psi}^*\varepsilon\hat{\psi}\right) = \left(\partial_x\hat{\psi}\right)^*\varepsilon\hat{\psi} + \hat{\psi}^*\varepsilon\partial_x\hat{\psi}$$

$$= \left(i\omega\hat{\psi} - iA\hat{\psi}\right)^*\hat{\psi} + \hat{\psi}^*\left(i\omega\hat{\psi} - iA\right)\hat{\psi}$$

$$= (-i\omega)\hat{\psi}^*\hat{\psi} + i\omega\hat{\psi}^*\hat{\psi} = 0.$$

$$+\underbrace{i\hat{\psi}A^*\hat{\psi} - i\hat{\psi}^*A\hat{\psi}}$$

Let's work ~~this~~ out carefully E_ω $\hbar=1$

Put $\phi(x) = \hat{\varphi}(x,\omega)$ for a solution of

$$\omega\phi = \begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix}\phi, \qquad \text{const. coeffs} \Rightarrow \text{try } \phi(x) = \begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix} e^{ikx}$$

$$\omega\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix} = \begin{pmatrix} k & 1 \\ 1 & -k\end{pmatrix}\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix} \qquad \begin{matrix}(\omega-k)\phi^1 = \phi^2 \\ (\omega+k)\phi^2 = \phi^1\end{matrix} \quad \therefore\ \omega^2 - k^2 = 1$$

two ind. solution

$$\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix}(x) = \begin{pmatrix}1\\ \omega-k\end{pmatrix} e^{ikx} c_1 + \begin{pmatrix}1\\ \omega+k\end{pmatrix} e^{-ikx} c_2$$

$$\begin{pmatrix} k & 1 \\ 1 & -k\end{pmatrix}\begin{pmatrix}1\\ \omega-k\end{pmatrix} = \begin{pmatrix}\omega\\ \omega^2-k\omega\end{pmatrix} = \begin{pmatrix}1\\ \omega-k\end{pmatrix}\omega$$

$$\begin{pmatrix} -k & 1 \\ 1 & +k\end{pmatrix}\begin{pmatrix}1\\ \omega+k\end{pmatrix} = \begin{pmatrix}\omega\\ \omega^2+k\omega\end{pmatrix} = \begin{pmatrix}1\\ \omega+k\end{pmatrix}\omega$$

$$\phi(x) = \begin{pmatrix}1\\ \omega-k\end{pmatrix} e^{ikx} c_1 + \begin{pmatrix}1\\ \omega+k\end{pmatrix} e^{-ikx} c_2$$

$$= \begin{pmatrix} 1 & 1 \\ \omega-k & \omega+k\end{pmatrix}\begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

$$\phi^*\Sigma\phi = \begin{pmatrix}c_1\\ c_2\end{pmatrix}^* \begin{pmatrix} e^{-ikx} & \\ & e^{ikx}\end{pmatrix} \boxed{\begin{pmatrix} 1 & \omega-k \\ 1 & \omega+k\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 1 & 1 \\ \omega-k & \omega+k\end{pmatrix}}$$

$$\times \begin{pmatrix} e^{ikx} & \\ & e^{-ikx}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

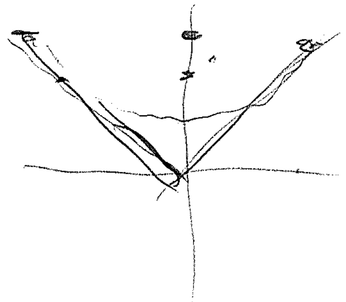
$$\begin{pmatrix} 1 & \omega-k \\ 1 & \omega+k\end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\omega+k & -\omega-k\end{pmatrix} = \begin{pmatrix} 1-(\omega-k)^2 & \underbrace{\cancel{1-\omega^2+k^2}}_{0} \\ \underbrace{\cancel{1-\omega^2+k^2}}_{0} & 1-(\omega+k)^2\end{pmatrix}$$

$$\boxed{\phi^*\Sigma\phi = \begin{pmatrix}c_1\\ c_2\end{pmatrix}^* \begin{pmatrix} 1-(\omega-k)^2 & 0 \\ 0 & 1-(\omega+k)^2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}} \quad \text{ind of } x$$

$\omega > 1$ $\qquad \sqrt{k^2+1} \pm k$

$$1-(\omega-k)^2 = 1-\omega^2+2k\omega-k^2 \Rightarrow 2k\omega-2k^2 = 2k(\omega-k)$$
$$1-(\omega+k)^2 = 1-\omega^2-k^2-2\omega k = -2k^2-2k\omega = -2k(\omega+k)$$

$$\omega = \sqrt{k^2+1}$$



$$\omega \geq \sqrt{k^2+1}$$

point if ω, k have same sign then
$$(\omega-k)^2 < 1 < (\omega+k)^2$$

You have to write up something about the Wronskian. Ultimate goal ~~is a take~~ is to understand projection method, inverse scattering transform through examples

Digress to analyze harmonic oscillator picture.

Recall a harmonic osc. given by a real vector space V, a symplectic form A on V, and a pos. def. symm. form S. $\quad S, A : V \longrightarrow V^*$

$A\dot{X} = S$ should be Ham. eqns.

~~Example $V = \mathbb{R}^2$~~

Example $\quad S = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 q^2$

Standard notation. $\quad H = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 q^2$

$$\dot{q} = \frac{\partial H}{\partial p} = p \qquad \dot{p} = -\frac{\partial H}{\partial q} = -\omega_0^2 q$$
$$\ddot{q} + \omega_0^2 q = 0 \qquad q = \mathrm{Re}(\alpha e^{i\omega_0 t})$$

~~[illegible]~~ Do examples to get the notation moving.

V vector space with coords. (q,p)

dual basis ∂_q, ∂_p. $A = dq\,dp$ $X = \dot{q}\partial_q + \dot{p}\partial_p$

$i_X(A) = \dot{q}\,dp - \dot{p}\,dq = dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp$

$\therefore \dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q}$ $\qquad Xf = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p}$

~~[illegible]~~ V is ~~[illegible]~~ phase space

V = phase space $= \mathbb{R}\partial_q \oplus \mathbb{R}\partial_p$

$V^* = \mathbb{R}q + \mathbb{R}p$

$A \in \Lambda^2 V^* \qquad A = q \wedge p$

$H \in S^2 V^* \qquad H = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 q^2$

~~[illegible]~~ Consider $\partial_t \phi = \partial_x \phi$, $\omega\hat{\phi} = \frac{1}{i}\partial_x \hat{\phi}$

$\phi(x,t) = \int e^{i\omega t}\hat{\phi}(x,\omega)\frac{d\omega}{2\pi}$. You have a vector space consisting of $\phi(x,t)$ satisfying

$\underbrace{\text{solutions}}_{}^{\phi(x,t)}$ of $\partial_t\phi = \partial_x\phi$ functions $\phi(x,0)$

$\hat{\phi}(x,\omega)$ of $\omega\hat{\phi} = \frac{1}{i}\partial_x\hat{\phi}$ functions $\hat{\phi}(x,\omega) =$.

The problem is to interpret ~~as~~ the wave eqn.

$\partial_t\psi = \begin{pmatrix}\partial_x & ih \\ ih & -\partial_x\end{pmatrix}\psi$ as a harmonic oscillator, with

Energy $\int \psi^*\psi\,dx$ and symplectic form related to $IH = \int \psi^* \varepsilon \psi\,dt$. The wave equation gives the time evolution, so ~~th~~ you have ?

Recall your viewpoint about a harmonic oscillators, namely, you have a real phase space with a time evolution operator X having spectral properties of a skew adjoint operator. Consider

$$X = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix} \quad \text{on} \quad \psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \end{pmatrix} \qquad x \in \mathbb{R}$$

Conjugation ~~[illegible]~~

$$({}^{\sigma}\psi)(x) = \begin{pmatrix} \overline{\psi^2(x)} \\ \overline{\psi^1(x)} \end{pmatrix}$$

$${}^{\sigma}(X\psi) = {}^{\sigma}\begin{pmatrix} \partial_x \psi^1 + ih\psi^2 \\ i\bar{h}\psi^1 - \partial_x \psi^2 \end{pmatrix} = \begin{pmatrix} -ih\overline{\psi^1} - \partial_x \overline{\psi^2} \\ \partial_x \overline{\psi^1} - i\bar{h}\,\overline{\psi^2} \end{pmatrix}$$

$$= \begin{pmatrix} -\partial_x \overline{\psi^2} - ih\overline{\psi^1} \\ -i\bar{h}\overline{\psi^2} + \partial_x \overline{\psi^1} \end{pmatrix} = \underbrace{\begin{pmatrix} -\partial_x & -ih \\ -i\bar{h} & \partial_x \end{pmatrix}}_{-X} \underbrace{\begin{pmatrix} \overline{\psi^2} \\ \overline{\psi^1} \end{pmatrix}}_{{}^{\sigma}\psi}$$

$\therefore\ \sigma(X\psi) = -X\,\sigma(\psi)$, so conjugation takes X into $-X$. ~~[illegible]~~

~~[illegible] the grid~~

~~[illegible]~~ to understand, to view $\partial_t u = \partial_x u$ as a harmonic oscillator. X = skew adjoint operator ∂_x, V_r = real valued functions $u(x)$, spectral decomp.

$$u(x) = \int_{-\infty}^{\infty} e^{ikx}\, \hat{u}(k)\, \frac{dk}{2\pi} \qquad u \text{ real means } \overline{\hat{u}(k)} = \hat{u}(-k)$$

$$= \int_0^{\infty} \left(e^{ikx}\hat{u}(k) + e^{-ikx}\hat{u}(-k)\right) \frac{dk}{2\pi}$$

So this decomposes phase space V_r into 2 planes

$$u(x,t) = \int_0^{\infty} \left(e^{ik(x+t)}\, \hat{u}(k) + e^{-ik(x+t)}\, \hat{u}(-k)\right) \frac{dk}{2\pi}$$

$$u(x+t) \longmapsto e^{ikt}\hat{u}(k)$$

$$\underset{\in V_r}{u(x)} \longrightarrow \hat{u}(k)$$

This takes care of time evolution. ~~[illegible]~~
The time evolution of X is such that $-X^2$ is diagonalizable with positive eigenvalues, ~~[illegible]~~ so you have polar decomp: $X = |X| J$, $J^2 = -1$ and $|X|$ gives the frequency of a mode.

But you're missing ~~[illegible]~~ a hermitian form on the eigenspaces giving the energy. Suppose the energy is $\int u(x)^2 dx = \int_0^\infty |\hat{u}(k)|^2 \frac{dk}{\pi}$. The symplectic form should be what? ~~[illegible]~~ The answer should arise from a geometric ~~[illegible]~~ 1-form on space time

$$\partial_t(u^2) = 2u\,\partial_t u = 2u\partial_x u = \partial_x \quad ?? \qquad H = \frac{p^2}{2} + \frac{\omega^2}{2} q^2$$

Standard quant. $\quad q = \mathrm{Re}\left(Ae^{-i\omega t}\right) \qquad \dot{q} = \frac{\partial H}{\partial p} = p$

$$p = \dot{q} = \mathrm{Re}\left(-i\omega A e^{-i\omega t}\right) = \mathrm{Im}\left(\omega A e^{-i\omega t}\right)$$

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) = \frac{\omega^2}{2}|A|^2 \qquad [p, q] = [\partial_x, x] = 1$$

$$H = a^* a + \text{const} \qquad [a, a^*] = 1.$$

$$a = r(\omega q + ip) \qquad a^* = r(\omega q - ip)$$

$$[a, a^*] = r^2(2\omega) \qquad r = \frac{1}{\sqrt{2\omega}}$$

$$a^* a = \frac{1}{2\omega}(\omega q - ip)(\omega q + ip) = \frac{1}{2\omega}\left(\omega^2 q^2 + p^2 - \omega\right)$$

$$\omega\left(a^* a + \frac{1}{2}\right) = \frac{1}{2}\left(\omega^2 q^2 + p^2\right) = H.$$

$$\parallel$$

$$\frac{\omega^2}{2}|A|^2 \qquad |A|^2 = \frac{2}{\omega}\left(a^* a + \frac{1}{2}\right) \qquad A = \sqrt{\frac{2}{\omega}}\, a$$

$$a = \sqrt{\frac{\omega}{2}}\, A$$

Energy $\int u(x)^2 dx = \int_0^\infty |\hat{u}(k)|^2 \frac{dk}{\pi}$

this should become $\int_0^\infty k a_k^* a_k \frac{dk}{\pi}$?

Let's use the following approach, classical approach. You have V_R real vector space with positive quadratic form energy H and time evolution X, assume X skew-symm. wrt H. To find a symplectic form S on V_R such that X = Hamiltonian flow corres to H.

$$\mathrm{Ham}(v, v') \qquad \mathrm{Sym}(v,v') \qquad X$$

$$\mathrm{Ham}(Xv, v') + \mathrm{Ham}(v, Xv') = 0, \text{ given}$$

$$\mathrm{Ham}(Xv, v) + \underbrace{\mathrm{Ham}(v, Xv)}_{\mathrm{Ham}(Xv,\, v)} = 0$$

$\therefore$ $\mathrm{Ham}(Xv, v')$ is skew-symm

Put $S(v,v') = H(Xv, v')$ No try

$$\boxed{S(Xv, v') = H(v,v')}$$

Assume

$$S(Xv, v') + S(v, Xv') = 0$$

$$S(Xv, v') = -S(v, Xv') = S(Xv', v)$$

$$S(\partial_x u, u_1) = \int u u_1 \, dx$$

You want $S(\sigma, X\sigma') = H(\sigma,\sigma')$

or $\boxed{S(\sigma, \dot\sigma') = H(\sigma, \sigma')}$ e.g.

$$\sigma^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot\sigma' = \sigma^t \begin{pmatrix} k & 0 \\ 0 & \frac{1}{m} \end{pmatrix} \sigma'$$

$$\begin{pmatrix} -\dot p \\ \dot q \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & \frac{1}{m} \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

So you want $\boxed{S(\sigma, X\sigma') = H(\sigma, \sigma')}$

$$H(u_1, u_2) = \int u_1 u_2 \, dx \qquad S(\partial_x u_1, \partial_x u_2) = \int (\partial_x u_1) u_2 \, dx$$

skew-symm

$$\| $$

$$S(u_1, \partial_x u_2) = \int u_1 u_2 \, dx$$

$$S(u_1, u_2) = \int u_1 \, \partial_x^{-1} u_2 \, dx \qquad (\partial_x^{-1} u)(x) = \int^x u(x')\,dx'$$

$$S(\sigma, X\sigma') = H(v, v')$$ H has the kernel

$$S(u_1, \partial_x u_2) = \int u_1 u_2 \, dx \qquad \delta(x-x')$$

$$S(u_1, u_2) = \int u_1 (\partial_x^{-1} u_2) \, dx$$ $\therefore$ S has the kernel

essentially the Heaviside function $H(x-x')$, made skew-symm:

$$S(x, x') = \frac{1}{2} \frac{x-x'}{|x-x'|} = \begin{cases} \frac{1}{2} & x > x' \\ -\frac{1}{2} & x < x'. \end{cases}$$

Thus it seems possible to view $\partial_t u = \partial_x u$ as a harmonic oscillator with energy $\frac{1}{2}\int u^2 dx$ and symplectic form $S(u_1, u_2) = \int u_1 (\partial_x^{-1} u_2) dx$. ~~how to proceed from here.~~

Maybe you should also have fermionic picture available.

Standard quantization

oscillator $H = \omega(a^* a + \frac{1}{2})$ $\quad [a, a^*] = 1.$

ferm. $H = \omega(\psi^* \psi - \frac{1}{2})$ $\quad \psi\psi^* + \psi^*\psi = 1$

~~to be more precise~~ kinematics say you have a real vector space of self adjoint operators: $\lambda a + \bar{\lambda} a^*$ in both cases, bosonic uses $[\ ,\ ]$, even commutator, ferm uses odd comm.

dynamics given by bracketing with a quadratic elt in the Weyl, Clifford alg

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Let's go back to $\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix} \psi$

you want h to be "general", to depend on x and to have varying phase. This is a wave equation, you want to quantize it somehow, extending what others have done. ~~This wave equation~~ ~~determines~~ What? It's called 2nd quantization, it's a space describing many particle system whose 1-particle piece is determined by the wave eqn.

to quantize $\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi$ means to construct a quantum state space for a many particle system, whose 1-particle subspace is determined by this wave equation. There are 2 possibilities — fermionic + bosonic Fock space

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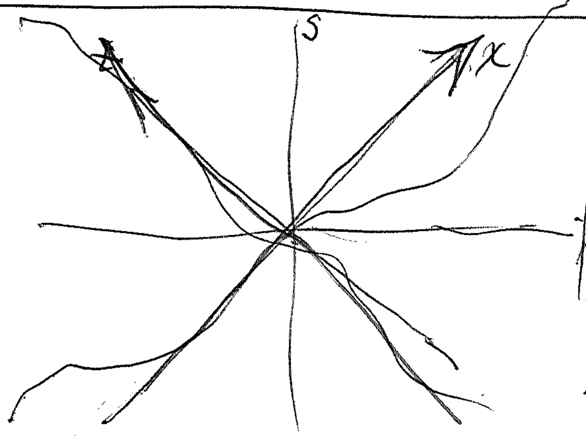
Lecture notes

$$\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi$$

$$(\partial_t - \partial_x)\psi^1 = ih\psi^2$$

$$(\partial_t + \partial_x)\psi^2 = i\bar{h}\psi^1$$

$$\boxed{\begin{aligned} -\partial_r \psi^1 &= ih\psi^2 \\ \partial_s \psi^2 &= i\bar{h}\psi^1 \end{aligned}}$$



$$\boxed{\begin{aligned} t &= -r+s \\ x &= r+s \end{aligned}}$$

$$\partial_r = -\partial_t + \partial_x \qquad \partial_r f = \partial_t f \overset{-1}{\frac{\partial t}{\partial r}} + \partial_x f \overset{1}{\frac{\partial x}{\partial r}}$$

$$\partial_s = \partial_t + \partial_x \qquad \partial_s f = \quad +1 \quad\quad +1$$

$$r = \frac{x-t}{2} \qquad s = \frac{x+t}{2}$$

$$\partial_t \psi = \varepsilon\partial_x \psi + iA\psi \qquad A = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}$$

$$\partial_t(\psi^*\psi) = (\varepsilon\partial_x\psi + iA\psi)^*\psi + \psi^*(\varepsilon\partial_x\psi + iA\psi) \qquad A^* = A$$

$$\partial_x\psi^*\varepsilon\psi + \cancel{\psi^*(-iA)\psi} + \psi^*\varepsilon\partial_x\psi + \cancel{\psi^* iA\psi}$$

$$\boxed{\partial_t(\psi^*\psi) = \partial_x(\psi^*\varepsilon\psi)}$$

$$\int \underbrace{\psi^*\psi\, dx + \psi^*\varepsilon\psi\, dt} \qquad \text{ind. of path.}$$

$$\psi^*\psi(dr+ds) + \psi^*\varepsilon\psi(-dr+ds)$$

$$= 2|\psi^1|^2 ds + 2|\psi^2|^2 dr$$

$$\cancel{+\partial_r(\bar{\psi}^1\psi^1) + \partial_s(\bar{\psi}^2\psi^2)}$$

$$\cancel{i\bar{h}\bar{\psi}^2\psi^1} + \bar{\psi}^2 i\bar{h}\psi^1$$

$$+\bar{\psi}^1(-ih\psi^2) \quad -ih\bar{\psi}^1\psi^2$$

$$-\partial_r \overline{\psi^1}\psi^1 = \overline{\psi^1}\, i h \psi^2 + \overline{i h \psi^2}\,\psi^1$$

$$\partial_s \overline{\psi^2}\psi^2 = \overline{\psi^2}\, i h \psi^1 + \overline{i h \psi^1}\,\psi^2$$

Green's fn. à la Riemann

$$\rho = \omega + k \qquad -\rho^{-1} = -\omega + k$$

$$\psi(x,t) = \int e^{ix\left(\frac{\rho-\rho^{-1}}{2}\right) - it\left(\frac{\rho+\rho^{-1}}{2}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

$$\psi(x,0) = \int e^{ixk}\left\{ \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} f(\omega+k) + \begin{pmatrix} 1 \\ \omega-k \end{pmatrix} f(-\omega+k) \right\} dk$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 \\ -\omega-k \end{pmatrix} = \begin{pmatrix} -\omega \\ \omega^2 + k\omega \end{pmatrix} = \begin{pmatrix} 1 \\ -\omega-k \end{pmatrix}(-\omega)$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 \\ \omega-k \end{pmatrix} = \begin{pmatrix} \omega \\ \omega^2 - k\omega \end{pmatrix} = \begin{pmatrix} 1 \\ \omega-k \end{pmatrix}(\omega)$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\omega-k & \omega-k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\omega-k & \omega-k \end{pmatrix}\begin{pmatrix} -\omega & 0 \\ 0 & \omega \end{pmatrix}$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\omega-k & \omega-k \end{pmatrix}\begin{pmatrix} -\omega & 0 \\ 0 & \omega \end{pmatrix}\begin{pmatrix} \omega-k & -1 \\ \omega+k & 1 \end{pmatrix}\frac{1}{2\omega}$$

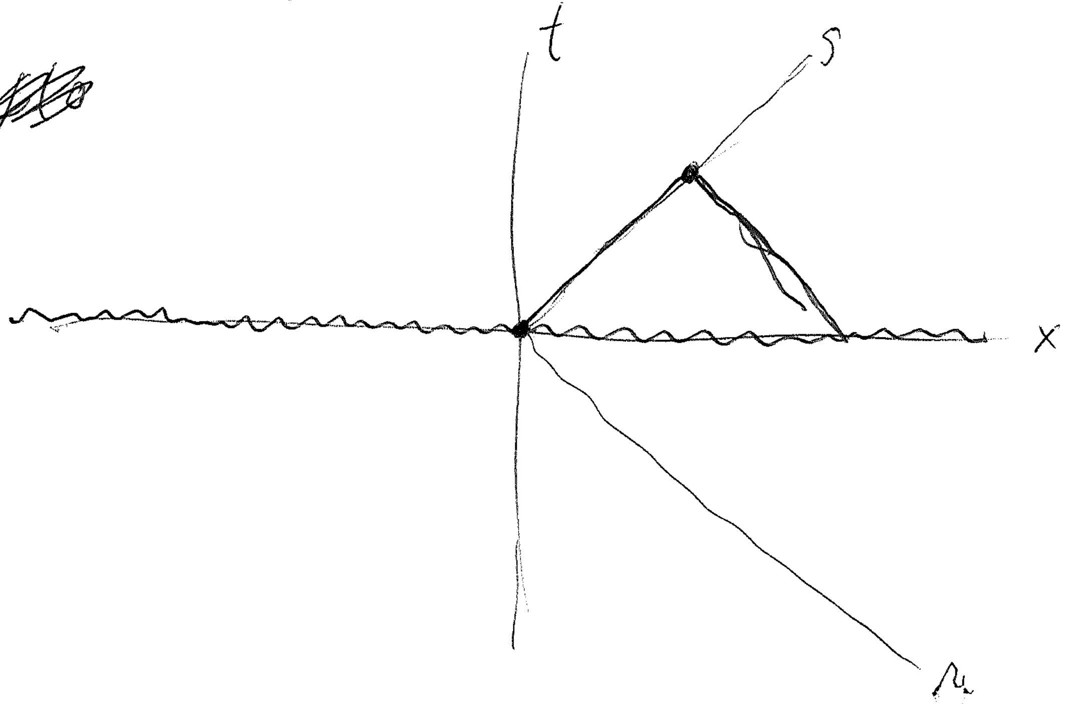
$$\begin{pmatrix} 1 \\ -\omega-k \end{pmatrix}\begin{pmatrix} \omega-k & -1 \end{pmatrix}\frac{1}{2\omega} = \begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix}\frac{1}{2\omega}$$

$$\psi(x,t) = \int e^{ixk}\left[\begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix}\frac{1}{2\omega}e^{-it\omega} + \begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix}\frac{1}{2\omega}e^{it\omega}\right]\hat{\psi}(k)\,\frac{dk}{2\pi}$$

$$\psi(x,t) = \int \frac{dk}{2\pi}\left\{ e^{i(kx-\omega t)}\frac{1}{2\omega}\begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix} + e^{i(kx+\omega t)}\frac{1}{2\omega}\begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix}\right] \times \int e^{-ikx'}\psi(x',0)\,dx'$$

Green's function

$$G(x,t) = \int \frac{dk}{2\pi}\, e^{i(kx-\omega t)}\, \frac{1}{2\omega} = \langle x | e^{tX} | 0 \rangle$$



$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi .$$ To solve Cauchy problem

$$\omega \hat{\psi} = \begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix} \hat{\psi} \qquad \omega \hat{\psi} = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \hat{\psi} \qquad \phi = \hat{\psi}$$

$$(\omega - k)\phi^1 = \phi^2$$
$$(\omega + k)\phi^2 = \phi^1 \qquad \omega^2 - k^2 = 1.$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \omega-k & -\omega-k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \omega-k & -\omega-k \end{pmatrix}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

$$\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \omega-k & -\omega-k \end{pmatrix}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}\begin{pmatrix} -\omega-k & -1 \\ -\omega+k & 1 \end{pmatrix}\frac{1}{-2\omega}$$

$$= \omega \begin{pmatrix} -\omega-k & -1 \\ -1 & -\omega+k \end{pmatrix}\frac{1}{-2\omega} + (-\omega)\begin{pmatrix} -\omega+k & 1 \\ 1 & -\omega-k \end{pmatrix}\frac{1}{-2\omega}$$

$$= \omega \begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix}\frac{1}{2\omega} + (-\omega)\begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix}\frac{1}{2\omega}$$

$$\psi(x,t) = e^{tX}\psi(x) = e^{it\begin{pmatrix}\frac{1}{i}\partial_x & 1\\ 1 & -\frac{1}{i}\partial_x\end{pmatrix}} \int e^{ikx}\hat{\psi}(k)\frac{dk}{2\pi}$$

$$= \int e^{ikx} e^{t\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}}\hat{\psi}(k)\frac{dk}{2\pi}$$

$$= \int e^{ikx}\left[\frac{e^{i\omega t}}{2\omega}\begin{pmatrix}\omega+k & 1\\ 1 & \omega-k\end{pmatrix} + \frac{e^{-i\omega t}}{2\omega}\begin{pmatrix}\omega-k & -1\\ -1 & \omega+k\end{pmatrix}\right]\hat{\psi}(k)\frac{dk}{2\pi}$$

Given $\psi(x,0) = \int e^{ikx}\hat{\psi}(k)\frac{dk}{2\pi}$

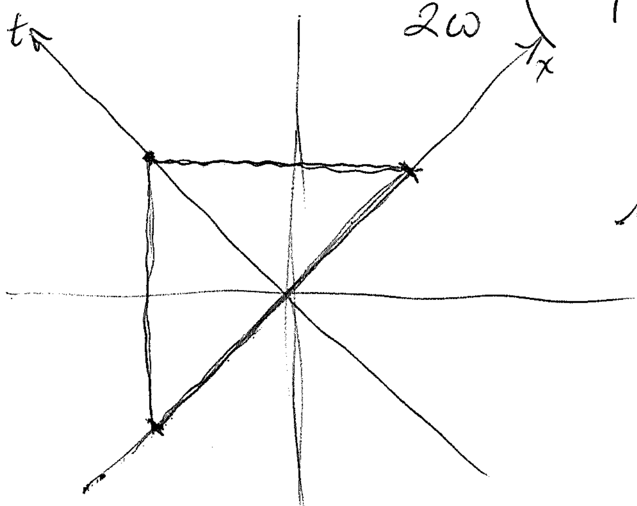
$$\psi(x,t) = e^{tX}\psi(x,0) = \int e^{ikx} e^{it\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}}\hat{\psi}(k)\frac{dk}{2\pi}$$

$$= \int\frac{dk}{2\pi} e^{ikx} e^{it\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}}\int dx'\, e^{-ikx'}\psi(x')\,dx'$$

$$\psi(x,t) = \int\frac{dk\,dx'}{2\pi} e^{ik(x-x')} e^{it\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}}\psi(x')$$

$$= \int dx'\, G(x-x',t)\,\psi(x')$$

$$G(x-x',t) = \int\frac{dk}{2\pi} e^{ikx} \underbrace{e^{it\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}}}_{\frac{e^{i\omega t}}{2\omega}\begin{pmatrix}\omega+k & 1\\ 1 & \omega-k\end{pmatrix} + \frac{e^{-i\omega t}}{2\omega}\begin{pmatrix}\omega-k & -1\\ -1 & \omega+k\end{pmatrix}} e^{-ikx'}$$



To see that $G(-x',t)$ is supported in $|x'| < t$

$$\frac{e^{i\omega t}+e^{-i\omega t}}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{e^{+i\omega t}-e^{-i\omega t}}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{k}{\omega}$$

$$+\frac{e^{i\omega t}-e^{-i\omega t}}{2\omega}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$=\cos(\omega t)\begin{pmatrix}1&0\\0&1\end{pmatrix}+i\frac{\sin\omega t}{\omega}\begin{pmatrix}k&1\\1&-k\end{pmatrix}$$

$$X=i\begin{pmatrix}k&1\\1&-k\end{pmatrix}$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots=\sum\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$e^{tX}=\underbrace{\sum_{n\geq 0}\frac{1}{(2n)!}t^{2n}\underbrace{X^{2n}}_{(i\omega)^{2n}}}_{\cos(\omega t)}+\underbrace{\sum_{n\geq 0}\frac{1}{(2n+1)!}t^{2n+1}\underbrace{X^{2n+1}}_{\omega(i\omega)^{2n}X}\omega}_{\frac{\sin(\omega t)}{\omega}X}$$

$$\boxed{e^{t\tilde{X}}=\cos(\omega t)+\frac{\sin(\omega t)}{\omega}\tilde{X}}\qquad \tilde{X}=i\begin{pmatrix}k&1\\1&-k\end{pmatrix}$$

$$\boxed{e^{it\begin{pmatrix}k&1\\1&-k\end{pmatrix}}=\cos(\omega t)I+i\frac{\sin(\omega t)}{\omega}\begin{pmatrix}k&1\\1&-k\end{pmatrix}}$$

So $$\psi(x,t)=e^{it\begin{pmatrix}\frac{1}{i}\partial_x&1\\1&-\frac{1}{i}\partial_x\end{pmatrix}}\int\frac{dk}{2\pi}e^{ikx}\int dx'\,e^{-ikx'}\psi(x')$$

$$=\int\frac{dk}{2\pi}dx'\,e^{ik(x-x')}e^{it\begin{pmatrix}k&1\\1&-k\end{pmatrix}}\psi(x')$$

$$\psi(0,t)=\int\frac{dk}{2\pi}dx'\,e^{-ikx'}e^{it\begin{pmatrix}k&1\\1&-k\end{pmatrix}}\psi(x')$$

$$=\int\frac{dk}{2\pi}e^{it\begin{pmatrix}k&1\\1&-k\end{pmatrix}}\hat{\psi}(k)$$

So if $\psi(x,0) = \int \frac{dk}{2\pi} e^{ikx} \hat{\psi}(k)$

then $\psi(0,t) = \int \frac{dk}{2\pi} \underbrace{e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}}_{\left(\cos(\omega t) I + i \frac{\sin(\omega t)}{\omega} \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\right)} \hat{\psi}(k)$

where $\omega = +\sqrt{k^2+1}$

Let's go the other way

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi \qquad \begin{pmatrix} \partial_x - \partial_t & i \\ -i & +\partial_x + \partial_t \end{pmatrix} \psi = 0$$

$$\partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi \qquad \psi(x,t) = \int \frac{d\omega}{2\pi} e^{it\omega} \hat{\psi}(x,\omega)$$

$$\partial_x \hat{\psi} = \begin{pmatrix} i\omega & -i \\ i & -i\omega \end{pmatrix} \hat{\psi} \qquad \frac{1}{i} \partial_x \hat{\psi} = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \hat{\psi}$$

$$\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} = \begin{pmatrix} \omega^2 - 1 & 0 \\ 0 & \omega^2 - 1 \end{pmatrix}$$

$$\hat{\psi}(x,\omega) = e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}} \psi \qquad ?$$

$$\hat{\psi}(x,\omega) = \int e^{-i\omega t} \psi(x,t)\, dt$$

$$\frac{1}{i} \partial_x \hat{\psi}(x,\omega) = \int e^{-i\omega t} \frac{1}{i} \partial_x \psi(x,t)\, dt$$

$$= \int e^{-i\omega t} \begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix} \psi(x,t)\, dt$$

$$\psi(x,t) = \int e^{i\omega t}\hat{\psi}(x,\omega)\frac{d\omega}{2\pi}$$

$$\frac{1}{i}\partial_t\psi(x,t) = \int e^{i\omega t}\,\omega\,\hat{\psi}(x,\omega)\frac{d\omega}{2\pi}$$

$$\begin{pmatrix}\frac{1}{i}\partial_t & -1\\ 1 & -\frac{1}{i}\partial_t\end{pmatrix}\psi(x,t) = \int e^{i\omega t}\begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}\hat{\psi}(x,\omega)\frac{d\omega}{2\pi}$$

$$\parallel$$

$$\frac{1}{i}\partial_x\psi(x,t) = \int e^{i\omega t}\frac{1}{i}\partial_x\hat{\psi}(x,\omega)\frac{d\omega}{2\pi}$$

$$\therefore\quad \frac{1}{i}\partial_x\hat{\psi}(x,\omega) = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}\hat{\psi}(x,\omega)$$

$$\hat{\psi}(x,\omega) = e^{ix\begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}}\hat{\psi}(0,\omega)$$

$$\therefore\quad \psi(x,t) = \int\frac{d\omega}{2\pi}e^{i\omega t}e^{ix\begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}}\int dt'\,e^{-i\omega t'}\psi(0,t')$$

$$\boxed{\psi(x,0) = \int\frac{d\omega}{2\pi}e^{ix\begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}}\int dt'\,e^{-i\omega t'}\psi(0,t')}$$

$$B = \begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}\qquad B^2 = \omega^2 - 1$$

$$e^{ixB} = \underbrace{\sum_{n\geq 0}\frac{(-1)^n}{(2n)!}\underbrace{(xB)^{2n}}_{x^{2n}(\omega^2-1)^n}}_{} + \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}\overbrace{(xB)^{2n}\,ixB}^{(x\sqrt{\omega^2-1})^{2n+1}\frac{ixB}{x\sqrt{\omega^2-1}}}$$

$$\cos(x\sqrt{\omega^2-1})\,I + i\frac{\sin(x\sqrt{\omega^2-1})}{\sqrt{\omega^2-1}}\begin{pmatrix}\omega & -1\\ 1 & -\omega\end{pmatrix}$$

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \qquad \begin{pmatrix} \partial_x - \partial_t & i \\ -i & +\partial_x + \partial_t \end{pmatrix} \psi = 0$$

$$\partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi \qquad \boxed{\frac{1}{i}\partial_x \psi = \begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix} \psi}$$

solve this DE with Cauchy data on $x = 0$.

$$\psi(x,t) = \underbrace{e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}} \psi(0,t)}$$

$$= e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}} \int e^{i\omega t} \hat{\psi}(\omega) \frac{d\omega}{2\pi} \qquad \hat{\psi}(\omega) = \int e^{-i\omega t} \psi(0,t)\, dt$$

$$= \int e^{i\omega t} \underbrace{e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}}} \hat{\psi}(\omega) \frac{d\omega}{2\pi}$$

$$B^2 = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}^2 = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} = \begin{pmatrix} \omega^2 - 1 & 0 \\ 0 & \omega^2 - 1 \end{pmatrix}$$

$$e^{ixB} = \underbrace{\sum_{n \geq 0} \frac{1}{(2n)!} (ixB)^{2n}} + \underbrace{\sum_{n \geq 0} \frac{1}{(2n+1)!} (ixB)^{2n}\, ixB}_{\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} \left(x\sqrt{\omega^2-1}\right)^{2n+1} \frac{iB}{\sqrt{\omega^2-1}}}$$

$$\underbrace{\sum_{n \geq 0} \frac{(-1)^n}{(2n)!} \left(x\sqrt{\omega^2 - 1}\right)^{2n} I} + \frac{i}{\sqrt{\omega^2 - 1}} \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} \left(x\sqrt{}\right)^{2n+1} B$$

$$\cos\left(x\sqrt{\omega^2 - 1}\right) I + \frac{\sin\left(x\sqrt{\omega^2 - 1}\right)}{\sqrt{\omega^2 - 1}}\, iB$$

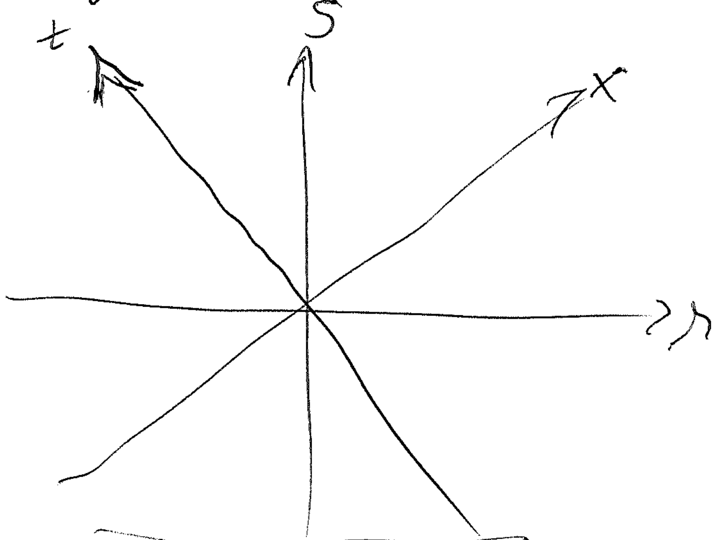
So what is important? You can have

$|\omega| < 1$ so that $k = \sqrt{\omega^2 - 1} = \pm i\sqrt{1 - \omega^2}$

You have to go over this.

Start again: Basic eqn. $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$

Shift to characteristic coords.



$\partial_r = -\partial_t + \partial_x$ | $t = -r + s$
$\partial_s = \partial_t + \partial_x$ | $x = r + s$

$r = \frac{x-t}{2}$

$s = \frac{x+t}{2}$

$$\boxed{\begin{aligned} -\partial_r \psi^1 &= i\psi^2 \\ \partial_s \psi^2 &= i\psi^1 \end{aligned}}$$

$$\int e^{i(r\rho + s\sigma)} f(\rho, \sigma) \frac{d\rho\, d\sigma}{(2\pi)^2}$$

$-\rho\psi^1 = \psi^2$
$\sigma\psi^2 = \psi^1$

$-\rho\sigma = 1$
$\sigma = -\rho^{-1}$

$$\int e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

describes grid space, or all solutions.

$$r\rho - s\rho^{-1} = \frac{x-t}{2}\rho - \frac{x+t}{2}\rho^{-1} = x\left(\frac{\rho - \rho^{-1}}{2}\right) - t\left(\frac{\rho + \rho^{-1}}{2}\right)$$

So you have this pretty picture of

solution to the DE.

$$\psi(x,t) = \int e^{i\left(x\underbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}_{k} - t\underbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}_{\omega}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

So now look at the things you were doing.

~~What is $\psi(x,0)$ Fourier~~ Look at the Cauchy problem along $t=0$.

$$\psi(x,0) = \int e^{ix\left(\frac{\rho-\rho^{-1}}{2}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

Can we find $f(\rho)$

$$\int e^{-ikx}\psi(x,0)\,dx = \int dx \int d\rho\; e^{ix\left(\frac{\rho-\rho^{-1}}{2} - k\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

$$= \int d\rho\; 2\pi\, \delta\left(\frac{\rho-\rho^{-1}}{2} - k\right) \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$$

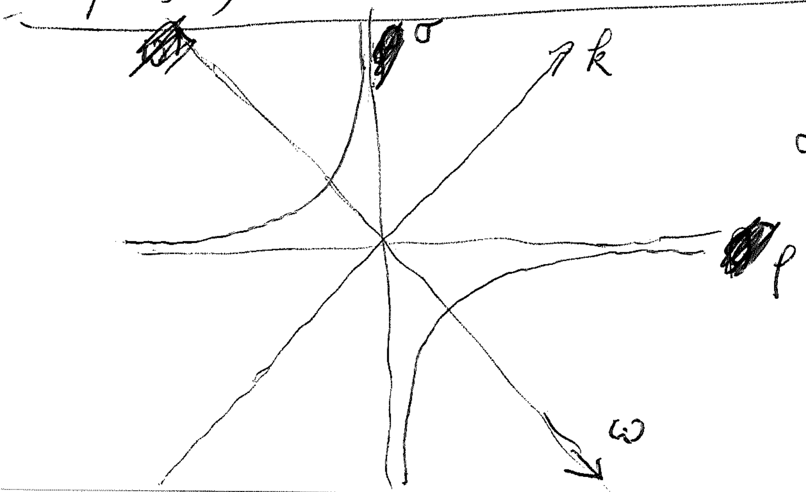
Problems. ~~But~~ with δ being a density

Instead look at Cauchy problem along $x=0$.

$$\psi(0,t) = \int e^{-it\left(\frac{\rho+\rho^{-1}}{2}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

~~Go back to~~ Go back to $\psi(x,0)$. Suppose

$\psi(x,0) = e^{ikx}$ $\quad k \in \mathbb{R}$



$$\rho = \omega + k$$
$$\sigma = -\rho^{-1} = -\omega + k$$

Repeat $\quad \partial_t\psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi \qquad \frac{1}{i}\partial_t\psi = \underbrace{\begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix}}\psi$

$$\frac{1}{i}\partial_x\psi = \begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}\psi$$

$$\begin{pmatrix} \frac{1}{i}\partial_x & 1 \\ 1 & -\frac{1}{i}\partial_x \end{pmatrix} \rightsquigarrow \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}$$

eigenv. $\lambda = \pm\sqrt{k^2+1}$

~~$\omega \in$ spectrum $\Longleftrightarrow$ $|\omega| \geq 1$.~~

$$\begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix} \rightsquigarrow \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$$

eigenvalues

$\lambda = \pm\sqrt{\omega^2-1}$

imaginary ~~not real~~ for $|\omega| < 1$.

Solve Cauchy problem

Given $\quad \psi(x,0) = \psi(x) \qquad x \in \mathbb{R}$

$$\psi(x) = \int e^{ikx}\hat{\psi}(k)\frac{dk}{2\pi} \qquad \hat{\psi}(k) = \int e^{-ikx}\psi(x)\,dx$$

$$\psi(x,t) = e^{tX}\psi(x) \qquad X = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}$$

$$= e^{tX}\int e^{ikx}\hat{\psi}(k)\frac{dk}{2\pi} = \int e^{ikx}e^{t\begin{pmatrix} ik & i \\ i & -ik \end{pmatrix}}\hat{\psi}(k)\frac{dk}{2\pi}$$

$$\boxed{\psi(x,t) = \int e^{ikx}e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}\hat{\psi}(k)\frac{dk}{2\pi}}$$

$\omega = \pm\sqrt{k^2+1}$

$$A = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \qquad A^2 = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2+1 \end{pmatrix} = \omega^2 I$$

$$e^{itA} = \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}t^{2n}\omega^{2n}I}_{\cos(\omega t)I} + \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}t^{2n}\omega^{2n}\,itA}_{\frac{\sin(\omega t)}{\omega}i\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}}$$

$$\psi(x) \mapsto \psi(0,t) = \iint \left\{ e^{i\omega t} \frac{1}{2\omega} \begin{pmatrix} \omega+k & 1 \\ 1 & \omega-k \end{pmatrix} + e^{-i\omega t} \frac{1}{2\omega} \begin{pmatrix} \omega-k & -1 \\ -1 & \omega+k \end{pmatrix} \right\} \hat{\psi}(k) \frac{dk}{2\pi}$$

So what else?

$$\psi(t) = \psi(0,t) \qquad \psi(x,t) = e^{ix\begin{pmatrix} \frac{1}{i}\partial_t & -1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}} \psi(t)$$

$$\psi(t) = \int e^{i\omega t} \hat{\psi}(\omega) \frac{d\omega}{2\pi} \qquad \hat{\psi}(\omega) = \int e^{-i\omega t} \psi(t)\, dt$$

$$\psi(x,t) = \int e^{i\omega t} e^{ix \underbrace{\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}}_{B}} \hat{\psi}(\omega) \frac{d\omega}{2\pi}$$

$$B^2 = (\omega^2 - 1) I \qquad k = \sqrt{\omega^2 - 1}$$

$$e^{ixB} = \cos(kx) I + \frac{\sin(kx)}{k} i \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \qquad 0$$

$$= e^{ikx} \frac{1}{2k} \begin{pmatrix} k+\omega & -1 \\ 1 & k-\omega \end{pmatrix} + e^{-ikx} \frac{1}{2k} \begin{pmatrix} k-\omega & 1 \\ -1 & k+\omega \end{pmatrix}$$

$$\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \begin{pmatrix} k+\omega & +1 \\ 1 & k+\omega \end{pmatrix} = \begin{pmatrix} \omega k + \overbrace{\omega^2 - 1}^{k^2} & -k \\ k & \underbrace{+1 - \omega^2}_{-k^2} - \omega k \end{pmatrix}$$

$$= \begin{pmatrix} k+\omega & +1 \\ 1 & k+\omega \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$$

Review: $\partial_t \psi = \begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}\psi \qquad \partial_x \psi = \begin{pmatrix}\partial_t & -i \\ i & -\partial_t\end{pmatrix}\psi$

$(\partial_t - \partial_x)\psi^1 = i\psi^2$ to solve the D.E.

$(\partial_t + \partial_x)\psi^2 = i\psi^1$ with $\psi(0,t) = \psi_0(t)$.

$$\psi(x,t) = e^{x\begin{pmatrix}\partial_t & -i \\ i & -\partial_t\end{pmatrix}}\psi_0(t) = \int e^{x(\;)} e^{i\omega t}\,\hat{\psi}_0(\omega)\,\frac{d\omega}{2\pi}$$

$$= \int e^{i\omega t}\, e^{ix\underbrace{\begin{pmatrix}\omega & -1 \\ 1 & -\omega\end{pmatrix}}_{B}}\,\hat{\psi}_0(\omega)\,\frac{d\omega}{2\pi}$$

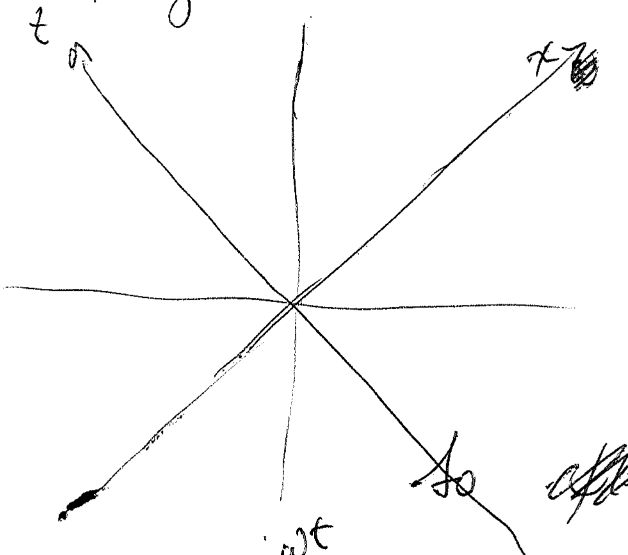
$$B^2 = (\omega^2-1)\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} \qquad e^{ixB} = \overbrace{\sum_{n\geq 0}\frac{(-1)^n}{(2n)!}(xB)^{2n}}^{x^{2n}(\omega^2-1)^n\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}} + \sum_{n\geq 0}\overbrace{\frac{(-1)^n}{(2n+1)!}}^{(x^2k^2)^n}(xB)^{2n+1}$$

$$e^{ixB} = \cos\big(x\overbrace{\sqrt{\omega^2-1}}^{k}\big)\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + i\,\frac{\sin(kx)}{k}\begin{pmatrix}\omega & -1 \\ 1 & -\omega\end{pmatrix}$$

$$= e^{ikx}\frac{1}{2k}\begin{pmatrix}k+\omega & -1 \\ 1 & k-\omega\end{pmatrix} + e^{-ikx}\frac{1}{2k}\begin{pmatrix}k-\omega & 1 \\ -1 & k+\omega\end{pmatrix}$$

Your problem is that if $-1 < \omega < 1$, then $k = \sqrt{\omega^2-1}$ is imaginary. ~~to take something imaginary~~

What might be an interesting $\psi_0(t)$? Id matrix.



$$\psi_0(t) = 1 \quad \forall t.$$

$$\psi_0(t) = \int e^{i\omega t}\,\hat{\psi}_0(\omega)\,\frac{d\omega}{2\pi}$$

$$\hat{\psi}_0(\omega) = 2\pi\,\delta(\omega)$$

$\omega^2 = k^2+1$

$0 = k^2+1$

$k = \pm i$

~~So~~

$$\psi(x,t) = \int e^{i\omega t}\left[\cos(kx)\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + i\,\frac{\sin(kx)}{k}\begin{pmatrix}\omega & -1 \\ 1 & -\omega\end{pmatrix}\right]\delta(\omega)\,d\omega$$

$$= \cosh(x) + i\sinh(x)\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$$

$$\psi(x,t) = \begin{pmatrix} \cosh x & -i\sinh x \\ i\sinh x & \cosh x \end{pmatrix}$$

$$\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\begin{pmatrix} \cosh x \\ i\sinh x \end{pmatrix} = \begin{pmatrix} \sinh x + i^2 \sinh x \\ i\cosh - i\cosh \end{pmatrix} = 0$$

$U(n,n)$ = autos of Krein space $H_+ \oplus H_- = \mathbb{C}^n \oplus \mathbb{C}^n$ with $IH(v) = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_+ \\ v_- \end{pmatrix}$ polarizations are the same as strict contractions $\gamma : H_+ \to H_-$. This is the unit disk model for the symmetric space. $U(n,n)/U(n)\times U(n)$. The boundary consists of contractions having some isometric part.

How is this related, if at all, to the CCR? There are commutators. Data should be a complex v.s. equipped with non-deg. skew-symm form ω and conjugation σ. Model is **bracket** and adjoint $*$. Fermionic version with anti-commutator and adjoint.

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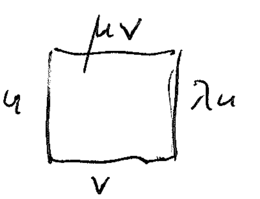
new project, back to discrete translation invariant grid space, to get an analog of ω, k



$$k\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

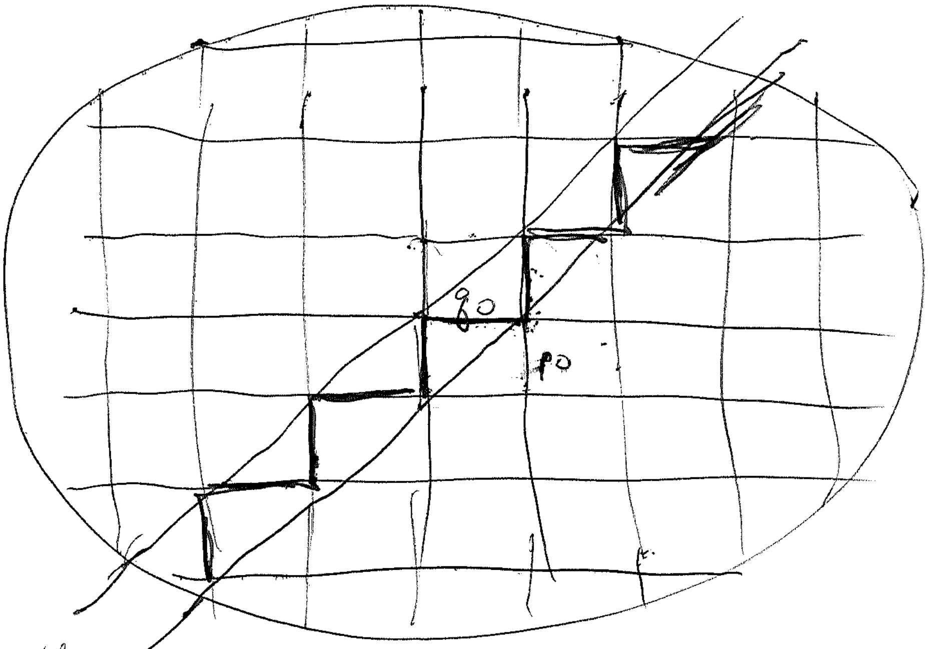
$$(k\lambda - 1)u = hv$$
$$(k\mu - 1)v = \bar{h}u$$

you need unitary operator $\quad \mu\lambda^{-1}$

You are interested in relating increasing and decreasing staircases.

Set up, go over $\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix} \psi$ $\quad h = h(x)$.

$$\omega \psi = \begin{pmatrix} \frac{1}{i}\partial_x & h \\ \bar{h} & -\frac{1}{i}\partial_x \end{pmatrix}$$



In the discrete case it's certainly clear that either staircase ╲ ╱ spans. But what happened in the continuous case.

$$\psi(x,t) = e^{t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}} \psi_0(x) = \int e^{ikx} e^{it\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}} \hat{\psi}_0(k) \frac{dk}{2\pi}$$

$$\psi(0,t) = \int \left(e^{i\omega t}\left(\quad \right) + e^{-i\omega t}\left(\quad \right) \right) \hat{\psi}_0(k) \frac{dk}{2\pi}$$

So the spectrum of $\psi(0,t)$, i.e. those ω occurring are $|\omega| > 1$.

$$\psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}} \psi_0(t) = \int e^{i\omega t} e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}} \hat{\psi}_0(\omega) \frac{d\omega}{2\pi}$$

$$\psi(x,0) = \int \left(e^{ikx}\left(\quad \right) + e^{-ikx}\left(\quad \right) \right) \hat{\psi}_0(\omega) \frac{d\omega}{2\pi}$$

Consider $(k\lambda-1)\psi^1 = h\psi^2$ $\quad k, h>0$, $k^2 = 1-h^2$

$(k\mu-1)\psi^2 = h\psi^1$

λ, μ unit shifts $\uparrow \rightarrow$ $\quad \tau = \mu\lambda^{-1}$ $\quad \xi = \lambda\mu$.

Go over Wronskian stuff.

First $V \simeq \mathbb{C}^2$ let it be equipped with a conjugation, eg $\sigma\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar z_2 \\ \bar z_1 \end{pmatrix}$ and a volume $\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$, eg. $v \wedge v' \longmapsto -\begin{vmatrix} v_1 & v_1' \\ v_2 & v_2' \end{vmatrix}$

Then $H(v, v') \overset{\text{def}}{=} \omega(\sigma v \wedge v')$ is sesq. and is hermitian symm. provided $\omega(\sigma v \wedge \sigma v') = -\overline{\omega(v \wedge v')}$

$$\overline{H(v', v)} = \overline{\omega(\sigma v' \wedge v)} = -\omega(v' \wedge \sigma v) = \omega(\sigma v \wedge v')$$
$$= H(v, v').$$

example $\omega(\sigma v \wedge v') = -\begin{vmatrix} \bar v_2 & v_1' \\ \bar v_1 & v_2' \end{vmatrix} = \bar v_1 v_1' - \bar v_2 v_2'$

Conversely given $H(v, v')$ herm. symm. and a conj. σ define $B(v, v') = H(\sigma v, v')$ $\quad$ B bilinear

Assume $H(\sigma v, \sigma v') = \overline{H(v, v')}$. Then

$$B(v', v) = H(\sigma v', \sigma(\sigma v)) = \overline{H(v', \sigma v)} = H(\sigma v, v')$$

$B(v, v') = H(\sigma v, v')$ sets up equiv between bilinear B and sesquilinear H.

$$B(\sigma v, \sigma v') = H(v, \sigma v')$$
$$\overset{\shortparallel}{\pm \overline{B(\sigma v', \sigma v)}} = \overline{H(v', \sigma v)}$$

$$B(\sigma\xi, \xi') = H(\sigma\xi, \xi')$$ equiv. between B bilinear, H sesquilin

Reality condition

$$\frac{B(\sigma\xi, \sigma\xi')}{B(\xi,\xi')} \qquad H$$

---

bilin equiv sesquilin

$$H(\xi, \xi') = B(\sigma\xi, \xi')$$

A

$$\overline{H(v, v')} = \bar{v}^t A v'$$

$$H(\sigma v, \sigma v') = v^t A \overline{v'}$$

<u>H real means</u>

$$\overline{H(\sigma\xi, \sigma\xi')} = B(\sigma^2\xi, \sigma\xi')$$
$$\| \qquad\qquad \|$$
$$H(\xi, \xi') = B(\sigma\xi, \xi')$$

<u>B real means</u>

$$\overline{B(\sigma\xi, \sigma\xi')} = B(\xi, \xi')$$

Assume H real and herm. skew symm.

$$H(\xi, \xi') + \overline{H(\xi', \xi)} = 0$$

$$B(\sigma\xi, \xi') + \underbrace{\overline{B(\sigma\xi', \xi)}}_{=\,B(\xi', \sigma\xi)}$$

$$\bar{A} = A$$

V complex vector space
σ conjugation on V
$B(\xi,\xi')$ skew-symm. bil. form on V
satisfying $B(\sigma\xi,\sigma\xi') = \overline{B(\xi,\xi')}$.

Put $H(\xi,\xi') = B(\sigma\xi,\xi')$.
Then H is sesqui-linear anti in ξ
lin in ξ'

Also
$$\overline{H(\xi,\xi')} = \overline{B(\sigma\xi,\xi')} = B(\sigma\sigma\xi,\sigma\xi')$$
$$= B(\xi,\sigma\xi') = -B(\sigma\xi',\xi) = -H(\xi',\xi)$$

so $iB(\sigma\xi,\xi')$ is hermitian symmetric when B is bilinear skew-symm, real wrt σ

$$\boxed{\overline{iB(\sigma\xi,\xi')} = -iB(\xi,\sigma\xi') = iB(\sigma\xi',\xi).}$$

B bilinear symm, real wrt σ, $\Rightarrow$
$$\overline{B(\sigma\xi,\xi')} = B(\xi,\sigma\xi') = B(\sigma\xi',\xi)$$

so $B(\sigma\xi,\xi')$ is herm. symm.

Now add $(B_{symm} + iB_{skew})(\sigma\xi,\xi')$ herm. symm.

~~$(v^* A v')$ = $v^t \bar{A} \bar{v'}$~~

$$\overline{\bar{v}^t A v'} = v^t \bar{A}\bar{v'}$$
$$= \bar{v'}^t \bar{A}^t v$$

Wronskian for $\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi$ $\quad h = h(x)$ 947

Conjugation. First recall as much as you can and then review notes. Given $\psi(x,t)$ a solution

$$(\sigma\psi)(x,t) = \begin{pmatrix} \psi^2(x,-t)^* \\ \psi^1(x,-t)^* \end{pmatrix} = \begin{pmatrix} \bar{\psi}^2 \\ \bar{\psi}^1 \end{pmatrix}(x,-t).$$

$$^{\sigma}(\partial_t \psi) = -\partial_{+t}\, {}^{\sigma}\psi \qquad {}^{\sigma}\left(\begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix}\psi\right) = \begin{pmatrix} -\partial_x & -ih \\ -i\bar{h} & \partial_x \end{pmatrix} {}^{\sigma}\psi$$

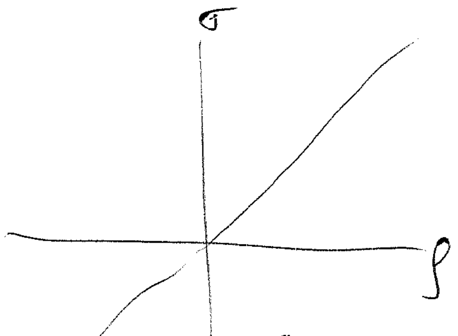
$$\psi(r,s) = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})}\begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

$$^{\sigma}\psi(r,s) = \int_{-\infty}^{\infty} e^{-i(r\rho - s\rho^{-1})}\begin{pmatrix} -\rho \\ +1 \end{pmatrix} \quad ?$$

$$-\rho\,\hat{\psi}^1 = \hat{\psi}^2 \qquad \psi(r,s) = \int e^{i(r\rho + s\sigma)}\begin{pmatrix} \hat{\psi}^1 \\ \hat{\psi}^2 \end{pmatrix}$$

$$\sigma\,\hat{\psi}^2 = \hat{\psi}^1$$

$$^{\sigma}\psi(r,s) = \int e^{-i\,{}^{\sigma}(r\rho + s\sigma)}\begin{pmatrix} \bar{\hat{\psi}}^2 \\ \bar{\hat{\psi}}^1 \end{pmatrix}$$



$$^{c}(r\rho + s\sigma)$$

$$t = -r + s$$
$$x = r + s$$

$$-t = -{}^{c}r + {}^{c}s$$
$$x = {}^{c}r + {}^{c}s$$

$$^{c}(r\rho + s\sigma) = s\sigma + r\rho$$

$$\frac{x-t}{2} = {}^{c}s = r$$

$$\frac{x+t}{2} = {}^{c}r = s$$

$$\left.\begin{array}{l} -\partial_r \psi^1 = i\psi^2 \\ \partial_s \psi^2 = i\psi^1 \end{array}\right| \mapsto \left(-\partial_s\right.$$

$$-\partial_r \psi^1(r,s) = i\psi^2(r,s)$$

$$-\partial_s \bar{\psi}^1(s,r) = -i\bar{\psi}^2(s,r)$$

$$-\partial_r \psi^1(r,s) = i\psi^2(r,s) \qquad +\partial_s \overline{\psi^1(s,r)} = +i\overline{\psi^2(s,r)}$$
$$\partial_s \psi^2(r,s) = i\psi^1(r,s) \qquad (-\partial_r)\overline{\psi^2(s,r)} = +i\overline{\psi^1(s,r)}$$

$\therefore$ If $\psi(r,s)$ is a soln so is $({}^c\psi)(r,s) = \begin{pmatrix} \overline{\psi^2(s,r)} \\ \overline{\psi^1(s,r)} \end{pmatrix}$

Check $\quad \psi(r,s) = \int e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho) \frac{d\rho}{\rho}$

$$\Big\downarrow$$

$$\begin{pmatrix} \overline{\psi^2(s,r)} \\ \overline{\psi^1(s,r)} \end{pmatrix} = \int e^{-i(s\rho - r\rho^{-1})} \begin{pmatrix} -\rho \\ 1 \end{pmatrix} \overline{f(\rho)} \frac{d\rho}{\rho}$$

$$\| $$

c seems to send $\rho = \omega + k$ to $\rho^{-1} = \omega - k$?

$$\int e^{i(r\rho^{-1} - s\rho)} \begin{pmatrix} -\rho \\ 1 \end{pmatrix} \overline{f(\rho)} \frac{d\rho}{\rho}$$

$$\|$$

change to +

$$\int e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} -\rho^{-1} \\ 1 \end{pmatrix} \overline{f(\rho^{-1})} \left(\cancel{-}\frac{d\rho}{\rho}\right)$$

$$\|$$

$$\int e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} (-\rho^{-1}) \overline{f(\rho^{-1})} \frac{d\rho}{\rho}$$

So it seems that conjugation sends $f(\rho)$ to $(-\rho^{-1})\overline{f(\rho^{-1})}$. Maybe can do simpler by considering exp. solution

$$\psi(r,s) = e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} \xmapsto{\ c\ } e^{i(r\rho^{-1} - s\rho)} \begin{pmatrix} -\rho \\ 1 \end{pmatrix}$$

$$\partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix} \psi \qquad \partial_x \psi = \begin{pmatrix} \partial_t & -ih \\ i\bar{h} & -\partial_t \end{pmatrix} \psi$$

conjugation $\quad ({}^c\psi)(x,t) = \begin{pmatrix} \overline{\psi^2} \\ \overline{\psi^1} \end{pmatrix}(x,-t).$

$$\left({}^c(\partial_t \psi)\right)(x,t) = \begin{pmatrix} \partial_t \overline{\psi^2}(x,-t) \\ \partial_t \overline{\psi^1}(x,-t) \end{pmatrix} = \begin{pmatrix} (\overline{\partial_t \psi^2}) \\ (\partial_t \overline{\psi^1}) \end{pmatrix}(x,-t).$$

$${}^c(\partial_t \psi) = \begin{pmatrix} (\overline{\partial_t \psi^2})(x,-t) \\ (\partial_t \overline{\psi^1})(x,-t) \end{pmatrix} = -\partial_t \begin{pmatrix} \overline{\psi^2}(x,-t) \\ \overline{\psi^1}(x,-t) \end{pmatrix} = -\partial_t {}^c\psi$$

$${}^c\left(\begin{pmatrix} \partial_x & ih(x) \\ i\bar{h}(x) & -\partial_x \end{pmatrix} \psi\right) = \begin{pmatrix} -\partial_x & -i\bar{h} \\ -ih & \partial_x \end{pmatrix} \begin{pmatrix} \overline{\psi^2}(x,-t) \\ \overline{\psi^1}(x,-t) \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi^1(x,t) \\ \psi^2(x,t) \end{pmatrix} \qquad {}^c\psi \overset{\text{def}}{=} \begin{pmatrix} \overline{\psi^2}(x,-t) \\ \overline{\psi^1}(x,-t) \end{pmatrix}$$

So the transfer matrix for $\partial_x \hat{\psi} = \begin{pmatrix} i\omega & -ih \\ i\bar{h} & -i\omega \end{pmatrix} \hat{\psi}$

lies in $SU(1,1)$, If $\phi(x,\omega)$ $\psi(x,\omega')$ two solutions, then

$$\partial_x \begin{vmatrix} \phi^1 & \psi^1 \\ \phi^2 & \psi^2 \end{vmatrix} = 0$$

$$\partial_x = i\omega\varepsilon + A \qquad A = A^*$$

$$\partial_x (\phi \wedge \psi) = \cancel{i\omega\varepsilon\phi \wedge \psi + \phi \wedge i\omega'\phi}$$

$$= i\omega\varepsilon\phi \wedge \psi + \phi \wedge i\omega'\varepsilon\psi = i(\omega+\omega')$$
$$+ A\phi \wedge \psi + \phi \wedge A\psi$$

$$\varepsilon\phi\wedge\psi + \phi\wedge\varepsilon\psi = 0 \qquad \text{as } tr(\varepsilon) = 0$$

$$\begin{vmatrix}\phi_1 & \psi_1\\ -\phi_2 & \psi_2\end{vmatrix} + \begin{vmatrix}\phi_1 & \psi_1\\ \phi_2 & -\psi_2\end{vmatrix} = 0$$

$$\partial_x(\phi\wedge\psi) = i\omega\varepsilon\phi\wedge\psi + \phi\wedge i\omega'\varepsilon\psi$$
$$= i(\omega-\omega')\varepsilon\phi\wedge\psi.$$

So given $\partial_x\phi_\omega = (i\omega\varepsilon + A)\phi_\omega$ $\quad tr\, A = 0 \quad$ $\partial_x\psi_{\omega'} = (i\omega'\varepsilon + A)\psi_{\omega'}$

get $$\boxed{\partial_x(\phi_\omega\wedge\psi_{\omega'}) = i(\omega-\omega')\underbrace{(\varepsilon\phi_\omega\wedge\psi_{\omega'})}_{\begin{vmatrix}\phi^1_\omega & \psi^1_{\omega'}\\ -\phi^2_\omega & \psi^2_{\omega'}\end{vmatrix}}}$$

<u>Note</u>: might be more useful than just the assertion for $\omega = \omega'$. In fact

$$\left[\frac{\phi_\omega\wedge\psi_{\omega'}}{i(\omega-\omega')}\right]_a^b = \int_a^b \varepsilon\phi_\omega\wedge\psi_{\omega'}\,dx \qquad \text{obviously useful}$$

Try to organize the ideas.

$$\partial_t\psi = \underbrace{\begin{pmatrix}\partial_x & ih\\ i\bar{h} & -\partial_x\end{pmatrix}}_{X}\psi \qquad h = h(x)$$

$$\partial_t(\psi^*\psi) = (X\psi)^*\psi + \psi^* X\psi$$
$$= (\varepsilon\partial_x\psi)^*\psi + \psi^*\varepsilon\partial_x\psi = \partial_x(\psi^*\varepsilon\psi)$$
$$\cancel{(iA\psi)^*\psi} + \cancel{\psi^*(iA\psi)}$$

what else? $\varepsilon\psi_\omega \wedge \psi_{\omega'} = \begin{vmatrix} \overline{\psi^2_\omega} & \psi^1_{\omega'} \\ -\overline{\psi^1_\omega} & \psi^2_{\omega'} \end{vmatrix}$

get notation straight.

Let $X\psi_\omega = i\omega\psi_\omega$ $\qquad X = \begin{pmatrix} \partial_x & ih \\ ih & -\partial_x \end{pmatrix} = \varepsilon\partial_x + A$

Then ${}^c(X\psi_\omega) = -X({}^c\psi_\omega)$ $\qquad {}^c(X\psi_\omega) = {}^c(i\omega\psi_\omega) = -i\bar\omega({}^c\psi_\omega)$

So $X({}^c\psi_\omega) = i\bar\omega({}^c\psi_\omega)$ $\qquad \therefore\ c : E_\omega \xrightarrow{\sim} E_{\bar\omega}$

$$-{}^c\psi_\omega \wedge \psi_{\omega'} = -\begin{vmatrix} \overline{\psi^2_\omega} & \psi^1_{\omega'} \\ \overline{\psi^1_\omega} & \psi^2_{\omega'} \end{vmatrix} = \psi^*_\omega \varepsilon \psi_{\omega'}$$

$$-\partial_x({}^c\psi_\omega \wedge \psi_{\omega'}) = \partial_x(\psi^*_\omega \varepsilon \psi_{\omega'})$$??

Wronskian fermionic oscillator.

bosonic oscillator – symplectic form for comm. relation

quad. form for the motion.

fermionic oscillator quadratic form for (anti-) comm relations

skew form for motion.

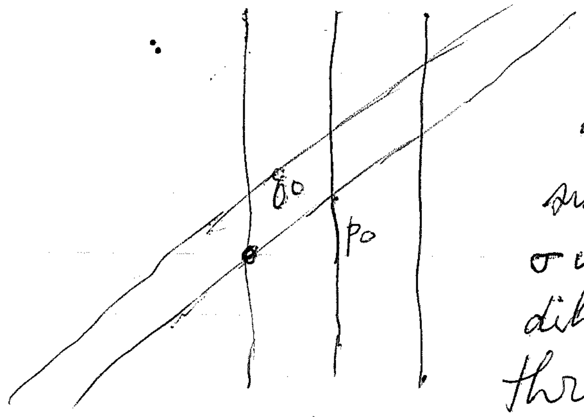
linear alg

decomposition into 2 planes is important,

you want to analyze a chain of coupled simple harmonic oscillators.

Idea: Getzler odd index theorem couples unbdd odd Dirac to an invertible g

Go back to grid space assoc. to disc DE | 952



because h_{mn} depends only on $m+n$ you get a conjugation, time reversal σ such that $\sigma p_0 = q_0$ and $\sigma u \sigma^{-1} = u^{-1}$. You have the dihedral group acting, reflections through $t=0$, $t=1$. really

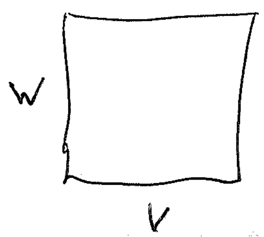
$m=n$ and $m=n-1$

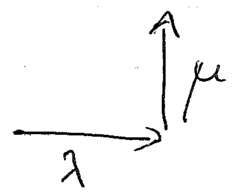
Original idea: $\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n u^{-n} \\ \bar{h}_n u^n & 1 \end{pmatrix} \begin{pmatrix} u^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$

Shows $\exists$ a $SU(1,1)$ structure on grid space

So you want to relate disc. DE to system on the $\mathbb{Z}$ tree. ~~Coupling partial~~ ~~limit~~ Translation invariant case.

$0 < h < 1$





$u = \mu\lambda^{-1}$

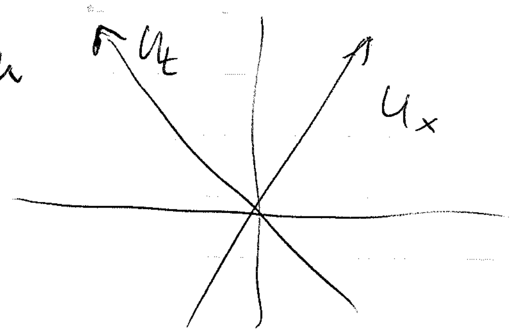
What kind of questions to ask?

Grid space
$$(k\lambda - 1) w = h v$$
$$(k\mu - 1) v = h w$$

$$(k\lambda-1)(k\mu-1) = h^2 = 1-k^2$$

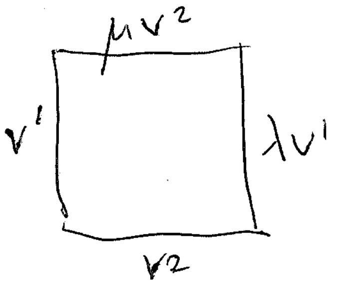
$$\mu = \frac{1}{k}\left(1 + \frac{1-k^2}{k\lambda-1}\right) = \frac{\lambda-k}{k\lambda-1} = \frac{(-\lambda)+k}{k(-\lambda)+1}$$

$u_t = \mu\lambda^{-1}$ $\qquad u_x = \lambda\mu$



Y want to look at grid space E as a rank 2 ~~free~~ module over $\mathbb{C}[u_x, u_x^{-1}]$, also over $\mathbb{C}[u_t, u_t^{-1}]$. Ask for the fibres. In the past you have used the fact that v is a cyclic vector for E as $\mathbb{C}[\lambda,\lambda^{-1}]$ module

Consider constant grid



$$\boxed{\begin{aligned}(k\lambda-1)v^1 &= hv^2\\ (k\mu-1)v^2 &= hv^1\end{aligned}}$$

$$k^2 = 1-h^2$$

$$(k\mu-1) = \frac{1-k^2}{k\lambda-1}$$

$$\mu = \frac{1}{k}\left(1+\frac{1-k^2}{k\lambda-1}\right) = \frac{\lambda-k}{k\lambda-1}$$

E = grid space = module over $\mathbb{C}[\mathbb{Z}\times\mathbb{Z}] = \mathbb{C}[\lambda,\lambda^{-1}]\otimes\mathbb{C}[\mu,\mu^{-1}]$ with 2 gen. v^1, v^2 subject to relations. Put $u = \mu\lambda^{-1}$ = time translation. Claim that E is free $\underbrace{\mathbb{C}[u,u^{-1}]}_{A}$-module w. gen. v^1, v^2.

Define λ, μ on $Av^1 + Av^2$.

$$\begin{pmatrix}\lambda v^1\\ \mu v^2\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ h & 1\end{pmatrix}\begin{pmatrix}v^1\\ v^2\end{pmatrix}$$

$$\lambda v^1 = \frac{1}{k}(v^1 + hv^2)$$

$$\lambda v^2 = \lambda\mu^{-1}(\mu v^2) = \overset{u^{-1}}{\lambda\mu^{-1}}\left(\frac{1}{k}hv^1 + \frac{1}{k}v^2\right)$$

$$\mu v^2 = \frac{1}{k}(hv^1 + v^2)$$

$$\mu v^1 = \mu\lambda^{-1}(\lambda v^1) = \overset{u}{\mu\lambda^{-1}}\left(\frac{1}{k}v^1 + \frac{h}{k}v^2\right)$$

$$\lambda\begin{pmatrix}v^1\\ v^2\end{pmatrix} = \begin{pmatrix}k^{-1}v^1 + k^{-1}hv^2\\ u^{-1}k^{-1}hv^1 + u^{-1}k^{-1}v^2\end{pmatrix} = \begin{pmatrix}k^{-1} & k^{-1}h\\ u^{-1}k^{-1}h & u^{-1}k^{-1}\end{pmatrix}\begin{pmatrix}v^1\\ v^2\end{pmatrix}$$

$$k^{-1}\begin{pmatrix}1 & 0\\ 0 & u^{-1}\end{pmatrix}\begin{pmatrix}1 & h\\ h & 1\end{pmatrix}$$

Similarly

$$\begin{pmatrix}\mu v^1\\ \mu v^2\end{pmatrix} = \begin{pmatrix}\mu\lambda^{-1}\lambda v^1\\ \cancel{\quad}\end{pmatrix}$$

$$\begin{pmatrix}\mu v^1\\ \mu v^2\end{pmatrix} = \begin{pmatrix}(\mu\lambda^{-1})\lambda v^1\\ \mu v^2\end{pmatrix} = \begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{k} & \frac{1}{k}h\\ \frac{1}{k}h & \frac{1}{k}\end{pmatrix}\begin{pmatrix}v^1\\ v^2\end{pmatrix}$$

$$\begin{pmatrix}\lambda v^1\\ \lambda v^2\end{pmatrix} = \begin{pmatrix}\lambda v^1\\ \lambda\mu^{-1}\mu v^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & u^{-1}\end{pmatrix}\begin{pmatrix}\frac{1}{k} & \frac{h}{k}\\ \frac{h}{k} & \frac{1}{k}\end{pmatrix}\begin{pmatrix}v^1\\ v^2\end{pmatrix}$$

Put $\kappa = \lambda\mu$. Claim $E = Bv^2 + B\lambda v^1$

$B = \mathbb{C}[\kappa, \kappa^{-1}]$.

$$\begin{tikzcd} \cdot \arrow[r, dash, "\mu v^2"] \arrow[d, dash, "v^1"'] & \cdot \arrow[d, dash, "\lambda v^1"] \\ \cdot \arrow[r, dash, "v^2"'] & \cdot \end{tikzcd}$$

$$\begin{pmatrix}\lambda^{-1}\lambda v^1\\ \lambda^{-1}v^2\end{pmatrix} = \begin{pmatrix}v^1\\ \kappa^{-1}\mu v^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \kappa^{-1}\end{pmatrix}\begin{pmatrix}v^1\\ \mu v^2\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & \kappa^{-1}\end{pmatrix}\begin{pmatrix}k & -h\\ h & k\end{pmatrix}\begin{pmatrix}\lambda v^1\\ v^2\end{pmatrix}$$

$$\begin{pmatrix}\lambda v^1\\ v^2\end{pmatrix} = \begin{pmatrix}k & h\\ -h & k\end{pmatrix}\begin{pmatrix}v^1\\ \mu v^2\end{pmatrix}$$

$$\begin{pmatrix}v^1\\ \mu v^2\end{pmatrix} = \begin{pmatrix}k & -h\\ h & k\end{pmatrix}\begin{pmatrix}\lambda v^1\\ v^2\end{pmatrix}$$

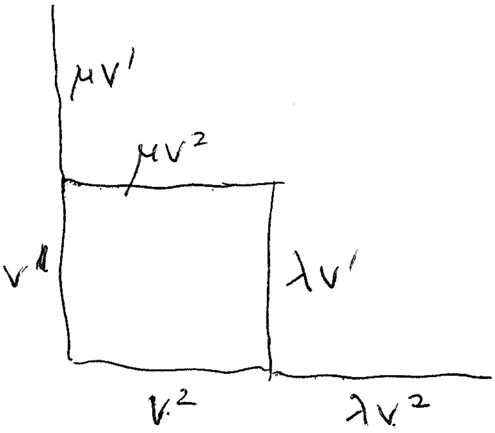
$$\begin{pmatrix}\mu\lambda v^1\\ \mu v^2\end{pmatrix} = \begin{pmatrix}\kappa & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}v^1\\ \mu v^2\end{pmatrix}$$

$$= \begin{pmatrix}\kappa & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}k & -h\\ h & k\end{pmatrix}\begin{pmatrix}\lambda v^1\\ v^2\end{pmatrix}$$

What have you accomplished?

Start again

$\tau = \mu\lambda^{-1}$

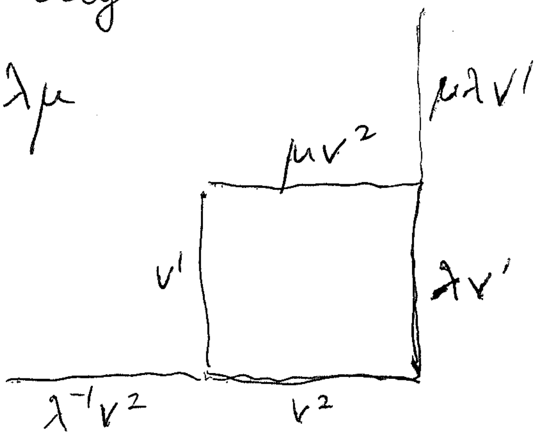


$$\lambda\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \lambda v^1 \\ \tau^{-1}\mu v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{pmatrix}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\frac{1}{k}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

$$\mu\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \tau\lambda v^1 \\ \mu v^2 \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{k}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

Similarly

$\kappa = \lambda\mu$



$$\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} k & h \\ -h & k \end{pmatrix}\begin{pmatrix} v^1 \\ \mu v^2 \end{pmatrix}$$

$$\begin{pmatrix} v^1 \\ \mu v^2 \end{pmatrix} = \begin{pmatrix} k & -h \\ h & k \end{pmatrix}\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix}$$

$$\lambda^{-1}\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} v^1 \\ \kappa^{-1}\mu v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa^{-1} \end{pmatrix}\begin{pmatrix} k & -h \\ h & k \end{pmatrix}\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix}$$

unitary

$$\mu\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \kappa v^1 \\ \mu v^2 \end{pmatrix} = \begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & -h \\ h & k \end{pmatrix}\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix}$$

Now what do you want to accomplish?

Solving the Cauchy problem – you want the discrete analog of what you did for $\partial_t\psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi$.

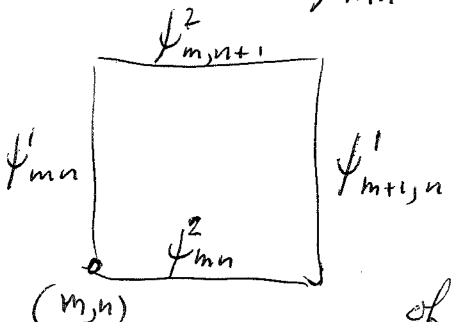
~~[illegible]~~ Cauchy problem for "$x=0$"
given initial data, ~~[illegible] i.e. coefficients i.e.~~
Wait: There are two viewpoints, first is to express any grid vector in terms of the descending staircase basis $\{\tau^n v^1, \tau^n v^2\}_{n\in\mathbb{Z}}$, the other is to use "solutions" of the grid eqns. – linear functions on grid space.

Recall formulas in the cont. case.

$$\psi(x,t) = e^{x\begin{pmatrix}\partial_t & -i \\ i & -\partial_t\end{pmatrix}} \psi(0,t)$$

$$= \int \frac{d\omega}{2\pi}\, e^{i\omega t}\, e^{ix\begin{pmatrix}\omega & -1 \\ 1 & -\omega\end{pmatrix}}\, \hat{\psi}_0(\omega)$$

Now you need to work with equations, I guess this means ψ^j_{mn} $j=1,2$. Try to make this work. Maybe look at exponential solutions $\psi_{mn} = \lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ where $\begin{pmatrix}\lambda v^1 \\ \mu v^2\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h \\ h & 1\end{pmatrix}\begin{pmatrix}v^1 \\ v^2\end{pmatrix}$



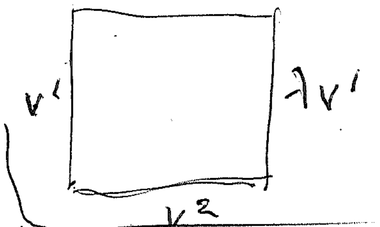
This is all very clear. What next? You want to work with analogs of x,t a κ τ $\kappa = \lambda\mu$, $\tau = \mu\lambda^{-1}$

This involves a lattice of index 2, but maybe this is not so bad since E is free of rank 2 over both $\mathbb{C}[\kappa,\kappa^{-1}]$ and $\mathbb{C}[\tau,\tau^{-1}]$. Perhaps things should be very simple.

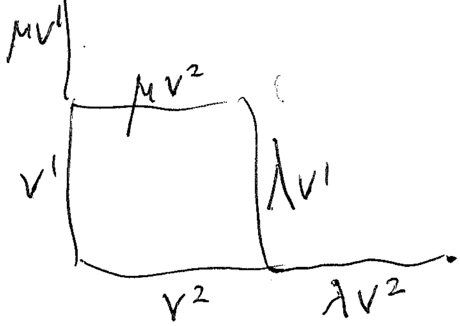
First discuss the correspondence on exp. solutions.

$$\psi_{mn} = \lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \qquad A = \mathbb{C}[\tau,\tau^{-1}] \quad \tau = \mu\lambda^{-1} \qquad \kappa = \lambda\mu \qquad B = \mathbb{C}[\kappa,\kappa^{-1}]$$

$$E = Av^1 \oplus Av^2 = B\lambda v^1 + Bv^2$$

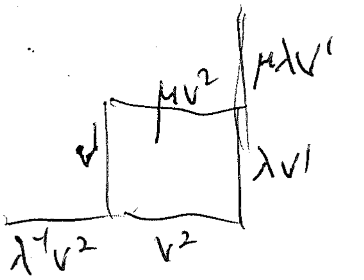


Review:



$$\lambda\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \lambda v^1 \\ \lambda\mu^{-1}\mu v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{pmatrix}\begin{pmatrix} \lambda v^1 \\ \mu v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{pmatrix}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\frac{1}{k}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

$$\mu\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \mu\lambda^{-1}\lambda v^1 \\ \mu v^2 \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{k}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$



$$\lambda^{-1}\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} v^1 \\ \lambda^{-1}\mu^{-1}\mu v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa^{-1} \end{pmatrix}\begin{pmatrix} k & -h \\ -h & k \end{pmatrix}\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix}$$

$$\mu\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v^1 \\ \mu v^2 \end{pmatrix} \overset{\begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix}}{=} \begin{pmatrix} k & -h \\ -h & k \end{pmatrix}\begin{pmatrix} \lambda v^1 \\ v^2 \end{pmatrix}$$

$$E = A_\tau v^1 + A_\tau v^2 = A_\kappa \lambda v^1 + A_\kappa v^2$$

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{pmatrix}\begin{pmatrix} k^{-1} & k^{-1}h \\ k^{-1}h & k^{-1} \end{pmatrix}$$

$$\mu = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k^{-1} & k^{-1}h \\ k^{-1}h & k^{-1} \end{pmatrix}$$

~~Whats~~

Express ~~Riemann~~ ~~Greens~~ function

~~$\partial_r \psi^1 = i\psi^2$~~

~~$\partial_s \psi^2 = i\psi^1$~~

~~Look~~ exponential solution – means linear functional on the grid space ~~...~~ which is an eigenfunction for the translation operators

Q: Can you solve the Cauchy problem for "$x = 0$"

i.e. take $f(\tau)v^1 + g(\tau)v^2$ anal. of $\begin{pmatrix} \psi^1(t) \\ \psi^2(t) \end{pmatrix} = \psi_0(t)$.

Then construct $\psi(x,t) = e^{cx\begin{pmatrix} \partial_t & -i \\ i & \partial_t \end{pmatrix}}\psi_0(t)$.

Program. ~~[illegible]~~ You know E is a free module over $A_\tau = \mathbb{C}[\tau,\tau^{-1}]$ with basis v^1, v^2, so ~~for~~ ~~any function~~ any solution ψ of the g.e. ?

You want the discrete analog of $\psi(x,t) = e^{x(\ \ )}\psi(0,t)$, should reduce to exponential ~~sol~~ solutions

Focus on exp. solns. characters for translation.

General eqn is
$$(k\lambda - 1)\psi^1 = h\psi^2$$
$$(k\mu - 1)\psi^2 = h\psi^1$$

where $\psi^1, v^2, \lambda, \mu$ are interpreted as numbers

corres
Soln. is $\psi_{mn} = \lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

restrict to $m + n = 0$ i.e. look at $\tau^n \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$, can you construct gen soln?

So given numbers for τ, v^1, v^2 can you construct ! numbers for $\lambda, \mu \ni$ $\qquad \tau = \mu\lambda^{-1}$

$$\left.\begin{aligned}(k\lambda - 1)v^1 &= hv^2 \\ (k\mu - 1)v^2 &= hv^1\end{aligned}\right) \Longrightarrow \quad \mu = \frac{\lambda - k}{k\lambda - 1} = \tau\lambda$$

$$(k\lambda - 1)(k\tau\lambda - 1) = 1 - k^2 \qquad \text{quad. eqn. for } \lambda:$$

$$k^2\tau\lambda^2 - (k\tau + k)\lambda + \cancel{1} = \cancel{1} - k^2$$

$$k^2\tau\,\lambda^2 - k(\tau+1)\lambda + k^2 = 0 \qquad (\tau+1)^2 - 4k^2\tau$$

$$k\tau\,\lambda^2 - (\tau+1)\lambda + k = 0 \qquad = \tau^2 + (2 - 4k^2)\tau + 1$$

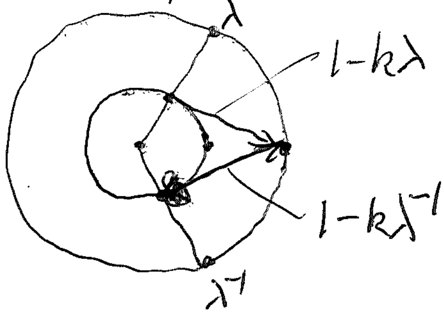
$$\lambda = \frac{\tau + 1 \pm \sqrt{(\tau+1)^2 - 4k^2\tau}}{2k\tau} \qquad \text{wrong thing to solve for.}$$

~~[illegible] Repeat [illegible] exponential~~

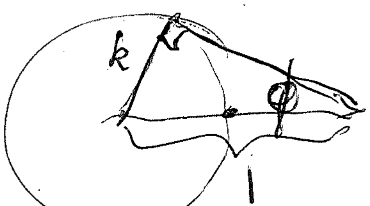
You know that E is a free module over A_τ with generators v^1, v^2. So for each specialization of τ to an elt of $\mathbb{C}^\times$, there is a ~~unique sol~~ 2 diml space of solutions of the grid equations for this value of τ. Since $\mu = \tau\lambda = \frac{\lambda-k}{k\lambda-1}$ is a quadratic equation for λ in terms of τ, there should be two values for λ corresp. to a value of τ, and corresp. values of μ. This is the analog of two wave vectors k for a given freq. ω.

Suppose we restrict $\lambda, \mu \in S^1$. Then

$$-\tau = -\mu\lambda^{-1} = +\frac{1-k\lambda^{-1}}{1-k\lambda}$$



as $\lambda = e^{i\theta}$ goes $0 \leq \theta \leq \pi$, ~~the [illegible] argument~~ of $\frac{1-k\lambda^{-1}}{1-k\lambda} = e^{i\phi}$ goes from $\phi = 0$ to a maximum when $\lambda \perp 1-k\lambda$.



$\sin\phi = k$

and then it decreases to $\phi = 0$ as $\theta = +\pi$.

Thus the ~~curve~~ map $\lambda \longmapsto -\tau = -\mu\lambda^{-1} = \frac{1-k\lambda^{-1}}{1-k\lambda}$ ~~will not cover [illegible]~~ from S^1 to S^1 is not surjective.

What about $\kappa = \lambda\mu = \lambda\frac{\lambda-k}{\lambda k-1}$ $\quad -\kappa = \lambda\frac{\lambda-k}{1-\lambda k}$

The map $\lambda \longmapsto -K$ from S^1 to S^1 has degree 2, hence is $2 \to 1$ except for ramification.

To understand IH better. Begin with ant. case:

$$\partial_t \psi = \underbrace{\begin{pmatrix} \partial_x & ih \\ +i\bar{h} & -\partial_x \end{pmatrix}}_{\varepsilon\partial_x + iA} \psi \implies \partial_t(\psi^*\psi) = (\varepsilon\partial_x\psi)^*\psi + \cancel{(iA\psi)^*\psi} + \psi^*(\varepsilon\partial_x\psi) + \cancel{\psi^* iA\psi}$$

$$= \partial_x(\psi^*\varepsilon\psi).$$

so $\psi^*\psi\, dx + \psi^*\varepsilon\psi\, dt$ is closed.

$\implies \int_{-\infty}^{\infty} (\psi^*\varepsilon\psi)(x,t)\, dt$ is ind of x when there is no problem with $\psi^*\psi$ at ∞.

precise

$$\partial_x \int_{-R}^{R} (\psi^*\varepsilon\psi)(x,t)\, dt = \int_{-R}^{R} \partial_t(\psi^*\psi)\, dt = \left[\psi^*\psi\right]_{t=-R}^{t=R}$$

so you need $\psi^*\psi(x,t)$ to have equal limits as $t \to \pm\infty$, ~~the sort of thing to go wrong when not small in resonant case.~~

So what is $\int_{-\infty}^{\infty} (\psi^*\varepsilon\psi)(x,t)\, dt = IH(\psi,\psi)$

$$\psi(x,t) = \int_{-\infty}^{\infty} e^{i\omega t}\hat{\psi}(x,\omega)\frac{d\omega}{2\pi} \qquad \hat{\psi}(x,\omega) = \int e^{-i\omega t}\psi(x,t)\,dt$$

$$IH(\psi,\psi) = \int_{-\infty}^{\infty} \psi_a(x,t)^*\,\varepsilon\,\psi(x,t)\, dt$$

$$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} e^{i\omega t}\,\psi_a(x,t)^*\varepsilon\,\hat{\psi}(x,\omega)\frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \left[\int dt\, e^{-i\omega t}\psi_a(x,t)\right]^* \varepsilon\,\hat{\psi}(x,\omega)\frac{d\omega}{2\pi}$$

$$IH\left(\psi_a(t), \psi_b(t)\right) = \int \psi_a(t)^* \varepsilon \psi_b(t)\, dt$$

$$= \int \left(\int e^{i\omega t} \hat{\psi}_a(\omega) \frac{d\omega}{2\pi} \right)^* \varepsilon \psi_b(t)\, dt$$

$$= \int \frac{d\omega}{2\pi} \hat{\psi}_a(\omega)^* \varepsilon \int e^{-i\omega t} \psi_b(t)\, dt$$

$$= \int \frac{d\omega}{2\pi} \hat{\psi}_a(\omega)^\dagger \varepsilon \hat{\psi}_b(\omega).$$

What ~~is~~ solution is critical?

$$\cancel{\partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi \qquad \psi = e^{i\omega t} \hat{\psi}}$$

$$\cancel{\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \hat{\psi} = 0} \qquad \cancel{\omega \hat{\psi}^1 = \hat{\psi}^2, \quad \omega \hat{\psi}^2 = \hat{\psi}^1 \quad \therefore \omega^2 = 1, \quad \omega = \pm 1.}$$

~~Better is~~ $\partial_t \psi = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \psi$

$$\psi = e^{t\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}} = e^{itB} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \sum \frac{(-1)^n}{(2n)!} t^{2n} + \sum \frac{(-1)^n}{(2n+1)!} t^{2n+1}\, i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{\psi(t) = (\cos t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi \qquad \partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi \qquad \partial_t \psi = 0$$

$$\psi(x) = e^{x\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \sum \frac{x^{2n}}{2n!} 1 + \sum \frac{x^{2n+1}}{(2n+1)!} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \cosh x + i \sinh x \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Solution of grid equations independent of ~~t~~ time, means only using characters $\ni \tau = 1$. 962

Exp Solution $\psi_{mn} = \lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ $\qquad \underline{\tau = \mu\lambda^{-1} = 1}$

$\mu = \dfrac{\lambda - k}{k\lambda - 1} = \tau\lambda \qquad (k\lambda - 1)(k\mu - 1) = (1-k^2) = h^2$

If $\lambda = \mu$, then $\qquad k\lambda - 1 = \pm h$

$\lambda = \dfrac{1 \pm h}{k}$, so $\psi_{mn} = \lambda^{m+n} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ where $\lambda = \dfrac{1+h}{k}$

or its inverse $\dfrac{1-h}{k}$. If $\tau = -1$, $(k\lambda - 1)(-k\lambda - 1) = h^2$

$-k^2\lambda^2 + 1 = 1 - k^2$, $\quad k^2\lambda^2 = k^2 \quad \lambda = \pm 1$ so

these are oscillatory.

$$(k\lambda - 1)(k\tau\lambda - 1) = k^2\tau\lambda^2 - k\tau\lambda - k\lambda + 1$$

$$(k\lambda - 1)(k\tau\lambda - 1) = h^2$$

$$k^2\tau\lambda^2 - (k\tau + k)\lambda + \cancel{1} = \cancel{1} - k^2$$

$$k\tau\lambda^2 - (\tau + 1)\lambda + k = 0$$

$$\text{disc} = (\tau+1)^2 - 4k^2 \qquad \text{Let } \tau = e^{2i\theta}$$

$$\left(\tau^{\frac{1}{2}}\lambda\right)^2 - \frac{1}{k}\left(\tau^{1/2} + \tau^{-1/2}\right)\left(\tau^{\frac{1}{2}}\lambda\right) + 1$$

$$\left(e^{i\theta}\lambda\right)^2 - \frac{2}{k}(\cos\theta)\left(e^{i\theta}\lambda\right) + 1 = 0$$

~~Use~~ you have oscillatory solns. $\underline{\text{for } -1 < \frac{\cos\theta}{k} < 1}$

What questions to ask?

To understand the grid space.

~~In the discrete case you have~~

Discuss the problems. Aim to understand well, properly the DE $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ +i & -\partial_x \end{pmatrix}$. Ultimately you want to make precise the "universal solution", "general solution" of this DE. This is a TVS whose dual should be a certain class of solutions.

First thing to do is to find exp. solns, which coord sys. x, t or r, s. You need a higher level of organization. There is a lot to review.

Conjugation (time reversal) on a grid space with $h_{mn} = h_{m+1\,n}$

grid relations are

$$\begin{array}{|c|c|} \hline h_{00} & h_{01} \\ \hline & h_{00} \\ \hline \end{array}$$

$$\begin{array}{ccc} & q_{m,n+1} & \\ p_{mn} & \boxed{h_{m+n}} & p_{m+1,n} \\ mn & q_{mn} & \end{array}$$

$$\begin{pmatrix} p_{m+1,n} \\ q_{m,n+1} \end{pmatrix} = \frac{1}{k_{m+n}} \begin{pmatrix} 1 & h_{m+n} \\ \bar{h}_{m+n} & 1 \end{pmatrix} \begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix}$$

define $\sigma\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \begin{pmatrix} q_{mn} \\ p_{mn} \end{pmatrix}$ on grid vector

$\sigma(c\xi) = \bar{c}\,\sigma(\xi) \qquad \sigma^2 = 1.$

Better to say $p^*_{mn} = q_{nm}$ for all m, n.

$$p^*_{m+1,n} = \frac{1}{k_{m+n}} \left(p_{mn} + h_{m+n} q_{mn} \right)$$

$$q_{n,m+1} = \frac{1}{k_{m+n}} \left(q_{nm} + \bar{h}_{m+n} p_{nm} \right) \quad ? \quad \forall m,n$$

$$q_{n,n+1} = \frac{1}{k_{n+m}} \begin{pmatrix} 1 & h_{n+m} \\ \bar{h}_{n+m} & 1 \end{pmatrix} \begin{pmatrix} p_{nm} \\ q_{nm} \end{pmatrix}$$

YES

~~You forgot that time reflection requires a t = constant line. You therefore want~~

$$\sigma \begin{pmatrix} p_{m+1,n} \\ q_{m,n+1} \end{pmatrix} = \frac{1}{k_{m+n}} \begin{pmatrix} 1 & h_{m+n} \\ \bar{h}_{m+n} & 1 \end{pmatrix} \sigma \begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} \quad ?$$

$$\begin{pmatrix} p_{n+1,m} \\ q_{n,m+1} \end{pmatrix} = \underset{\displaystyle \frac{1}{k_{n+m}} \begin{pmatrix} 1 & h_{n+m} \\ h_{n+m} & 1 \end{pmatrix}}{\underset{\Vert}{\rule{4em}{0.4pt}}} \begin{pmatrix} p_{nm} \\ q_{nm} \end{pmatrix} \quad ?$$

---

straighten out Wronskian.

Think this out clearly.

First part which you now understand I think relates a volume $\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$ on a 2 dim'l v.s. V with conjugation σ, ω satisf. real condition $\omega(\sigma v \wedge \sigma v') = \overline{\omega(v \wedge v')}$ to a herm. form H on V sat $H(\sigma v, \sigma v') = \overline{H(v, v')}$

NO

Point. Let V be a $\mathbb{C}$- v.s. with conjugation σ; so that $V = \mathbb{C} \otimes_{\mathbb{R}} V^\sigma$, let H be a herm. bil. form on V sat $\overline{H(v, v')} = H(\sigma v, \sigma v')$

Then $H(v, v') = A_{sym}(\sigma v, v') + i A_{sk}(\sigma v, v')$

$$\overset{\shortparallel}{v^* H v'} = (\sigma v)^t H v'$$

$H^* = H$ rel. to a real basis

Mistake $\left(B_{sym} + i B_{sk}\right)(\sigma v, v)$

back to Wronskian ideas. V complex vector space, σ conjugation, B skew ~~form~~ complex bilinear-symm. form $B(\sigma v_1, \sigma v_2) = \overline{B(v_1, v_2)}$. Then ~~$H(\sigma v, v') = iB(v, v$~~

$H(v, v') = iB(\sigma v, v')$ $\qquad \overline{H(v, v')} = -i\,\overline{B(\sigma v, v')} = -iB(v, \sigma v')$

$= iB(\sigma v', v) = H(v', v)$

Also you have that $\sigma v = v \Rightarrow H(v, v) = iB(\sigma v, v)$

$= iB(v, v) = 0$

Special case where you have a $B_{sk}(v, v')$ skew-symm. bilinear and define $H(v, v') = i B_{sk}(\sigma v, v')$ to get a hermitian form vanishing on real ~~vectors~~ lines, lines in V^σ.

There's a lot of stuff to work on here

But back to $V \simeq \mathbb{C}^2$ with σ eg $\sigma\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \bar{v}_2 \\ \bar{v}_1 \end{pmatrix}$

~~Then~~ Given $\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$ real [illegible]

$\omega(\sigma v, \sigma v') = \overline{\omega(v, v')}$. $\qquad$ e.g. $\omega(v \wedge v') = \begin{vmatrix} v_1 & v_1' \\ v_2 & v_2' \end{vmatrix} i$

Then $i\,\omega(\sigma v \wedge v) = -\begin{vmatrix} \bar{v}_2 & v_1 \\ \bar{v}_1 & v_2 \end{vmatrix} = |v_1|^2 - |v_2|^2$

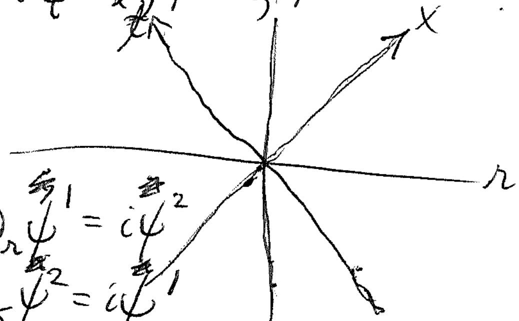
$U(1,1)$ = autos of $\mathbb{C}^2$ commuting with σ

~~[illegible]~~

perhaps also in the continuous case you can show that the Hilbert space completion is given by the characteristic lines $r=0$ or $s=0$.

Consider then $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix} \psi$ $\quad (\partial_t - \partial_x)\psi^1 = i\psi^2$
$(\partial_t + \partial_x)\psi^2 = i\psi^1$

$\partial_r = -\partial_t + \partial_x$ $\qquad t = -r+s$
$\partial_s = \partial_t + \partial_x$ $\qquad x = r+s$



$-\partial_r \psi^1 = i\psi^2$
$\partial_s \psi^2 = i\psi^1$

So $\psi = \int e^{i(r\rho + s\sigma)} \hat{\psi} \frac{d\rho\, d\sigma}{(2\pi)^2}$

$\begin{matrix} -\rho\hat{\psi}^1 = \hat{\psi}^2 \\ \sigma\hat{\psi}^2 = \hat{\psi}^1 \end{matrix} \Rightarrow (-\rho\sigma + 1)\hat{\psi} = 0$ $\quad\Big|\quad$ $\psi(r,s) = \int \frac{d\rho}{2\pi} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)$

Better viewpoint is that an exponential solution has form $e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix}$ const. $\quad \rho \in \mathbb{C}^\times$

and the oscillatory exponential solns. corresp to $\rho \in \mathbb{R}^\times$. Question –

Suppose you try the Cauchy problem the curve $s=0$. You suppose given $\psi_0(r)$ and you want to find $f(\rho)$ such that

$$\psi_0(r) = \int e^{ir\rho} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho) \frac{d\rho}{2\pi}$$

This looks over determined because $f(\rho)$ should be $\hat{\psi}_0(\rho) = \int e^{-ir\rho} \psi_0(r)\, dr$

It seems OK because the grid equations sa

$$-\partial_r \psi^1(r,s) = i\,\psi^2(r,s)$$
$$\partial_s \psi^2(r,s) = i\,\psi^1(r,s)$$

so $\psi^2(r,0) = i\partial_r \psi^1(r,0)$ i.e.

$$\hat{\psi}_0^2(\rho) = -\rho\,\hat{\psi}_0^1(\rho).$$

So what do you learn? Namely a solution $\psi(r,s)$ is ~~equation~~ equivalent to ~~the~~ function $\psi^1(r,0)$, by the recipe.

$$\psi(r,s) = \int \frac{d\rho}{2\pi} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} \int e^{-ir'\rho}\, \psi^1(r',0)\, dr'$$

~~Formulas~~ There seems to be something simple happening here. You ~~d~~ are ~~looking~~ representing solutions by single functions of ρ, really, as linear combinations of $e^{ir\rho}$ $r \in \mathbb{R}$, which means you are looking at grid space as a module over translations in the r direction, in discrete case looking at E as a $\mathbb{C}[\lambda, \lambda^{-1}]$-module free of rank 1.

What about $\| \|^2$ and IH. ~~Also~~

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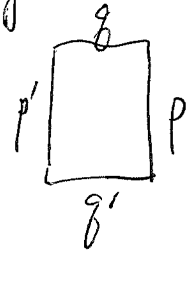
digression.



$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} k^{-1} & k^{-1}h \\ k^{-1}\bar{h} & k^{-1} \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix}$$

$W(p,q) \simeq W(p',q')$
Weyl algs.

$C(p,q') \simeq C(p',q)$
Cliff algebras

First point to examine, recall, is how to ~~combine~~ CCR CAR supersymmetrically.

have various models. Look at the standard bosonic Hilb. space for the harmonic oscillator.

$$\int e^{-|z|^2} |f(z)|^2 \frac{dz}{\pi}$$

$a^* = z, \quad a = \partial_z$

$$\|z^n\|^2 = (1, a^n (a^*)^n 1) = n!$$

$$f(w) = \sum c_n w^n$$

Point evaluator.

$$f(z) = \sum c_n z^n$$

$$(z^n | f) = c_n n! \qquad c_n = \left(\frac{z^n}{n!} \Big| f\right).$$

$$f(w) = \sum w^n \left(\frac{z^n}{n!} \Big| f\right) = \sum \left(\frac{\bar{w}^n z^n}{n!} \Big| f\right)$$

$$= \left(e^{\bar{w}z} \big| f\right).$$

$\bar{\partial}$ complex

$$f(z) \longmapsto \partial_z f \, dz$$

For several variable $z_1, \dots, z_r$ you have $a_j = \partial_{z_j}$, $a_j^* = z_j$. Not clear what ~~to~~ do about $\omega(a^* a + \frac{1}{2})$ and $\omega(b^* b - \frac{1}{2})$ for the Hamiltonians

Let try p's q's, terrible conflict in notation

There's a problem getting ~~started~~, which can perhaps be sorted out. Bosonic picture

You have a complex 2 dim'l vector space V of ~~dim 2 with~~ ~~basis a, a^*~~ 1st order operators, conjugation $*$, skew symm. form given by $[\ ,\ ]$, and a Hamiltonian given by symmetric form. Enough time waste back to grid space.

Witten example $e^{-\frac{1}{2}x^2} d e^{\frac{1}{2}x^2} = d + x$

Finish up Wronskian related ~~stuff~~.

$$-\partial_r \psi^1 = i\psi^2 \qquad -\rho\hat{\psi}^1 = \hat{\psi}^2$$
$$\partial_s \psi^2 = i\psi^1 \qquad \sigma\hat{\psi}^2 = \hat{\psi}^1$$

leads to $\rightarrow$ exp. solns. $\psi(r,s) = e^{i(r\rho - s\rho^{-1})}\begin{pmatrix}1\\-\rho\end{pmatrix}$ const.

What is your aim? ~~What is~~ ~~To work out~~

In the discrete case you form grid space E - gen. + relations, understand ~~its~~ E structure as module over the group ring $\mathbb{C}[\lambda, \mu, \lambda^{-1}, \mu^{-1}]$ of translations. e.g. rank 1 over $\mathbb{C}[\lambda, \lambda^{-1}, (\lambda-k)^{-1}, (\lambda-k^{-1})^{-1}]$ with $\mu = \frac{\lambda - k}{k\lambda - 1}$, but also comes with ~~these~~ two hermitian forms. Note: E is not a free module of rank 1 over $\mathbb{C}[\lambda, \lambda^{-1}]$, ~~so might~~ but this is ~~~~ in some sense becomes true if ~~you~~ restrict $\lambda \in S^1$.

What's the ~~corresp~~ analogous situation in the cont case - ~~~~

You want, instead of ~~polys~~ rational functions in λ, some ~~sort~~ kind of entire functions with ~~~~ a growth condition.

Your aim ~~...~~ now is to find an entire function version of grid space in the continuous case.

Review transition from disc to cont.

$$V = \mathbb{C}^2 \qquad \ell_z = \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix} \subset V$$

$s(z) = \begin{pmatrix} z \\ 1 \end{pmatrix} f(z)$ section (local) rational of $\mathcal{O}(-1)$.

$$-s \wedge ds = -\begin{vmatrix} zf & d(zf) \\ f & df \end{vmatrix} = +f^2 dz$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C}).$$

$$(g^*f)(z) = f\left(\frac{az+b}{cz+d}\right).$$

$$g^*(dz) = d\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)\,a\,dz - (az+b)\,c\,dz}{(cz+d)^2} = \frac{(ad-bc)\,dz}{(cz+d)^2}$$

$$(g^*s)(z) = g^{-1}(s(gz))$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} f\left(\frac{az+b}{cz+d}\right)$$

$$= \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} az+b \\ cz+d \end{pmatrix}}_{\begin{pmatrix} z \\ 1 \end{pmatrix}} \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right)$$

~~so who~~

examine ~~the correspondence~~

$$z \longmapsto \frac{-z+k}{-kz+1} = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}(-z)$$

$$= \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}(z)$$

Let's recall, review the limit process:

$$(k\lambda-1)v^1 = bv^2$$
$$(k\mu-1)v^2 = \bar{b}v^1$$

$$\mu = \frac{1}{k}\left(1+\frac{1-k^2}{k\lambda-1}\right) = \frac{\lambda-k}{k\lambda-1}$$

Want horizontal translations to be continuous, v^1 stays a unit vector, ~~[illegible]~~ but v^2 becomes a δ-fn.

Idea: Take unit v^2 and ~~[illegible]~~ subdivide into $\frac{1}{\varepsilon}$ orth. vectors of norm $\sqrt{\varepsilon}$.

$$k_\varepsilon = \sqrt{1-|b|^2\varepsilon} \simeq 1-\tfrac{1}{2}|b|^2\varepsilon$$

$$(k_\varepsilon\lambda^\varepsilon-1)v^1 = b\sqrt{\varepsilon}\, v^2\sqrt{\varepsilon}$$
$$(k_\varepsilon\mu_\varepsilon-1)v^2\sqrt{\varepsilon} = \bar{b}\sqrt{\varepsilon}\, v^1$$

$$\frac{1}{\varepsilon}\left(k_\varepsilon\lambda^\varepsilon - 1\right) = \frac{\left(1-\frac{1}{2}|b|^2\varepsilon\right)\left(1+\varepsilon i\xi\right)-1}{\varepsilon} = -\tfrac{1}{2}|b|^2+i\xi$$

$$\left(-\tfrac{1}{2}|b|^2+i\xi\right)v^1 = bv^2$$
$$(\mu-1)v^2 = \bar{b}v^1$$
$$v^1 = \frac{b}{s-a}v^2$$

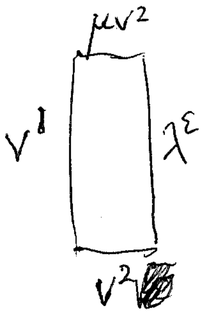
$$\mu = 1+\frac{|b|^2}{-\frac{1}{2}|b|^2+i\xi} = \frac{\frac{1}{2}|b|^2+i\xi}{-\frac{1}{2}|b|^2+i\xi} = \frac{s+a}{s-a}$$

You are aiming for a class of entire function of s, no. A class of meromorphic functions of s possible poles at $\pm a$. Instead of k, k^{-1} you will have $i\xi = \pm ia$.

$$\lambda^n \rightsquigarrow \lambda^{\varepsilon\frac{x}{\varepsilon}} = e^{i\xi x} = e^{sx}$$

Grid space should consist of ~~[illegible]~~?

Consider grid cont in r direction and discrete in s direction. Grid equations.



$$\begin{pmatrix}\lambda^\varepsilon v^1\\ \mu_\varepsilon (v^2\sqrt{\varepsilon})\end{pmatrix} = \frac{1}{k_\varepsilon}\begin{pmatrix}1 & b\sqrt{\varepsilon}\\ \bar b\sqrt{\varepsilon} & 1\end{pmatrix}\begin{pmatrix}v^1\\ v^2\sqrt{\varepsilon}\end{pmatrix} \qquad a = \tfrac{1}{2}|b|^2$$

$$(k_\varepsilon \lambda^\varepsilon - 1) v^1 = b v^2 \varepsilon \qquad (-a + i\xi) v^1 = b v^2$$

$$(k_\varepsilon \mu_\varepsilon - 1) v^2 = \bar b v^1 \qquad (\mu - 1) v^2 = \bar b v^1$$

$$\mu = 1 + \frac{\bar b b}{i\xi - a} = \frac{i\xi + a}{i\xi - a}$$

when you use the $SU(1,1)$ form v^1, $v^2\sqrt{\varepsilon}$ are unit vectors so $\|v^2\| = \frac{1}{\sqrt{\varepsilon}}$. $a = \frac{1}{2}|b|^2$

$$\boxed{\left.\begin{aligned}(-a + i\xi) v^1 &= b v^2\\ (\mu - 1) v^2 &= \bar b v^1\end{aligned}\right\} \Longrightarrow \mu = \frac{i\xi + a}{i\xi - a}}$$

Suppose now you want vertical translations to be continuous.

First start from
$$(k\lambda - 1) v^1 = h v^2$$
$$(k\mu - 1) v^2 = \bar h v^1$$

idea to replace h by $b\varepsilon$

$$\left(\frac{k_\varepsilon \lambda^\varepsilon - 1}{\varepsilon}\right) v^1 = b v^2 \qquad k_\varepsilon = \sqrt{1 - |b|^2 \varepsilon^2}$$

$$\left(\frac{k_\varepsilon \mu^\varepsilon - 1}{\varepsilon}\right) v^2 = \bar b v^1$$

limiting case is
$$i\xi v^1 = b v^2$$
$$i\eta v^2 = \bar b v^1$$
$$\boxed{-\xi\eta = |b|^2}$$

So what next? replace μ by μ^ε, v^1 by $v^1\sqrt{\varepsilon}$, $b \mapsto b\sqrt{\varepsilon}$ and $a \mapsto a\varepsilon$

$$(i\xi - a\varepsilon) v^1 \cancel{\sqrt{\varepsilon}} = b\cancel{\sqrt{\varepsilon}}\, v^2 \qquad i\xi v^1 = b v^2$$

$$\left(\frac{\mu^\varepsilon - 1}{\varepsilon}\right) v^2 = \bar{b} \cancel{\sqrt{\varepsilon}}\, v^1 \cancel{\sqrt{\varepsilon}} \qquad i\eta v^2 = \bar{b} v^1 \quad \text{OK}$$

---

grid equations

$$k_\varepsilon \psi^1(r+\varepsilon, \ell) - \psi^1(r,\ell) = b\varepsilon\, \psi^2(r,\ell)$$

$$k_\varepsilon \psi^2(r, \ell+1) - \psi^2(r,\ell) = \bar{b}\psi^1(r,\ell)$$

$$\boxed{\begin{aligned} (\partial_r - a)\psi^1(r,\ell) &= b\psi^2(r,\ell) \\ \psi^2(r,\ell+1) - \psi^2(r,\ell) &= \bar{b}\psi^1(r,\ell) \end{aligned}} \qquad a = \tfrac{1}{2}|b|^2$$

Check this by putting $\psi(r,\ell) = e^{ir\rho}\mu^\ell \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

$$(i\rho - a) v^1 = b v^2$$
$$(\mu - 1) v^2 = \bar{b} v^1 \qquad \mu = 1 + \frac{|b|^2}{i\rho - a} = \frac{i\rho + a}{i\rho - a}$$

$$\psi(r,\ell) = e^{ir\rho}\left(\frac{i\rho + a}{i\rho - a}\right)^\ell \begin{pmatrix} \frac{b}{i\rho - a} \\ 1 \end{pmatrix} v^2(\rho)$$

change ℓ to n

$$\psi(r,n) = \int e^{ir\rho}\left(\frac{i\rho + a}{i\rho - a}\right)^n \begin{pmatrix} \frac{b}{i\rho - a} \\ 1 \end{pmatrix} v^2(\rho)$$

should yield ? Point is that $\forall \rho \in \mathbb{C} - \{\pm ia\}$ you get an exponential solution of the grid eqns.

$$e^{ir\rho}\left(\frac{i\rho + a}{i\rho - a}\right)^n \begin{pmatrix} \frac{b}{i\rho - a} \\ 1 \end{pmatrix}$$

unique up to a nonzero scalar factor

Now do spectral theory. ~~the spectrum~~ What do you mean? Assume you knew what the grid space E is. It is a module for the group $\mathbb{R}\times\mathbb{Z}$ of translations, so for each character $\chi : (r,n) \mapsto e^{ir\rho}\mu^n$ you get a quotient space E_χ of E, universal for these types of solutions. You know that $E_\chi \neq 0 \implies \mu = \frac{i\rho+a}{i\rho-a}$ then E_χ is 1-diml.

So the hypothetical grid space E should be sections of a trivial line bundle over $\mathbb{C}-\{\pm ia\}$. You probably want meromorphic functions and some control over growth

But you saw there is a minimal answer to this question, namely the smallest space of functions on $\mathbb{C}-\{\pm ia\}$ containing rational fns. regular off $\pm ia$ and stable under mult by $e^{ir\rho}$ $r\in\mathbb{R}$.

example.
$$e^{ir\rho}\frac{1}{\rho-ia} = \frac{e^{ir(ia)}}{\rho-ia} + \frac{e^{ir\rho}-e^{ir(ia)}}{\rho-ia}$$

$i\rho \mapsto s.$
$$e^{rs}\frac{1}{s+a} = \frac{e^{-ra}}{s+a} + \frac{e^{rs}-e^{-ra}}{s+a}$$

So basically you take all expon. e^{rs} $r\in\mathbb{R}$ and apply $\left(\partial_r^2-a^2\right)^{-1}$, possible some Green's fn.

This might involve Hadamard's finite part if you are lucky. ~~You have to~~

First discuss rational functions regular outside $z = a, b$. basis $\left(\frac{z-a}{z-b}\right)^n \quad n \in \mathbb{Z}$,

another basis is $\frac{1}{(z-a)^n}, \; \frac{1}{(z-b)^{n+1}} \quad n \geq 0$.

Look at grid equations.

$$(-a+z)v^1 = bv^2 \qquad\qquad \mu = 1 + \frac{\overbrace{|b|^2}^{2a}}{z-a} = \frac{z+a}{z-a}$$
$$(\mu - 1)v^2 = \bar{b}v^1$$

You are getting exponential solution

$$\psi(r,n) = e^{rz}\left(\frac{z+a}{z-a}\right)^n \begin{pmatrix} \frac{b}{z-a} \\ 1 \end{pmatrix}$$

so putting $r = 0$, you need both $\left\{\left(\frac{z+a}{z-a}\right)^n, \; n \in \mathbb{Z}\right\}$ and $\left\{\frac{(z+a)^n}{(z-a)^{n+1}}, \; n \in \mathbb{Z}\right\}$ as bases it seems.

Point $\quad 1 + \frac{2a}{z-a} = \frac{z+a}{z-a}$

so that $\quad 2a\left(\frac{z+a}{z-a}\right)^n \frac{1}{z-a} = \left(\frac{z+a}{z-a}\right)^{n+1} - \left(\frac{z+a}{z-a}\right)^n$

I am still a bit puzzled.

$$\frac{1}{(z-a)^n}e^{rz} \qquad\qquad \frac{e^{rz}}{z-a} = \frac{e^{rz} - e^{ra}}{z-a} + \frac{e^{ra}}{z-a}$$

$r=1$

$$\frac{e^z}{z-a} = \frac{e^a}{z-a} + \frac{e^z-e^a}{z-a}$$

$$\frac{e^z-e^a}{z-a} = e^a\left(\frac{e^{z-a}-1}{z-a}\right)$$

$$= e^a\int_0^1 e^{t(z-a)}\,dt = \int_0^1 e^{a-ta}\,e^{tz}\,dt$$

This involves the expotentials e^{tz} for $0\le t\le 1$

$$\frac{e^{rz}}{z-a} = \frac{e^{ra}}{z-a} + \underbrace{\frac{1}{z-a}\left(e^{rz}-e^{ra}\right)}_{e^{ra}\underbrace{\left(\frac{e^{r(z-a)}-1}{z-a}\right)}_{\int_0^r e^{t(z-a)}\,dt}}$$

Maybe you want to involve $(\partial_r - a)^{-1}$, treat r as important variable, z as a constant.

$$(\partial_r - a)^{-1}\,e^{rz} = \frac{e^{rz}}{z-a}$$

Better is to say that

$$(\partial_r - a)\left(\frac{e^{rz}}{z-a}\right) = \frac{ze^{rz}-ae^{rz}}{z-a} = e^{rz}$$

You have functions of r related by F.T. 977
to functions of ρ. 01273 722999 14:45

$$\psi(r) = \int e^{ir\rho}\,\hat{\psi}(\rho)\,\frac{d\rho}{2\pi}$$

Start again. Derive grid equations for $\psi(r,n)$.

$$k_\varepsilon\,\psi^1(r+\varepsilon,n) - \psi^1(r,n) = b\varepsilon\,\psi^2(r,n)$$

$$k_\varepsilon\,\psi^2(r,n+1) - \psi^2(r,n) = \bar{b}\,\psi^1(r,n)$$

$$\boxed{\begin{aligned}(\partial_r - a)\,\psi^1(r,n) &= b\,\psi^2(r,n)\\ \psi^2(r,n+1) - \psi^2(r,n) &= \bar{b}\,\psi^1(r,n)\end{aligned}}$$

$k_\varepsilon = \sqrt{1-|b|^2\varepsilon}$
$= 1 - a\varepsilon$

$a = \frac{1}{2}|b|^2$

$$\psi(r,n) = e^{ir\rho}\mu^n\begin{pmatrix}v^1\\ v^2\end{pmatrix}$$

$$(i\rho - a)\,v^1 = b v^2$$
$$(\mu - 1)\,v^2 = \bar{b} v^1$$

$$\mu = 1 + \frac{2a}{i\rho - a} = \frac{i\rho + a}{i\rho - a}$$

exp. solutions ~~x~~ $\psi(r,n) = e^{ir\rho}\left(\frac{i\rho+a}{i\rho-a}\right)^n\begin{pmatrix}\frac{b}{i\rho-a}\\ 1\end{pmatrix}\cdot \text{const}$

~~[illegible]~~ Let's imitate ~~[illegible]~~ the cont. case if possible. $\psi(r,s) = e^{i(r\rho+s\sigma)}\begin{pmatrix}v^1\\ v^2\end{pmatrix}$

$$-\partial_r\psi^1 = i\psi^2$$
$$\partial_s\psi^2 = i\psi^1$$

$$-\rho v^1 = v^2$$
$$\sigma v^2 = v^1$$

$\therefore$ $\psi(r,s)$ $= e^{i(r\rho - s\rho^{-1})}\begin{pmatrix}1\\ -\rho\end{pmatrix}$ const

---

Given $\psi(r,0) = \int e^{ir\rho}\begin{pmatrix}1\\ -\rho\end{pmatrix}\hat{\psi}_0^1(\rho)\,\frac{d\rho}{2\pi}$ then
$\psi(r,s) = e^{+s\partial_r^{-1}}\psi(r,0)$

Given $\psi(r,0) = \int e^{ir\rho}\left(\frac{b}{i\rho - a}\right)\hat{\psi}^2(\rho,0)\,\frac{d\rho}{2\pi}$

then

$$\boxed{\psi(r,n) = \left(\frac{\partial_r + a}{\partial_r - a}\right)^n \psi(r,0)}$$

need to check this.

$$(\partial_r - a)\psi^1(r,n) = b\psi^2(r,n)$$

$$\psi^2(r,n+1) - \psi^2(r,n) = \bar{b}\psi^1(r,n) = \frac{2a}{\partial_r - a}\psi^2(r,n)$$

$$\psi^2(r,n+1) = \frac{\partial_r + a}{\partial_r - a}\psi^2(r,n)$$

$$\underbrace{\psi^1(r,n) = \frac{b}{\partial_r - a}\psi^2(r,n)}_{\text{true for } n=0}$$

so what seems ~~to~~ happen? You need to understand the operator $\frac{\partial_r + a}{\partial_r - a}$ which should be unitary provided ρ is kept real.

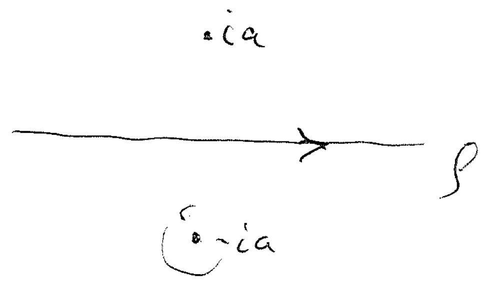
Also $\frac{\partial_r + a}{\partial_r - a} = 1 + \frac{2a}{\partial_r - a}$

Simplest function ~~seems to~~ $\psi^2(r,0)$ seems to be $\delta(r)$, whence $\widehat{\psi^2}(\rho,0) = 1$

$$\widehat{\psi^1}(\rho,0) = \frac{b}{i\rho - a}$$

decays in UHP if $r>0$

$$\int e^{ir\rho}\frac{b}{i\rho + ia}\frac{d\rho}{2\pi i}$$



$$= \begin{cases} 0 & r > 0 \\ -e^{ra} & r < 0 \end{cases}$$

Go over this stuff.

$$(k_\varepsilon \lambda^\varepsilon - 1) v^1 = b\sqrt{\varepsilon}\, v^2\sqrt{\varepsilon}$$

$$(k_\varepsilon \mu - 1) v^2\sqrt{\varepsilon} = \bar{b}\sqrt{\varepsilon}\, v^1$$

$$\begin{pmatrix} \lambda^\varepsilon v^1 \\ \mu v^2 \sqrt{\varepsilon} \end{pmatrix} = \frac{1}{k_\varepsilon}\begin{pmatrix} 1 & b\sqrt{\varepsilon} \\ \bar{b}\sqrt{\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2\sqrt{\varepsilon} \end{pmatrix}$$

$$(-a + i\rho) v^1 = b v^2$$

$$(\mu - 1) v^2 = \bar{b} v^1$$

$$\mu = 1 + \frac{\overset{2a}{\cancel{|b|^2}}}{i\rho - a} = \frac{i\rho + a}{i\rho - a}$$

$$(\partial_r - a)\psi^1(r, n) = b\psi^2(r, n)$$

$$\psi^2(r, n+1) - \psi^2(r, n) = \bar{b}\psi^1(r, n) = \frac{\overset{2a}{\cancel{\bar{b}b}}}{\partial_r - a}\psi^2(r, n)$$

$$\boxed{\begin{aligned} \psi^2(r, n+1) &= \frac{\partial_r + a}{\partial_r - a}\psi^2(r, n) \\ \psi^1(r, n) &= \frac{b}{\partial_r - a}\psi^2(r, n) \end{aligned}}$$

If you start with $\psi^2(r, 0) = \delta(r)$

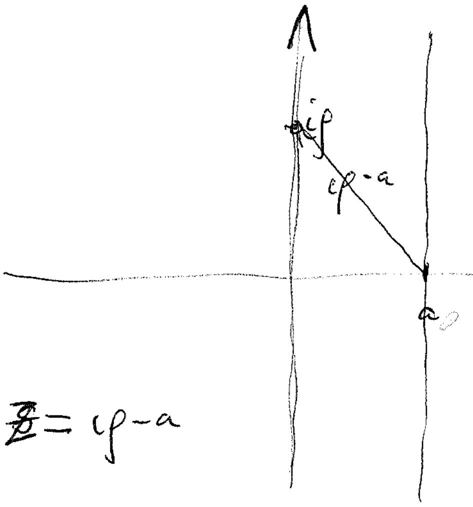
if $r > 0$ decays in UHP

$$\psi(r, n) = \int e^{ir\rho}\left(\frac{i\rho + a}{i\rho - a}\right)^n \begin{pmatrix} \frac{b}{i\rho - a} \\ 1 \end{pmatrix} \frac{d\rho}{2\pi}$$

For $n \geq 0$ $\quad \rho = -ia$ sing. $\quad \psi(r, n) = 0$ $\quad n \geq 0,\ r > 0$

For $n < 0$ $\quad \rho = ia$ sing $\quad \psi(r, n) = 0$ $\quad n < 0,\ r < 0$

$$e^{-ar}\psi(r,n) = \int_{-a+i\infty} e^{(i\rho-a)r}\left(\frac{i\rho+a}{i\rho-a}\right)^n\left(\frac{b}{i\rho-a} \atop 1\right)\frac{d(i\rho-a)}{2\pi i}$$



$$= \int_{-a+i\infty} e^{zr}\left(\frac{z+2a}{z}\right)^n\left(\frac{b}{z} \atop 1\right)\frac{dz}{2\pi i}$$

$z = i\rho - a$

if $r>0$, e^{rz} decays in LHP giving

if $r<0$, e^{zr} ———— RHP

giving a polynomial in r.

$$\left(1+\frac{2a}{z}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\frac{(2a)^k}{z^k}$$

$$\oint e^{zr}\frac{1}{z^k}\frac{dz}{2\pi i} = \frac{r^{k-1}}{(k-1)!}$$

$$\therefore\ \psi(r,n) = \begin{cases} e^{ar}\left(\text{poly of degree} \le k \text{ in } r\right) & r<0 \\ 0 & r>0 \end{cases}$$

---

next make σ direction cont.

$$\psi(r,n) = \int_{-\infty}^{\infty} e^{ir\rho}\left(\frac{i\rho+a}{i\rho-a}\right)^n\left(\frac{b}{i\rho-a} \atop 1\right)\frac{d\rho}{2\pi}$$

$$(i\rho-a)\overset{v'_\varepsilon\sqrt{\varepsilon}}{v^1} = \overset{b\sqrt{\varepsilon}}{b}\,v^2\ {}^{v^2_\varepsilon}$$

$$(\mu-1)v^2 = \overset{b\sqrt{\varepsilon}}{b}\,v^1\ {}^{v'_\varepsilon\sqrt{\varepsilon}}$$

$$a_\varepsilon = \frac{1}{2}|b|^2\varepsilon$$

$$\mu_\varepsilon^{\frac{1}{\varepsilon}} = \left(\frac{i\rho+a}{i\rho-a}\right)^{1/\varepsilon} = \left(1+\frac{2a_\varepsilon}{i\rho-a_\varepsilon}\right)^{1/\varepsilon}$$

$$= \exp\left\{\frac{\overset{|b|^2}{2a'_0}}{i\rho}\right\}$$

Start with $\cancel{(k_\varepsilon \lambda^\varepsilon - 1)\, v^1_\varepsilon = \underset{b\sqrt{\varepsilon}}{h_\varepsilon} v^2_\varepsilon} = v^2\sqrt{\varepsilon}$

$\cancel{(k_\varepsilon \mu^\varepsilon - 1)\, \underset{v^2\sqrt{\varepsilon}}{v^2_\varepsilon} = \underset{\bar{b}\sqrt{\varepsilon}}{\bar{h}_\varepsilon} \underset{v^1}{v^1_\varepsilon}}$

$$(k\lambda - 1)\, v^1 = h v^2$$
$$(k\mu - 1)\, v^2 = \bar{h} v^1$$

these quantities depend on ε

$k_\varepsilon \quad h_\varepsilon \quad v^1_\varepsilon \quad v^2_\varepsilon \quad \lambda_\varepsilon = \lambda^\varepsilon \quad \mu_\varepsilon$

$1 - a\varepsilon \quad b\sqrt{\varepsilon} \quad v^1 \quad v^2\sqrt{\varepsilon} \quad e^{i\rho\varepsilon} \quad \mu$

$$\frac{1-k_\varepsilon}{\varepsilon} \to a \qquad \frac{h_\varepsilon}{\sqrt{\varepsilon}} \to b, \quad \frac{\bar{h}_\varepsilon}{\sqrt{\varepsilon}} \to \bar{b} \qquad \therefore\ a = \tfrac{1}{2}|b|^2$$

$$\frac{v^1}{\varepsilon} \to v^1 \qquad \frac{v^2_\varepsilon}{\sqrt{\varepsilon}} \to v^2$$

$$(i\rho - a)\, v^1 = b v^2$$
$$(\mu - 1)\, v^2 = \bar{b} v^1$$

$$\frac{v^1_\varepsilon}{\sqrt{\varepsilon}} \to v^1 \qquad v^2_\varepsilon \to v^2$$

$$\frac{b_\varepsilon}{\sqrt{\varepsilon}} \to 1 \qquad a_\varepsilon = \tfrac{1}{2}|b_\varepsilon|^2$$

$$\frac{a_\varepsilon}{\varepsilon} \to \frac{1}{2}$$

$$(i\rho - a_\varepsilon)\frac{v^1_\varepsilon}{\sqrt{\varepsilon}} = \frac{b_\varepsilon}{\sqrt{\varepsilon}} v^2_\varepsilon \quad \rightsquigarrow \quad i\rho\, v^1 = v^2$$

$$\left(\frac{\mu_\varepsilon - 1}{\varepsilon}\right) v^2_\varepsilon = \frac{\bar{b}_\varepsilon}{\sqrt{\varepsilon}} \frac{v^1_\varepsilon}{\sqrt{\varepsilon}} \to v^1 \quad \rightsquigarrow \quad \mu'_0\, v^2 = v^1$$

$$\mu_\varepsilon = \frac{i\rho + a_\varepsilon}{i\rho - a_\varepsilon}$$

digress on harmonic oscillators. Consider a real symplectic vector space V_r^{2n}. The point is that ~~the~~ infinitesimal symplectic transf. on V_r are given by quadratic forms on V_r. ~~The~~

$$\mathrm{Lie}\left(Sp(2n,\mathbb{R})\right) = \underline{\text{quadratic forms on } \mathbb{R}^{2n}}$$

$$\dim = \frac{2n(2n+1)}{2} = n(2n+1)$$

$$\dim Sp(2n,\mathbb{R}) = 2n + 2n-1 + \dim Sp(2n-2,\mathbb{R})$$

$$= 2n\left(\frac{2n+1}{2}\right) = n(2n+1).$$

Consider real ~~orth~~ Euclidean space V_r^{2n}, inf. orth. transf given by skew-symm bilinear forms.

$$\dim \mathrm{Lie}\left(O(2n)\right) = \frac{2n(2n-1)}{2} = n(2n-1)$$

$$\dim O(m) = m-1 + \dim O(m-1)$$

$$= \frac{m(m-1)}{2}$$

Suppose given on V_r^{2n} a symp. form ω and a pos. def. quadratic form H. Possible structure has

~~$\dim = \frac{2n(2n-1)}{2} + \frac{2n(2n+1)}{2}$~~

$$\dim = \frac{2n(2n+1)}{2} + \frac{2n(2n-1)}{2} = \frac{2n(4n)}{2} = 4n^2 = (2n)^2$$

which is the same as the dim $GL(2n,\mathbb{R})$. This general linear group acts ~~by~~ as usual on bilinear forms. There are n real frequencies.

V real v.s., two nondegenerate bilinear forms

ω skew-symm. $\qquad \omega : V \xrightarrow{\sim} V'$

H symm. $\qquad H : V \xrightarrow{\sim} V'$

from this data you get $H^{-1}\omega : V \circlearrowleft$ and its inverse $\omega^{-1}H$ $\qquad (\omega^{-1}H)^t = H^t(\omega^t)^{-1} = H(-\omega)^{-1} = -(H\omega^{-1})$

Thus the operator $\omega^{-1}H$ on V and $H\omega^{-1}$ on V' are related by negative transpose. If you use H ~~a basis~~ to identify V ~~and~~ V', the X

Repeat: Given V a vector space, 1) $\omega : V \longrightarrow V'$ a non deg skew-symm. bilinear form $\quad \omega^t = -\omega$

$\exists$ equivalence between symm. bilinear forms $H : V \longrightarrow V'$ $\quad H^t = H$ and endos X of V preserving ω : means $X^t\omega + \omega X = 0$, given by $X = \omega^{-1}H$

Proof: Let $X = \omega^{-1}H$. Then

$H = \omega X$, $\quad H^t = X^t(-\omega) = -X^t\omega$ so you have $H = H^t \iff \omega X = -X^t\omega$ i.e. $X^t\omega + \omega X = 0$.

2) Given $H : V \xrightarrow{\sim} V'$ $H^t = H$ a non deg symm bil. form, Then have equiv. between endos X of V pres. H : means $X^tH + HX = 0$, and skew symm. bil f. $\omega : V \to V'$ given by $\omega = HX$

because $\omega^t = (HX)^t = X^t H$, $\omega = HX$ so

$\omega^t + \omega = X^t H + HX$ whence ω skew symm

$\Leftrightarrow X$ preserves H. Something very puzzling here is the occurrence of both X, X^{-1} being in the Lie alg ~~pre~~ of ~~autos~~ inf. autos preserving H and ω.

At the moment we have $V \underset{\omega}{\overset{H}{\longrightarrow}} V'$

so we have two endo of V, $X = \omega^{-1}H$ and $H^{-1}\omega = X^{-1}$!

I am normally used to ~~[illegible]~~ $\omega X = H$.

~~$X^t\omega = -X^t\omega^t = (\omega X)^t = -H^t = -H = -\omega X$~~

Check that
$$\begin{cases} X^t\omega + \omega X = 0 \\ X^t H + HX = 0. \end{cases}$$

first one easy: $H = \omega X \qquad H^t = X^t(-\omega)$

$$\therefore\ 0 = H - H^t = \omega X + X^t\omega$$

$X = \omega^{-1}H \qquad$ ~~$X^tH + HX = \underbrace{(\omega^{-1}H)^t}\,H + H(\omega^{-1}H) = 0$~~

$0 = (\omega X + X^t\omega)X = HX + X^tH$ ~~$H^t(-\omega)^{-1}H$~~

~~[illegible]~~ $\qquad$ Puzzle: $X^t\omega + \omega X = 0$

$\Longrightarrow \omega X^{-1} + \underbrace{(X^t)^{-1}}_{(X^{-1})^t}\omega = 0$

$$V \xrightarrow[\underset{H}{\sim}]{\overset{\omega}{\sim}} V' \qquad \begin{matrix} \omega^t = -\omega \\ H^t = H \end{matrix}$$

Define $X = \omega^{-1}H : V \circlearrowleft$. Then $\omega X = H = H^t$ $= (\omega X)^t = -X^t\omega$, so $X^t\omega + \omega X = 0$. Also

$$X^t H = X^t \omega X = -(\omega X)X = -HX \Rightarrow X^t H + HX = 0.$$

All this assumes only invertibility of ω.

Because X preserves H it is skew-symmetric for the inner product defined by H i.e.

$$\xi^* HX\eta + (X\xi)^t H\eta = 0$$

Also

$$\xi^t \omega X\eta = \xi^t H\eta \quad \text{is symmetric.}$$

Aim? Because X preserves H it ~~acts~~ extends to a derivation on the Clifford algebra arising from from (V, H), so $\mathrm{Ad}(e^{tX})$ gives a time evolution on Cliff. V itself sits inside Cliff generating it.

Because X on V preserves ω it

Repeat. $V \underset{H}{\overset{\omega}{\rightrightarrows}} V'$ $\omega(v_1, v_2) =$

$\iota_{v_2} \iota_{v_1} \omega = \omega(v_1, v_2)$ $\omega(v_1)$

Let V be a f.d. vector space, V^* its dual $\omega \in \Lambda^2 V^*$ $H \in S^2 V^*$. Interpret ω as skew-symm. bil. form. $\omega(u,v) = \iota_v \iota_u \omega$, so if $\omega = \lambda \wedge \mu$, then $(\lambda\wedge\mu)(u,v) = \iota_v\iota_u(\lambda\wedge\mu)$

$$= \begin{vmatrix} \iota_u\lambda & \iota_v\lambda \\ \iota_u\mu & \iota_v\mu \end{vmatrix}$$

$B \in V^* \otimes V^*$ $\langle u\otimes v \mid \lambda \otimes \mu \rangle$?

$B(u,v)$ bilinear form on V

$\tilde{B} : V \longrightarrow V^*$, $\tilde{B}u$ think vectors + ~~matrices~~ matrices

$B(u,v) = u^t B v$ $\mathbb{C} \xrightarrow{v} V \xrightarrow{B} V^* \xrightarrow{u^t} \mathbb{C}$

$B(u,v) = v^t B^t u = B^t(v,u)$ $\mathbb{C} \xrightarrow{u} V \xrightarrow{B^t} V^* \xrightarrow{v^t} \mathbb{C}$

$B(v,u) = v^t B u = (v^t B u)^t = u^t B^t v$

So $B(u,v) = B(v,u) \iff B = B^t$

Let $X = \omega^{-1} H : V \to V' \to V$

$X^t = H^t (\omega^t)^{-1} = H(-\omega^{-1}) = -H\omega^{-1}$

$X^t\omega + \omega X = -H\omega^{-1}\omega + \omega\omega^{-1}H = 0$ | $X^t H = -H\omega^{-1}H = -HX$

So how to set up? First point is that given ω invertible, then X sat $X^t\omega + \omega X = 0$ are uniquely rep. $X = \omega^{-1}H$ with $H = H^t$.

Why: ~~$(\omega X)^t = X^t(-\omega) = \omega X$~~ | ~~Put $H = \omega X$~~

Thus X sats $X^t\omega + \omega X = 0 \iff (\omega X)^t = \omega X$.

in this case $X^t(\omega X) + (\omega X)X = (X^t\omega + \omega X)X = 0$.

Sim. If H inv. then any X sats $X^tH + HX = 0$ is uniquely representable $X = H^{-1}A$ with $A^t = -A$.

$$(HX)^t = X^tH = -HX \quad \Big| \quad X^tH + HX = 0 \iff (HX)^t = -HX$$

in this case $X^tA + AX = X^tHX + HXX = 0$.

Now comes the interesting point, an asymmetry. Suppose given both ω, H invertible. Then there are two choices $\omega^{-1}H$ or $H^{-1}\omega$ for X. In fact in the usual ~~theory~~ quantization one replaces X by its phase. So you form X^2. Now

$$X^t\omega + \omega X = 0 \implies (X^2)^t\omega = X^tX^t\omega = -X^t\omega X = \omega X^2$$

So all odd powers of X preserve ω, H.

So now examine quantization. You have V with ω, H and you ~~put $X = \omega$~~ define X by $\omega X = H$, get a flow e^{tX} on V preserving the forms ω, H. The other choice for the flow is to use $X^{-1} = H^{-1}\omega$. Is there something neutral, like $\frac{X}{|X|}$?

Quantization. Again start with V, ω, H
How am I to proceed? Associated to the symplectic form ω is ~~a~~ Weyl(V,ω), and to the quad. form H is Cliff(V,H). X is a derivation of these, so you get 1-param. groups of autos. e^{tX}.

Cliff is generated by ψ_v $v \in V$ subject to the CAR
$\psi_v \psi_{v'} + \psi_{v'} \psi_v = 2H(v,v')$ equiv. $\psi_v^2 = H(v,v)$
It has an increasing filtration ~~w~~whose gr is ΛV and there should be a canonical linear isom
Cliff $\xrightarrow{\sim} \Lambda V$ defined by making Cliff act on ΛV
$\psi_v \longmapsto e_v + \iota_v$ and ~~then~~ taking the action on $1 \in \Lambda^0 V$.
So you ~~then~~ have $\Lambda^2 V$ naturally embedded in Cliff.

Similarly you have $S^2 V$ naturally embedded in ~~W~~ Weyl(V,ω). Question: Can you naturally assoc. ~~elements~~ an element ~~one~~ of $\Lambda^2 V$ corresp to $\omega \in \Lambda^2 V^*$, and an elt of $S^2 V$ corresp. to ~~H~~ $H \in S^2(V^*)$? Something ~~is~~ seems obvious, namely, if you choose kinematics (V,H) i.e. Cliff picture, then you should use $H : V \xrightarrow{\sim} V^*$ to transport $\omega \in \Lambda^2 V^*$ to ~~$H^{-1} \otimes H^{-1}$~~ $(\Lambda^2 H^{-1})(\omega) \in \Lambda^2 V$ to get dynamics.

Consider Weyl(V,ω) $[\phi_v, \phi_{v'}] = \omega(v,v')$
where $\omega \in \Lambda^2 V^*$ is a non-deg skew-symm b.f.
Suppose $H \in S^2 V^*$ is ~~pos def. quadr~~ a symm. bil. form on V.

what is the ~~[illegible]~~ question. Inside Weyl is the subspace spanned by $\{\phi_v^2 \mid v\in V.\}$ which is isom. to S^2V. ~~From H you get~~

You begin with V, ω $\qquad$ ω non. deg.

$$\{X\in End(V) \mid X^t\omega + \omega X = 0\} \cong \{H\in S^2V^*\} \quad \{H\in Hom(V, V^*)\ \ H^t = H.$$

$$\cancel{X}_H = \omega^{-1}H \longleftarrow H$$

$$X \longmapsto \omega X$$

To find quadratic elt $Y\in Weyl(V)$ such that $[Y, \phi_{v_1}] = \phi_{Xv_1}$. Exs. $Y = \phi_v^2$

$$[\phi_v^2, \phi_{v_1}] = \phi_v\,\omega(v, v_1) + \omega(v, v_1)\phi_v$$

$$= \phi_v\ 2\omega(v, v_1)$$

V vector space with bilinear forms $\omega, H : V\longrightarrow V^*$ $\omega^t = -\omega$, $H^t = H$. Assume ω non deg, put $X = \omega^{-1}H : V\longrightarrow V$,

$$\left|\begin{array}{l}\omega X = H = H^t = -X^t\omega \implies X^t\omega + \omega X = 0\\ \omega X X = -X^t\omega X \implies HX + X^tH = 0.\end{array}\right.$$

Notice $\cancel{H}\ H = 0 \implies X = 0$. Form $Weyl(V) \cong \mathbb{R} + V + S^2V + .$

Weyl gen. by linear $\phi : V\longrightarrow Weyl$ $\quad v\mapsto \phi_v$ $\quad [\phi_v, \phi_{v'}] = v^t\omega v'$

Weyl $= \mathbb{R}\oplus\phi V\oplus$ span of $\phi_v\phi_{v'} + \phi_{v'}\phi_v$

$$[(\phi_v\phi_{v'} + \phi_{v'}\phi_v), \phi_{v''}] = \phi_v\, v'^t\omega v'' + \phi_v'\, v^t\omega v'' + \phi_{v'}\, v^t\omega v'' + \phi_v\, v'^t\omega v''$$

$$\boxed{\left[\tfrac{1}{2}\phi_{v_1}^2, \phi_v\right] = \phi_{v_1}(v_1^t\omega v)}$$

At the moment for each ~~as~~ $H = \frac{1}{2}\phi_v^2 \in S^2V$

you get the op. $\phi_{v'} \longmapsto [H,\phi_{v'}] = \phi_v\, v^t\omega v'$

So to $\frac{1}{2}v_1^2 \in S^2V$ you get ?

$Q = \frac{1}{2}\phi_x^2$ $\qquad [\frac{1}{2}\phi_x^2, \phi_v] = \phi_x(x^t\omega v)$.

$$v \longmapsto x(x^t\omega v)$$

$$\begin{array}{ccccc} & & V\otimes V^* & & \\ & & \cup & & \\ S^2V & \longrightarrow & \mathfrak{sp}(V) & \longrightarrow & S^2(V^*) \\ \frac{x^2}{2} & \longmapsto & (v \mapsto x(x^t\omega v)) & & \end{array}$$

Gets clearer. You want to introduce Lie $Sp(V) = \mathfrak{sp}(V)$

$$\begin{array}{ccccc} S^2V & \longrightarrow & \mathfrak{sp}(V) & \longrightarrow & S^2V^* \\ \| & & \| & & \| \\ \text{~~spanned~~ gen. by} & & \{X \in \mathcal{L}(V) \mid X^t\omega + \omega X = 0\} & & \{H : V \to V^* \mid H^t = H\}. \end{array}$$

$v \mapsto \frac{1}{2}\phi_v^2 \longmapsto (v' \longmapsto v(v^t\omega v'))$

$$\begin{array}{ccc} \mathbb{C} & \xleftarrow{\ v^t\ } & \\ \downarrow v & & \\ V & \xrightarrow{\ \omega\ } & V^* \end{array}$$

means $[\frac{1}{2}\phi_v^2, \phi_{v'}] = \phi_v\, v^t\omega v'$

So $$\begin{array}{ccccc} S^2V & \longrightarrow & \mathfrak{sp}(V) & \longrightarrow & \overset{H}{S^2V^*} \\ \frac{v^2}{2} & \longmapsto & v v^t\omega & \longmapsto & \omega v v^t \omega = -\omega v(\omega v)^t \end{array}$$

$$\frac{1}{2}\sum v_i^2 \longmapsto \omega \sum v_i v_i^t \omega$$

$$\frac{1}{2}\sum v_i^2 \searrow \sum v_i v_i^t\omega = X \nearrow \omega \sum v_i v_i^t \omega$$

$v \longmapsto v^t\omega : V \xrightarrow{\omega} V^* \xrightarrow{v^t} \mathbb{C}$

$V \xrightarrow{\sim} V^*$

Start with V, $\omega : V \xrightarrow{\sim} V'$ $\qquad \omega^t = -\omega$

have isom.

$$S^2V \longrightarrow sp(V,\omega) \longrightarrow S^2V^*$$

$$\begin{array}{ccc} \Psi & X & \longmapsto \omega X \\ \frac{1}{2}\left(\phi_{v_1}\phi_{v_2} + \phi_{v_2}\phi_{v_1}\right) & & \end{array}$$

goes to the operator

$$[\phi_{v'}, \phi_v] = v'^t\omega v$$

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "v"'] & \\ V \arrow[rr, "\omega"] & & V^* \arrow[ul, "v^t"'] \end{tikzcd}$$

$$\phi_v \longmapsto \tfrac{1}{2}\left[\phi_{v_1}\phi_{v_2} + \phi_{v_2}\phi_{v_1}, \phi_v\right]$$

$$= \Big(v_1^t\omega v\, \phi_{v_2} + v_2^t\omega v\, \phi_{v_1} + \phi_{v_1} v_2^t\omega v + \phi_{v_2} v_1^t\omega v\Big)/2$$

$$= \phi_{v_1} v_2^t\omega v + \phi_{v_2} v_1^t \omega v$$

Map is $\qquad v \longmapsto v_1\, v_2^t\omega v + v_2\, v_1^t\omega v$

$$v_1\cdot v_2 \longmapsto v_1 \otimes \overbrace{v_2^t\omega}^{\in V^*} + v_2 \otimes v_1^t\omega$$

$$\longmapsto \omega v_1 \otimes v_2^t\omega + \omega v_2 \otimes v_1^t \omega \qquad ?$$

$$= \underbrace{\omega v_1}_{\in V^*} \otimes \underbrace{(-\omega v_2)^t}_{V^*} + \omega v_2 \otimes (-\omega v_1)^t.$$

$$S^2 V \longrightarrow sp(V, \omega)$$

$$v_1 \cdot v_2 \longmapsto v_1 \otimes v_2^t \omega + v_2 \otimes v_1^t \omega \in V \otimes V^*$$

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "v"'] & \\ V \arrow[rr, "\omega"] & & V^* \arrow[ul, "v^t"'] \end{tikzcd}$$

$$(v^t \omega)^t = \omega^t v = -\omega v$$

$$v^t \omega : V \xrightarrow{\omega} V^* \longrightarrow \mathbb{C}$$

$$(v^t \omega)^t : V^* \xleftarrow{-\omega} V \xleftarrow{v} \mathbb{C}$$

Go over again V, ω non deg $= -\omega^t$,

$$X \longmapsto \omega X$$

$$S^2 V \longrightarrow sp(V, \omega) \xrightarrow{\sim} S^2 V^*$$

$$sp(V,\omega) = \{X \in V \otimes V^* \mid X^t \omega + \omega X = 0\}$$

$$S^2 V \hookrightarrow Weyl(V, \omega)$$

$$v_1 v_2 \longmapsto \tfrac{1}{2}(\phi_{v_1} \phi_{v_2} + \phi_{v_2} \phi_{v_1}) \rightsquigarrow ($$

$$S^2 V \times V \longrightarrow V$$

$$\left[\tfrac{1}{2}(\phi_{v_1}\phi_{v_2} + \phi_{v_2}\phi_{v_1}), \phi_v\right] = \phi_{v_1} v_2^t \omega v + \phi_{v_2} v_1^t \omega v = {}^{\phi}X_{v_1 v_2}(v)$$

$$X_{v_1 v_2}(v) = v_1 v_2^t \omega v + v_2 v_1^t \omega v$$

$$\therefore X_{v_1 v_2} = v_1 \otimes v_2^t \omega + v_2 \otimes v_1^t \omega$$

$$H_{v_1 v_2} = \omega X_{v_1 v_2} = \omega v_1 \otimes v_2^t \omega + \omega v_2 \otimes v_1^t \omega \in S^2(V^*)$$

$$= -\omega v_1 \otimes (\omega v_2)^t - \omega v_2 \otimes (\omega v_1)^t$$

$X_{v_1 v_2} = v_1\,\omega(v_2, -) + v_2\,\omega(v_1, -)$?

$\omega X_{v_1 v_2} = \omega v_1\,\omega(v_2, -) + \omega v_2\,\omega(v_1, -)$

$$S^2 V \hookrightarrow \mathrm{Weyl}(V, \omega)$$

$$v_1 v_2 \xmapsto{\alpha} \tfrac{1}{2}(\phi_{v_1}\phi_{v_2} + \phi_{v_2}\phi_{v_1})$$ call this element $\alpha(v_1 v_2)$.

Then $[\alpha(v_1 v_2), \phi_v] = \phi_{v_1}\,\omega(v_2, v) + \phi_{v_2}\,\omega(v_1, v)$.

$\parallel$

$\phi_{v_1 \omega(v_2, v) + v_2 \omega(v_1, v)}$ This should be

$\phi_{X_{v_1 v_2}(v)}$ where $X_{v_1 v_2}(v) = v_1\,\omega(v_2, v) + v_2\,\omega(v_1, v)$

$X_{v_1 v_2} \in sp(V, \omega)$? | $X_{v_1 v_2} = v_1 \otimes v_2^t \omega + v_2 \otimes v_1^t \omega$

$$\omega X_{v_1 v_2} = \omega v_1 \otimes v_2^t \omega + \omega v_2 \otimes v_1^t \omega = H_{v_1 v_2}$$

Given an element of $S^2 V \subset V \otimes V$ you can view it as a map from V^* to V then compose with $\omega: V \to V^*$

Go to $\mathrm{Cliff}(V, H)$ $\{\psi_{v_1}, \psi_{v_2}\} = v_1^t H v_2$

$$\Lambda^2 V \xrightarrow{\sim} o(V, H) \xrightarrow{\sim} \Lambda^2 V^*$$

$$v_1 \wedge v_2 \mapsto \tfrac{1}{2}[\psi_{v_1}, \psi_{v_2}]$$

$$\mathrm{ad}(v_1 \wedge v_2)\, v \mapsto \tfrac{1}{2}[\psi_{v_1}\psi_{v_2} - \psi_{v_2}\psi_{v_1}, \psi_v] = \psi_{v_1}\, v_2^t H v - \psi_{v_2}\, v_1^t H v$$

$$v_1^t H v\, \psi_{v_2} + v_2^t H v\, \psi_{v_1}$$

so ad ~~action of~~ $\phi_{\Lambda^2 V}$ on ϕ_V gives the operator

$$X_{v_1 \wedge v_2} : v \longmapsto v_1 v_2^t H v - v_2 v_1^t H v$$

$$\therefore X_{v_1 \wedge v_2} = v_1 \otimes v_2^t H - v_2 \otimes v_1^t H$$

~~OK. ... What~~ Start again. First suppose given $V, H : V \xrightarrow{\sim} V^*$ $H^t = H$. Then have isos.

$$X \longmapsto HX$$

$$o(V, H) \xrightarrow{\sim} \Lambda^2 V^* = \{\omega : V \to V^* \mid \omega^t = -\omega\}$$

$$\| $$

$$\{X \in V \otimes V^* \mid X^t H + HX = 0\}$$

$$\therefore \omega = HX$$
$$H^{-1}\omega = X$$

viewpoint might be ~~that since ω can~~ to concentrate on a general skew-symm. ω even $\omega = 0$.

What about quantization. Because you deal with $o(V, H)$ and symm. H you use $Cliff(V, H)$ ~~and $S^2 V = \{[\phi_{v_1}, \phi_{v_2}] \subset Cliff\}$~~ ~~containing the~~ the CAR alg, equipped with $\phi(V)$ $\phi(S^2 V)$ with the various brackets. $\{\phi_{v_1}, \phi_{v_2}\} = v_1^t H v_2$.

define $S^2 V, V$

Another point: Given ω non-degenerate you get $\omega : V \xrightarrow{\sim} V^*$ and hence can transport ω to V^* to get $V^* \longrightarrow V^*$

Just what maps are naturally around

V, ω skew, nondeg.

$$X \longmapsto \omega X$$

$$S^2V \xrightarrow{\alpha} sp(V,\omega) \xrightarrow{\sim} S^2V^*$$

$$\| $$

$$\{X \in V\otimes V^* \mid \underbrace{X^t\omega + \omega X}_{=0}\}$$

$W(V)$ $\quad [\phi_{v_1}, \phi_{v_2}] = v_1^t \omega v_2$

$$[\{\phi_{v_1}, \phi_{v_2}\}, \phi_v] = \{v_1^t\omega v, \phi_{v_2}\} + \{\phi_{v_1}, v_2^t \omega v\}$$

$$= 2\left(\phi_{v_1} v_2^t \omega v + \phi_{v_2} v_1^t \omega v\right)$$

$$= 2\, \phi_{v_1 v_2^t \omega v + v_2 v_1^t \omega v}$$

define $\alpha_{v_1 v_2}$ $\cancel{\alpha(v_1, v_2)} = v_1 v_2^t \omega + v_2 v_1^t \omega \in V\otimes V^*$

$$\omega\alpha_{v_1v_2} = \omega v_1 v_2^t \omega + \omega v_2 v_1^t \omega \in V^*\otimes V^*$$

So is $v_2^t\omega = \omega v_2$?

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "v_1"'] & \\ V \arrow[rr, "\omega"] & & V^* \arrow[ul, "v_2^t"'] \end{tikzcd}$$

$$\mathbb{C} \xrightarrow{v_1} V \xrightarrow{\omega} V^* \xrightarrow{v_2^t} \mathbb{C}$$

NO $\quad \omega v_1 : \mathbb{C} \longrightarrow V^*$

$\quad v_2^t \omega : V \longrightarrow \mathbb{C}$

so $(v_2^t \omega)^t = -\omega v_2$?

$v_1 v_2^t : V^* \longrightarrow \mathbb{C} \longrightarrow V$

$v_2 v_1^t : V^* \longrightarrow \mathbb{C} \longrightarrow V$

There is this confusion. Something about identifying v with the map $\mathbb{C} \to V$ $c \mapsto cv$ doesn't work. Given $v_1 \otimes v_2 \in V \otimes V$ you want to ~~use~~ apply ω to v_2 to get an element of $V \otimes V^*$.

$$\frac{1}{2}\left[\phi_{v_1}\phi_{v_2}, \phi_v\right] = \phi_{v_1} v_1^t \omega v$$

So what is the operator $v \mapsto v_1 \cdot v_1^t \omega v$.

need notation to clarify this

Problem seems to be this: to relate ωv_1 with $v_1^t \omega$. You have endom. $v \mapsto v_1 v_1^t \omega v$

$$\mathbb{C} \xrightarrow{v} V \xrightarrow{\omega} V^* \xrightarrow{v_1^t} \mathbb{C} \xrightarrow{v_1} V$$

What you need is $v_1 v_2^t : V^* \longrightarrow \mathbb{C} \longrightarrow V$

and then you ~~apply~~ compose with ω on both sides $\omega v_1 v_2^t \omega : V \xrightarrow{\omega} V^* \xrightarrow{v_2^t} \mathbb{C} \xrightarrow{v_1} V \xrightarrow{\omega} V^*$

This gives you $\omega v_1 \in V^*$ and $v_2^t \omega = (-\omega v_2)^t : V \to \mathbb{C}$

$$V \otimes V^* \longrightarrow \mathrm{End}(V)$$

$$v \otimes \lambda \longmapsto v\lambda$$

$$V \otimes V \xrightarrow{1 \otimes \omega} V \otimes V^* \longrightarrow \mathrm{End}(V)$$

$$v_1 \otimes v_2 \longmapsto v_1 \otimes \omega v_2 \longmapsto v_1 \omega v_2$$

Confusion source. Given v_2 you get the element $\omega v_2 \in V^*$ which ~~can be which gives rise~~ is identified with a map $\mathbb{C} \xrightarrow{\omega v_2} V^*$, which in turn has a

---

Start again. Given $v \in V$ (equiv. $v : \mathbb{C} \to V$) you get $\omega v \in V^*$ (equiv. $\omega v : \mathbb{C} \to V \xrightarrow{\omega} V^*$). Now ~~$\omega v$ pairs~~ an element $\lambda \in V^*$ (equiv. a map $\lambda : \mathbb{C} \to V^*$) can be interpreted as the map $\lambda^t : V \to \mathbb{C}$. So you need your notation to distinguish the element $\lambda \in V^*$ from the map $\lambda^t : V \to \mathbb{C}$. So lets see if we can get things correctly written

$$V \otimes V \xrightarrow{1 \otimes \omega} V \otimes V^* \longrightarrow V^* \otimes V^*$$

---

e.g.

$$V \otimes V^* \longrightarrow \mathrm{End}(V) \quad \text{should be written}$$

$$v_1 \otimes \lambda \longmapsto v_1 \lambda^t$$

$$v_1 \otimes v_2 \longmapsto v_1 \otimes \omega v_2 \longmapsto v_1 (\omega v_2)^t \longmapsto \omega v_1 (\omega v_2)^t$$

We have the problem of straightening

$$\begin{tikzcd}
\mathrm{End}(V) \arrow[r] \arrow[d, equal] & V^* \otimes V^* \\
V \otimes V^* \arrow[ur, "\omega \otimes 1"'] &
\end{tikzcd}$$

So go back to $\left\{ X \in \underset{\mathrm{Hom}(V,V)}{\underline{\mathrm{End}\, V}} \right\} \xrightarrow{\omega\cdot} \mathrm{Hom}(V, V^*)$

$$X \longmapsto \omega X$$

$$v_i \lambda_i^t$$

Start again to straighten out the notation.

$$V \qquad V \xrightarrow{B} V^*$$

A bilinear form $B(v_1, v_2)$ on V is rep.

$\underline{v_1^t B v_2}$: $\mathbb{C} \xrightarrow{v_2} V \xrightarrow{B} V^* \xrightarrow{v_1^t} \mathbb{C}$

$$W \otimes V^* \xrightarrow{\sim} \mathrm{Hom}(V, W)$$

$$\omega \otimes \lambda \longmapsto \omega \lambda^t : V \to \mathbb{C} \to W$$

$$V^* \otimes V^* \xrightarrow{\sim} \mathrm{Hom}(V, V^*)$$

$$\lambda_1 \otimes \lambda_2 \longmapsto \lambda_1 \lambda_2^t : V \xrightarrow{\lambda_2^t} \mathbb{C} \xrightarrow{\lambda_1} V^*$$

Get to the point which is the Lie alg. st. on $S^2 V$

What is left to be done? Poisson bracket

We know that

$$S^2 V \longrightarrow sp(V) \xrightarrow{\sim} S^2(V^*) \subset \mathrm{Hom}(V, V^*)$$

$$X \longmapsto \omega X = H$$

I need to understand this better

$$S^2V \ni \frac{v^2}{2} \longmapsto \left[\tfrac{1}{2}\phi_v^2, \phi_x\right] = \phi_v\, v^t\omega x$$

get assoc. to $\frac{v^2}{2} \in S^2V$ the operator $v v^t \omega \in \mathrm{End}(V)$

the image of $v \otimes \lambda^t$ where $\lambda = -\omega v$

$\lambda^t = (-\omega v) = v^t(-\omega)^t = v^t\omega$

$X = vv^t\omega$

This is simplectic because $\omega X = \omega v v^t \omega$ is symm.

$(\omega v v^t \omega)^t = \omega^t v v^t \omega^t = (\pm 1)^2\, \omega v v^t \omega.$

So to complete the picture you want the map

$$S^2V \longrightarrow sp(V) \longrightarrow S^2(V^*) \hookrightarrow V^* \otimes V^*$$

$$\tfrac{1}{2}v^2 \longmapsto v v^t\omega \longmapsto \omega v v^t \omega = -\omega v \otimes \omega v$$

What is the map $\tfrac{1}{2}v^2 \longmapsto vv^t \in \mathrm{Hom}(V^*, V)$

$$vv^t : V^* \xrightarrow{v^t} \mathbb{C} \xrightarrow{v} V$$

There's a canonical iso

$$V \otimes V \xrightarrow{\sim} \mathrm{Hom}(V^*, V)$$

$$v_1 \otimes v_2 \longmapsto v_1 v_2^t : V^* \xrightarrow{v_2^t} \mathbb{C} \xrightarrow{v_1} V$$

$$v_1 v_2 = \tfrac{1}{2}(v_1+v_2)^2 - \tfrac{1}{2}v_1^2 - \tfrac{1}{2}v_2^2$$

$$\longmapsto \tfrac{1}{2}(v_1+v_2)(v_1+v_2)^t - \tfrac{1}{2}v_1v_1^t - \tfrac{1}{2}v_2v_2^t$$

$$= \tfrac{1}{2}(v_1v_2^t + v_2v_1^t) \longmapsto \tfrac{1}{2}(\omega v_1 v_2^t\omega + \omega v_2 v_1^t \omega)$$

$$\lambda \in V^* \equiv \mathbb{C} \xrightarrow{z \,\mapsto\, z\lambda} V^*$$

$$\lambda^t : V \longrightarrow \mathbb{C}$$