Analyze M the v.s. gen. by edges, relations from □'s

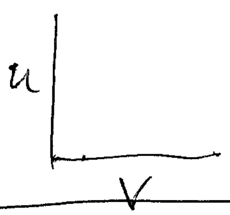
$M' =$ ~~[illegible]~~ the first quadrant subspace. Know that $\{\lambda^n u\}_{n\geq 0}$ $\{\mu^n v\}_{n\geq 0}$ is a basis for M', in fact orthonormal for Krein structure. ~~But Let~~

Regard M' as $\mathbb{C}[\lambda,\mu] = \mathbb{C}\left[\frac{k\lambda-1}{h}, \frac{k\mu-1}{\bar h}\right]$ module, it has ~~[illegible]~~ generators u, v satisfying $\frac{k\lambda-1}{h}u = v$ $\frac{k\mu-1}{\bar h}v = u$

$Xu = v$, $Yv = u$ whence $XY-1$ kills M'. ~~[illegible]~~

~~[illegible]~~ So X^{-1} $\exists$ on M' and $M' = \mathbb{C}[X,X^{-1}]u$, since M' inf dim. must have M' free over $\mathbb{C}[X,X^{-1}]$ with generator u, which agrees with $(\lambda^n u)_{n\geq 0}$, $(\mu^n v)_{n\geq 0}$ being a basis. ~~[illegible]~~ Now localize wrt λ, μ (which act injectively on M') to get $M \simeq \mathbb{C}[X, X^{-1}, \lambda^{-1}, \mu^{-1}]$

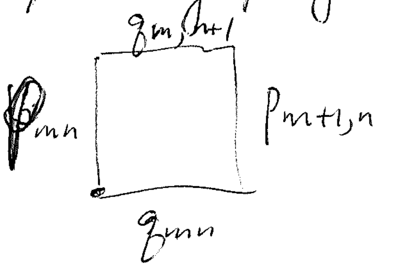
Another point: The sequence ~~[illegible]~~ $(\mu^n u)_{n\in\mathbb{Z}}$ which is orthonormal ~~is an~~ for the Hilbert space structure, actually generates the Hilbert space completion of M. In effect v is not perp to u



Better is that ~~[illegible]~~

$$v = \frac{\bar h}{k\mu-1}u = -\sum_{n\geq 0} \bar h k \mu^n u$$

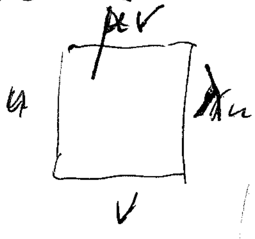
Yesterday progress made.



M generators p_{mn}, q_{mn} $(m,n)\in\mathbb{Z}\times\mathbb{Z}$

relations for each square

Better M $\mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$-module with gen. u, v relations



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

M^q $\mathbb{C}[\lambda,\mu]$-mod ~~submodule~~ gen. u, v

$$\frac{k\lambda-1}{h}u = v$$
$$\frac{k\mu-1}{\bar h}v = u.$$

~~Relation~~ $\mathbb{C}\left[\frac{k\lambda-1}{h}, \frac{k\mu-1}{\bar h}\right]$-module

$Xu = v$ $Yv = u$

domain ~~[illegible]~~ mult. by λ, μ inj.

In the end you ~~...~~ want what?
Structure of M'.
Hilbert, Krein structure.

~~use u as generator~~ ~~$M = \mathbb{C}[X, X^{-1}][\lambda^{-1}, \mu^{-1}]$~~

discuss Hilb. space structure $\mu^n u$, $n \in \mathbb{Z}$ is an orth basis, also $\lambda^n v$, $n \in \mathbb{Z}$. Point $\frac{k\mu - 1}{\hbar} v = u$

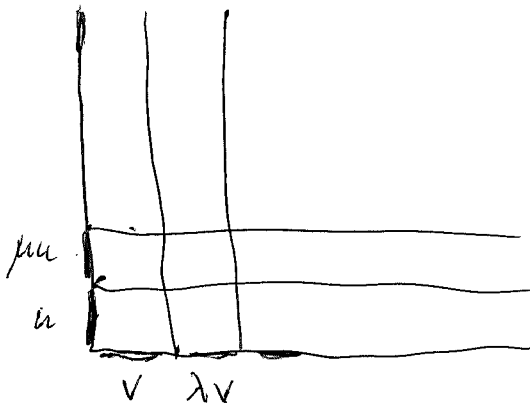
$$\Rightarrow v = \frac{\hbar}{k\mu - 1} u = -\frac{\hbar}{1 - k\mu} = -\hbar \sum_{n \geq 0} k^n \mu^n u \Big)$$

Something interesting happens for the Krein form. $(\mu^n u)_{n \in \mathbb{Z}}$ is an orthonormal family of ~~orthog~~ vectors of norm 1
$(\lambda^n v)_{n \in \mathbb{Z}}$ ______ -1.

so it seems that the Krein "completion" of M is much bigger than the Hilbert completion. ~~This is a puzzle because~~

Question: You know $M' \simeq \mathbb{C}[X, X^{-1}] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} X^n$

In fact the situation is very elliptic curve suggestive with $k = q = e^{2\pi i \tau}$

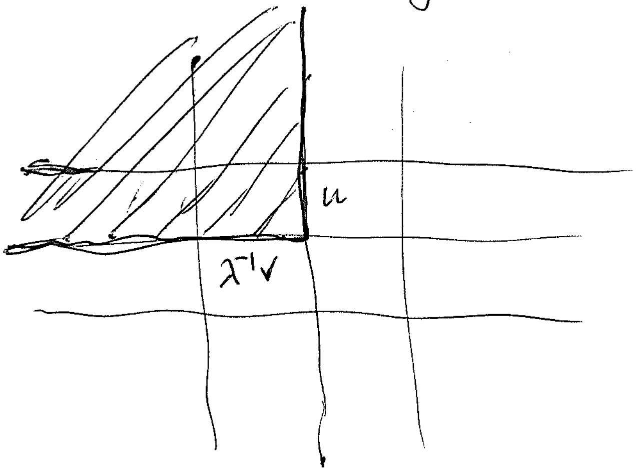


the actual M' looks like a scattering ~~situation~~ except that $\mu \neq \lambda^{-1}$

$$\cdots + u^{-1} V^- + \underbrace{X + V_1^+}_{\substack{\parallel \\ V^- + uX}} + u V^+ + \cdots$$

Idea is to construct an analogue of scattering with a Krein form present

~~by~~ Perhaps ~~[illegible]~~ adopting the first quadrant picture is not good. Instead treat the const coeff system as stationary in time. Descending staircases are then good for Krein form. This will give a different picture of M. Instead of the generators λ, μ for the group $\mathbb{Z} \times \mathbb{Z}$ of translations you use

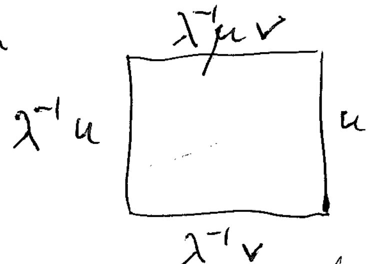


$\overleftarrow{\lambda^{-1}}, \mu\uparrow$ and $z = \lambda^{-1}\mu$ is time translation. $\nwarrow$

Before calculating discuss theory. ~~Primary interest~~ To calculate M is interesting algebra, but you ultimately want the spectrum to lie in $S^1 \times S^1$, I think. On the Hilbert side you get a repn of ~~[illegible]~~ $\mathbb{Z} \times \mathbb{Z}$, ~~cycli~~ unitary cyclic so get measure with support a circle, ~~and~~ ~~[illegible]~~ = Lebesgue measure. Then M appears as a countable dimens. subspace of functions on the circle. You want a similar picture on the Krein side, but ~~these~~ ~~[illegible]~~ things have to be different

$M'' = \mathbb{C}[\lambda^{-1}, \mu]$ module gen. by u and $\lambda^{-1}v$

relation



$$\begin{pmatrix} u \\ \lambda^{-1}\mu v \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1}u \\ \lambda^{-1}v \end{pmatrix}$$

same relation ~~by~~ (mult by λ.)

mult. old relations by λ^{-1}. $\quad \frac{k\lambda - 1}{h}u = v \qquad \frac{k\mu - 1}{\bar{h}}v = u$

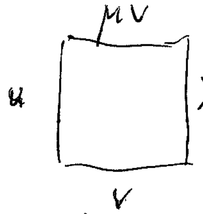
$\frac{k - \lambda^{-1}}{h}u = \lambda^{-1}v \qquad \frac{k\mu - 1}{\bar{h}}(\lambda^{-1}v) = \lambda^{-1}u \qquad$ ~~$\frac{k(\lambda^{-1}\mu) - \lambda^{-1}}{h}$~~

relations are $\frac{k-\lambda^{-1}}{h}u = \lambda^{-1}v \qquad \frac{k\mu-1}{\bar h}(\lambda^{-1}v) = \lambda^{-1}u$

$$\Longrightarrow \quad \frac{k-\lambda^{-1}}{h}\,\frac{k\mu-1}{\bar h}\,\lambda^{-1}v = \frac{k-\lambda^{-1}}{h}\,\lambda^{-1}u = \lambda^{-1}(\lambda^{-1}v).$$

$$\therefore \quad \frac{k-\lambda^{-1}}{h}\,\frac{k\mu-1}{\bar h} = \lambda^{-1}$$

M = module over $\mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$ gen. by u, v



subject to $\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$ $\Bigg|$ $\frac{k\lambda-1}{h}u = v$, $\frac{k\mu-1}{\bar h}v = u$

You found that $M = \cancel{\mathbb{C}[\lambda,\mu][\lambda^{-1},\mu^{-1},X^{-1}]}$

$= \left(\mathbb{C}[\mathbb{Z}\times\mathbb{Z}]/(XY-1)\right)u$ $\quad X = \frac{k\lambda-1}{h} \quad Y = \frac{k\mu-1}{\bar h}$

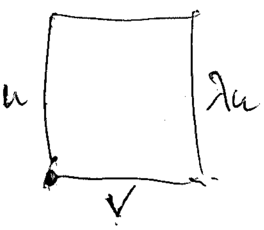
You want to calculate

Problem. In the Hilbert picture understand the unitary going between the orth bases $(\lambda^n v)_{n\ge 0}$ and $(\mu^n u)_{n\ge 0}$

better might be to ~~understand~~ find the operator λ in terms of the orth basis $(\mu^n u)_{n\in\mathbb{Z}}$.

$\frac{k\lambda-1}{h}u = v$

$\frac{k\mu-1}{\bar h}v = u$



So $v = \frac{\bar h}{k\mu-1}u = -\bar h\sum_{n\ge 0}k^n\mu^n u$

$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \qquad \lambda u = \frac{1}{k}u + \frac{h}{k}\left(\frac{\bar h}{k\mu-1}u\right)$$

$$= \frac{1}{k}\left(u + \frac{|h|^2}{k\mu-1}u\right)$$

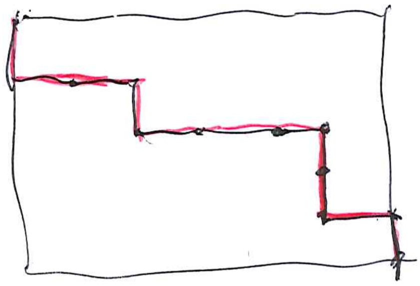
$$= \frac{1}{k}\left(\frac{\overbrace{k\mu-1+|h|^2}^{k\mu-k^2}}{k\mu-1}\right)u = \frac{\mu-k}{k\mu-1}u$$

One more time. You have this basic module M over the group ring $\mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$, generators u, v ~~subject to~~ and relations $\frac{k\lambda-1}{h}u = v$ $\frac{k\mu-1}{\bar{h}}v = u$ which leads to an interesting additive basis for M consisting of the elements $\lambda^n u$, $n\in\mathbb{Z}$ and

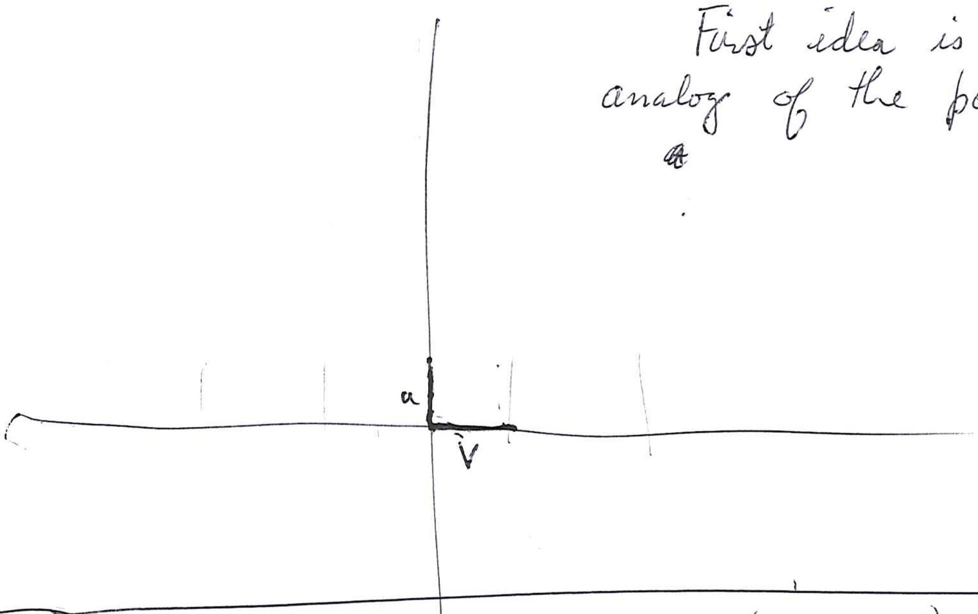
$$(\lambda-k)^{-n}u \quad n\geq 1 \qquad\qquad \mu^n u,\ n\in\mathbb{Z} \quad \text{overlap at } n=0.$$

$$(\lambda-k^{-1})^{-n}u \quad n\geq 1 \qquad\qquad \left(\frac{\lambda-k}{k\lambda-1}\right)^n u,\ n\in\mathbb{Z}$$

Perhaps you should go over the Krein structure. Basically there is ~~a~~ this hermitian form on M ~~preserve this measure~~ such that for any rectangle in the grid + decreasing staircase gives an orthogonal sequence with norms $\begin{cases} +1 & \text{vertical} \\ -1 & \text{horizontal}\end{cases}$



First idea is to look at the analog of the pos. def. function on $\mathbb{Z}\times\mathbb{Z}$



Discuss cont. case. $\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$ $\quad \lambda = e^{\varepsilon L}$, $\mu = e^{\varepsilon M}$ $\quad h \mapsto h\varepsilon$

$$(k\lambda-1)u = (e^{\varepsilon L}-1)u = \varepsilon L u + O(\varepsilon^2) \qquad Lu = hv$$
$$(k\mu-1)v = \varepsilon M v + O(\varepsilon^2) \qquad Mv = \bar{h}u$$

put $h=1$.

Then the "module" E you seek has generators u, v relns $Lu = v$, $Mv = u$. Thus $LM = 1$. You need to

~~To~~ make the nature of L, M precise. This $\mathcal{E}$ is noncommutal in spirit. How to view $\mathcal{E}$?

Suppose you start with the ~~[illegible]~~ Lie group $\mathbb{R}\times\mathbb{R}$ of translations, typical elt $\lambda^r\mu^s$

The group ring is $L^1(\mathbb{R})$ and it acts on $L^2(\mathbb{R})$ by convolution. Fourier transform converts $L^1(\mathbb{R})$ to ~~[illegible]~~ a subring of cont. fns on $\hat{\mathbb{R}}=\mathbb{R}$ vanishing at ∞. ~~Thus~~ The reduced C^* alg $C_r(\mathbb{R})$ is probably ~~the~~ $C_0(\hat{\mathbb{R}})$ 0 = vanish at ∞.

So how to proceed? ~~In your situation you have~~ but you also have the ~~[illegible]~~ Schwartz picture - smooth subrings.

Formulate solution – $\overbrace{\text{an } \mathcal{A} \text{ module } \mathcal{E}}^{\text{consists of (1)}}$, $\mathcal{A}$ is the group ring chosen for $\mathbb{R}\times\mathbb{R}$, i.e. $\mathcal{E}$ is a vector space with action of the translation group,
(2) u, v equivariant maps $\mathcal{A}\longrightarrow\mathcal{E}$

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discrete case $\quad A = \mathbb{C}[\mathbb{Z}\times\mathbb{Z}]\quad$ gp ring of translation gp.

$$\mathcal{E} = Au\oplus Av\Big/\left(\frac{k\lambda-1}{h}u=v,\ \frac{k\mu-1}{h}=u\right)$$

$\mathbb{C}[\lambda,\mu][\lambda^{-1},\mu^{-1}]$

Study via F.T. A becomes the alg of Laurent polys. ~~which can be viewed as~~ view A as functions on $\{(\lambda,\mu)\in\mathbb{C}^\times\times\mathbb{C}^\times\}$. Calculate

$$\mathcal{E}\xleftarrow[\sim]{u}\left(A\Big/\left(\frac{k\lambda-1}{h}\frac{k\mu-1}{h}-1\right)A\right)$$

. Continue the calc. to get $\quad \mathcal{E}\xleftarrow[u]{\sim}B \qquad B=\mathbb{C}[\lambda,\lambda^{-1},(\lambda-k^{-1})^{-1},(\lambda-k)^{-1}]$.

You ~~[illegible]~~ ought to be able to compute the Krein form in terms of this description.

Continuous case. Take group ring to be $\mathcal{A} = \mathcal{S}(\mathbb{R}\times\mathbb{R})$ under convolution. Define E ~~to be universal~~ as follows. Start with $\mathcal{A}\oplus\mathcal{A}$ as $\mathcal{A}$-module and introduce relations

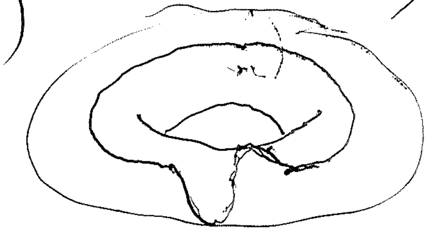
$$\begin{pmatrix} Xf \\ Yg \end{pmatrix} = \begin{pmatrix} ig \\ if \end{pmatrix}$$

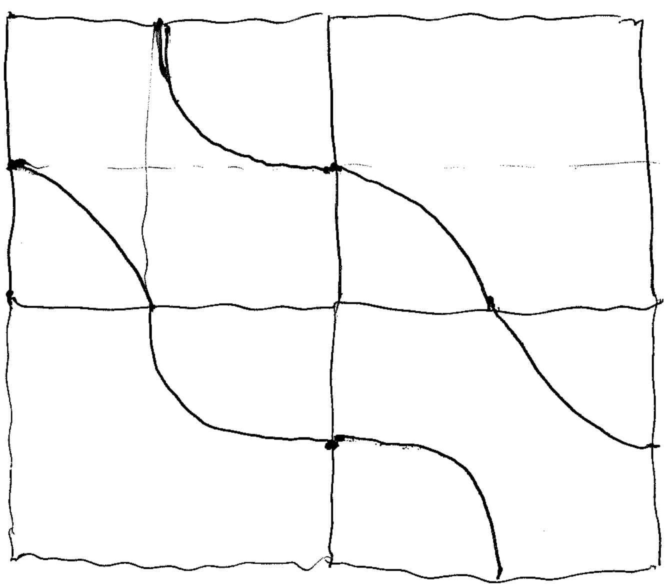
Use x,y coord on $\mathbb{R}\times\mathbb{R}$, $X=\partial_x$, $Y=\partial_y$. These relations are const coeff (no x,y only ∂_x, ∂_y) so the quotient is an $\mathcal{A}$-module, eg. space with $\mathbb{R}\times\mathbb{R}$ action. Now analyze via F.T. $\mathcal{A}$ becomes the Schwartz space on the dual $\mathcal{A}\oplus\mathcal{A}$ becomes $\begin{pmatrix} f(\xi,\eta) \\ g(\xi,\eta)\end{pmatrix}$

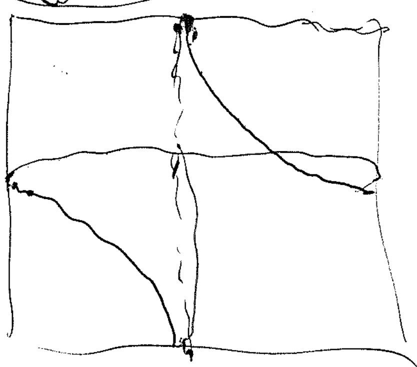
and the relations become $\xi f = g$, $\eta g = f$. So you expect the quotient to be $\cong \mathcal{A}/(\xi\eta-1)\mathcal{A}$.

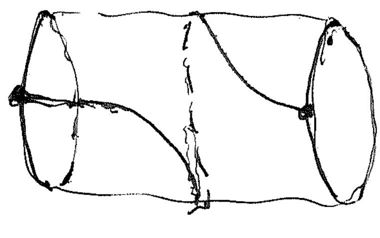
~~you expect~~ So what is $\mathcal{S}(\mathbb{R}\times\mathbb{R})/(\xi\eta-1)$?

What is $\mathcal{S}(\mathbb{R}\times\mathbb{R})/(\xi\eta-1)\,\mathcal{S}(\mathbb{R}\times\mathbb{R})$



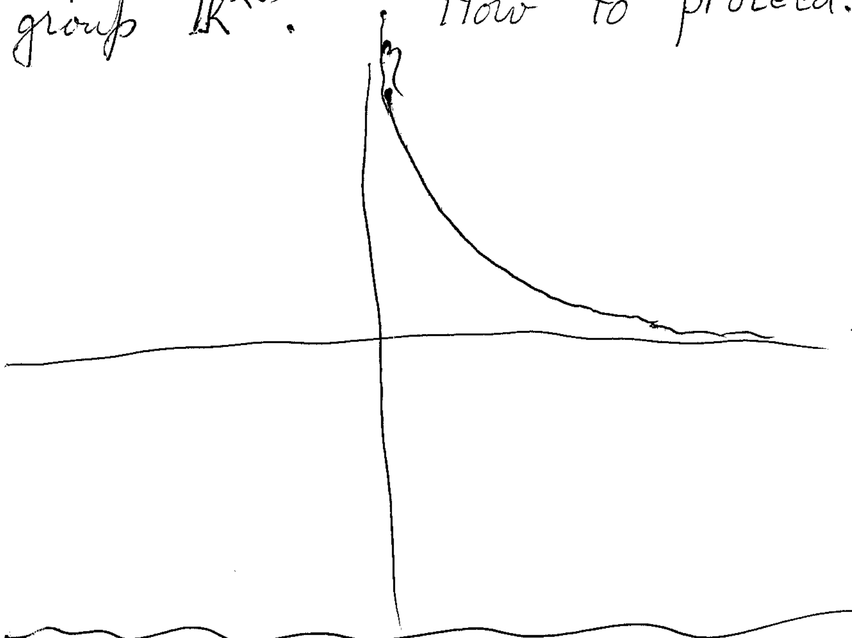


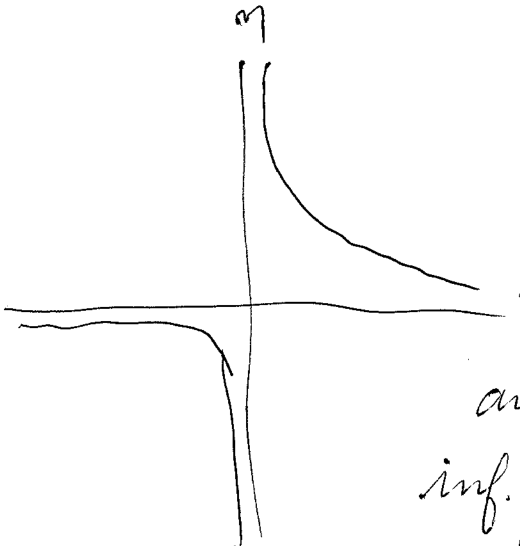




It seems that $\mathcal{S}(\mathbb{R}\times\mathbb{R})/(\xi\eta-1)\mathcal{S}(\mathbb{R}\times\mathbb{R})$ will turn out to be the space of Schwartz functions on the curve $\xi\eta=1$. This curve has 2 components.

But there may be a problem, ~~[illegible]~~ or opportunity, with this curve being the multiplicative group $\mathbb{R}^{\times}$. How to proceed?





Let $f(\xi,\eta)\in\mathcal{S}(\mathbb{R}\times\mathbb{R})$

Does $f(\xi,\xi^{-1})\in\mathcal{S}(\mathbb{R}_{\neq 0})$?

i.e. does $f(\xi,\xi^{-1})$ extend to an element of $\mathcal{S}(\mathbb{R})$ vanishing to inf. order at $x=0$. This should be true because $\mathcal{S}(\mathbb{R}\times\mathbb{R})$ should be ~~subalg~~ subring of $C^\infty(S^1\times S^1)$ cons. of f vanishing to inf. order on $(S^1\times\infty)\cup(\infty\times S^1)$ and because ~~when we pull back~~ the curve $\xi\eta=1$ closes in $S^1\times S^1$ to a smooth circle cutting $S^1\vee S^1$ transversally. need to check this.

Here's the problem: Look at $\mathbb{R}_{\geq 0}$ with coord $x = e^t$ where $\underset{=\log x}{t} \in \mathbb{R}$. Let

$y = g(x) = g(e^t) = f(t)$. Then $y \to 0$ as $x \downarrow 0$ and as $t \downarrow -\infty$

You are interested in rapid decrease. But you've got x, t wrong order. t is the Mellin transform variable. $t = e^x$ as $x \to -\infty$, $t \xrightarrow{\text{exp.}} 0$.

Let's ~~~~ approach the analysis by examples if possible. The idea is ~~to~~ start from the assumption that

$\mathcal{S}(\mathbb{R}\times\mathbb{R})/(\xi\eta - 1)\mathcal{S}(\mathbb{R}\times\mathbb{R})$ is somehow $\simeq$ $\mathcal{S}(\mathbb{R}) + \mathcal{S}(\mathbb{R})$.

Another idea: First get straight the isom of $\mathcal{S}(\mathbb{R})$ with smooth fns. on S^1 vanishing at -1.

Continuous ~~case~~ limit.

$$z^\varepsilon = e^{i\omega dx} = 1 + i\omega dx$$

$$\begin{pmatrix} p_{n\varepsilon} \\ q_{n\varepsilon} \end{pmatrix} = \frac{1}{k_{n\varepsilon}} \begin{pmatrix} 1 & h_{n\varepsilon}\varepsilon \\ \bar{h}_{n\varepsilon}\varepsilon & 1 \end{pmatrix} \begin{pmatrix} z^\varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{(n-1)\varepsilon} \\ q_{(n-1)\varepsilon} \end{pmatrix}$$

$$\psi_x = \begin{pmatrix} 1 & h_x dx \\ \bar{h}_x dx & 1 \end{pmatrix} \begin{pmatrix} 1 + i\omega dx & 0 \\ 0 & 1 \end{pmatrix} \psi_{x-dx}$$

$$\partial_x \psi_x = \begin{pmatrix} i\omega & h_x \\ \bar{h}_x & 0 \end{pmatrix} \psi_x \qquad \partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} i\omega & h_x \\ \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\partial_x \underbrace{\begin{pmatrix} e^{-i\frac{\omega}{2}x} p_x \\ e^{i\frac{\omega}{2}x} q_x \end{pmatrix}}_{\Psi} = \begin{pmatrix} -i\frac{\omega}{2} & 0 \\ 0 & i\frac{\omega}{2} \end{pmatrix} \Psi +$$

$$\partial_x \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda & h \\ \bar{h} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\partial_x \begin{pmatrix} e^{-\frac{\lambda}{2}x} p \\ e^{-\frac{\lambda}{2}x} q \end{pmatrix} = \begin{pmatrix} e^{-\frac{\lambda}{2}x}(\lambda p + h q) \\ e^{-\frac{\lambda}{2}x}(\bar{h} p) \end{pmatrix} + \begin{pmatrix} -\frac{\lambda}{2} e^{-\frac{\lambda}{2}x} p \\ -\frac{\lambda}{2} e^{-\frac{\lambda}{2}x} q \end{pmatrix}$$

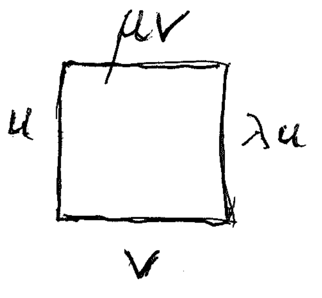
$$= \begin{pmatrix} \frac{\lambda}{2} & h \\ \bar{h} & -\frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} e^{-\frac{\lambda x}{2}} p \\ e^{-\frac{\lambda}{2}x} q \end{pmatrix}$$

$$\begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{2} & h \\ -\bar{h} & +\frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

$$\begin{pmatrix} \partial_x & -h \\ \bar{h} & -\partial_x \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \frac{\lambda}{2} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

What is the cont. case?

$$\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$



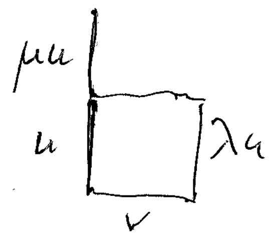
$$\frac{(k\lambda - 1)}{h} u = v \qquad \frac{k\mu - 1}{\bar{h}} v = u$$

$h \mapsto h\varepsilon$. $\lambda \mapsto \lambda^{\varepsilon}$ $\qquad \frac{\lambda^{\varepsilon} - 1}{h\varepsilon}$ $\qquad \lambda = e^{L}$

$k \mapsto \sqrt{1 - |h\varepsilon|^2} = 1$ $\qquad \frac{e^{\varepsilon L} - 1}{h\varepsilon} = \frac{L}{h}$ $\qquad h=1$

Cont. case becomes $Lu = iv$

What is your description in the disc. case.



orth. basis $(\mu^n u)_{n\in\mathbb{Z}}$

$$\frac{k\lambda-1}{h}u = v \qquad \frac{k\mu-1}{\bar h}v = u$$

$$v = \frac{\bar h}{k\mu-1}u = -\bar h\sum_{n\geq 0} k^n\mu^n u$$

Set up isom.

$$k\lambda-1 = \frac{k(\mu-k)-(k\mu-1)}{k\mu-1} = \frac{|h|^2}{k\mu-1}$$

$$\boxed{\begin{aligned} u &\longmapsto 1 \in L^2(S^1) \\ \mu &\longmapsto z \\ \lambda = \frac{\mu-k}{k\mu-1} &\longmapsto \frac{z-k}{kz-1} = \begin{pmatrix}1 & k\\ k & 1\end{pmatrix}(-z) \\ v &\longmapsto \frac{\bar h}{kz-1}\end{aligned}}$$

It should be simple to do cont. limit. Too ~~much~~ many threads at the moment. Begin with

$$\partial_t\psi = \begin{pmatrix}\partial_x & im\\ im & -\partial_x\end{pmatrix}\psi$$

wave equation constant coeffs.

Constant coeffs $\Longrightarrow$ look for exp. solutions $\psi = e^{i(\omega t+\xi x)}\begin{pmatrix}u\\ v\end{pmatrix}$

$$\omega\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}\xi & m\\ m & -\xi\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$\frac{\omega-\xi}{m}u = v$$

$$\frac{\omega+\xi}{m}v = u$$

$$\begin{vmatrix}\xi-\omega & m\\ m & -\xi-\omega\end{vmatrix} = \omega^2-\xi^2-m^2 = 0$$

$$\frac{\omega^2-\xi^2}{m^2} = 1$$

Put $m=1$. Where are we? ~~You have a spectral curve~~ You've found exponential solutions. An exp. solution consists of a pair (ω,ξ) $\ni$ $\omega^2 = 1+\xi^2$, and a pair $\begin{pmatrix}u\\ v\end{pmatrix}$ $\ni$ $(\omega-\xi)u = v$, $(\omega+\xi)v = u$. So the set of exp. solutions is a line bundle over this spectral curve.

$$\psi(x,t) = \int e^{i(\omega t + \xi x)} \begin{pmatrix} 1 \\ \omega - \xi \end{pmatrix} u(\xi)$$

$$\psi(x,t) = \int \frac{d\xi}{2\pi} e^{i(\omega t + \xi x)} \begin{pmatrix} 1 \\ \omega - \xi \end{pmatrix} u^+(\xi) + e^{i(-\omega t + \xi x)} \begin{pmatrix} 1 \\ -\omega - \xi \end{pmatrix} u^-(\xi)$$

where $\omega = +\sqrt{1+\xi^2}$

$$\psi(x,0) = \int \frac{d\xi}{2\pi} e^{i\xi x} \left\{ \begin{pmatrix} 1 \\ \omega - \xi \end{pmatrix} u^+(\xi) + \begin{pmatrix} 1 \\ -\omega - \xi \end{pmatrix} u^-(\xi) \right\}$$

$$\begin{vmatrix} 1 & 1 \\ \omega - \xi & -\omega - \xi \end{vmatrix} = -2\omega$$

You want the Hilbert space, (energy norm) on the space of solutions of the wave equation. The above formula for solutions in terms of functions $u^+(\xi), u^-(\xi)$ doesn't seem to help. It might help to write

$$\psi(x,t) = \int \frac{d\xi}{2\pi} e^{i\xi x} \left\{ e^{i\omega t} \underbrace{\begin{pmatrix} 1 \\ \omega - \xi \end{pmatrix}} u^+(\xi) + e^{-i\omega t} \underbrace{\begin{pmatrix} -\omega + \xi \\ 1 \end{pmatrix}} u^-(\xi) \right\}$$

different from above

eigenvectors for $\begin{pmatrix} \xi & 1 \\ 1 & -\xi \end{pmatrix}$ + eigenvalue $\omega, -\omega$

$+\sqrt{1+\xi^2}$

This setup with x, t seems ~~unnecessarily~~ complicated. You want to use characteristics coords $t \pm x$, dually replace ξ, ω by $\omega \pm \xi$

$$\begin{pmatrix} \partial_t & \cancel{\partial_t} \\ 0 & \partial_t \end{pmatrix} \psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \psi \qquad \psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

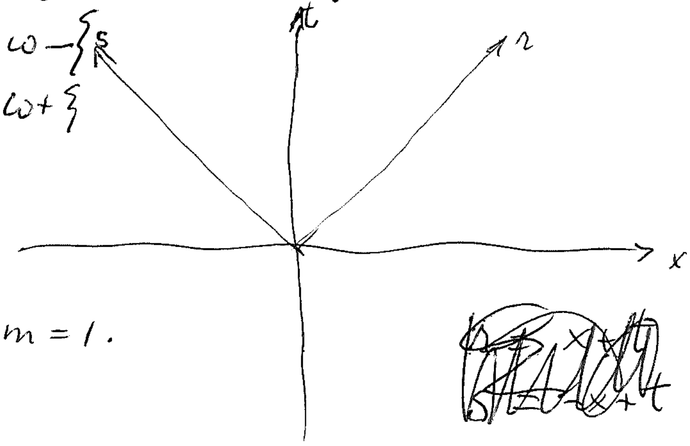
$$\boxed{\begin{aligned} \tfrac{1}{i}(\partial_t - \partial_x)\psi^1 &= m\psi^2 \\ \tfrac{1}{i}(\partial_t + \partial_x)\psi^2 &= m\psi^1 \end{aligned}}$$

change from ~~...~~

$\partial_s = \partial_t - \partial_x \qquad \partial_t + \partial_x = \partial_r$

$\sigma = \omega - \xi$

$\rho = \omega + \xi$



$$\frac{(\omega - \xi)}{m}\psi^1 = \psi^2$$

$$\left(\frac{\omega + \xi}{m}\right)\psi^2 = \psi^1$$

put $m = 1$.

$\sigma\psi^1 = \psi^2$

$\rho\psi^2 = \psi^1 \qquad \therefore \quad \sigma = \rho^{-1}$.

~~So~~ You are calculating the exponential solutions to the equation $\frac{1}{i}\partial_s \psi^1 = \psi^2 \qquad \frac{1}{i}\partial_r \psi^2 = \psi^1$.

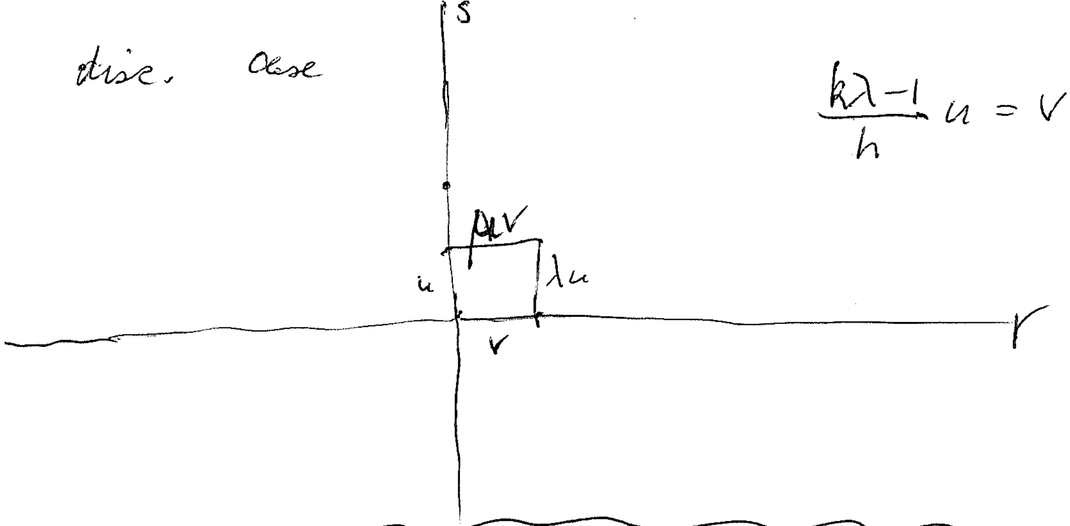
~~Philosophy ... Program: Start with~~

Program: Constant coeffs. DE.

space time vector space V, 2 comp. first order partial DE, translation invariant, solutions form representation of ~~...~~ translation grp $(V, +)$, ~~...~~ study via characters, i.e. look at exp solns., philosophy of universal solution, module over group ring.

Repeat: space time vector sp V, 2 comp 1st order DE which is transl. inv., ~~...~~ to study solutions of the DE, these form a repn of transl. gp. $(V, +)$, ~~...~~ or a module over grp ring, ~~...~~ look at exponential solutions, ~~...~~ alg. geom. picture – ~~...~~ variety with line bundle. Somewhere inside this scene is a Hilbert space. ~~...~~

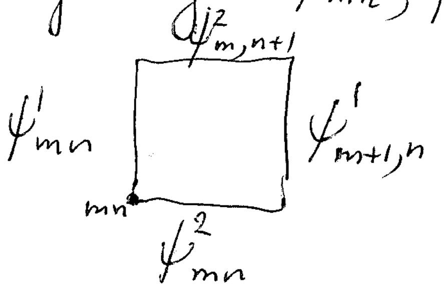
disc. case



$$\frac{k\lambda-1}{h}u = v \qquad \frac{k\mu-1}{\bar h}v = u$$

Begin again the discrete case

$\mathcal{E}$ v. space gen. by ψ^1_{mn}, ψ^2_{mn} $m,n \in \mathbb{Z}\times\mathbb{Z}$ subject to relations



$$\begin{pmatrix}\psi^1_{m+1,n}\\ \psi^2_{m,n+1}\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}\psi^1_{mn}\\ \psi^2_{mn}\end{pmatrix}$$

translation gp $\mathbb{Z}\times\mathbb{Z}$ acts on $\mathcal{E}$ $\qquad \psi_{mn} = \lambda^m\mu^n\psi_{00}$

$A = \mathbb{C}[\mathbb{Z}\times\mathbb{Z}] = \mathbb{C}[\lambda,\mu][\lambda^{-1},\mu^{-1}]$

$$A \xrightarrow{\begin{pmatrix}\frac{k\lambda-1}{h}\\ -\frac{k\mu-1}{\bar h}\end{pmatrix}} \begin{matrix}A\\ \oplus\\ A\end{matrix} \xrightarrow{(u\ \ v)} \mathcal{E}$$

NO

$$\frac{k\lambda-1}{h}u = v \qquad \frac{k\mu-1}{\bar h}v = u.$$

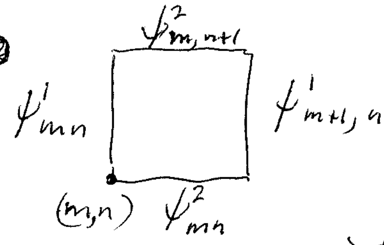
$$\begin{matrix}A\\ \oplus\\ A\end{matrix} \xrightarrow{\begin{pmatrix}\frac{k\lambda-1}{h} & -1\\ -1 & \frac{k\mu-1}{\bar h}\end{pmatrix}} \begin{matrix}A\\ \oplus\\ A\end{matrix} \xrightarrow{(u\ \ v)} \mathcal{E} \longrightarrow 0 \qquad \text{exact}$$

So $\mathcal{E}$ killed by $\frac{(k\lambda-1)(k\mu-1)}{|h|^2} - 1 = XY - 1$ $\qquad B = A/A(XY-1)$

$$\begin{matrix}B\\ \oplus\\ B\end{matrix} \xrightarrow{\begin{pmatrix}X & -1\\ -1 & Y\end{pmatrix}} \begin{matrix}B\\ \oplus\\ B\end{matrix} \xrightarrow{(u\ \ v)} \mathcal{E} \longrightarrow 0 \qquad \text{exact}$$

$\therefore$ $\mathcal{E} \simeq uB$.

discrete case.

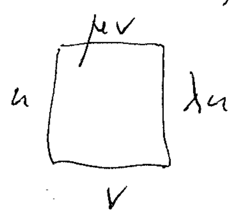


E = grid space gen. by edge relations from squares.

E has pos. def inner product where exhaustive increasing staircases are orthonormal bases. E has an indefinite inner product where decreas. staircases are orth. bases.

Action of $A = \mathbb{C}[\mathbb{Z}\times\mathbb{Z}] = \mathbb{C}[\lambda,\mu,\lambda^{-1},\mu^{-1}]$. $\lambda^m\mu^n\psi_{m_1,n_1} = \psi_{m+m_1,n+n_1}$

$u = \psi^1_{00}$ $\quad v = \psi^2_{00}$



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$\underline{E}$ = A-module gen by u, v relations $\overset{X}{\frac{k\lambda-1}{h}}u = v$

$\overset{Y}{\frac{k\mu-1}{h}}v = u$. $\quad E$ killed by $\frac{k\lambda-1}{h}\frac{k\mu-1}{h} = 1$.

E is $B = \underbrace{A/A(XY-1)}_{\mathbb{C}[X,Y]}$ mod. gen u no relation

$$\mu = \frac{1}{k}\left(1+\frac{1-k^2}{k\lambda-1}\right) = \frac{-\lambda+k}{-k\lambda+1}$$

You want to get this in the right form. ~~Basically~~ Ultimate assertion is? What should be the philosophy about the universal solution? You have a diff eqn with const. coefficients, translation gp $\mathbb{Z}\times\mathbb{Z}$, group ring A, universal A-module E,

A solution with values in a vector space V is same as a linear map $E \to V$.

A solution with values in an A-module M is same as an $\underline{A}$-linear map. $E \to M$

Start again with what is mind.

traditional approch. look for exponential solutions

~~Assume~~ S

Start with a solution (ψ_{mn}) take F.T.

$$\hat\psi(\lambda,\mu) = \sum_{m,n\in\mathbb{Z}} \lambda^{-m}\mu^{-n}\psi_{mn}$$

$$\sum \lambda^{-m}\mu^{-n}\begin{pmatrix}\psi^1_{m+1,n}\\ \psi^2_{m,n+1}\end{pmatrix} = \begin{pmatrix}\lambda\hat\psi^1\\ \mu\hat\psi^2\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}\hat\psi^1\\ \hat\psi^2\end{pmatrix}$$

so you get $\frac{k\lambda-1}{h}\hat\psi^1 = \hat\psi^2$ $\quad \frac{k\mu-1}{\bar h}\hat\psi^2 = \hat\psi^1$

implies $\left(\frac{k\lambda-1}{h}\,\frac{k\mu-1}{\bar h} - 1\right)\hat\psi = 0.$

So $\hat\psi$ not a function (on $S^1\times S^1$) but rather a distribution with support in the curve $\mu = \frac{-\lambda+k}{k(-\lambda)+1}$ ~~solution~~

to 636

From 637

Summary: discrete case $\psi_{mn} = \lambda^m\mu^n\binom{u}{v}$ $\quad \frac{k\lambda-1}{h}u = v \quad \frac{k\mu-1}{\bar h}v = u$

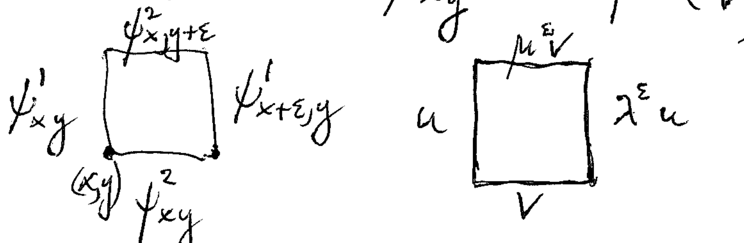
Model inside $L^2(S^1)$, namely

λ = mult by z $\qquad v = 1$

μ = mult by $\frac{z-k}{kz-1}$ $\qquad u = \frac{h}{kz-1}$

Because the Hilb. space completion $\bar E$ admits $(\lambda^n v)_{n\in\mathbb{Z}}$ as an orth basis which should corresponds to the orth basis $(z^n)_{n\in\mathbb{Z}}$ of $L^2(S^1)$, one finds $\bar E \simeq L^2(S^1)$.

Cont. case. $\psi_{xy} = \lambda^x\mu^y\binom{u}{v}$ relations are



$$\begin{pmatrix}\lambda^\varepsilon u\\ \mu^\varepsilon v\end{pmatrix} = \begin{pmatrix}1 & h_c\varepsilon\\ \bar h_c\varepsilon & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

here $\varepsilon^2=0$ $\lambda^\varepsilon = 1+\varepsilon X$, $\mu^\varepsilon = 1+\varepsilon Y$ and we get $Xu = h_c v$ $\quad Yv = \bar h_c u$ since $\partial_x\lambda^x = X\lambda^x$ the cont. eqn. is $\boxed{\partial_x\psi^1_{xy} = h_c\psi^2_{xy} \qquad \partial_y\psi^2_{xy} = \bar h_c\psi^1_{xy}}$

To 638

Review: $\psi_{mn} = \lambda^m \mu^n \begin{pmatrix} u \\ v \end{pmatrix}$ $\qquad \frac{k\lambda-1}{h} u = v \qquad \frac{k\mu-1}{\bar h} v = u$

rep in $L^2(S^1)$. Let λ = mult. by z $\qquad v = 1$

$k\lambda - 1 = kz - 1 \qquad \mu$ = mult. by $\frac{z-k}{kz-1} \qquad u = \frac{h}{kz-1}$

Then $k\mu - 1 = k\frac{z-k}{kz-1} - 1 = \frac{\cancel{kz} - k^2 - \cancel{kz} + 1}{kz-1}$

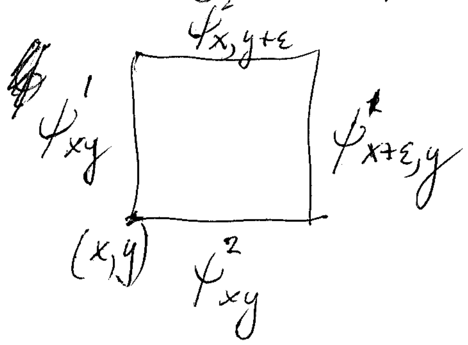
and $(k\lambda-1)(k\mu-1) = kz - k^2 - kz + 1 = 1 - k^2$.

Better would have been to calculate

$$\frac{k\lambda-1}{h}u = \frac{kz-1}{h}\frac{h}{kz-1} = 1 = v$$

$$\frac{k\mu-1}{\bar h}v = \frac{1}{\bar h}\left(k\left(\frac{z-k}{kz-1}\right) - 1\right) = \frac{1}{\bar h}\frac{1-k^2}{kz-1} = \frac{h}{kz-1} = u.$$

$$\psi_{xy} = \lambda^x \mu^y \begin{pmatrix} u \\ v \end{pmatrix} \qquad \lambda = e^X \qquad \mu = e^Y$$



$$\begin{pmatrix} \lambda^\varepsilon u \\ \mu^\varepsilon v \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2\varepsilon^2}}\begin{pmatrix} 1 & h\varepsilon \\ \bar h\varepsilon & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} Xu \\ Yv \end{pmatrix} = \begin{pmatrix} 0 & h \\ \bar h & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

should be same as

$$\partial_x u = hv$$
$$\partial_y v = \bar h u$$

In fact the inf equation is precisely

$$\boxed{\begin{aligned} \partial_x \psi^1_{xy} &= h_e \psi^2_{xy} \\ \partial_y \psi^2_{xy} &= \bar h_e \psi^1_{xy} \end{aligned}}$$

$$\psi_{xy} = \lambda^x \eta^y \begin{pmatrix} u \\ v \end{pmatrix}$$

$\lambda^x = e^{i\xi x}$. $\qquad$ on $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$

$\mu^y = e^{i\eta y}$.

what is η

$$\partial_x \psi^1_{xy} = \partial_x(\lambda^x \mu^y u) = i\xi\psi^1_{xy} = h\psi^2_{xy}$$

$$\partial_y \psi^2_{xy} = \partial_y(\lambda^x \mu^y V) = \lambda^x (i\eta)\mu^y V = i\eta \psi^2_{xy}$$

So get $\qquad i\xi \psi^1_{xy} = h\psi^2_{xy} \qquad i\xi u = h_c V$

$$i\eta \psi^2_{xy} = \bar{h}\psi^1_{xy} \qquad i\eta V = \bar{h}_c u$$

$\therefore\ -\xi\eta = |h_c|^2$ $\longrightarrow$ So you conclude $\eta = -\xi^{-1}$

Say $|h_c| = 1$. ~~[illegible]~~

Go back ~~[illegible]~~ and check.

$$\boxed{\left.\begin{aligned} \lambda^x &= \text{mult by } e^{i\xi x} \\ \mu^y &= \text{mult by } e^{-i\xi^{-1} y} \\ V &= 1 \\ u &= \frac{1}{i\xi} \end{aligned}\right) \text{ in } L^2\left(\mathbb{R}, \frac{d\xi}{2\pi}\right)}$$

$$\psi_{xy} = \lambda^x \mu^y \begin{pmatrix} u \\ V \end{pmatrix} = e^{i(\xi x - \xi^{-1} y)} \begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix}$$

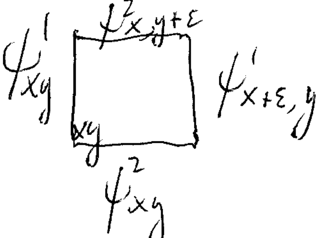
$$\cancel{\partial_x \psi^1_{xy} = e^{i(\xi x - \xi^{-1} y)} \ldots i\xi \ldots i\xi \psi^2_{xy}}$$

$$\cancel{\partial_y \psi^2_{xy} = e^{i(\xi x - \xi^{-1} y)} (-i\xi^{-1}) = i(-\xi^{-1}) e^{i(\xi x - \xi^{-1} y)} \quad = i\eta \psi^2_{xy}}$$

$$\partial_x \psi^1_{xy} = i\xi\, e^{i(\xi x - \xi^{-1} y)} \frac{1}{i\xi} = e^{i(\xi x - \xi^{-1} y)} = \psi^2_{xy}$$

$$\partial_y \psi^2_{xy} = i(-\xi^{-1})\, e^{i(\xi x - \xi^{-1} y)}\, 1 = e^{i(\xi x - \xi^{-1} y)} \frac{1}{i\xi} = \psi^1_{xy}$$

Back to 635

Review. 

$$\begin{pmatrix}\psi^1_{x+\varepsilon,y}\\ \psi^2_{x,y+\varepsilon}\end{pmatrix} = \begin{pmatrix}1 & h'\varepsilon\\ \bar{h}'\varepsilon & 1\end{pmatrix}\begin{pmatrix}\psi^1_{xy}\\ \psi^2_{xy}\end{pmatrix}$$

$$\partial_x \psi^1_{xy} = h'\psi^2_{xy} \qquad \psi_{xy} = \underbrace{\lambda^x\mu^y}_{\text{translation operator}}\begin{pmatrix}u\\ v\end{pmatrix} \qquad \partial_x\lambda^x = \lambda^x(i\xi)$$
$$\partial_y \psi^2_{xy} = \bar{h}'\psi^1_{xy} \qquad\qquad \partial_y\mu^y = \mu^y(i\eta)$$

$$i\xi u = h'v \qquad -\xi\eta = |h'|^2 \qquad h'=1$$
$$i\eta v = \bar{h}' u$$

have ~~sol~~ Solution values in $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$ $\quad \lambda^x = {}_{\text{mult by}}\, e^{i\xi x} \quad v = 1$

$$\mu^y = \cancel{\quad} e^{-i\xi^{-1}y} \qquad u = \frac{1}{i\xi}$$

$$\psi_{xy} = \lambda^x\mu^y\begin{pmatrix}u\\ v\end{pmatrix} = e^{i(\xi x - \xi^{-1}y)}\begin{pmatrix}\frac{1}{i\xi}\\ 1\end{pmatrix}$$

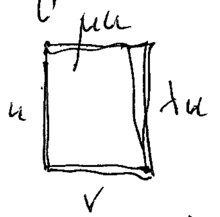
To obtain vectors in the Hilbert space $\bar{E}$ you use convolution

$$\int dx\,dy\,\rho(x,y)\,\psi_{xy} = \underbrace{\int dx\,dy\,\rho(x,y)e^{i(\xi x-\xi^{-1}y)}}_{\hat{\rho}(-\xi,\,\xi^{-1})\begin{pmatrix}\frac{1}{i\xi}\\ 1\end{pmatrix}}\begin{pmatrix}\frac{1}{i\xi}\\ 1\end{pmatrix}$$

what was the idea to explore. The idea is ~~that~~ the ~~spectral~~ ~~on~~ growth conditions on the spectral curve.

---

indefinite hermitian form. $H(\xi',\xi) \qquad \xi',\xi \in E.$



$$k\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \qquad \text{~~date~~}$$

H determined by ~~$H(v, \lambda^m\mu^n v)$~~ g

Go back. E ~~is a module of~~ is acted upon by the group $\{\lambda^m\mu^n \mid m,n\in\mathbb{Z}\}$ of translations, ~~and~~ the form H is invariant, i.e. the operators $\lambda^m\mu^n$ are (pseudo) unitary so $H(g_1\xi_1, g\xi) = H(\xi_1, g_1^{-1}g\xi)$

Also $\exists$ cyclic vector v. ~~then~~ so that H is determined by $H(v, gv)$, ~~this~~ fn. on group. = linear fnl on gp alg.

~~...~~ picture of E, certain rational functions of z

λ = mult by z $\qquad v = 1$

μ = mult by $\frac{z-k}{kz-1}$ $\qquad u = \frac{h}{kz-1}$

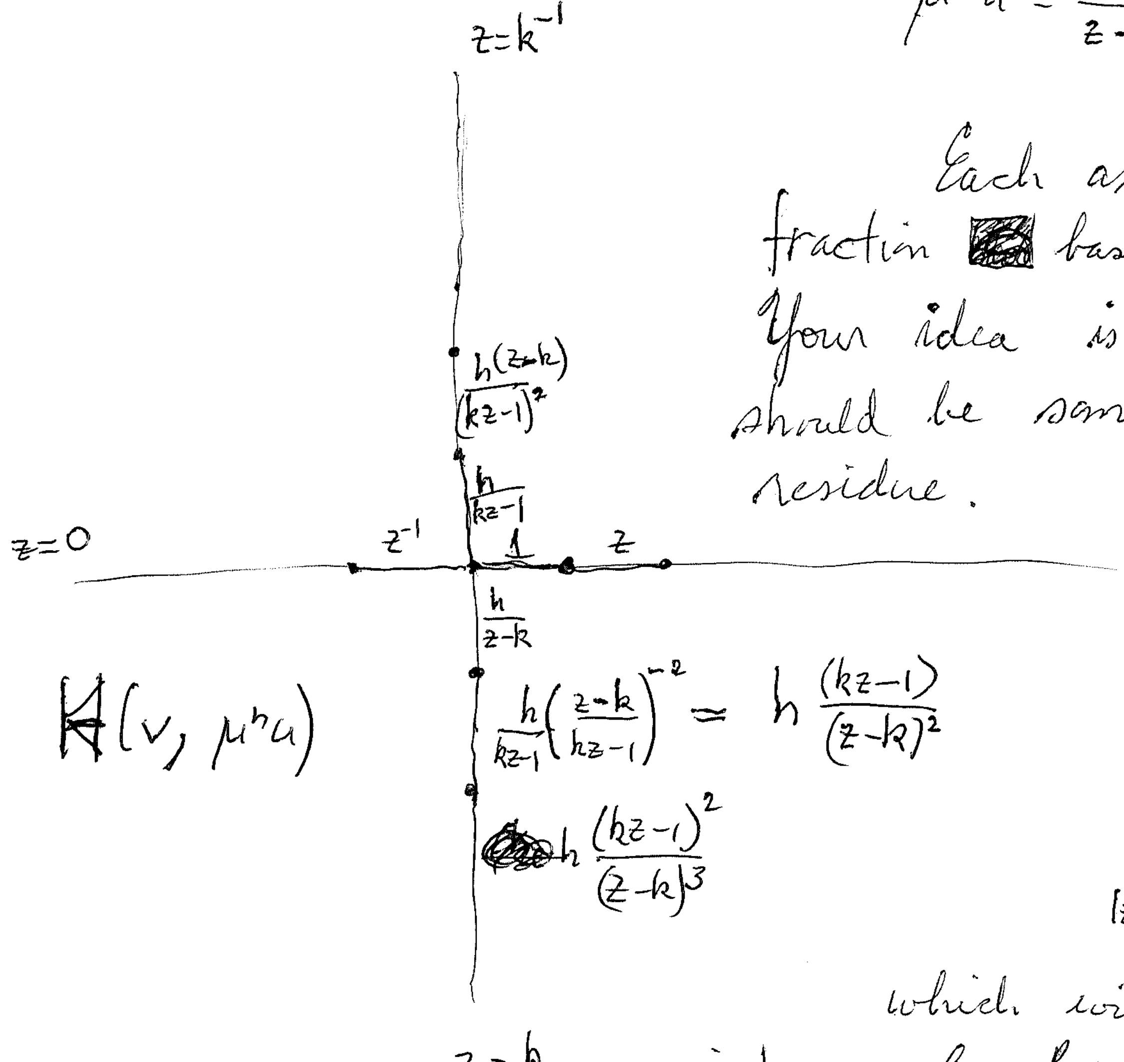
~~...~~ You want a linear functional on this space of rational functions, which should be $f \mapsto H(v, fv)$. The hermitian form should be

$$H(f_1 v, f_2 v) = H(v, f_1^* f_2 v)$$

You know basis for E is $(\lambda^m v)_{m\in\mathbb{Z}}$ and

$$\left(\mu^n u = h\frac{(z-k)^n}{(kz-1)^{n+1}}\right)_{n\in\mathbb{Z}}$$

$$\mu^{-1}u = \frac{h}{z-k}$$



Each axis gives partial fraction basis.

Your idea is that the integral should be some kind of residue.

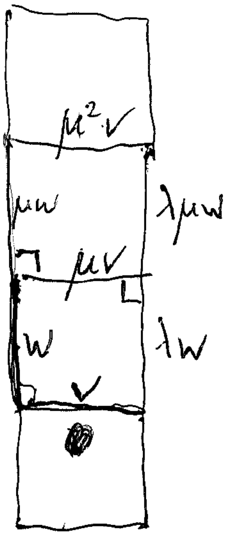
$H(v, \mu^n u)$

Example

$$\oint_{|z|=1} \frac{dz}{2\pi i z}$$

which will be expressible in terms of Res_0 Res_k (or Res_∞, $\mathrm{Res}_{k^{-1}}$)

Need $H(v, \mu^{n} w) = H(\mu^{n} v, w)$



$$\begin{pmatrix} \mu v \\ \lambda w \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}$$

$$\mu v = \underbrace{\frac{1}{k} v + \frac{h}{k} w}$$ $$1 = \frac{1}{k^2} - \frac{|h|^2}{k^2}$$

$$\mu^2 v = \frac{1}{k}\mu v + \frac{h}{k}\mu w$$

$$= \frac{1}{k^2} v + \frac{h}{k^2} w + \frac{h}{k}\mu w$$

$$H(\mu v, w) = \cancel{H\left(\frac{1}{k} v, w\right)} + H\left(\frac{h}{k} w, w\right)$$

$$= \frac{\bar{h}}{k}(-1)$$

$$\mu^3 v = \frac{1}{k^2}\left(\frac{1}{k} v + \frac{h}{k} w\right) + \frac{h}{k^2}\mu w + \frac{h}{k^3}\mu^2 w$$

$$= \frac{1}{k^3} v + \frac{h}{k^3} w + \frac{h}{k^2}\mu w + \frac{h}{k^3}\mu^2 w$$

$$H(\mu^2 v, w) = -\frac{\bar{h}}{k^2}$$

$$H(\mu^3 v, w) = -\frac{\bar{h}}{k^3}$$

So it seems that $H(v, \mu^{-n} w) = \begin{cases} 0 & -n \geq 0 \\ -\frac{\bar{h}}{k^n} & n > 0. \end{cases}$

~~Also~~ combine with

$$H(v, \lambda^n v) = \delta_n$$

$$\cancel{H(w, \mu^n w) = -\delta_n}$$

so what?

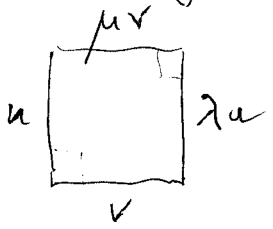
~~So~~ You have basis $\lambda^m v$, $\mu^n w$

and the linear functional $H(v, -)$

E of $\lambda^m\mu^n$ and V, W $\qquad \lambda = \text{mult by } z \qquad V = 1$

$\mu = \frac{z-k}{kz-1} \qquad W = \frac{h}{kz-1}$

Start again. E module ~~[illegible]~~ for group $\mathbb{Z}\times\mathbb{Z}$ etc.



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$\frac{k\lambda-1}{h}u = v \qquad \frac{k\mu-1}{\bar h}v = u$$

$$E = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}\lambda^m v + \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\mu^n u$$

model for E: $\qquad E' = \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}] \subset \mathbb{C}(z)$

$\lambda = \text{mult by } z \qquad v = 1$

$\mu = \text{mult by } \frac{z-k}{kz-1} \qquad u = \frac{h}{kz-1}$

Clearly E' is a $\mathbb{Z}\times\mathbb{Z}$ module

$$\frac{k\lambda-1}{h}u = \frac{kz-1}{h}\,\frac{h}{kz-1} = 1 = v$$

$$\frac{k\mu-1}{\bar h}v = \frac{1}{\bar h}\left(k\frac{z-k}{kz-1}-1\right)1 = \frac{1}{\bar h}\left(\frac{\overbrace{\cancel{kz}-k^2-\cancel{kz}+1}^{h\bar h}}{kz-1}\right) = \frac{h}{kz-1} = u$$

$$\bigoplus_{m\in\mathbb{Z}} \mathbb{C}\lambda^m v = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}z^m$$

$$\mu^n u = \left(\frac{z-k}{kz-1}\right)^n\frac{h}{kz-1} = h\frac{(z-k)^n}{(kz-1)^{n+1}}$$

$$\bigoplus_{n\geq 0} \mathbb{C}\mu^n u = \bigoplus_{n\geq 0}\mathbb{C}\frac{1}{(kz-1)^{n+1}}$$

$$\bigoplus_{n<0} \mathbb{C}\mu^n u = \bigoplus_{p\geq 0}\mathbb{C}\frac{1}{(z-k)^{p+1}}$$

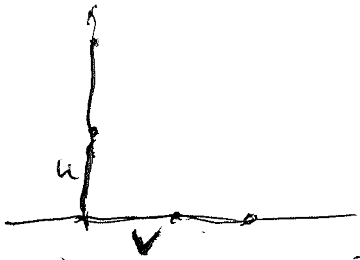
$$\mu^n u = h\frac{(kz-1)^{-n-1}}{(z-k)^{-n}} \qquad n\leq -1$$

Now you want $H(\xi, \xi')$ <u>hermitian form on E</u>

any (exhaustive) decreasing staircase gives an orthonormal basis ± 1 $|{-1}$

$$H\left(\lambda^m\mu^n\begin{pmatrix}v\\ u\end{pmatrix}, \lambda^{m'}\mu^{n'}\begin{pmatrix}v\\ u\end{pmatrix}\right)$$

$$H(v, \lambda^n v) = \delta_n$$
$$H(v, \mu^n u) = 0 \qquad n \geq 0$$



One point is that $H(\xi', \xi) = H(g\xi', g\xi) \quad \forall g \in \mathbb{Z}\times\mathbb{Z}$

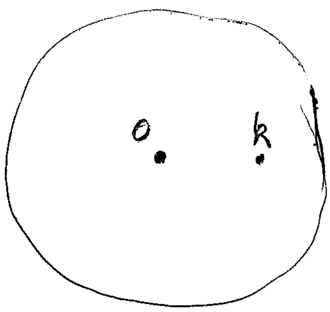
$$H(g\xi', \xi) = H(\xi', g^{-1}\xi)$$
$$\therefore \quad H(a\xi', \xi) = H(\xi', a^*\xi)$$
$$H(b'v, bv)$$

So ultimately it seems that if $f, g \in \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$

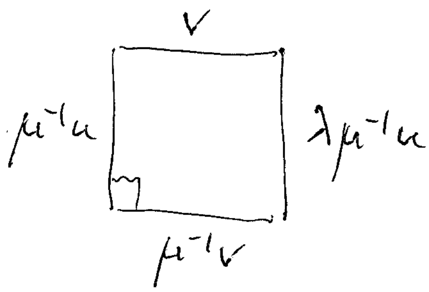
then $H(f, g) = H(1, f^* g)$

After $\int z^n = \delta_n$

$$\int \underbrace{h\frac{(z-k)^n}{(kz-1)^{n+1}}}_{\text{all the rational fns. having only } k^{-1} \text{ as pole}} = 0 \qquad n \geq 0.$$



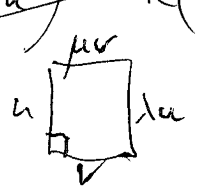
$$\frac{h}{z-k} = \mu^{-1}u = \frac{kz-1}{z-k}\,\frac{h}{kz-1}$$



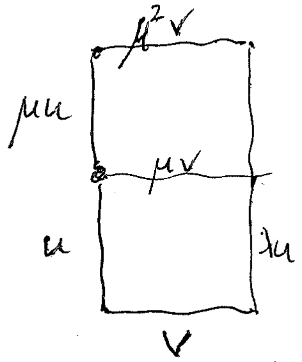
$$\mu v = \frac{\bar h}{k}u + \frac{1}{k}v$$
$$v = \frac{\bar h}{k}\mu^{-1}u + \frac{1}{k}\mu^{-1}v$$

$$\cancel{\begin{pmatrix} v \\ \lambda\mu^{-1}u \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix}}$$

$$H(v, \mu^{-1}u) = H(\mu v, u) = -\frac{h}{k}$$



You want $H(v, \mu^{-n}u) = H(\mu^n v, u)$



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$\boxed{\mu v = \frac{\bar h}{k}u + \frac{1}{k}v}$$

$$\mu^2 v = \frac{\bar h}{k}\mu u + \frac{1}{k}\mu v$$

$$= \frac{\bar h}{k}\mu u + \frac{1}{k}\left(\frac{\bar h}{k}u + \frac{1}{k}v\right)$$

$$\boxed{\mu^2 v = \frac{\bar h}{k}\mu u + \frac{\bar h}{k^2}u + \frac{1}{k^2}v}$$

$$\boxed{\mu^3 v = \frac{\bar h}{k}\mu^2 u + \frac{\bar h}{k^2}\mu u + \frac{\bar h}{k^3}u + \frac{1}{k^3}v}$$

$$\boxed{\mu^4 v = \frac{\bar h}{k}\mu^3 u + \frac{\bar h}{k^2}\mu^2 u + \frac{\bar h}{k^3}\mu u + \frac{\bar h}{k^4}u + \frac{1}{k^4}v}$$

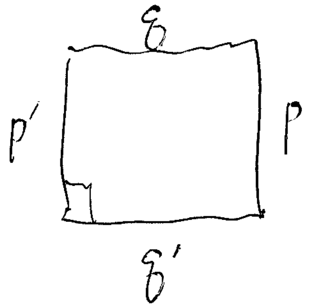
$$H(\mu^4 v, u) = H\left(\frac{\bar h}{k^4}u, u\right) = -\frac{h}{k^4}$$

$$\therefore \quad H(v, \mu^{-n}u) = -\frac{h}{k^n} \qquad n \geq 1$$

$$h\frac{(kz-1)^{n-1}}{(z-k)^n} \qquad \mu^{-n}u = \left(\frac{z-k}{kz-1}\right)^{-n}\frac{h}{kz-1} = h\frac{(kz-1)^{n-1}}{(z-k)^n}$$

$$h\frac{(-1)^{n-1}}{(-k)^n} = -\frac{h}{k^n}$$

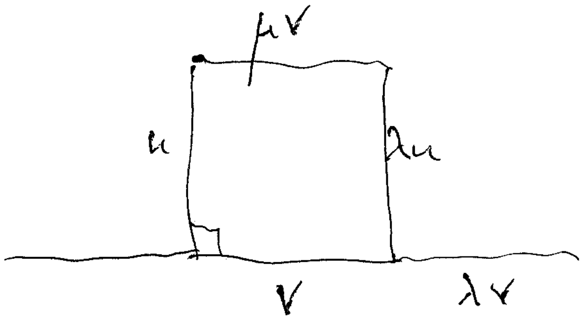
It looks as if $\quad H(v, fv) = \int_{|z|=r<k} f\,\frac{dz}{2\pi i z}$



$$\begin{pmatrix}p\\q\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}p'\\q'\end{pmatrix} \qquad k=\sqrt{1-|h|^2}$$

$$\begin{pmatrix}p\\q'\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar h & k\end{pmatrix}\begin{pmatrix}p'\\q\end{pmatrix}$$

$$\begin{pmatrix}p\\q\end{pmatrix}^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}p'\\q'\end{pmatrix}^*\frac1k\begin{pmatrix}1&h\\\bar h&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac1k\begin{pmatrix}1&h\\\bar h&1\end{pmatrix}\begin{pmatrix}p'\\q'\end{pmatrix}$$



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac1k\begin{pmatrix}1&h\\\bar h&1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$$

$$(k\lambda-1)u = hv$$
$$(k\mu-1)v = \bar h u$$

model. $\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$.

$\lambda =$ mult by z $\qquad v = 1$

$\mu =$ mult by $\frac{z-k}{kz-1}$ $\qquad u = \frac{h}{kz-1}$

$$(k\lambda-1)u = (kz-1)\frac{h}{kz-1} = h = hv$$
$$(k\mu-1)v = \left(k\frac{z-k}{kz-1}-1\right) = \frac{\cancel{kz}-k^2-\cancel{kz}+1}{kz-1} = \frac{h\bar h}{kz-1} = \bar h u$$

You can obviously change v,u by mult. by something like z!! ~~NO! NO! NO!~~

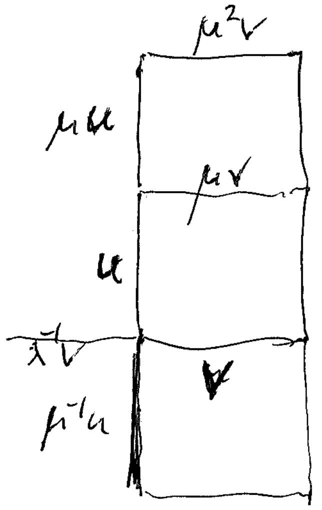
$$\mu v = \frac{\bar h}{k}u + \frac1k v \qquad H(v,u)=0$$

$H(v,\mu^{-1}u) = ?$ $\qquad v = \frac{\bar h}{k}\mu^{-1}u + \frac1k \mu^{-1}v$

$\|$

$$H(v,v) = \frac{\bar h}{k}H(v,\mu^{-1}u) + \frac1k H(v,\mu^{-1}v)$$

$$H(\mu v,u) = H\left(\frac{\bar h}{k}u + \cancel{\frac1k v}, u\right) = -\frac{h}{k}$$



Find the solution

$H(v,-)$

$H(v,\lambda^n v) = \delta_n$

$H(v,\mu^n u) = 0 \qquad n \geq 0.$

$\mu v = \frac{\bar h}{k}u + \frac{1}{k}v$

$H(\mu v, u) = \frac{\bar h}{k}(-1)$

$H(\mu^2 v, u) = \frac{h}{k^2}(-1)$

$H(v, \mu^{-n}u) = H(\mu^n v, u)$

$\mu^2 v = \frac{\bar h}{k}\mu u + \frac{\bar h}{k^2}u + \frac{1}{k^2}v$

$\mu^3 v = \frac{\bar h}{k}\mu^2 u + \frac{\bar h}{k^2}\mu u + \frac{h}{k^3}u + \frac{1}{k^3}v$

$$H(v,\mu^{-n}u) = H(\mu^n v, u) = \begin{cases} 0 & n \leq 0 \\ -\frac{h}{k^n} & n \geq 1 \end{cases}$$

$$\mu^{-n}u = \left(\frac{z-k}{kz-1}\right)^{-n}\frac{h}{kz-1} = h\frac{(kz-1)^{n-1}}{(z-k)^n} \qquad n \geq 1$$

$H(v,\mu^{-n}u) = -\frac{h}{k^n} \qquad n \geq 1$ = values of $h\frac{(kz-1)^{n-1}}{(z-k)^n}$ at $z=0$

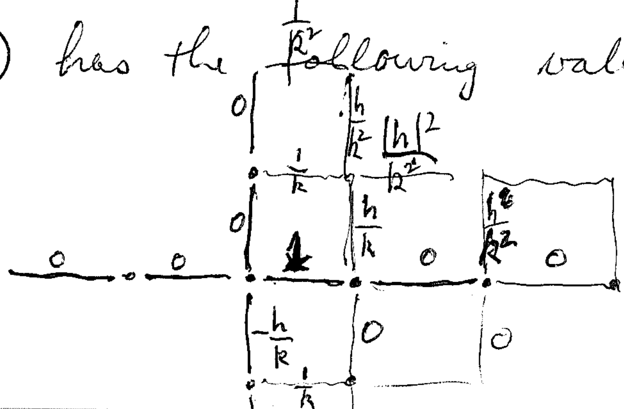
$H(v,\mu^{+n}u) = 0 \qquad n \geq 0$

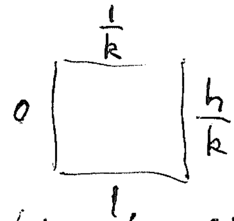
$\mu^n u = h\frac{(z-k)^n}{(kz-1)^{n+1}}$

$u = \frac{h}{kz-1}$ value at $z=0$ of $-h$

$\mu u = \frac{h(z-k)}{(kz-1)^2}$ value $-hk$ at $z=0$

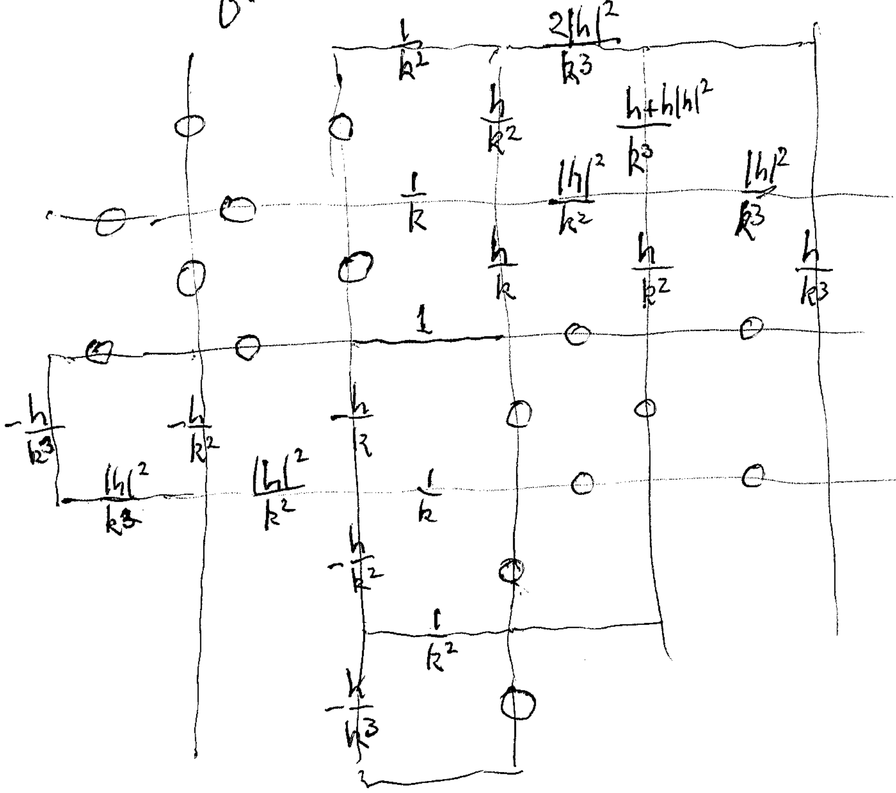
So $H(v,-)$ has the following values

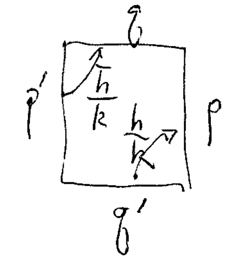




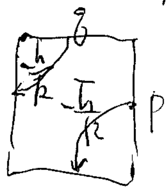
$$\begin{pmatrix} \frac{1}{k} & \frac{h}{k} \\ \frac{\bar h}{k} & \frac{1}{k} \end{pmatrix}\begin{pmatrix} \frac{h}{k} \\ 0 \end{pmatrix}$$

indefinite





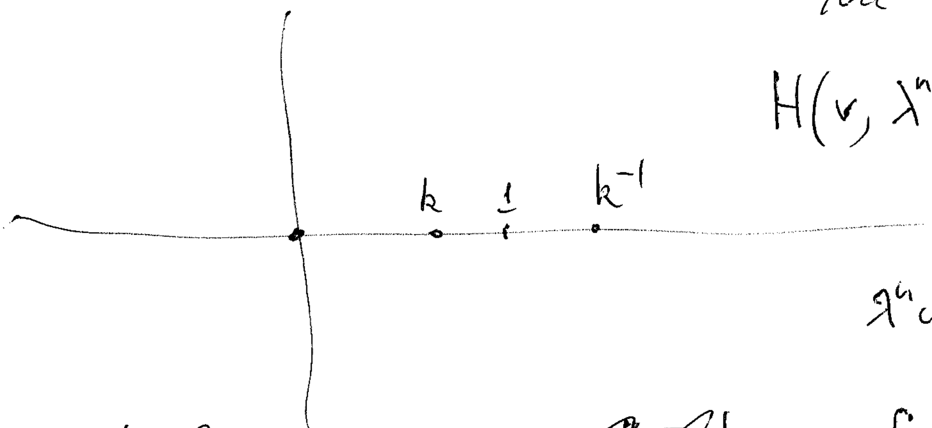
$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} \frac{1}{k} & -\frac{h}{k} \\ -\frac{\bar h}{k} & \frac{1}{k} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$



Look at
$$H(v, \mu^n v) = \frac{1}{k^{|n|}} \qquad \mu^n v = \left(\frac{z-k}{kz-1}\right)^n$$
$$H(v, \mu^n u) = \begin{cases} 0 & n \geq 0 \\ -\frac{h}{k^{-n}} & n < 0 \end{cases} \qquad \mu^n u = h \frac{(z-k)^n}{(kz-1)^{n+1}}$$

dealing with rational functions poles at $0, \infty, k, k^{-1}$.

You want to understand
$$H(v, \lambda^n u) = \begin{cases} 0 & n \leq 0 \\ \frac{h}{k^n} & n \geq 1. \end{cases}$$

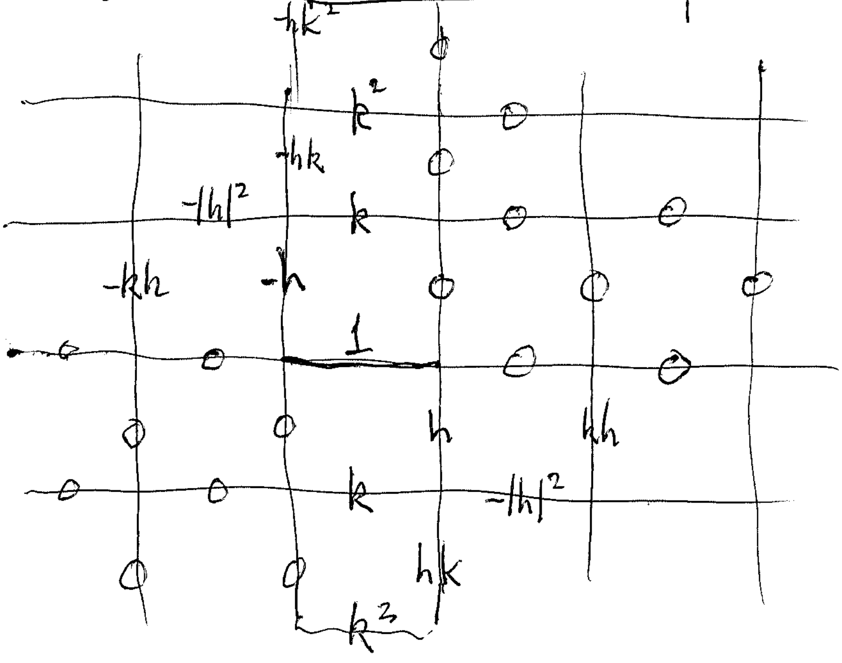


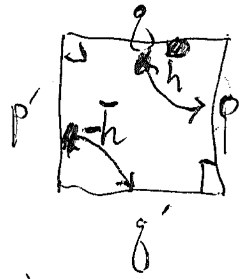
$$\lambda^n u = z^n \frac{h}{kz-1}$$

$$\oint z^n \frac{h}{kz-1}\, dz$$

~~try taking~~

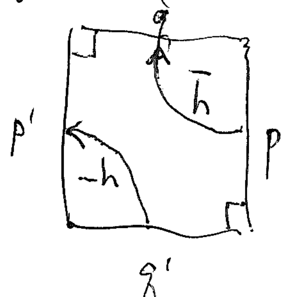
Look at the positive inner product $(v|-)$





$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar h & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

$$\begin{pmatrix} p' \\ q \end{pmatrix} = \begin{pmatrix} k & -h \\ \bar h & k \end{pmatrix}\begin{pmatrix} p \\ q' \end{pmatrix}$$



$(v|\mu^n v) = k^{|n|}$?

$$\oint_{|z|=1} \left(\frac{z-k}{kz-1}\right)^n \frac{dz}{2\pi i z} = \left(\frac{-k}{-1}\right)^n \qquad n \geq 0.$$

$$\oint_{|z|=1} \left(\frac{kz-1}{z-k}\right)^n \frac{dz}{2\pi i z} = \cancel{\oint_{|z|=1} \frac{(kz^{-1}-1)^n}{(z^{-1}-k)^n} \frac{dz}{2\pi i z}} = k^n$$

analytic except at $z=k$ inside

push contour out

$\forall m,\ (v|\lambda^m v) = \delta_m \qquad \int z^m \frac{d\theta}{2\pi} = \delta_n$ ✓

$$(v|\mu^n u) = \int \underbrace{h\frac{(z-k)^n}{(kz-1)^{n+1}}}_{\text{analyt in } |z|<1 \text{ except at } z=0} \frac{dz}{2\pi i z} = h\frac{(-k)^n}{(-1)^{n+1}} = -hk^n$$

$n \geq 0$

Go back to the ~~indefinite~~ indefinite case. You hope to choose a closed contour C so that $H(v, fv)$ is given by $\oint f \frac{dz}{2\pi i z}$. The contour is a homology class in $\mathbb{C} - \{0, k, k^{-1}\}$ whose first homology gp should be $\mathbb{Z}^3$ giving the residues at $0, k, k^{-1}$. The form of the differential namely $\frac{dz}{2\pi i z}$ is suggested by $H(v, \lambda^m v) = \oint z^m \frac{dz}{2\pi i z} = \delta_m$. This integral does the correct thing when ~~there are no~~ f has no poles at k, k^{-1}. Work out residues

Other basis elements are $\mu^n u$

$$\lambda^n u = z^n \frac{h}{kz-1} \qquad H(v, \lambda^n u) = \begin{cases} 0 & n \leq 0 \\ \frac{h}{k^n} & n \geq 1 \end{cases}$$

$$\operatorname*{res}_{z=k^{-1}} \left\{ z^n \frac{h}{kz-1} \frac{dz}{z} \right\} = k^{-n+1} \frac{h}{k} = \frac{h}{k^n} \quad \forall n$$

$$\operatorname*{res}_{z=0} \left\{ \underbrace{z^n \frac{h}{kz-1} \frac{dz}{z}} \right\} = \begin{cases} 0 & \text{if } n \geq 1 \\ -h & n = 0 \\ -hk^{-n} & n \leq 0. \end{cases}$$

$$\frac{-h}{z^{1-n}} \frac{1}{1-kz} dz$$

$$= -\frac{h}{z^{1-n}} \sum_{q \geq 0} k^q z^q \, dz$$

$$\operatorname*{res}_{z=0} \left\{ -h \frac{\sum k^q z^q}{z^{-n}} \frac{dz}{z} \right\} = -hk^{-n} \qquad n \leq 0.$$

$$\left\{ \operatorname*{res}_{z=k^{-1}} + \operatorname*{res}_{z=0} \right\} \left\{ z^n \frac{h}{kz-1} \frac{dz}{z} \right\} = \begin{cases} \frac{h}{k^n} + 0 & n \geq 1 \\ \frac{h}{k^n} - hk^{-n} & n \leq 0 \end{cases}$$

$$H(v, \underbrace{\mu^n u}) = \begin{cases} 0 & n \geqslant 0 \\ -\dfrac{h}{k^{-n}} & n < 0 \end{cases}$$

$$\operatorname{res}_{z=0} \left\{ h \frac{(z-k)^n}{(kz-1)^{n+1}} \frac{dz}{z} \right\} = h \frac{(-k)^n}{(-1)^{n+1}} = -hk^n \qquad \forall n \in \mathbb{Z}.$$

$$\operatorname{res}_{z=k^{-1}} \left\{ \qquad \right\} \overset{\text{if } n<-1}{=} \operatorname{res}_{z=k^{-1}} \left\{ h \frac{(kz-1)^{\overset{\geqslant 0}{-n-1}}}{(z-k)^{-n}} \frac{dz}{z} \right\}$$

$$= 0 \quad \text{if} \quad -n-1 > 0 \quad \text{i.e. } n < -1$$

$$= 0 \quad \text{if} \quad n = -1.$$

because $\dfrac{h}{(z-k)z}$ is analytic at $z = k^{-1}$!

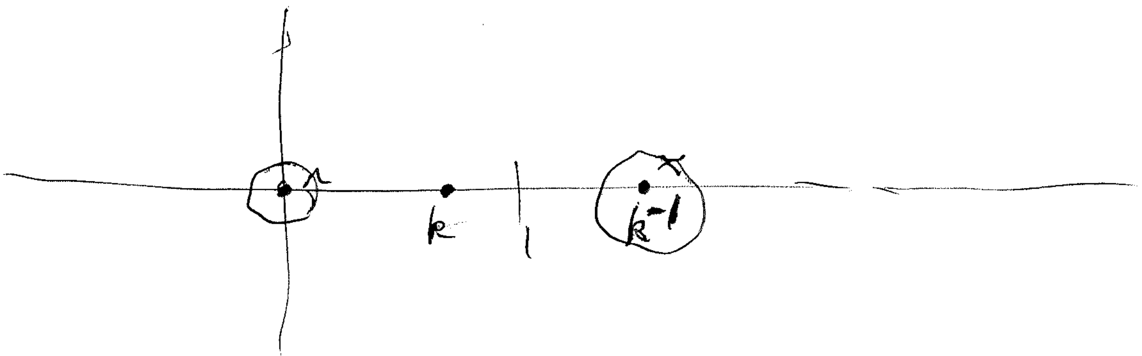
$$\operatorname{res}_{z=0} + \operatorname{res}_{z=k^{-1}} = -\frac{h}{k^{-n}} + 0 = -hk^n \qquad \text{OK.}$$

for $n \leqslant -1$

Remains $\operatorname{res}_{z=k^{-1}} \left\{ h \dfrac{(z-k)^n}{(kz-1)^{n+1}} \dfrac{dz}{z} \right\}$ for $n \geqslant 0$.

$$= -\Big(\operatorname{res}_{z=0} + \underset{\overset{\shortparallel}{0}}{\operatorname{res}_{z=k}} + \underset{\overset{\shortparallel}{0}}{\underbrace{\operatorname{res}_{z=\infty}}}\Big) = -\operatorname{res}_{z=0}$$

$$\operatorname{res}_{z=0} + \operatorname{res}_{z=k^{-1}} = -\left(\operatorname{res}_{z=k} + \operatorname{res}_{z=\infty}\right)$$



Continuous case. Review.

$$\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_y \end{pmatrix} \psi_{xy} = \begin{pmatrix} 0 & h_0' \\ \bar{h}_0' & 0 \end{pmatrix} \psi_{xy}$$

$\psi^2_{x,y+\varepsilon}$

ψ^1_{xy} $\quad$ $\psi^1_{x+\varepsilon,y}$

$\psi^2_{x,y}$

$$\psi_{xy} = \lambda^x \mu^y \begin{pmatrix} u \\ v \end{pmatrix} \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \psi_{00}$$

repn. in $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$ $\qquad$ $\lambda^x =$ mult by $e^{i\xi x}$ $\qquad$ $v = 1$

$\mu^y =$ ——— $e^{-i\xi^{-1} y}$ $\qquad$ $u = \frac{1}{i\xi}$

$$\begin{pmatrix} i\xi \, u \\ i\eta \, v \end{pmatrix} = \begin{pmatrix} h_0' v \\ \bar{h}_0' u \end{pmatrix} \qquad h_0' = 1. \qquad \cancel{= i\xi^{-1} = u?}$$

$$H(u, fv) = \operatorname{res}_{\{0, k^{-1}\}} \Big(\underbrace{\left(\frac{h}{kz-1}\right)^* (f)}_{\frac{\bar{h}}{kz^{-1}-1} f} \frac{dz}{z} \Big)$$

$$= \operatorname{res}_{\{0, k^{-1}\}} \left(\frac{\bar{h}}{k-z} f \, dz \right)$$

$$= + \operatorname{res}_{\{k, \infty\}} \left(\frac{\bar{h}}{z-k} f \, dz \right) \qquad fv = \frac{h}{kz-1}$$

$$= \operatorname{res}_{\{k, \infty\}} \left(\frac{|h|^2}{(z-k)(kz-1)} dz \right) = \frac{|h|^2}{k^2-1} = -1.$$

$$\cancel{H(u, A^n u) = \operatorname{res}_{\{0, k^{-1}\}} \left(\frac{\bar{h}}{k-z} z^n \frac{h}{kz-1} dz \right)}$$

$$\cancel{= \operatorname{res}_{\{k, \infty\}} \left(\frac{|h|^2 z^n}{(z-k)(kz-1)} dz \right)}$$

$$\cancel{= \operatorname{res}_\infty + \frac{|h|^2 k^n}{k^2-1}} \qquad = -k^n$$

0 if

$$H(u, fv) = \operatorname{res}_{\{0,k^{-1}\}}\left(\left(\frac{h}{kz-1}\right)^* f\,\frac{dz}{z}\right)$$

$$H(u, \mu^n u) = \operatorname{res}_{\{0,k^{-1}\}}\left(\frac{\bar h}{k-z}\left(\frac{z-k}{kz-1}\right)^n \frac{h}{kz-1}\,dz\right)$$

$$= \operatorname{res}_{\{k,\infty\}}\left(|h|^2\,\frac{(z-k)^{n-1}}{(kz-1)^{n+1}}\,dz\right) \qquad \operatorname{res}_\infty = 0$$

$$\operatorname{res}_k = 0 \quad \text{if} \quad n \geq 1 \qquad \left(\text{if } n=0 \quad \frac{|h|^2}{k^2-1} = -1\right)$$

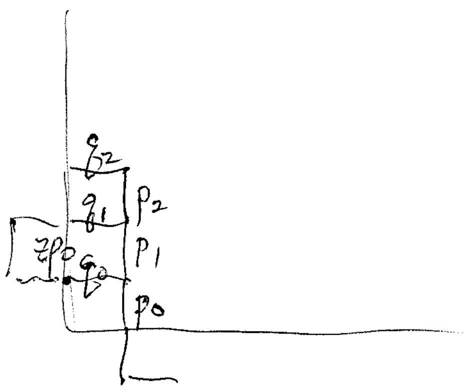
$$H(u, \mu^{-n} u) = \operatorname{res}_{\{0,k^{-1}\}}\left(\frac{\bar h}{k-z}\,\frac{(kz-1)^{n-1}}{(z-k)^n}\,\frac{h}{\cancel{kz-1}}\,dz\right)$$

$$\operatorname{res}_0 = 0 \qquad \operatorname{res}_{k^{-1}} = 0 \quad \text{if} \quad n \geq 1$$

~~[illegible] point to see that [illegible]~~

Look at continuous case – try following idea, ~~namely~~

$$\operatorname{res}_0 + \operatorname{res}_{k^{-1}} = \underbrace{\operatorname{res}_0 + \operatorname{res}_k}_{\displaystyle\int_{\xi\in\mathbb{R}} \frac{d\xi}{2\pi}} + \underbrace{\operatorname{res}_{k^{-1}} - \operatorname{res}_k}_{\text{might have a meaning as } h\to 0 \text{ so } k\uparrow 1}$$



$\psi_{mn} = \lambda^m \mu^n \begin{pmatrix} u \\ v \end{pmatrix}$ $\quad \begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$\frac{k\lambda - 1}{h} u = v \qquad \frac{k\mu - 1}{h} v = u.$ $\qquad (k\lambda - 1)(k\mu - 1) = 1 - k^2$

$$\mu = \frac{1}{k}\left(1 + \frac{1-k^2}{k\lambda - 1}\right) = \frac{\lambda - k}{k\lambda - 1}$$

model $\quad \lambda =$ mult by z $\qquad u = \frac{h}{kz-1}$

$\mu =$ mult by $\frac{z-k}{kz-1}$ $\qquad v = 1 \qquad \frac{dz}{z}$

want linear functions $L = \operatorname{res}_{k^{-1}} - \operatorname{res}_k$ on $\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$.

Go back over ~~the~~ your basis

$\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$ by partial fractions has the basis $\underbrace{z^m}_{\lambda^m v}$, $m \in \mathbb{Z}$ and $\underbrace{h\frac{(z-k)^n}{(kz-1)^{n+1}}}_{\mu^n u}$ $n \in \mathbb{Z}$.

~~Examine~~ $\qquad \operatorname{res}_0 + \operatorname{res}_k = \oint_{|z|=1}$ $\qquad v = (-h)\frac{1}{1-k\mu} u = \sum_{n\geq 0}(-h)k^n \mu^n u$

$$(v|\mu^n u) = \int_{|z|=1} h \frac{(z-k)^n}{(kz-1)^{n+1}}\frac{dz}{2\pi i z} = \begin{cases} 0 & n \leq -1 \\ -hk^n & n \geq 0 \end{cases}$$

$\therefore (\mu^n u|v) = -\bar{h}k^n \quad n \geq 0$

$(v|\mu^n u) = -hk^n \quad n \geq 0$

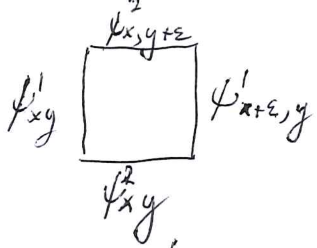
$$(v|\lambda^m v) = \int z^m \frac{dz}{2\pi i z} = \delta_m$$

You want to look seriously at

$$\underbrace{(\operatorname{res}_{k^{-1}} - \operatorname{res}_k)}\left(\oint \frac{dz}{z}\right)$$

Is it possible to ~~generalize~~ extend your basis. Recall the continuous version

Cont. version



$$\begin{pmatrix}\psi^1_{x+\varepsilon,y}\\ \psi^2_{x,y+\varepsilon}\end{pmatrix} = \begin{pmatrix} 0 & h\varepsilon \\ \bar h\varepsilon & 0\end{pmatrix}\begin{pmatrix}\psi^1_{xy}\\ \psi^2_{xy}\end{pmatrix}$$

~~Now suppose~~

$$\psi_{xy} = \lambda^x \mu^y \begin{pmatrix} u \\ v\end{pmatrix} \qquad \partial_x \psi^1 = h\psi^2$$

$$\partial_x \psi^1 = \lambda^x\mu^y \begin{pmatrix} i\xi u \\ i\eta v\end{pmatrix} \qquad \begin{aligned} i\xi u &= hv \\ i\eta v &= \bar h u \\ -\xi\eta &= |h|^2\end{aligned} \qquad \partial_y\psi^2 = \bar h \psi^1$$

put $h = 1$.

model $\quad L^2\left(\mathbb{R}, \frac{d\xi}{2\pi}\right)$

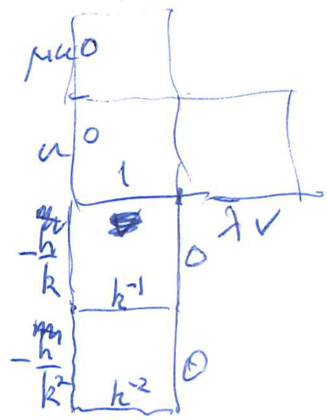
$\lambda^x =$ mult by $e^{i\xi x}$ $\qquad v = 1$

$\mu^y =$ ——— $e^{i\eta y}$ $\qquad u = \frac{1}{i\xi}$ instead of $u = \frac{h}{kz-1}$

can understand by $\quad \frac{h\varepsilon}{kz^{\varepsilon}-1} \approx \frac{h}{i\xi} = \frac{1}{i\xi}$.

~~orthonormal basis~~ In discrete case you have the basis cons. of

$$\begin{cases} \lambda^m v & m \in \mathbb{Z} \\ \mu^n u & n \in \mathbb{Z}.\end{cases}$$



From your viewpoint what's important is the linear fnl $H(v, -)$. Recall calculation

$$H(v, \lambda^m v) = \delta_m$$

$$H(v, \mu^n u) = \begin{cases} 0 & \text{for } n \geq 0 \\ -\frac{h}{k^{-n}} & n \leq -1\end{cases}$$

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h \\ \bar h & 1\end{pmatrix}\begin{pmatrix}-\frac{h}{k} \\ +\frac{1}{k}\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{k^2}\\ -\frac{h}{k^2}\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & -h \\ -\bar h & 1\end{pmatrix}\begin{pmatrix}k^{-1} \\ 0\end{pmatrix}$$

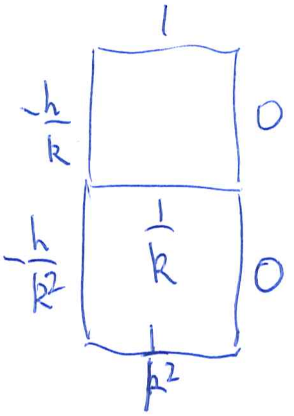
and the check.

$$\left(\operatorname{res}_0\right)\left(h\frac{(z-k)^n}{(kz-1)^{n+1}}\frac{dz}{z}\right) = h\frac{(-k)^n}{(-1)^{n+1}} = -hk^n \qquad \forall n \in \mathbb{Z}.$$

$$\left(\operatorname{res}_\infty\right)\left(h\frac{(z-k)^n}{(kz-1)^{n+1}}\frac{dz}{z}\right) = 0$$

$$\left(\operatorname{res}_0 + \operatorname{res}_k\right)\left(\;-\!-\;\right) = \frac{1}{2\pi i}\oint_{|z|=1} h\frac{(z-k)^n}{(kz-1)^{n+1}}\frac{dz}{z} = \begin{cases} -hk^n & n \geq 0 \\ 0 & n \leq -1\end{cases}$$

$$\frac{(kz-1)^{-n-1}}{(z-k)^{-n}}\frac{dz}{z}$$



$$\begin{pmatrix} 0 \\ \bullet \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} ? \\ k^{-1} \end{pmatrix}$$

$$\begin{pmatrix} ? \\ k^{-1} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & -h \\ -h & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \bullet \end{pmatrix}$$

$$\begin{pmatrix} \\ \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & -h \\ -h & 1 \end{pmatrix}\begin{pmatrix} -\frac{h}{k} \\ \frac{1}{k} \end{pmatrix}$$

$$-\operatorname{res}_{\{0,k,\infty\}} = \operatorname{res}_{k^{-1}}\left(h \frac{(z-k)^n}{(kz-1)^{n+1}} \frac{dz}{z} \right)$$

$$\|$$

$$-\operatorname{res}_{\{0,k\}} = \begin{cases} hk^n & n \geq 0 \\ 0 & n \leq -1 \end{cases}$$

~~Results~~ too hard.

$$\operatorname{res}_0 = -hk^n$$

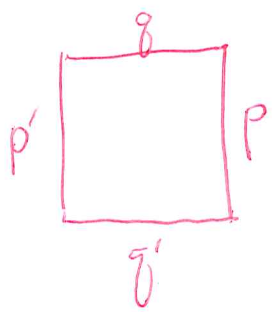
$$\operatorname{res}_\infty = 0$$

$$\operatorname{res}_k = \begin{cases} 0 & n \geq 0 \\ hk^n & n \leq -1 \end{cases}$$

$$\operatorname{res}_{k^{-1}} = \begin{cases} hk^n & n \geq 0 \\ 0 & n \leq -1 \end{cases}$$

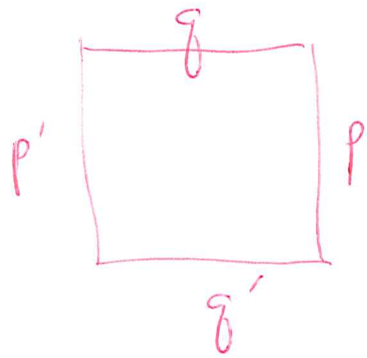
$$\operatorname{res}_{k^{-1}} - \operatorname{res}_k = \begin{cases} hk^n & n \geq 0 \\ -hk^n & n \leq -1 \end{cases}$$

So what about today's lecture



$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

So what is the point? You want transfer isos.

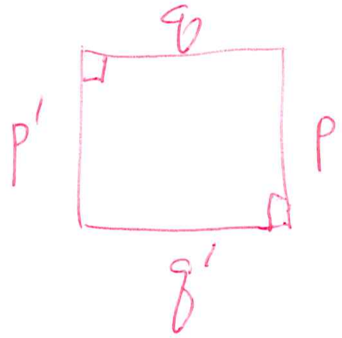


$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

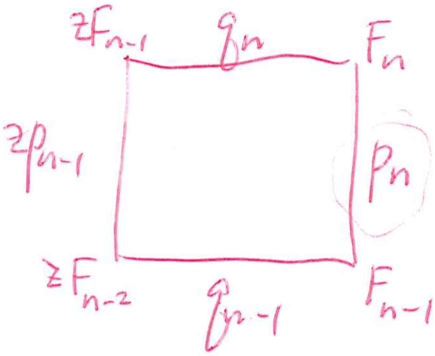
$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

transfer map.

Take the unitary picture



$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k \end{pmatrix}$$



$$p_n = a z p_{n-1} + b q_n$$

$$\boxed{q_{n-1} = c z p_{n-1} + d q_n}$$

$\delta > 0$

$$\boxed{\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}}$$

$d > 0$

$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} z p_{n-1} \\ q_n \end{pmatrix}$$

Start again

$$\begin{array}{ccccc} zF_{n-1} & & q_n & & F_n \\ & \square & & & \\ zp_{n-1} & & & & p_n \\ & & & & \\ zF_{n-2} & & q_{n-1} & & F_{n-1} \end{array}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{u(1,1)} \begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix} \qquad a, d > 0.$$

Is there an alternative to calculating residues?

~~Ha~~ $(k\lambda - 1)(k\mu - 1) = 1 - k^2$. ~~Id~~ Idea is to understand H on the first quadrant OK what's new.

$$\mathbb{C}[\lambda,\mu]u + \mathbb{C}[\lambda,\mu]v \Big/ \left(\frac{k\lambda-1}{h}u = v,\ \frac{k\mu-1}{h}v = u \right)$$

~~Wrong~~

$$H(v, \cancel{\ \ } v) = 1$$

$$H\left(v, \frac{k\lambda-1}{h} v\right) = -\frac{1}{h}$$

$$H\left(v, \left(\frac{k\lambda-1}{h}\right)^2 v\right) = \left(-\frac{1}{h}\right)^2$$

$$H\left(v, \left(\frac{k\lambda-1}{h}\right)^n v\right) = \left(-\frac{1}{h}\right)^n \quad \text{for } n \geq 0.$$

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variables

$$H(v, u) = 0$$

$$H\left(v, \left(\frac{k\mu-1}{h}\right)^n u\right) = 0 \qquad n \geq 0.$$

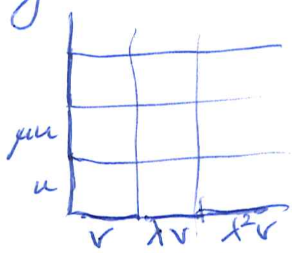
Check this $\left(res_0 + res_{k^{-1}}\right)\left\{ f(z)\frac{dz}{z} \right\} = H(v, fv)$

$$\begin{array}{l} res_0 \\ + \\ res_{k^{-1}} \end{array} \left\{ \left(\frac{kz-1}{h}\right)^n \frac{dz}{z} \right\} = \left(-\frac{1}{h}\right)^n \qquad n \geq 0 \qquad \text{since anal at } k^{-1}$$

$$\text{''} \quad -\cancel{res_k} - res_\infty = 0 \qquad \text{if} \quad n \leq -1$$

~~[scribbled out]~~ Somehow you need to get a picture of the first quadrant, that emphasizes the operators $X = \frac{k\lambda - 1}{h}$ $Y = \frac{k\mu - 1}{\bar{h}}$. It might not work because of $*$. ~~Thus~~ Your idea is that H ~~is determined by~~ reduces to the linear functional $H(v, -)$.

~~Th~~ approach



~~[scribbled out]~~ H is well understood with this basis: ~~[scribbled out] basis~~ elements ~~form~~ orthonormal basis up to signs. Algebraically it's a mess so far.

Standard repn $\quad \lambda = z. \quad v = 1 \qquad u \in H$!!

$\mu = \frac{\cancel{z} - k}{kz - 1}. \quad u = \frac{h}{kz-1}$

Instead of λ, μ you want to use ~~the~~ the operators $X = \frac{k\lambda - 1}{h}$ $Y = \frac{k\mu - 1}{\bar{h}}$ $\ni$ $Xu = v$, $Yv = u$

$\lambda = \frac{1 + hX}{k}$ Possibly you need a different viewpoint. The obvious symmetries λ, μ of the grid ~~may be replaced~~ might not be convenient. ~~[scribbled out]~~ We have to ~~[scribbled out]~~ get used to X^* Y^*.

$X = \frac{k\lambda - 1}{h} \qquad X^* = \frac{k\lambda^{-1} - 1}{\bar{h}}$

$XX^* = \frac{k^2 + 1 - k(\lambda + \lambda^{-1})}{|h|^2} \qquad Y = \frac{k\mu - 1}{\bar{h}} = \frac{h}{k\lambda - 1}$

What do we have? $\mathbb{C}[X, X^{-1}]$ with a hermitian form.

$X = \frac{k\lambda - 1}{h}$ $\quad \lambda = \frac{1+hX}{k}$ $\quad$ injective on $\mathbb{C}[X]$

$X^{-1} = Y = \frac{k\mu - 1}{\bar h}$ $\quad \mu = \frac{1+\bar h X^{-1}}{k}$ $\quad$ ———— $\mathbb{C}[X^{-1}]$

$\frac{k\mu - 1}{\bar h} v = u$ $\quad \frac{k\lambda - 1}{h} u = v$

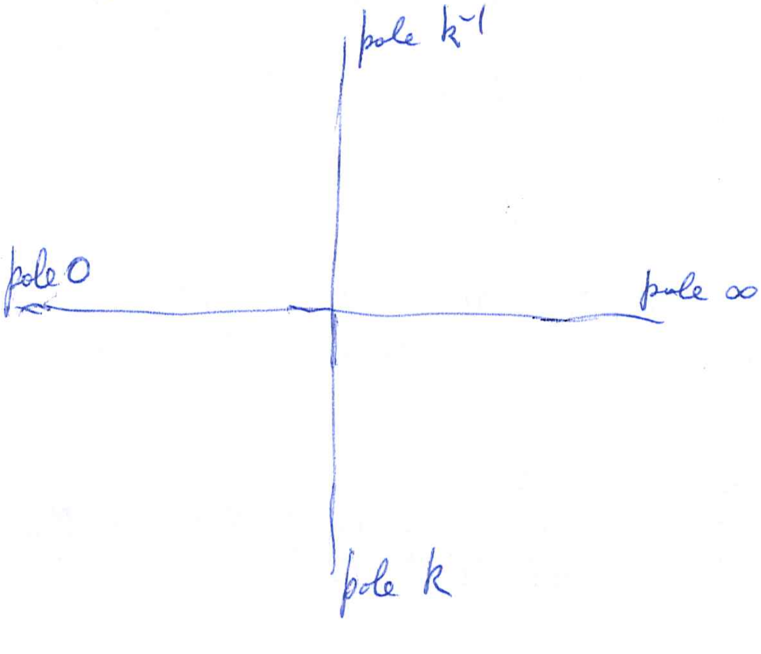
$\therefore \quad u = \frac{h}{k\lambda - 1} v$

$X = \frac{k\lambda - 1}{h}$ = mult by $\frac{kz-1}{h}$

$$Y = \frac{k\mu - 1}{\bar h} = \frac{1}{\bar h}\left(k\frac{z-k}{kz-1} - 1\right) = \frac{1-k^2}{\bar h(kz-1)} = \frac{h}{kz-1} = X^{-1}!$$

The first quadrant space has the basis

$X^n v = X^{n-1} u \quad n \in \mathbb{Z}.$ $\qquad X^n v = \frac{(kz-1)^n}{h^n}$



$\lambda^{-1} v \qquad \mu^{-1} u$

$X^* = \frac{k\lambda^{-1} - 1}{\bar h} \qquad Y^* = \frac{k\mu^{-1} - 1}{h}$

$$\mu^{-n} u = h\frac{(z-k)^{-n}}{(kz-1)^{n+1}} = h\frac{(kz-1)^{n-1}}{(z-k)^n}$$

$\mu^{-1} u = \frac{h}{z-k} \qquad \lambda^{-1} v = z$

So $\frac{z-k}{hz} = \frac{1-k\lambda^{-1}}{h}$

$$X = \frac{k\lambda - 1}{h} \qquad X^* = \frac{k\lambda^{-1} - 1}{\bar h} \qquad \mu = \frac{\lambda - k}{k\lambda - 1}$$

$$\lambda X^{-1} X^* = \lambda \frac{k\lambda^{-1} - 1}{k\lambda - 1} \frac{h}{\bar h} = \left(-\frac{h}{\bar h}\right) \frac{\lambda - k}{k\lambda - 1} = -\frac{h}{\bar h}\mu$$

$$H(\mu^n u, \lambda^m v) = H(\overset{X^{-1}v}{u}, \mu^{-n}\lambda^m v) = H(v, (X^{-1})^* \mu^{-n}\lambda^m v)$$

$$X^{-1} = Y = \frac{k\mu - 1}{\bar h} \qquad Y^* = \frac{k\mu^{-1} - 1}{h} \qquad = H\left(v, \frac{k\mu^{-1} - 1}{h} \mu^{-n}\lambda^m v\right)$$

$$\frac{k\mu^{-1} - 1}{h} = \frac{1}{h}\left(k\left(\frac{kz - 1}{z - k}\right) - 1\right) = \frac{k^2 z - k - z + k}{h} = -\bar h z$$

~~So what next ???????~~

$$\left(\frac{k\mu - 1}{\bar h}\right)^* = \frac{k\mu^{-1} - 1}{h} = \frac{1}{h}\left(k\frac{kz - 1}{z - k} - 1\right)$$

$$= \frac{1}{h}\left(\frac{k^2 z - \cancel{k} - z + \cancel{k}}{z - k}\right) = \frac{-(1 - k^2)}{h} \frac{z}{z - k}$$

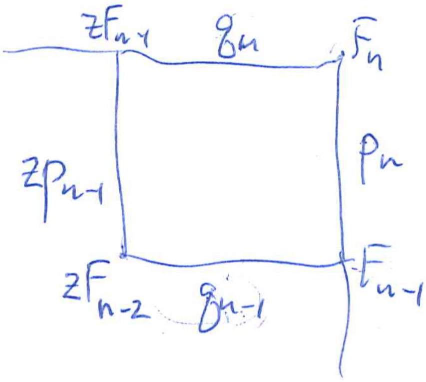
$$= \frac{-\bar h z}{z - k} = \frac{-\bar h}{1 - kz^{-1}}$$

$$\left(\frac{k\lambda - 1}{h}\right)^* = \frac{k\lambda^{-1} - 1}{\bar h} = \frac{k - z}{\bar h z} = \frac{-(1 - kz^{-1})}{\bar h} \qquad \text{YES.}$$

$$X = \frac{kz - 1}{h} \qquad X^{-1} = Y = \frac{h}{kz - 1}$$

$$X^* = \frac{kz^{-1} - 1}{\bar h} = -\frac{z - k}{\bar h z} \qquad Y^* = \frac{\bar h}{kz^{-1} - 1} = -\frac{\bar h z}{z - k}$$

Concentrate



~~Let~~

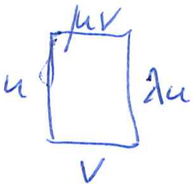
$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix}$$

with $a, d > 0$..

But

$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} \text{unitary} \end{pmatrix} \begin{pmatrix} zp_{n-1} \\ q_n \end{pmatrix}$$

What is your aim? To work on the discrete case until it generalizes to the cont. case. Discuss discrete case results.



$$\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

E = $\mathbb{C}[\lambda, \mu, \lambda^{-1}, \mu^{-1}]$ mod gen. by u, v subject to

$$\frac{k\lambda - 1}{h} u = v$$

$$\frac{k\mu - 1}{\bar{h}} v = u$$

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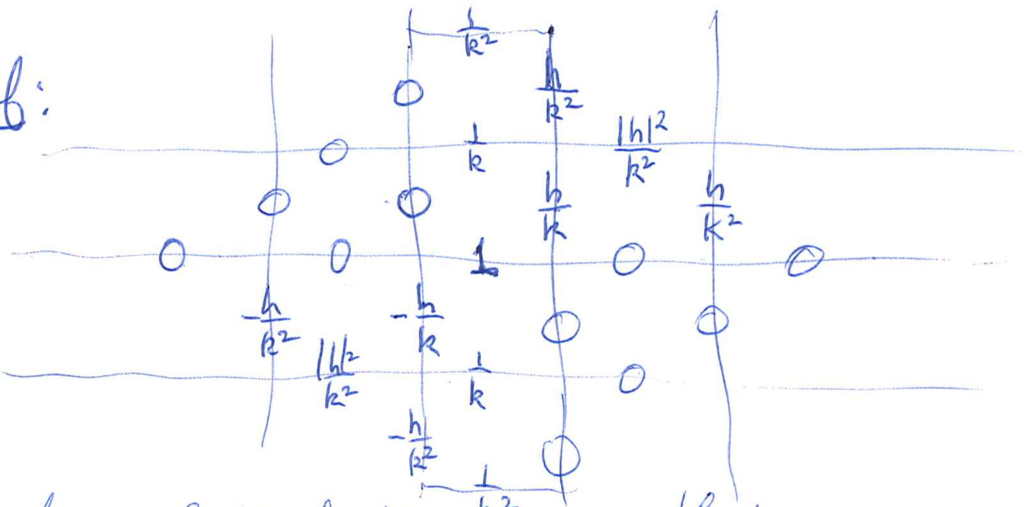
basis of E consisting of $\lambda^m v$, $\mu^n u$ $m, n \in \mathbb{Z}$

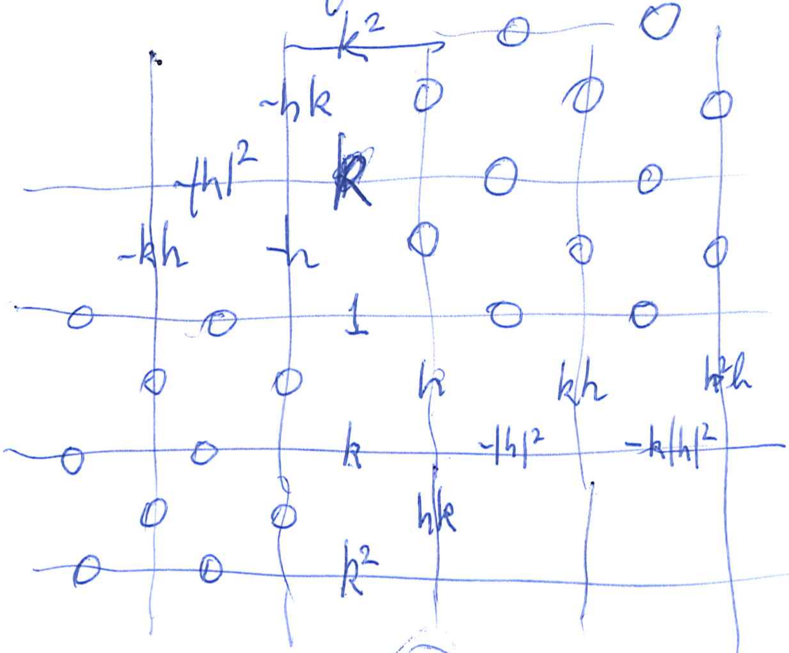
Green's function idea for $H(v, -)$. Actually this linear functional ~~...~~ on E is ~~...~~ equivalent to a solution of the equations ~~...~~, but I think it's possible to give ~~the~~ vanishing conditions:

indef:



Clearly what's happening is that you ~~are~~ given initial data on one decreasing staircase.

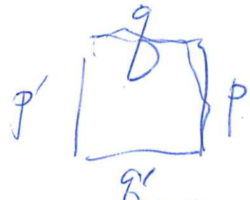
positive herm. form.



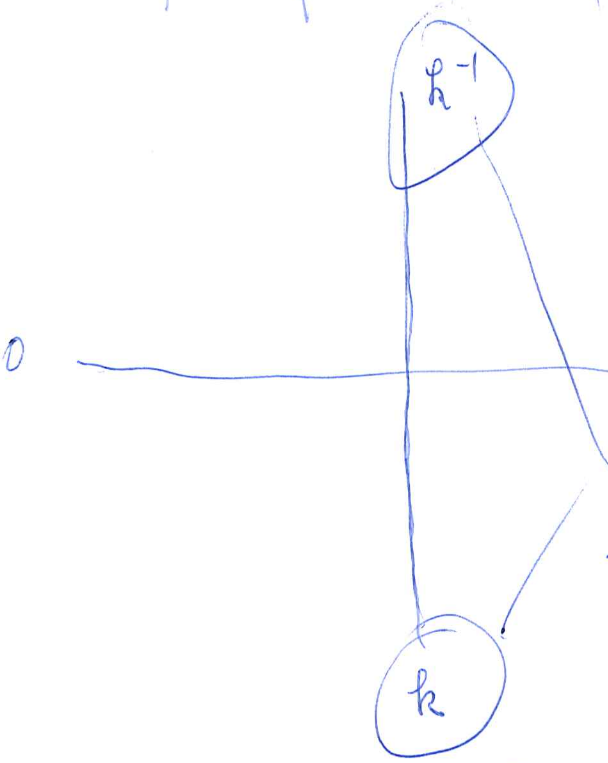
point is that reflecting ~~naturally~~ the squares in the vertical direction means

$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$



$$\begin{pmatrix} p' \\ q \end{pmatrix} = \begin{pmatrix} k & -h \\ \bar{h} & k \end{pmatrix}\begin{pmatrix} p \\ q' \end{pmatrix}$$



interchange these two residues it seems

~~We have~~

What are invertibles in $\mathbb{C}[\lambda, \lambda^{-1}, (\lambda-k)^{-1}, (k\lambda-1)^{-1}]$

$\{c\,\lambda^m(\lambda-k)^n(\lambda-k^{-1})^p\}$

Cont. case $\quad \partial_x \psi^1 = h\,\psi^2 \qquad i\xi u = h v$

$\qquad\qquad\ \ \partial_y \psi^2 = \bar{h}\,\psi^1 \qquad i\eta v = \bar{h} u$

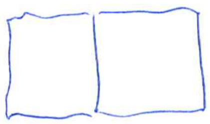
$-\xi\eta = |h'|^2$

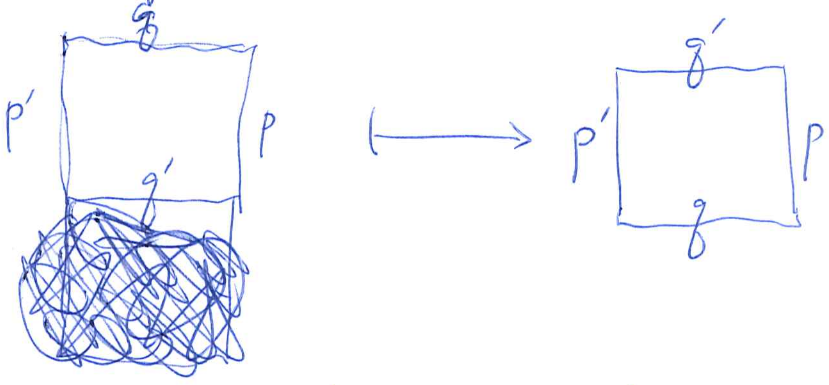
can take $h' = 1$.

$\partial^2_{xy}\psi^i = \psi^i$ ~~treat as a~~

You want the continuous analog.
It seems like you want solutions

Grid space generated by edges with relations from the squares. Reflect thru x axis. better $y=\frac{1}{2}$





$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -h & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

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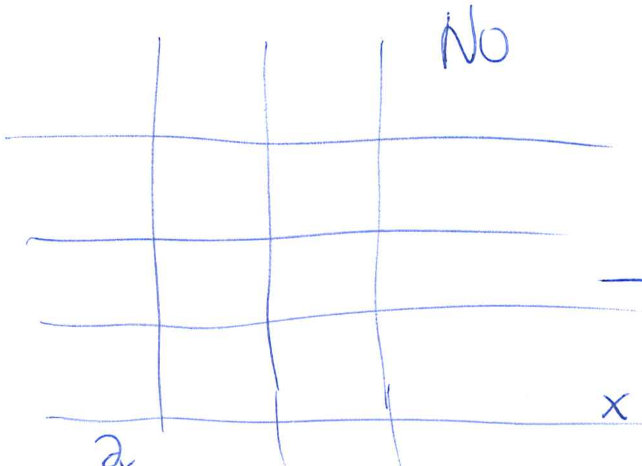
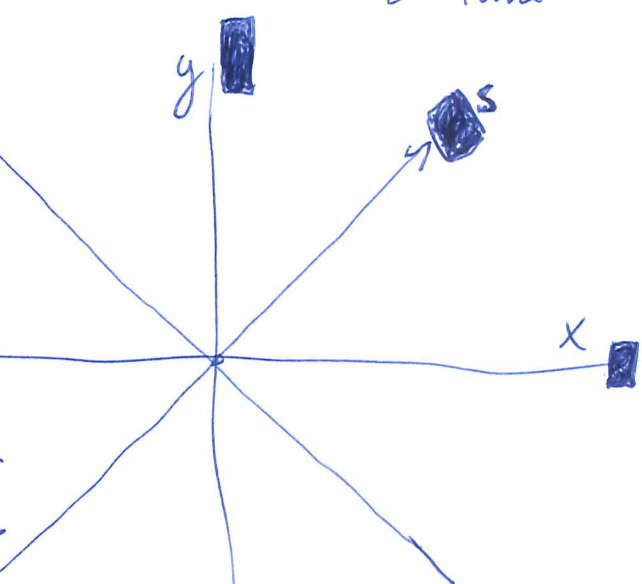
for the continuous case you want? $t=$ time



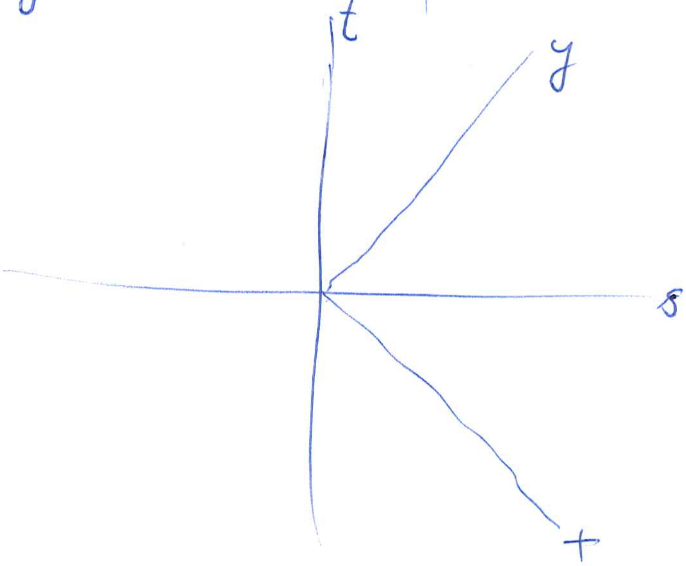


$x = s - t$

$y = s + t$

$$\overbrace{(-\partial_t + \partial_s)}^{\partial_x} u = v$$

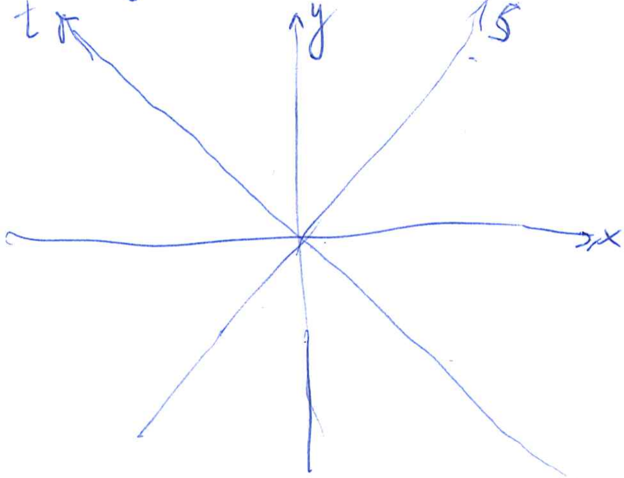
$$\underbrace{(\partial_t + \partial_s)}_{\partial_y} v = u$$



$$\partial_x = \overbrace{\frac{\partial t}{\partial x}}^{-1} \partial_t + \overbrace{\frac{\partial s}{\partial x}}^{+} \partial_s$$

$$\partial_y = \frac{\partial t}{\partial y} \partial_t + \frac{\partial s}{\partial y} \partial_s$$

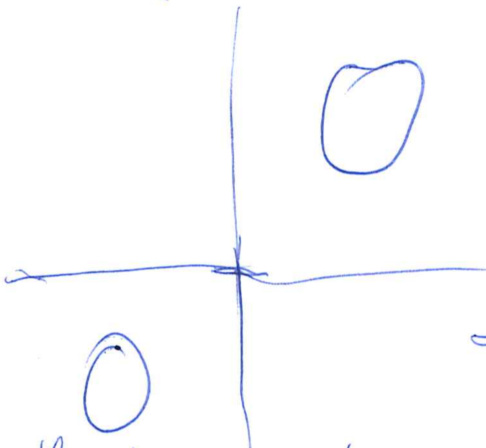
So you wish to solve $(-\partial_t + \partial_s)u = v$
$(\partial_t + \partial_s)v = u$



You need

Anyway go back to $\partial_x u = v$, $\partial_y v = u$. You want solution with given initial values on $x=y$.

Perhaps simpler is support. You look for a solution with orth properties. Solutions are understood via Fourier transform. The linear functionals $(v|-)$ $H(v,-)$ on E maybe can be understood in terms of the ξ model via support.



But solution should be

$$\psi_{xy} = \int e^{i(\xi x - \xi^{-1} y)} \begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix} (?) \frac{d\xi}{2\pi}$$

If you set $x = y$

I think you know the answer in the pos. def. case

Limit of $\displaystyle\oint_{|z|=1} f(z)\frac{dz}{2\pi i z}$. Let

IDEA: You need to work symmetrically maybe, otherwise how do you see the cycle splitting into + − components

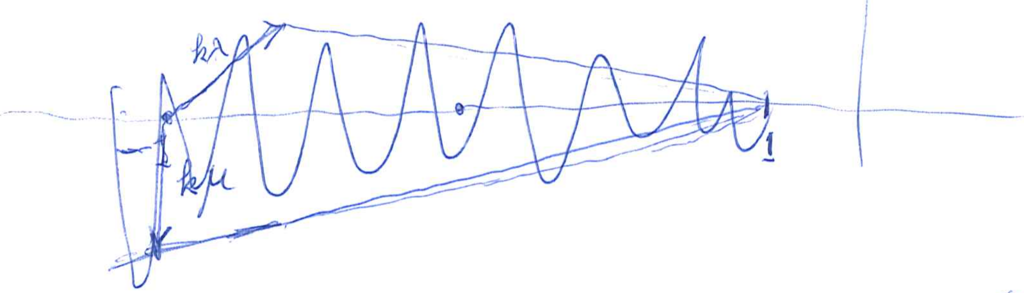
$$\frac{k\lambda-1}{h}\,\frac{k\mu-1}{\bar h}=1. \qquad (k\lambda-1)(k\mu-1)=1-k^2$$

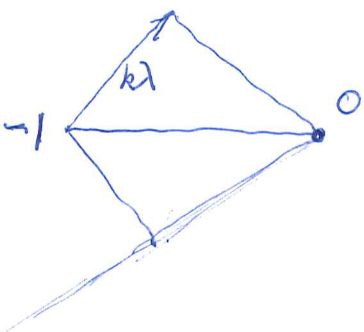
$$\frac{e^{i\xi\varepsilon}-k^{-1}}{h'\varepsilon}\,\frac{e^{i\eta\varepsilon}-k^{-1}}{\bar h'\varepsilon}=\frac{1}{k^2}$$

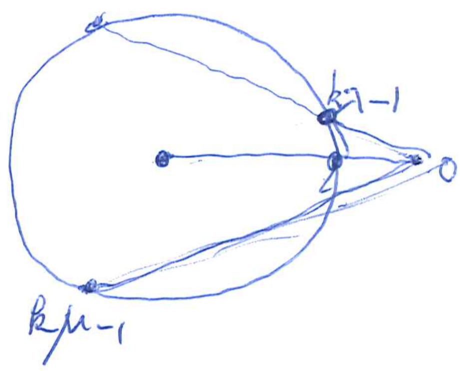
$$\downarrow$$

$$i\xi\; i\eta = 1$$

so what to do?







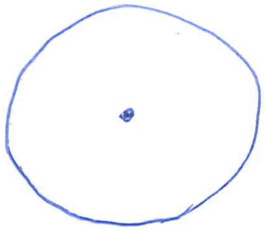
Want to analyze positive def. product

$$(v|\lambda^m\mu^n v)=\oint_{|z|=1} z^m\left(\frac{z-k}{kz-1}\right)^n\frac{dz}{2\pi i z}$$

$$z=e^{i\xi\varepsilon} \qquad \frac{dz}{iz}=d\xi\,\varepsilon$$

$$=\int_{\xi=-\frac{2\pi}{\varepsilon}}^{\xi=+\frac{2\pi}{\varepsilon}} e^{i(\xi m\varepsilon)}\left(\frac{e^{i\xi\varepsilon}-k}{ke^{i\xi\varepsilon}-1}\right)^n\frac{\varepsilon\, d\xi}{2\pi}$$

$$\left(\frac{e^{i\xi\varepsilon}-\sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2}\,e^{i\xi\varepsilon}-1}\right)^{x/\varepsilon} \sim \frac{1+i\xi\varepsilon-1}{}$$



$$\log\left(e^{i\xi\varepsilon}-\sqrt{1-\varepsilon^2}\right)^{x/\varepsilon} = \ ?$$

$$(v|\lambda^m\mu^n v) = \int z^m\left(\frac{z-k}{kz-1}\right)^n \frac{dz}{2\pi i z} \qquad z=e^{i\xi}$$

$$e^{(i\xi\varepsilon)m} \quad e^{(i\eta\varepsilon)n}$$

$$\frac{e^{i\xi\varepsilon}-\sqrt{1-|h'|^2\varepsilon^2}}{\sqrt{1-|h'|^2\varepsilon^2}\,e^{i\xi\varepsilon}-1} = \frac{1+i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}-\left(1-\frac{1}{2}|h'|^2\varepsilon^2\right)}{\left(1-\frac{1}{2}|h'|^2\varepsilon^2\right)\left(1+i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}\right)-1}$$

$$= \frac{i\xi\varepsilon+\frac{1}{2}(i\xi\varepsilon)^2+\frac{1}{2}|h'|^2\varepsilon^2}{i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}-\frac{1}{2}|h'|^2\varepsilon^2}.$$

$$= \frac{i\xi-\frac{\xi^2}{2}\varepsilon+\frac{|h'|^2}{2}\varepsilon}{i\xi-\frac{\xi^2}{2}\varepsilon-\frac{|h'|^2}{2}\varepsilon} = \frac{1-\frac{\varepsilon}{2i\xi}+\frac{|h'|^2}{2i\xi}\varepsilon}{1-\frac{\varepsilon}{2i\xi}-\frac{|h'|^2}{2i\xi}\varepsilon}$$

$$= \left(1-\frac{\varepsilon}{2i\xi}+\frac{|h'|^2\varepsilon}{2i\xi}\right)\left(1+\frac{\varepsilon}{2i\xi}+\frac{|h'|^2}{2i\xi}\varepsilon\right)$$

$$= \left(1+\cancel{\frac{\varepsilon}{2i\xi}}+\frac{|h'|^2}{2i\xi}\varepsilon\right)-\cancel{\frac{\varepsilon}{2i\xi}}+\frac{|h'|^2\varepsilon}{2i\xi}$$

$= 1+\frac{|h'|^2}{i\xi}\varepsilon$ raised to the $n=\frac{y}{\varepsilon}$ power is $e^{\frac{|h'|^2}{i\xi}y}$

Review. $(v|\lambda^{m}\mu^{n}v) = \int\limits_{|z|=1} z^{m}\left(\frac{z-k}{kz-1}\right)^{n}\frac{dz}{2\pi i z}$?

$\lambda = z^{\varepsilon} = e^{i\xi\varepsilon}$

$z = e^{i\xi}$

$\frac{dz}{z} = i\,d\xi$

$$(v|\lambda^{m\varepsilon}\mu^{n\varepsilon}v) = \int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} (z^{\varepsilon})^{m}\left(\frac{z^{\varepsilon}-k}{kz^{\varepsilon}-1}\right)^{n}\frac{d\xi}{2\pi}$$

$$k = \sqrt{1-|h'|^2\varepsilon^2} = 1-\frac{1}{2}|h'|^2\varepsilon^2$$

$$\frac{e^{i\xi\varepsilon}-k}{ke^{i\xi\varepsilon}-1} = \frac{1+i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}-1+\frac{1}{2}|h'|^2\varepsilon^2}{\left(1-\frac{1}{2}|h'|^2\varepsilon^2\right)\left(1+i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}\right)-1}$$

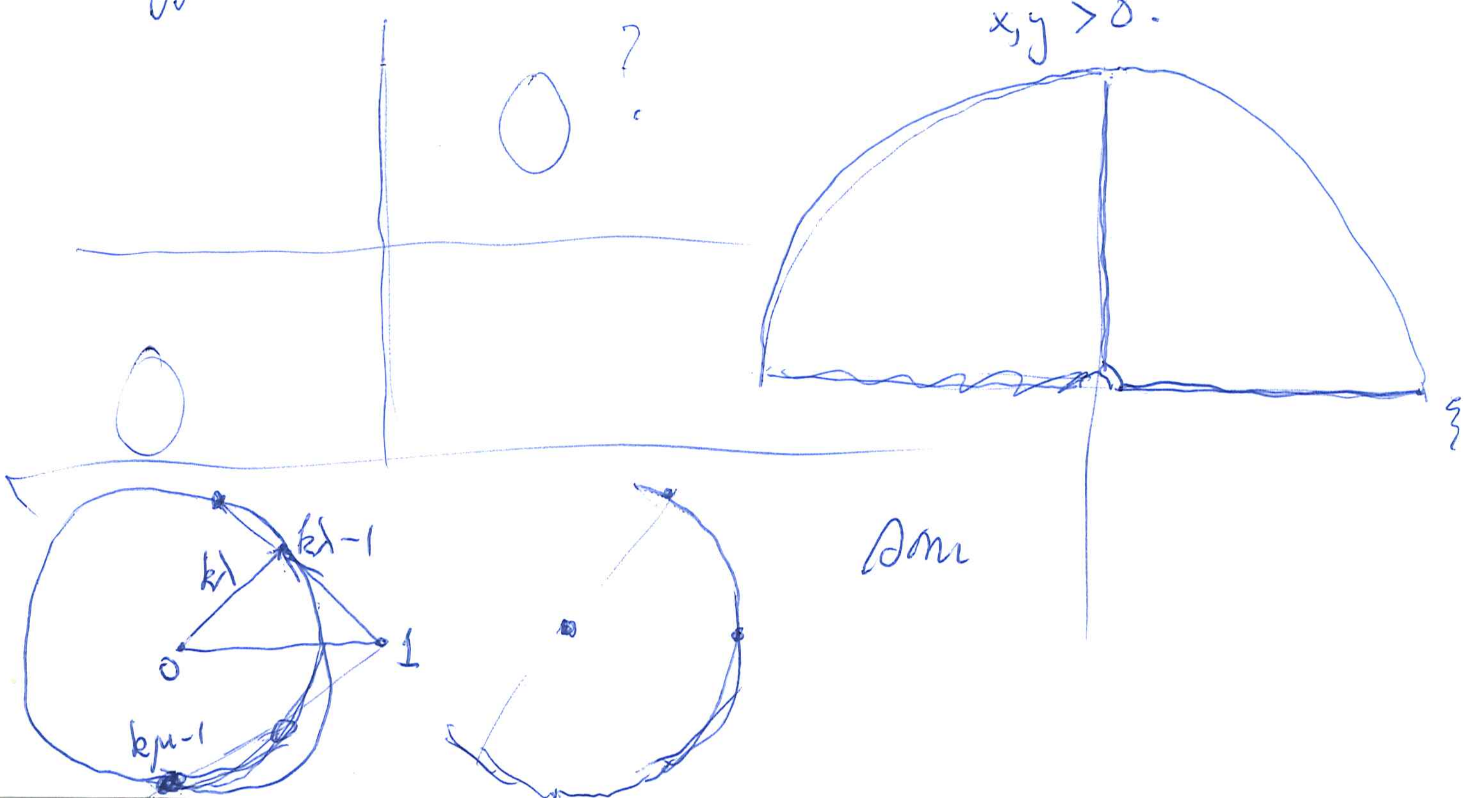
$$= \frac{i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}+\frac{1}{2}|h'|^2\varepsilon^2}{i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}-\frac{1}{2}|h'|^2\varepsilon^2} = \frac{1+\frac{i\xi\varepsilon}{2}+\frac{1}{2}\frac{|h'|^2\varepsilon}{i\xi}}{1+\frac{i\xi\varepsilon}{2}-\frac{1}{2}\frac{|h'|^2\varepsilon}{i\xi}}$$

$$= \frac{1+\frac{1}{2}\frac{|h'|^2\varepsilon}{i\xi}}{1-\frac{1}{2}\frac{|h'|^2\varepsilon}{i\xi}} = \left(1+\frac{|h'|^2\varepsilon}{i\xi}\right) + O(\varepsilon^2)$$

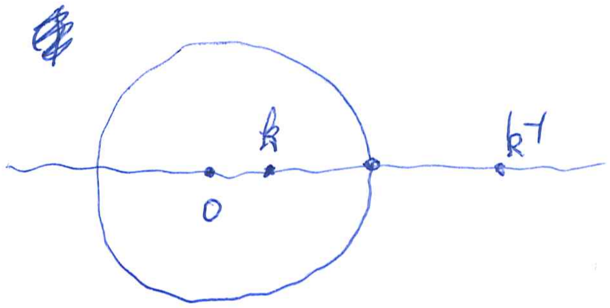
$i(\xi x-\xi^{-1}y)$ if $|h'|=1$.

$$(v|\lambda^{x}\mu^{y}v) = \int_{-\infty}^{\infty} e^{i\xi x+\frac{|h'|^2}{i\xi}y}\frac{d\xi}{2\pi}$$

Check support



can you compute $res_{k^{-1}}$ in the cont. limit

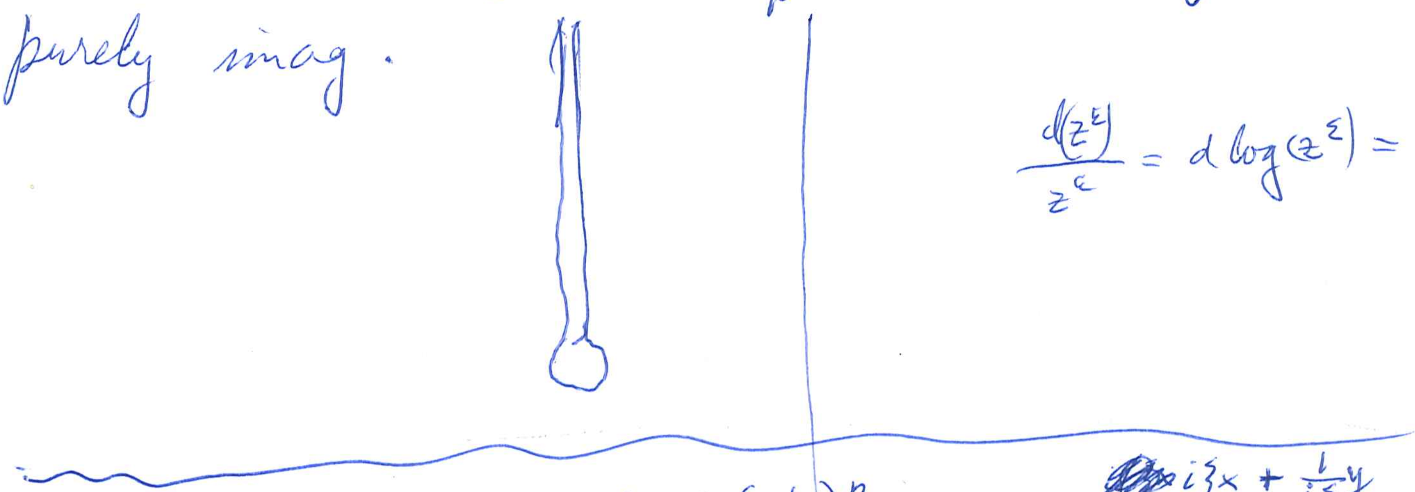


You are still integ. $z^m\left(\frac{z-k}{kz-1}\right)^n \frac{dz}{2\pi i z}$

the basic substitution is replacing z by $z^\varepsilon = e^{i\xi\varepsilon}$

h by $h'\varepsilon$ Try a different contour for ξ, purely imag.

$$\frac{d(z^\varepsilon)}{z^\varepsilon} = d\log(z^\varepsilon) = \varepsilon\frac{dz}{z}$$



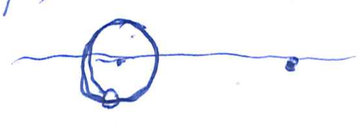
Space of functions

Linear fnl.

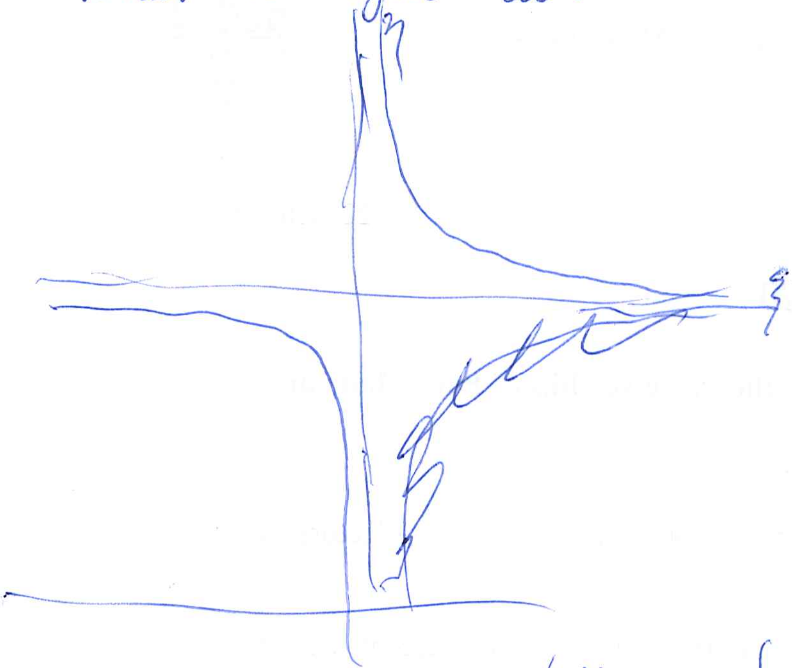
$$(z^\varepsilon)^m\left(\frac{z^\varepsilon-k}{kz^\varepsilon-1}\right)^n \longrightarrow e^{i\xi x+\frac{1}{i\xi}y}$$

$$\int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} (z^\varepsilon)^m\left(\frac{z^\varepsilon-k}{kz^\varepsilon-1}\right)^n \frac{dz}{2\pi i z} \longrightarrow \int_{-\infty}^{\infty} e^{i(\xi x-\xi^{-1}y)}\frac{d\xi}{2\pi}$$

Some insight might be gained by calculating the linear fnls. res_0, res_k, $res_{k^{-1}}$, res_∞.
The functions behave well. What about the traces. Another viewpoint: Think of the torus $\{(\lambda,\mu)\in S^1\times S^1\}$ and the circle ~~curve~~ $\{(\lambda,\mu) \mid \frac{(k\lambda-1)}{h}\frac{(k\mu-1)}{h}=1.\}$



Some things are happening around $z=1$.
What can you do.



Question: Can you split $\int \frac{d\xi}{2\pi}$ into two parts related to res_0, res_k. There is an obvious splitting into $\xi>0, \eta<0$ and $\xi<0, \eta>0$.

Consider then

$$\int_0^\infty e^{i(\xi x - \xi^{-1} y)} \begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix} \frac{d\xi}{2\pi}$$

$$\mathrm{Re}\left(i\xi x + \frac{\bar\xi}{i|\xi|^2}y\right) = -\mathrm{Im}(\xi)x + \mathrm{Im}(\bar\xi)\frac{1}{|\xi|^2}y$$

$$= -\mathrm{Im}(\xi)\left(x + \frac{1}{|\xi|^2}y\right)$$

Check. $$\mathrm{Re}\left(i\xi x + \frac{\bar\xi}{i|\xi|^2}y\right) = -\mathrm{Im}(\xi)\left(x + \frac{1}{|\xi|^2}y\right)$$

$$= -\mathrm{Im}\left(\frac{\xi}{|\xi|}\right)\left(|\xi|x + |\xi|^{-1}y\right)$$

You have to consider both $\xi>0, <0$.
push imag. part

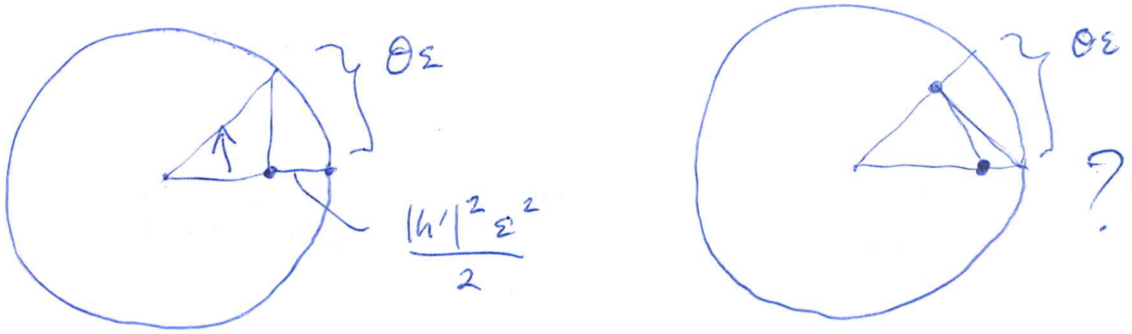
~~Hlld~~ Yesterdays computation

$$(v, \lambda^m \mu^n v) = \oint_{\cancel{|z|=1}\,-\pi}^{\pi} z^m \left(\frac{z-k}{kz-1}\right)^n \frac{d\theta}{2\pi}$$

$z^\varepsilon = e^{i\theta\varepsilon}$

$$(v, \lambda^{\varepsilon m} \mu^{\varepsilon n} v) = \int_{-\pi/\varepsilon}^{\pi/\varepsilon} z^{\varepsilon m} \left(\frac{z^\varepsilon - k}{kz^\varepsilon - 1}\right)^{\frac{1}{\varepsilon} n\varepsilon} \frac{d\theta}{2\pi}$$

$$\cancel{\frac{z^\varepsilon - k}{kz^\varepsilon - 1} = k \frac{k^{-1} e^{i\theta\varepsilon} - 1}{k e^{i\theta\varepsilon} - 1}} \qquad \frac{z^\varepsilon - k}{kz^\varepsilon - 1} = -z^\varepsilon \frac{1 - z^{-\varepsilon} k}{1 - z^\varepsilon k}$$



$$\left(\frac{z^\varepsilon - k}{kz^\varepsilon - 1}\right)^{1/\varepsilon} = \frac{e^{i\xi\varepsilon} - \sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2}\, e^{i\xi\varepsilon} - 1} = \frac{i\xi\varepsilon + \frac{(i\xi\varepsilon)^2}{2} + \frac{1}{2}\varepsilon^2}{i\xi\varepsilon + \frac{(i\xi\varepsilon)^2}{2} - \frac{1}{2}\varepsilon^2}$$

$$(v, \lambda^x \mu^y v) = \int_{-\infty}^{\infty} e^{i(\xi x - \xi^{-1} y)} \frac{d\xi}{2\pi}$$

Look: ~~Replace~~ $$\frac{z^\varepsilon - k}{z^\varepsilon - 1} = \frac{i\xi\varepsilon + i\xi\varepsilon}{i\xi\varepsilon + \frac{(i\xi\varepsilon)^2}{2}}$$?

$$\frac{z^\varepsilon - k}{z^\varepsilon - k^{-1}} = \frac{z^\varepsilon - k^{-1} + k^{-1} - k}{z^\varepsilon - k^{-1}} = 1 + \frac{k^{-1} - k}{z^\varepsilon - k^{-1}} \qquad z^2$$

$$\begin{array}{ccc} & \psi^2_{x,y+\varepsilon} & \\ \psi^1_{xy} & \boxed{} & \psi^1_{x+\varepsilon,y} \\ & \psi^2_{xy} & \end{array} \qquad \psi_{xy} = \lambda^x \mu^y \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\partial_x \psi^1 = \psi^2 \qquad i\xi u = v$$
$$\partial_y \psi^2 = \psi^1 \qquad i\eta v = u$$

~~General solution universal soln.~~

discrete case E gen. u, v over $\mathbb{C}[\lambda, \mu, \lambda^{-1}, \mu^{-1}] = \mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$

$$\frac{k\lambda - 1}{h} u = v, \quad \frac{k\mu - 1}{h} v = u.$$

Realization $E = \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$.

$\lambda =$ mult by z $\qquad u = \frac{h}{kz-1}$

$\mu =$ mult by $\frac{z-k}{kz-1}$ $\qquad v = 1$

Philosophy is to start with grid space $E = \mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$-module gen u, v reln ... Redundant generators $\lambda^m \mu^n \begin{pmatrix} u \\ v \end{pmatrix}$. Cont. version requires group ring of $\mathbb{R}\times\mathbb{R}$ – at least L^1, might be able to handle $\mathcal{S}(\mathbb{R}\times\mathbb{R})$. ~~So smooth~~

whence of. λ^x, μ^y and $i\xi, i\eta$

partial def of $\mathcal{E}$: module for gp $\mathbb{R}\times\mathbb{R}$, gen u, v relations $i\xi u = v$, $i\eta v = u$.

Realization: smooth functions $f(\xi)$ on $\mathbb{P}^1\mathbb{R}$ vanishing to ∞ order at $\xi = 0, \infty$.

$\lambda^x =$ mult. by $e^{i\xi x}$ $\qquad u = \frac{1}{i\xi}$

$\mu^y =$ —— $e^{-i\xi^{-1} y}$ $\qquad v = 1$

$$\psi(x,y) = \int_{-\infty}^{\infty} e^{(i\xi)x + \left(\frac{1}{i\xi}\right)y} \begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix} f(\xi) \frac{d\xi}{2\pi}$$

Analog of

$$\psi_{mn} = \int z^m \left(\frac{\cancel{k}z-k}{kz-1}\right)^n \begin{pmatrix} \frac{h}{kz-1} \\ 1 \end{pmatrix} f(z) \frac{dz}{2\pi i z}$$

where $f \in \mathbb{C}[\cancel{z}, z^{-1}, (z-k)^{-1}, (\cancel{k}z-1)^{-1}]$.

Integral equation. $\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_y \end{pmatrix} \psi = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix} \psi$

Puzzle about Green's fn (fund. solution) and the solns of the homogeneous equation you seek related to hermitian forms.

Green's function calc. for $\begin{pmatrix} \partial_x & -1 \\ -1 & \partial_y \end{pmatrix} \psi = \delta(x)\delta(y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\hat{\psi}(\xi,\eta) = \begin{pmatrix} i\xi & -1 \\ -1 & i\eta \end{pmatrix}^{-1} = \frac{1}{-\xi\eta - 1} \begin{pmatrix} i\eta & 1 \\ 1 & i\xi \end{pmatrix}$$

So

$$\cancel{G(x,y) =} \int e^{i(\xi x + \eta y)} \frac{(-1)}{\xi\eta + 1} \begin{pmatrix} i\eta & 1 \\ 1 & i\xi \end{pmatrix} \frac{d\xi\, d\eta}{(2\pi)^2}$$

If we try to evaluate by ~~[illegible]~~ doing the η integral possibly by residues, then something like

$$\int \frac{d\xi}{2\pi} e^{i\xi x} \underbrace{\int \frac{d\eta}{2\pi i} e^{i\eta y} \frac{\eta}{\xi\eta + 1}}_{e^{i(-\xi^{-1})y} \frac{(-\xi^{-1})}{\xi}}$$

← if the integral is the residue at the simple pole

arises which is worse than $\int \frac{d\xi}{2\pi} e^{i(\xi x - \xi^{-1} y)} \frac{1}{i\xi}$ before

So go back to disc. case

$$(v|\lambda^m\mu^n\binom{v}{u}) = \oint_{|z|=1} z^m\left(\frac{z-k}{kz-1}\right)^n \begin{pmatrix}\frac{h}{kz-1}\\ 1\end{pmatrix}\frac{dz}{2\pi i z}$$

$$\left(v\middle|\lambda^{\varepsilon m}\mu^{\varepsilon n}\binom{u}{v}\right) = \int_{\xi=-\frac{\pi}{\varepsilon}}^{\xi=\frac{\pi}{\varepsilon}} z^{\varepsilon m}\left(\frac{z^\varepsilon-k_\varepsilon}{k_\varepsilon z^\varepsilon-1}\right)^n\begin{pmatrix}\frac{h'\varepsilon}{k_\varepsilon z^\varepsilon-1}\\ 1\end{pmatrix}\frac{dz}{2\pi i z}$$

See if it possible to keep track of $\mathrm{res}_{k^{-1}}$ or res_0.
So what's happening to the functions?

$$\frac{z^\varepsilon-k_\varepsilon}{k_\varepsilon z^\varepsilon-1} = \left(\frac{1-k_\varepsilon z^{-\varepsilon}}{k_\varepsilon z^\varepsilon-1}\right) z^\varepsilon$$

$$k_\varepsilon z^\varepsilon - 1 = \left(1-\tfrac{1}{2}\varepsilon^2\right)\left(1+i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}+\dots\right)-1$$

$$= i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}-\frac{1}{2}\varepsilon^2$$

$$= i\xi\varepsilon+\left(-\tfrac{1}{2}\right)(\xi^2+1^2)\varepsilon^2 + O(\varepsilon^3)$$

so compare

$$\frac{e^{i\xi\varepsilon}-\sqrt{1-|h'\varepsilon|^2}}{\sqrt{1-|h'\varepsilon|^2}\left(e^{i\xi\varepsilon}\right)-1} = \frac{\cancel{1}+i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}-\cancel{1}+\frac{1}{2}|h'|^2\varepsilon^2+O(\varepsilon^3)}{\cancel{1}+i\xi\varepsilon+\frac{(i\xi\varepsilon)^2}{2}-\cancel{1}-\frac{1}{2}|h'|^2\varepsilon^2+O(\varepsilon^3)}$$

$$k_\varepsilon z^\varepsilon-1 = i\xi\varepsilon-\tfrac{1}{2}(\xi^2+1)\varepsilon^2+O(\varepsilon^3)$$

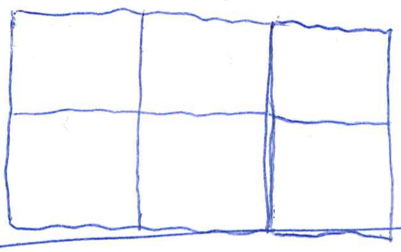
$$k_\varepsilon z^{-\varepsilon}-1 = -i\xi\varepsilon-\tfrac{1}{2}(\xi^2+1)\varepsilon^2+O(\varepsilon^3)$$

$$\frac{1-k_\varepsilon z^{-\varepsilon}}{k_\varepsilon z^\varepsilon-1} = \frac{i\xi\varepsilon+\frac{1}{2}(\xi^2+1)\varepsilon^2+O(\varepsilon^3)}{i\xi\varepsilon-\frac{1}{2}(\xi^2+1)\varepsilon^2+O(\varepsilon^3)}$$

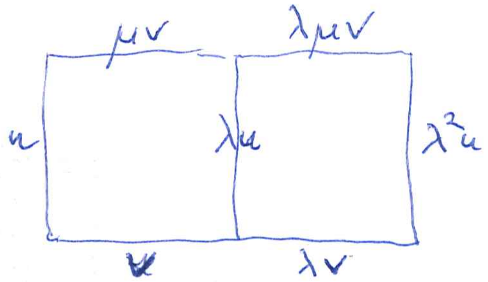
$$\left(\frac{1-k_\varepsilon z^{-\varepsilon}}{k_\varepsilon z^{\varepsilon}-1}\right)^{\frac{1}{\varepsilon}} \longrightarrow e^{\frac{\xi^2+1}{i\xi}} \cdot e^{i\xi} = e^{\frac{1}{i\xi}}$$

~~Probably~~ Probably you need to write everything in terms of $e^{i\xi}$

Play periodic games



Consider constant h grid but ~~with~~ with action of $2\mathbb{Z}\times\mathbb{Z}$. The grid space should decompose according to the charaters of $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z} = \mathbb{Z}/2$.



$$\begin{pmatrix}p'\\ q'\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar{h} & k\end{pmatrix}\begin{pmatrix}p'\\ q\end{pmatrix}$$

find a basis. ~~Assuming~~

$$\lambda u = \frac{1}{k}(u + hv)$$

$$\lambda^2 u = \frac{1}{k}(\lambda u + h\lambda v) = \frac{1}{k}\left(\frac{1}{k}u + \frac{h}{k}v + \frac{h}{}\lambda v\right)$$

$$\lambda^2 u = \frac{1}{k^2}u + \frac{h}{k^2}v + \frac{h}{k}\lambda v$$

$$(k^2\lambda^2 - 1)u = hv + hk\lambda v$$

$$(1-|h|^2)^2 + |h|^2(1-|h|^2) + \|h\|^2$$

$$u = k^2\lambda^2 u - hk\lambda v - hv$$

$$(k\lambda+1)(k\lambda-1)u = (k\lambda+1)hv$$

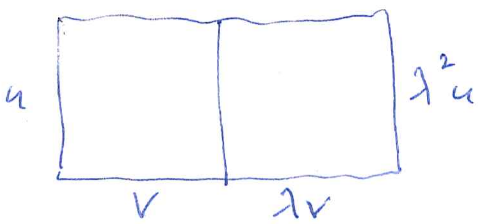
$$k^2(\lambda^2 u) = u + \underbrace{(k\lambda+1)}_{\| \quad k^2|h|^2+|h|^2 \quad \|} hv$$

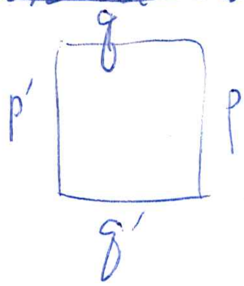
$$\frac{1-|h|^2(k^2+1)}{k^4} = \frac{k^2-|h|^2k^2}{k^4} = 1$$

$$\lambda^2 u = \frac{1}{k^2}u + \frac{h(k\lambda+1)v}{k^2} = \frac{1}{k^2}u + \frac{\sqrt{|h|^2(k^2+1)}}{k^2}\frac{hk\lambda v + hv}{\sqrt{|h|^2k^2+|h|^2}}$$

Hecke operators?

Basic idea is that the grid space is generated by u and translations. so replace λ by λ^2.





$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

$h = (q|p)$

$(q'|p') = -h$

$$(k^2\lambda^2)u - u = (k\lambda+1)\overbrace{(k\lambda-1)u}^{hv} = hk\lambda v + hv$$

$$k^2\lambda^2 u = u + hv + hk\lambda v$$

$$1 - |h|^2 - |h|^2k^2 = (1-|h|^2)k^2 = k^4.$$

keep u but replace v by $v' = hv + hk\lambda v$

$$\lambda^2 u = \underbrace{\frac{1}{k^2}u}_{} + \frac{hv+hk\lambda v}{k^2}$$

$k' = k^2$

h'

$$\frac{1}{k'}u + \frac{h'}{k'}v'$$

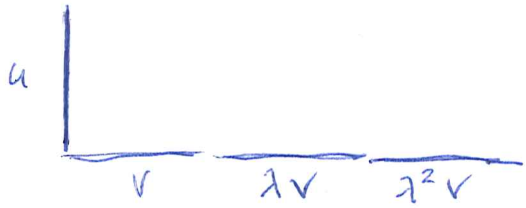
$$v' = \frac{(v+k\lambda v)}{\sqrt{1+k^2}}$$

$$\|v+k\lambda v\|^2 = \sqrt{1+k^2}$$

$$\lambda^2 u = \underset{\overset{\|}{k'}}{\frac{1}{k^2}}u + \underset{\overset{\|}{h'}}{\frac{h\sqrt{1+k^2}}{k^2}} \cdot \frac{v+k\lambda v}{\sqrt{1+k^2}}$$

Check this over

$\cancel{(k^d\lambda^d - 1)u = (k^{d-1}\lambda^{d-1} - 1)}$



$\lambda^d u \quad ((k\lambda)^d - 1)u = ((k\lambda)^{d-1} + \dots + k\lambda + 1)\underbrace{(k\lambda - 1)u}_{hv}$

$$v' = \frac{((k\lambda)^{d-1} + \dots + 1)v}{\sqrt{(k^{d-1})^2 + \dots + k^2 + 1^2}}$$

Yes!

$$\lambda^d u = \frac{1}{k^d}u + \frac{h\sqrt{1^2 + k^2 + \dots + k^{2d-2}}}{k^d}v'$$

Check $\quad 1 - \overbrace{|h|^2}^{1-k^2}(1 + k^2 + (k^2)^2 + \dots + (k^2)^{d-1}) = 1 - (1 - (k^2)^d) = (k^2)^d$!!

I think what you want to do is to ~~take~~ modify the above ~~so~~ as to take the continuous limit in the horizontal direction. Thus you will have symmetry group $\mathbb{R} \times \mathbb{Z}$ and character group $\mathbb{R} \times S^1$. You have ~~to deform the business~~ to understand what happens to the relation. You expect universal solution $\psi_{xn} = e^{i\xi x}\mu^n \begin{pmatrix} u \\ v \end{pmatrix}$, here $\lambda^x = e^{i\xi x}$ is an 1-par group operator, μ an operator. The character group is $\mathbb{R} \times S^1 = \{(\xi, \omega) \mid \xi \in \mathbb{R}, \omega \in S^1\}$. Now using functions of (ξ, ω) such as $e^{i\xi x}\omega^n$ and linear combination. One hint: See if the Hilbert space is ~~$L^2(\mathbb{R}$~~ $L^2(\mathbb{R}, \frac{d\xi}{2\pi})v$, that is, u can be written as a function of ξ. Recall that before $u = \frac{h}{kz-1}v$

$$\frac{h'\varepsilon}{k_\varepsilon z^\varepsilon - 1} = \frac{h'\varepsilon}{(1 - |h|^2\varepsilon^2)(1 + i\xi\varepsilon) - 1} = \frac{h'}{i\xi}$$

$(k e^{i\xi}-1)(k\mu-1) =$

$\mu = \frac{z-k}{kz-1}$

Somehow you expect the curve $w = \frac{1+i\xi}{1-i\xi}$

so the question is how might this arise from $w = \frac{z-k}{kz-1}$?

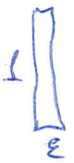
$$w = \frac{e^{i\xi\varepsilon} - k_\varepsilon}{k_\varepsilon e^{i\xi\varepsilon}-1}$$

so try $k_\varepsilon = \sqrt{1-\varepsilon} = 1-\frac{1}{2}\varepsilon + O(\varepsilon^2)$

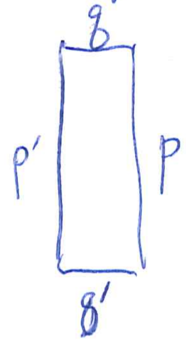
$$w = \frac{\cancel{1}+i\xi\varepsilon - \cancel{1}+\frac{1}{2}\varepsilon}{\cancel{1}-\frac{1}{2}\varepsilon+i\xi\varepsilon-\cancel{1}} = \frac{i\xi+\frac{1}{2}}{i\xi-\frac{1}{2}}$$

which is fine

the point maybe is that the for rectangles



the width has $\|\ \|^2=\varepsilon$. Given $f(x)$ on $0\le x<1$, if you divide the interval into ε steps and approximate f on a subinterval by its average, then you weight the interval by $\sqrt{\varepsilon}$ to get the L^2 norm.



$$|p|^2 + t^2|g'|^2 = |p'|^2 + t^2|g|^2$$

$$\begin{pmatrix} p \\ g \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g} \begin{pmatrix} p' \\ g' \end{pmatrix}$$

$$g^* \begin{pmatrix} 1 & 0 \\ 0 & -t^2 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -t^2 \end{pmatrix}$$

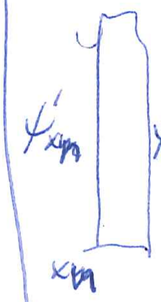
$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = g^* = \begin{pmatrix} 1 & 0 \\ 0 & -t^2 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -t^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} d & -b \\ t^2 c & -t^2 a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{t^2} \end{pmatrix} = \begin{pmatrix} d & b t^{-2} \\ t^2 c & a \end{pmatrix}$$

replace t^2 by t.

$$g = \frac{1}{k}\begin{pmatrix} 1 & ht^{-1} \\ t\bar{h} & 1 \end{pmatrix} \qquad \psi_{xm} = e^{i\xi x}\omega^m \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} e^{i\xi\varepsilon} u \\ \omega v \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & ht^{-1} \\ t\bar{h} & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$



$$\begin{pmatrix} \psi^1_{x+\varepsilon,m} \\ \psi^2_{x,m+1} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & ht^{-1} \\ t\bar{h} & 1 \end{pmatrix}\begin{pmatrix} \psi^1_{xm} \\ \psi^2_{xm} \end{pmatrix}$$

$$(ke^{i\xi\varepsilon} - 1)\, u = ht^{-1} v \qquad k = \sqrt{1-|h|^2}$$

$$(k\omega - 1)\, v = \bar{h} t\, u \qquad h = h'\sqrt{\varepsilon}$$

$$\left(1 - \tfrac{1}{2}|h'|^2\varepsilon\right)(1 + i\xi\varepsilon)\, u = h'\sqrt{\varepsilon}\, t^{-1} v$$

$$\left(1 - \tfrac{1}{2}|h'|^2\varepsilon\right)\omega$$

Try again $\qquad \omega = \dfrac{e^{i\xi\varepsilon} - k_\varepsilon}{k_\varepsilon e^{i\xi\varepsilon} - 1} \longrightarrow \dfrac{i\xi - k_0'}{k_0' + i\xi}$

$$(k_\varepsilon e^{i\xi\varepsilon} - 1)\, u = hv \qquad (k_\varepsilon \omega - 1)\, v = \bar{h} u$$

$$(k_\varepsilon e^{i\xi\varepsilon} - 1)(k_\varepsilon \omega - 1) = |h|^2 = 1 - k_\varepsilon^2$$

$$(k\lambda - 1)(k\omega - 1) = 1 - k^2$$

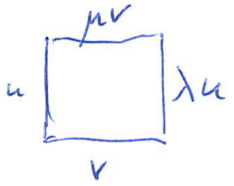
$$\omega = \frac{1}{k}\left(\frac{1-k^2}{k\lambda - 1} + 1\right) = \frac{1}{k}\,\frac{\cancel{1} - k^2 + k\lambda - \cancel{1}}{k\lambda - 1} = \frac{\lambda - k}{k\lambda - 1}$$

So it's clear that $|h|^2 = c\varepsilon$. What about u, v.

~~If you want~~ Clearly you need $h = \varepsilon$ $\quad \bar{h} = \text{const.}$

Review what you learned yesterday about making the grid continuous in the horizontal direction

Making the grid continuous in the horizontal direction. Begin with the discrete case



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$(k\lambda-1)u = hv$$
$$(k\mu-1)v = \bar h u$$
$$(k\lambda-1)(k\mu-1) = |h|^2 = 1-k^2$$
$$\mu = \frac{\lambda-k}{k\lambda-1}$$

$$\psi_{mn} = \lambda^m\mu^n\begin{pmatrix}u\\ v\end{pmatrix}$$

Better: replace λ by $\lambda^\varepsilon = e^{i\xi\varepsilon}$, keep μ the same, let k, h, $\bar h$ depend on ε. Will see necess to make $h, \bar h$ independent.

$$\mu_\varepsilon = \frac{e^{i\xi\varepsilon} - k_\varepsilon}{k_\varepsilon e^{i\xi\varepsilon}-1} \xrightarrow{\text{Hopital}} \frac{i\xi - k_0'}{k_0' + i\xi}$$

assuming $k_\varepsilon = 1 + k_0'\varepsilon + O(\varepsilon^2)$.

$$\frac{(k_\varepsilon e^{i\xi\varepsilon}-1)u}{h_\varepsilon} = v \implies \frac{k_0'+i\xi}{h_0'}u = v$$

$$(k_\varepsilon\mu_\varepsilon - 1)v = \bar h_\varepsilon u \implies \underbrace{\left(\frac{i\xi-k_0'}{k_0'+i\xi}-1\right)}v = \bar h_0 u$$

$$\frac{i\xi - k_0' - k_0' - i\xi}{k_0'+i\xi} = \frac{-2k_0'}{k_0'+i\xi}$$

Thus $\quad \frac{k_0'+i\xi}{h_0'}u = v \qquad \frac{-2k_0'}{k_0'+i\xi}v = \bar h_0 u \qquad -2k_0' = h_0'\bar h_0$

Here you want $h_\varepsilon = h_0'\varepsilon + O(\varepsilon^2)$ to apply Hopital

and here You want $\bar h_\varepsilon = \bar h_0 + O(\varepsilon)$. Thus

$$k_\varepsilon = \sqrt{1-h_\varepsilon\bar h_\varepsilon} = 1 - \tfrac12 h_0'\varepsilon\bar h_0 \qquad k_0' = -\tfrac12 h_0'\bar h_0$$

You can't keep $\bar h_\varepsilon$ = conj. of h_ε.

So ~~inst~~ we expect exponential solutions

$$\psi_{x,n} = \lambda^x \mu^n \begin{pmatrix} u \\ v \end{pmatrix} = e^{i\xi x} \left(\frac{i\xi - k_0'}{k_0' + i\xi}\right)^n \begin{pmatrix} \frac{h_0'}{k_0'+i\xi} \\ 1 \end{pmatrix}$$

What are the corresponding equations?

$$\partial_x \psi^1_{x,n} = e^{i\xi x} \mu^n \left(i\xi \frac{h_0'}{k_0'+i\xi} \right)$$

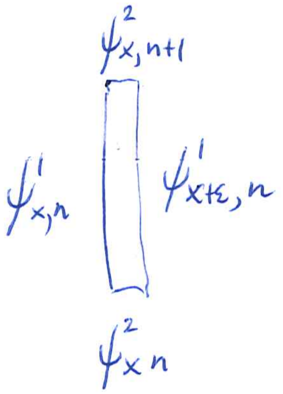
$$\frac{i\xi h_0'}{k_0'+i\xi} = \left(\frac{k_0'+i\xi - k_0'}{k_0'+i\xi}\right) h_0' = h_0' - \frac{k_0' h_0'}{k_0'+i\xi}$$

$$= h_0' \left(\psi^2_{x,n}\right) - k_0' \psi^1_{x,n}$$

$$\psi^2_{x,n+1} = e^{i\xi x} \mu^n \left(\frac{i\xi - k_0'}{k_0'+i\xi}\right)$$

$$\frac{i\xi - k_0'}{k_0'+i\xi} = 1 + \frac{\overset{h_0'\bar{h}_0}{-2k_0'}}{k_0'+i\xi}$$

$$= \psi^2_{x,n} + \bar{h}_0 \psi^1_{x,n}$$



$$\begin{pmatrix} \psi^1_{x+\varepsilon,n} \\ \psi^2_{x,n+1} \end{pmatrix} = \begin{pmatrix} 1 - k_0'\varepsilon & h_0'\varepsilon \\ \bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} \psi^1_{xn} \\ \psi^2_{xn} \end{pmatrix}$$

$$\partial_x \psi^1_{xn} = -k_0' \psi^1_{xn} + h_0' \psi^2_{xn}$$

$$\psi^2_{x,n+1} = \bar{h}_0 \psi^1_{xn} + \psi^2_{xn}$$

$$\psi_{xn} = \lambda^x \mu^n \begin{pmatrix} u \\ v \end{pmatrix}$$

$$(i\xi + k_0') u = h_0' v$$

$$(\mu - 1) v = \bar{h}_0 u$$

$$(\mu - 1)(i\xi + k_0') = h_0' \bar{h}_0 = -2k_0'$$

$$\mu = \frac{-2k_0'}{i\xi + k_0'} + 1 = \frac{i\xi - k_0'}{k_0' + i\xi_0}$$

$$\psi_{x,n} = \lambda^x \mu^n \begin{pmatrix} u \\ v \end{pmatrix} = e^{i\xi x} \left(\frac{i\xi - k_0'}{k_0' + i\xi} \right)^n \begin{pmatrix} \frac{h_0'}{k_0' + i\xi} \\ 1 \end{pmatrix}$$

$$\partial_x \psi_{x,n} = e^{i\xi x} \mu^n \; i\xi \begin{pmatrix} \frac{h_0'}{k_0' + i\xi} \\ 1 \end{pmatrix}$$

$$\partial_x \psi^1_{x,n} = \text{---} \quad i\xi \left(\frac{h_0'}{k_0' + i\xi} \right) = h_0' \psi^2_{x,n} - k_0' \psi^1_{x,n}$$

$$\psi^2_{x,n+1} = \text{---} \left(\frac{i\xi - k_0'}{k_0' + i\xi} \right) = \text{---}$$

$$\frac{i\xi}{k_0' + i\xi} = 1 + \frac{-k_0'}{k_0' + i\xi}$$

$$\left(\partial_x + k_0' \right) \psi^1_{x,n} = e^{i\xi x} \mu^n h_0' = h_0' \psi^2_{x,n}$$

$$\left(\partial_x - k_0' \right) \psi^1_{x,n} = e^{i\xi} \mu^n (i\xi - k_0') \frac{h_0'}{k_0' + i\xi} = e^{i\xi x} \mu^{n+1} h_0'$$

$$\left(\partial_x - k_0' \right) \psi^1_{x,n} = h_0' \psi^2_{x,n+1}$$

$$\boxed{\begin{aligned} (i\xi + k_0')\, u &= h_0' v \\ (i\xi - k_0')\, u &= h_0' \mu v \end{aligned}} \qquad \mu = \frac{i\xi - k_0'}{k_0' + i\xi}$$

$$1 - \underbrace{k_0'\varepsilon}_{-\frac{1}{2}\bar{h}_0 h_0'} - \bar{h}_0 h_0' \varepsilon$$

$$\begin{pmatrix} \psi^1_{x+\varepsilon,n} \\ \psi^2_{xn} \end{pmatrix} = \begin{pmatrix} k_\varepsilon & h_0'\varepsilon \\ -\bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} \psi^1_{xn} \\ \psi^2_{x,n+1} \end{pmatrix}$$

$$1 \underbrace{- \tfrac{1}{2} \bar{h}_0 h_0'}_{k_0} \varepsilon = k_\varepsilon$$

Review. $\psi_{mn} = \lambda^m \mu^n \begin{pmatrix} u \\ v \end{pmatrix}$ $\qquad \begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

Universal Solution in $SL(2,\mathbb{Z})$

$$(k\lambda - 1)u = hv \qquad (k\mu - 1)v = \bar h u$$

$$(k\lambda-1)(k\mu-1) = h\bar h = 1-k^2$$

$$\mu = \frac{1}{k}\left(1 + \frac{1-k^2}{k\lambda-1}\right) = \frac{1}{k}\left(\frac{k\lambda - 1 + 1 - k^2}{k\lambda - 1}\right) = \frac{\lambda - k}{k\lambda - 1}$$

$$\psi_{xn} = e^{i\xi x}\mu^n \begin{pmatrix} u \\ v \end{pmatrix} \qquad (k_\varepsilon \lambda^\varepsilon - 1)u = h_\varepsilon v$$

$$\boxed{(k_0' + i\xi)u = h_0' v}$$

$$\mu_\varepsilon = \frac{e^{i\xi\varepsilon} - k_\varepsilon}{k_\varepsilon e^{i\xi\varepsilon} - 1} \longrightarrow \frac{i\xi - k_0'}{i\xi + k_0'}$$

$$k_\varepsilon = 1 + k_0'\varepsilon + O(\varepsilon^2)$$
$$h_\varepsilon = h_0'\varepsilon + O(\varepsilon^2)$$

$$\cancel{(k_\varepsilon e^{i\xi\varepsilon} - 1)v = \bar h_\varepsilon u}$$

$$(k_\varepsilon \mu_\varepsilon - 1)v = \bar h_\varepsilon u \qquad \bar h_\varepsilon = \bar h_0 + O(\varepsilon)$$

$$\bar h_0 u = \left(\frac{i\xi - k_0'}{i\xi + k_0'} - 1\right)v = \frac{-2k_0'}{i\xi + k_0'}v \qquad \Bigg| \qquad \bar h_0 \underbrace{(i\xi + k_0')u}_{h_0' v} = -2k_0' v$$

$$k_0' = -\tfrac{1}{2}\bar h_0 h_0'$$

which is consistent with $k_\varepsilon = \sqrt{1 - \bar h_\varepsilon h_\varepsilon} = 1 - \frac{1}{2}\bar h_0 h_0' \varepsilon$

You end up with the eqns.

$$\boxed{\begin{aligned} (i\xi + k_0')u &= h_0' v \\ \mu &= \frac{i\xi - k_0'}{i\xi + k_0'} \end{aligned}}$$

$$\mu - 1 = \frac{-2k_0'}{i\xi + k_0'} = \frac{\bar h_0 h_0'}{i\xi + k_0}$$

$$(\mu - 1)v = \bar h_0 u$$

$$(i\xi + k_0')\mu u = (i\xi - k_0')u$$

$$\mu(i\xi + k_0')u = \mu h_0' v$$

$$\boxed{\begin{aligned} (i\xi + k_0')u &= h_0' v \\ (i\xi - k_0')u &= \mu h_0' v \\ \mu &= \frac{i\xi - k_0'}{i\xi + k_0'} \end{aligned}}$$

$$(\partial_x + k_0')\psi^1_{xn} = h_0'\psi^2_{xn}$$

$$(\partial_x - k_0')\psi^1_{xn} = h_0'\psi^2_{x,n+1}$$

$$\frac{\psi^1_{x+\varepsilon,n} - \psi^1_{x,n}}{\varepsilon} + k_0'\psi^1_{x,n} = h_0'\psi^2_{xn}$$

$$\psi^1_{x+\varepsilon,n} = (1 - k_0'\varepsilon)\psi^1_{x,n} + (h_0'\varepsilon)\psi^2_{x,n}$$

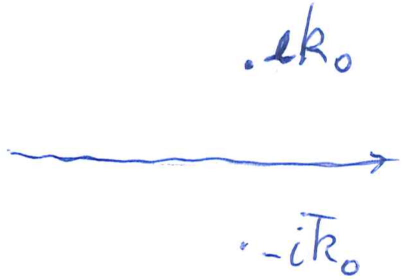
$$h_0'\psi^2_{x,n+1} = \frac{\psi^1_{x+\varepsilon,n} - \psi^1_{x,n}}{\varepsilon} - k_0'\psi^1_{x,n}$$

$$\varepsilon h_0'\psi^2_{x,n+1} = \psi^1_{x+\varepsilon,n} - \psi^1_{x,n} - \varepsilon k_0'\psi^1_{xn} \quad ?$$

Realize in $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$. $\quad \lambda^x = e^{i\xi x}$, $\quad \mu = \frac{i\xi - k_0'}{i\xi + k_0'}$, $\; u = \frac{h_0'}{i\xi + k_0'}$ $\quad V = 1$

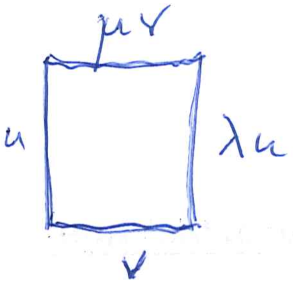
$$u = \frac{h_0'}{i\xi + k_0'}$$

$$u = \frac{h_0'}{i\xi + k_0'} \qquad \cancel{\int \frac{h_0'\,\overline{h_0'}}{i\xi + k_0}}$$



$$\int_{-\infty}^{\infty} \frac{1}{-i\xi + \overline{k_0}}\,\frac{1}{i\xi + k_0}\,\frac{d\xi}{2\pi}$$

$$= \frac{1}{-i(ik_0) + \overline{k_0}} = \frac{1}{k_0 + \overline{k_0}}$$



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$(k\lambda-1)u = hv$$
$$(k\mu-1)v = \bar h u$$

$$(k\lambda-1)(k\mu-1) = h\bar h = 1-k^2$$

$$\mu = \frac{-\lambda+k}{-k\lambda+1} = \frac{(-\lambda)+k}{k(-\lambda)+1} = \begin{pmatrix}1 & k\\ k & 1\end{pmatrix}(-\lambda)$$

Look at first quadrant grid space

$$\mathbb{C}[\lambda,\mu]u + \mathbb{C}[\lambda,\mu]v \Big/ \left(\frac{k\lambda-1}{h}u = v,\ \frac{k\mu-1}{\bar h}v = u\right)$$

$$= \mathbb{C}[X,Y]/(XY-1) \qquad X = \frac{k\lambda-1}{h} \qquad Y = \frac{k\mu-1}{\bar h}$$

$$\left(\mu = \frac{\lambda-k}{k\lambda-1}\right)$$

$$k_\varepsilon = (1-|h'|^2\varepsilon^2)^{1/2} = 1+\tfrac{1}{2}|h'|^2\varepsilon^2$$

$$\mu_\varepsilon = \frac{e^{i\xi\varepsilon}-k_\varepsilon}{k_\varepsilon e^{i\xi\varepsilon}-1} \longrightarrow \frac{e^{i\xi\varepsilon}i\xi - |h'|^2\varepsilon}{|h'|^2\varepsilon\, e^{i\xi\varepsilon} + k_\varepsilon e^{i\xi\varepsilon} i\xi}$$

Computations.

$$u = \frac{h}{k\lambda-1}v = -\sum_{n=0}^{\infty} h k^n \underbrace{\lambda^n v}_{\text{orthonormal set.}}$$

ℓ^2 sequence $\displaystyle\sum_{n=0}^{\infty}|h|^2k^{2n} = \frac{|h|^2}{1-k^2} = 1$

$$\mu^n u = \frac{(\lambda-k)^n}{(k\lambda-1)^{n+1}}v$$

$$(u|\mu^n u) = \int \overline{\frac{h}{kz-1}}\left(\frac{z-k}{kz-1}\right)^n \frac{h}{kz-1}\frac{dz}{2\pi i z}$$

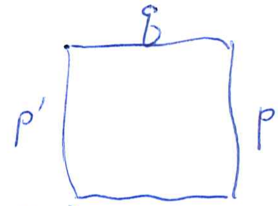
$$= \int \frac{\bar h}{k-z}\,\frac{(z-k)^n h}{(kz-1)^{n+1}}\frac{dz}{2\pi i} = -\int |h|^2\frac{(z-k)^{n-1}}{(kz-1)^{n+1}}\frac{dz}{2\pi i}$$

$n=0$

$$-\int |h|^2 \frac{1}{(z-k)(kz-1)}\frac{dz}{2\pi i} = \frac{-|h|^2}{k^2-1} = 1$$

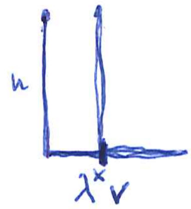
$$\exp\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}t = \begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix} = \frac{1}{\sqrt{1-h^2}}\begin{pmatrix}1 & h\\ h & 1\end{pmatrix}$$

$$= \cosh t\begin{pmatrix}1 & \tanh t\\ \tanh t & 1\end{pmatrix}$$



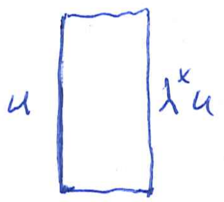
$$\begin{pmatrix}1 & h\\ h & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} \qquad \begin{pmatrix}p\\ q'\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar{h} & k\end{pmatrix}\begin{pmatrix}p'\\ q\end{pmatrix} \qquad q' \qquad (q'|p') = (-\bar{h}p'|p') \quad -h$$

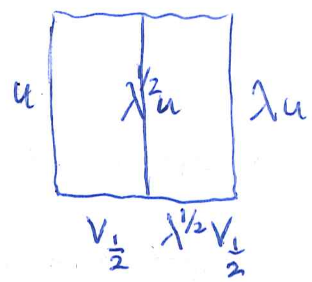
So what

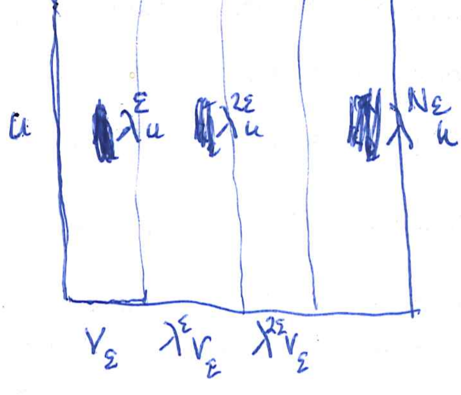


adjoin a unit vector u to $L^2(\mathbb{R})$, you want continuous in the horizontal direction and discrete vertically. You can project u onto $L^2(\mathbb{R})$. It seems simple enough

$$\int_0^\infty \lambda^x v(\lambda^x v|u)\,dx \longmapsto \int_0^\infty e^{i\xi x}$$





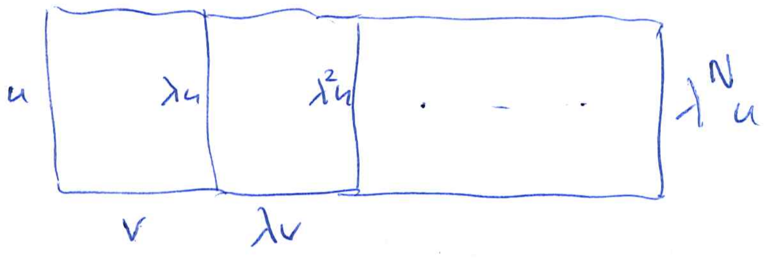


what should be the relation between $u, \lambda^\varepsilon u, v_\varepsilon$

need $v_\varepsilon \perp \lambda^\varepsilon u$

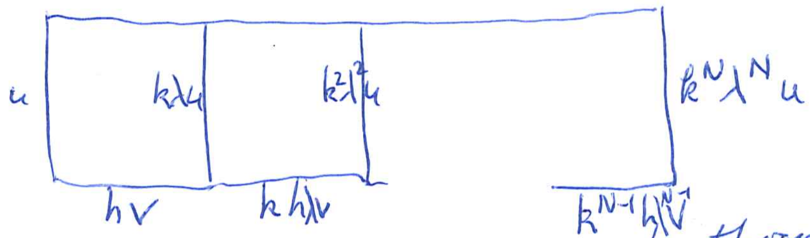
in

~~First~~ Let's see if you can analyze this sensibly. Begin with a situation to iterate.



$$(k^N\lambda^N - 1)u = \sum_{j=0}^{N-1} (k\lambda)^j \underbrace{(k\lambda - 1)u}_{hv} = \sum_{j=0}^{N-1} k^j h v$$

$$\sum_{j=0}^{N-1} |k^j h|^2 = \frac{1-k^{2N}}{1-k^2}|h|^2$$



second picture actual orthogonal projections without being made unit vectors.

now replace λ by λ^ε where $\varepsilon = \frac{1}{N}$. ~~What happens to~~ It should be clear what happens.

u splits orthogonally into

$$\underbrace{\lim \sum_{j=0}^{N} h_N k_N^j \lambda^{j\varepsilon} v} + \lim k_N^N \lambda^{N\varepsilon} u$$

$$\| \qquad\qquad$$

this should become an integral

$$(\lim k_N^N)\lambda u$$

means $k_\varepsilon = 1 + k_0'\varepsilon + O(\varepsilon^2)$.

$$e^{k_0'}\lambda \qquad k_0' < 0$$

$$\sum_{j=0}^{N} h_\varepsilon (k_\varepsilon^N)^{\frac{j}{N}} \lambda^{\frac{j}{N}} v$$

$$\int_0^1 h' e^{k_0' x} \underbrace{\lambda^x v} \, dx$$

$$\int_0^1 |h' e^{k_0' x}|^2 dx = |h'|^2 \frac{\left[e^{2k_0' x}\right]_0^1}{2k_0'} =$$

$u \quad k\lambda^{\varepsilon}u \quad k^2\lambda^{2\varepsilon}u \quad \cdots \quad k^N\lambda^{N\varepsilon}u$

$-hv \quad -hk\lambda v \quad \cdots \quad -hk^{N-1}\lambda^{\varepsilon(N-1)}v$

$$\begin{pmatrix} k\lambda^{\varepsilon}u \\ k\mu v \end{pmatrix} = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$(k\lambda - 1)u = hv \qquad u = \dfrac{hv}{k\lambda - 1} = \cdots$

$$u = \sum_{j=0}^{N-1} h\, k^j \lambda^{\varepsilon j} v + k^N \lambda^{N\varepsilon} u$$

orthogonal direct sum.

$$\sum_{j=0}^{N-1} |h k^j|^2 = |h|^2 \frac{1-k^{2N}}{1-k^2} = 1 - k^{2N}$$

$a > 0$

$k = \sqrt{1-|h|^2}$ What goes on? You need $k^{\frac{1}{\varepsilon}} \to e^{-a}$

then $k^N \lambda^{N\varepsilon} u = k^{\frac{x}{\varepsilon}} \lambda^x u \longrightarrow e^{-ax}\lambda^x u$

Next thing you want to understand is the sum.

now $v, \lambda^{\varepsilon}v, \lambda^{2\varepsilon}v, \ldots\ldots$ is an orthonormal sequence but you need convergence to a δ function type orthogonal basis.

$$k^j \lambda^{\varepsilon j} = \left(k^{\frac{1}{\varepsilon}}\right)^x \lambda^x \longrightarrow e^{-ax}\lambda^x$$

$$\|hv\|^2 = |h|^2 = 1 - k^2 \qquad k =$$

~~assume~~ get constants straight!!

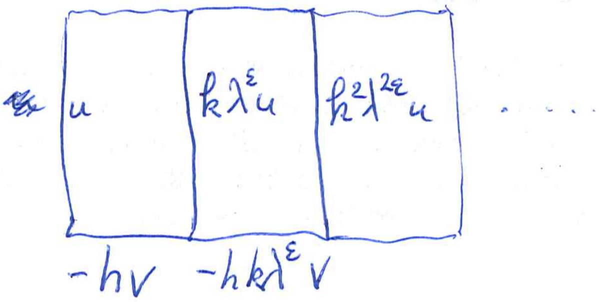
$k = \sqrt{1-|h|^2} = 1 - a\varepsilon \qquad |h|^2 = a\varepsilon \qquad a > 0.$

say $h = \sqrt{a\varepsilon} \qquad k^{\frac{1}{\varepsilon}} = (1-a\varepsilon)^{\frac{1}{\varepsilon}} \to e^{-a}$

$$\sum_{j=0}^{N} \left(k^{\frac{1}{\varepsilon}}\lambda\right)^{\varepsilon j} hv \rightsquigarrow \int_0^1 e^{-ax}\lambda^x \frac{dx}{\varepsilon} \; (?)$$

perhaps the idea is that v_ε in $L^2(\mathbb{R}, dx)$ is to be $\chi_{[0,\varepsilon]}$ normalized ~~[illegible]~~ to a unit vector i.e. $v_\varepsilon = \frac{1}{\sqrt{\varepsilon}} \chi_{[0,\varepsilon]}$.

Review.



$$(k\lambda^\varepsilon - 1)u = hv$$

$$u = -hv + k\lambda^\varepsilon u$$

$$= -hv - h(k\lambda^\varepsilon)v + (\ )^2 u$$

$$\left((k\lambda^\varepsilon)^N - 1\right)u = \sum_{j=0}^{N-1} (k\lambda^\varepsilon)^j (k\lambda^\varepsilon - 1)u = \sum_{j=0}^{N-1} (k\lambda^\varepsilon)^j h v$$

$$u = -\sum_{j=0}^{N-1} h (k\lambda^\varepsilon)^j v + (k\lambda^\varepsilon)^N u$$

assume $k = 1 - a\varepsilon + O(\varepsilon^2)$. Then

$$k^{\frac{1}{\varepsilon}} \longrightarrow e^{-a} \quad \text{as } \varepsilon \downarrow 0.$$

$$(k\lambda^\varepsilon)^j = \left(k^{\frac{1}{\varepsilon}}\lambda\right)^{j\varepsilon} \longrightarrow e^{-ax}\lambda^x$$

~~[illegible] to make~~

Problem is to get $\sum_{j=0}^{N-1} h\left(k^{\frac{1}{\varepsilon}}\lambda\right)^{\varepsilon j} v$ to have limit (cont. limit)

$$\int_0^{y=N\varepsilon} dx\, e^{-ax}\lambda^x \tilde{v} \qquad \tilde{v} = ? \text{ like } \delta$$

Other point: $h = b\sqrt{\varepsilon}$ $\qquad k = \sqrt{1-|h|^2} = \sqrt{1-|b|^2\varepsilon}$

$= 1 - \frac{1}{2}|b|^2\varepsilon \quad = 1 - a\varepsilon$

$$\therefore \boxed{2a = |b|^2.}$$

$$\begin{pmatrix} \lambda^\varepsilon u \\ \mu\sqrt{\varepsilon}\, v \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & b\sqrt{\varepsilon} \\ \bar{b}\sqrt{\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} u \\ \sqrt{\varepsilon}\, v \end{pmatrix}$$

$$(k\lambda^\varepsilon - 1)\, u = b\,\varepsilon\, v \implies \boxed{(-a + i\xi)u = bv}$$

$$(k\mu - 1)\cancel{\sqrt{\varepsilon}}\, v = \bar{b}\cancel{\sqrt{\varepsilon}}\, u \qquad \boxed{(\mu - 1)v = \bar{b}u}$$

$$\boxed{\mu = \frac{i\xi + a}{i\xi - a}}$$

$$(k\mu - 1)(k\lambda^\varepsilon - 1) = |b|^2\varepsilon = 1 - k^2$$

$$\mu = \frac{\lambda^\varepsilon - k}{k\lambda^\varepsilon - 1} \longrightarrow \frac{i\xi + a}{i\xi - a}$$

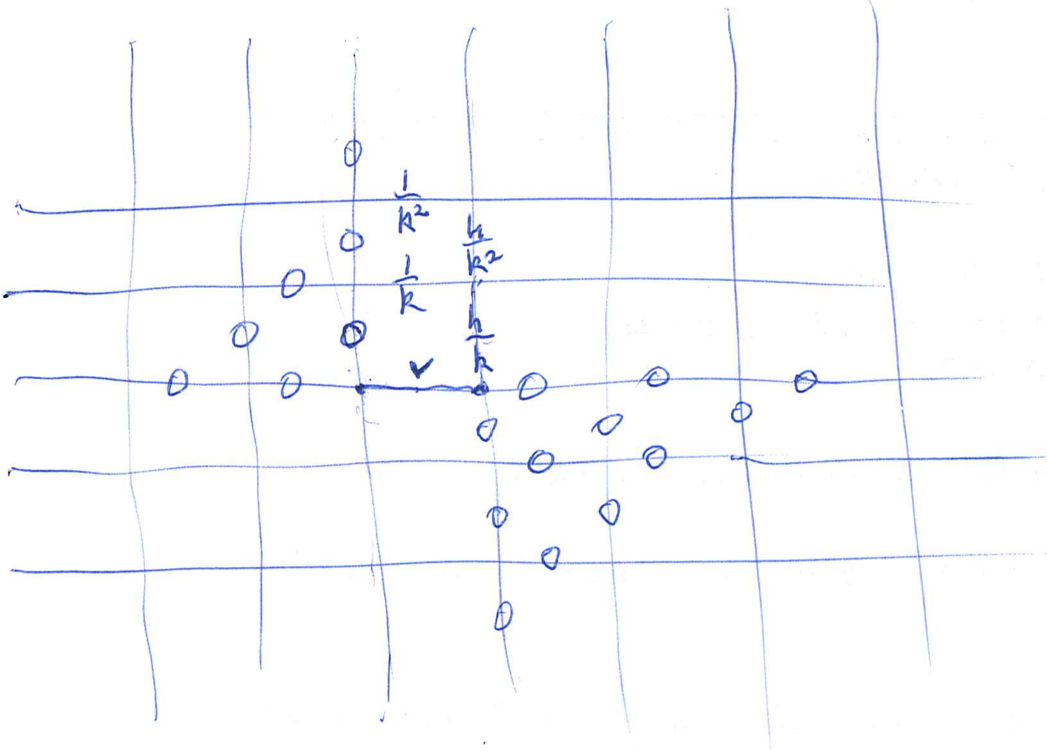
$$\mu - 1 = \frac{2a}{i\xi - a} \qquad \frac{2a}{i\xi - a} v = \bar{b}u$$

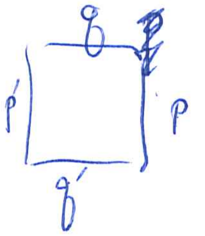
$$(i\xi - a)u = \frac{2a}{\bar{b}} v = bv$$

Representation in $L^2(\mathbb{R}, \frac{d\xi}{2\pi})$

$$\boxed{\begin{aligned} \lambda^x &= \text{mult by } e^{i\xi x} \\ \mu &= \text{mult by } \frac{i\xi + a}{i\xi - a} \\ v &= 1 \\ u &= \frac{b}{i\xi - a} \qquad 2a = |b|^2 \end{aligned}}$$

For tomorrow need to calculate $H(v, -)$

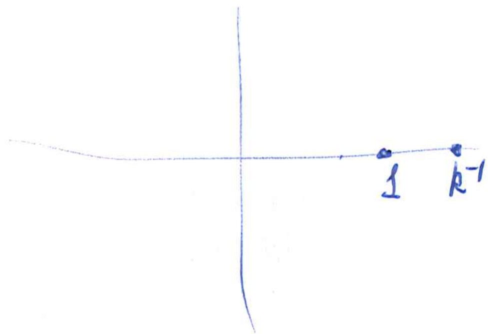




$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$H(v, \mu^n \lambda u) = \begin{cases} 0 & n \leq -1. \\ \frac{h}{k^{n+1}} & n \geq 0 \end{cases}$$

$$H(v, \lambda^m v) = \delta_m$$



$$\operatorname{res}_0\left(z^m \frac{dz}{2\pi i z}\right) = \delta_m$$

$$\operatorname{res}_{k^{-1}}\left(z^m \frac{dz}{2\pi i z}\right) = 0 \quad \forall m.$$

$$\operatorname{res}_0\left(\frac{(z-k)^n h}{(kz-1)^{n+1}} \frac{dz}{2\pi i}\right) = 0 \qquad -\operatorname{res}_\infty\left(\frac{(z-k)^n h}{(kz-1)^{n+1}} \frac{dz}{2\pi i}\right)$$

$$\operatorname{res}_{k^{-1}}\left(\frac{(z-k)^n h}{(kz-1)^{n+1}} \frac{dz}{2\pi i}\right) = 0 \quad n \leq -1 \qquad = -\operatorname{res}_\infty\left(\frac{z^n h}{(kz)^{n+1}} \frac{dz}{2\pi i}\right)$$

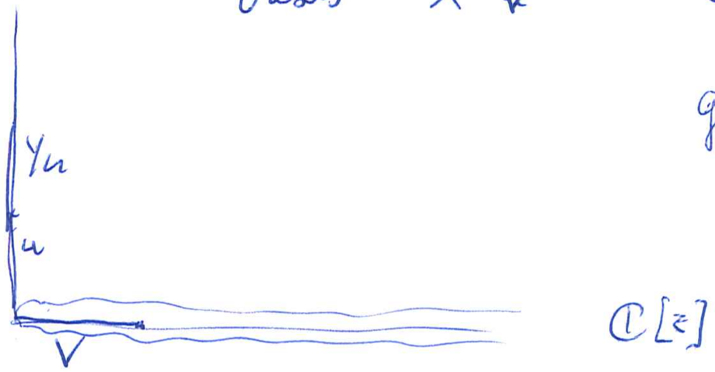
$$n \geq 0 \qquad = \frac{h}{k^{n+1}}$$

first quadrant $\quad X = \frac{k\lambda - 1}{h} \quad Y = \frac{k\mu - 1}{h} \quad XY = 1.$ $\quad$ 690

$Xu = v \qquad Yv = u \qquad z$

basis $\quad X^n v = \frac{(kz-1)^n}{h^n} \quad n \geq 0.$

get $\mathbb{C}[z]$



$$X^{-n-1} v = \left(\frac{h}{k\lambda - 1}\right)^{n+1}$$

NOTICE: The first quadrant has ~~zo~~ poles at ∞ and k^{-1}.

$$w = \frac{z-k}{kz-1}$$

$$dw = \frac{(kz-1)dz - (z-k)k\,dz}{(kz-1)^2}$$

$$= \frac{(-1+k^2)\,dz}{(kz-1)^2}$$

~~total~~ ~~zeros~~ ~~zeros and~~

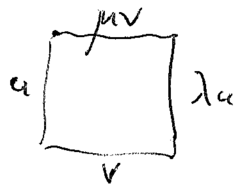
Point get subspaces ~~to~~ $\quad L(+D) \quad D$ divisor

First quadrant = subspace of rational functions with poles ~~in~~ $\{\infty, \; k^{-1}\}$, ~~then $kz-1$~~

want a simple zero at k^{-1} and simple at ∞ $\quad kz-1.$

third quad. $\quad 0, k \qquad (kz-1)^* = kz^{-1} - 1 = \frac{k-z}{z}$

some other ideas. Form a grid space with a general $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$



$$\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$(\lambda - a) u = b v$$
$$(\mu - d) v = c u$$

$$(\lambda - a)(\mu - d) = cb$$

$$\mu = d + \frac{bc}{\lambda - a} = \frac{-d\lambda + (ad - bc)}{-\lambda + a}$$

$$= \frac{d(-\lambda) + \Delta}{(-\lambda) + a} = \underbrace{\begin{pmatrix} d & \Delta \\ 1 & a \end{pmatrix}}(-\lambda)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta \bar{d} & \Delta \bar{c} \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \Delta \bar{b} & \Delta \bar{a} \end{pmatrix}$$

$$\begin{pmatrix} d & \Delta \\ 1 & a \end{pmatrix} = \begin{pmatrix} \Delta \bar{a} & \Delta \\ 1 & a \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{a} & 1 \\ 1 & a \end{pmatrix} \frac{1}{\sqrt{|a|^2 - 1}}$$

Start again. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1) \quad \therefore \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{c} \\ c & d \end{pmatrix}$

$$\mu = \frac{d\lambda - \Delta}{\lambda - a}$$

$$\lambda = a + \frac{bc}{\mu - d} = \frac{a\mu - \Delta}{\mu - d} = \Delta \, \frac{\bar{d}\mu - 1}{\mu - d}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} \frac{\Delta}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

Questions to ask. You have a map from $U(1,1)$ to fractional linear transf.

$$U(1,1) \longrightarrow GL(2,\mathbb{C})/\text{scalars}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -\Delta \\ 1 & -a \end{pmatrix}$$

$$i\Delta^{1/2}\begin{pmatrix} -i\Delta^{1/2}\bar a & i\Delta^{1/2} \\ -i\Delta^{-1/2} & +i\Delta^{-1/2} a \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \Delta\bar c & \Delta\bar a \end{pmatrix} \qquad \begin{pmatrix} d & -\Delta \\ 1 & -a \end{pmatrix} = \begin{pmatrix} \Delta\bar a & -\Delta \\ 1 & -a \end{pmatrix} \leftarrow \text{is this in } U(1,1)/\text{scalars}$$

Repeat.

$$\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad \begin{aligned} (\lambda - a)u &= bv \\ (\mu - d)v &= cu \end{aligned}$$

$$(\lambda - a)(\mu - d) = bc \qquad \mu = d + \frac{bc}{\lambda - a} = \frac{d\lambda - \Delta}{\lambda - a}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -\Delta \\ 1 & -a \end{pmatrix}$$

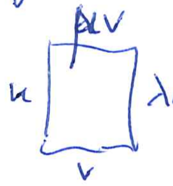
$$\begin{vmatrix} d & -\Delta \\ 1 & -a \end{vmatrix} = -da + \Delta = -bc$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{i\theta}\bar d & e^{i\theta}\bar c \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -e^{i\theta} \\ 1 & -e^{i\theta}\bar d \end{pmatrix}$$

$$\begin{pmatrix} d & -e^{i\theta} \\ 1 & -e^{i\theta}\bar d \end{pmatrix} = ie^{\frac{i\theta}{2}}\begin{pmatrix} -ie^{-\frac{i\theta}{2}}d & ie^{\frac{i\theta}{2}} \\ -ie^{-\frac{i\theta}{2}} & ie^{+\frac{i\theta}{2}}\bar d \end{pmatrix}$$

Discuss what you've found. Somehow your grid spaces ~~allow you to ca~~ produce

You have lots to work on.



$$k\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \qquad \begin{aligned}(k\lambda-1)u &= hv\\ (k\mu-1)v &= \bar h u\end{aligned}$$

$$(k\lambda-1)(k\mu-1) = 1-k^2$$

$$\mu = \frac{1}{k}\left(1+\frac{1-k^2}{k\lambda-1}\right) = \frac{\lambda-k}{k\lambda-1}$$

$$\mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]$$
$$\|$$

E A-module generators u, v relations

$\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$ make this an A-module with

$\lambda =$ mult. by z let $u = \dfrac{\bar h}{kz-1}$

$\mu = \dfrac{z-k}{kz-1}$ $v = 1$.

So how to think. Best way to proceed is analytic. Form $\bar E$ completion of E for pos. def. inner product. Closed subspace

$\overline{\mathbb{C}[\lambda,\lambda^{-1}]v} \subset \bar E$

How to proceed analytically. Form $\bar E$, know λ, μ preserve pos. def $(\ |\ )$ on E hence extend uniquely to unitary operators on $\bar E$. $k\lambda - 1$ is bounded & invertible on $\bar E$ (geom. series since $|k|<1$), so

$\mu = \dfrac{\lambda-k}{k\lambda-1}$ on E $\Rightarrow$ same on $\bar E$.

Consider closed subspace

$E' = \overline{\mathbb{C}[\lambda,\lambda^{-1}]v} \subset \bar E$ spanned by orth. set $(\lambda^m v)_{m\in\mathbb{Z}}$

Can say relations $\begin{aligned}(k\lambda-1)u &= hv\\ (k\mu-1)v &= \bar h u\end{aligned}$ hold on $\bar E$.

$$(k\lambda-1)(k\mu-1) = 1-k^2$$

$$\therefore\ \mu = \frac{\lambda-k}{k\lambda-1}$$ ~~go on~~

Cons. E' = closed sub. spanned by. $E' \simeq L^2(S^1, \frac{dz}{2\pi i z})$

$\lambda =$ mult by z $E' = \dots$ $\lambda^m v$ — z^m

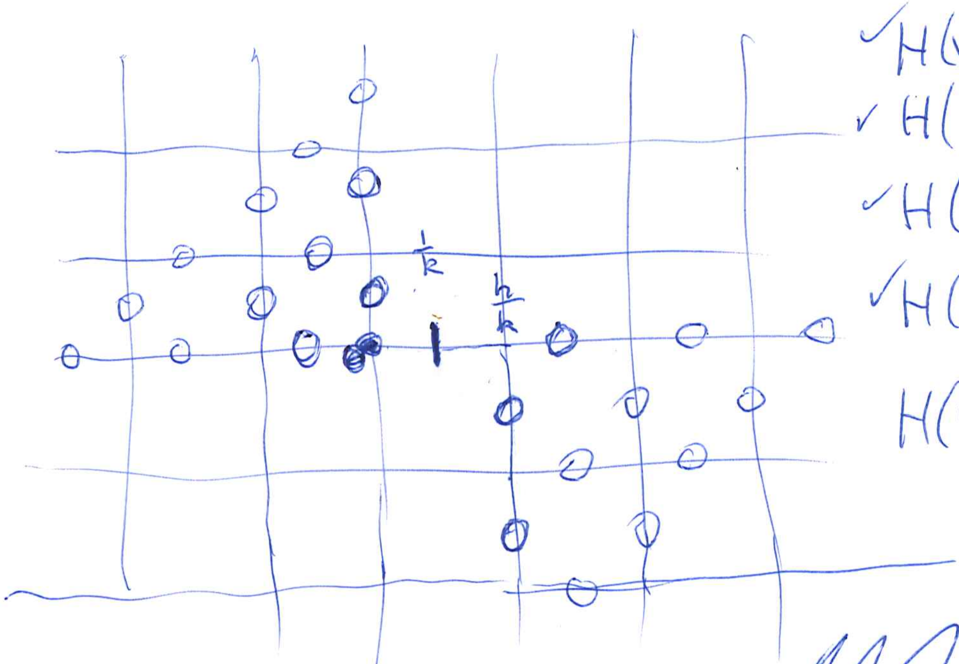
Let's analyze $H(\ ,\ )$. Start with E with this hermitian form. You know $\bar{E}$ is incompatible with H, because $H(u,u) = -1$ and $u = \frac{h}{kz-1} = -\sum h k^n z^n$.

Better: There is a Cauchy sequence f_n in E such that $f_n \longrightarrow u$ in $\bar{E}$ such that $H(f_n|f_n) = (f_n|f_n) \longrightarrow +1$ and $H(u,u) = -1$

So what is a suitable viewpoint

Review calc. of $H(v, -)$, this is a lin. fnl on the grid space, $\therefore$ is a solution of the grid equations:

$$\psi_{mn} = \begin{pmatrix} H(v, \lambda^m \mu^n u) \\ H(v, \lambda^m \mu^n v) \end{pmatrix}$$



$\checkmark H(v, \lambda^m v) = 0 \quad m \leq -1$

$\checkmark H(v, \mu^n u) = 0 \quad n \geq 0$

$\checkmark H(v, v) = 1$

$\checkmark H(v, \lambda^m v) = 0 \quad m \geq 1$

$H(v, \lambda \mu^n u) = 0 \quad n \leq -1$

$m \geq 1$

$$\operatorname{res}_{\{0,k^{-1}\}} \left(z^m \frac{dz}{2\pi i z} \right) = \delta_m$$

$$\operatorname{res}_{\{0,k^{-1}\}} \left(\underbrace{\frac{(z-k)^n}{(kz-1)^{n+1}}}_{\text{reg. at } k \text{ for } n\geq 0} h \frac{dz}{2\pi i z} \right) = \int_{|z|=R} \text{———} = 0$$

$-n\xi+1$

$$\operatorname{res}_{\{0,k^{-1}\}}\left(\cancel{f}\left(\frac{kz-1}{z-k}\right)^{-n}\frac{h}{kz-1}\frac{dz}{2\pi i\cancel{z}}\right)=0$$

Now you ~~have to~~ want to take a continuous limit.

try horizontally first. means $\lambda\mapsto\lambda^{\varepsilon}$ $\quad h\mapsto b\sqrt{\varepsilon}$

$v_{\varepsilon}\mapsto v\sqrt{\varepsilon}$ $\qquad u_{\varepsilon}=\dfrac{h}{k_{\varepsilon}\lambda-1}v\longmapsto$ $\qquad k=\sqrt{1-a\varepsilon}$

Again. $\boxed{\lambda\mapsto\lambda^{\varepsilon},\quad h\mapsto b\sqrt{\varepsilon},\quad v\mapsto v\sqrt{\varepsilon}}$ $\quad k\to\sqrt{1-|b|^2\varepsilon}$

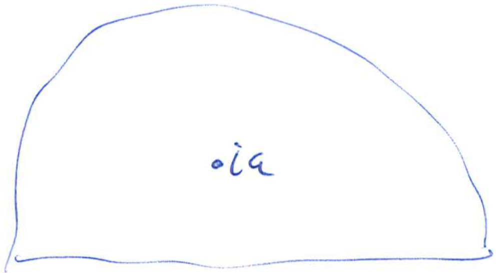
$2a=|b|^2$

$$u=\frac{h}{k\lambda-1}v\longmapsto u=\frac{b\sqrt{\varepsilon}}{(1-\frac{1}{2}|b|^2\varepsilon)\lambda^{\varepsilon}-1}v\sqrt{\varepsilon}\longrightarrow\frac{bv}{i\xi-a}$$

$$\mu=\frac{\lambda-k}{k\lambda-1}\longmapsto\mu=\frac{\lambda^{\varepsilon}-k_{\varepsilon}}{k_{\varepsilon}\lambda^{\varepsilon}-1}\longrightarrow\frac{i\xi+a}{i\xi-a}$$

Use the Hilbert space $L^2\left(\mathbb{R},\frac{d\xi}{2\pi}\right)$. Get rep.

$\lambda^{x}=e^{i\xi x}$	$\longmapsto$	mult by $e^{i\xi x}$
μ	$\longmapsto$	$\dfrac{i\xi+a}{i\xi-a}$
u	$\longmapsto$	$\dfrac{b}{i\xi-a}$
v	$\longmapsto$	1



$$\int\left|\frac{b}{i\xi-a}\right|^2\frac{d\xi}{2\pi}=\int\frac{|b|^2}{(i\xi-a)(-i\xi-a)}\frac{d\xi}{2\pi}$$

$$=\frac{|b|^2\cancel{2\pi i}}{+2a\,\cancel{(+i)}\,\cancel{2\pi}}=1.$$