so next you try for the $\mathcal{A}$-module
= group ring of $\mathbb{R}\times\mathbb{Z}$. $=\{\lambda^x\mu^n \mid x\in\mathbb{R}, n\in\mathbb{Z}\}$

idea $\quad \Theta(v|-) - H(v,-)$

$\mathrm{res}_{\{0,k\}} - \mathrm{res}_{\{0,k^{-1}\}} = \mathrm{res}_{\{k\}} - \mathrm{res}_{\{k^{-1}\}}$ note this vanishes on $\mathbb{C}[\lambda,\lambda^{-1}]v$

In the situation horiz cont. you expect this difference to be replaced by $\mathrm{res}_{\{ia\}} - \mathrm{res}_{\{-ia\}}$) and hopefully this in turn has a limit as the vertical direction takes the cont. limit.

Yesterday found nice things - continuous limit from $L^2(S^1)$ to $L^2(\mathbb{R})$ having a ~~...~~ geometric interpretation. Discuss invariantly. The point is that attached to an oriented circle C in $S^2=\mathbb{CP}^1$ you have an intrinsic Hilbert space of L^2 sections of $\mathcal{O}(-1)$, ~~...~~ which comes with ~~...~~ Hilbert-Hardy splitting. ~~...~~
Choose an interior point a in C get rational function z ~~...~~ zero at a pole at reflected point, modulus 1 on C. ~~...~~ get ~~...~~: coordinate unique up to $e^{i\theta}$ have rotational group acting, H ~~splitting~~ splits according ~~...~~ to characters, half integers. ~~...~~

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focus on $H(v,-)$. You want to calculate it carefully. It is a linear functional on the grid module. Your first problem is maybe to find ~~...~~ a good group ring for $\mathbb{R}$, ~~...~~ a ring of functions on the dual. ~~...~~ Let $\mathcal{A}$ be this sought for group ring. It should consist of functions (dists.) of $x\in\mathbb{R}$, ~~...~~ e.g. $C_c^\infty(\mathbb{R})$ under convolution If $f\in\mathcal{A}$ strictly $\int dx\, f(x)\lambda^x$ lin. comb. of gp elts.

corresp. to mult by $\hat{} \int dx\, f(x) e^{i\xi x}$ on $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$. The idea now is that $\mathcal{A}v + \mathbb{C}[\mu, \mu^{-1}]u$ is $\mathcal{E}$ the desired grid space. This generalizes ~~the desired~~ the result that in the discrete case the grid space E has the basis $(\lambda^m v)_{m\in\mathbb{Z}}$, $(\mu^n u)_{n\in\mathbb{Z}}$. $\mathcal{A}$ tells you how the "δ functions" $\lambda^x v$ have to be combined.

Q: How big must $\mathcal{A}$ be so that $\mathcal{E}$ is closed under the operators λ^x, μ? You want $\mathcal{E}$ to be a subspace of $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$. ~~Thus see~~ See what you need. $v=1$, $u = \frac{b}{i\xi - a}$, $\mu = \frac{i\xi + a}{i\xi - a}$. $\lambda^x = e^{i\xi x}$. It seems clear that you can treat x formally, following the way the cont. limit ~~wa~~ was obtained.

$$\begin{pmatrix} \lambda^\varepsilon u \\ \mu\, v\sqrt{\varepsilon} \end{pmatrix} = \frac{1}{1 - a\varepsilon} \begin{pmatrix} 1 & b\sqrt{\varepsilon} \\ \bar{b}\sqrt{\varepsilon} & 1 \end{pmatrix} \begin{pmatrix} u \\ v\sqrt{\varepsilon} \end{pmatrix} \qquad a = \frac{|b|^2}{2}$$

$$\left((1-\varepsilon a)\lambda^\varepsilon - 1\right) u = b\sqrt{\varepsilon}\, v\sqrt{\varepsilon} \longrightarrow (i\xi - a) u = b v$$

$$\left((1-\varepsilon a)\mu - 1\right) v\sqrt{\varepsilon} = \bar{b}\sqrt{\varepsilon}\, u \Longrightarrow (\mu - 1) v = \bar{b} u$$

$$\longrightarrow \mu = 1 + \frac{\overset{2a}{|b|^2}}{i\xi - a} = \frac{i\xi + a}{i\xi - a}$$

So your first (or formal) version for $\mathcal{E}$ will be generated by operators ξ, μ elements u, v with properties $(i\xi - a) u = b v$, $i\xi \pm a$ invertible, $(\mu - 1) v = \bar{b} u$.

~~It seems that ... you are getting ...~~

want $\mathcal{E} \hookrightarrow L^2(\mathbb{R}, \frac{d\xi}{2\pi})$

$u \longmapsto \frac{b}{i\xi - a} \qquad \mu \mapsto \frac{i\xi + a}{i\xi - a}$

so far you're getting the vertical orth basis $\mu^n u$.

V not in $\mathcal{E}$, but ~~when you remove~~ $(\mu-1)v = \sigma u$ and $\mu - 1$ in the aug ideal. So for you have rational functions of ξ with only sings. at $\xi = \pm ia$. and vanishing at $\xi = \infty$. $e^{i\xi x} V$

~~So ffa~~

$e^{i\xi x} \frac{1}{i\xi - a}$ Go over the ideas.

In the discrete case you found E inside $L^2(S^1, \frac{dz}{2\pi i z})$ as the subspace of rational functions having poles $\in \{0, \infty, k, k^{-1}\}$. You now want to find $\mathcal{E}$ inside $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$ as certain meromorphic functions with poles $\in \{\pm ia\}$. You want to replace $\mathbb{C}[\lambda, \lambda^{-1}]v$ by the span of exponential functions $\lambda^x v = e^{i\xi x}$ where span means Fourier transf. of fn (dist) with compact support.

Look at $u = \frac{h}{kz - 1} = -\sum_{n\geq 0} hk^n z^n$, cont. analog of,

$$u = \frac{b}{i\xi - a} = -b\int_0^\infty e^{i\xi x} e^{-ax}\,dx$$

$$\lambda^t u = e^{i\xi t}\frac{b}{i\xi - a} = \int_0^\infty e^{i\xi(x+t)}(-b)e^{-ax}\,dx$$

$$= \int_t^\infty e^{i\xi x'}(-b)e^{-a(x'-t)}\,dx'$$

$$= e^{at}\int_t^\infty e^{i\xi x'}(-b)e^{-ax'}\,dx'$$

$$e^{at}\lambda^t u - u = \int_t^\infty e^{i\xi x}(-b)e^{-ax}\,dx - \int_0^\infty e^{i\xi x}(-b)e^{-ax}\,dx$$

$$= \int_0^t e^{i\xi x}\, b e^{-ax}\,dx$$

Repeat this. To describe "the" grid space as a space of meromorphic functions of ξ inside $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$. "Vertically" you the elements $\mu^n u$ $n \in \mathbb{Z}$ realized by rational functions $\frac{(i\xi+a)^n b}{(i\xi-a)^{n+1}}$ which form a basis for the rational functions ~~modulo constants~~ having having ~~poles~~ sing. $\subset \{\pm ia\}$ and vanishing at ∞.

Now act on this by λ^x

The important point is to express

$$u \underbrace{\boxed{}}_{x} e^{-ax}\lambda^x u$$

---

ideas from walk. $s = i\xi$ Also ~~replace~~ the algebraic group ring you seek, (the analog of $\mathbb{C}[\lambda,\lambda^{-1}]$?) ~~is found~~ is probably to be found by subtracting singularities of $\lambda^x u = \frac{e^{sx}b}{s-a}$, etc ~~to get~~ using the basis $\mu^n u$ for rational fns of s van. at ∞ ~~and~~ sing only at $\pm a$. So what you are doing is to add to $\mathbb{C}[\mu,\mu^{-1}]u$ = span of $\frac{1}{(s-a)^{n+1}}, \frac{1}{(s+a)^{n+1}}$ $n \geq 0$, just those entire functions of s needed so that the translation operators λ^x are defined.

Suppose $x \in \mathbb{Z}\varepsilon$.

~~[illegible]~~ Single x.

$$e^{sx}\frac{1}{(s-a)^{n+1}} \qquad n \geq 0 \,.\, ?$$

$$e^{sx}\frac{1}{s-a} = e^{ax}\frac{e^{(s-a)x}}{s-a} = \frac{e^{ax}}{s-a} + \underbrace{\frac{e^{sx}-e^{ax}}{s-a}}_{\text{regular}}$$

$$e^{sx}\frac{1}{(s-a)^2} = e^{ax}\frac{e^{(s-a)x}}{(s-a)^2} = \frac{e^{ax}}{(s-a)^2}\left(1+(s-a)x + e^{(s-a)x} - 1 - (s-a)x\right)$$

$$= \frac{e^{ax}}{(s-a)^2} + \frac{e^{ax}x}{s-a} + \frac{e^{ax}\left(e^{(s-a)x} - 1 - (s-a)x\right)}{(s-a)^2}$$

regular at a
hence entire.

However

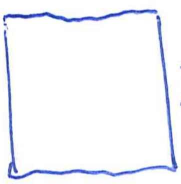
$$e^{sx} = e^{ax}\left(1 + \frac{x}{1!}(s-a) + \frac{x^2}{2!}(s-a)^2 + \cdots\right)$$

$$\therefore\ e^{sx}\frac{1}{(s-a)^{n+1}} = \frac{e^{ax}}{(s-a)^{n+1}} + \frac{e^{ax}x}{1!}\frac{1}{(s-a)^n} + \frac{e^{ax}x^2}{2!}\frac{1}{(s-a)^{n-1}} + \cdots$$

~~So for each x, n this~~

~~[illegible]~~ Probably there's a ~~[illegible]~~ remainder formula for the resulting entire fn.

Obvious question at this point is ~~[illegible]~~ what about the grid space generated by χ^x, μ, u



u $\quad$ $\chi^x u$

$$s = i\xi \qquad \frac{ds}{2\pi i} = \frac{d\xi}{2\pi}$$

Taylor with remainder.

$$f(x) - f(0) = \int_0^1 f'(tx)\, x\,dt \qquad \left[f(tx)\right]_{t=0}^{t=1} = f(x) - f(0)$$

$$\int_0^1 \partial_t f(tx)\,dt = \int_0^1 f'(tx)\,x\,dt$$

$$=$$

$$\int_0^1 \underbrace{f'(tx)}_{u}\, \underbrace{x\,dt}_{dv} = \left[\underbrace{f'(tx)}_{u}\, \underbrace{xt}_{v}\right]_0^1 - \int_0^1 f''(tx)\,x^2\,dt$$

$$f'(x)x$$

$$\int_0^1 f'(tx)\,x\,dt = \left[f'(tx)\,x(t-1)\right]_0^1 - \int_0^1 f''(tx)\,x\,x(t-1)\,dt$$

$$= + f'(0)(+x)$$

$$f(x) - f(0) - f'(0)x = \int_0^1 f''(tx)\, x^2(1-t)\,dt$$

$$\cancel{= \left[f''(tx)\,x^2\left(t - \frac{t^2}{2}\right)\right]_0^1 - \int f'''(x)\,x^3\left(t - \frac{t^2}{2}\right)dt}$$

$$= \left[-f''(tx)x^2\,\frac{(1-t)^2}{2}\right]_0^1 + \int_0^1 f'''(tx)\,x^3\,\frac{(1-t)^2}{2}\,dt$$

$$= f''(0)\,\frac{x^2}{2!}$$

$$\int_0^1 f^{(n+1)}((1-t)x)\, x^{n+1}\,\frac{t^n}{n!}\,dt$$

$$= \underbrace{\left[-f^{(n)}((1-t)x)\; x^n\,\frac{t^n}{n!}\right]_0^1} + \int_0^1 f^{(n)}((1-t)x)\; x^n\,\frac{t^{n-1}}{(n-1)!}\,dt$$

$$= -f^{(n)}(0)\,\frac{x^n}{n!}$$

So $$f(x) = f(0) + \frac{f'(0)x}{1!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \int_0^1 f^{(n+1)}((1-t)x)\, x^{n+1}\frac{t^n}{n!}\,dt$$

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + \int_0^1 e^{x-tx}\frac{x^{n+1}t^n}{n!}\,dt$$

$$e^{sx} = e^{ax}\, e^{(s-a)x}$$
$$= e^{ax}\left(1 + (s-a)x + \int_0^1 e^{(1-t)(s-a)x}((s-a)x)^{n+1}\frac{t^n}{n!}\,dt\right)$$

$$e^x - e^y = \int_0^1 e^{(1-t)x+ty}(y-x)\,dt$$
$$= \underbrace{\left[e^{(1-t)x+ty}\,t(y-x)\right]_0^1}_{e^y(y-x)} - \int_0^1 e^{(1-t)x+ty}\,t(y-x)^2\,dt$$

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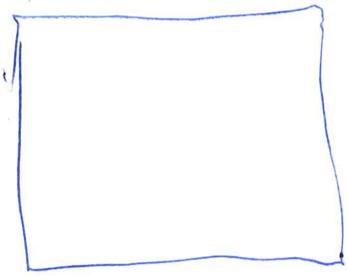
u  $\lambda^x u$

$$u = \frac{b}{s-a}v \qquad \mu = \frac{s+a}{s-a}$$

$$e^{sx}\frac{b}{s-a}$$

$$\frac{e^{sx} - e^{ax}}{s-a} \qquad \left[e^{tx}\right]_a^s = \int_a^s e^{tx}x\,dt$$

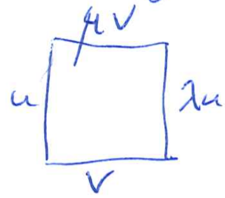
$$u = \frac{b}{s-a} \qquad u - e^{-ax}\lambda^x u$$
$$= \frac{b}{s-a}\left(1 - e^{-ax+sx}\right)$$

not making use of $$\frac{1}{s-a} = \int_0^\infty e^{-(s-a)t}\,dt$$
$$= \int_0^x e^{-(s-a)t}\,dt + \underbrace{\int_x^\infty e^{-(s-a)t}\,dt}_{\frac{e^{-(s-a)x}}{s-a}}$$

$$\frac{1 - e^{-(s-a)x}}{s-a} = \int_0^x e^{-(s-a)t}\,dt$$

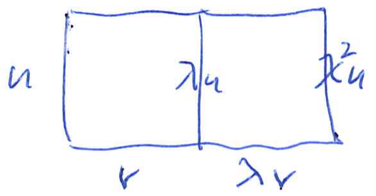
Is there a systematic way to ~~[illegible]~~ subdivide grid spaces? ~~[illegible]~~

Start again. ~~[illegible]~~



$$(k\lambda-1)u = hv$$
$$(k\mu-1)v = \bar{h}u$$

$$(k\lambda+1)(k\lambda-1)u = h(k\lambda+1)v$$



$$k^2\lambda^2 u - u = \cancel{(k\lambda-1)(k\lambda+1)u}$$

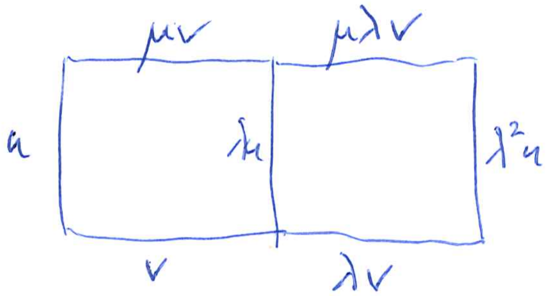
$$\left((k\lambda)^2-1\right)u = (h+hk\lambda)v$$

$$u = hv + hk\lambda v + k^2\lambda^2 v$$

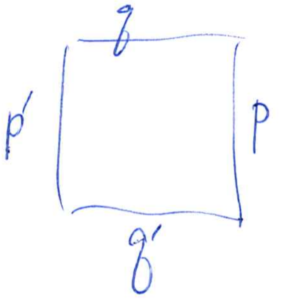
$$1 \qquad |h|^2+|h|^2k^2+k^4 = |h|^2+k^2\overbrace{\left(|h|^2+k^2\right)}^{1} = 1.$$

$$\|(h+hk\lambda)v\|^2 = |h|^2+|h|^2k^2 = \left|h(1+k^2)^{1/2}\right|^2$$

$$\begin{pmatrix}\lambda^2 u \\ \mu\frac{v+k\lambda v}{(1+k^2)^{1/2}}\end{pmatrix} = \frac{1}{k^2}\begin{pmatrix}1 & h(1+k^2)^{1/2} \\ \bar{h}(1+k^2)^{1/2} & 1\end{pmatrix}\begin{pmatrix}u \\ \frac{v+k\lambda v}{(1+k^2)^{1/2}}\end{pmatrix} \qquad ?$$



$$\lambda^2 u = k\lambda u + h\mu\lambda v$$
$$= k(ku+h\mu v) + h\mu\lambda v$$
$$= k^2 u + \mu\underbrace{(khv+h\lambda v)}_{\mu h(kv+\lambda v)}$$



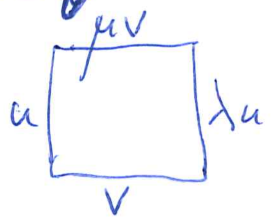
$$\begin{pmatrix}p \\ q'\end{pmatrix} = \begin{pmatrix}k & h \\ -\bar{h} & k\end{pmatrix}\begin{pmatrix}p' \\ q\end{pmatrix}$$

$$k^4+|h|^2(k^2+1) = 1$$

$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \qquad \begin{pmatrix}\lambda^2 u\\ \lambda\mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}\lambda u\\ \lambda v\end{pmatrix}$$

$$\lambda^2 u = \frac{1}{k}\lambda u + \frac{h}{k}\lambda v = \frac{1}{k}\left(\frac{1}{k}u + \frac{h}{k}v\right) + \frac{h}{k}\lambda v$$

$$\lambda^2 u = \frac{1}{k^2}u + \frac{1}{k^2}hv + \frac{h}{k}\lambda v$$



$$\begin{pmatrix}\lambda u\\ v\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar{h} & k\end{pmatrix}\begin{pmatrix}u\\ \mu v\end{pmatrix} \qquad \begin{pmatrix}\lambda^2 u\\ \lambda v\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar{h} & k\end{pmatrix}\begin{pmatrix}\lambda u\\ \lambda\mu v\end{pmatrix}$$

$$\begin{aligned}\lambda^2 u &= k\lambda u + h\lambda\mu v\\ &= k(ku + h\mu v) + h\lambda\mu v\\ &= k^2 u + kh\mu v + h\lambda\mu v\end{aligned}$$

$$\frac{1}{k^2}\lambda^2 u = u + \frac{h}{k}\mu v + \frac{h}{k^2}\mu\lambda v$$

$$\lambda^2 u = \frac{1}{k^2}u + \frac{h}{k}\left(k^{-1}v + \lambda v\right)$$

not proportional.

$$\lambda^2 u = k^2 u + h\mu(kv + \lambda v)$$

doesn't work.

So now return to

$$\begin{pmatrix}\lambda^{\varepsilon} u\\ \mu v\varepsilon^{1/2}\end{pmatrix} = \frac{1}{(1-|b|^2\varepsilon)^{1/2}}\begin{pmatrix}1 & b\varepsilon^{1/2}\\ \bar{b}\varepsilon^{1/2} & 1\end{pmatrix}\begin{pmatrix}u\\ v\varepsilon^{1/2}\end{pmatrix}$$

$2a = |b|^2$. $\qquad 1 - a\varepsilon$

$$\mu = 1 + \frac{2a}{s-a} = \frac{s+a}{s-a}$$

$$\left((1-a\varepsilon)e^{s\varepsilon} - 1\right)u = bv\varepsilon \implies (s-a)u = bv$$

$$\left((1-a\varepsilon)\mu - 1\right)v = \bar{b}a \qquad (\mu-1)v = \bar{b}u$$

~~Relations are operators e^{sx} ...~~ translation operators are $\lambda^x \mu^n$ relations $\mu = \frac{s+a}{s-a}$ $\lambda^x = e^{sx}$

$u = \frac{b}{s-a} v$. ~~that~~ Realize in $L^2(i\mathbb{R}, \frac{ds}{2\pi i})$ $s = i\xi$

with $u = \frac{b}{s-a}$ $\|u\|^2 = \int_{-i\infty}^{i\infty} \frac{|b|^2}{(-s-a)(s-a)} \frac{ds}{2\pi i}$

$$= \frac{|b|^2}{2a} = 1$$

$$(u|\mu^n u) = \int_{-i\infty}^{i\infty} \frac{\bar{b}}{(-1)(s+a)} \left(\frac{s+a}{s-a}\right)^n \frac{b}{s-a} \frac{ds}{2\pi i} = 0$$

$n \geq 1$

Program. So far in the grid space you have $\mathbb{C}[\mu, \mu^{-1}]u$ $\mu^n u = \frac{(s+a)^n b}{(s-a)^{n+1}}$ rat. fn of s sings $\subset \{\pm a\}$ vanish at ∞.

Need analog of $\mathbb{C}[\lambda, \lambda^{-1}]v$, to get entire functions of s. ~~As you need to get entire fns.~~ Idea is to ~~form~~ add what you need so as to get something closed under e^{sx} for all x. And you want to know whether you can do it with a given discrete spacing.

Want ~~$e^s/\ldots$~~.

$$e^s \frac{1}{s-a} = \frac{e^a}{s-a} + \frac{e^s - e^a}{s-a}$$

$$e^s \frac{1}{(s-a)^2} = e^a \left(\frac{e^{s-a}}{(s-a)^2}\right) =$$

$$e^{s-a} = 1 + (s-a) + \left(e^{s-a} - 1 - (s-a)\right)$$

$$\frac{e^s}{(s-a)^2} = \frac{e^a}{(s-a)^2} + \frac{e^a}{s-a} + \frac{e^{s-a} - 1 - (s-a)}{(s-a)^2} e^a$$

Is there some way to write this? What else?

$\lambda = e^s$

Your idea is to take $\lambda^m = e^{sm}$ $m \in \mathbb{Z}$, then consider the span of $\boxed{\lambda^m \mu^n u}$

$$e^{ms}\frac{(s+a)^n}{(s-a)^{n+1}}b \qquad \text{equiv.} \qquad e^{ms}\frac{1}{(s-a)^{n+1}},\ e^{ms}\frac{1}{(s+a)^{n+1}}$$

$n \geq 0$.

So what are you getting?

You are now following two paths - constructing the grid space - the answer is the space of the functions $e^{sx}\frac{1}{(s-a)^{n+1}}$, $e^{sx}\frac{1}{(s+a)^{n+1}}$ $n \geq 0$, $x \in \mathbb{R}$

but you want to split off $x = 0$, i.e. you want the entire functions within this span, and this should be a convolution algebra of continuous functions (may continuous piecewise polynomial functions) of compact support. The other path involves restricting x to $\mathbb{Z}$, you want to see how close this is to a discrete grid.

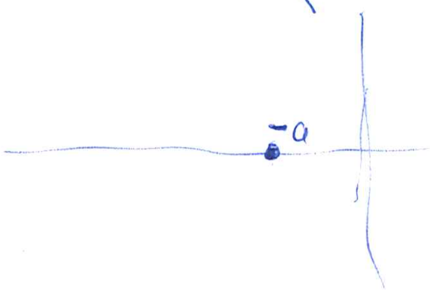
Point. dividing by $s \mp a$ amounts to inverting $\partial_x \pm a$. So $\frac{1}{s+a} \longleftrightarrow \begin{cases} e^{-ax} & x > 0 \\ 0 & x < 0 \end{cases}$

$$\frac{1}{2\pi i}\int e^{st}\frac{1}{s+a}\,ds$$

$$\parallel$$

$$\begin{cases} e^{-at} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$\int_0^\infty e^{-st}e^{-at}\,dt = \frac{1}{s+a}$$



So basically you have the operation on entire functions of dividing by $s - a$, then removing the pole. $f(s) \longmapsto \dfrac{f(s)-f(a)}{s-a}$

$$e^s \longmapsto \frac{e^s - e^a}{s-a} \longmapsto \frac{\frac{e^s-e^a}{s-a} - e^a}{s-a}$$

$$e^s \longmapsto \frac{e^s - e^{-a}}{s+a} \longmapsto \frac{\frac{e^s-e^{-a}}{s+a} - e^{-a}}{s+a}$$

no sing. for $Re(s) > 0$. e^{ts} decays as $Re(s) \to +\infty$ $t<0$

~~[illegible]~~.

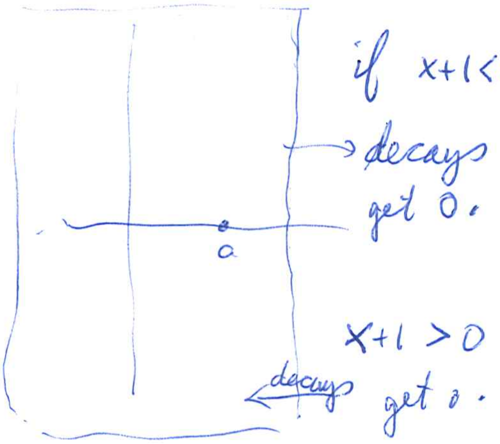
You don't see support yet.

$$e^s \longmapsto \frac{e^s - e^a}{s-a} \quad \text{entire function of } s$$

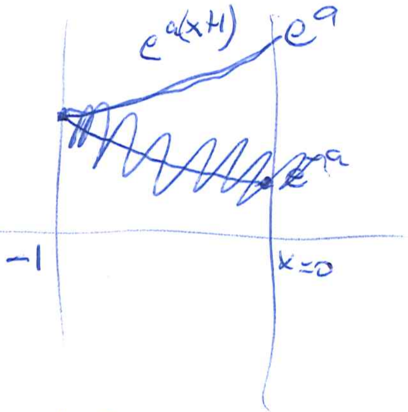
$$\phi(x) = \int_{-i\infty}^{i\infty} e^{sx} \frac{e^s - e^a}{s-a} \frac{ds}{2\pi i}$$

$$(\partial_x - a)\phi(x) = \int_{-i\infty}^{i\infty} \left(e^{s(x+1)} - e^{sx+a}\right) \frac{ds}{2\pi i}$$

$$= \delta(x+1) - e^a \delta(x)$$



$$\phi(x) = \begin{cases} e^{a(x+1)} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

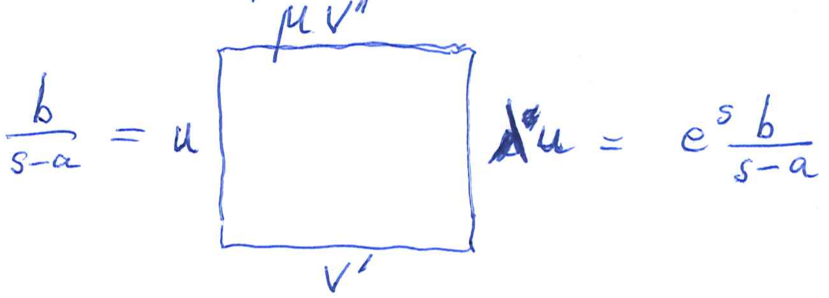


set the problem up as a differential equ.

$$e^{sx} \frac{1}{(s+a)^n} \qquad \text{think about what you want}$$

check carefully that you do not get a grid space by restricting to the translation subgroup: $\lambda^m \mu^n u$

say $\lambda = e^s$, $\mu = \frac{s+a}{s-a}$



$u, \lambda u$ generate a 2 dim'l subspace, V' is a suitable linear combination of $u, \lambda u$ which is represented by an entire function of s. So

$$V' = \lambda u - e^a u = \frac{b}{s-a}(e^s - e^a)$$

V'' is a suitable linear combination of $\lambda u, u$ which should be rep g where g is entire. So we want

$$\frac{s+a}{s-a} g = c_1 \frac{1}{s-a} + c_2 \frac{e^s}{s-a} \implies c_1 + c_2 e^{-a} = 0$$

$$c_1 = -c_2 e^{-a}$$

$$\therefore \frac{s+a}{s-a} g = c_2 \frac{-e^{-a} + e^s}{s-a}$$

$$g = c_2 \frac{e^s - e^{-a}}{s+a}$$

$$e^{(s-a)x} = 1 +$$

So $V' = \text{const}\, \frac{e^s - e^a}{s-a}$ | $V'' = \text{const}\, \frac{e^s - e^{-a}}{s+a}$

So V', V'' not in same line, and it doesn't work, just like in the discrete case on p704

You would like to pin down the functions $\phi(x)$ of x which are the Fourier transforms ~~of~~ the entire functions

$$\frac{e^{sx}}{(s-a)^n} - e^{ax}\left(\frac{1}{(s-a)^n} + \frac{x}{(s-a)^{n-1}} + \frac{x^2}{2!(s-a)^{n-2}} + \cdots + \frac{x^{n-1}}{(n-1)!(s-a)}\right)$$

$\cancel{e^{s} \cdots e^{st} \,\delta(t-1)\, dt}$?

$$\frac{e^{s}-1-s}{s^2} = \int_0^\infty e^{-st}\,\phi_2(t)\,dt$$

$$\frac{e^{s}-1-s}{s} = \int_0^\infty s e^{-st}\phi_2\,dt = \left[-e^{-st}\phi_2\right]_0^\infty + \int_0^\infty e^{-st}\phi_2'\,dt$$

$$= -\phi_2(0) + \int_0^\infty e^{-st}\phi_2'\,dt$$

$$e^{s}-1-s = -\phi_2(0)s + \left[-e^{-st}\phi_2'\right]_0^\infty + \int_0^\infty e^{-st}\phi_2''\,dt$$

$$= -\phi_2(0)s - \phi_2'(0) + \int_0^\infty e^{-st}\phi_2''\,dt.$$

$\therefore \quad \phi_2(0)=1 \qquad \phi_2'(0)=1 \qquad \phi_2''=\delta(t+1).$?

$$\frac{e^{s}-1-s}{s^2} = \int_0^\infty e^{st}\,\phi(t)\,dt$$

$$\frac{e^{s}-1-s}{s} = \int_0^\infty s e^{st}\phi(t)\,dt = \int_0^\infty\left(\frac{d}{dt}\left[e^{st}\phi(t)\right] - e^{st}\phi'\right)dt$$

$Re(s)<0$

$$= \left[e^{st}\phi(t)\right]_0^\infty - \int_0^\infty e^{st}\phi'\,dt$$

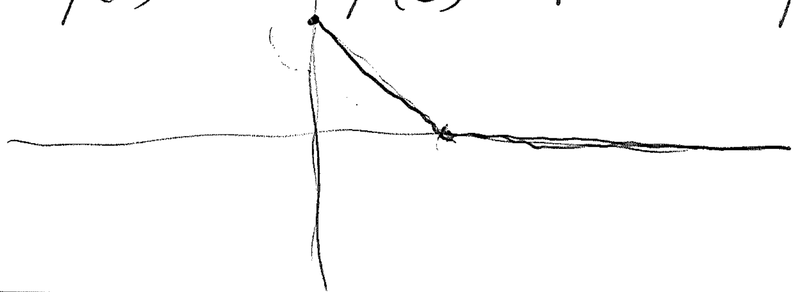
$$= -\phi(0) - \int_0^\infty e^{st}\phi'\,dt$$

$$e^{s}-1-s = -s\phi(0) - \int_0^\infty\left(\frac{d}{dt}\left[e^{st}\phi'\right] - e^{st}\phi''\right)dt$$

$$= -s\phi(0) - \left[e^{st}\phi'\right]_0^\infty + \int_0^\infty e^{st}\phi''\,dt$$

$$= -s\phi(0) + \phi'(0) + \int_0^\infty e^{st}\phi''\,dt$$

$\therefore \quad \phi(0)=1 \qquad \phi'(0)=-1 \qquad \phi''=\delta(t-1).$



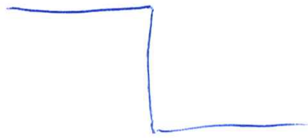
$$\int_0^1 e^{st}(1-t)\,dt = \int_0^1 \left[\partial_t\left(\frac{e^{st}}{s}(1-t)\right) + \frac{e^{st}}{s}\right] dt$$

$$= +\left[\frac{e^{st}}{s}(1-t)\right]_0^1 + \int_0^1 \frac{e^{st}}{s}\,dt$$

$$= -\frac{1}{s} + \frac{e^{s}-1}{s^2} = \frac{e^s-1-s}{s^2}$$

$$e^s = \int e^{st}\,\delta(t-1)\,dt$$

$$\frac{e^s-1}{s} = \int_0^1 e^{st}\,H(1-t)$$



$$\frac{e^s-1-s}{s^2} = \int_0^1 e^{st}(1-t)\,dt$$

Taylor

$$\int_0^1 \frac{d}{dt} f(tx)\,dt$$

$$f(x)-f(0) = \int_0^1 f'(tx)x\,dt$$

$$= \int_0^1 \left\{\frac{d}{dt}\left(f'(tx)\,x(t-1)\right) - f''(tx)x^2(t-1)\right\} dt$$

$$= \left[f'(tx)\,x(t-1)\right]_0^1 - \int_0^1 f''(tx)x^2(t-1)\,dt$$

$$f(x)-f(0)-f'(0)x = \int_0^1 f''(tx)\,x^2(1-t)\,dt$$

$a+t(s-a)$

$$= \left[f''(tx)x^2\frac{(1-t)^2}{2}(-1)\right]_0^1 + \int_0^1 f'''(tx)\,x^3\frac{(1-t)^2}{2!}\,dt$$

$$e^s - e^a - e^a(s-a) = \int_0^1 e^{ts+(1-t)a}(s-a)^2(1-t)\,dt$$

$$f(s)-f(a)-f'(a)(s-a) = \int_0^1 f''(a+t(s-a))\,(s-a)^2(1-t)\,dt$$

$$\frac{e^s-e^a-e^a(s-a)}{(s-a)^2} = \int_0^1 e^{a+t(s-a)}(1-t)\,dt$$

$f(a+t(x-a))(1-t$?

$\cancel{a+t(x-a)}$

$a+(1-t)(x-a) = x+t(a-x)$

$= ta+(1-t)x$

$\cancel{f(x+t)}$

$\partial_t^n f(a+t(x-a))\frac{(1-t)^n}{n!}$

$$\left[f(tx)(1-t)\right]_0^1 = \int_0^1 \underline{\frac{d}{dt}\left(f(tx)(1-t)\right)} \, dt$$

$-f(0)$

$f'(tx)x(1-t)+f(tx)(-1)$

$\cancel{f(t)x}$ f

$\int_0^1 D^{n+1}(\dots)$

$\cancel{D^k f(tx) \quad D^{-k}\delta(t-1)}$

$\int_0^1 e^{ts} \quad e^{a(1-t)}(1-t)\,dt$

piecewise pol

convolve

$\int_0^1 e^{(1-t)s} \quad e^{at}t\,dt$

$$f(x+1) - f(x) = \left(e^{D} - 1\right) f$$

$$= D\left(\frac{e^{D}-1}{D}\right) f$$

$$= D^2$$

$$f(x+\omega) - f(x) = \left(e^{\omega D} - 1\right) f = \frac{e^{\omega D}-1}{\omega D}\, \omega D f$$

$$\partial_\omega \frac{e^{\omega D}-1}{D} = \frac{e^{\omega D} D}{D} = e^{\omega D}$$

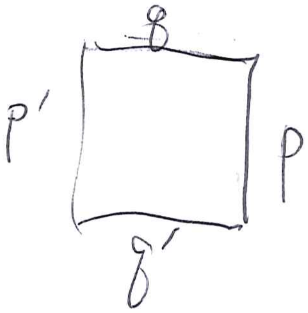
$$\therefore \quad \frac{e^{\omega D}-1}{D} = \int_0^\omega e^{tD}\, dt$$

$$f(x+\omega) - f(x) = \int_0^\omega f'(x+t)\, dt$$

$$\partial_\omega\left(\frac{e^{\omega D} - 1 - \omega D}{D^2}\right) = \frac{e^{\omega D} D - D}{D^2} = \frac{e^{\omega D}-1}{D}$$

$$\partial_\omega^2\left(\qquad\qquad\right) = e^{\omega D}$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$$



$$q \wedge p \overset{?}{=} p' \wedge q'$$

$$q \wedge p = (cp' + dq') \wedge (ap' + bq')$$

$$= (cb - da)\, p' \wedge q'$$

Analytic version

E = A-module with gen. u, v relns.

$$(k\lambda - 1)u = hv \qquad (k\mu - 1)v = \bar{h}u$$

know E has $(\ |\ )$ defined preserved by λ, μ

$\exists$ any exhaustive ascending staircase is an orth. basis.

~~So $(\lambda^m v)_{m\in\mathbb{Z}}$ is orth. set~~. $\bar{E}$ is Hilb. space with E as dense subspace.

Let $\bar{E}$ be completion, $*$ λ, μ extend ~~uniquely~~ uniquely by cont. to unitary operators on $\bar{E}$.

$k\lambda - 1$, $\lambda - k$ invertible bdd ops., $\mu = \frac{\lambda - k}{k\lambda - 1}$, ~~Claim~~ Note

$(\lambda^m v)_{m\in\mathbb{Z}}$ orth ~~basis for~~ set in $\bar{E}$. ~~By Claim~~ Consider closure $\mathbb{C}[\lambda, \lambda^{-1}]v$, call this F. ~~Then~~ F is preserved ~~closed~~ under $\lambda, \lambda^{-1}, \mu, \mu^{-1}$ contains v and $u = \frac{h}{k\lambda - 1}v$, hence F contains E so $F = \bar{E}$. But $F \cong L^2(S^1)$ with $\lambda^m v \leftrightarrow z^m$

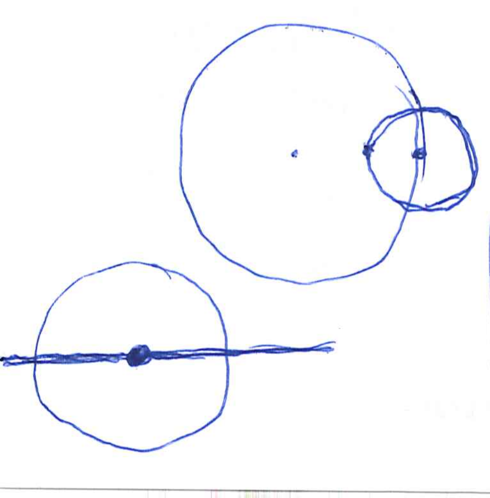
Formula for $(\ |\ )$ is

$$\cancel{f(\lambda)} \quad (f(\lambda)v \mid g(\lambda)v) = (v \mid f(\lambda)^* g(\lambda) v)$$

$$= \int_{S^1} f(z)^* g(z) \frac{dz}{2\pi i z}$$

$$\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$$

$$\int_{S^1} \frac{dz}{2\pi i z} = \operatorname{Res}_{\{0,k\}}$$



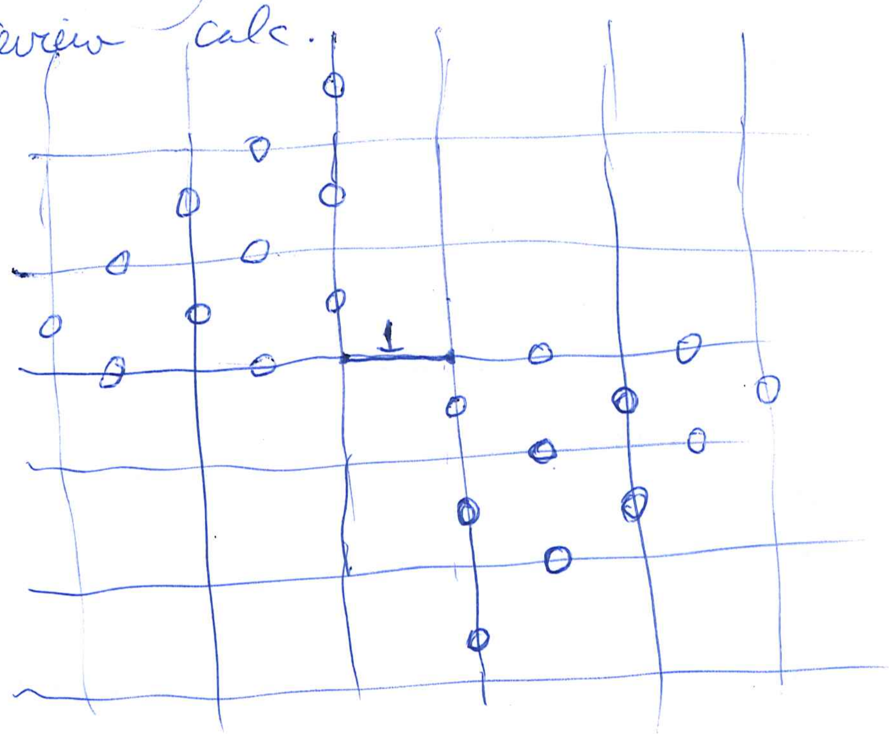
Philosophy (adelic?) There should be a completion appropriate to $H(\ ,\ )$. This is what adeles do. But it is a linearly compact top.

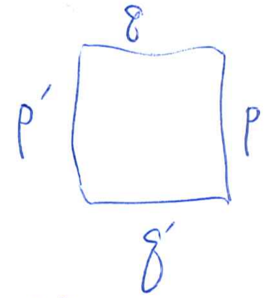
What's natural, customary is to consider the ring $\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (k-1)^{-1}]$ inside the product of the ~~four~~ local fields at these four points. Except you also have

the complementary
∧ product of regular functions at these points.

~~completion~~ appropriate completion, the idea gets clearer. We have those four points on RS, ~~[illegible]~~ ~~circles~~ remove disks around them, replace the ring B by holom. functions on the RS with boundary

need to calculate $H(V, -)$
review calc.





$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

basis $\quad (\lambda^m v)_{m\in\mathbb{Z}} \qquad (\mu^n u)_{n\geq 0} \qquad (\lambda\mu^n u)_{n<0}$

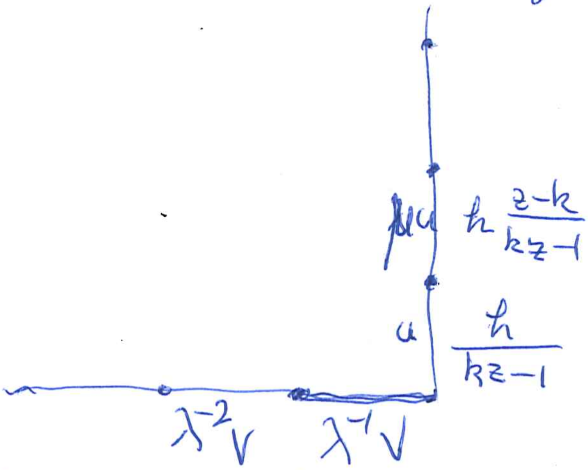
$$\mathrm{Res}_{\{0,k^{-1}\}}\left(z^m \frac{dz}{2\pi i z}\right) = \delta_m \qquad ?$$

$$\mathrm{Res}_{\{0,k^{-1}\}}\left(\frac{(z-k)^n h}{(kz-1)^{n+1}} \frac{dz}{2\pi i z}\right) = 0 \qquad n\geq 0 \qquad ?$$

$$\mathrm{Res}_{\{0,k^{-1}\}}\left(\cancel{z}\, \frac{(kz-1)^{-n-1} h}{(z-k)^{-n}} \frac{dz}{2\pi i \cancel{z}}\right) = 0 \qquad n\leq -1 \quad ?$$

$$-n \geq 1$$

Look at the 2nd quadrant



Look at $IH(\mu, \lambda^{-n}v)$

Look at the linear fnl $IH(\mu, -)$ and restrict it to the horizontal subspace

It's clear that you get

$$u = \frac{h}{kz-1} = \frac{h}{kz}\,\frac{1}{1-\frac{1}{kz}} = \sum_{n\geq 1} h\,\frac{1}{(kz)^n}$$

when $|kz| > 1$ i.e. $|z| > k^{-1}$.

Check $\cancel{\mathrm{Res}_{\{0,k^{-1}\}}\left(\frac{\bar{h}}{kz^{-1}-1}\, z^{+n}\, \frac{dz}{2\pi i z}\right) = IH(\mu, \lambda^{n}v)}$

$$IH(u, \lambda^{-n}v) = IH\left(\frac{h}{k\lambda-1}v, \lambda^{-n}v\right)$$

$$= \mathrm{Res}_{\{0,k^{-1}\}}\left(\frac{\bar{h}}{k-z}\, z^{-n}\, \frac{dz}{2\pi i}\right) = 0 \qquad -n \geq 0$$

$$= \frac{\bar{h}}{k}\, \mathrm{Res}_{\{0\}}\left(\sum_{j\geq 0}\left(\frac{z}{k}\right)^j z^{+n}\, \frac{dz}{2\pi i}\right) = \frac{\bar{h}}{k\, k^{n-1}} = \frac{\bar{h}}{k^{+n}} \qquad \frac{-n \leq -1}{n \geq 1}$$

∴ formally:

$$u = \frac{h}{k\lambda - 1} = \sum_{n\geq 1} \lambda^{-n}v\, \overline{IH(u, \lambda^{-n}v)}$$

$$= \sum_{n\geq 1} \lambda^{-n}v\, \frac{h}{k^n} = \sum_{n\geq 1} h\, \frac{1}{(kz)^n}$$

Discussion

~~[illegible]~~ The idea was to understand
~~[illegible]~~ E together with IH ~~[illegible]~~ as the direct sum of the horizontal subspace $\mathbb{C}[\lambda,\lambda^{-1}]v$ on which $IH > 0$ and ~ vertical " $\mathbb{C}[\mu,\mu^{-1}]u$ —— $IH < 0$, glued together by the projection from one to the other. (Recall in the scattering situation the formula involving $\begin{pmatrix} 1 & b \\ \bar{b} & -1 \end{pmatrix}$.)

---

notes on cont. limit.

$$k = \sqrt{1-|b|^2\varepsilon} = 1 - \underbrace{\tfrac{1}{2}|b|^2}_{a}\varepsilon + O(\varepsilon^2)$$

$$(k\lambda^\varepsilon - 1)u = bv\varepsilon$$
$$(k\mu - 1)v = \bar{b}u$$

$$\boxed{\begin{aligned}(i\xi - a)u &= bv \\ (\mu - 1)v &= \bar{b}u\end{aligned}}$$

$$\lambda^x = e^{i\xi x}$$

$$~~[illegible]~~ \quad (\mu-1)(i\xi - a) = |b|^2 = 2a$$

$$\boxed{\mu = 1 + \frac{2a}{i\xi - a} = \frac{i\xi + a}{i\xi - a}}$$

repn. in $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$

$\lambda^x \longmapsto$ mult. by $e^{ix\xi}$

$\mu \longmapsto$ mult by $\dfrac{i\xi + a}{i\xi - a}$

$u \longmapsto \dfrac{b}{i\xi - a}$

$v \longmapsto 1$.

$$\int \rho(x)\,dx\,\lambda^x \longmapsto \int \rho(x) e^{i\xi x} dx = \hat{\rho}(\xi)$$

problem with explaining what E is. Alg. it will be spanned by functions $e^{sx}\dfrac{1}{(s \pm a)^n}$

$x \in \mathbb{R}$, $n \geq 1$

$$f(x) - f(a) = \int_0^1 (-1)\,\partial_t f(ta + (1-t)x)\, dt$$

$$= \int_0^1 f'(ta+(1-t)x)(x-a)\, dt$$

$$\frac{f(x)-f(a)}{x-a} = \int_0^1 f'(ta+(1-t)x)\, dt$$

$$= \Big[f'(ta+(1-t)x)\,t \Big]_0^1 + \int_0^1 f''(ta+(1-t)x)(x-a)\,t\, dt$$

$$\frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)^2} = \int_0^1 f''(ta+(1-t)x)\,t\,dt$$

$$= \Big[f''(\text{——})\frac{t^2}{2} \Big]_0^1 + \int_0^1 f'''(\quad)(x-a)\frac{t^2}{2}\,dt$$

$$\frac{f(s)-f(a)-\cdots-f^{(n-1)}(a)\dfrac{(s-a)^{n-1}}{(n-1)!}}{(s-a)^n} = \int_0^1 f^{(n)}(ta+(1-t)s)\frac{t^{n-1}}{(n-1)!}\,dt$$

$$e^{sx} = e^{ax} + xe^{ax}(s-a) + x^2 e^{ax}\frac{(s-a)^2}{2}$$

$$+ x^n e^{ax}\frac{(s-a)^n}{n!} + x^{n+1}(s-a)^{n+1}\int_0^1 e^{((1-t)s+ta)x}\frac{t^n}{n!}\,dt$$

$$\frac{e^{sx} - e^{ax} - xe^{ax}(s-a) - \cdots - x^n e^{ax}\dfrac{(s-a)^n}{n!}}{(s-a)^{n+1}} = \int_0^1 e^{(ta+(1-t)s)x}\frac{t^n}{n!}\,dt$$

Now do you have any way to see IH. Let's go over the limiting process.

~~Do everything inside~~

Go back to your idea of constructing the grid space by hand, ultimately you get functions of $i\xi = s$

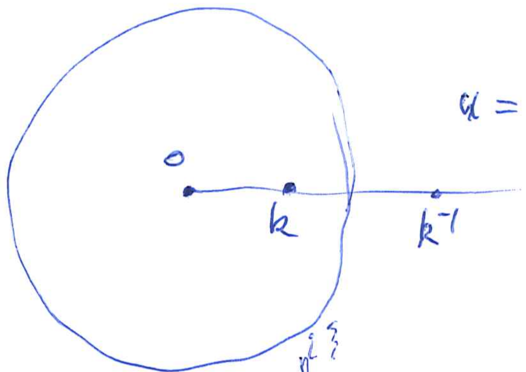
vertical space of $(\mu^n u)_{n\in\mathbb{Z}}$

$$\| $$

$$b\frac{(s+a)^n}{(s-a)^{n+1}} \qquad n\in\mathbb{Z}$$

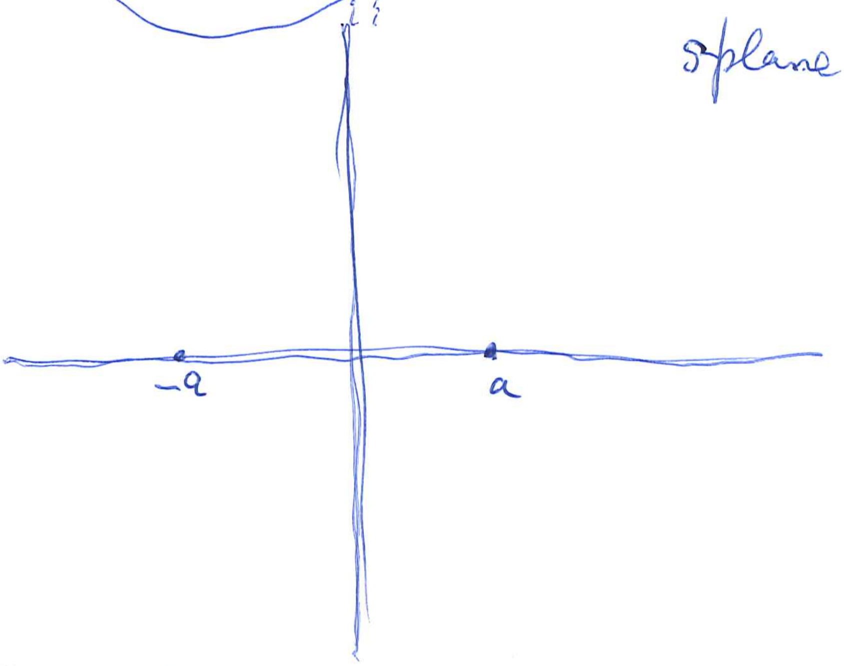
span of $\frac{1}{(s-a)^{n+1}}, \frac{1}{(s+a)^{n+1}} \qquad n\geq 0.$

$$u = \frac{bv\varepsilon}{k\lambda^{\varepsilon}-1} \longrightarrow \frac{b}{-a+i\xi} \simeq \frac{b}{s-a}$$



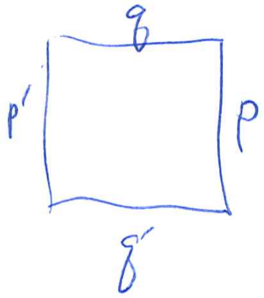
$$u = \frac{b}{kz-1} \longrightarrow \frac{b}{s-a}$$

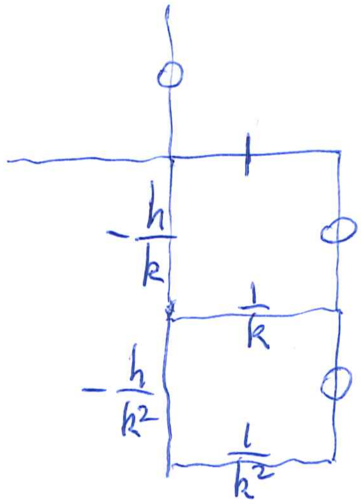
IH(v, —)



Your idea is to use splitting

$$E = \mathbb{C}[\lambda, \lambda^{-1}]v \oplus \underline{\mathbb{C}[\mu, \mu^{-1}]u}$$





$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \frac{1}{h} \begin{pmatrix} 1 & -h \\ -\frac{1}{h} & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$H(v, \mu^n u) = \begin{cases} 0 & n \geq 0 \\ -\frac{h}{k^{-n}} & n \leq -1 \end{cases}$$

$$\mathrm{res}_{\{0, k^{-1}\}} \left(h \frac{(z-k)^n}{(kz-1)^{n+1}} \frac{dz}{2\pi i z} \right)$$

$$\left\{ \begin{array}{ll} \mathrm{res}_0 = h \dfrac{(-k)^n}{(-1)^{n+1}} = -hk^n & \forall n \\ \mathrm{res}_{k^{-1}} = 0 \quad \text{if } n \leq -1 \end{array} \right.$$

$n \geq 0$ $\quad \mathrm{res}_k = 0$, $\mathrm{res}_\infty = 0$ $\quad \therefore \mathrm{res}_{\{k^{-1}, 0\}} = 0.$

So now go over to the cont. case.

So what happens? You have a horizontal space of entire functions of the form

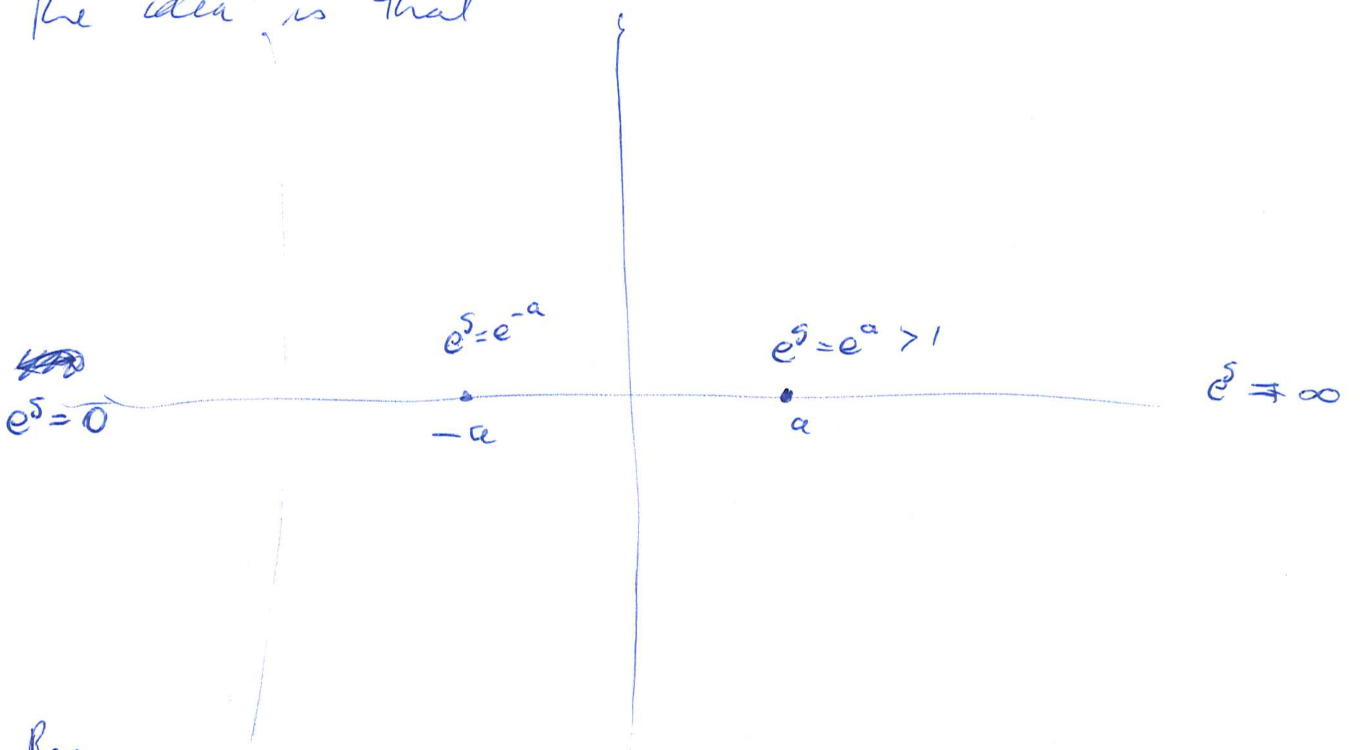
$$\int_{|x|<M} f(x)\, e^{sx}\, dx$$

vertical space of

$$\frac{1}{(s-a)^{n+1}}, \ \frac{1}{(s+a)^{n+1}} \qquad n \geq 0.$$

~~So wh~~

The idea is that



Res_0

$$IH(v, \mu^n u) \overset{?}{=} \mathrm{Res}_{\{-\infty, a\}} \left(b \frac{(s+a)^n}{(s-a)^{n+1}} \frac{ds}{2\pi i} \right)$$

Apparently you have a problem with $n=0$, because the contour is not closed. Problem is where v sits relative to the vertical axis.

$$\int_0^1 e^{\overset{s-t(s-a)}{(ta+(1-t)s)}} \frac{t^n}{n!}\, dt = e^s \int_0^1 e^{-t(s-a)} \frac{t^n}{n!}\, dt$$

$$\frac{1}{m!}(-t)^m (s-a)^m$$

Taylor with remainder $\qquad ta + (1-t)x$ $\qquad$ 721

$$f(x) - f(a) = -\int_0^1 \frac{d}{dt} f\big(x + t(a-x)\big)\, dt$$

$$= \int_0^1 f'\big(x + t(a-x)\big)(x-a)\, dt$$

$$= (x-a)\int_0^1 f'\big(ta + (1-t)x\big)\, dt$$

$$= (x-a)\left[\, \big[f'(x_t)\, t\big]_0^1 + \int_0^1 f''(x_t)(x-a)\, t\, dt \right]$$

$$f(x) - f(a) - (x-a)f'(a) = (x-a)^2 \underbrace{\int_0^1 f''(x_t)\, t\, dt}$$

$$\left[f''(x_t)\frac{t^2}{2}\right]_0^1 + (x-a)\int_0^1 f'''(x_t)\frac{t^2}{2}\, dt$$

$$f(x) = \sum_{j=0}^{n-1} f^{(j)}(a)\frac{(x-a)^j}{j!} + \left(\int_0^1 f^{(n)}(x_t)\, t^n\, dt\right)\frac{(x-a)^{n-1}}{(n-1)!} \qquad ?$$

$$f(x) = f(a) + f'(a)(x-a) + (x-a)^2\int_0^1 f''(x_t)\, t\, dt$$

$$= f(a) + f'(a)(x-a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{}\int_0^1 f'''(x_t)\frac{t^2}{2!}\, dt$$

$$x_t = (1-t)a + tx \qquad \partial_t x_t = x - a$$

$$f(x) - f(a) = \int_0^1 dt\; \partial_t f(x_t) = \int_0^1 dt\; f'(x_t)(x-a)$$

$$= \underbrace{\Big[-(1-t)f'(x_t)(x-a)\Big]_0^1}_{f'(a)(x-a)} + \underbrace{\int_0^1 dt\,(1-t)f''(x_t)(x-a)^2}$$

$$= \underbrace{\left[-\frac{(1-t)^2}{2}f''(x_t)(x-a)^2\right]_0^1}_{f''(a)\frac{(x-a)^2}{2!}} + \int_0^1 dt\,\frac{(1-t)^2}{2!}f'''(x_t)(x-a)^3$$

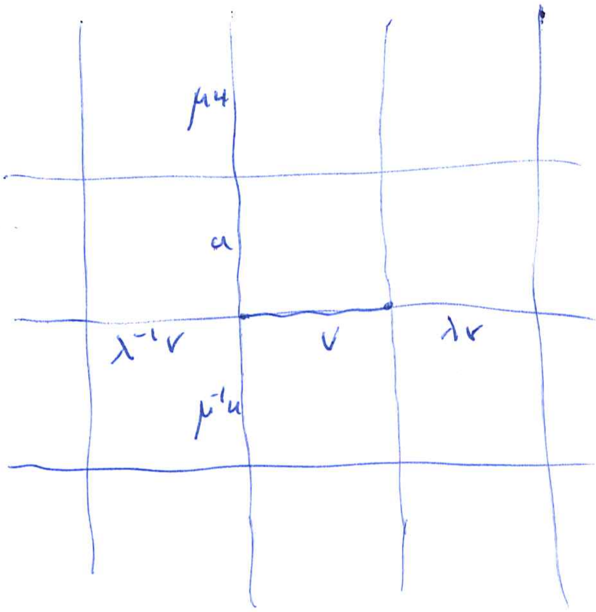
~~Sp~~ cont. limit.

$$(k\lambda - 1)u = hv$$
$$(k\mu - 1)v = \bar{h}u$$

$$(k\lambda-1)(k\mu-1) = 1-k^2$$
$$\mu = \frac{1}{k}\left(1 + \frac{1-k^2}{k\lambda-1}\right) = \frac{\lambda-k}{k\lambda-1}$$

$$\left(\begin{array}{lcl} \lambda & \longmapsto & \text{mult by } z \\ \mu & \longmapsto & \dfrac{z-k}{kz-1} \\ v & \longmapsto & 1 \\ u & \longmapsto & \dfrac{h}{kz-1} \end{array}\right.$$

$$E \hookrightarrow L^2\left(S^1, \frac{dz}{2\pi i z}\right)$$
$$\cup$$
$$\mathbb{C}[\cancel{z}, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$$



horizontal subspace $\mathbb{C}[z,z^{-1}]$

$\mathbb{C}[\lambda,\lambda^{-1}]v = \mathbb{C}[z,z^{-1}]$

vertical subspace

$\mathbb{C}[\mu,\mu^{-1}]u$ = span of $\frac{1}{(kz-1)^{n+1}}, \frac{1}{(z-k)^{n+1}}$ for $n \geq 0$.

$$\mu^n u = h\frac{(z-k)^n}{(kz-1)^{n+1}} \qquad n=0$$

$$u = \frac{b}{s-a}$$

$$IH(u, fv) = \mathrm{Res}_{\{?\}}\left(\frac{\bar{b}}{-(s+a)} f(s)\frac{ds}{2\pi i}\right)$$

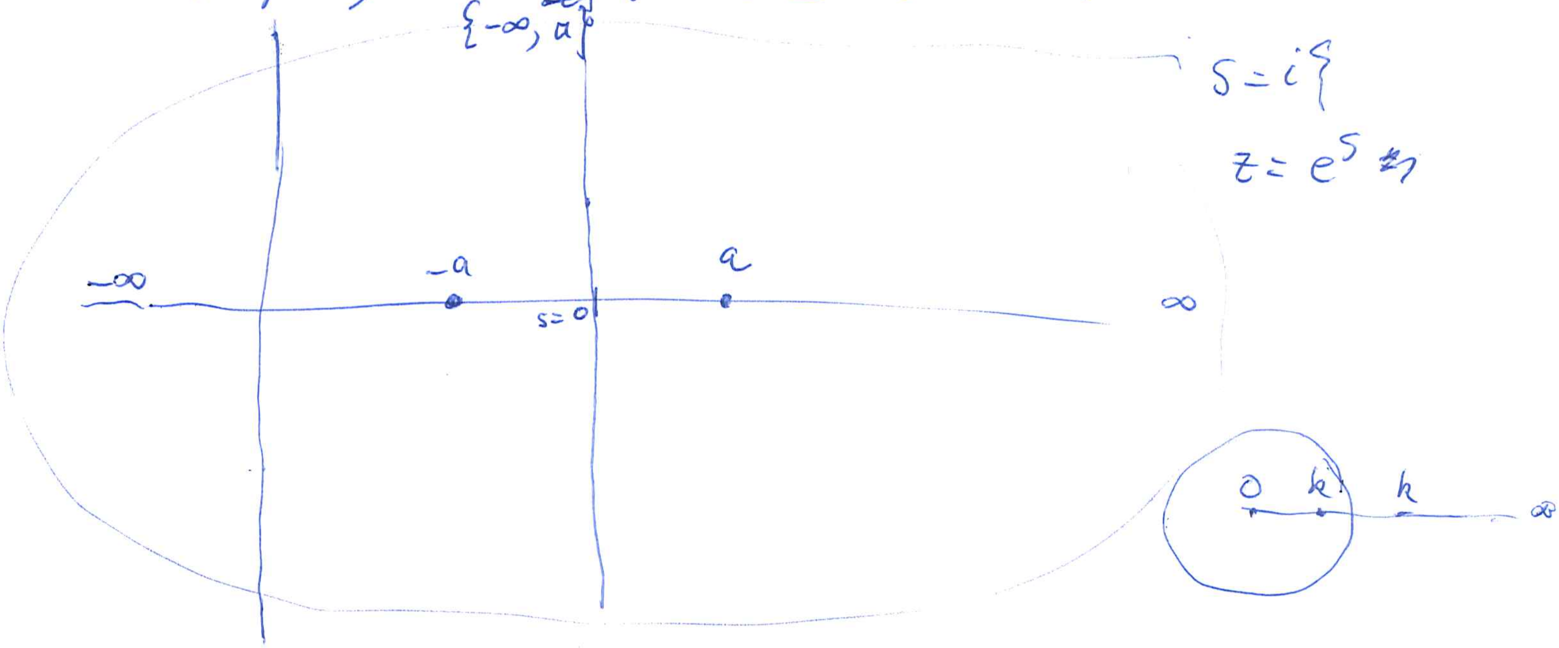
$$IH(u, \mu^n u) = \mathrm{Res}_{\{-\infty, a\}}\left(\frac{|b|^2}{-(s+a)}\frac{(s+a)^{n-1}}{(s-a)^{n+1}}\frac{ds}{2\pi i}\right) \quad ?$$

$n \geq 1$ OK

$n \leq -1$ OK

$s = i\xi$

$z = e^s$



idea do the limit within the Hilbert space, really the Fourier transform side.

Begin with $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$ and the data

$$\lambda^\varepsilon \longmapsto e^{i\xi\varepsilon} \qquad \lambda^x \longmapsto \left(e^{i\xi\varepsilon}\right)^{\frac{x}{\varepsilon}}$$

$$\mu \longmapsto \frac{e^{i\xi\varepsilon} - k}{ke^{i\xi\varepsilon} - 1}$$

$$\nu \longmapsto 1$$

$$u \longmapsto \frac{h}{ke^{i\xi\varepsilon} - 1}$$

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first review spinors. V 2 diml over $\mathbb{C}$ equipped ~~with [illegible] and constant~~ volume $\Lambda^2 V \ni \omega \neq 0$. Then $\mathcal{O}(-1) \otimes_{\mathcal{O}} \mathcal{O}(-1) \simeq \Omega^1$ over $\mathbb{P}(V)$ so get intrinsic thing.

Bellman Idea central extension of $\mathbb{Z}/N\mathbb{Z} \times \mu_N$ finite Heisenberg group.

What happens really ~~is coherent sta~~

Let's see if we can find the answers. You're dealing with a fixed Hilbert space attached to a given circle in the **Riemann sphere**. And you can view it "from" an interior point or a boundary point. An interior point leads to a unitary operator multiplication by $\frac{z-a}{1-\bar{a}z}$, ~~and a boundary point leads~~ there's a filtration around by order of poles. You probably have two S^1's acting which might mean that the result depends on h or maybe k, should be $0 \le k < 1$. ~~But the~~
You also have outside + inside S^1?

$$\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$\lambda u = au + bv$
$\mu v = cu + dv$

$(\lambda - a)u = bv$
$(\mu - d)v = cu$

$$\boxed{(\lambda - a)(\mu - d) = bc}$$

$$\mu = d + \frac{bc}{\lambda - a} = \frac{\lambda d - \Delta}{\lambda - a} \qquad \Delta = ad - bc$$

$$\lambda = a + \frac{bc}{\mu - d} = \frac{a\mu - \Delta}{\mu - d} \qquad \begin{pmatrix} a & -\Delta \\ 1 & -d \end{pmatrix}$$

$\Delta = e^{i\phi}$

$+d \quad -\Delta$

$+1 \quad -a$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta \bar{d} & \Delta \bar{c} \\ c & d \end{pmatrix} \qquad |d|^2 - |c|^2 = 1$$

$$\frac{1}{|c|}\begin{pmatrix} \Delta \bar{d} & -\Delta \\ 1 & -d \end{pmatrix} = \frac{(-i)\Delta^{1/2}}{|c|} \underbrace{\begin{pmatrix} i\Delta^{1/2}\bar{d} & -i\Delta^{1/2} \\ i\Delta^{-1/2} & -i\Delta^{-1/2} d \end{pmatrix}}_{\text{has det } = |d|^2 - 1 = |c|^2}$$

$\cancel{(k\lambda^\varepsilon - 1)(k\mu - 1)}$

$$k = (1 - \underbrace{\tfrac{1}{2}|b|^2}_{a}\varepsilon)$$

$$\underbrace{(k\lambda^\varepsilon - 1)}_{\varepsilon} u = b v \qquad (k\mu - 1)v = \bar{b} u$$

$$\mu = \frac{1}{k}\left(1 + \frac{1 - k^2}{k\lambda^\varepsilon - 1}\right) = \frac{\lambda^\varepsilon - k}{k\lambda^\varepsilon - 1} \longrightarrow \frac{i\xi + a}{-a + i\xi}$$

const
grid equation
$$\begin{pmatrix} \psi^1_{m+1,n} \\ \psi^2_{m,n+1} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \psi^1_{mn} \\ \psi^2_{mn} \end{pmatrix}$$

$$\begin{pmatrix} \psi^1_{x+\varepsilon,y} \\ \psi^2_{x,y+\varepsilon} \end{pmatrix} = \frac{1}{k_\varepsilon}\begin{pmatrix} 1 & m\varepsilon \\ \bar{m}\varepsilon & 1 \end{pmatrix}\begin{pmatrix} \psi^1_{xy} \\ \psi^2_{xy} \end{pmatrix}$$

$$k_\varepsilon = (1-|b|^2\varepsilon^2)^{1/2} = 1+O(\varepsilon^2)$$

$$\begin{pmatrix} \cancel{\psi^1_{xy}} + \varepsilon\partial_x\psi^1_{xy} \\ \cancel{\psi^2_{xy}} + \varepsilon\partial_y\psi^2_{xy} \end{pmatrix} = \cancel{(1+O(\varepsilon))}\begin{pmatrix} \cancel{\psi^1_{xy}} + \varepsilon b\psi^2_{xy} \\ \varepsilon\bar{b}\psi^1_{xy} + \cancel{\psi^2_{xy}} \end{pmatrix}$$

$$\boxed{\begin{aligned} \partial_x\psi^1 &= m\psi^2 \\ \partial_y\psi^2 &= \bar{m}\psi^1 \end{aligned}}$$
grid equations.

$$\begin{pmatrix} \psi^1_{xy} \\ \psi^2_{xy} \end{pmatrix} = e^{i\xi x}e^{i\eta y}\begin{pmatrix} u \\ v \end{pmatrix}$$
exp. solns.

$$i\xi u = mv \qquad i\eta v = \bar{m}u$$

$$-\xi\eta = |m|^2 \qquad m=1, \qquad i\xi u = v$$

exp. solns. $\psi_{xy} = e^{i(\xi x - \xi^{-1}y)}\begin{pmatrix} u \\ v \end{pmatrix}$

general solns $\psi_{xy} = \int_{-\infty}^{\infty} e^{i(\xi x-\xi^{-1}y)}\begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix} f(\xi)\,\frac{d\xi}{2\pi}$

Hilbert space completion should be $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$ with $\lambda^x = e^{i\xi x}$ $\mu^y = e^{-i\xi^{-1}y}$

$v = 1$ $u = \frac{1}{i\xi}$ You don't know how to make sense of these see how this works. What is horizontal space? In disc. case $\lambda^m v \rightsquigarrow z^m$, so horizontal space should consist of $\int dx\, f(x)\, e^{i\xi x} v$ with $f(x) \in L^2(\mathbb{R}, dx)$.

You have relation $i\xi u = v \qquad u = \frac{1}{i\xi} v$

$$\int_0^\infty e^{i\xi x}\, dx$$

$$u = \frac{h}{kz-1} v = -h \sum_{n\geq 0} k^n z^n v$$

$$u = -\int_0^\infty e^{(-\delta + i\xi)x}\, dx$$

$$= \underbrace{-\frac{1}{\delta - i\xi}} = \frac{1}{i(\xi + i0^+)}$$

$$\frac{1}{i\xi - 0^+} = \frac{1}{i(\xi + i0^+)} \qquad \xi = -i0^+$$

<u>real line</u> $\quad f(x)\begin{pmatrix} x \\ 1 \end{pmatrix}$ section of $\mathcal{O}(-1)$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* f(x)\begin{pmatrix} x \\ 1 \end{pmatrix} = f\left(\frac{ax+b}{cx+d}\right) \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} x \\ 1 \end{pmatrix}}$$

$$\cancel{s(x)} = \begin{pmatrix} x \\ 1 \end{pmatrix} \qquad \mathbb{R} \longrightarrow \qquad \begin{pmatrix} ax+b \\ cx+d \end{pmatrix} = \frac{1}{cx+d}\begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* f(x)\begin{pmatrix} x \\ 1 \end{pmatrix} = \frac{1}{cx+d} f\left(\frac{ax+b}{cx+d}\right)\begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix} \text{ — NO}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* f(x)\, dx = f\left(\frac{ax+b}{cx+d}\right) \underbrace{d\left(\frac{ax+b}{cx+d}\right)}$$

$$\frac{(cx+d)a - (ax+b)c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$$

next need to explain ~~SES~~

$z \qquad 0 \to L \hookrightarrow \mathbb{C}^2 \twoheadrightarrow Q \to 0$

$$0 \to l_z \hookrightarrow \mathbb{C}^2 \to \mathbb{C}^2/l_z \to 0$$

$$l_z \otimes \mathbb{C}^2/l_z \longrightarrow \Lambda^2\mathbb{C}^2 = \mathbb{C}.$$

$\ell_z = \binom{z}{1}\mathbb{C}$ tang. vector $\varepsilon\Delta z$

take point $z \in S^2$, ~~tang.~~ $a\binom{z}{1} \in \ell_z$ $\left| a\binom{z}{1} \quad a\binom{\varepsilon}{0} \right| = -a^2\varepsilon$

$$\cancel{f(z)\binom{z}{1}} \quad - \begin{vmatrix} f(z)z & f(z)dz + f'(z)zdz \\ f(z) & f'(z)zdz \end{vmatrix} = +f(z)^2 dz$$

$$\therefore \quad f(z)\binom{z}{1} \longmapsto f(z)^2 dz$$

$$\mathcal{O}(-1)^{\otimes 2} \longrightarrow \Omega^1$$

$\mathbb{R}$ $\quad f(x)\binom{x}{1}$ $\quad f(x)^2 dx.$ $\longrightarrow$

$L^2(\mathbb{R}, dx)$ with $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* f\right)(x) = \frac{1}{cx+d} f\left(\frac{ax+b}{cx+d}\right)$
$\in SL^2(\mathbb{R})$

$|z|=1.$

$$f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \xmapsto{Q\,d\theta} f(z)^2 dz$$
$$= f(e^{i\theta})^2 e^{i\theta} i\, d\theta$$

$z = e^{i\theta}$
$dz = zi\,d\theta$

$\cancel{f(e^{i\theta})\begin{pmatrix} e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix}}$

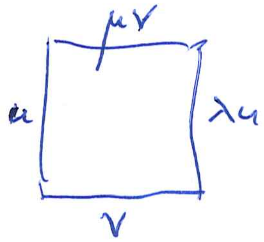
$$\left(f(e^{i\theta})e^{\frac{i\theta}{2}}\right)\begin{pmatrix} e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix} \longmapsto$$

$$\begin{vmatrix} i & 1 \\ -i & 1 \end{vmatrix} = 2i$$

$$z = \frac{1+ix}{1-ix} = \frac{1-(-ix)}{1+(-ix)} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}(-ix)$$

$$-ix = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} z = \frac{-z+1}{z+1} \qquad x = \frac{1}{i}\,\frac{z-1}{z+1}$$

grid eqn. for gen. u(1,1)-matr



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$(\lambda-a)u = bv$$
$$(\mu-d)v = cu$$

$\Delta = ad-bc$

$$\mu = d + \frac{bc}{\lambda-a} = \frac{d\lambda-\Delta}{\lambda-a} = \underbrace{\begin{pmatrix} d & -\Delta\\ 1 & -a\end{pmatrix}}(\lambda)$$

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} \Delta & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} \bar d & \bar c\\ c & d\end{pmatrix} = \begin{pmatrix} \Delta\bar d & \Delta\bar c\\ c & d\end{pmatrix}$$

$|\Delta| = 1$

$|d|^2 - |c|^2 = 1.$

$$\begin{vmatrix} d & -\Delta\\ 1 & -\Delta\bar d\end{vmatrix} = \Delta - |d|^2\Delta = -\Delta|c|^2$$

$$\begin{pmatrix} i\Delta^{-\frac12}\dfrac{d}{|c|} & -i\dfrac{\Delta^{+1/2}}{|c|}\\ i\dfrac{\Delta^{-1/2}}{|c|} & -i\Delta^{1/2}\dfrac{\bar d}{|c|}\end{pmatrix}$$

SU(1,1)

$$\begin{pmatrix} \alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}$$

$|\alpha|^2 - |\beta|^2 = 1$

$$\begin{pmatrix} e^{-i\phi} & 0\\ 0 & e^{i\phi}\end{pmatrix}\begin{pmatrix} d & -\Delta\\ 1 & -\Delta\bar d\end{pmatrix} = \Big($$

$$\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad \begin{matrix} (\lambda - a) u = b v \\ (\mu - d) v = c u \end{matrix}$$

$$\mu = d + \frac{bc}{\lambda - a} = \frac{d\lambda - \Delta}{\lambda - a} = \begin{pmatrix} d & -\Delta \\ 1 & -a \end{pmatrix} (\lambda)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta \bar{d} & \Delta \bar{c} \\ c & d \end{pmatrix} \qquad \begin{vmatrix} d & -\Delta \\ 1 & -\Delta \bar{d} \end{vmatrix} = -\Delta |d|^2 + \Delta = \Delta(-|d|^2 + 1) = \Delta(-|c|^2)$$

$$\begin{pmatrix} -i\Delta^{-1/2} \frac{d}{|c|} & \frac{i\Delta^{1/2}}{|c|} \\ -i\Delta^{-\frac{1}{2}} \frac{1}{|c|} & i\Delta^{1/2} \frac{\bar{d}}{|c|} \end{pmatrix} \in SU(1,1)$$

$$\frac{e^{-i\theta} e^{+i\phi}}{|c|}$$

$$\|$$

$$\begin{pmatrix} -i\Delta^{-\frac{1}{2}} e^{i\phi} \frac{|d|}{|c|} & i\Delta^{1/2} \frac{1}{|c|} \\ -i\Delta^{-1/2} \frac{1}{|c|} & i\Delta^{1/2} e^{-i\phi} \frac{|d|}{|c|} \end{pmatrix}$$

$$\|$$

$$\underbrace{\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}}_{\text{arb. elt of } K} \begin{pmatrix} \frac{|d|}{|c|} & \frac{1}{|c|} \\ \frac{1}{|c|} & \frac{|d|}{|c|} \end{pmatrix} \underbrace{\begin{pmatrix} -i\Delta^{-1/2} & 0 \\ 0 & i\Delta^{1/2} \end{pmatrix}}_{\text{arb. element of } \mathbb{K}}$$

$$z = \frac{1+ix}{1-ix} = \frac{1-(-ix)}{1+(-ix)} \qquad \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}(x) = z$$

$$-ix = \frac{1-z}{1+z}$$

$$\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}^{*} f(x) \begin{pmatrix} x \\ 1 \end{pmatrix} = g(z) \begin{pmatrix} z \\ 1 \end{pmatrix}$$

where $g(z) = \frac{1}{1-ix} f\left(\frac{1+ix}{1-ix}\right)$

$$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \frac{|d|}{|c|} & \frac{1}{|c|} \\ \frac{1}{|c|} & \frac{|d|}{|c|} \end{pmatrix} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$$

$$= \frac{1}{|c|} \begin{pmatrix} e^{i(\phi+\psi)}|d| & e^{i(\phi-\psi)} \\ e^{-i(\phi-\psi)} & e^{-i(\phi+\psi)}|d| \end{pmatrix}$$

$$\frac{e^{-i\theta}}{|c|} \begin{pmatrix} d & -\Delta \\ 1 & -\Delta\bar{d} \end{pmatrix} = \begin{pmatrix} e^{-i\theta+i\phi}\frac{|d|}{|c|} & \\ e^{-i\theta}\frac{1}{|c|} & \end{pmatrix}$$

$$\begin{pmatrix} e^{i\phi}|d| & e^{2i\theta} \\ 1 & e^{2i\theta} e^{-i\phi}|d| \end{pmatrix} \frac{1}{e^{i\theta}|c|} = \begin{pmatrix} e^{i\theta}\frac{1}{|c|} \\ e^{i\theta-i\phi}\frac{|d|}{|c|} \end{pmatrix}$$

$$\frac{e^{-i\theta}}{|c|} \begin{pmatrix} d & -\Delta \\ 1 & -\Delta\bar{d} \end{pmatrix} = \frac{e^{-i\theta}}{|c|} \begin{pmatrix} e^{i\phi}|d| & e^{2\theta} \\ 1 & e^{2i\theta} e^{-i\phi}|d| \end{pmatrix}$$

$$\cancel{\neq \begin{pmatrix} e^{-i\theta} & \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} \frac{|d|}{|c|} & \\ & \end{pmatrix} \begin{pmatrix} e^{i\phi} & \\ & \end{pmatrix}}$$

$$= \begin{pmatrix} e^{-i\theta+i\phi}|d| & e^{i\theta} \\ e^{-i\theta} & e^{i\theta-i\phi}|d| \end{pmatrix}$$

$r = \frac{|d|}{|c|}$ $\quad s = \frac{1}{|c|}$

$s r = |d|$ $\quad \frac{1}{s} = |c|$

$|d| = \frac{r}{s}$

There are all kinds of things to understand, but maybe the discrete grid space is a good place to start – realizing a fractional linear transf. on S^1!

E has two descriptions

$$\overline{\mathbb{C}[\lambda,\lambda^{-1}]v} = \bar{E} = \overline{\mathbb{C}[\mu,\mu^{-1}]u}$$

I think you want to use spin str.

Try invariantly without choosing origin but you still have λ, μ so There is E defined with generators and relations, naturally module over

$$A\Big/\big((k\lambda-1)(k\mu-1) = 1-k^2\big).$$

carries pos-def and indef. herm. forms ~~stable under~~ preserved by translation.

Possibility: Instead of picking an edge and getting an isom with $L^2(S^1)$, you pick a vertex and direction and get an isomorphism with L^2(anti-periodic functions on S^1)

Let's follow the obvious. Use grid space

$$\mathbb{C}[\lambda,\lambda^{-1},(\lambda-k)^{-1},(k\lambda-1)^{-1}] \xrightarrow{\sim} \frac{\mathbb{C}[\lambda,\mu,\lambda^{-1},\mu^{-1}]}{(k\lambda-1)(k\mu-1)=1-k^2} \xleftarrow{\sim} \mathbb{C}[\mu,\mu^{-1},(\mu-k)^{-1},(k\mu-1)^{-1}]$$

these are the operators at our disposal $\quad$ $*$-algebra

These are our operators. The $*$ comes from inversion in the circle which is given. Next I need vectors and these should lie where?

Intrinsically you have? It should be possible for you to say this clearly. You are given $\mathcal{O}(-1) \otimes \mathcal{O}(-1) \simeq \Omega^1$, so you can take rational sections of $\mathcal{O}(-1)$ regular in the circle and form their hermitian inner product.

Riemann sphere.

There should be obvious sections of $\mathcal{O}(-1)$ locally, namely you have a hermitian form on V. so you can projects onto l_z ~~[illegible]~~

$$l_z = \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix} \subset \mathbb{C}^2$$

$$H\left(\begin{pmatrix} z_1 \\ z_0 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_0 \end{pmatrix}\right) = \bar{z}_1 w_1 - \bar{z}_0 w_0$$

you need to know the hermitian form restricted to the line $\mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix}$ is non degenerate

i.e. $H\left(\begin{pmatrix} z_1 \\ z_0 \end{pmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix}\right) = \bar{z}_1 z - \bar{z}_0 \neq 0.$

No you need to know that $H\left(\begin{pmatrix} z \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix}\right) = |z|^2 - 1 \neq 0.$

? How to process.

$\mathcal{O}(-1) = \{ (z,$ $\qquad ($

Have $\quad \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix} \subset \mathbb{C}^2 \qquad \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbb{C}^2$

Take $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$

$$\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & z \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$x = -b$

$$z \longmapsto (-b)\begin{pmatrix} z \\ 1 \end{pmatrix}$$

$f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$ rational section of $\mathcal{O}(-1)$.

~~maybe what's happening is that~~

This is a section of $\mathcal{O}(-1)$, if it regular on the circle it has a norm.

What is the local norm^2 of $\begin{pmatrix} z \\ 1 \end{pmatrix}$ You are stupid.

$\begin{pmatrix} z \\ 1 \end{pmatrix} \in l_z$

$$\begin{tikzcd} l_z \arrow[r, phantom, "\subset"] \arrow[rr, bend right, "dz"'] & \mathbb{C}^2 \arrow[r] & \mathbb{C}^2/l_z \end{tikzcd}$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} \wedge \begin{pmatrix} dz \\ 0 \end{pmatrix} = -dz$$

$f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$ $\qquad f(z+\varepsilon)\begin{pmatrix} z+\varepsilon \\ 1 \end{pmatrix}$

$$\| $$

$$(f(z)+f'(z)\varepsilon)\begin{pmatrix} z+\varepsilon \\ 1 \end{pmatrix} =$$

Given $s(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \in l_z \subset \mathbb{C}^2$

tangent vector ∂_z

$$s \wedge \partial_z s = \begin{vmatrix} fz & f'(z)z + f(z) \\ f & f'(z) \end{vmatrix} = -f(z)^2$$

So up to sign you have.

$$s \longmapsto -s \wedge ds = f^2 dz$$

$$s = f\begin{pmatrix} z \\ 1 \end{pmatrix}$$

---

use unit circle, line

do the real line first. Section of $\mathcal{O}(-1)$ has the form $s(x) \in f(x)\begin{pmatrix} x \\ 1 \end{pmatrix}$ ~~suppose~~ say f rational fn. regular on $\mathbb{R}$ regular at ∞ i.e. $f(x)x$ reg. at ∞.

take $$-s \wedge ds = -\begin{vmatrix} fx & (f'(x)x + f(x))\,dx \\ f & f'(x)\,dx \end{vmatrix} = +f^2 dx$$

Thus we find $f\begin{pmatrix} x \\ 1 \end{pmatrix}$ is real when f is real and you get $L^2(\mathbb{R}, dx)$ action of $SL(2,\mathbb{R})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* f(x)\begin{pmatrix} x \\ 1 \end{pmatrix} = f\left(\frac{ax+b}{cx+d}\right)\frac{1}{cx+d}\begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}$$

---

next take $|z| = 1$ for your circle

$$s(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \in l_z \subset \mathbb{C}^2$$

$$s \wedge ds = i\begin{vmatrix} fz & df\,z + f\,dz \\ f & df \end{vmatrix} = -i f^2 dz$$

$$z = e^{i\theta} \qquad dz = e^{i\theta} i\,d\theta = z i\,d\theta$$

$$-i f^2 dz = +z f^2 d\theta = (z^{1/2} f)^2 d\theta$$

$$z = \frac{1+ix}{1-ix} = \frac{1-(-ix)}{1+(-ix)} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}(-ix)$$

$$-ix = \begin{pmatrix} +1 & -1 \\ -1 & -1 \end{pmatrix}(z) = \frac{1-z}{1+z} \qquad \frac{dz}{iz} = \frac{2\,dx}{\cancel{(1+ix)(1-ix)}\ 1+x^2}$$

$$z = \frac{1+ix}{1-ix} \qquad dz = \frac{(1-ix)i\,dx - (1+ix)(-i)dx}{(1-ix)^2} = \frac{2i\,dx}{(1-ix)^2}$$

The important thing is that if

$$s(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \in l_z \subset \mathbb{C}^2$$

then $s \wedge ds = \begin{vmatrix} fz & d(fz) \\ f & df \end{vmatrix} = \begin{vmatrix} fz & f\,dz \\ f & 0 \end{vmatrix} = -f^2 dz$

but we apply this to $z = e^{i\theta}$ $\quad dz = ie^{i\theta}d\theta$

$$s \wedge ds = -ie^{i\theta}f^2\,d\theta$$

so s is real when $-ie^{i\theta}f^2 \geq 0$

or $\quad (-i)^{1/2}(e^{i\theta/2}f) \in \mathbb{R}$

Still problems! Let's build in both calculations. In other works you have both a circle coordinate and a line coordinate, say arising from grid continuous in one direction.

so what is the philosophy? In the discrete case ~~the~~ the grid space is an $A = \mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$-module

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So let's start with $L^2(\mathbb{R}, dx)$ ~~by~~

hence $f(x) \longleftrightarrow s(x) = f(x)\begin{pmatrix} x \\ 1 \end{pmatrix}$ $\qquad -s\wedge ds = -\begin{vmatrix} fx & dfx + fdx \\ f & df \end{vmatrix} = f^2 dx$

$$\downarrow$$

$$f\left(\frac{ax+b}{cx+d}\right)\frac{1}{cx+d}\begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix} \quad \text{NO}$$

$s(x) = f(x)\begin{pmatrix} x \\ 1 \end{pmatrix} \in l_x \subset \mathbb{C}^2$

$z = \frac{1+ix}{1-ix} = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}(x)$

$x = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} z = \frac{z - 1}{iz+i}$

$= \frac{1}{i}\,\frac{z-1}{z+1}$

$$\begin{array}{ccc} S^2\times\mathbb{C}^2 & \xrightarrow{g^{-1}} & S^2\times\mathbb{C}^2 \\ \uparrow g^*s & & \uparrow s \\ x \in S^2 & \xrightarrow{g} & S^2 \\ x & \longmapsto & z \end{array}$$

g

$g \circ s \circ g^{-1}$

$z = \frac{1-(-ix)}{1+(-ix)}$

$\therefore -ix = \frac{1-z}{1+z}$

$$z = gx = \frac{ax+b}{cx+d}$$

$$s(z) = f\left(\frac{ax+b}{cx+d}\right)\begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix} = \frac{1}{cx+d} f\left(\frac{ax+b}{cx+d}\right)\begin{pmatrix} ax+b \\ cx+d \end{pmatrix}$$

$$g^{-1}s(gx) = \frac{1}{cx+d} f\left(\frac{ax+b}{cx+d}\right).$$

So say it right. If you have a change of variable $w = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g} z$, then for a function $f = f(w)$ you have $(g^*f)(z) = f(gz) = f\left(\frac{az+b}{cz+d}\right)$ and for a section of $\mathcal{O}(-1)$: $s = s(w) = f(w)\begin{pmatrix} w \\ 1 \end{pmatrix}$ you get

$$g^{-1}(g^*f)(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f\left(\frac{az+b}{cz+d}\right)\begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} = f\left(\frac{az+b}{cz+d}\right)\frac{1}{cz+d}\, g^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

Let's now go between

$$\begin{array}{ccccc} \tilde f(z)\begin{pmatrix} z \\ 1\end{pmatrix} & S^2\times\mathbb{C}^2 & \xrightarrow{g\times g} & S^2\times\mathbb{C}^2 & f(x)\begin{pmatrix} x \\ 1\end{pmatrix} \\ \uparrow & \uparrow & & \uparrow & \uparrow \\ z & S^2 & \xrightarrow[\begin{pmatrix} a & b \\ c & d\end{pmatrix}]{} & S^2 & x \\ & \overset{\omega}{z} & & x & \end{array}$$

So what you seem to ~~so~~ get is that $f(x)\begin{pmatrix} x \\ 1\end{pmatrix}$ goes to

$$\frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right)\begin{pmatrix} z \\ 1\end{pmatrix}$$

$$\tilde f(z)\begin{pmatrix} z \\ 1\end{pmatrix} = g^{-1} f\left(\frac{az+b}{cz+d}\right)\begin{pmatrix} \frac{az+b}{cz+d} \\ 1\end{pmatrix}$$

$$\frac{az+b}{cz+d} = x \longleftarrow\!\shortmid\; z$$

$$X = \frac{z-1}{iz+i}$$

$$f(x) \longmapsto \tilde f(z) = \frac{1}{cz+d} f\left(\frac{az+b}{cz+d}\right)$$

$$dx \longmapsto d\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)}{(cz+d)^2}\,dz$$

$$f(x)^2 dx \xmapsto[\text{vol not preserved}]{\text{approx because}} \tilde f(z)^2\,dz = \frac{1}{(cz+d)^2} f\left(\frac{az+b}{cz+d}\right)^2 dz$$

$$\xrightarrow{\text{actual}} f\left(\frac{az+b}{cz+d}\right)^2 \frac{ad-bc}{(cz+d)^2}\,dz$$

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 1 & -1 \\ i & i\end{pmatrix}$$

$$ad-bc = 2i$$

$$\therefore\ f(x)^2 dx \longmapsto \underbrace{(ad-bc)}_{2i}\,\tilde f(z)^2 dz$$

$$2i\,\tilde f(z)^2\,dz \quad \text{is} \geq 0$$

means

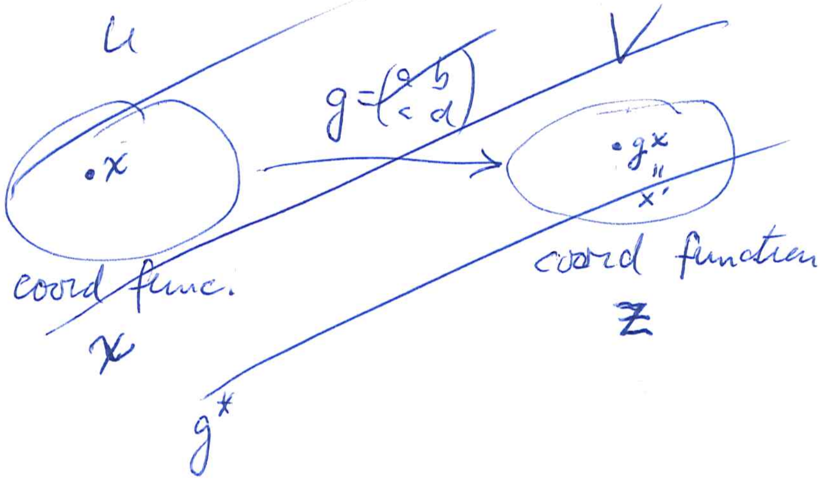
$$\tilde f(z)^2\; 2i\,z\,i\,d\theta$$

You ask when

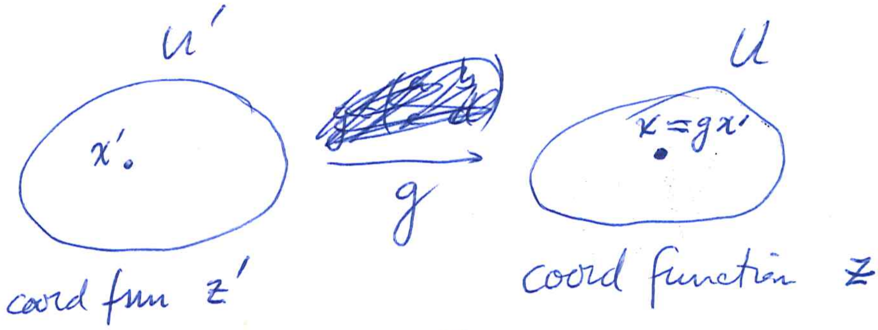
$$2i\,\frac{1}{(iz+i)^2} f\left[\frac{z-1}{iz+i}\right]^2 z\,i\,d\theta \quad \geq 0$$

$$\frac{2}{(z+1)^2} f(x)^2\,z\,d\theta \geq 0$$

$$\frac{2}{}$$

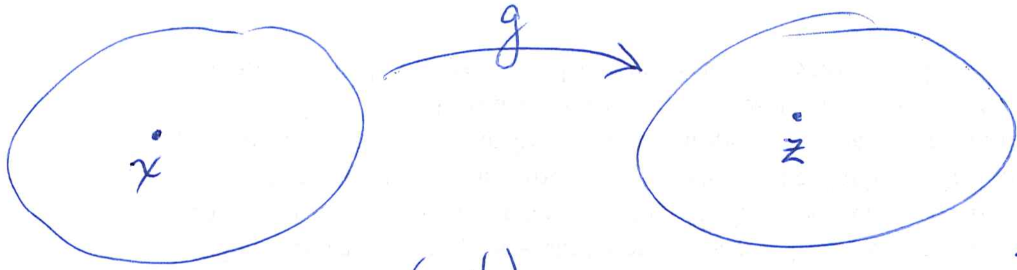


$$z' = zg$$



$$z' = zg \xleftarrow{g^*} z$$

$=$



$$x = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}(z)$$

$$cx + d = c\,\frac{dz - b}{-cz + a} + d = \frac{\Delta}{-cz+a}$$

$$z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x)$$

$$l_x = \mathbb{C}\begin{pmatrix} x \\ 1 \end{pmatrix} \qquad l_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix}\mathbb{C} = \begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}\mathbb{C}$$

$$(g^*f)(x) = f\left(\frac{a\cancel{z}x+b}{c\cancel{z}x+d}\right) \qquad f(z)$$

$$(g^*s)(x) \qquad s(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} ax+b \\ cx+d \end{pmatrix} = \frac{1}{cx+d}\begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}$$

$$(cx+d)\begin{pmatrix} x \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}$$

so $\frac{\Delta}{-cz+a}\begin{pmatrix}\frac{dz-b}{-cz+a}\\ 1\end{pmatrix} \longleftarrow \begin{pmatrix} z \\ 1\end{pmatrix}$

This is how to proceed, namely, the ~~[illegible]~~ specific Cayley Transform between $\mathbb{R}$ and S^1.

$\mathbb{R}$ $\qquad\qquad S^1 \quad \underbrace{\begin{pmatrix} a & b \\ c & d\end{pmatrix}} \qquad \Delta = ad-bc = 2i$

$x \longmapsto z = \frac{1+ix}{1-ix} = \begin{pmatrix} i & 1 \\ -i & 1\end{pmatrix}(x) \qquad x = \begin{pmatrix} 1 & -1 \\ i & i\end{pmatrix}$

$l_x = \begin{pmatrix} x \\ 1\end{pmatrix}\mathbb{C} \subset \mathbb{C}^2 \qquad l_z = \begin{pmatrix} z \\ 1\end{pmatrix}\mathbb{C} \subset \mathbb{C}^2$

$l_x \xrightarrow{g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}} l_z$

$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x \\ 1\end{pmatrix}\mathbb{C} = \begin{pmatrix} ax+b \\ cx+d\end{pmatrix}\mathbb{C} = \begin{pmatrix}\frac{ax+b}{cx+d} \\ 1\end{pmatrix}\mathbb{C}$

$f(x)\begin{pmatrix} x \\ 1\end{pmatrix} \xrightarrow{g} f(x)\begin{pmatrix} ix+1 \\ -ix+1\end{pmatrix} = f(x)(-ix+1)\begin{pmatrix} z \\ 1\end{pmatrix}$

$x \longrightarrow gx = \frac{1+ix}{1-ix} = z$

$1 - i\left(\frac{z-1}{iz+i}\right) = 1 - \frac{z-1}{z+1} = \frac{2}{z+1}$

$f(x)\begin{pmatrix} x \\ 1\end{pmatrix} \longmapsto f\left(\frac{z-1}{iz+i}\right)\frac{2i}{iz+i}\begin{pmatrix} z \\ 1\end{pmatrix}$?

why do you have $\qquad$ You expect

$f(x)\begin{pmatrix} x \\ 1\end{pmatrix} \longrightarrow f\left(\frac{ax+b}{cx+d}\right)\frac{1}{cx+d}$

OKAY

Consider the action of $GL(2,\mathbb{C})$ on $L = \mathcal{O}(-1) \subset \mathbb{P}^1 \times \mathbb{C}^2$. A point of L is a pair $\left(z, t\binom{z}{1}\right)$. Under $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ this goes to $\left(\frac{az+b}{cz+d}, t\binom{az+b}{cz+d}\right)$. A rational section of L over $\mathbb{P}^1$ ~~is a pair (~~ has generic graph $\left\{\left(z, f(z)\binom{z}{1}\right)\right\}$ ~~under~~ which goes to

---

$GL(2,\mathbb{C})$ acts on $L = \left\{(z, v) \,\middle|\, \begin{matrix} z \in \mathbb{P}^1 \\ v \in l_z \end{matrix}\right\} \subset \mathbb{P}^1 \times \mathbb{C}^2$ so it acts on rational sections, these are ~~subspaces~~ ~~so~~ subsets ,

---

~~here's the problem~~ ~~Su~~

$$L = \{(z, v) \mid v \in l_z\} \subset \mathbb{P}^1 \times \mathbb{C}^2$$

Given a fn. $f(z)$ on $\mathbb{P}^1$ you get a left $GL(2,\mathbb{C})$ action on functions by $(g \cdot f)(z) = f(g^{-1}z)$. and a right action without the inverse. If $f: \mathbb{P}^1 \longrightarrow \mathbb{C}^2$ then you get left action by $g(f) = g_V f g^{-1}$

Take f to be $f(z)\binom{z}{1}$ $\qquad f($

$$g_V\left(f(z)\binom{z}{1}\right) = f(z)\binom{az+b}{cz+d}$$

$$f(g^{-1}z)\binom{g^{-1}z}{1} = f\left(\frac{dz-b}{-cz+a}\right)\begin{pmatrix} \frac{dz-b}{-cz+a} \\ 1 \end{pmatrix}$$

so $g_V f(g^{-1}z)\begin{pmatrix} g^{-1}z \\ 1 \end{pmatrix} = f\left(\frac{dz-b}{-cz+a}\right)\begin{pmatrix} a\frac{dz-b}{-cz+a}+b \\ c\frac{dz-b}{-cz+a}+d \end{pmatrix}$

$$= f\left(\frac{dz-b}{-cz+a}\right)\begin{pmatrix} \frac{(ad-bc)z}{-cz+a} \\ \frac{ad-bc}{-cz+a} \end{pmatrix} = \frac{ad-bc}{-cz+a} f\left(\frac{dz-b}{-cz+a}\right)\begin{pmatrix} z \\ 1 \end{pmatrix}$$

what lesson to learn.

---

setting $\quad V = \mathbb{C}^2$ acted on by $GL(2,\mathbb{C})$

$$L = \{ (z, v) \mid v \in l_z \} \subset \mathbb{P}^1 \times V$$

given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, ~~and~~ U open in $\mathbb{P}^1$, get

$$g^* : \mathcal{O}(U) \longrightarrow \mathcal{O}(g^{-1}U) \qquad (g^* f)(w) = f(gw)$$

want

$$g^* : \Gamma(U, L) \longrightarrow \Gamma(g^{-1}U, L) \qquad g^*(s) = g_V^{-1} s g$$

$$s(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \longmapsto g_V^{-1} f(gw)\begin{pmatrix} gw \\ 1 \end{pmatrix} = f(gw)\, g_V^{-1}\begin{pmatrix} gw \\ 1 \end{pmatrix}$$

$z = gw = \dfrac{aw+b}{cw+d} \qquad g_V\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \begin{pmatrix} gw \\ 1 \end{pmatrix}(cz+d)$

$$\therefore \quad g_V^{-1}\begin{pmatrix} gw \\ 1 \end{pmatrix} = \frac{1}{cz+d} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

or $\quad g^*\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} \frac{aw+b}{cw+d} \\ 1 \end{pmatrix} = \dfrac{1}{cw+d} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} aw+b \\ cw+d \end{pmatrix}}_{\begin{pmatrix} w \\ 1 \end{pmatrix}}$

Think about what to do!

Ultimately you have $\quad \mu = \dfrac{\lambda - k}{k\lambda - 1}$ and also

$$z = \frac{i\xi + a}{i\xi - a} = \frac{\xi - ia}{\xi + ia}$$

~~The idea next is to~~

You know how to define the Hilbert space on a circle - (what about an immersed circle e.g. ∞?). Go over carefully $s(z) = f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$

$$s \wedge ds = \begin{vmatrix} fz & df\, z + f dz \\ f & df \end{vmatrix} e_1 \wedge e_2 = f(z)^2 dz\,(-e_1 \wedge e_2)$$

use $|z| = 1$ $\quad z = e^{i\theta}$

$$f(e^{i\theta})^2 e^{i\theta} i\, d\theta \;(-e_1 \wedge e_2)$$

If we take the simplest volume elt. $-ie_1 \wedge e_2$, then we obtain $\left(f(e^{i\theta})e^{i\theta/2}\right)^2 d\theta$ and ~~this~~ is ≥ 0 when $f(e^{i\theta})e^{i\theta/2}$ is real-valued, which, if it is continuous means that it vanishes somewhere

Describe merom. sections of L. Since $L \simeq \mathcal{O}(-1)$ numbers of zeroes - number of poles $= -1$,

$1\begin{pmatrix} z \\ 1 \end{pmatrix}$ has simple pole at ∞

$z^{-1}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix}$ has simple pole at 0.

~~simple pole at 0~~ We know $f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$ is real valued on S^1 when $f(z)z^{1/2}$ is a real-valued function, i.e. when $\overline{f(z)}\, z^{-1/2} = f(z)z^{1/2}$

or $\overline{f(z)} = z f(z)$ or $f^* = zf$

R.V.

$f\begin{pmatrix} z \\ 1 \end{pmatrix}$ is real when $f^2 z\, d\theta \geq 0$

$f\begin{pmatrix} z \\ 1 \end{pmatrix}$ ———— $f^2 z \geq 0$

i.e. $f z^{1/2} \in \mathbb{R}$

~~Better~~ is $c\begin{pmatrix} z \\ 1 \end{pmatrix}$ is real when $c^2 z \geq 0$ i.e. $c z^{1/2} \in \mathbb{R}$.

$$\begin{array}{ccc} c\begin{pmatrix} z \\ 1 \end{pmatrix} & \longmapsto & c z^{1/2} \\ & & \downarrow \\ \bar{c} z^{-1}\begin{pmatrix} z \\ 1 \end{pmatrix} & \longmapsto & \bar{c} z^{-1/2} \end{array}$$

$$\therefore \quad \left(f\begin{pmatrix} z \\ 1 \end{pmatrix}\right)^* = f^* z^{-1}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

and so $\left(f\begin{pmatrix} z \\ 1 \end{pmatrix} \,\middle|\, g\begin{pmatrix} z \\ 1 \end{pmatrix} \right) = f^* z^{-1} g\, z\, d\theta = f^* g\, d\theta$

$\dfrac{1}{kz-1}\begin{pmatrix} z \\ 1 \end{pmatrix}$ simple pole at k^{-1}

$\dfrac{1}{z-k}\begin{pmatrix} z \\ 1 \end{pmatrix}$ ———— at k

$$\left(\frac{1}{z-k}\begin{pmatrix} z \\ 1 \end{pmatrix}\right)^* = \frac{1}{z^{-1}-k}\begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} = \frac{z}{1-kz}\begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} = \frac{1}{1-kz}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

$\underbrace{A \otimes_A M}$ $\xrightarrow{\sim} M$ ~~stables~~

naturally a module over $\mathrm{Hom}_{A^{op}}(A,A) = \{ f : A \to A \mid f \text{ add.}, f(aa') = f(a)a' \}$

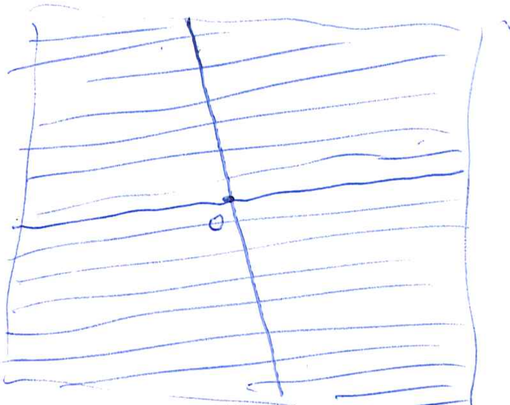
Basic facts.

$M \xrightarrow{f} M'$ nil isom. $\Longrightarrow A^{(2)} \otimes_A M \xrightarrow{\sim} A^{(2)} \otimes_A M'$

$\forall M$ $A^{(2)} \otimes_A M$ is firm and $A^{(2)} \otimes_A M \longrightarrow M$

is a nil isom.

$$\underset{\text{firm}}{\mathcal{M}(A)} \hookrightarrow \underbrace{\mathrm{Mod}(\tilde{A})}_{A\text{-mods}}$$



hereditary C^* subalg gen by cc^*

apparently there's a key result of Kasparov about C^*-alg. extensions $B \longrightarrow C \longrightarrow A$ which are invertible.

for sum of extensions (assuming B stable for this to be def'd, also need nuclear hyp. on A and/or B). ~~Kontraktion~~

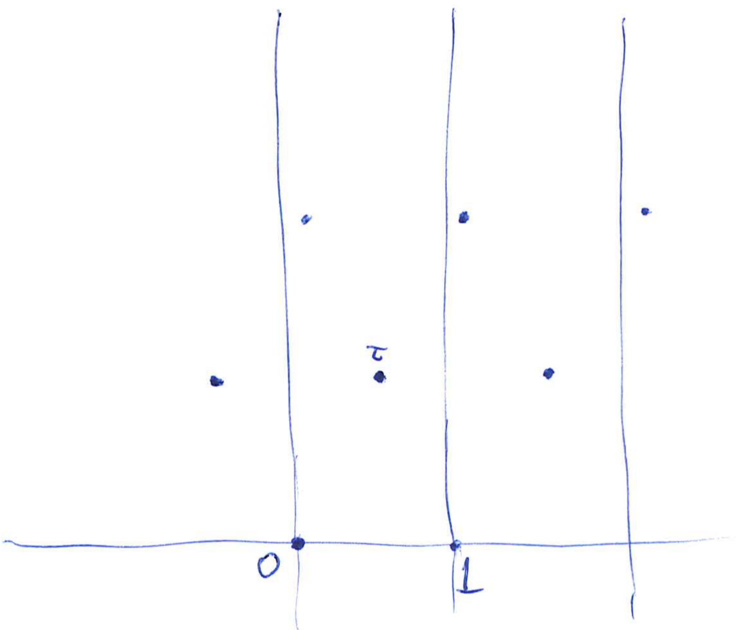
~~Key~~ An extension absorbing when isom. to its sum with any trivial extn. (trivial means $A \to M(B)/B$ lifts to $A \to M(B)$.

$$\begin{array}{ccccc} B & \longrightarrow & C & \longrightarrow & A \\ \| & & \downarrow & \text{cart} & \downarrow \\ B & \longrightarrow & M(B) & \longrightarrow & M(B)/B \end{array}$$

$\therefore$ got ~~a~~ section hom. $A \to C$.

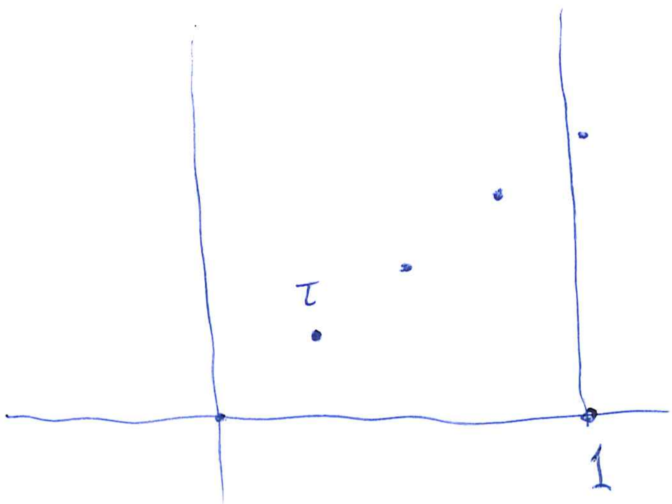
Irving Adler. Phyllotaxis

lattice in $\mathbb{C}/\mathbb{Z}$, points model leaves on a stem
Lift to a lattice in $\mathbb{C}$, containing $\mathbb{Z}$, assume the lattice, call it L, has $\mathbb{Z}$ as a summand.
This means $L \cap \mathbb{R} = \mathbb{Z}$, no ring of leaves around the cylinder. Suppose then $L = \mathbb{Z} + \mathbb{Z}\tau$ with $\mathrm{Im}(\tau) > 0$. Can also suppose $0 \leq \mathrm{Re}(\tau) < 1$. Ignore $\mathrm{Re}(\tau) = 0$.
Picture

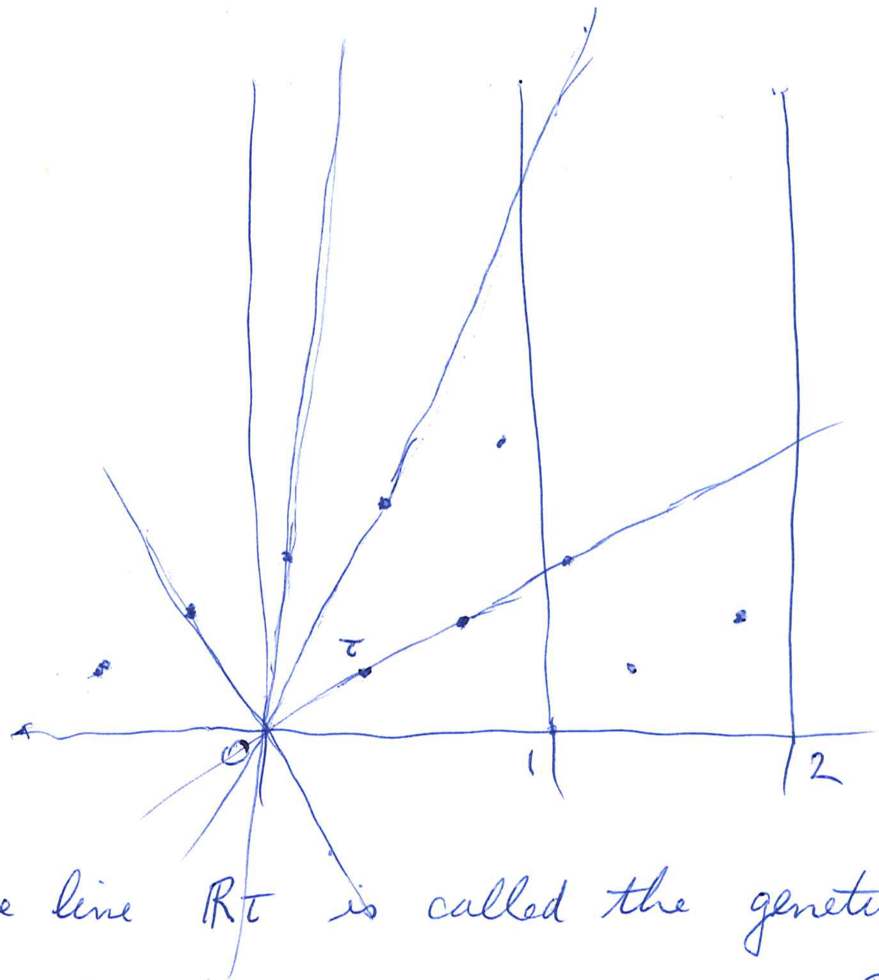


It seems τ is uniquely determined. $\tau = \overset{\text{divergence}}{d} + i\overset{\text{rise}}{r}$

This picture is bad because typically ~~the imaginary part of~~ τ is small



Better picture



The line $\mathbb{R}\tau$ is called the genetic spiral

right parastichy - take a line $\mathbb{R}z$ with slope >0 which intersects L, whence $L \cap \mathbb{R}z$ is a direct summand of L, then a right parastichy is a coset of $L \cap \mathbb{R}z$ in L, left parastichy means slope <0. Actually the slope should be ≥ 2 (resp ≤ -2).

$mx - ny = 1 = \begin{vmatrix} m & y \\ n & x \end{vmatrix}$ If $|x| < n$ roughly then $|y| < m$ roughly IAS4

$1 \le x < n \qquad 0 < y = \frac{mx-1}{n} < m$

Exp. Fmla 1951 Weil quadratic form

mult. form uses Mellin transf – also FT form

$f(x)$ on $(0,\infty)$ $\qquad f^*(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ $\qquad$ even / odd $\; f(x) = \pm\frac{1}{x} f\left(\frac{1}{x}\right)$

$f \in C_0^\infty$ to begin with $\qquad \tilde{f}(s) = \int_0^\infty f(x)\, x^{s-1} dx$ $\qquad$ Exp Fmla

$$T(f) = \left(\int_0^\infty f(x)\,dx + \int_0^\infty f^*(x)\,dx - \sum_1^\infty \Lambda(n) \{ f(n) + f^*(n) \} - (\log(4\pi) + \gamma) f(1) - \int_1^\infty \left\{ f(x) + f^*(x) - \frac{2}{x} f(1) \right\} \frac{x\,dx}{x^2 - 1} \right) = \sum_\rho \hat{f}(\rho)$$

secretly Trace

$$-(\log 4 + \gamma)\frac{f(1)}{1} \int_1^\infty = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\right)\left(\frac{w}{2}\right) \hat{f}(w)\, \cancel{x}^{-w}\, dw$$

$$(f * g)(x) = \int_0^\infty f\left(\frac{x}{y}\right) g(y) \frac{dy}{y} \qquad (f * \overline{g^*})(x) \stackrel{?}{=} \int_0^\infty f(xy)\, \overline{g(y)}\, dy$$

$RH \iff T[f * \bar{f}^*] \ge 0$ suitable f (beyond C_0^∞ in Weil's class)

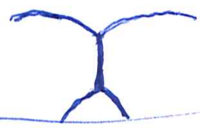
Study this quadratic form

$$\widetilde{(f * \bar{f}^*)}(s) = \tilde{f}(s)\, \overline{\tilde{f}}(1-s) = |\tilde{f}(\tfrac{1}{2} + it)|^2$$

Bombieri says formula has nice interpretation in function field case – feels it is right object to study.

$f(x) \longmapsto a^{\frac{1}{2}} f(ax)$ preserves quadratic form

Problem 1: Study infimum of $T[f * \bar{f}^*]$ on classes of test functions.



You have trouble realizing what you remember because you consider elements of $\mathbb{Q} \cup \infty$.

Idea of defining a Spectrum using valuation-type functions building of an adic vector space

Look at an oriented circle on the Riemann sphere. Corresponding Hilbert space of sections of $\mathcal{O}(-1)$. Hardy space. Do you have a 2-dimensional real symplectic vector space acting on Hardy space. It would have to be connected with the 2 diml complex space V given. Use loop group picture, so functions on the should give multiplication operators.

Ideas central extension of loop group given by dilogarithm $\exp \int \log f \, d\log g$ of Deligne.

Go back to RS, $\mathcal{O}(-1)$ sections are

$f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$ points of $\mathcal{O}(-1)(z)$ are $c\begin{pmatrix} z \\ 1 \end{pmatrix}$. If δz is a tangent vector at z, get symm. form on fibre

$$-\begin{vmatrix} f(z)\,z & f'(z)\delta z\, z + f(z)\,\delta z \\ f(z) & f(z)\,\delta z \end{vmatrix} = +f(z)^2\delta z$$

Get real structure on fibre: $f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$ real when $f(z)^2\delta z \geq 0$.

$\delta z = |\delta z| e^{i\phi}$ $\qquad f(z)^2 e^{i\phi} = \left(f(z)e^{i\phi/2}\right)^2 \geq 0$

$f(z)e^{i\phi/2} \in \mathbb{R}$. $\qquad$ Inv.

Take S^1 $z = e^{i\theta}$

$\delta z = \underbrace{ie^{i\theta}}_{e^{i\phi}}\, d\theta$ $|\delta z|$

$$\begin{array}{ccc} f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} & \longmapsto & f(z)e^{i\phi/2} \\ \downarrow * & & \downarrow \text{conj.} \\ e^{-i\phi}\,\overline{f(z)}\begin{pmatrix} z \\ 1 \end{pmatrix} & \longmapsto & \overline{f(z)}\,e^{-i\phi/2} \end{array}$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^* = -ie^{-i\theta}\begin{pmatrix} z \\ 1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} = -i\frac{1}{z}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

~~somehow this i should~~

IASG

Inner product

$$\left(f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \right)^{*}, \left(g(z) \begin{pmatrix} z \\ 1 \end{pmatrix} \right) \mapsto e^{-i\phi}\, \overline{f(z)}\, g(z)\, \underbrace{\overline{\delta z}}_{e^{i\phi}\,\delta\theta}$$

$$= \overline{f(z)}\, g(z)\, \delta\theta$$

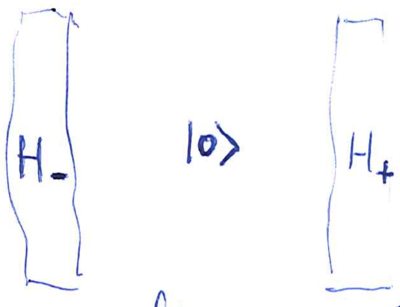
$$\left(\begin{pmatrix} z \\ 1 \end{pmatrix}^{*}, \begin{pmatrix} z \\ 1 \end{pmatrix} \right) \longmapsto$$

$$\parallel \qquad\qquad\qquad iz\,\delta\theta$$

$$\left(+\frac{1}{iz}\begin{pmatrix} z \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix} \right) \longmapsto +\frac{1}{iz}\, \overline{\delta z} = \delta\theta$$

Go over your ideas. The polarized Hilb. space you start with is L^2 sections of $\mathcal{O}(-1)$ over the circle, with Hardy polarization. You quantize, form Fock space. Certain things become operators. $\Lambda \bar{H}_- \otimes \Lambda H_+$ = Fock space

It seems that bosonic operators, the current operators are of degree 0 on Fock space. Are there functions on the circle?



You want to look at functions on S^1 with the skew form $\int f\,dg$. Functions modulo constants, constants become relevant when you move between different charge.

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{OU}(1,1) \qquad \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}(z) = e^{2it} z$$

IAS 7

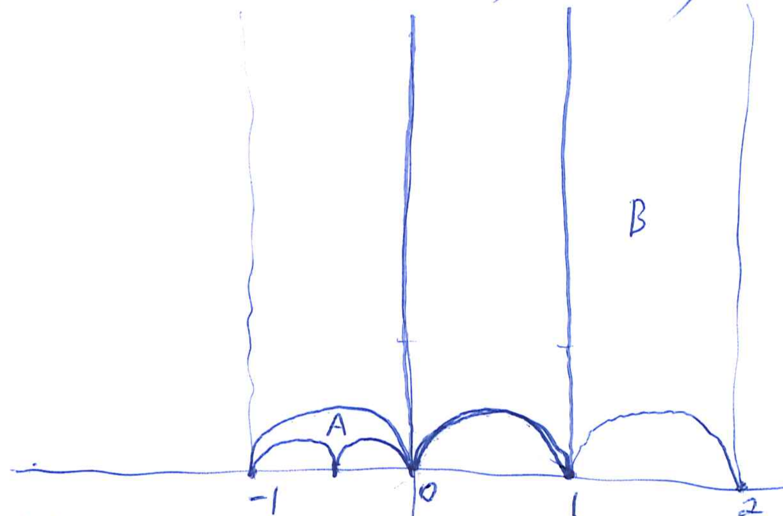
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} f\left(\frac{az+b}{cz+d}\right)\begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$$

$$= \underbrace{f\left(\frac{az+b}{cz+d}\right)\frac{1}{cz+d}}\ \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} az+b \\ cz+d \end{pmatrix}}_{\begin{pmatrix} z \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}^* f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} = \underbrace{f(e^{2it})\frac{1}{e^{-it}}}_{\text{frequencies } 2n+1.}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

Continued fractions & the UHP. Start with $\mathbb{Z}^2$, then direct summands of rank 1 of $\mathbb{Z}^2$ have the form $\mathbb{Z}\binom{m}{n}$ where $(m,n)=1$, so "$\mathbb{P}^1(\mathbb{Z})$" $= \mathbb{Q}\cup\{\infty\}$, $\mathbb{Z}\binom{m}{n} \longmapsto \frac{m}{n}$. Define a simplicial complex X of dim 2, the vertices are these summands of $\mathbb{Z}^2$, two vertices form a 1-simplex iff $\mathbb{Z}^2$ is the direct sum of the summands, and a 2-simplex consists of 3 summands such that each pair is a 1-simplex. So $\mathbb{Z}\binom{1}{0}=\infty$, $\mathbb{Z}\binom{0}{1}=0$ make a 1-simplex. Assume $\mathbb{Z}\binom{m}{n}$ can be added to these to get a 2-simplex, i.e. $\begin{vmatrix} 1 & m \\ 0 & n \end{vmatrix} = \pm 1$ and $\begin{vmatrix} 0 & m \\ 1 & n \end{vmatrix} = -m = \pm 1$. Thus there are two possibilities namely $\mathbb{Z}\binom{m}{n} = \mathbb{Z}\binom{1}{1}, \mathbb{Z}\binom{-1}{1}$.

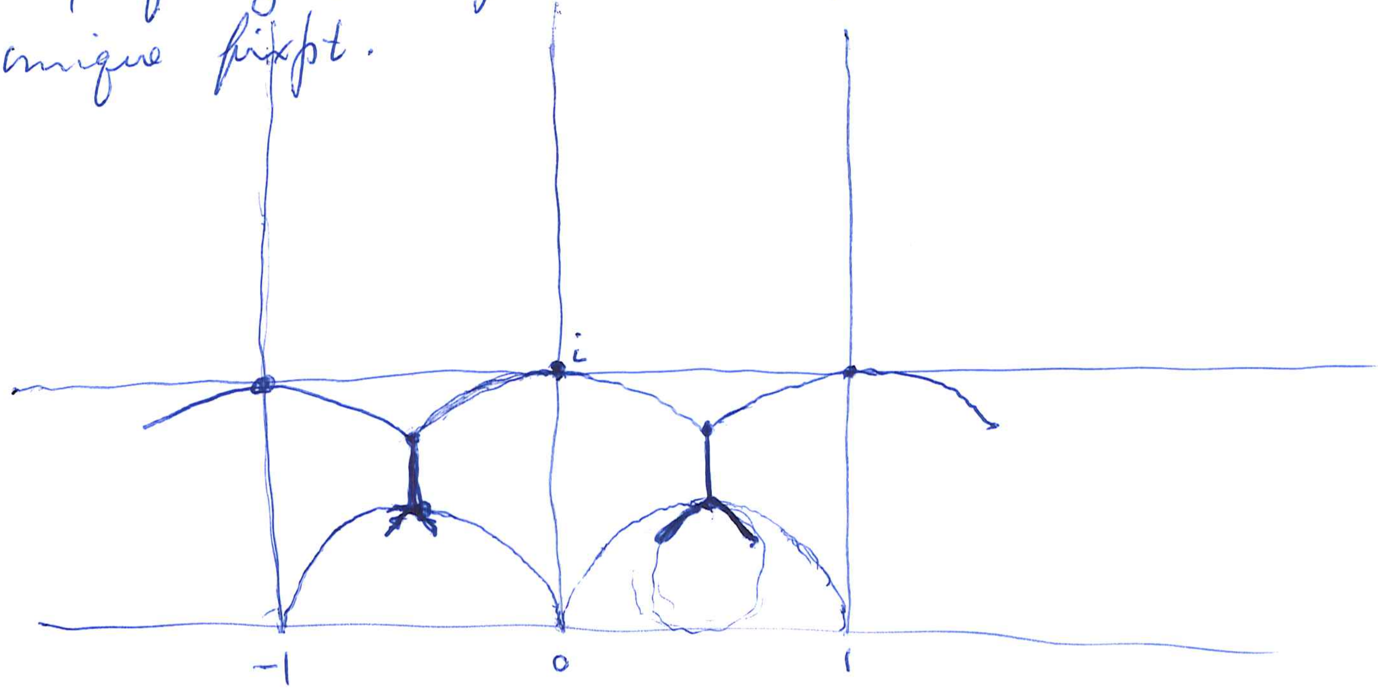
Picture in UHP



A, B reflections of each other thru -1.

This gives a triangulation of the UHP preserved by $PSL_2(\mathbb{Z})$. One can deform this 2 complex to a 1-diml simplicial complex by replacing each 1-simplex and 2-simplex by its "center" – the symmetry group of any 1-simplex is $\mathbb{Z}/2$ and of any 2-simplex is $\mathbb{Z}/3$, the center is the unique fixpt.

6:30 Harry's car



You need to understand the metaplectic representation

Question: Can you attach a metaplectic repn to any oriented circle on the RS so as to have imaginary time evolution. One problem is that you need a double covering of $SU(1,1) = SL_2(\mathbb{R})$ to get the group to act on the representation.

So what's going on.

$$\partial_t \psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \psi \qquad \begin{aligned} (\partial_t - \partial_x)\psi^1 &= im\psi^2 \\ (\partial_t + \partial_x)\psi^2 &= im\psi^1 \end{aligned}$$

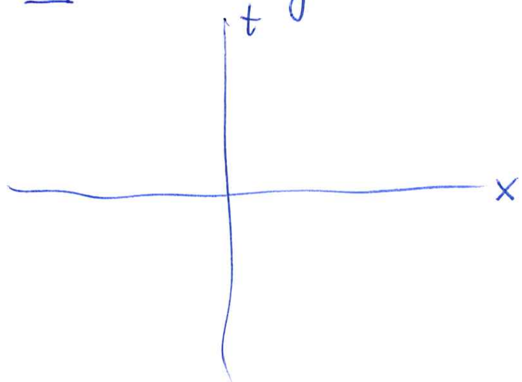
$$\begin{aligned} (\omega - k)\psi^1 &= m\psi^2 \\ (\omega + k)\psi^2 &= m\psi^1 \end{aligned} \qquad \omega = \pm\sqrt{m^2+k^2}$$

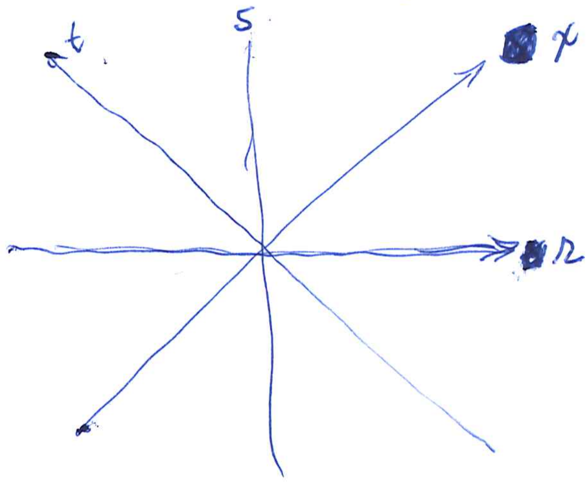
$$\psi(x,t) = \int_{-\infty}^{\infty} e^{+i(\omega_k t + kx)} \begin{pmatrix} \frac{1}{\omega - k} \\ 1 \end{pmatrix} v(k) \frac{dk}{2\pi} \quad \text{NO}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \left(e^{i\omega_k t} \begin{pmatrix} \overset{\omega_k + k}{\frac{1}{\omega_k - k}} \\ 1 \end{pmatrix} v(k) + e^{-i\omega_k t} \begin{pmatrix} \overset{-(\omega_k - k)}{\frac{1}{-\omega_k - k}} \\ 1 \end{pmatrix} w(k) \right)$$

two ~~[illegible]~~ functions of k needed for a point in phase space

Something doesn't make sense. You have described solutions of the equation by 2 functions of k. but if ~~[illegible]~~ change coordinates





$$\begin{aligned} x &= r + s \\ t &= -r + s \end{aligned}$$

$$\begin{aligned} -\partial_r \psi^1 &= im\psi^2 \\ \partial_s \psi^2 &= im\psi^1 \end{aligned}$$

$$\begin{aligned} -\xi u &= v \\ \eta v &= u \end{aligned}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial t}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}$$

$$\int_{-\infty}^{\infty} e^{i(\xi r - \xi^{-1} s)} \begin{pmatrix} \frac{-1}{\xi} \\ 1 \end{pmatrix} v(\xi) \frac{d\xi}{2\pi}$$

one function on the line

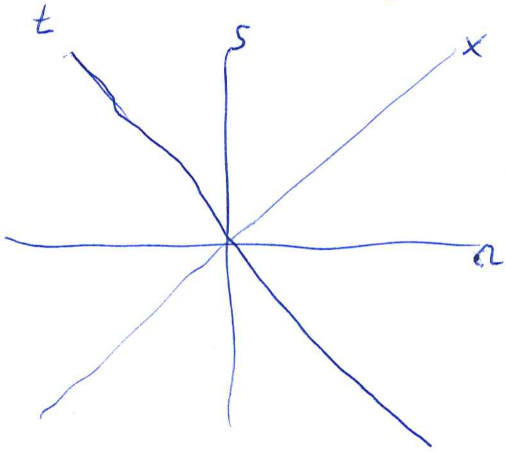
IAS 10

$\partial_t \psi = \begin{pmatrix}\partial_x & im\\ im & -\partial_x\end{pmatrix}\psi$ Fourier transform in x

$\partial_t \hat\psi = i\begin{bmatrix}k & m\\ m & -k\end{bmatrix}\hat\psi$ $\hat\psi(t,k) = \exp\left(i\begin{pmatrix}k & m\\ m & -k\end{pmatrix}t\right)\hat\psi(0,k)$

general solution

where $\hat\psi(0,k)$ is a pair of functions of k. Energy norm $\|\psi\|^2 = \int \psi^*\psi\,dx = \int \hat\psi^*\hat\psi\,\frac{dk}{2\pi}$.



$x = r+s$
$t = -r+s$

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}1 + \frac{\partial f}{\partial t}(-1)$

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}1 + \frac{\partial f}{\partial t}1$

$a\ -\partial_r\psi^1 = (\partial_t - \partial_x)\psi^1 = i\psi^2$

$\partial_s\psi^2 = (\partial_t + \partial_x)\psi^2 = i\psi^1$

Look for exp. solutions $e^{i(\rho r+\sigma s)}\begin{pmatrix}u\\ v\end{pmatrix}$

F.T. equations $\left.\begin{aligned}-\rho u &= v\\ \sigma v &= u\end{aligned}\right\} \Rightarrow (1-\rho\sigma)v = 0$

Work in $\mathbb{P}^2(\mathbb{R})$? You have a curve, a conic section, a line bundle over it.

Apparently

$\begin{matrix}\omega+k & -1\\ 1 & \omega+k\end{matrix}$

$\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \omega\begin{pmatrix}u\\ v\end{pmatrix}$

$(\omega-k)u = v$
$(\omega+k)v = u$

$k\omega + \frac{k^2+1}{\omega^2}$ $-k\omega + \frac{k^2+1}{\omega^2}$

$\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}\omega+k\\ 1\end{pmatrix},\ \begin{pmatrix}-\omega+k\\ 1\end{pmatrix}$

$$\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix}\begin{pmatrix}\omega+k & -\omega+k\\ 1 & 1\end{pmatrix} = \begin{pmatrix}\omega+k & -\omega+k\\ 1 & 1\end{pmatrix}\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix}$$

$$\begin{pmatrix}k & 1\\ 1 & -k\end{pmatrix} = \begin{pmatrix}\omega+k & -\omega+k\\ 1 & 1\end{pmatrix}\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix}\begin{pmatrix}1 & \omega-k\\ -1 & \omega+k\end{pmatrix}\frac{1}{2\omega}$$

$$\exp\left(it\begin{pmatrix}k&1\\1&-k\end{pmatrix}\right) = \frac{1}{2\omega}\begin{pmatrix}\omega+k & -\omega+k\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{it\omega}&0\\0&e^{-it\omega}\end{pmatrix}\begin{pmatrix}1&\omega-k\\-1&\omega+k\end{pmatrix}$$

here $\omega = \omega_k = +\sqrt{1+k^2}$.

$$K(t,x) = \int \frac{dk}{2\pi}\begin{pmatrix}\omega+k & -\omega+k\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{i(\omega t+kx)}&0\\0&e^{i(-\omega t+kx)}\end{pmatrix}\begin{pmatrix}1&\omega-k\\-1&\omega+k\end{pmatrix}$$

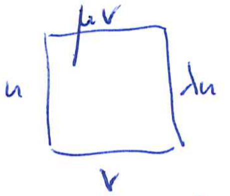
should be the solution of D.E. with $K(0,x) = \delta(x)\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

Discuss philosophy. The 2 to 1 problem you are wrestling with occurs already in the discrete case, namely, with the two orthonormal bases for the grid Hilbert space — the horizontal line, the staircase corresp to $t=0$.

Review situation. $\partial_t\psi = \begin{pmatrix}\partial_x & im\\ im & -\partial_x\end{pmatrix}\psi$ | $\partial_t\hat\psi = i\begin{pmatrix}k&m\\m&-k\end{pmatrix}\hat\psi$

$\hat\psi(t,k) = \exp\left(i\begin{pmatrix}k&m\\m&-k\end{pmatrix}t\right)\hat\psi(0,k)$. Question. Is there a Green's function in the hyperbolic case — of course, it should be a fundamental solution, but what are the boundary conditions? For $\partial_t - A$, Green's functions are $e^{tA}(H(t)+\text{const})$. ~~for what is it the On~~ constant means matrix, e.g. Feynman type bdry conditions.

Focus on what to do. Go back to the discrete case



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1&h\\ \bar h&1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$$

$(k\lambda-1)u = hv$
$(k\mu-1)v = \bar h u$

$$\mu = \frac{1}{k}\left(1+\frac{1-k^2}{kz-1}\right) = \frac{z-k}{kz-1}$$

$$\psi_{mn} = \int z^m\left(\frac{z-k}{kz-1}\right)^n\begin{pmatrix}\frac{h}{k\lambda-1}\\1\end{pmatrix} f(z)\,\frac{dz}{2\pi i z}$$

orth basis $\{(\lambda\mu)^p v,\ (\lambda\mu)^p\lambda u\}$ or $\{(\lambda\mu)^p u,\ (\lambda\mu)^p\mu v\}$

also an alg basis for the grid space

IAS 12

$z\left(\frac{z-k}{kz-1}\right)$ two k's better $z\left(\frac{z-k}{kz-1}\right)$

$z = e^{i\xi}$? Review the continuous limit.

$$\lambda^m\mu^n = (z)^{\frac{m}{\varepsilon}}\left(\frac{z^\varepsilon - k}{kz^\varepsilon - 1}\right)^{\frac{n}{\varepsilon}} \quad ?$$

λ^m replaced by $\lambda^x = (\lambda^\varepsilon)^{\frac{x}{\varepsilon}}$ $\qquad \lambda^\varepsilon = e^{i\xi\varepsilon}$

μ^n —— $\mu^y = \mu_\varepsilon^{\frac{y}{\varepsilon}}$ $\qquad \mu_\varepsilon = \frac{\lambda^\varepsilon - k_\varepsilon}{k_\varepsilon\lambda^\varepsilon - 1}$

$$(\mu_\varepsilon)^{\frac{1}{\varepsilon}} = \left(\frac{e^{i\xi\varepsilon} - (1-\varepsilon^2|h|^2)^{1/2}}{(1-\varepsilon^2|h|^2)^{1/2}e^{i\xi\varepsilon} - 1}\right)^{\frac{1}{\varepsilon}}$$

$$= \left(\frac{\cancel{1} + i\xi\varepsilon - \frac{\xi^2}{2}\varepsilon^2 - \cancel{1} + \frac{\varepsilon^2|h|^2}{2}}{\cancel{1} + i\xi\varepsilon - \frac{\xi^2}{2}\varepsilon^2 - \frac{|h|^2}{2}\varepsilon^2 - \cancel{1}}\right)^{1/\varepsilon}$$

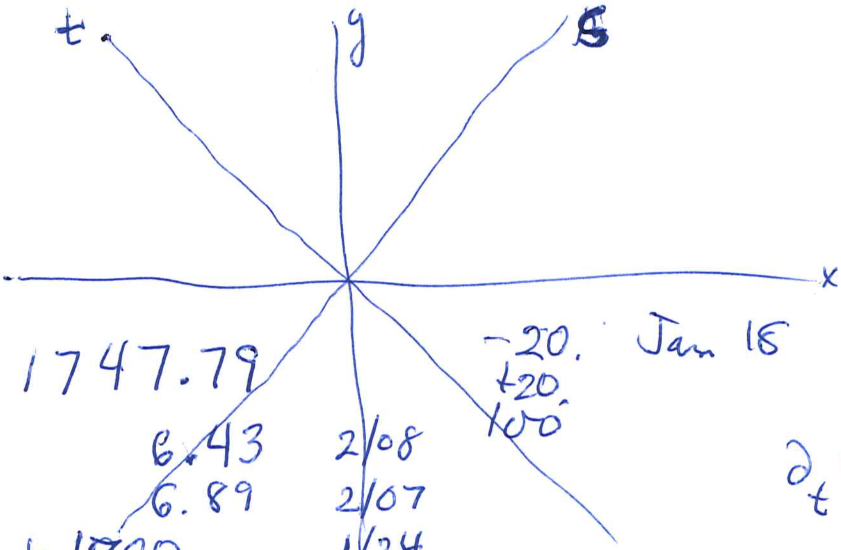
$$= \left(\frac{1 - \frac{\xi\varepsilon}{2i} + \frac{|h|^2}{2i\xi}\varepsilon}{1 - \frac{\xi\varepsilon}{2i} - \frac{|h|^2}{2i\xi}\varepsilon}\right)^{1/\varepsilon} \rightarrow$$

$$\rightarrow e^{-\frac{\xi}{2i} + \frac{|h|^2}{2i\xi}} \Big/ e^{-\frac{\xi}{2i} - \frac{|h|^2}{2i\xi}} = e^{\frac{|h|^2}{i\xi}}$$

so $(\lambda\mu)^x = (\lambda^\varepsilon\mu_\varepsilon)^{\frac{x}{\varepsilon}} \longrightarrow e^{\left(i\xi + \frac{|h|^2}{i\xi}\right)x}$

$$(v|\lambda^{m+1}\mu^m u) = \int z^{m+1}\left(\frac{z-k}{kz-1}\right)^m \frac{dz}{2\pi i z} = 0 \qquad m \geq 0$$

$$(v|\lambda^m\mu^m v) = \int z^m\left(\frac{z-k}{kz-1}\right)^m \frac{dz}{2\pi i z} = 0 \qquad m > 0$$



IAS 13

$s = x + y$

$t = -x + y$

$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\partial_x = \partial_s - \partial_t$

$\partial_y = \partial_s + \partial_t$

−20. Jan 18
+20.
100

1747.79

6.43 2/08
6.89 2/07
+1700. 1/24
+160
316

$$\partial_t \psi = \begin{pmatrix} \partial_s & im \\ im & -\partial_s \end{pmatrix} \psi$$

$(\partial_t - \partial_s)\psi^1 = im\psi^2$

$(\partial_t + \partial_s)\psi^2 = im\psi^1$

$$\boxed{\begin{aligned} -\partial_x \psi^1 &= im\psi^2 \\ \partial_y \psi^2 &= im\psi^2 \end{aligned}}$$

m=1: $-\xi\hat{\psi}^1 = \hat{\psi}^2$

$\eta\hat{\psi}^2 = \hat{\psi}^1$

Picture of Hilbert space as $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$,
translation operators $\lambda^x = e^{ix\xi}$ $\mu^y = e^{iy\eta} = e^{-iy\xi^{-1}}$

1747.79
13.32
1761.11

6.43
6.89
9.85
4.45
27.62

1747.79
14.30
1733.49

13.90
4.30
2.96
.35
21.51

	88.00
Cap + Cross	2.26
lunch San	90.26
phone	21.51
	111.77

$L^2(\mathbb{R}, \frac{d\xi}{2\pi})$ "horizontal" picture of the grid Hilbert space
finite energy solutions of the wave equation should be linear functionals on this Hilbert space. How?
The universal solution should be $\psi_{xy} = \lambda^x \mu^y \begin{pmatrix} u \\ v \end{pmatrix}$
$= e^{i(\xi x - \xi^{-1} y)} \begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix}$, this is not in the grid Hilbert space, but presumably should yield solutions of the wave equation when a sufficiently nice linear functional is applied.

$\xi x - \xi^{-1} y = (\xi \;\; -\xi^{-1}) \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \frac{s-t}{2} \\ \frac{s+t}{2} \end{pmatrix}$$

$$\xi x - \xi^{-1} y = \frac{1}{2}\begin{pmatrix} \xi & -\xi^{-1} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\xi-\xi^{-1}}{2} & -\frac{\xi+\xi^{-1}}{2} \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}$$

$$e^{i(\xi x - \xi^{-1} y)} = e^{i\left(s\frac{\xi-\xi^{-1}}{2} - t\frac{\xi+\xi^{-1}}{2}\right)}$$

$$\Psi(t,s) = \int \frac{d\xi}{2\pi}\, e^{i\left(s\frac{\xi-\xi^{-1}}{2} - t\frac{\xi+\xi^{-1}}{2}\right)} \begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix} f(\xi)$$

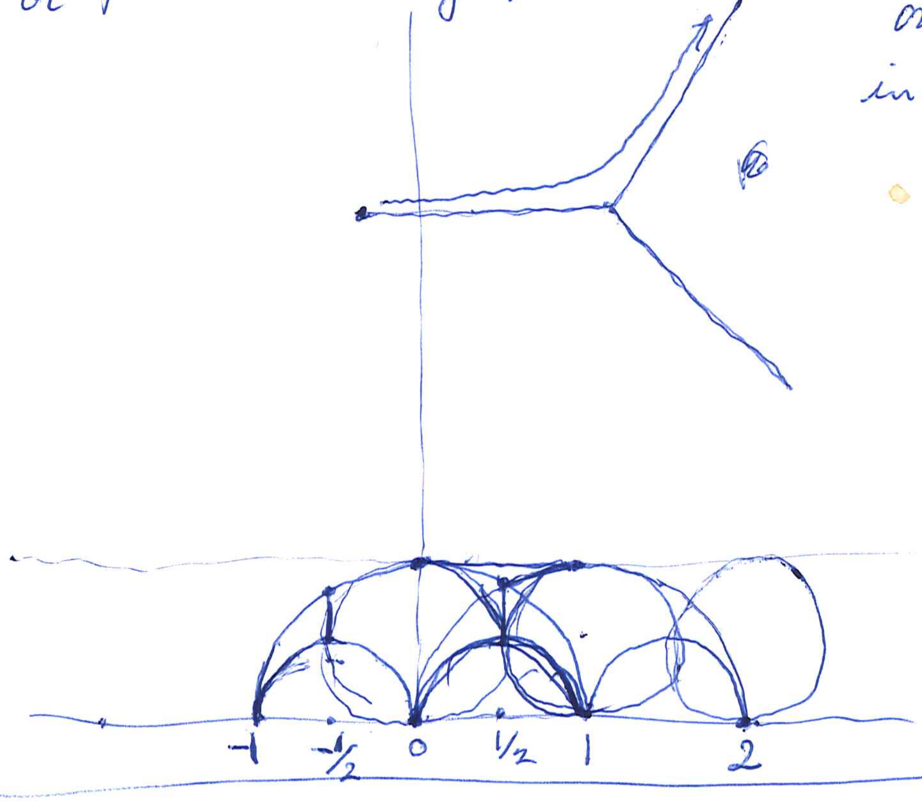
This should be a finite energy solution of the wave equation $\partial_t \Psi = \begin{pmatrix} \partial_s & i \\ i & -\partial_s \end{pmatrix}\Psi$. Now take $t=0$.

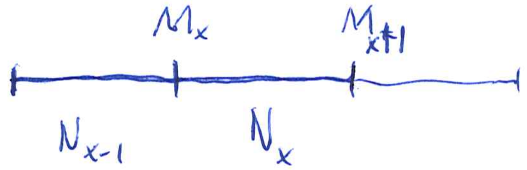
$$\Psi(0,s) = \int \frac{d\xi}{2\pi}\, e^{is\frac{\xi-\xi^{-1}}{2}} \begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix} f(\xi)$$

You see now how you get two functions of k from $k = \frac{\xi-\xi^{-1}}{2}$ because this mapping maps both $\{\xi>0\}$ and $\{\xi<0\}$ isom. to $\mathbb{R}$.

Look at $PSL(2,\mathbb{Z})$ tree. Algebraic study (based on Waldhausen): Consider local system on the tree – when acyclic, get Waldhausen filtration whose quotients are obviously acyclic. Get Nil-K stuff. Because algebraically acyclic get a nilpotent operator. Is there a Hilbert space analog? Vaguely remember no reflection in the algebraic case.

Is there some ~~[illegible]~~ analog of grid space for the $PSL_2(\mathbb{Z})$ tree? Yes – you have ~~[illegible]~~ constructed something which leads to ~~[illegible]~~ a cubic curve, ~~[illegible]~~ involving the ribbon graph for the $PSL_2(\mathbb{Z})$ tree, edge ~~becomes a pair~~ has 2 sides corresponding to the 2 orientations of the edge, there seems to be a flow on the edges of the ribbon graph whose orbits are $\mathbb{Z}$-graphs. one for each point in $\mathbb{Q} \cup \{\infty\}$.

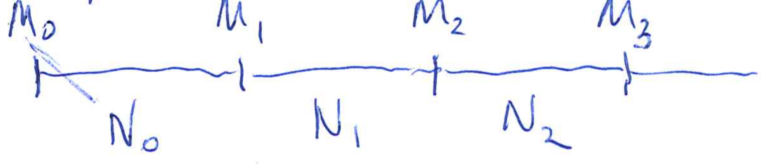




$$\cdots M_{x-1} \oplus M_x^- \qquad M_x^+ \oplus M_{x+1} \oplus M_{x+2} \oplus \cdots$$

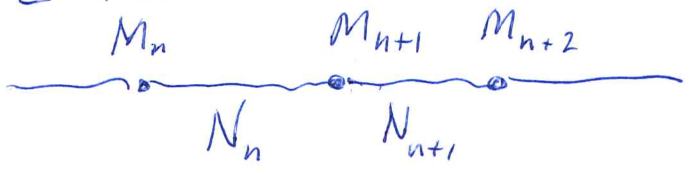
$$\cdots \oplus N_{x-1} \qquad N_x \oplus N_{x+1} \oplus N_{x+2}$$

1st example $\mathbb{N}$ tree.

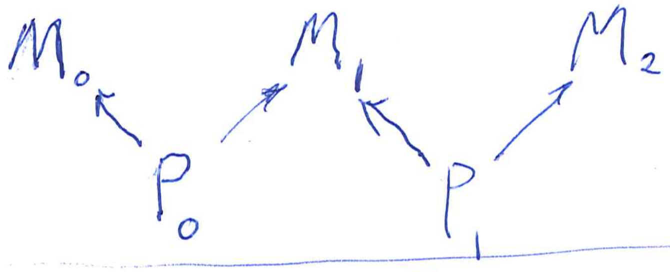


$$\begin{array}{c} M[z] \\ \uparrow {\scriptstyle az-b} \\ N[z] \end{array}$$

Review carefully an acyclic coeff. system on the $\mathbb{Z}$-tree

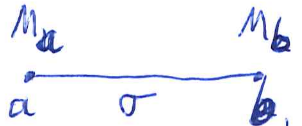


You want to analyze an acyclic coeff system on a tree.



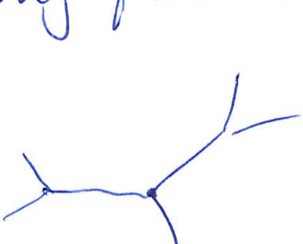
Let M_v be the system^F at the vertex v.
Issuing from v are branches b, ~~around v~~

$$C_1(F) = \bigoplus_{\text{edges } \sigma} P_\sigma \xrightarrow[\sim]{\partial} \bigoplus_v M_v$$



What you need is to write what occurs to you. The first step is to ~~look at~~ pick a vertex v, look at $\partial^{-1}(M_v)$
$= Z_1(T, T-\{v\})$ = 1 chains whose ∂ has support v.

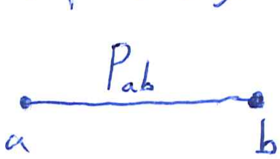
But " splits as a direct sum over the branches β issuing from v.
$$M_v \rightleftarrows \bigoplus Z_1(\beta, \beta-\{v\})$$



So $M_v = \bigoplus_\beta M_v^\beta$ ~~branch issuing~~

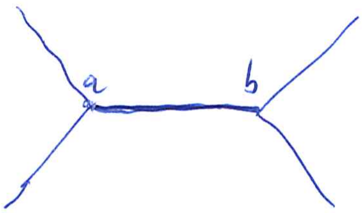
where $M_v^\beta = Z_1(\beta, \beta-\{v\})$.

Now take an edge



P_{ab} is a summand of $C_1(T)$ hence is mapped isom ~~by~~ ∂ to a summand of $C_0(T)$ contained in $M_a \oplus M_b$

Split $M_a = M_a^- \oplus M_a^{ab}$ $\quad M_b = M_b^{ab} \oplus M_b^+$



$$\begin{tikzcd} M_a \oplus M_b & Z_1(T, T-[a,b]) \supset P_{ab} \arrow[l, "\partial"', "\sim"] \arrow[d, "S"] \\ & M_a^{ab} \oplus M_b^{ab} \end{tikzcd}$$

1-chains which are cycles except at a, b
splits into 1-chains which are cycles except ~~at~~ at a
direct sum with ———————— b.
?

P_{ab} is included in $Z_1(T, T-a) \oplus Z_1(T, T-b)$

Almost correct. You should find that P_{ab} is isomorphic to $M_a^{ab} \oplus M_b^{ab}$

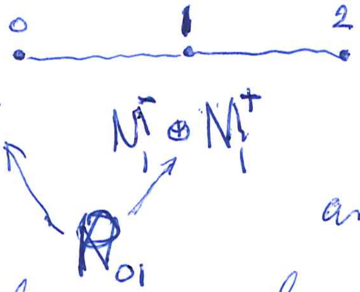
Try again You have a tree T with a coeff. system M, P which is acyclic: $C_1 = \bigoplus_{\text{edge}} P_\sigma \xrightarrow[\sim]{\partial} C_0 = \bigoplus_{\text{vertex}} M_v$

$\partial^{-1} M_v = \bigoplus_{\substack{\text{branches issuing}\\ \text{from } v}} Z_1(\beta, \beta - \{v\})$

$M_a \oplus M_b \xleftarrow{\sim}$ 1-chains which are cycles off a, b $= P_{ab} \oplus \bigoplus_{\substack{\beta \text{ issuing from } a\\ \beta \neq ab}} Z_1(\beta, \beta - a) \oplus \bigoplus_{\substack{\beta \text{ iss. from } b\\ \beta \neq ab}} Z_1(\beta, \beta - b)$

so you get $P_{ab} \xrightarrow{\sim} M_a^{ab} \oplus M_b^{ab}$

Try $\mathbb{N}$ tree



What seems to happen for the $\mathbb{Z}$-tree is that each M_a and $P_{a,a+1}$ splits into $\pm$ parts and roughly we have isos. $M_a^+ \xleftarrow{\sim} P_{a,a+1}^+$, $P_{a,a+1}^- \to M_{a+1}^-$

free situation then will involve a direct sum situation. Need examples to get feeling, intuition.

One dimensional. What about $az - b : X \otimes \Lambda \xrightarrow{\sim} Y \otimes \Lambda$

$\Lambda = \mathbb{C}[z, z^{-1}]$. This is all related to Kronecker modules.

Try for simply "$\mathbb{Z}$" examples, want $\mathbb{Z}$ to act. $M_0 = M_0^- \oplus M_0^+$ two lines, P_{01} should also be two lines. Simplest case is where there is no linking between $\pm$

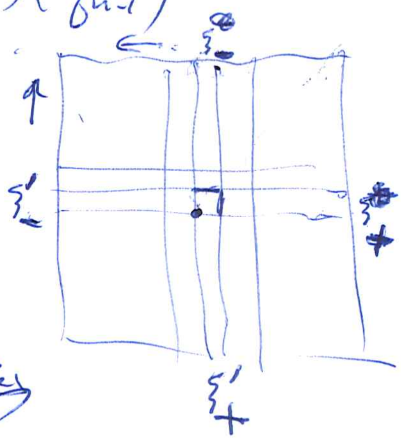
$$\begin{tikzcd} M_0 \arrow[rr, "\sim"', "z"] & & M_1 = zM_0 \\ & P_{01} \arrow[ul, "-b"] \arrow[ur, "za"'] & \end{tikzcd}
\qquad
\begin{tikzcd} M_0[z,z^{-1}] \\ P_{01}[z,z^{-1}] \arrow[u, "za-b"'] \end{tikzcd}$$

where $a,b : P_{01} \to M_0$

---

ideas for Monday's lecture $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$. Consider associated grid space E.

$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$



$$\sum |h_n|^2 < \infty \iff \lim_{n \to \pm\infty} \begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} \ \exists$$

to show $u^n \xi_\pm$, $u^n \xi'_\pm$ orth. sets.

$$\sum |h_n| < \infty \implies$$

~~\$~~ 2.57 coins	90.26	coffee Newark Airport
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Ideas for future − take $\mathbb{Z}$-tree

a translation equivariant finite diml coefficient system is the same thing as a Kronecker module, which yields a 1 diml chain complex of "vector bundles" over RS pure of slopes −1, 0. The chain complex associated to the coeff sys is the complex of finite free $\mathbb{C}[z,z^{-1}]$ modules − these vector bundle rest. to $\mathbb{C}^\times$. Acyclicity means the canon resolution of a torsion sheaf supported at $0, \infty$.

Q: Can you get an L^2 version?

Notice that an equivariant coeff system yields two K-modules + small one

$$\begin{tikzcd} P_{01} \arrow[r, "a"] \arrow[dr, "b"'] & M_0 \arrow[d, "z"] \\ & M_1 \end{tikzcd}$$

and large one $C_1 = \bigoplus_n P_{n,n+1} \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \bigoplus_n M_n = C_0$. Actually you seem to get a K-module from an oriented simplicial 1-complex.

Observe that a coefficient system on an oriented graph yields a K-module.

Study carefully an equiv. coeff system on the $\mathbb{Z}$-graph. The Bass fund. thm. You want to understand clearly the algebraic situation, then look for Hilbert versions.

$C_0 = \bigoplus_{n\in\mathbb{Z}} u^n M_0$ $\qquad C_1 \simeq \bigoplus_{n\in\mathbb{Z}} u^n P_{01}$ $\qquad a, b : P_{01} \to M_0$

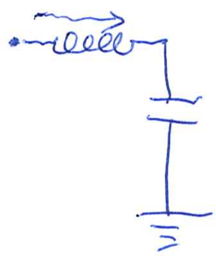
$$\begin{tikzcd} M_0 \arrow[rr] & & uM_0 \\ & P_{01} \arrow[ul, "b"] \arrow[ur, "ua"'] & \end{tikzcd}$$

$\partial = ua - b : C_1 \to C_0$

assume $\partial = ua - b$ invertible

The real question is whether there can be any reflection. It seems that you get a canonical splitting of the K-module into left and right movers. Clear from Bott's residue $\int (az-b)^{-1} a\,dz$. But you believe these are examples with reflection. Sort this puzzle out.

Segments of transmission line for the edges, connected with reflection at the vertices. Review transmission line.



$$E_x - E_{x+dx} \simeq (l\,dx)(+\dot{I})$$

$$I_x - I_{x+dx} = c\,dx(\dot{E})$$

$$-\partial_x E = l\dot{I} \qquad -\partial_x I = c\dot{E}$$

$$(-\partial_x)^2 E = (-\partial_x)(l\dot{I}) = lc\,\ddot{E}$$

$$(\partial_x + \partial_t)(E+I) = 0 \qquad E+I = Ae^{\lambda(x-t)}$$

$$(\partial_x - \partial_t)(E-I) = 0 \qquad E-I = Be^{\lambda(x+t)}$$

junction.

I think you should first get the algebra straight. Start with acyclic coeff system

$$C_0 = \bigoplus_{n\in\mathbb{Z}} M_n \qquad C_1 = \bigoplus_{n\in\mathbb{Z}} P_{n,n+1}$$

$$\partial = \partial_0 - \partial_1 : C_1 \xrightarrow{\sim} C_0$$

$$\begin{tikzcd}[column sep=small]
C_0: & \cdots \oplus & M_n & \oplus & M_{n+1} & \oplus \cdots \\
C_1 \arrow[u, "S"] & P_{n-1,n} \arrow[ur] \arrow[u] & & P_{n,n+1} \arrow[ul, "\partial_1"'] \arrow[ur, "\partial_0"] & & P_{n+1,n+2} \arrow[ul] \arrow[ur]
\end{tikzcd}$$

$$M_{-1} \oplus M_0 \oplus M_1$$

$$+\; P_{-2,-1} + P_{-1,0} \oplus \operatorname{Ker}\partial_0(P_{01})$$

Improve notation.

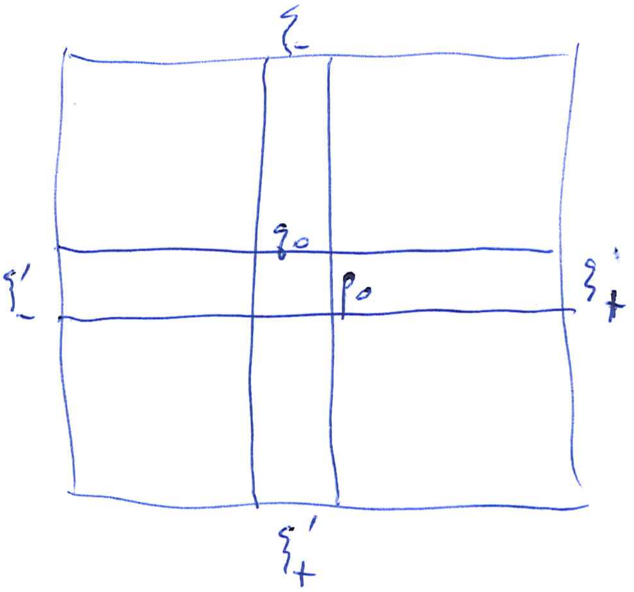
$$C_0 = \cdots \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus \cdots$$

$$C_1 = \cdots \oplus P_{-1} \oplus P_0 \oplus P_1 \oplus \cdots$$

$$C_1^{\leq 0} = \cdots \oplus P_{-2} \oplus P_{-1} \oplus P_0 \qquad C_0^{\leq 0} = \cdots \oplus M_{-1} \oplus M_0$$

So $\partial : C_1^{\leq 0} \hookrightarrow C_0^{\leq 1}$

You have $\partial^{-1} M_1$ ~~[illegible]~~



analysis end.

~~[illegible]~~ ~~Cauchy sequence~~

$$\lim_{n\to\infty} \prod_{j=1}^{n} g_j \ \exists ?$$

$\cancel{\prod_{j=1}^{n} g_j}$

$$\| g_1 \cdots g_m - g_1 \cdots g_m g_{m+1} \cdots g_{m+k} \|$$

$$= \| g_1 \cdots g_m (1 - g_{m+1} \cdots g_{m+k}) \|$$

$$\leq \|g_1\| \cdots \|g_m\| \, \| 1 - g_{m+1} \cdots g_{m+k} \|$$

$\cancel{\| g_1 \cdots g_m - g_1 \cdots g_n \| = \| \sum g_1}$

$$1 - g_{m+1} \cdots g_{m+k} = 1$$

$\cancel{\prod_{j=1}^{n}(1+a_j)}$ $\quad g_1 \cdots g_n - 1 \quad$ $\delta(g_1 g_2) = g_1\,\delta g_2 + \delta g_1$

$$\cancel{= (g_1-1)g_2 \cdots g_n + g_1(g_2-1)g_3 \cdots g_n + \cdots + g_1 \cdots g_{n-1}(g_n-1)}$$

$$\cancel{g_1 g_2 - 1 = (g_1-1)g_2 + g_2 - 1}$$

$$\cancel{g_1 \cdots g_n - 1 = (g_1-1)g_2 \cdots g_n + g_2 - 1}$$

$$g_1 g_2 - 1 = g_1 - 1 + g_1(g_2-1)$$

$$g_1 g_2 \cdots g_n - 1 = g_1 - 1 + g_1(g_2-1) + g_1 g_2 (g_3-1) + \cdots + g_1 \cdots g_{n-1}(g_n-1)$$

$$\|g_1 \cdots g_n - 1\| \leq \|g_1 - 1\| + \|g_1\|\|g_2-1\| + \cdots + \|g_1\| \cdots \|g_{n-1}\|\,\|g_n - 1\|$$

$$g_j - 1 = a_j \qquad g_j = 1 + a_j$$

$$\underbrace{\leq \|a_1\| + (1+\|a_1\|)\|a_2\| + (1+\|a_1\|)(1+\|a_2\|)\|a_3\| + (1+\|a_1\|)\cdots(1+\|a_{n-1}\|)\|a_n\|}_{\prod_{i=1}^{n}(1+\|a_i\|) - 1}$$

$$\cancel{(1+x_1)\cdots(1+x_n)}$$

$$g_i = 1 + x_i \qquad x_i = g_i - 1.$$

$$g_1 \cdots g_n = x_1 g_2 \cdots g_n + g_2 \cdots g_n$$

$$= x_1 g_2 \cdots g_n +$$

$$\underline{g_1 \cdots g_{n-1} + g_1 \cdots g_{n-1} x_n}$$

$$g_1 \cdots g_{n-2} x_{n-1}$$

$$g_1 \cdots g_n = 1 + \sum_{i=1}^{n} g_1 \cdots g_{i-1} x_i$$

$$g_1 g_2 = g_1 + g_1 x_2$$
$$= 1 + x_1 + g_1 x_2$$

$$\|g_1 \cdots g_n - 1\| \leq 1 + \sum_{i=1}^{n} \|g_1\| \cdots \|g_{i-1}\|\,\|x_i\| = \|g_1\| \cdots \|g_n\| - 1$$

$g_i = 1 + x_i$

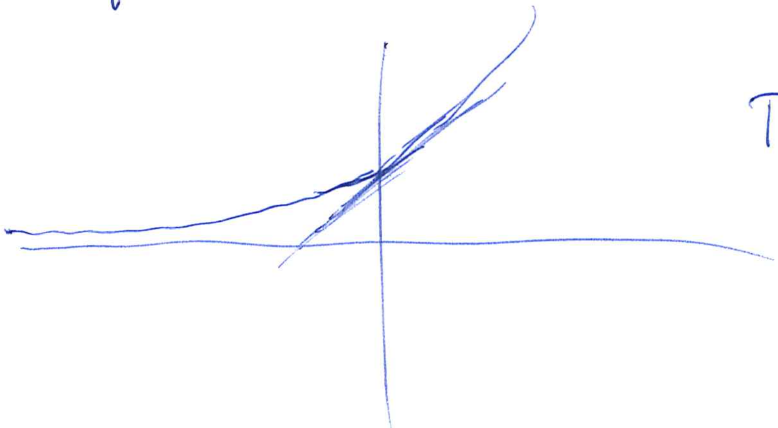
$$g_1 \cdots g_n = (1+x_1) \cdots (1+x_n)$$

$$= \sum_{I \subset \{1,\dots,n\}} x_I \qquad I = i_1 < \dots < i_p$$

$$\|x\|_I = \|x_{i_1}\| \cdots \|x_{i_p}\|$$

$$\| g_1 \cdots g_n - 1 \| \leq \sum_{I \neq \phi} \|x_I\| \leq \sum_{I \neq \phi} \|x\|_I = \prod (1 + \|x_i\|) - 1.$$

But for real numbers $\quad 1 + x \leq e^x$



$$\prod (1 + \|x_i\|) \leq e^{\sum \|x_i\|}$$

back to $\mathbb{Z}$-tree |

$z^2 V$

zW

$z V$

W

V

$z^{-1} W$

$z^{-1} V$

Your aim should be to ~~fytwhd~~ see whether acyclic coefficient systems produce interesting Hilbert space examples.

$z^2 V$

zW $\qquad$ zV

W $\qquad$ V

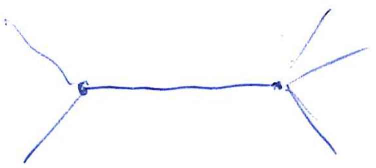
$z^{-1}W$ $\qquad$ $z^{-1}V$

Instead of acyclic systems over $\mathbb{C}[z, z^{-1}]$, consider those over the unit circle, i.e. you want $az-b$ to be invertible for $z \in S^1$. Let's see if we can produce some interesting examples, finite diml. W, V finite diml same dimension $az-b : W \longrightarrow V$ ~~[illegible]~~ an isom for $z \in S^1$. Projectors $P_\pm$ arising from residue.

$V = V_+ \oplus V_-$ $\qquad$ $V_+ = P_+ V$ $\qquad$ $P_+ = \oint \frac{1}{2\pi i} \, d(az-b)/(az-b)$

Things are going to split.

Today will be one of interruption



key point is that at a vertex v the system M_v splits according to the edges e issuing from v, for each $\quad v \quad w \quad x$

parametrix idea.

notation $$M_a = \bigoplus_{b \in \mathrm{Star}(a)} M_{ab}$$

$$\partial^{-1} M_{ab} = \{ \alpha \in C_1(B_{ab}) \mid \partial\alpha \in M_a \}$$

$$C_1(T) = \bigoplus_{b \in \mathrm{Star}(a)} C_1(B_{ab})$$

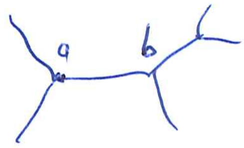
$$Z_1(B_{ab}, B_{ab} - a)$$

$$\partial^{-1} M_a = Z_1(T, T - a) = \bigoplus_{b \in St(a)} Z_1(B_{ab}, B_{ab} - a)$$

$$C_1(T-a) \qquad C_1(T) \qquad C_1(T, T-a) \qquad ?$$

I think I now understand the decomposition. Notation T for tree, ab for oriented edge B_{ab} for the branch of vertices c such that the path from a to c contains b.

$$C_1(T) = \bigoplus_{b \in St(a)} C_1(B_{ab})$$



$$\begin{array}{c} C_1(T) \\ \partial \big\downarrow S \\ 0 \longrightarrow M_a \longrightarrow C_0(T) \longrightarrow C_0(T)/M_a \longrightarrow 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \parallel \end{array}$$

Go over what you can do: Homogeneous $\mathbb{Z}$-tree

First of all you can consider the $\mathbb{Z}$-tree ?

Let's ~~go over~~ review the electrical ~~analogy~~ picture, unit segments of transmission line coupled together. What I would like is some analogy, link, between the acyclic coeff system on the trees and ~~the end~~ some kind of spectral splitting.

You want to link acyclic coeff systems on a tree to Hilbert space, in fact to the kind of Hilbert space with unitary operator you have constructed, you should probably work with the $\mathbb{Z}$-tree

discuss similarities, a coefficient system is a family of abelian gps (or vector spaces) M_n for each ~~vertex~~ $n \in \mathbb{Z}$, and $P_{n,n+1}$ for each edge together with boundary ops. $M_n \xleftarrow{\partial_1} P_{n,n+1} \xrightarrow{\partial_0} M_{n+1}$, ~~the~~ a discrete Dirac equation assigns a vector space V_n to each vertex n, and an isomorphism $V_n \xrightarrow{\sim} V_{n+1}$ for each edge $n,n+1$, possibly the ~~vector space~~ isomorphism depends on a spectral parameter z. a key feature is the asymptotics of a solution of the equation, this affects Green's fn.

First case to analyze: ~~[illegible]~~ Stationary, translation equivariant. You have a const coeff 1st order difference equation.

Coeff. system, acyclic, on the $\mathbb{Z}$-tree. For each vertex n you have $M_n = M_n^- \oplus M_n^+$

$$\begin{tikzcd} P^+_{n,n+1} \arrow[r, "d_e"] \arrow[rr, bend right, "\sim"'] & M_n \arrow[r] & M_n^+ \end{tikzcd}$$

$$P_{n,n+1} = P^+_{n,n+1} \oplus P^-_{n,n+1}$$

$$\begin{tikzcd} P^+_{n,n+1} \arrow[r, "d_r"] \arrow[rr, bend right, "=0"'] & M_{n+1} \arrow[r] & M^-_{n+1} \end{tikzcd}$$

where

$$\begin{tikzcd} P_{n,n+1} \arrow[r, "{d_e, d_r}"] \arrow[dr, "\sim"'] & M_n \oplus M_{n+1} \arrow[d] \\ & M_n^+ \oplus M_{n+1}^- \end{tikzcd}$$

so $P_{n,n+1}$ splits where

$$\begin{tikzcd} P^+_{n,n+1} \arrow[d, "\wr"] & \subset & P_{n,n+1} \arrow[d, "\wr"] & \supset & P^-_{n,n+1} \arrow[d, "\wr"] \\ M_n^+ & \hookrightarrow & M_n^+ \times M^-_{n+1} & \hookleftarrow & M^-_{n+1} \end{tikzcd}$$

Review ~~[illegible]~~ Acyclic Coeff sys. V_n $W_{n,n+1}$ on $\mathbb{Z}$ tree ~~acyclic~~ has polarization into $\pm$. ~~describe~~ + type means the ~~[illegible]~~ Greens operator ∂^{-1} is supported on ~~one side~~ the plus side. ~~Example~~ Picture ~~Graded~~

$$\begin{tikzcd} & V_n \\ W_n = V_n \arrow[ur, "1"] \arrow[dr, "-\nu"'] & \\ & V_{n+1} \end{tikzcd}$$

So you have a "graded" v.s. $V = \overset{(?)}{\bigoplus_{n \in \mathbb{Z}}} V_n$

and $W = V$ ~~and~~ $\partial^e = 1$ $\partial^r = \nu$.

$\partial = 1 - \nu$ where ν has degree 1

and $(1-\nu)^{-1}$ ~~[illegible]~~ is invertible.

equivariant case $V_n = z^n V_0$, ~~for~~ maybe different period, periodic case. ~~to~~

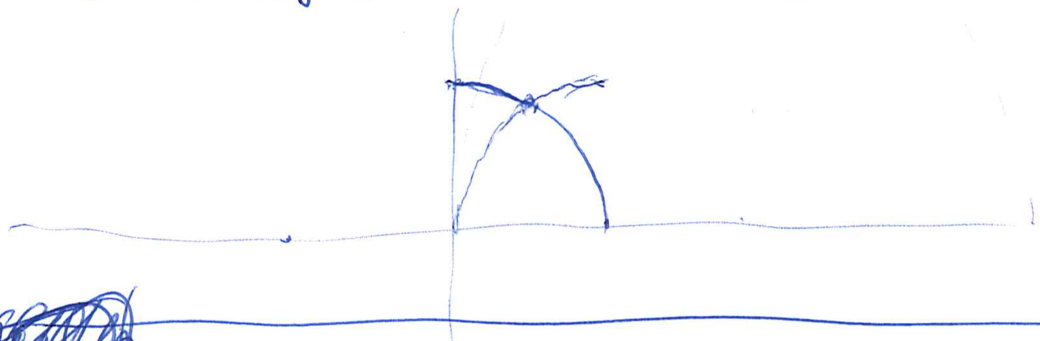
The next step? ~~[illegible]~~ Try to link ~~[illegible]~~ grid spaces to acyclic coeff sys, bifiltration?

Look at ~~the~~ a transmission line ~~equations~~ with jolts at each integer. ~~[illegible]~~ ~~[illegible]~~ Piecewise linear path, polygonal path in $SU(1,1)$. The problem, obstruction, difficulty involves $z^{1/2}$ & Hilbert space. Principle, insight should be that ~~there is a Hilb~~ a sequence of $U(1,1)$ matrices will give a Hilbert space. | No δ

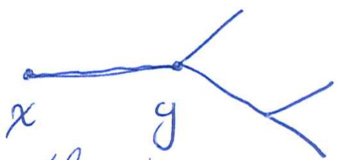
Look at constant ~~[illegible]~~ coeff cases, aim to bring in ~~[illegible]~~ $\{\pm 1\}$ in some way. ~~[illegible]~~

Points: An acyclic coeff system on a tree leads to a decomposition of ~~the~~ each vertex space V_x indexed by $st(x)$ and a comp. of each edge space $W_{\{x,y\}}$ ~~according to each~~ indexed by the vertices x, y. ~~[illegible]~~
Is there a global decomp? ~~[illegible]~~ each oriented edge (x,y) produces ~~[illegible]~~ $P^+_{(x,y)} \xrightarrow{\sim} M_{(x,y)}$

Look at $\mathbb{Z}$-tree again - you have V^+_n. ~~[illegible]~~
Use homotopy. ~~[illegible]~~ Simplest situation is where the ~~[illegible]~~ coeff system has $\nu = 0$, direct sum of $\oplus W^+_{n,n+1} \xrightarrow[d]{\sim} V^+_n$. Try for a filtration
You basically have for each oriented edge (x,y) ~~a subc~~ a subcomplex - the branch starting with ~~[illegible]~~ x then y. You have ~~an~~ partial ordering on oriented edges.

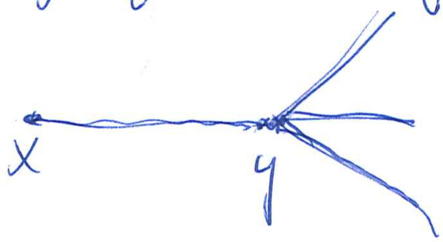


Go back to a general tree and acyclic coeff system. The upshot is a decomposition of C_0 indexed by oriented edges, other way to say it, a decomp. of each V_x into V_{xy}, $y \in St(x)$, and a corresp decomp of C_1. ~~opposite~~ oriented edge ess. same as a branch



partial order of oriented edges, there's a corresp. filtration. Acyclic subcomplex B_{xy} consisting of lifts of V_{zw}

for $y \le z < w$, $z \neq x$



So what do we have?

~~Next idea is whether this train track idea trees~~ Does an acyclic coeff sys. lead to a difference equation, recursion relation.

The idea is to ~~interpret a corresp.~~ link (tree + acyclic coeff system) to difference eqns + recursion relations. You start with the vector space V_n $n \in \mathbb{Z}$ and $W_{n,n+1} \xrightarrow{(d^2, d^1)} V_n \times V_{n+1}$ correspondences $\forall n$.

Simple Examples: Translation-invariant. all $V_n = V$ Correspondence given by an operator A on V. Propagation:

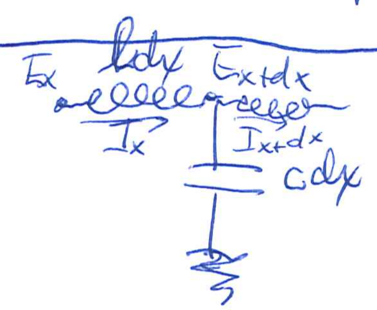
$$\xrightarrow{A} V \xrightarrow{A} V \xrightarrow{A}$$

~~Assume~~ This is a ~~standard diff~~ linear recursion relation $\psi_{n+1} = A\psi_n$. Euler method for solns. $\psi_n = \lambda^n v$ $\quad (\lambda - A)v = 0.$

V finite dim

Some ideas. If all $V_n = V$, then it makes sense to consider decaying solutions. You can consider $L^2(S^1, V)$ instead of $\mathbb{C}[z, z^{-1}] \otimes V$ and pose acyclicity. Invertibility of $z - A$ on $L^2(S^1, V)$ means spectrum of A is off S^1.

example



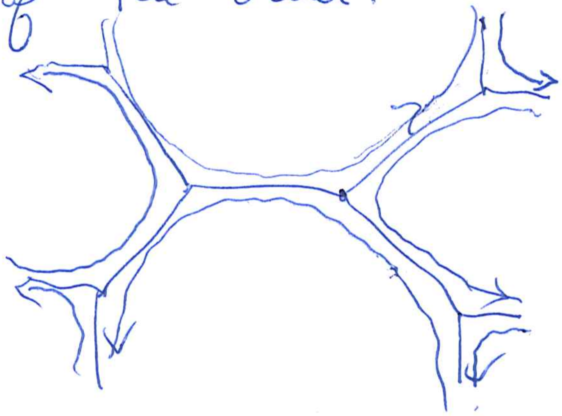
$$I_x - I_{x+dx} = c\,dx\, \partial_t E_x$$
$$E_x - E_{x+dx} = l\,dx\, \partial_t I_x$$
$$-\partial_x I = c\, \partial_t E \qquad \text{speed}$$
$$-\partial_x E = l\, \partial_t I \qquad = \frac{1}{\sqrt{lc}}$$

$$\partial_t E + \partial_x I = 0 \qquad (\partial_t + \partial_x)(E+I) = 0$$
$$\partial_t I + \partial_x E = 0 \qquad (\partial_t - \partial_x)(E-I) = 0$$

$$\begin{pmatrix} E+I \\ E-I \end{pmatrix} = \begin{pmatrix} Ae^{s(t-x)} \\ Be^{s(t+x)} \end{pmatrix}$$

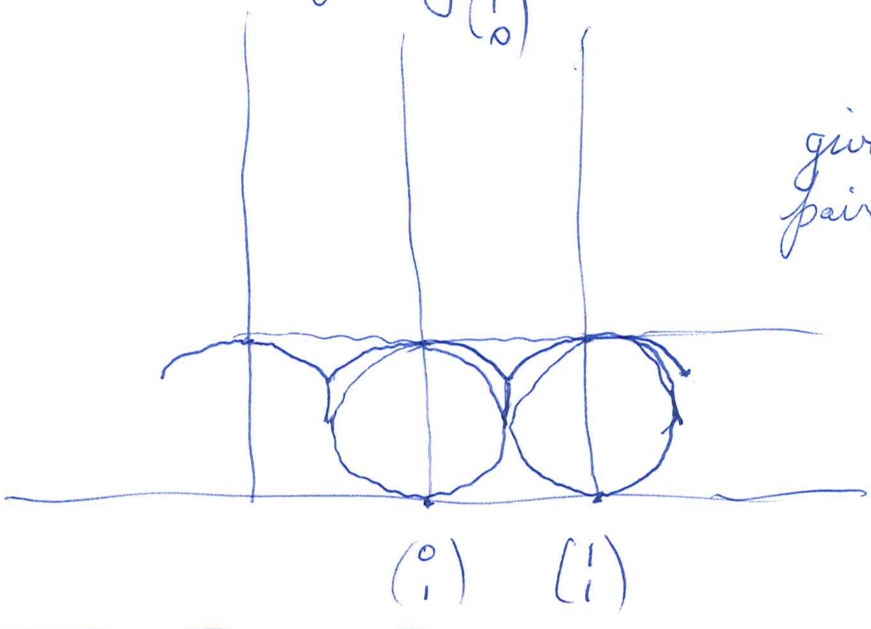
So what are you hoping to do? Make a link between $\mathbb{Z}$-tree and grid spaces.

Imagine a coeff system on the $PSL_2(\mathbb{Z})$ tree. There are 2 versions of this tree, one is the barycentric subdivision of the other.



trivalent graph ~~every~~ cyclic order at each vertex, ribbon graph

These circles hopefully corresp to rational nos. Clear from



Each 1-simplex gives an opposing pair of rational nos.

$$\begin{vmatrix} m & x \\ n & y \end{vmatrix} = \pm 1$$

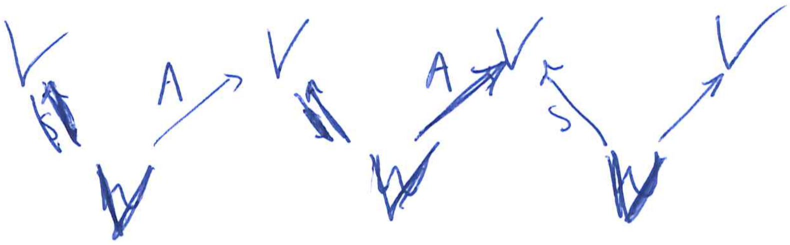
Look at the tree. ~~ribbon graph~~ Consider an acyclic coeff system. Do you allow subdivisions?

~~So what actually~~ ~~Where now~~

You have difficulty linking acyclic coeff sys. to recursion relation. Why? because transmission is nilpotent and there is no reflection. But the main idea of Green's function should work. Let's study the analogy on the $\mathbb{Z}$-tree. Direct product of chain groups? try on the $\mathbb{Z}$-tree.

$$\prod_n W_{n,n+1} \rightrightarrows \prod_n V_n$$

Ex. . .



Useful example. When is $1-A$ onto?

$$(1-zA)\sum_{n\in\mathbb{Z}} z^n v_n = \sum_n \left(z^n v_n - z^{n} A v_{n-1}\right)$$

$$= \sum_n z^n\left(v_n - A v_{n-1}\right)$$

If A is invertible, then you have a local system, which means for finite chains that $H_1 = 0$, $H_0 = V$ since the tree is contractible. Opposite for inf chains.

You should focus on Green's function, analogy with first order DE on the line.

$\mathbb{Z}$ tree - coeff syst is family of V_n together with ~~...~~ correspondences $W_{n,n+1} \to V_n$, $W_{n,n+1} \to V_{n+1}$

If acyclic, then it ~~splits...~~ has a polarization

$$+ \quad W^+_{n,n+1} \xrightarrow{\sim} V^+_n, \quad W^+_{n,n+1} \to V^+_{n+1}$$

So $C_0 = \bigoplus_{n\in\mathbb{Z}} V^+_n$

$C_1 \simeq \bigoplus_{n\in\mathbb{Z}} V^+_n$

and $\partial = 1 - \nu$ where ν is a degree $+1$ op on the graded v.s. $\oplus V^+_n$ such that $(1-\nu)$ is invertible. You get interesting variants by ~~use ... other~~ replacing $\bigoplus_n$ with TVS versions.

Homogeneous case: equivariant coeff system on the $\mathbb{Z}$ tree. $C_0 = \oplus z^n V$

$C_1 = \oplus z^n W$ $\qquad C_1 \xrightarrow{za-b} C_0$

$$W \xrightarrow{b} V, \quad W \xrightarrow{za} zV$$

Consider $PSL(2,\mathbb{Z})$ tree $\quad \Gamma = PSL(2,\mathbb{Z}) \simeq \mathbb{Z}/2 * \mathbb{Z}/3$

Γ acts simply transitively on oriented edges of T. So you can describe an acyclic equivariant coeff system as a vector space V together with two maps $V \underset{b}{\overset{a}{\rightrightarrows}} V$ satisfying a nilpotence condition $\forall v \ \exists N$ such that $(\mathbb{Z}a + \mathbb{Z}b)^N v = 0$.

So our branch has a tensor alg structure

Connection with O_2 Cuntz alg?

Consider the dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$ acting on the $\mathbb{Z}$-tree

the two reflections ~~???~~ being around 0 and $\frac{1}{2}$. What does an equivariant acyclic coeff ~~??~~ syst look like? decomp according to oriented edges. You get a vector space $V = V^+_0$ attached to the oriented edge and a map ~~?~~ $V \xrightarrow{\nu} V$ ~~which should must~~ ~~be nilpotent ???~~ satisfying $\forall v\ \exists N \ni \nu^N v = 0$

(Check this. Suppose $(1-z\nu)^{-1} v = \sum_{n\geq 0}^{N} z^n v_n$

$v_n = \nu v_{n-1} = \cdots = \nu^n v_0 \qquad v_0 = v$);

Now what do you need to do???

Basically you now have good control over acyclic coeff systems on trees, but you are missing how to link them to difference equations; the link should involve the Green's functions idea.

A simple example: Iterating an ~~correspondence~~ operator A

~~Here you have $W \underset{b}{\overset{a}{\rightrightarrows}} V$~~

$$\begin{tikzcd} {} \arrow[r] & V \arrow[r, "zA"] \arrow[d, equal] & zV \arrow[r, "zA"] \arrow[d, equal] & z^2V \\ & W \arrow[ur, "1-zA"'] & zV \arrow[ur, "1-zA"'] & \end{tikzcd}$$

You assume $(1-zA)^{-1}\ \exists$ on $V[z, z^{-1}]$. get splitting. The difference equation is also clear

$z^n \psi_n \in z^n V \qquad A\psi_n = \psi_{n+1}$.

~~It is possible~~

~~analytic~~

$$z^{-1}V \xrightarrow{zA} V \xrightarrow{zA} zV \quad ?$$

corresp.

$$\begin{tikzcd}[column sep=small]
z^{-1}V & & V & & zV & & z^2V \\
& z^{-1}W \arrow[ul, "-b"] \arrow[ur, "az"'] & & W \arrow[ul, "-b"] \arrow[ur, "az"'] & & zW \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

At the moment you have some control over acyclic coeff systems. Fix $\mathbb{Z}$ tree with translation action.

$$\begin{tikzcd}[column sep=small]
z^{-1}V & & V & & zV & & z^2V \\
& & & W \arrow[ul, "b"] \arrow[ur, "za"'] & & &
\end{tikzcd}$$

assume $\quad za - b : W[z, z^{-1}] \longrightarrow V[z, z^{-1}] \quad$ invertible

$$V^+ = \{ v \mid (za-b)^{-1} v \in W[z] \}$$

$$V^- = \{ v \mid (za-b)^{-1} v \in W[z^{-1}]z^{-1} \}$$

$$G : V \longrightarrow W[z, z^{-1}] \qquad G_n :$$

$$W[z,z^{-1}] = z^{-1}W[z^{-1}] \oplus W[z]$$

$$G = G^{<0} + G^{\geq 0} = \sum_{n \in \mathbb{Z}}{}' G_n z^n \qquad G_n : V \to W$$

$$1 = DG = DG^{<0} + DG^{\geq 0} = \underbrace{D\pi^{<0}D^{-1}}_{\text{projectors on } C_0} + D\pi^{\geq 0}D^{-1}$$

$$= G^{<0}D + G^{\geq 0}D$$

Ultimately you get two maps

$V \underset{G_{-1}}{\overset{G_0}{\rightrightarrows}} W$ $\qquad a, b \in \mathrm{Hom}(W, V)$

$\qquad\qquad\qquad c, d \in \mathrm{Hom}(V, W)$

$$\begin{tikzcd}[column sep=small]
z^{-1}V & & V & & zW \\
& z^{-1}W \arrow[ul, "b"] \arrow[ur, "za"] & \oplus & W \arrow[ul, "b"'] \arrow[ur, "za"'] &
\end{tikzcd}$$

$za - b$

Yes!!! OKAY.

Consider K-module $W \underset{b}{\overset{a}{\rightrightarrows}} V$ $\ni$ assoc. equivariant coefficient system is acyclic. $W[z, z^{-1}] \xrightarrow[\sim]{za-b} V[z, z^{-1}]$. Assertion: the K-module has canonical splitting into 2 types

$b_+ : W_+ \xrightarrow{\sim} V_+$ and $a_+ b_+^{-1}$ nilp on V_+

$a_- : W_- \xrightarrow{\sim} V_-$ and $b_- a_-^{-1}$ —— V_-.

$\exists$ Topological variant where instead of $\mathbb{C}[z, z^{-1}]$ you use $C(S^1)$. Atiyah-Bott residue gives splitting.

IDEA: Use $\mathbb{Z}/2 * \mathbb{Z}/2$ equivariance on the $\mathbb{Z}$-tree. This should amounts to $\mathbb{Z}/2\mathbb{Z}$ acting on the spaces W, V. Maybe you can bring in conjugation and relate to quaternions or $SU(1,1)$?

In the top variants the two cases are

right moving $\quad b : W \xrightarrow{\sim} V$ and ab^{-1} contraction

left —— $\quad a : W \xrightarrow{\sim} V$ and ba^{-1} ——.

e.g. if $za - b = z - T$ then you split

$V = W$ into eigenspace corresp to roots inside and outside S^1 (assume none on S^1). It look like you get some sort of L^2 Green's function.

$$\begin{tikzcd} V^+ \arrow[rr, bend left, "\text{nilp}"] & & zV^+ \\ & W^+ \arrow[ul, "\cong"] \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} z^{-1}V^- & & V^- \arrow[ll, bend right, "\text{nilp}"'] \\ & z^{-1}W^- \arrow[ul] \arrow[ur, "\sim"] & \end{tikzcd}$$

$$\begin{tikzcd} V^+ \oplus V^- & & zV^+ \oplus zV^- \\ & W^+ \oplus W^- \arrow[ul] \arrow[ur] & \end{tikzcd}$$

iso $\oplus$ nilp   nilp $\oplus$ iso

~~[illegible]~~ You still have the problem of reflection. Let's explore the Hilbert space aspects. This time take ~~a~~ a torsion K-module better, a K-module acyclic over S^1. So what do you have? A vector space V and a subspace W of $V \times V$ of the same dimension as V. There's an obvious spectrum here related to your Cayley transform paper! But it might not be suitable for eigenvalues of a correspondence (e.g. operator). Dihedral group. So what do I do?

First look carefully at an equivariant coeff system on the $\mathbb{Z}$-tree with dihedral group action.

$s_0(x) = -x$

$s_{\frac{1}{2}}(x) = 1 - x$

$s_{\frac{1}{2}} s_0(x) = 1 - (-x) = 1 + x$

$s_0 s_{\frac{1}{2}}(x) = -(1-x) = -1 + x$

What is an equiv. coeff sys.: Two classes of fixpts. $\mathbb{Z}$, $\mathbb{Z}+\frac{1}{2}$,

the group $\mathbb{Z}/2 * \mathbb{Z}/2$ has an tree assoc. to this amalgamated product rep.

---

group $\mathbb{Z}/2 \ltimes \mathbb{Z}$ $\quad C_0 = \mathbb{C}[z,z^{-1}]\otimes V$

$s(z^n v) = z^{-n} s(v)$. $\quad C_1 = \mathbb{C}[z,z^{-1}]z^{\frac{1}{2}}\otimes W$

$\partial = -z^{-\frac{1}{2}}b + z^{\frac{1}{2}}a : C_1 \longrightarrow C_0$.

$\cancel{s\partial s^{-1} = z^{\frac{1}{2}} sbs^{-1} + z^{-\frac{1}{2}} sas^{-1}}$

$\cancel{\partial(z^{\frac{1}{2}}\otimes w) = -1\otimes bw + z}$

$\partial(z^{\frac{1}{2}}w) = z^{\frac{1}{2}}\left(-z^{-\frac{1}{2}}b + z^{\frac{1}{2}}a\right)w = -bw + zaw$

$s\,\partial(z^{\frac{1}{2}}w) = -sbs^{-1}sw + z^{-1}sas^{-1}sw$

$s\partial z^{1/2}s^{-1}w = -sbs^{-1}w + z^{-1}sas^{-1}w$

$\therefore \quad \partial z^{\frac{1}{2}} = (-b+za) : W \longrightarrow V[z,z^{-1}]$

$s\partial z^{\frac{1}{2}}s^{-1} = -(sbs^{-1}) + z^{-1}(sas^{-1})$

---

so what to do? $\quad \mathbb{C}[z,z^{-1}]\otimes W \longrightarrow \mathbb{C}[z,z^{-1}]\otimes V$

$z^n w \longmapsto z^n(-bw + zaw)$

anyway $\Gamma = \mathbb{Z}\rtimes\mathbb{Z}/2$ $\quad$ subdivide

$\mathbb{C}[\Gamma]\otimes_s W$

$s(z^n w) = z^{1-n} sw$ $\quad$ s = reflection about $\frac{1}{2}$

$s(z^n v) = z^{1-n} sv$ $\quad$ screwy $\quad s(z^0 w) = z^1(sw)$

$\Gamma = \langle s_0, s_{\frac{1}{2}} \rangle$ $\qquad s_0 : n \longmapsto -n$

$\qquad\qquad\qquad\qquad s_{\frac{1}{2}} : n \longmapsto 1-n$

$$\mathbb{C}[\Gamma] \otimes_{s_{\frac{1}{2}}} W \qquad\qquad \mathbb{C}[\Gamma] \otimes_{s_0} V$$

embed W into $\mathbb{C}[\Gamma] \otimes_{s_0} V = \bigoplus z^n V$

$$s_{\frac{1}{2}} = z s_0$$

$$\begin{tikzcd}
w \arrow[r, mapsto] \arrow[d, "s_{\frac{1}{2}}"] & -bw + zaw \arrow[d, mapsto] \\
s_{\frac{1}{2}} w \arrow[r, mapsto] & -z s_0 b w + \underbrace{z s_0 (z a w)}_{z z^{-1} s_0 a w}
\end{tikzcd}$$

$$-z s_0 b w + z s_0(zaw) = s_0 a w - z s_0 b w$$

$$s_{\frac{1}{2}} w \longmapsto -b s_{\frac{1}{2}} w + z a s_{\frac{1}{2}} w$$

so $\quad s_0 a = -b s_{\frac{1}{2}}$

$\qquad -s_0 b = a s_{\frac{1}{2}}$

$$\boxed{\begin{aligned} s_0 a s_{\frac{1}{2}}^{-1} &= -b \\ s_0 b s_{\frac{1}{2}}^{-1} &= -a \end{aligned}}$$

What does this mean? Original idea ~~[illegible]~~ was an iterated correspondence. Wagner situation

Start again. A K-module is a self correspondence $W \rightrightarrows V$. Acyclicity $\Longrightarrow$ $\dim(W) = \dim(V)$ in fin. diml case

First idea today. Look at ~~[illegible]~~ $C_1 \to C_0$ as analog of grid space so that linear functions on it are "solutions" repeat simple ideas

$$\mathbb{C}[\mathbb{Z}]\otimes W \longrightarrow \mathbb{C}[\mathbb{Z}]\otimes V$$

$$w \longmapsto -bw + zaw$$

$$V \xleftarrow{b} W \xrightarrow{a} V$$

acyclic $\Longrightarrow \dim(W) = \dim(V)$.

Prop: $\exists$ ~~canonical~~ unique splitting

$$\begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix} \xleftarrow[b]{\text{iso} \,\oplus\, \text{nilp}} \begin{matrix} W^+ \\ \oplus \\ W^- \end{matrix} \xrightarrow[a]{\text{nilp} \,\oplus\, \text{iso}} \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$$

$SL(2,\mathbb{Z})$ tree. 2 dim simplicial complex, vertices are $\mathbb{P}^1\mathbb{Q} = \mathbb{P}^1\mathbb{Z} = \mathbb{Q}\cup\{\infty\} = \{\mathbb{Z}\binom{m}{n}\subset\mathbb{Z}^2 \ni m,n$ rel. prime$\}$. Two vertices $\mathbb{Z}\binom{m}{n}$, $\mathbb{Z}\binom{x}{y}$ form a 1-simplex $\Longleftrightarrow \begin{vmatrix} m & x \\ n & y \end{vmatrix} = \pm 1$, three dist.? vertices form a 2 simplex when each pair is a 1-simplex.

$$\left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 1 \qquad \begin{vmatrix} 1 & x \\ 0 & y \end{vmatrix} = y \;/\; \begin{vmatrix} 0 & x \\ 1 & y \end{vmatrix} = -x$$

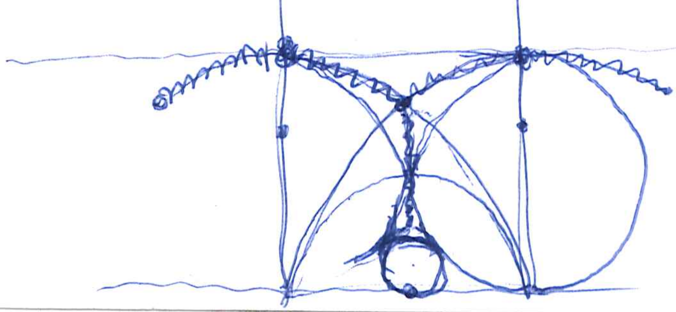
assume $\left(\binom{1}{0}, \binom{0}{1}, \binom{x}{y}\right)$ is a 2 simplex $|y| = |x| = 1$.

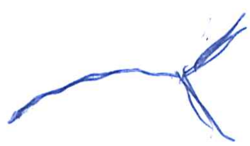
two poss. up to sign $\mathbb{Z}\binom{x}{y} = \mathbb{Z}\binom{1}{1}$ or $\mathbb{Z}\binom{-1}{1}$

transverse (extended) rational numbers.

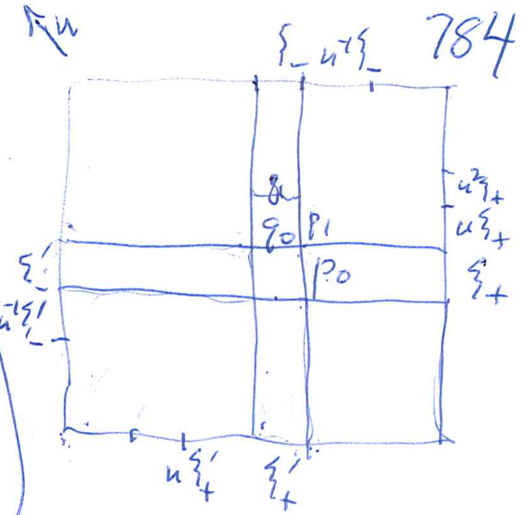
Tree = dual complex for the triangulation | ~~rational no.~~

A transverse pair of rational number





scattering again (Transfer matrix picture)



$$\begin{pmatrix} z^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \begin{pmatrix} z^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \underbrace{\begin{array}{cc} zH_- & H_+ \\ \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \\ zH_- & H_+ \end{array}} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \underbrace{\begin{array}{cc} H_+ & H_- \\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \\ zH_+ & zH_- \end{array}} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

cont. loop on S^1 values in $SU(1,1)$

$H_+ = \mathbb{C}[z] \subset L^2(S^1)$

$H_- = z^{-1}\mathbb{C}[z^{-1}]$

$zH_- = \mathbb{C}[z^{-1}] = H_+^*$

Q abs. convergent F.S.?

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \cancel{a^l} & \frac{b^l}{d} \\ \cancel{c^l} & \frac{d^l}{d} \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

$$\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ 0 & d \end{pmatrix}$$

$$\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \cancel{\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}} =$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \underbrace{\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a^l - \frac{b^l c}{d} & \frac{b^l}{d} \\ c^l - \frac{d^l c}{d} & \frac{d^l}{d} \end{pmatrix} = \frac{1}{d} \begin{pmatrix} a^l d - b^l c & b^l \\ c^l d - d^l c & d^l \end{pmatrix} = \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{a} \begin{pmatrix} \overset{a^l}{ad^r - cb^r} & -b^r \\ \underset{c^l}{-ac^r + ca^r} & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix}$$

$$\boxed{\begin{aligned} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} &= \begin{pmatrix} a^l_m & b^l_m \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \\ &= \frac{1}{d} \underbrace{\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}}_{\text{has det } = d} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a} \underbrace{\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}}_{\text{has det } = a} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} \end{aligned}}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} & H_+ \end{array} \\ \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \\ \begin{array}{cc} zH_- & \end{array} \end{array} \begin{pmatrix} \xi'_- \\ \xi_+ \end{pmatrix} = \begin{array}{c} \begin{array}{cc} & H_- \end{array} \\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \\ \begin{array}{cc} zH_+ & \end{array} \end{array} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \underbrace{\frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}}_{\in \begin{pmatrix} H_+ & H_+ \\ H_+ & H_+ \end{pmatrix}} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \underbrace{\frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}}_{\begin{pmatrix} zH_- & H_- \\ H_- & zH_- \end{pmatrix}} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

Check

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \underbrace{\begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}}_{\frac{1}{d}\begin{pmatrix} 1^\vee & b^\vee \\ -c & 1^\vee \end{pmatrix}} \frac{1}{d} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

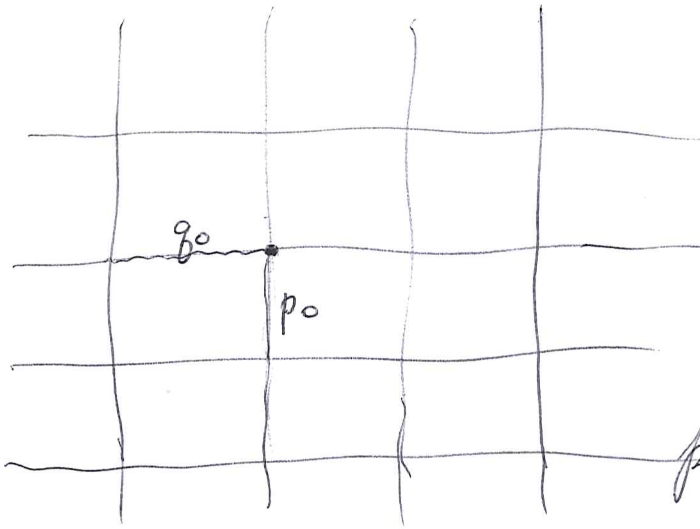
$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix}\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}$$

Understand inverse scattering again.
First explore idea of center. Basically ~~you want smaller~~ it shows that the $z^{1/2}$ shouldn't arise. So what? Try again
Begin with $\mathbb{Z}$-tree ~~Which~~ The uncertainty in $z^{1/2}$ can ~~should~~ be eliminated by an uncertainty

in the space containing $\binom{p_n}{q_n}$ for n odd.

What is the good viewpoint ??

First understand completaty the translation invariant $\mathbb{Z}$-tree. This should link easily to translation invariant grid spaces.

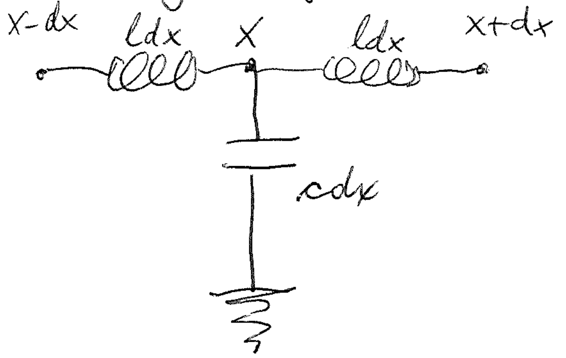


You need to couple Hilbert spaces together. Things like continued fractions and the Schur expansion

Let's suggest a program. You have the idea of coupling equal segments of transmission line with $SU(1,1)$ matrices at each junction. You believe this gives a Hilbert space with a unitary operator. Actually you get a 1-parameter unitary group, but if you ~~let~~ take unit time, ~~this becomes ideas~~ then you can somehow collapse what happens on each each segment.

There's another way maybe to look at coupled transmission lines, namely, use ~~differ~~ different impedances for the segments, but keep the junctions simple. It's harder then to see a $\mathbb{Z}$ action.

Let's look at this varying impedance, but keeping signal speed = 1.



$$E_x - E_{x+dx} = l\,dx\,\dot{I}_x$$

$$I_{x-dx} - I_x = c\,dx\,\dot{E}_x$$

$$-\partial_x E = l\,\partial_t I$$

$$-\partial_x I = c\,\partial_t E$$

$$+\partial_x^2 E = -l\,\partial_x\partial_t I = lc\,\partial_t^2 E \qquad \text{speed } \frac{1}{\sqrt{lc}}$$

so take $\quad c = l^{-1}$.

$$\partial_x E + l$$

$$\partial_x E + l\,\partial_t I = 0 \qquad\qquad \partial_x E + l s I = 0$$

$$\partial_x I + l^{-1}\partial_t E = 0 \qquad\qquad \partial_x I + l^{-1} s E = 0$$

$$\cancel{\begin{pmatrix} E \\ I \end{pmatrix} = e^{ikx}\begin{pmatrix} E_0 \\ I_0 \end{pmatrix}}$$

$$\partial_x\left(\frac{1}{\sqrt{l}}E\right) + \partial_t\left(\sqrt{l}\,I\right) = 0$$

$$\partial_x\left(\sqrt{l}\,I\right) + \partial_t\left(\frac{1}{\sqrt{l}}E\right) = 0$$

$$(\partial_x + \partial_t)\left(\frac{1}{\sqrt{l}}E + \sqrt{l}\,I\right) = 0$$

$$(-\partial_x + \partial_t)\left(-\frac{1}{\sqrt{l}}E + \sqrt{l}\,I\right) = 0$$

$$\begin{pmatrix} \frac{1}{\sqrt{l}} & \sqrt{l} \\ -\frac{1}{\sqrt{l}} & +\sqrt{l} \end{pmatrix}\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} Ae^{-sx} \\ Be^{sx} \end{pmatrix} e^{st}$$

Repeat. $\partial_x E + \ell \partial_t I = 0$ $\quad$ $\partial_x(gE) + \partial_t(g^{-1}I) = 0$ $\quad g = \frac{1}{\sqrt{\ell}}$

$\partial_x I + \ell^{-1} \partial_t E = 0$ $\quad$ $\partial_x(g^{-1}I) + \partial_t(g\,E) = 0$

$$(\partial_x + \partial_t)(gE + g^{-1}I) = 0$$

$$(-\partial_x + \partial_t)(gE - g^{-1}I) = 0$$

solutions $\boxed{\begin{pmatrix} gE + g^{-1}I \\ -gE + g^{-1}I \end{pmatrix} = \begin{pmatrix} Ae^{-sx} \\ Be^{sx} \end{pmatrix} e^{st}}$

assuming time dependence e^{st}

to start with E_0, I_0 at $x = 0$. Then

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} g_0 & g_0^{-1} \\ -g_0 & +g_0^{-1} \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix} \qquad \begin{pmatrix} A_x \\ B_x \end{pmatrix} = \begin{pmatrix} e^{-sx} & 0 \\ 0 & e^{sx} \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \quad 0 \le x \le 1$$

so

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} g_0 & g_0^{-1} \\ -g_0 & +g_0^{-1} \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} g_0 & g_0^{-1} \\ -g_0 & +g_0^{-1} \end{pmatrix}^{-1} \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} g_0 & g_0^{-1} \\ -g_0 & +g_0^{-1} \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

$$\begin{pmatrix} g_0 & g_0^{-1} \\ -g_0 & g_0^{-1} \end{pmatrix}$$

$$\begin{pmatrix} E_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} g_1 & g_1^{-1} \\ -g_1 & +g_1^{-1} \end{pmatrix}^{-1} \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} g_1 & g_1^{-1} \\ -g_1 & +g_1^{-1} \end{pmatrix} \begin{pmatrix} g_0 & g_0^{-1} \\ -g_0 & +g_0^{-1} \end{pmatrix}^{-1} \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

Still puzzled. $\qquad$ What you are doing looks real, yet you expect $SU(1,1)$.

E_0, I_0

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} g & -g \\ g^{-1} & g^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} g + g^{-1} \end{pmatrix}$$

This looks like the Lorentz group.

Repeat. $\partial_x E + \ell \partial_t I = 0$ $\qquad \partial_x(\ell^{-1/2} E) + \partial_t(\ell^{1/2} I) = 0$

$\partial_x I + \ell^{-1} \partial_t E = 0$ $\qquad \partial_x(\ell^{1/2} I) + \partial_t(\ell^{-1/2} E) = 0$

$g = \ell^{1/2}$

$$\partial_x(gE) + \partial_t(g^{-1} I) = 0 \qquad (\partial_x + \partial_t)(gE + g^{-1} I) = 0$$

$$\partial_x(g^{-1} I) + \partial_t(gE) = 0 \qquad (\partial_x - \partial_t)(gE - g^{-1} I) = 0.$$

$$\begin{pmatrix} g & g^{-1} \\ -g & g^{-1} \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} A e^{-sx} \\ B e^{sx} \end{pmatrix} e^{st}$$

$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$ initial values at $x = 0$, then

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} g & g^{-1} \\ -g & g^{-1} \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix} \text{ propagates to}$$

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} g & g^{-1} \\ -g & g^{-1} \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

which corresp to

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \underbrace{\begin{pmatrix} g & g^{-1} \\ -g & g^{-1} \end{pmatrix}^{-1}} \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \underbrace{\begin{pmatrix} g & g^{-1} \\ -g & g^{-1} \end{pmatrix}} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

$$\begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

$$\underbrace{\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-s} & e^{-s} \\ -e^{s} & e^{s} \end{pmatrix}}$$

$$= \begin{pmatrix} \dfrac{e^{-s} + e^{s}}{2} & \dfrac{e^{-s} - e^{s}}{2} \\ \dfrac{e^{-s} - e^{s}}{2} & \dfrac{e^{-s} + e^{s}}{2} \end{pmatrix} = \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix}$$

This reminds me ~~of~~ of earlier work, namely where the h_n are real.

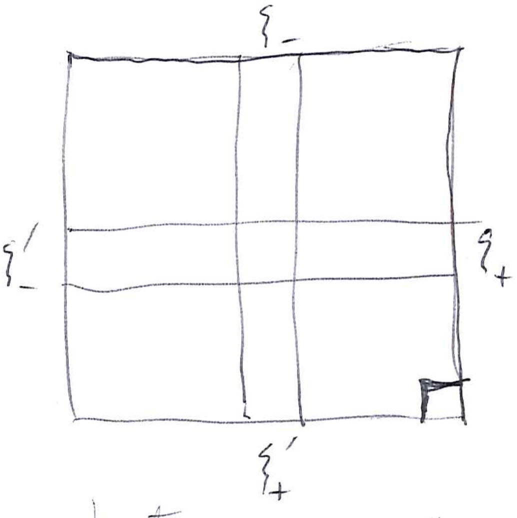
$$\begin{pmatrix}1&1\\-1&1\end{pmatrix}^{-1}\frac{1}{k}\begin{pmatrix}1&h\\h&1\end{pmatrix}\begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}$$

$$\begin{pmatrix}1&h\\h&1\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}=\begin{pmatrix}1-h&1+h\\h-1&h+1\end{pmatrix}=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1-h&0\\0&1+h\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}1&1\\-1&1\end{pmatrix}^{-1}\frac{1}{k}\begin{pmatrix}1&h\\h&1\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}=\begin{pmatrix}\frac{1-h}{k}&0\\0&\frac{1+h}{k}\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}z&z\\-z^{-1}&z^{-1}\end{pmatrix}=\begin{pmatrix}\frac{z+z^{-1}}{2}&\frac{z-z^{-1}}{2}\\\frac{z-z^{-1}}{2}&\frac{z+z^{-1}}{2}\end{pmatrix}$$

basically equivalent, namely $\quad z=e^{-s} \quad \frac{1-h}{k}=g$



picture

Consider b bdd measurable on S^1.

$$IH(f\xi_+ + g\xi_-)$$

β bdd meas. fun. on S^1 $\quad \|\beta\|_\infty < 1$

$$\|\xi_+ f+\xi_- g\|^2=(\xi_+ f+\xi_- g\mid \xi_+ f+\xi_- g)$$

$$=\|f\|^2+\|g\|^2+\cancel{(\xi_+ f\mid(\xi_-(\tfrac{c}{d})+\xi_+}$$

$$(\xi_+ f\mid\xi_- g)+(\xi_- g\mid\xi_+ f)$$

$$\begin{pmatrix}\xi_+\\\xi_-\end{pmatrix}=\begin{pmatrix}\frac{1}{d}&\frac{b}{d}\\-\frac{c}{d}&\frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\\xi'_+\end{pmatrix}$$