conjugation σ, volume $\omega \in \Lambda^2 V$ such that $\sigma(\omega) = -\omega$ yields a herm. form $K(\xi, \eta) = \frac{\sigma(\xi)\wedge\eta}{\omega}$

$$\sigma(\xi)\wedge\eta = K(\xi,\eta)\,\omega \implies \underset{\substack{\parallel \\ -\sigma(\eta)\wedge\xi}}{\xi\wedge\sigma(\eta)} = \overline{K(\xi,\eta)}(-\omega)$$

$$\therefore\ K(\eta,\xi) = \overline{K(\xi,\eta)}$$

Let σ, σ' be conjugations $\ni$ $\sigma(\omega) = -\omega$

let $g\sigma = \sigma'$ Then $g(\sigma\omega) = \sigma'\omega$

$g\,\sigma(g) = 1$ $g(\omega) = \omega$

i.e. $\det(g) = 1$.

σ conj. g auto. $g\sigma$ anti-linear

so $g\sigma g\sigma = g\,\sigma(g)$ is linear.

Given K and ω define σv by $K(v, w) = \frac{\sigma v \wedge w}{\omega}$

i.e.

$$\bar{V} \xrightarrow{\sim} V^* \xleftarrow{\sim} V$$

$$v \mapsto K(v, -)$$

$$\left(v' \mapsto \frac{w \wedge v'}{\omega}\right) \leftarrow\!\shortmid w$$

This gives an anti-linear isom. σ on V; when is $\sigma^2 = 1$? Choose $e_\pm$

$K(e_+, e_+) = 1$

$K(e_+, e_-) = 0$

$K(e_-, e_-) = -1$.

~~First case~~ suppose $\omega = r\, e_+ \wedge e_-$ $r \in \mathbb{C}^\times$.

$$K(\omega,\omega) = \begin{vmatrix} K(re_+, re_+) & K(e_-, re_+) \\ K(re_+, e_-) & K(e_-, e_-) \end{vmatrix} = \begin{vmatrix} |r|^2 & 0 \\ 0 & -1 \end{vmatrix} = -|r|^2$$

K $\qquad \sigma v \wedge v' = K(\sigma, v') r e_+ \wedge e_-$

$$\sigma(fe_+ + ge_-) \wedge (\phi e_+ + \psi e_-) = \cancel{f}$$

$$= (\bar f \sigma(e_+) + \bar g \sigma(e_-)) \wedge (\phi e_+ + \psi e_-)$$

$$K(fe_+ + ge_-, \phi e_+ + \psi e_-) r e_+ \wedge e_-$$

$$= (\bar f \phi - \bar g \psi) r e_+ \wedge e_-$$

$$= (-\bar f r e_- - \bar g r e_+) \wedge (\phi e_+ + \psi e_-)$$

$$\therefore \boxed{\sigma(fe_+ + ge_-) = -\bar g r e_+ - \bar f r e_-}$$

$$\sigma\begin{pmatrix} f \\ g \end{pmatrix} = -r\begin{pmatrix} \bar g \\ \bar f \end{pmatrix} \qquad \begin{pmatrix} f \\ g \end{pmatrix} = (-\bar r)(-r)\begin{pmatrix} f \\ g \end{pmatrix}$$

so $|r| = 1$.

Repeating this. Given K, ω choose $e_\pm$ and let r : $\omega = r e_+ \wedge e_-$

$$K(v, \omega') = \frac{\sigma(v) \wedge \omega'}{\omega}$$

$$\sigma(v) \wedge v' = K(v, v') r e_+ \wedge e_-$$

~~$\sigma(fe_+ + ge_-) \wedge$~~

$$\sigma(e_\pm) \wedge (\phi e_+ + \psi e_-) = \underbrace{K(e_\pm, \phi e_+ + \psi e_-)}_{\substack{\phi \\ -\psi}} r e_+ \wedge e_-$$

$$\sigma(e_+) = -r e_- \qquad \sigma(e_-) = -r e_+$$

Question: When is this $\sigma^2 = -1$. $\sigma^2(e_+) = \sigma(-re_-) = -\bar{r}(-re_+)$

So $|r|^2 = 1$. So where are we?

Consider now our $M =$

$$K(v, v')\,\omega = \sigma(v)\wedge v'$$

$$\underbrace{K(e_+, \phi e_+ + \psi e_-)\,\omega}_{\phi\, r e_+\wedge e_-} = \sigma(e_+)\wedge(\phi e_+ + \psi e_-)$$

$$= -re_-\wedge(\phi e_+ + \psi e_-)$$

$$K(e_-, \phi e_+ + \psi e_-)\,\omega = \sigma(e_-)\wedge(\phi e_+ + \psi e_-)$$

$$\overset{\parallel}{-\psi\, re_+\wedge e_-} = -re_+\wedge(\phi e_+ + \psi e_-)$$

$$\therefore \quad \sigma(fe_+ + ge_-) = \bar{f}(-re_-) + \bar{g}(-re_+)$$

$$\sigma^2 = 1 \iff |r|^2 = 1.$$

Quantization of DD?

cont. case $L^2(\mathbb{R}, dx)$ with translation inf. gen. ∂_x | F.T. picture $L^2(\mathbb{R}, \frac{dk}{2\pi})$, Hardy space

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$E = \xi_+ L^2 + \xi_- L^2 \qquad \underset{\sim}{K}(\xi_+ f + \xi_- g) = |f|^2 - |g|^2$

$$(\xi_+ f_1 + \xi_- g_1)\wedge(\xi_+ f_2 + \xi_- g_2) = (\xi_-\wedge\xi_+)(g_1 f_2 - f_1 g_2)$$

$$\sigma(\xi_+ f + \xi_- g) = \xi_+\bar{g} + \xi_-\bar{f}$$

Then $\underset{\sim}{K}\left(\sigma(\xi_+ f_1 + \xi_- g_1), \xi_+ f_2 + \xi_- g_2\right)$

$= \underset{\sim}{K}\left(\xi_+ \bar{g}_1 + \xi_- \bar{f}_1, \xi_+ f_2 + \xi_- g_2\right)$

$= g_1 f_2 - f_1 g_2$ so that $K(v, v')\omega = \sigma v \wedge v'$

$$Wr(\xi_+ f + \xi_- g, \xi_+ \phi + \xi_- \psi) = -\begin{vmatrix} f & \phi \\ g & \psi \end{vmatrix}$$

$$Wr(\sigma(\xi_+ f + \xi_- g), \xi_+ \phi + \xi_- \psi) = -\begin{vmatrix} \bar{g} & \phi \\ \bar{f} & \psi \end{vmatrix}$$

$$\underset{\sim}{K}(\xi_+ f + \xi_- g, \xi_+ \phi + \xi_- \psi) = \bar{f}\phi - \bar{g}\psi$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$ assume $\sigma \xi'_- = \xi'_+$

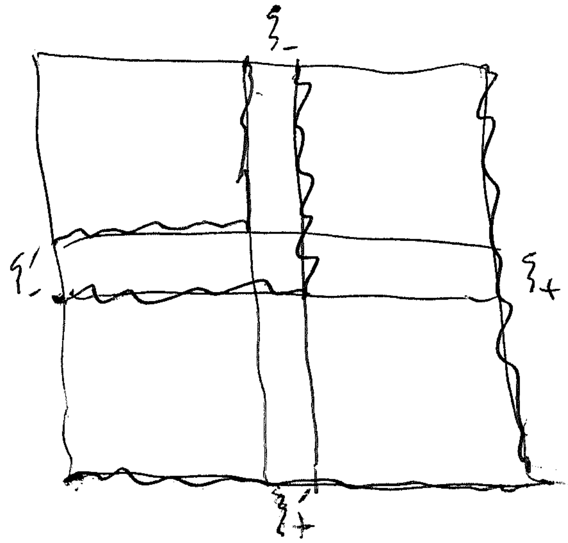
Then $\sigma p_0 = \overline{b^l}\,\xi'_- + \overline{a^l}\,\xi'_+$ so $\sigma p_0 = q_0$ iff $\overline{b^l} = c^l$ and $\overline{a^l} = d^l$

$$Wr(p_0, q_0) = -\begin{vmatrix} a^l & b^l \\ c^l & d^l \end{vmatrix} \underbrace{Wr(\xi'_-, \xi'_+)}_{-1}$$

$Wr(\xi, \eta) \in A$ defined to be skew-symm. A bilinear such that $Wr(z^{-n} p_n, q_n) = -1$.

Q. Is it possible to show $h_n d(0)$ is approximated by $\int b z^n$, better, is h_n approx by $\dfrac{\int b z^n}{d(0)}$.

$\dfrac{1}{d(0)} = \prod_{n \in \mathbb{Z}} k_n$. This factor should compensate for $|h_n| < 1$.



$$\xi'_- zH_+ + \xi_- zH_+$$

$$\cap$$

$$\xi'_- H_+ + \xi_- H_+$$

$$\cap$$

$$\xi'_- L^2 + \xi_- L^2$$

$\underset{\sim}{K}(\xi'_-) = 1$

$\underset{\sim}{K}(\xi_-) = -1$

$$\underset{\sim}{K}(\xi_-, \xi'_-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b$$

$$\tilde{\xi}_+ = \xi'_-(1 - \overset{\overset{zH_+}{\cup}}{f}) + \xi_-(\overset{\overset{L^2}{\cup}}{-g}) \qquad \tilde{f} = 1 - f.$$

$$\begin{pmatrix} \pi_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1-f \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \pi_1 \bar{b} \\ b\varepsilon_1 & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

~~$bf = g$~~

~~$(1 + \pi_1 \bar{b} b) f$~~

$f + \pi_1 \bar{b} g = 0 \qquad -g - b\varepsilon_1 \pi_1 \bar{b} g = b$

$(1 + b\varepsilon_1 (b\varepsilon_1)^*) g = -b$

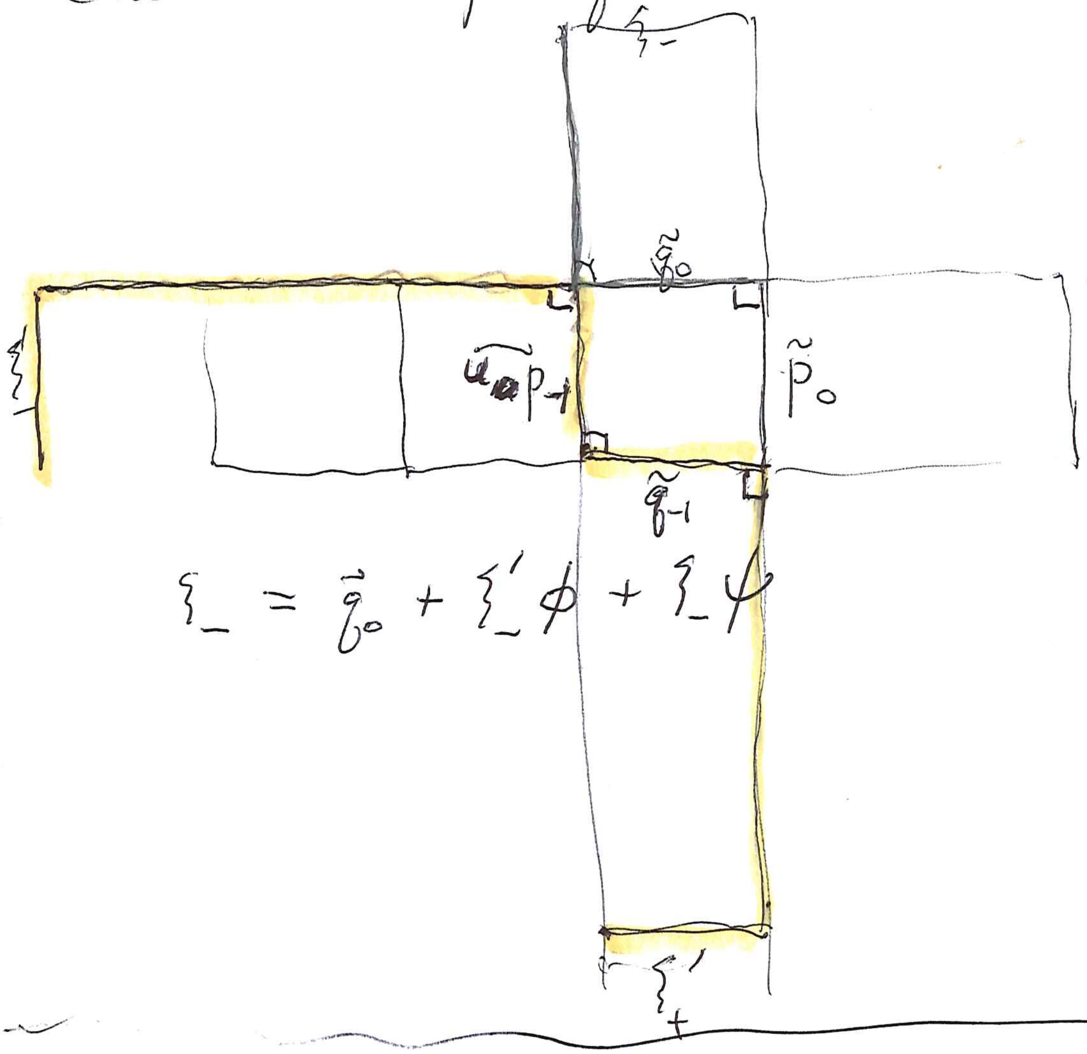
$bf + g = 0 \qquad -f - \pi_1 \bar{b} g = 0$

$\tilde{f} - \pi_1 \bar{b} g = 1$

$\tilde{f} + \pi_1 \bar{b} b \tilde{f} = 1$

$\pi_1 (\tilde{f} + |b|^2 \tilde{f}) = 0 \qquad (1 + |b|^2) \tilde{f} \in zH_-$

Construct the p-sequence



$$\xi_- = \tilde{q}_0 + \xi'_- \phi + \xi_- \psi$$

$$\underset{\sim}{K}(\xi_+ f + \xi_- g) = |f|^2 - |g|^2$$

$$\underset{\sim}{K}(\sigma(\xi_+ f + \xi_- g)) = \underset{\sim}{K}(\xi_+ \bar{g} + \xi_- \bar{f}) = |\bar{g}|^2 - |\bar{f}|^2 = -|f|^2 + |g|^2$$

$$\underset{\sim}{K}(\sigma(\xi_+ f + \xi_- g), \xi_+ f + \xi_- g) = \underset{\sim}{K}((\xi_+ \bar{g} + \xi_- \bar{f}), \xi_+ f + \xi_- g)$$

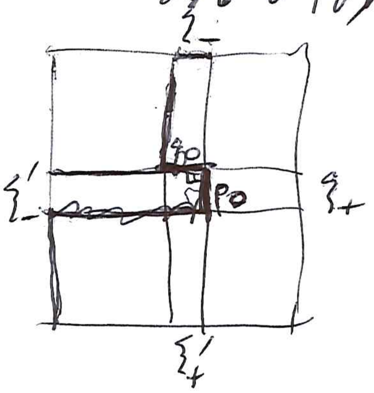
$$= \bar{g} f - \bar{f} g = 0 .$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

~~$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$~~

~~$\sigma(\xi'_-) = \sigma(d\xi_- - b\xi_+) = \bar{d}\xi_+ - \bar{b}\xi_- \overset{?}{=}$~~

go ~~back~~ to [illegible]



$$\begin{pmatrix}p_0\\ q_0\end{pmatrix}=\begin{pmatrix}a^l & \overset{H_+}{b^l}\\ \underset{zH_-}{c^l} & d^l\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}=\begin{pmatrix}d^r & \overset{H_-}{-b^r}\\ \underset{zH_+}{-c^r} & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$=\frac{1}{d}\begin{pmatrix}d^r & b^l\\ -c^r & d^l\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}=\frac{1}{a}\begin{pmatrix}a^l & -b^r\\ c^l & a^r\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}=\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}\qquad\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\underbrace{Wr(\xi_+,\xi_-)}_{-1}=\begin{vmatrix}a & b\\ c & d\end{vmatrix}\underbrace{Wr(\xi'_-,\xi'_+)}_{-1}$$

$$Wr(\xi_+,\xi'_+)=Wr(a\xi'_-+b\xi'_+,\xi'_+)=a\underbrace{Wr(\xi'_-,\xi'_+)}_{-1}$$

$$\underbrace{Wr(p_0,q_0)}_{-1}=\frac{1}{a^2}\begin{vmatrix}a^l & -b^r\\ c^l & a^r\end{vmatrix}\underbrace{Wr(\xi_+,\xi'_+)}_{-a}$$

$$\therefore\ \begin{vmatrix}a^l & -b^r\\ c^l & a^r\end{vmatrix}=a$$

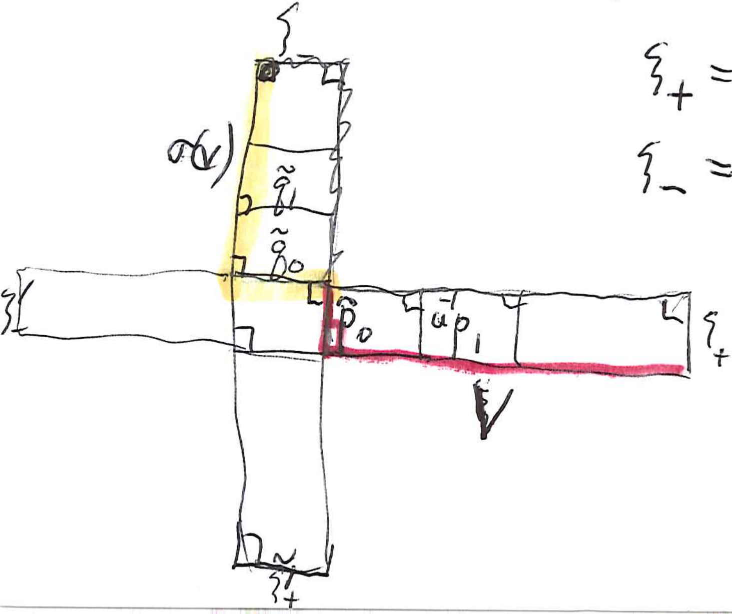
Check: $\begin{pmatrix}a & \\ & \end{pmatrix}=\begin{pmatrix}a^r & b^r\\ & \end{pmatrix}\begin{pmatrix}a^l & \\ c^l & \end{pmatrix}$

~~Suppose~~ Start w:

$$\xi_+=\frac{1}{d}\xi'_-+\frac{b}{d}\xi_-$$

$$\xi_+=\tilde{p}_0+v$$

$$\xi_-=\tilde{q}_0+\sigma(v)$$



$$h_0 = (q_0|p_0) = (\xi_-' c^\ell + d^\ell \xi_+' \,|\, d^r \xi_+ - b^r \xi_-)$$

$$= (c^\ell \xi_-' \,|\, d^r \underbrace{\xi_+}_{\frac{1}{d}\xi_-' + \frac{b}{d}\xi_-}) - (d^\ell \underbrace{\xi_+'}_{-\frac{c}{d}\xi_-' + \frac{1}{d}\overline{\xi_-}} \,|\, b^r \xi_-)$$

$$= \int \overline{c^\ell}\,\frac{d^r}{d} - \cancel{\int \frac{d^\ell}{d}\, b^r}$$

$$= \int b^\ell \frac{d^r}{d} = \frac{b^\ell(0)\, d^r(0)}{d(0)} = \frac{b^\ell(0)}{d^\ell(0)}.$$

$$h_0 = (q_0|p_0) = \left(-\frac{c^r}{d}\xi_-' + \frac{d^\ell}{d}\xi_- \,\middle|\, \frac{a^\ell}{a}\xi_+ - \frac{b^r}{a}\xi_+' \right)$$

---

go over Wr computation.

$$\underset{\sim}{K}(\sigma(\xi), \eta) = Wr(\xi, \eta) \quad \Rightarrow \quad Wr(\xi_+, \xi_-) = -1$$

$$Wr(\xi_+ f + \xi_- g, \xi_+\phi + \xi_-\psi) = \begin{vmatrix} f & \phi \\ g & \psi \end{vmatrix} Wr(\xi_+, \xi_-)$$

$$\underset{\sim}{K}(\sigma(\xi_+ f + \xi_- g), \xi_+\phi + \xi_-\psi)$$

$$= \underset{\sim}{K}(\xi_+ \bar{g} + \xi_- \bar{f}, \xi_+\phi + \xi_-\psi) = \bar{\bar{g}}\phi - \bar{\bar{f}}\psi = g\phi - f\psi$$

$$p_0 = \frac{1}{d}(d^r \xi_-' + b^\ell \xi_-)$$

$$\|p_0\|^2 = \int\left(\left|\frac{d^r}{d}\right|^2 + \left|\frac{b^\ell}{d}\right|^2\right)$$

$$\begin{pmatrix} & \\ & d \end{pmatrix} = \begin{pmatrix} & 1 \\ c^r & d^r \end{pmatrix}\begin{pmatrix} & b^\ell \\ & d^\ell \end{pmatrix}$$

$$|b^\ell|^2 = b^\ell c^\ell = -1 + a^\ell d^\ell$$

$$\|p_0\|^2 = \int \left|\frac{dr}{d}\right|^2 + \left|\frac{bl}{d}\right|^2 = \int \left|\frac{al}{a}\right|^2 + \left|\frac{-br}{a}\right|^2$$

~~$\pm\int \overline{\left(\frac{dr}{d}\right)}\left(\frac{al}{a}\right) + \overline{\left(\frac{bl}{d}\right)}\left(\frac{-br}{a}\right)$~~

$\qquad\qquad\qquad\qquad H_+ \qquad H_-$

~~$\frac{dr}{d}(0)\ \frac{al}{a}(\infty)$~~

~~$\frac{1}{d(0)}$~~

Question: Define the herm. operator B by

$$K(\xi, \eta) = (\xi | B\eta)$$
$$K(\xi, \xi) = (\xi | B\xi)$$

$$K(\xi'_- f + \xi_- g, \;-\;) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

So $\quad B(\xi'_- f + \xi_- g) = \begin{pmatrix} \xi'_- & \xi_- \end{pmatrix} \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$

~~Perhaps the point is~~ The problem is whether the Hilbert space ~~inner~~ scalar product has a local version. ~~that does it~~ You are talking about the space of sections of a rank 2 vector bundle over S^1. You have 2 bases (ξ'_-, ξ_-) and (ξ_+, ξ'_+) related the S-matrix which is unitary over the circle. You get a herm. scalar product on the rank 2 vector bundle by requiring either to be an orthonormal frame.

$p_0 = \frac{d^r}{d}\xi'_- + \frac{b^\ell}{d}\xi_-$ so $1 = \|p_0\|^2 = \int \left|\frac{d^r}{d}\right|^2 + \left|\frac{b^\ell}{d}\right|^2$

can you check this somehow?

$$p_0 = \frac{d^r}{d}\left(d\xi_+ - b\xi_-\right) + \frac{b^\ell}{d}\xi_-$$

$$= d^r\xi_+ + \left(-\frac{d^r b}{d} + \frac{b^\ell}{d}\right)\xi_-$$

$$\|p_0\|^2 = \int \begin{pmatrix} d^2 \\ (-d^r b + b^\ell)\frac{1}{d} \end{pmatrix}^* \underbrace{\begin{pmatrix} 1 & \frac{\bar{b}}{d} \\ \frac{b}{d} & 1 \end{pmatrix} \begin{pmatrix} d^r \\ (-d^r b + b^\ell)\frac{1}{d} \end{pmatrix}}_{\left(d^r + \frac{\bar{b}}{d}(-d^r b + b^\ell) \right.}$$

$$\begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^\ell & a^\ell \end{pmatrix} \begin{pmatrix} d^r & b^\ell \\ -c^r & d^\ell \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} \frac{1}{d} \begin{pmatrix} d^\ell & -b^\ell \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^\ell & a^\ell \end{pmatrix}$$

$$\frac{-b^\ell + bd^r}{d} = b^r$$

$$p_0 = d^r\xi_+ + \left(\frac{b^\ell - bd^r}{d}\right)\xi_- = d^r\xi_+ - b^r\xi_-$$

$$\|p_0\|^2 = \int \begin{pmatrix} d^r \\ -b^r \end{pmatrix}^* \begin{pmatrix} 1 & \frac{\overline{b}}{d} \\ \frac{b}{d} & 1 \end{pmatrix} \begin{pmatrix} d^r \\ -b^r \end{pmatrix}$$

$$= \int \begin{pmatrix} a^r & -c^r \end{pmatrix} \begin{pmatrix} d^r - \overline{\beta} b^r \\ \beta d^r - b^r \end{pmatrix}$$

$$d^r - \frac{c}{a} b^r = \frac{1}{a}(a d^r - c b^r) = \frac{a^\ell}{a}$$

$$\frac{b}{d} d^r - b^r = \frac{1}{d}(b d^r - d b^r) = \frac{b^\ell}{d}$$

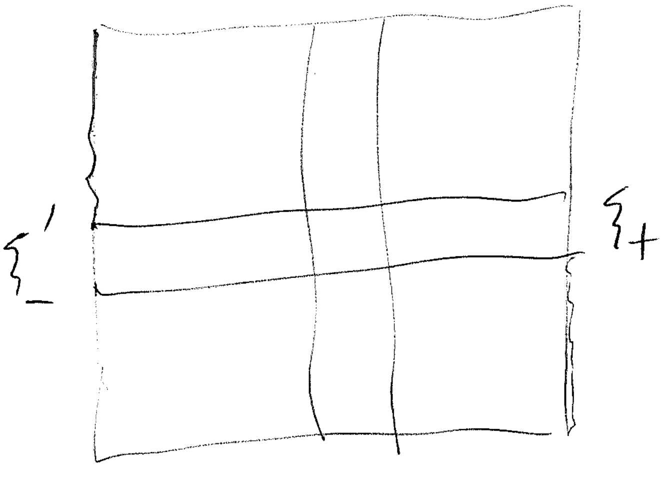
$$\|p_0\|^2 = \int \frac{a^r a^\ell}{a} - \frac{c^r b^\ell}{d} = \frac{a^r a^\ell}{a}(\infty) = 1.$$

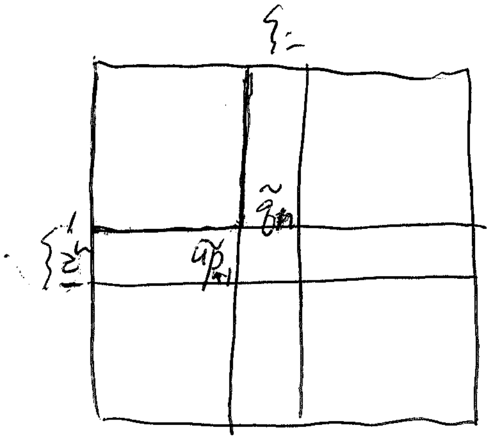
ξ_0

$d\xi_+ - b\xi_-$

$$\|p_0\|^2 = \left(\frac{d^r}{d} \boxed{\xi'_-} + \frac{b^\ell}{d} \xi_- \,\middle|\, \frac{a^\ell}{a} \xi_+ - \frac{b^r}{a} \xi'_+ \right)$$

$$=$$





$$\widetilde{up}_{n+1} = \xi'_-\overset{z^{n+1}H_+}{(z^n - f)} + \xi_-\overset{zH_+}{(-g)}$$

$$\tilde{g}_n = \xi'_-(-\phi) + \xi_-(1-\psi)$$

$$\begin{pmatrix}\pi_{n+1} & 0\\ 0 & \pi_1\end{pmatrix}\begin{pmatrix}1 & \bar{b}\\ b & -1\end{pmatrix}\begin{pmatrix}z^n - f & -\phi\\ -g & 1-\psi\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1 & T^*\\ T & -1\end{pmatrix}\begin{pmatrix}f & \phi\\ g & \psi\end{pmatrix} = \begin{pmatrix}0 & \pi_{n+1}\bar{b}\\ \pi_1(bz^n) & 0\end{pmatrix}$$

$$T\phi = \psi$$

$$(1+T^*T)\phi = \pi_{n+1}\bar{b}$$

$$f = -T^*g$$

$$(-1-TT^*)g = \pi_1(bz^n)$$

$$\xi'_- z^n = \widetilde{up}_{n-1} + \underbrace{\xi'_- f + \xi_- g}$$

$$K(\xi'_- f + \xi_- g) = \int \begin{pmatrix}f\\ g\end{pmatrix}^* \begin{pmatrix}1 & T^*\\ T & -1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix} = \int g^*(-1-TT^*)g$$

$$= -(\|g\|^2 + \|f\|^2)$$

$$\|\widetilde{up}_{n-1}\|^2 = 1 + \|g\|^2 + \|f\|^2$$

$$K(\xi'_-\phi + \xi_-\psi) = \int \begin{pmatrix}\phi\\ \psi\end{pmatrix}^* \begin{pmatrix}1 & T^*\\ T & -1\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \int \phi^*(1+T^*T)\phi$$

$$= \|\phi\|^2 + \|\psi\|^2$$

$$\xi_- = \tilde{g}_n + \xi'_-\phi + \xi_-\psi$$

$$-1 = K(\tilde{g}_n) + \|\phi\|^2 + \|\psi\|^2 \qquad K(\tilde{g}_n) = -\|\tilde{g}_n\|^2$$

$$K(\tilde{g}_n, \widetilde{up}_{n-1}) = \int \begin{pmatrix}-\phi\\ 1-\psi\end{pmatrix}\begin{pmatrix}1 & \bar{b}\\ b & -1\end{pmatrix}\begin{pmatrix}z^n - f\\ -g\end{pmatrix} = \int bz^n - bf + g$$

$$= \int bz^n + \underbrace{\int bT^*g}$$

$$\int (\pi_{n+1}\bar{b})^* T^*g = \int \phi^*(1+T^*T)T^*g$$

$$\int b \overset{\pi_{n+1} b \varepsilon_1}{T^*} g = \int (\pi_{n+1} \bar{b})^* T^* g = \int ((1+T^*T)\phi)^* T^* g$$

$$= \int \phi^* (1+T^*T) T^* g = \int \phi^* T^* (\pi_1 (b z^n))$$

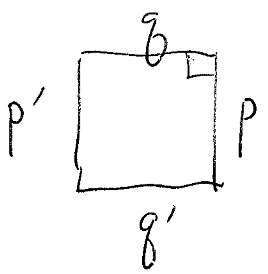
$$\int (\pi_{n+1} \bar{b})^* T^*$$

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$p' = -\frac{h}{k} q + \frac{1}{k} p \quad \tilde{p}'$$

$$~~p = \frac{h}{k} q' + \frac{1}{k} p'~~$$

$$\|\tilde{p}\| = \frac{1}{k} \|p\|$$



$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$K(q, p') = K\left(\frac{\bar{h}}{k} p' + \frac{1}{k} q', p'\right) = \frac{h}{k}$$

$$q = \frac{\bar{h}}{k} p' + \frac{1}{k} q' \quad \tilde{q}' \qquad \|\tilde{q}'\| = \frac{1}{k} \|q\|.$$

$$\therefore \|\tilde{q}_n\| = \frac{1}{k_{n+1}} \frac{1}{k_{n+2}} \cdots \qquad \|\tilde{p}_{n-1}\| = \frac{1}{k_{n-1} k_{n-2} \cdots}$$

$$K(\tilde{q}_n, \widetilde{u p}_{n-1}) = \|\tilde{q}_n\| \, K(q_n, u p_{n-1}) \, \|\widetilde{u p}_{n-1}\| = \frac{h_n}{\prod_j k_j}$$

$$\text{"} \qquad \int b z^n - \int b f$$

$$\int b f = \int \bar{b}^* f = \int \pi_{n+1}(\bar{b})^* f$$

$f = -T^* g = +T^* (1 + TT^*)^{-1} \pi_1 (b z^n)$. The problem here is:

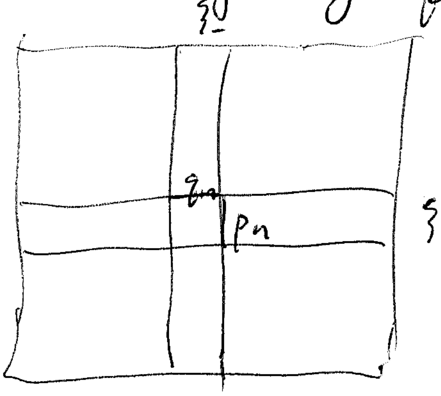
$$b = \sum b_j z^{-j} \qquad \pi_1 (b z^n) = \sum_{n > j} b_j z^{n-j}, \quad \|\pi_1 (b z^n)\|^2 = \sum_{j<n} |b_j|^2$$

$$\bar{b} = \sum \bar{b}_j z^j \qquad \pi_{n+1}(\bar{b}) = \sum_{j > n} \bar{b}_j z^j \qquad \|\pi_{n+1}(\bar{b})\|^2 = \sum_{j > n} |b_j|^2$$

So error = $\int \pi_{n+1}(\bar{b})^* f$ where

$$f = T^*(1+TT^*)^{-1}\pi_1(bz^n) \qquad T = \pi_1 b\,\varepsilon_{n+1} : z^{n+1}H_+ \to zH_+$$

Review everything ~~[illegible]~~



$$\tilde{p}_n = \xi_+(z^n - \overset{z^{n+1}H_+}{f}) + \xi_-(-\overset{zH_-}{g})$$

$$\tilde{q}_n = \xi_+(-\phi) + \xi_-(1-\psi)$$

$$\begin{pmatrix}\pi_{n+1} & 0\\ 0 & \pi_-\end{pmatrix}\begin{pmatrix}1 & \bar{\beta}\\ \beta & 1\end{pmatrix}\begin{pmatrix}z^n - f & -\phi\\ -g & 1-\psi\end{pmatrix} = 0$$

$$\begin{pmatrix}1 & T^*\\ T & 1\end{pmatrix}\begin{pmatrix}f & \phi\\ g & \psi\end{pmatrix} = \begin{pmatrix}0 & \pi_{n+1}(\bar{\beta})\\ \pi_-(\beta z^n) & 0\end{pmatrix}$$

$$f + T^*g = 0 \qquad T\phi + \psi = 0 \qquad \begin{matrix}T = \pi_-\beta\varepsilon_{n+1}\\ T^* = \pi_{n+1}\bar{\beta}\varepsilon_-\end{matrix}$$

$$(1 - TT^*)g = \pi_-(\beta z^n) \qquad (1 - T^*T)\phi = \pi_{n+1}(\bar{\beta})$$

$$\|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix}f\\ g\end{pmatrix}^* \begin{pmatrix}1 & \bar{T}^*\\ T & 1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix} = \int g^* \overset{(1-TT^*)g}{\pi_-(\beta z^n)} = \|g\|^2 - \|f\|^2$$

$$\|\xi_+\phi + \xi_-\psi\|^2 = \int \begin{pmatrix}\phi\\ \psi\end{pmatrix}^* \begin{pmatrix}\pi_{n+1}(\bar{\beta})\\ 0\end{pmatrix} = \int \phi^*(1-T^*T)\phi = \|\phi\|^2 - \|\psi\|^2$$

$$\xi_+ z^n = \tilde{p}_n + \xi_+ f + \xi_- g \qquad 1 = \|\tilde{p}_n\|^2 + \overbrace{\|g\|^2 - \|f\|^2}^{\geq 0}$$

$$\xi_- = \tilde{q}_n + \xi_+\phi + \xi_-\psi \qquad 1 = \|\tilde{q}_n\|^2 + \|\phi\|^2 - \|\psi\|^2$$

$$\underbrace{(\xi_- | \xi_+ z^n)} = \underbrace{(\tilde{q}_n | \tilde{p}_n)} + \int \begin{pmatrix}\cancel{\phi}\\ \psi\end{pmatrix}^* \begin{pmatrix}\cancel{1} & \cancel{\bar{T}^*}\\ T & 1\end{pmatrix}\begin{pmatrix}\overset{-T^*g}{\cancel{f}}\\ g\end{pmatrix}$$

$$\int \beta z^n = \|\tilde{q}_n\|\|\tilde{p}_n\|\, h_n + \int \psi^* \underbrace{(1 - TT^*)g}_{\pi_-(\beta z^n)} \qquad \psi = -T(1-T^*T)^{-1}\underset{\pi_{n+1}(\bar{\beta})}{[illegible]}$$

$$\int \beta z^n - h_n = \big(\underbrace{\|\tilde{q}_n\|\|\tilde{p}_n\|}\, - 1\big) h_n + \int \psi^* \pi_-(\beta z^n)$$

$$\|g\|^2 - \|T^* g\|^2 = \|\phi\|^2 - \|T\phi\|^2$$

$$= \int g^* \pi_-(\beta z^n)$$

$$g = (1 - TT^*)^{-1} \pi_-(\beta z^n) \qquad \psi = -T(1 - T^*T)\pi_{n+1}(\bar\beta)$$

$$f = -T^*(1 - TT^*)^{-1} \pi_-(\beta z^n) \qquad \phi = (1 - T^*T)\pi_{n+1}(\bar\beta)$$

$$\|g\|^2 - \|f\|^2 = \big(g, (1 - TT^*) g\big) = \int g^* \pi_-(\beta z^n)$$

$$= \int \pi_-(\beta z^n)^* (1 - TT^*)^{-1} \pi_-(\beta z^n)$$

$$\leq \frac{1}{1 - \|T\|^2} \sum_{j > n} |\beta_j|^2$$

$$\beta = \sum \beta_j z^{-j}$$

$$\beta z^n = \sum \beta_j z^{n-j}$$

$$\pi_-(\beta z^n) = \sum_{j > n}' \beta_j z^{n-j}$$

$$\|\pi_-(\beta z^n)\|^2 = \sum_{j > n} |\beta_j|^2$$

$$\bar\beta = \sum \bar\beta_j z^j$$

$$\pi_{n+1}(\bar\beta) = \sum_{j > n} \bar\beta_n z^j \qquad \|\pi_{n+1}(\bar\beta)\|^2 = \sum_{j > n} |\beta_j|^2$$

$$\|\phi\|^2 - \|T\phi\|^2 = \int \phi^* (1 - T^*T)\phi$$

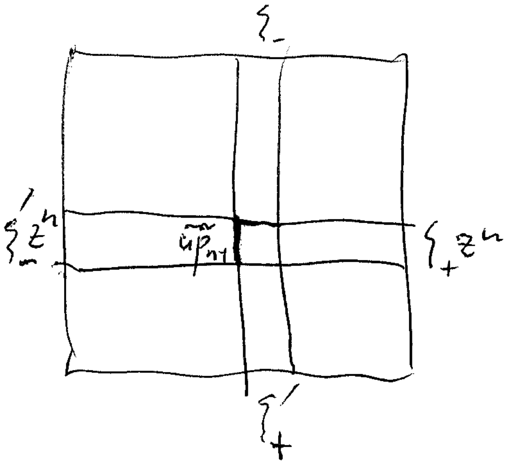
$$= \int \phi^* \pi_{n+1}(\bar\beta) = \int \pi_{n+1}(\bar\beta)^* (1 - T^*T)^{-1} \pi_{n+1}(\bar\beta)$$

$$\leq \frac{1}{1 - \|T\|^2} \sum_{j > n} |\beta_j|^2$$

$$\left| \int \beta z^n - h_n \right| \leq \frac{1}{1 - \|T\|^2} \sum_{j > n} |\beta_j|^2 + \frac{\|T\|}{1 - \|T\|^2} \sum_{j > n} |\beta_j|^2$$

~~Important [illegible] are the extremale~~

So what about the K calculation



$$\widetilde{up}_{n-1} = \xi'_-\overbrace{(z^n - f)}^{z^{n+1}H_+} + \xi_-\overbrace{(-g)}^{zH_+}$$

$$\tilde{g}_n = \xi'_-(-\phi) + \xi_-(1-\psi)$$

$$\begin{pmatrix}\pi_{n+1} & 0\\ 0 & \pi_1\end{pmatrix}\begin{pmatrix}1 & \bar{b}\\ b & -1\end{pmatrix}\begin{pmatrix}z^n-f & -\phi\\ -g & 1-\psi\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$T = \pi_1 b\,\varepsilon_{n+1} : z^{n+1}H_+ \to zH_+$$

$$\begin{pmatrix}1 & T^*\\ T & -1\end{pmatrix}\begin{pmatrix}f & \phi\\ g & \psi\end{pmatrix} = \begin{pmatrix}0 & \pi_{n+1}(\bar{b})\\ \pi_1(bz^n) & 0\end{pmatrix}$$

$$f + T^*g = 0 \qquad T\phi = \psi$$

$$(-1 - TT^*)g = \pi_1(bz^n) \qquad (1+T^*T)\phi = \pi_{n+1}(\bar{b})$$

$$K(\xi'_- f + \xi_- g) = \int \begin{pmatrix}f\\ g\end{pmatrix}^*\begin{pmatrix}0\\ \pi_1(bz^n)\end{pmatrix} = \int g^*\overbrace{\pi_1(bz^n)}^{(-1-TT^*)g} = -(\|g\|^2 + \|f\|^2)$$

$$K(\xi'_-\phi + \xi_-\psi) = \int \phi^*\pi_{n+1}(\bar{b}) = \int \phi^*(1+T^*T)\phi = \|\phi\|^2 + \|\psi\|^2$$

$$\xi'_- z^n = \widetilde{up}_{n-1} + \xi'_- f + \xi_- g \qquad 1 = K(\widetilde{up}_{n-1}) - \|g\|^2 - \|f\|^2$$

$$\xi_- = \tilde{g}_n + \xi'_-\phi + \xi_-\psi \qquad -1 = K(\tilde{g}_n) + \|\phi\|^2 + \|\psi\|^2$$

$$K(\xi_-, \xi'_- z^n) = K(\tilde{g}_n, \widetilde{up}_{n-1}) + \int\begin{pmatrix}\phi\\ \psi\end{pmatrix}^*\begin{pmatrix}1 & T^*\\ T & -1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}$$

$$\|$$

$$\int\begin{pmatrix}0\\ 1\end{pmatrix}^*\begin{pmatrix}1 & \bar{b}\\ b & -1\end{pmatrix}\begin{pmatrix}z^n\\ 0\end{pmatrix} \qquad \psi = T(1+T^*T)^{-1}\pi_{n+1}(\bar{b})$$

$$\|$$

$$\int bz^n = \|\tilde{g}_n\|\,\|\widetilde{up}_{n-1}\|\,\overbrace{K(g_n, up_{n-1})}^{\frac{h_n}{k_n}} + \int \psi^*\pi_1(bz^n)$$

$$b = \sum b_j z^{-j} \qquad \pi_1(bz^n) = \sum_{n>j} b_j z^{n-j} \qquad \|\pi_1(bz^n)\|^2 = \sum_{j<n}|b_j|^2$$

$$\bar{b} = \sum \bar{b}_j z^{+j} \qquad \pi_{n+1}(\bar{b}) = \sum_{j>n}\bar{b}_j z^j \qquad \|\pi_{n+1}(\bar{b})\|^2 = \sum_{j>n}|b_j|^2$$

So what can we say about $\|\psi^*\|$?
You want to control $\int \psi^* \pi_1(bz^n)$ as $n \to +\infty$
but $\|\pi_1(bz^n)\| \to \|b\|$, so you have to control $\|\psi\|$.

$$\psi = T\phi = T(1+T^*T)^{-1}\pi_{n+1}(\bar{b}) \qquad T = \pi_1 b \varepsilon_{n+1}$$

$$\|\psi\| \leq \|T\| \frac{1}{1+\|T\|^2} \|\pi_{n+1}(\bar{b})\|$$

$$\begin{matrix} 1 & -T^* \\ T & 1 \end{matrix} \qquad\qquad \frac{1+X}{1-X}$$

$$T = \pi_1 b \varepsilon_{n+1} \qquad \|T\| \leq \|\pi_1\| \, \|b\|_\infty \, \|\varepsilon_{n+1}\| = \|b\|_\infty$$

$$\|\psi\| \leq \|T\| \, \underbrace{\|(1+T^*T)^{-1}\|}_{\leq 1} \, \|\pi_{n+1}(\bar{b})\|$$

$$\therefore \|\psi\| \leq \|b\|_\infty \left(\sum_{j>n} |b_j|^2\right)^{1/2}$$

$$\frac{\int bz^n}{d(0)} = h_n + \frac{1}{d(0)} \int \psi^* \pi_1(bz^n)$$

$$d(0) = \|\tilde{q}_n\| \frac{1}{k_n} \|\tilde{\varphi}_{n-1}\| = \prod_{j \in \mathbb{Z}} \left(\frac{1}{1-|h_j|^2}\right)^{1/2}$$

also know
$$\|\tilde{q}_n\| = (1+\|\phi\|^2+\|\psi\|^2)^{1/2}$$
$$\|\tilde{\varphi}_{n-1}\| = (1+\|f\|^2+\|g\|^2)^{1/2} \qquad \frac{1}{d(0)}$$

So error $= \left|\frac{1}{d(0)} \int \psi^* \pi_1(bz^n)\right| \leq \frac{1}{d(0)} \underbrace{\|\psi\|}_{\leq \|b\|_\infty \left(\sum_{j>n}|b_j|^2\right)^{1/2}} \underbrace{\|\pi_1(bz^n)\|}_{\leq \|b\|_2}$

$$\leq \frac{\|b\|_\infty \|b\|_2}{d(0)} \left(\sum_{j>n} |b_j|^2\right)^{1/2}$$

Repeat. Improve $K(\xi_-, \xi'_- z^n) = \int b z^n$

$$up_{n-1} = \xi'_- s(z^n - f) + \xi_- s(-g) \qquad s = \|\tilde{u}p_{n-1}\|^{-1}$$

$$q_n = \xi'_- t(-\phi) + \xi_- t(1-\psi) \qquad t = \|\tilde{q}_n\|^{-1}$$

$$\begin{pmatrix} up_{n-1} \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} d_n^2 & b_n^\ell \\ -c_n^2 & d_n^\ell \end{pmatrix}\frac{1}{d}$$

$$K(\xi_+ f + \xi_- g, \xi_+ \phi + \xi_- \psi) = \bar{f}\phi - \bar{g}\psi$$

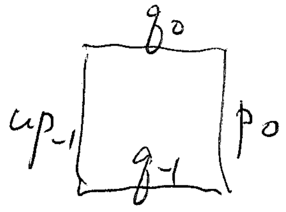
$$\sigma(\xi_+ f + \xi_- g) \wedge (\xi_+ \phi + \xi_- \psi) =$$

$$(\xi_- \bar{f} + \xi_+ \bar{g}) \wedge (\xi_+ \phi + \xi_- \psi) = (\bar{f}\phi - \bar{g}\psi)(\xi_- \wedge \xi_+)$$

$$\boxed{\sigma(v) \wedge v' = K(v, v')\,\omega}$$

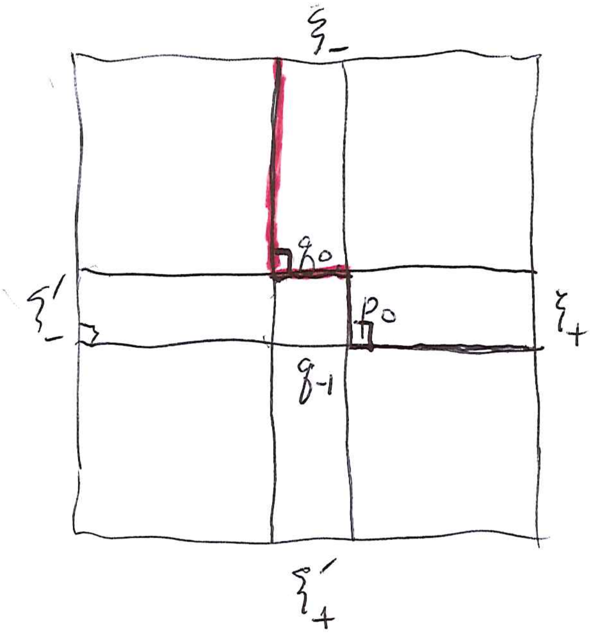
$$v \wedge \sigma(v') = \overline{K(v, v')}\ \sigma(\omega)$$

$$-\sigma(v') \wedge v = K(v', v)\ -\omega$$



$$\sigma q_{-1} = u p_{-1}$$

$$\begin{pmatrix} d & -c \\ b & -a \end{pmatrix}$$



$$p_0 = \frac{d^r}{d}\xi'_- + \frac{b^l}{d}\xi_-$$

$$\sigma(p_0) = \frac{b^l}{d}\xi_+ + \frac{d^r}{d}\xi'_+$$

$$= \frac{c^l}{a}\xi_+ + \frac{a^r}{a}\xi'_+ = q_0$$

$$K(\xi_-, \xi'_- z^n) = \int \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* \begin{pmatrix} 1 & 5 \\ b & -1 \end{pmatrix} \begin{pmatrix} z^n \\ 0 \end{pmatrix}$$

$$= \int b z^n$$

$$\xi'_+ = -\frac{c}{d}\xi'_- + \frac{1}{d}\xi_-$$

$$\underset{\sim}{K}(\xi_-, \xi'_+) = \underset{\sim}{K}\left(\xi_-, -\frac{c}{d}\xi'_- + \frac{1}{d}\xi_-\right)$$

$$\shortparallel$$

$$-\frac{1}{d(0)}$$

$$\xi'_- = d\xi_+ - b\xi_-$$

$$\underbrace{\underset{\sim}{K}\left(\xi_-, \left(-\frac{c}{d}\right)(\cancel{d}\xi_+ - b\xi_-)\right)}_{-\frac{c}{d}(-b)(-1)} + \lambda$$

$$-\frac{1}{d}(1 + bc) = -\frac{1}{d}(ad) = -a \qquad -\frac{1}{d}$$

$$\underset{\sim}{K}(\xi_-, \xi'_+) = \underset{\sim}{K}(\xi_-, -c\xi_+ + a\xi_-) = \boxed{-a}$$

$$\underset{\sim}{K}(\xi_+, \xi'_+) = \underset{\sim}{K}(\xi_+, -c\xi_+ + a\xi_-) = -c$$

$$\underset{\sim}{K}(\xi_+, \xi'_-) = \underset{\sim}{K}(\xi_+, d\xi_+ - b\xi_-) = d$$

$$\underset{\sim}{K}(\xi_-, \xi'_-) = \underset{\sim}{K}(\xi_-, d\xi_+ - b\xi_-) = b$$

So what is the [illegible]? You want ~~to find~~ a conceptual explan[illegible] why h_n is approximated by $\frac{\int b z^n}{d(0)}$ in some asymptotic sense.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \cdots \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \cdots$$

so if you ~~multiply~~ divide by $d(0) = \prod \frac{1}{k_n}$ you get

$$\frac{1}{d(0)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \cdots \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \cdots$$

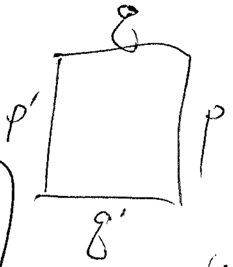
to first order in h this is

$$\frac{1}{d(0)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \sum_n h_n z^{-n} \\ \sum_n \bar{h}_n z^{-n} & 1 \end{pmatrix}$$

This doesn't say much because $d(0) = 1$ to first order. IDEA. $d(0) = a(0) > 0$.

Write up an account of DDE's.

Identities.



$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \frac{ab-cd}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

$|k|^2 = 1 - |h|^2$

$$\cancel{\frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h}\frac{k}{\bar{k}} & 1 \end{pmatrix}} \qquad \begin{pmatrix} k & h \\ -\bar{h}\frac{k}{\bar{k}} & k \end{pmatrix}$$

$\frac{1}{d} q = \frac{c}{d} p' + \cancel{d} q'$

$q' = \frac{1}{d} q - \frac{c}{d} p'$

$p = a p' + b\left(\frac{1}{d} q - \frac{c}{d} p'\right)$

$$SU(1,1) \subset U(2)$$

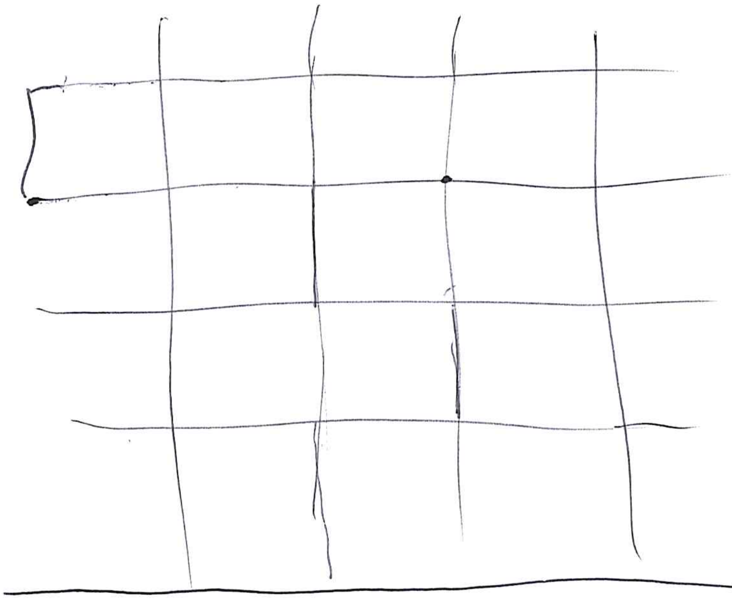
$$\left\{\begin{pmatrix} a & \overset{a}{b} \\ \bar{b} & \bar{a} \end{pmatrix} \middle|\, |a|^2 - |b|^2 = 1\right\}. \qquad \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2) \,\middle|\, \alpha = \delta \neq 0\right\}.$$

When is $\begin{pmatrix} k & h \\ h' & k \end{pmatrix} \in U(2)$?

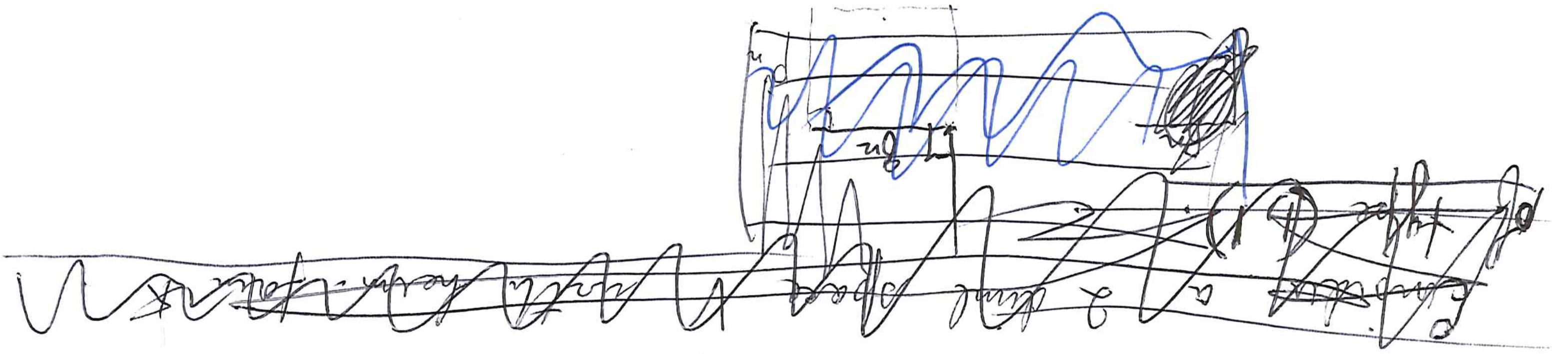
$$|k|^2+|h|^2=1 \qquad \bar{k}h'+\bar{h}k=0$$

$$h' = -\bar{h}\frac{k}{\bar{k}}$$

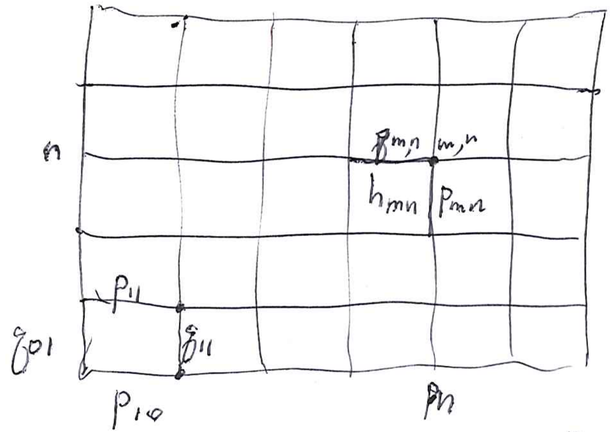
corresp. $\begin{pmatrix} \frac{1}{\bar{k}} & \frac{h}{k} \\ +\frac{\bar{h}}{\bar{k}} & \frac{1}{k} \end{pmatrix}$



~~[illegible]~~ I think that the Wronskian stuff makes sense only with u around, certainly the local $K_m(\xi,\xi') = \frac{\tau(\xi)\wedge\xi'}{\omega}$ seems restricted to rank 2. Check that the global K is defined for ~~any~~ a grid with arb. h's.

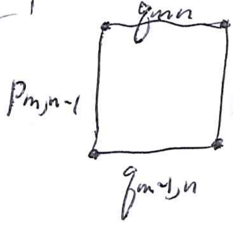


Finite case: rectangle



Define a vector space E by generators and relation. p_{mn}, q_{mn} $m,n \in \mathbb{Z}$

relations are



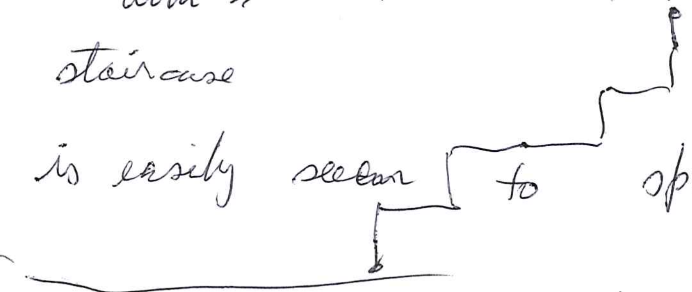
$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k_{mn}} \begin{pmatrix} 1 & h_{mn} \\ \bar{h}_{mn} & 1 \end{pmatrix} \begin{pmatrix} p_{m,n-1} \\ q_{m-1,n} \end{pmatrix}$$

Consider ~~$m \le M, \ n \le N$~~

$(0,0) < (m,n) \le (M,N)$

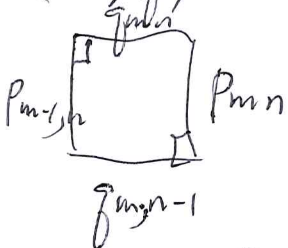
no. of gen. $(M+1)(N+1)$
$= MN + M + N + 1$
no of relations: = no of rectangles
$= MN$

dim $\geq M+1+N$. But any staircase



is easily seen to span and it has card. $M+N+1$.

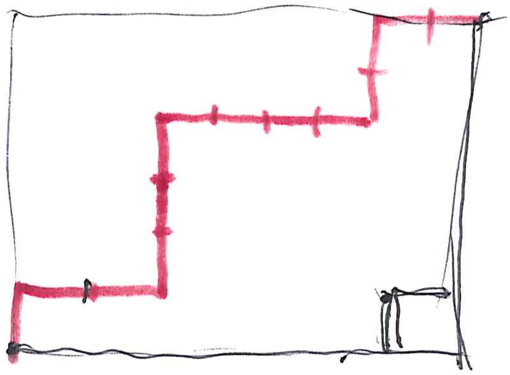
~~Define~~ Have

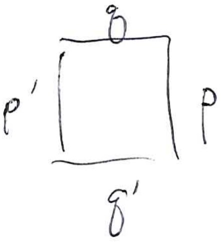


$$\begin{pmatrix} p_{mn} \\ q_{m,n-1} \end{pmatrix} = \begin{pmatrix} k_{mn} & h_{mn} \\ -\bar{h}_{mn} & k_{mn} \end{pmatrix} \begin{pmatrix} p_{m-1,n} \\ q_{m,n} \end{pmatrix}$$

Claim ! hermitian inner product such that any staircase yields an orth basis.

Claim ∃! hermitian form such that any descending staircase is orth with $K(p's) = 1$ $K(q's) = -1$.





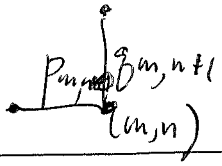
$K(p,p) = 1 = -K(q,q)$

and $\begin{pmatrix} p^* \\ q^* \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$

$p' = \frac{1}{k}(p - hq)$

$K(p') = \frac{1}{k^2} K(p-hq, p-hq) = \frac{1}{k^2}(1-|h|^2) = 1$. same true for p', q'.

~~$K\begin{pmatrix}p_{mn}\\ q_{mn}\end{pmatrix}$~~ $\quad K\left(p_{jk}, \begin{matrix}p_{mn}\\ q_{mn}\end{matrix}\right) = 0 \quad \begin{matrix} j<m \\ k<n\end{matrix}$



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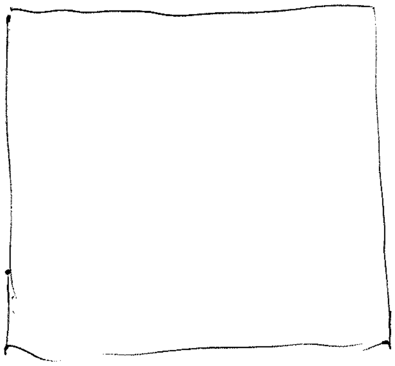
digress to cont. case $\quad \partial_x \begin{pmatrix} p_x \\ q_x\end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar h_x & 0\end{pmatrix}\begin{pmatrix}p_x\\ q_x\end{pmatrix}$

group $\mathbb{R}$ $\quad u^t \quad t\in\mathbb{R}$. Want universal solution of this DE with values in ~~[illegible]~~ a module over ~~the~~ group ring - some sort of ~~[illegible]~~ convolution alg^a on $\mathbb{R}$. Because this is an O.D.E. first order the module should have rank 2, with basis ~~[illegible]~~ p_x, q_x for each x.

Continuous case ~~[illegible]~~ things should be simpler since $\bar H_+ = H_-$



$$\partial_x \begin{pmatrix} \cancel{z}^{-x}p_x \\ q_x\end{pmatrix} = \begin{pmatrix} -\cancel{\lambda u^{-x}p_x} + u^{-x}(\cancel{\lambda p_x} + h q_x) \\ \bar h\, p_x \end{pmatrix}$$

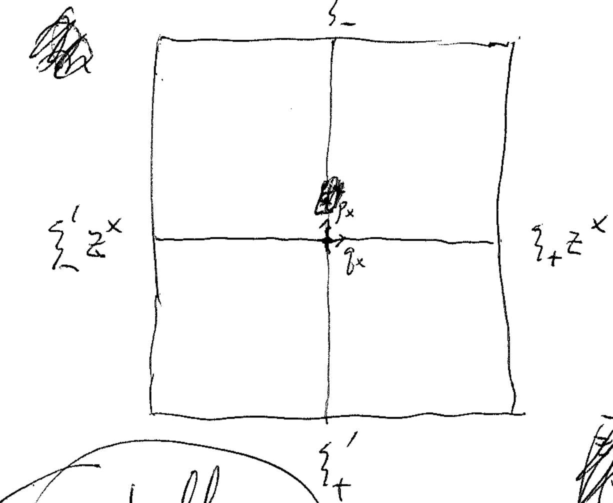
$$= \begin{pmatrix} 0 & h u^{-x} \\ \bar h \cancel{z}^{x} & 0\end{pmatrix}\begin{pmatrix}\cancel{z}^{-x}p_x\\ q_x\end{pmatrix}$$

~~[illegible]~~

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \mathop{T}_{+\infty}^{\infty} \exp\begin{pmatrix}0 & h_x\cancel{z}^{-x}\\ \bar h_x \cancel{z}^{x} & 0\end{pmatrix}$$

~~existence~~ solution of ODE review later.

$L^2 = H_+ \oplus H_- \qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \begin{pmatrix}1+H_- & L^2\\ L^2 & 1+H_+\end{pmatrix}$



$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a^l & \overset{H_+}{b^l} \\ \underset{H_-}{c^l} & d^l \end{pmatrix}\begin{pmatrix} \xi'_- z^x \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & \overset{H_-}{-b^r} \\ \underset{H_+}{-c^r} & a^r \end{pmatrix}\begin{pmatrix} \xi_+ z^x \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi'_- z^x \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ z^x \\ \xi'_+ \end{pmatrix}$$

$$p_x = \xi_+ z^x(\overset{H_+}{1-f}) + \xi_-(\overset{H_-}{-g})$$

$$q_x = \xi_+ z^x(\overset{H_+}{-\phi}) + \xi_-(\overset{H_-}{1-\psi})$$

x stuff all wrong.
OK for x=0

$$\begin{pmatrix} \pi_+ & 0 \\ 0 & \pi_- \end{pmatrix}\begin{pmatrix} 1 & \overline{\beta z^{-x}} \\ \beta z^x & 1 \end{pmatrix}\begin{pmatrix} 1-f & -\phi \\ -g & 1-\psi \end{pmatrix} = 0$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$T_x = \pi_- \beta_x \varepsilon_+$

$$\begin{pmatrix} 1 & T_x^* \\ T_x & 1 \end{pmatrix}\begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 0 & \pi_+(\overline{\beta z^{-x}}) \\ \pi_-(\beta z^x) & 0 \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

need $\quad z^{-x}p_x, \overset{q_x}{\vee} \perp (\xi_+ z^x H_+ + \xi_- H_-)$

$$\boxed{\begin{aligned} f + T^* g &= 0 \\ (1 - TT^*)g &= \pi_-(\beta_x) \end{aligned}}$$

$$\| \xi_+ z^x f + \xi_- g \|^2 =$$

$$= \left\| \left(\xi'_- \frac{1}{d} + \xi_- \frac{b}{d}\right) z^x f + \xi_- g \right\|^2$$

$$= \left\| \frac{1}{d} z^x f \right\|^2 + \left\| \frac{b}{d} z^x f + g \right\|^2$$

$$= \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} \overset{1}{|\beta|^2 + \frac{1}{|d|^2}} & \overline{\beta z^x} \\ \beta z^x & 1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

$$\phi + T_x^* \psi = \pi_+(\overline{\beta_x})$$

$$\boxed{T_x \phi + \psi = 0}$$

$$\boxed{(1 - T^*T)\psi = \pi_+(\overline{\beta_x})}$$

$$\therefore \quad \boxed{\beta_x = \beta z^x}$$

$h_x = (q_x | p_x)$

But $\quad \xi_+ z^x = p_x + \xi_+ z^x f + \xi_- g$

$\quad\quad\;\; \xi_- = q_x + \xi_+ z^x \phi + \xi_- \psi$

$$(\xi_- | \xi_+ z^x) = (\xi_- | p_x) + (\xi_- | \xi_+ z^x f + \xi_- g)$$

$(q_x | p_x)$

$$\int \beta z^{x} = h_x = \left(\xi_- \middle| \xi_+ z^{x} f + \xi_- g\right) = \int \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{*} \begin{pmatrix} 1 & \overline{\beta_x} \\ \beta_x & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$= \int \beta_x f + \cancel{g} = \int \left(\overline{\beta_x}\right)^{*} f = \int \left(\pi_+(\overline{\beta_x})\right)^{*} f$$

$$= \int \left(\pi_+(\overline{\beta_x})\right)^{*} (-T^{*})(1 - TT^{*})^{-1} \pi_-(\beta_x)$$

β_x lousy notation

better $\beta^{(x)} = \beta z^{x}$

same calc.

$$\beta z^{x} = \int \beta_y z^{x-y}\, dy$$

$$\pi_-(\beta z^{x}) = \int_{y > x}^{\infty} \beta_y z^{x-y}$$

$$\|\pi_-(\beta z^{x})\|^2 = \int_{x}^{\infty} |\beta_y|^2 dy$$

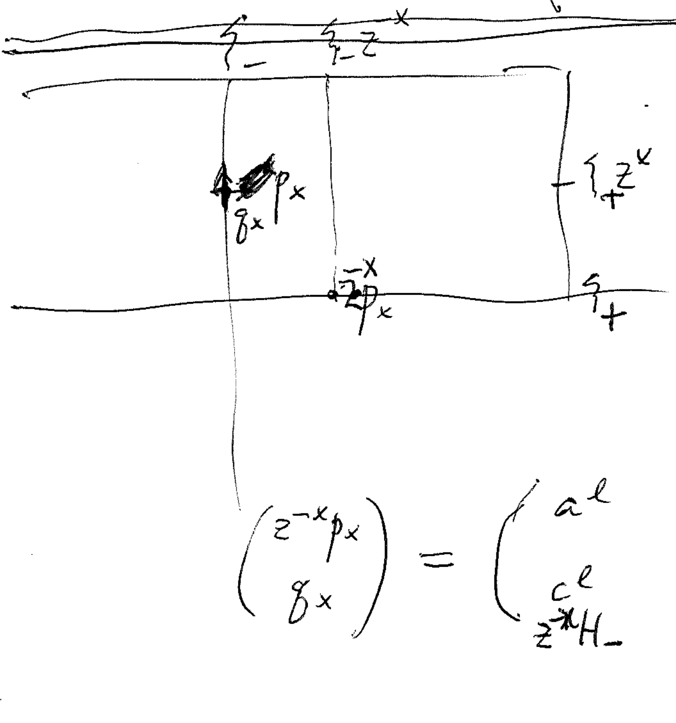
$$\overset{\shortparallel}{\|\pi_+(\overline{\beta z^{x}})\|^2}$$

$$\overline{\beta} = \int \overline{\beta_y} z^{y}\, dy$$

$$\overline{\beta z^{x}} = \int \overline{\beta_y} z^{y-x}\, dy \qquad \pi_+\left(\overline{\beta z^{x}}\right) = \int_{y>x} \overline{\beta_y} z^{y-x}$$

Can you derive the diffl. equation?

It's clear that $\left(\qquad \right)$



$$\cancel{\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a_x^{l} & b_x^{l} \\ c_x^{l} & d_x^{l} \end{pmatrix}} \quad z^{-x}H_-$$

$$\begin{pmatrix} z^{-x}p_x \\ q^{x} \end{pmatrix} = \begin{pmatrix} d_x^{r} & -b_x^{r} \\ -c_x^{r} & d_x^{r} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

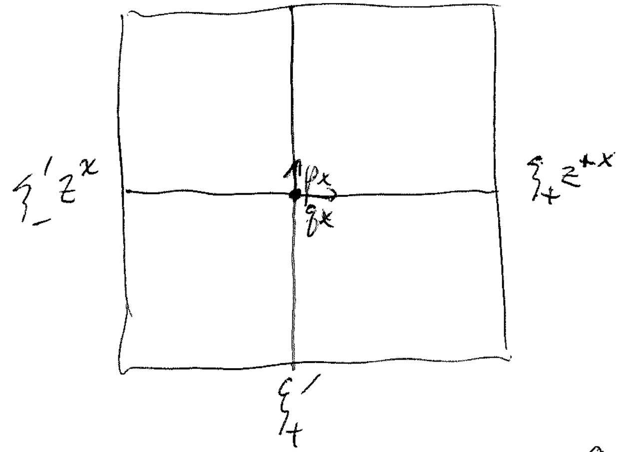
$$\qquad\qquad z^{x}H_+ \quad z^{-x}H_-$$

$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} a^{l} & \overset{z^{x}H_+}{b^{l}} \\ \underset{z^{-x}H_-}{c^{l}} & d^{l} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \begin{pmatrix} d^{r} & -b^{r} \\ \underset{z^{x}H_+}{-c^{r}} & a^{r} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \xi_+(1-f) + \xi_- z^{-x}(-g) \\ \xi_+(-z^x\phi) + \xi_-(1-\psi) \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \xi_+ z^x(1-f) + \xi_-(-g) \\ \xi_+ z^x(-\phi) + \xi_-(1-\psi) \end{pmatrix}$$

So apparently the eqns. are OKAY



$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \xi_+ z^x(1-\overset{H_+}{f}) + \xi_-(-\overset{H_-}{g}) \\ \xi_+ z^x(-\underset{H_+}{\phi}) + \xi_-(1-\underset{H_-}{\psi}) \end{pmatrix}$$

$$\|\xi_+ z^x f + \xi_- g\|^2 = \|f\|^2 + \|g\|^2$$

$$(\xi_- g \mid \xi_+ z^x f) + \text{c.c.}$$

$$\int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \overline{\beta z^x} \\ \beta z^x & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \left(\left(\int \bar{g}\,\beta z^x f + \int \bar{f}\,\overline{\beta z^x} g \right.\right.$$

leads to $\overbrace{\begin{pmatrix} 1 & T_x^* \\ T_x & 1 \end{pmatrix}}^{\text{op on } H_+ \oplus H_-} \begin{pmatrix} f_x & \phi_x \\ g_x & \psi_x \end{pmatrix} = \begin{pmatrix} 0 & \pi_+(\overline{\beta z^x}) \\ \Pi_-(\beta z^x) & 0 \end{pmatrix}$

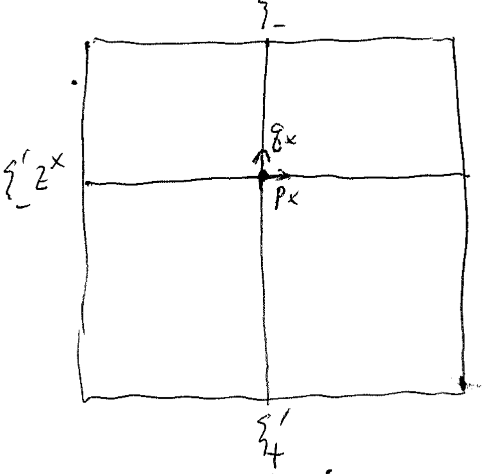
$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 1-f_x & -g_x \\ -\phi_x & 1-\psi_x \end{pmatrix} \begin{pmatrix} \xi_+ z^x \\ \xi_- \end{pmatrix}$$

$$\boxed{\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 1-f_x & -z^{-x}g_x \\ -\phi_x z^x & 1-\psi_x \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}}$$

$$\begin{pmatrix} d_x^n & -b_x^n \\ -c_x^n & a_x^n \end{pmatrix}$$

But the real question is how to handle other twists. You change β to βz^x $\quad z^x = e^{\lambda x}$ $\lambda \in i\mathbb{R}$. [redacted] For KdV you want $e^{\sum_n x_n \lambda^n}$ Check this: In the discrete case you have $|\beta|$ fixed but you change the phase i.e. $\beta(z) \longmapsto e^{f-\bar{f}}\beta$

Review: ~~Go back to KdV~~



$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \xi_+ z^x \overset{H_+}{(1-f_x)} + \xi_- \overset{H_-}{(-g_x)} \\ \xi_+ z^x \underset{H_+}{(-\phi_x)} + \xi_- \underset{H_-}{(1-\psi_x)} \end{pmatrix}$$

$$\begin{pmatrix} \pi_+ & 0 \\ 0 & \pi_- \end{pmatrix} \begin{pmatrix} 1 & \bar{\beta} z^{-x} \\ \beta z^x & 1 \end{pmatrix} \begin{pmatrix} 1-f_x & -\phi_x \\ -g_x & 1-\psi_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & T_x^* \\ T_x & 1 \end{pmatrix} \begin{pmatrix} f_x & \phi_x \\ g_x & \psi_x \end{pmatrix} = \begin{pmatrix} 0 & \pi_+(\bar{\beta} z^{-x}) \\ \pi_-(\beta z^x) & 0 \end{pmatrix} \qquad T_x = \pi_- \beta z^x \varepsilon_+$$

$$\beta = \int \beta_y z^{-y} \qquad \beta z^x = \int \beta_y z^{x-y} \qquad \pi_-(\beta z^x) = \int_{y>x} \beta_y z^{x-y}$$

$$\bar{\beta} = \int \bar{\beta}_y z^y \qquad \bar{\beta} z^{-x} = \int \bar{\beta}_y z^{y-x} \qquad \pi_+(\bar{\beta} z^{-x}) = \int_{y>x} \bar{\beta}_y z^{y-x}$$

Now you want to forget the z^x, rather replace it by a ~~the map to~~ phase function $e(z)$ $S^1 \longrightarrow U(1)$. Geometry? ~~So basically you win~~

Next point $\quad \beta(k) \rightsquigarrow h_x$ transform

Point is that a ~~vector field~~ flow on the β's goes into a flow on the h's.

Example. To first order ~~[illegible]~~ you should have the F.T. First you need.

$$h_x = (g_x|p_x) = (\xi_-|p_x) = \left(\xi_- \middle| \xi_+ z^x(1-f_x) + \xi_-(-\overset{\overset{H_-}{\circ}}{g_x})\right)$$

$$= (\xi_-|\xi_+ z^x) + (\xi_-|\overset{H_+}{\xi_+} z^x(-f_x)) + \underset{\substack{\| \\ 0}}{(\xi_-|\xi_-(-g_x))}$$

by convention, regularization?

$$= \underbrace{\int \beta z^x}_{\beta_x} - \underbrace{\int \beta z^x f_x}_{\underbrace{\int \pi_+(\bar\beta z^{-x})^* f_x}_{\text{3rd order in } \beta.}}$$

$$f_x \approx -T^* g_x$$
$$= -T^*(1-TT^*)\pi_-(\beta z^x)$$
2nd order in β.

Thus to ~~2nd order~~

$$h_x - \beta_x = O(\beta^3)$$

The point now is that $\beta \longmapsto \beta_x = \int \beta z^{+x}$ WAIT.

~~$\beta(k)$~~ $z^x = e^{ikx}$ $\beta(k) \longmapsto \hat\beta_x = \int \beta(k) e^{ikx} \frac{dk}{2\pi}$

Then $\partial_x \hat\beta = (ik\beta)^{\wedge}$

~~[illegible]~~ Find viewpoint! You have transform β ~~x~~ $\rightsquigarrow h$ so a flow on the β's should translates to a flow on h's. $\dot h = F(h)$. ~~[illegible]~~

~~[illegible]~~ Situation: suppose you ~~[illegible]~~ have a small variation $\delta\beta$ of β, what is corresp δh.

This brings up the Green's fn. again. $(\lambda-u)^{-1}$,

This somehow relates to splitting

Discuss ~~[illegible]~~ Green's function. u unitary on the Hilbert space $E = \xi_+ L^2 + \xi_- L^2$ with

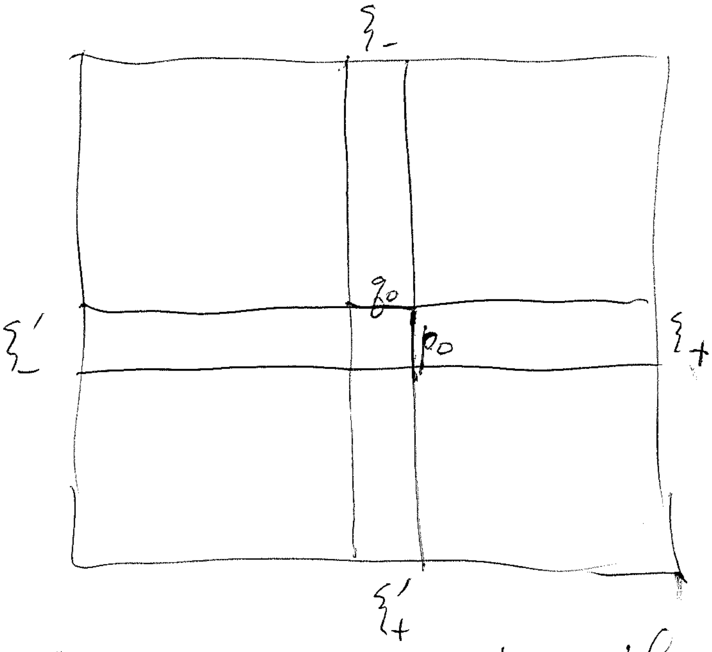
$$\|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} . \Longrightarrow (\lambda - u)^{-1} \text{ invertible on } E$$

for $\lambda \notin S^1$. What does thats look like i.e.

$(\lambda - u)^{-1} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ For $|\lambda| > 1$ given by by

$$(\lambda - u)^{-1} = \sum_{n \geq 0} \frac{u^n}{\lambda^{n+1}}$$

Recall



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} \overset{H_+}{d^r} & b^\ell \\ \underset{zH_+}{-c^r} & d^\ell \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

It should be significant that this matrix is analytic for $|z| < 1$.

Let's go back to the basic equations relating $\beta(z)$ to h_n, try to understand variationally. Maybe work out the ~~[illegible]~~ K-picture in the ~~[illegible]~~ continuous setting. Given b define

$$\underset{\sim}{K}(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\sigma(\xi'_- f + \xi_- g) = \xi'_+ \bar{f} + \xi_+ \bar{g}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

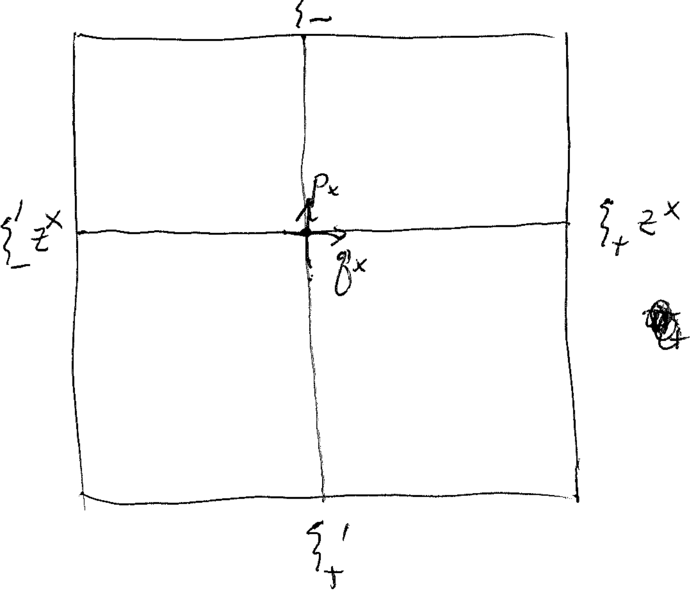
Then $\sigma(\xi'_- f + \xi_- g) \wedge (\xi'_- f + \xi_- g)$

$$= (\xi'_+ \bar{f} + \xi_+ \bar{g}) \wedge (\xi'_- f + \xi_- g) =$$

$$\left(\bar f \xi'_+ + (a\xi'_- + b\xi'_+)\bar g\right) \wedge \left(\xi'_- f + (c\xi'_- + d\xi'_+) g\right)$$

$$= \begin{vmatrix} \bar f + b\bar g & dg \\ a\bar g & f + cg \end{vmatrix} \xi'_+ \wedge \xi'_- = \begin{pmatrix} |f|^2 & \bar f \bar b g \\ \bar g b f & |b|^2|g|^2 - \overset{|d|^2}{\wedge}|g|^2 \end{pmatrix}^{\wedge}$$

$$= \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \underset{\sim}{K}\left(\xi' f + \xi g\right)$$



$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \xi'_- z^x \overset{H_+}{(1-f)} + \xi_- \overset{H_+}{(-g)} \\ \xi'_- z^x(-\phi) + \xi_-(1-\psi) \end{pmatrix}$$

$$\int \begin{pmatrix} \cancel{z^x} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} \overset{1}{\cancel{z^x}} & \bar b z^{-x} \\ b z^x & -1 \end{pmatrix} \begin{pmatrix} \cancel{z^x}(1-f) & -\cancel{z^x}\phi \\ -g & 1-\psi \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & T_x^* \\ T_x & -1 \end{pmatrix} \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 0 & \pi_+(\bar b z^{-x}) \\ \pi_+(b z^x) & 0 \end{pmatrix}$$

$$T_x = \pi_+ b z^x \varepsilon_+$$

What is h_x?

~~Quantum~~
Go back to pos. def case

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \xi_+ z^x \overset{H_+}{(1-f)} + \xi_- \overset{H_-}{(-g)} \\ \xi_+ z^x(-\phi) + \xi_-(1-\psi) \end{pmatrix}$$

$$\|$$

$$\begin{pmatrix} z^x(1-f) & -g \\ z^x(-\phi) & 1-\psi \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} 1 & T_x^* \\ T_x & 1 \end{pmatrix} \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 0 & \pi_+(\bar\beta z^{-x}) \\ \pi_-(\beta z^x) & 0 \end{pmatrix}$$

$$T_x = \pi_- \beta z^x \varepsilon_+$$

$$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 1-f & -g z^{-x} \\ -z^x \phi & 1-\psi \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

Problem of deriving the DDE. Is it clear that f_x, g_x, ϕ_x, ψ_x are smooth functions of x. with values in H_+, H_-, H_+, H_-? Look at $T_x = \pi_+ b z^x \varepsilon_+$ on H_+. What is the problem? Look at the operator bz^x on L^2, $z^x = e^{ikx}$, multip op. Not C^1 because $\partial_x(z^x) = ikz^x$ is unbdd. WAIT, you multiply by b.

Back to the K-picture

$$\begin{pmatrix} 1 & T_x^* \\ T_x & -1 \end{pmatrix}\begin{pmatrix} f_x & \phi_x \\ g_x & \psi_x \end{pmatrix} = \begin{pmatrix} 0 & \pi_+(\bar{b} z^{-x}) \\ \pi_+(bz^x) & 0 \end{pmatrix}$$

$$T_x = \pi_+ b z^x \varepsilon_+ \qquad \begin{pmatrix} z^{-x}\rho_x \\ \varrho_x \end{pmatrix} = \underbrace{\begin{pmatrix} \cancel{z}(1-f_x) & -g_x z^{-x} \\ z^x(-\phi_x) & 1-\psi_x \end{pmatrix}}_{\frac{1}{d}\begin{pmatrix} d^r_x & b^\ell_x \\ -c^r_x & d^\ell_x \end{pmatrix} \in \begin{pmatrix} \tilde{H}_+ & H_+ z^{-x} \\ z^x H_+ & \tilde{H}_+ \end{pmatrix}}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

~~Exactly~~ What do you hope is true? about $f_x, g_x, \phi_x, \psi_x \in H_+$. You expect that if $b(k)$ ~~decays~~ $\in$ Schwartz space, then it should be OKAY; ~~So~~ you want f_x, g_x, ϕ_x, ψ_x to be nice functions of both x, k.

Point is the mult. operator $bz^x = b(k)e^{ikx}$

$$(\partial_x)^n(bz^x) = b(k)(ik)^n e^{ikx}$$

as a bdd operator on $L^2(\mathbb{R}, \frac{dk}{2\pi})$ this mult. by bz^x

$$x \longmapsto (bz^x \cdot) \in \mathcal{L}(L^2) \qquad \text{smooth map.}$$

$$x \longmapsto \pi_+ b z^x \varepsilon_+ \in \mathcal{L}(H_+)$$

$$\pi_+(bz^x) = \pi_+\left(\sum_y \hat{b}_y z^{-y} z^x\right) = \int_{y<x} \hat{b}_y z^{x-y}\,dy$$

$$\pi_+(\bar{b}z^{-x}) = \pi_+\int \bar{\hat{b}}_y z^{y-x} = \int_{y>x} \bar{\hat{b}}_y z^{y-x}\,dy$$

you need these in L^2 and you want them to be smooth in x. Since true for bz^x then OKAY after applying projection. π_+.

$$(1+X)\varepsilon = \begin{pmatrix} 1 & +T^* \\ T & -1 \end{pmatrix} \qquad \begin{pmatrix} f \\ g \end{pmatrix} = \varepsilon(1+X)^{-1}\begin{pmatrix} 0 \\ T\hat{1} \end{pmatrix}$$

$$= \varepsilon\frac{1-X}{(1-X)(1+X)}\begin{pmatrix} 0 \\ T\hat{1} \end{pmatrix}$$

$$= \frac{1}{1-X^2}\begin{pmatrix} 1 & T^* \\ +T & -1 \end{pmatrix}\begin{pmatrix} 0 \\ T\hat{1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{1+T^*T} & 0 \\ 0 & \frac{1}{1+TT^*} \end{pmatrix}\begin{pmatrix} T^*T\hat{1} \\ -T\hat{1} \end{pmatrix}$$

$$f + T^*g = 0$$

$$-g - TT^*g = T\hat{1}$$

$$g = -\frac{1}{1+TT^*}T\hat{1}$$

$$f = T^*\frac{1}{1+TT^*}T\hat{1}$$

$$\begin{pmatrix} f_x \\ g_x \end{pmatrix} = \begin{pmatrix} (1+T_x^*T_x)^{-1}T_x^*T_x\hat{1} \\ -(1+T_xT_x^*)^{-1}T_x\hat{1} \end{pmatrix}$$

provided T_x is smooth in x for this need $\|k^n b(k)\|_\infty < \infty$ $\forall n$ then f_x, g_x, ϕ_x, ψ_x smooth in x.

so what about potential.

So ask about variation!!! Really what is important is δT. To get DE you use bz^x and $\delta(bz^x) = bz^x ik\delta x$. Significance here is that $kH_+ \not\subset H_+$. Recall that for $\kappa \in UHP$ you have $H_+ \supset \frac{1}{k+\kappa} H_+$. Wait. Recall that $L^2 = H_+ \oplus H_-$ for $\mathbb{R}$ is essentially the same as $L^2 = H_+ z^{1/2} \oplus H_- z^{+1/2}$ for S^1, by C.T. $z = \frac{k-i}{k+i}$

So you have something

So you are concerned with $\delta b = bk^n$

$$\delta T = \pi_+ b k^n \varepsilon_+$$

$$\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix} \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 0 & \pi_+(\bar{b}) \\ \pi_+(b) & 0 \end{pmatrix}$$

$$\begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \pi_+(\bar{b}) \\ \pi_+(b) & 0 \end{pmatrix}$$

$$\delta \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = - \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \overset{\pi_+ \delta\bar{b}\, \varepsilon_+}{\delta T^*} \\ \underbrace{\delta T}_{\pi_+ \delta b\, \varepsilon_+} & 0 \end{pmatrix} \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \pi_+(\bar{b}) \\ \pi_+(b) & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \pi_+(\delta\bar{b}) \\ \pi_+(\delta b) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix} \delta \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = - \begin{pmatrix} 0 & \delta T^* \\ \delta T & 0 \end{pmatrix} \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} + \begin{pmatrix} 0 & \pi_+(\delta\bar{b}) \\ \pi_+(\delta b) & 0 \end{pmatrix}$$

$$\begin{pmatrix}1 & T^* \\ T & -1\end{pmatrix} \delta\begin{pmatrix} f & \phi \\ g & \psi\end{pmatrix} = \begin{pmatrix}1 & \tilde{T}^* \\ \tilde{T} & -1\end{pmatrix}\begin{pmatrix}1-f & -\phi \\ -g & 1-\psi\end{pmatrix}$$

$$\begin{pmatrix}1 & T^* \\ T & -1\end{pmatrix}\begin{pmatrix} f & \phi \\ g & \psi\end{pmatrix} = \begin{pmatrix}0 & T^*1 \\ T\hat{1} & 0\end{pmatrix}$$

Situation: Have $E = \xi'_- L^2 + \xi_- L^2$ herm. inner prod. indef. ~~inner product~~ $\begin{pmatrix}1 & \bar{b} \\ b & -1\end{pmatrix}$ subspace $\xi'_- H_+ + \xi_- H_+$ K-orthogonal splitting, ~~now~~ want δb. You have to vary

$$X = \begin{pmatrix}0 & -T^* \\ T & 0\end{pmatrix} \qquad 1+X$$

Start again. You want some understanding of the geometry

$(1+X)\varepsilon$ is selfadjoint: $\left((1+X)\varepsilon\right)^* = \varepsilon(1-X) = (1+X)\varepsilon$

$\overset{\shortparallel}{\begin{pmatrix}1 & T^* \\ T & -1\end{pmatrix}}$ geometry: You have certain things fixed namely $E = \xi'_- L^2 \oplus \xi_- L^2$ ~~with bifiltration~~

$$F_{nm}E = \xi'_- z^n H_+ \oplus \xi_- z^m H_+$$

~~discrete/close~~ ~~Think through~~ ~~Transfer matrix~~ ~~again~~ discrete case. Given $b = \sum b_j z^{-j}$

Given $b = \sum b_j z^{-j} \in C^\infty(S^1)$ you know how to reconstruct (h_n). You want to understand δh corresp. δb. Guess that you want to use $b^{-1}\delta b$ roughly. Look at stuff remaining fixed. K will vary, what about σ

Review calculation: $K(v,v)\omega = \sigma(v)\wedge v \qquad \sigma(\omega) = -\omega$

Idea: Suppose you want δb corresp to δh.

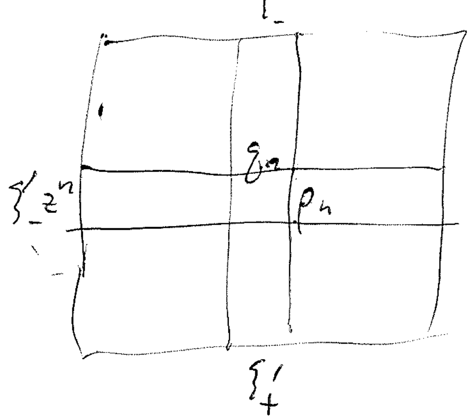
~~Perturbation theory should~~ $\delta b = \sum_n \frac{\partial b}{\partial h_n} \delta h_n$

$$b = \cdots\cdots \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \cdots\cdots$$

There should be k_n^2 etc

$$\frac{\delta}{\delta h_n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_n^r & +b_n^r \\ +c_n^r & d_n^r \end{pmatrix} \begin{pmatrix} 0 & \overset{1}{\cancel{z^{-n}}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -h_n z^{-n} \\ -\bar{h}_n \cancel{z^n} & z^{-n} \end{pmatrix} \begin{pmatrix} a_n^l & \overset{z^{-n}H_+}{b_n^l} \\ \underset{z^{n+1}H_-}{c_n^l} & d_n^l \end{pmatrix}$$

$$= \begin{pmatrix} a_n^r & +b_n^r \\ +c_n^r & d_n^r \end{pmatrix} \begin{pmatrix} a_n^l - h_n z^{-n} c_n^l \\ b_n^l - h_n z^{-n} d_n^l \end{pmatrix}$$



$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n^l & b_n^l \\ c_n^l & d_n^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_n^l & \overset{H_+}{z^n b_n^l} \\ \underset{zH_-}{z^{-n} c_n^l} & d_n^l \end{pmatrix} \begin{pmatrix} \xi'_- z^n \\ \xi'_+ \end{pmatrix}$$

$$\frac{\delta}{\delta h_n} \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}$$

$$\frac{\delta}{\delta h_n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_n^r & b_n^r \\ c_n^r & d_n^r \end{pmatrix} \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} a_{n-1}^l & b_{n-1}^l \\ c_{n-1}^l & d_{n-1}^l \end{pmatrix}$$

$$\partial_h \frac{1}{\sqrt{1-|h|^2}} = \partial_h \left(1-|h|^2\right)^{-1/2} = \left(+\tfrac{1}{2}\right)\left(1-|h|^2\right)^{-3/2}\left(+\bar{h}\right) = \frac{1}{2k}\frac{\bar{h}}{1-|h|^2}$$

$$\partial_h \frac{h}{\sqrt{1-|h|^2}} = \frac{1}{\sqrt{1-|h|^2}}\left(1 + \frac{\frac{1}{2}\bar{h}}{1-|h|^2}\right)$$

You need to use d.

$$d\left(\frac{1}{k}\begin{pmatrix}1 & hz^{-n}\\ \bar{h}z^{n} & 1\end{pmatrix}\right) = \frac{1}{k}\begin{pmatrix}0 & dh\,z^{-n}\\ d\bar{h}\,z^{n} & 0\end{pmatrix} + d\left(\frac{1}{k}\right)\begin{pmatrix}1 & hz^{-n}\\ \bar{h}z^{n} & 1\end{pmatrix}$$

$$d\left(\frac{1}{k}\right) = d\left((1-|h|^2)^{-1/2}\right) = \left(-\frac{1}{2}\right)(1-|h|^2)^{-3/2}(dh\,\bar{h}+h\,d\bar{h})$$

$\int\Phi$ ~~So it seems that~~

---

idea: allow variations ~~such th~~ fixing $|h_n|$, ~~just like you~~ only phase variations of (h_n), just like you restrict to phase variations in b.

So back to factorization etc. ~~[illegible]~~

You start with $b(z)$, know time translation is $b \mapsto bz^n$, but now you want the rest. Phase variations think loop group $g(z) = \underbrace{cz^n}_{S^1\times\mathbb{Z}} \underbrace{\exp\sum_{n=1}^{\infty} a_n z^n - \bar{a}_n z^{-n}}_{\frac{f(z)}{\overline{f(\bar{z})}}}$

simplest seems $g(z) = e^{az-\bar{a}z^{-1}}$

Want to treat as a variation

First look at $b \longmapsto e^{i\theta}b$. You already know that $b \longmapsto z^n b$ is time translation. In general if $\phi: S^1 \longrightarrow S^1$, then $\begin{pmatrix}\phi & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\phi & 0\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}a & \phi b\\ c\phi^{-1} & d\end{pmatrix}$ }

So you have action of loop group on the space of b's, commuting flows. Pretty clear that you get $(h_n) \longmapsto (e^{i\theta}h_n)$ on the level of the potentials.

$$h_0 = (q_0|p_0) = \frac{(\tilde{q}_0|\tilde{p}_0)}{\|\tilde{q}_0\|\,\|\tilde{p}_0\|}$$

$$\begin{pmatrix}\tilde{p}_0\\ \tilde{q}_0\end{pmatrix} = \begin{pmatrix}1-f & -\phi\\ -g & 1-\psi\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}1 & T_x^*\\ \underset{\underset{\pi_-\beta\,\xi_+}{\shortparallel}}{T_x} & 1\end{pmatrix}\begin{pmatrix}f & \phi\\ +g & \psi\end{pmatrix} = \begin{pmatrix}0 & \pi_+(\bar{\beta})\\ \pi_-(\beta) & 0\end{pmatrix}$$

$f + T^*g = 0$

$(1-TT^*)g = \pi_-(\beta)$

$$(\tilde{g}_0|\tilde{p}_0) = (\xi_-|\tilde{p}_0) = \int \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1-f \\ -g \end{pmatrix} = \int \beta - \beta f - g$$

$$f = -T^* g = -T^*(1-TT^*)^{-1}\pi_-(\beta) = -(1-T_x^*T_x)^{-1}\underbrace{T_x^*}_{e^{-i\theta}}\underbrace{\pi_-(\beta)}_{e^{i\theta}}$$

$$\therefore \quad h_0 \longmapsto e^{i\theta}h_0$$

So now you want to look at the loops

$e^{az-\bar{a}z^{-1}}$, ~~on b.~~ i.e. $b \longmapsto e^{\lambda z-\lambda z^{-1}}b$

also $\beta \longmapsto e^{\lambda z-\bar{\lambda}z^{-1}}\beta$

$$\delta\beta = (\lambda z - \bar{\lambda}z^{-1})\beta$$

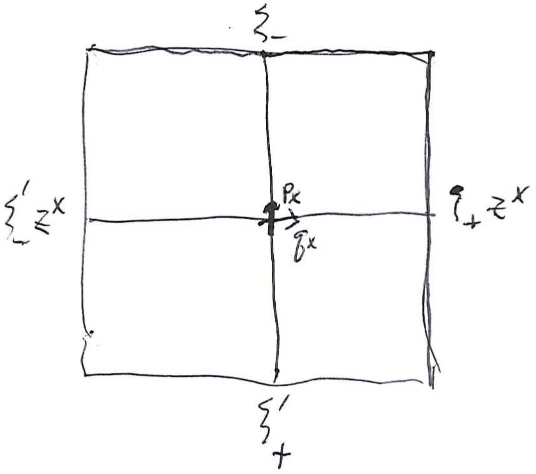
Question: How does this affect ~~the~~ (h_n)?

$$\begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 1 & T_x^* \\ \underset{\pi_-\beta\varepsilon_+}{\underset{\shortparallel}{\boxed{T_x}}} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \pi_+(\bar{\beta}) \\ \pi_-(\beta) & 0 \end{pmatrix}$$

$\pi_-\beta e^{\lambda z-\bar{\lambda}z^{-1}}\varepsilon_+$ Question whether there's something interesting you can do by putting $e^{\lambda z}$ with ε_+ and $e^{\bar{\lambda}z^{-1}}$ with π_-. But you should see if this is so by working infinitesimally.

$$T = \pi_-\beta e \qquad\qquad H_+ \xhookrightarrow{\varepsilon_+} L^2 \xrightarrow{\pi_-} H_-$$

$$\begin{tikzcd}
H_- \arrow[r, "\varepsilon_-"] & L^2 \arrow[r, "\pi_+"] \arrow[d] & H_+ \arrow[r, "\varepsilon_+"] \arrow[d, "e^{\lambda z}"] & L^2 \arrow[r, "\pi_-"] \arrow[d, "e^{\lambda z}"] & H_- \\
 & {} & H_+ \arrow[r, "\varepsilon_+"] & L^2 &
\end{tikzcd}$$



$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \overset{H_+}{1-f} & \overset{H_+}{-g} \\ \underset{H_+}{-\phi} & \underset{H_+}{1-\psi} \end{pmatrix} \begin{pmatrix} \xi'_- z^x \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} u^{-x} p_x \\ q_x \end{pmatrix} = \underbrace{\begin{pmatrix} 1-f & -g z^{-x} \\ -\phi z^x & 1-\psi \end{pmatrix}}_{\frac{1}{d}\begin{pmatrix} d^r_x & b^l_x \\ -c^r_x & d^l_x \end{pmatrix}} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\partial_x \begin{pmatrix} b^l_x \\ d^l_x \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 & h_x z^{-x} \\ \bar{h}_x z^x & 0 \end{pmatrix} \begin{pmatrix} b^l_x \\ d^l_x \end{pmatrix}$$

One problem here is $\begin{pmatrix} 1-f \\ -\phi \end{pmatrix}$ not $\begin{pmatrix} 1-f \\ -g \end{pmatrix}$

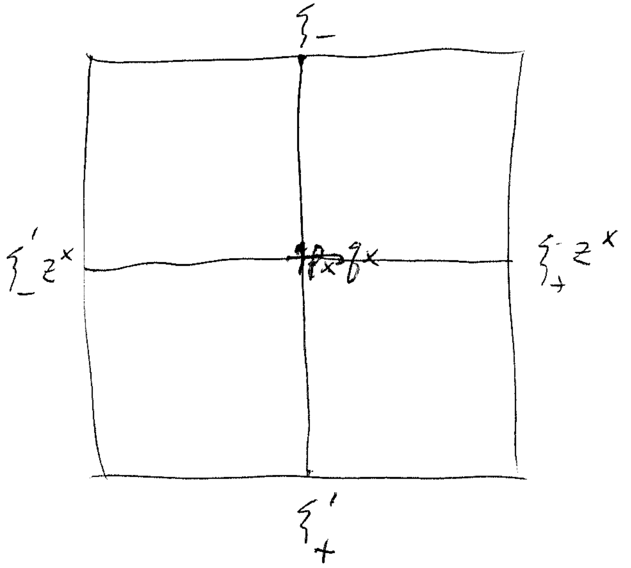
You've set up recovery via orthogonal projection, i.e. splitting ~~via~~ wrt a herm. form. suitably non-deg

$$K\Big(\overset{\xi'_- z^x H_+}{H_+ \xi_-},\ \overbrace{\xi'_- z^x (1-f) + \xi_-(-g)}^{p_x}\Big)$$

$$\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} z^{-x} \\ b z^x & -1 \end{pmatrix} \begin{pmatrix} 1-f \\ -g \end{pmatrix} = 0.$$

Try again. What's important? In the continuous case, you add to $L^2 = H_+ \oplus H_-$ the functions $1, k, k^2, k^3, k^4, \ldots$. Nothing quite the same in the discrete case, unless you ~~fix~~ allow a ~~sig~~ singularity somewhere.

Back to the cont. case.



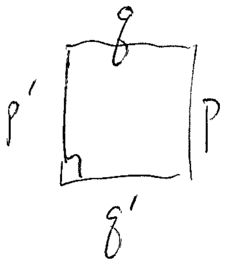
$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 1-f & -g \\ -\phi & 1-\psi \end{pmatrix} \begin{pmatrix} \xi'_+ z^x \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \underbrace{\begin{pmatrix} 1-f & -z^{-x} g \\ -\phi z^x & 1-\psi \end{pmatrix}}_{} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\frac{1}{d} \begin{pmatrix} d^r_x & b^{\ell}_x \\ -c^r_x & d^{\ell}_x \end{pmatrix} \quad z^{-x} H_+ \quad z^x H_+$$

$\therefore$ you have integral equations for these entries. You need to calculate ∂_x

There are problems here because



$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$K(q, p') = K\left(\frac{h}{k} p' + \frac{1}{k} q', p'\right) = \frac{h}{k} \overbrace{K(p', p')}^{1} = \frac{h}{k}$$

$$K(q', p) = K\left(q', \frac{1}{k} p' + \frac{h}{k} q'\right) = -\frac{h}{k}$$

apparently there's a jump.

$$K(q_x, p_x) = \int \begin{pmatrix} -\phi z^x \\ 1-\psi \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} (1-f) z^x \\ -g \end{pmatrix} = \int \begin{pmatrix} -\phi_x \\ 1-\psi_x \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} z^{-x} \\ b z^x & -1 \end{pmatrix} \begin{pmatrix} 1-f_x \\ -g_x \end{pmatrix}$$

$$= \int \left(b z^x (1-f_x) + g_x\right)$$

$$K(q_y, p_x) = \int \begin{pmatrix} -\phi_y \\ 1-\psi_y \end{pmatrix}^* \begin{pmatrix} z^{-y} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} z^{x} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-f_x \\ -g_x \end{pmatrix}$$

$$\cancel{= \int \begin{pmatrix} -\phi_y \\ 1-\psi_y \end{pmatrix}^* \begin{pmatrix} z^{x-y} & z^{-y}\bar{b} \\ bz^{x} & -1 \end{pmatrix} \begin{pmatrix} 1-f_x \\ -g_x \end{pmatrix}}$$

$$\cancel{= \int bz^x(1-f_x) + g_x = \int (-\phi_y)^* z^{x-y} + (1-\psi_y)^* bz^x}$$

if $x<y$ this is in $\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^{*c}$ so you get zero

$$= \int \begin{pmatrix} -\phi_y \\ 1-\psi_y \end{pmatrix}^* \begin{pmatrix} z^{x-y} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{b}z^{-x} \\ bz^{x} & -1 \end{pmatrix} \begin{pmatrix} 1-f_x \\ -g_x \end{pmatrix}$$

$$= \int \begin{pmatrix} -\phi_y \\ 1-\psi_y \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b}z^{-y} \\ bz^{y} & -1 \end{pmatrix} \underbrace{\begin{pmatrix} z^{x-y} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-f_x \\ -g_x \end{pmatrix}}_{\text{if } x>y \text{ this is in } \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \text{ so get } 0}$$

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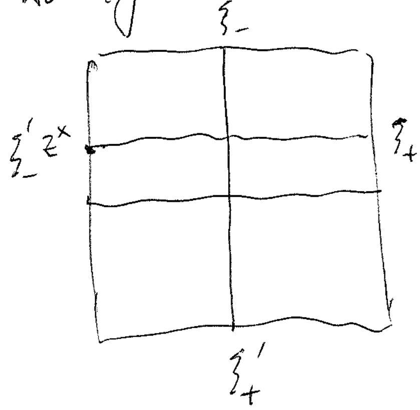
do again



$$p_x = \xi' z^x(1-f) + \xi_-(-g)$$

$$q_x = \xi'_- z^x(-\phi) + \xi_-(1-\psi)$$

$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \xi'_- & \xi \end{pmatrix} \begin{pmatrix} 1-f & -\phi z^x \\ -gz^{-x} & 1-\psi \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 1-f & -g \\ -\phi & 1-\psi \end{pmatrix} \begin{pmatrix} \xi'_- z^x \\ \xi_- \end{pmatrix}$$

$$\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b}z^{-x} \\ bz^{x} & -1 \end{pmatrix} \begin{pmatrix} 1-f & -\phi \\ -g & 1-\psi \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & T_x^* \\ T_x & -1 \end{pmatrix} \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 0 & \pi_+(\bar{b}z^{-x} \\ \pi_+(bz^x) & 0 \end{pmatrix}$$

You have to analyze carefully.

$$p_x = \xi'_- z^x(1-f) + \xi_-(-g) \quad \perp \quad \left(\xi'_- z^x H_+ + \xi_- H_+\right)$$

$$\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar b z^{-x} \\ b z^x & -1 \end{pmatrix} \begin{pmatrix} 1-f \\ -g \end{pmatrix} = 0 .$$

note that $p_x \notin \overbrace{\left(\xi'_- L^2 + \xi_- L^2\right)}^{E}$, because $\xi'_- z^x \notin E$.

meaning: $K\left(\xi'_- z^x, \begin{pmatrix} f \\ g \end{pmatrix}\right)$ is a linear functional on $\xi'_- z^x H_+ + \xi_- H_+$ densely defined.

$$= \int \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 & \bar b z^{-x} \\ b z^x & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$= \int f - \bar b z^{-x} g \ ?$$

~~There is a problem here~~ in the continuous ~~situation~~

So you've reached a difficult spot. Before this you understood ~~factorization~~ splitting pretty well. Splitting means constructing $\forall x$ the orthogonal projection of $\xi'_- L^2 + \xi_- L^2$ onto $\xi'_- z^x H_+ + \xi_- H_+$. This amounts to invertibility of the operator $\begin{pmatrix} 1 & T_x^* \\ T_x & -1 \end{pmatrix}$ on $\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$ where $T_x = \pi_+ b z^x \varepsilon_+$, true for any bdd T

$$\left(\begin{pmatrix} 1 \\ f \end{pmatrix} V\right)^\perp = \left\{ \begin{pmatrix} v \\ w \end{pmatrix} \,\middle|\, \begin{pmatrix} v \\ w \end{pmatrix}^* \begin{pmatrix} v' \\ f(v') \end{pmatrix} = 0 \quad \forall v' \in V \right\}$$

$$(v|v') + (w|fv') = 0$$

$$\Gamma_f \subset \begin{matrix} V \\ \oplus \\ W \end{matrix}$$

take $w \in W$, write $\begin{pmatrix} 0 \\ w \end{pmatrix} = \begin{pmatrix} v_0 \\ f v_0 \end{pmatrix} + \begin{pmatrix} -v_0 \\ w_1 \end{pmatrix}$

where $\begin{pmatrix} -v_0 \\ w_1 \end{pmatrix} \perp \Gamma_f$ i.e. $(v_0|v) = (w_1|fv)$

It works. $\Gamma_f \subset \begin{matrix}V\\ \oplus\\ W\end{matrix}$ Given $w \in W$ write

$$\begin{pmatrix}0\\ w\end{pmatrix} = \begin{pmatrix}v_0\\ fv_0\end{pmatrix} + \begin{pmatrix}-v_0\\ w_1\end{pmatrix} \quad \text{with} \begin{pmatrix}-v_0\\ w_1\end{pmatrix} \in \Gamma_f^{\perp}$$

Then ~~[illegible] g(w)~~ v_0 uniquely determined so define $g : W \to V$ by $g(w) = v_0$ g is linear.

$$\|w\|^2 \geqslant \|g(w)\|^2 + \|fg(w)\|^2 \qquad \text{so } \|g\| \leqslant 1.$$

$$\Gamma_f = \left\{ \begin{pmatrix}v\\ fv\end{pmatrix} \middle| v \in W \right\} \subset \begin{matrix}V\\ \oplus\\ W\end{matrix}$$

~~Given w can write~~

$$\cancel{\begin{pmatrix}0\\ w\end{pmatrix} = \begin{pmatrix}g(w)\\ f(g(w))\end{pmatrix} + \begin{pmatrix}-g(w)\\ h(w)\end{pmatrix}}$$

Given w $\exists!$ v_0 ~~[illegible]~~ such that

$$\begin{pmatrix}0\\ w\end{pmatrix} = \begin{pmatrix}v_0\\ fv_0\end{pmatrix} + \begin{pmatrix}-v_0\\ w - fv_0\end{pmatrix} \quad \text{and} \quad \boxed{\begin{pmatrix}-v_0\\ w - fv_0\end{pmatrix} \in \Gamma_f^{\perp}}$$

Given w $\exists!$ v_0 s.t.

$$\begin{pmatrix}0\\ w\end{pmatrix} - \begin{pmatrix}v_0\\ fv_0\end{pmatrix} \in \Gamma_f^{\perp} \quad \Bigg| \quad \left(\begin{matrix}v\\ fv\end{matrix} \middle| \begin{matrix}v_0\\ fv_0 - w\end{matrix}\right) = 0$$

~~[illegible]~~.

$$\Gamma_T^{\perp} = \left\{ \begin{pmatrix}v\\ w\end{pmatrix} \middle| (v'|w) = -(Tv'|w) \right\}$$

if $w = 0$, then $v = 0$.

so that $\Gamma_T^{\perp}$ is the graph of a partial defd $W \to V$

But $\Gamma_T^{\perp} \subset \begin{pmatrix} V \\ Z \end{pmatrix} \Rightarrow \Gamma_T \supset \begin{pmatrix} 0 \\ Z^{\perp} \end{pmatrix} \Rightarrow Z^{\perp} = 0$
$\Rightarrow Z = W.$

---

need to work out continuous version of the filtration. ~~linear~~

A filtration is a family of closed subspaces linearly ordered by inclusion. ~~A~~ A family of (orthog) projections, linearly ordered, get a Boolean algebra ~~which is necessarily~~ commutative.

Start again: Hilbert space ~~H~~ H, family of closed subspaces V_α linearly ordered by inclusion. ~~If~~ E_α = proj on V_α. Then $E_\alpha E_\beta = E_\beta E_\alpha = \begin{cases} E_\alpha & \text{if } V_\alpha \subset V_\beta \\ E_\beta & \text{if } V_\beta \subset V_\alpha \end{cases}$

~~[illegible]~~ ~~Introduce~~ If the family is finite then you have a split filtration, ~~a bunch~~ finite orthogonal decomposition. Assume 0, H belong to family. The E_α's should generate a commutative ~~[illegible]~~ von Neumann alg. Other ideas - make the filtration closed under sup's inf's.
~~Make the filtration~~

$x^2 = x$ $\quad (x+y)^2 = x^2 + xy + yx + y^2$ $\quad xy + yx = 0$

~~$x+y$~~ $\quad \therefore 2x = 0$ $\quad \therefore xy = -yx = yx.$

Have atoms. Get a Boolean alg of projections, probably spectrum

Basically you end up with the idea that a filtration is essentially equivalent to a self adjoint op. Can you work with this?

- What is the continuous version of your bifiltrations?
- half line $L^2(R_{>0}, dx) + L^2(R_{>0}, dy)$

these Hilbert spaces are glued together via a contraction.

The problem? When you come to the cont. case you replace orthonormal sequences by delta functions, your understanding of δ functions is incomplete. On the filtration level things are somewhat clear. NO, there's some ~~serious~~ analysis to straighten out. Figure out what you need. Start with a Hilbert space H and a filtration by closed subspaces. When do we get an isomorphism of this data with the model $L^2([0,1], dx)$ and the subspaces of functions with support in $[0,x]$ $0 < x \leq 1$. This is like trying to characterize $[0,1]$ as a poset.

Let's return to H and filtration $\mathcal{F}$

First understand filtration

Problem: Continuous version of bifiltration Grid of unit vectors, orthonormal bases. and staircases. ~~Let's begin with~~

First ~~study~~ filtration: H a Hilbert space $\mathcal{F}$ a linearly ordered family of closed subspaces including 0 and H. $\mathcal{F} = \{H_i, i \in I\}$

E_i = orth proj. on H_i. Given $i, j \in I$

$i \neq j$ either $H_i < H_j \Rightarrow E_i E_j = E_j E_i = E_i$

or $H_i > H_j \Rightarrow$ ——— $= E_j$

So you have a commuting family of projections in H. In general a commuting family ~~Suppose [illegible] finite~~ of projections generates a Boolean algebra of projections, analogy a family of subsets generates a Boolean algebra by introducing complements. Wrong direction

Suppose $\mathcal{F}$ finite: $0 = H_0 < H_1 < \cdots < H_n = H$.
Split filtration orthogonally $H = \bigoplus_{i=1}^{n} V_i$ $\quad V_i = H_i \ominus H_{i-1}$

$e_i = E_i - E_{i-1}$ orth. proj on V_i. Alg gen. by the E_i is $\bigoplus_{i=1}^{n} \mathbb{C} e_i$

Mult. one condition. Assume given a cyclic vector ξ, one such that $\{E_i \xi\}$ generates H.

Let I be the poset of subspaces in the filtration. $i \longmapsto \|E_i\xi\|^2$ should be strictly increasing

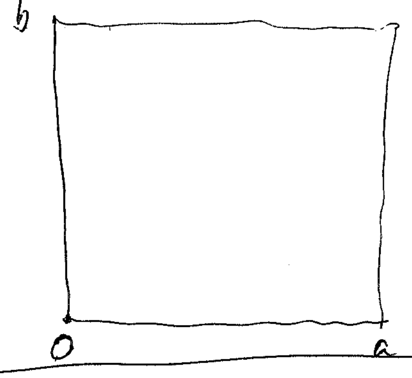
---

Start again with H a Hilbert space
H_i, $i \in I$ a family of closed subspaces, linearly ordered. closed under sups + infs. Assume $\exists\ \xi$ unit vector in H such that $\{E_i \xi\}_{i \in I}$ generates H. I order complete

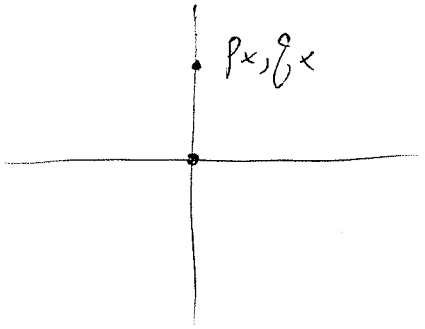
Then you get $i \longmapsto \|E_i\xi\|^2 = (\xi | E_i \xi)$, $I \longrightarrow [0,1]$ strictly monotone, i.e. $i < j \implies f(i) < f(j)$. What's the possible behavior?? The map is injective, so the question is whether surjective Also it's continuous wrt sups + infs.

~~It seems clear that~~

Basic problem. Start with ~~[illegible]~~ H obtained by glueing $L^2(0,a)$ and $L^2(0,b)$ together using a contraction operator $\beta : L^2(0,a) \to L^2(0,b)$



~~If you should~~ Do a constant coeff ~~case~~ example.



$$\partial_x \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 0 & h z^{-x} \\ \bar{h} z^{x} & 0 \end{pmatrix} \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix}$$

$$\|$$

$$\cancel{\begin{pmatrix} -ik z^{-x} \partial_x p_x \\ \partial q_x \end{pmatrix}} \qquad \begin{pmatrix} z^{-x} \partial_x p_x + (-ik) z^{-x} p_x \\ \partial_x q_x \end{pmatrix}$$

$$\cancel{z^{-x}} \partial_x p_x = ik \cancel{z^{-x}} p_x + h \cancel{z^{-x}} q_x$$

$$\partial_x q_x = \bar{h} p_x$$

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h \\ \bar{h} & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

$$\partial_x \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda & h \\ h & -\lambda \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \qquad \begin{pmatrix} \partial_x & -h \\ +h & -\partial_x \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix}$$

~~[illegible] $p' = \lambda p + hq$~~

$$(\partial - \lambda) p = hq$$

$$(\partial + \lambda) q = hp$$

$$\partial(p+q) - \lambda(p-q) = h(p+q) \qquad (\partial + h)(p-q) = \lambda(p+q)$$

$$\partial(p-q) - \lambda(p+q) = h(-p+q) \qquad (\partial - h)(p+q) = \lambda(p-q)$$

$$(\partial^2 - h^2)(p \pm q) = \lambda^2 (p \pm q).$$

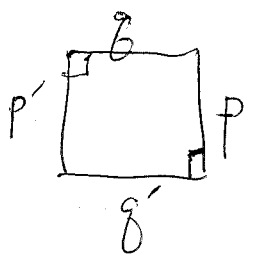
$$\begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix} \qquad \begin{aligned} p_x + q_x &= A e^{\sqrt{h^2+\lambda^2}\,x} \\ p_x - q_x &= B e^{-\sqrt{h^2+\lambda^2}\,x} \end{aligned}$$

$$\lambda = ik \qquad \sqrt{h^2 - k^2} = ik\sqrt{1+\left(\frac{h}{ik}\right)^2}$$

Bessel fn. stuff.

Why am I doing this example? You want to get an example not ~~[illegible]~~ part of scattering, to study the situation locally in space time. The aim is to find, construct the ~~grid~~ ~~cen~~ cont analogue of the grid of unit vectors in the discrete case.

discrete



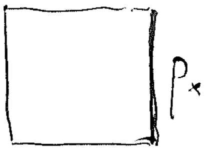
$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

$$(q|p) = (q|\cancel{k}p' + hq) = h$$

$$(q'|p') = (-\bar{h}p' + \cancel{k}q|p') = -h$$

cont.



What is the issue? There is a Hilbert space around to be constucted from a potential function h_x, ~~which~~ [illegible] using reflecting, somehow corresponding to the D.E. $\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix}\begin{pmatrix} p_x \\ q_x \end{pmatrix}$. $\begin{matrix} p_x \\ q_x \end{matrix}$ not in the Hilbert space. But you want something more general connected with a bifilt.

I want to consider constant coefficients. You should have the translation group as symmetries. In ~~the~~ other words, not just stationary under time translation but also space translation.

Start with the D.E. in the form

$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} \lambda & h \\ \bar{h} & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix} \qquad \text{NO}$$

Let's try to keep things geometric. Basic Dirac equation is

$$\partial_t \psi = \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \psi$$

solution has left and right moving wave forms.

$$(\partial_t - \partial_x)\psi_1 = 0 \implies \psi_1 = f(x+t)$$

$$(\partial_t + \partial_x)\psi_2 = 0 \implies \psi_2 = g(x-t)$$

~~Constant~~ gauge field

$$\partial_t \psi = \begin{pmatrix} \partial_x & h_x \\ -\bar{h}_x & -\partial_x \end{pmatrix} \psi$$

form is skew-adj.

Suppose h constant, make a constant phase adjustment to ψ_1 to make h purely imag?

$$\psi = e^{i(\omega t + kx)} \Psi$$

$$i\omega \Psi = \begin{pmatrix} ik & h \\ -\bar{h} & -ik \end{pmatrix} \Psi$$

$$\begin{vmatrix} ik - i\omega & h \\ -\bar{h} & -ik - i\omega \end{vmatrix} = k^2 - \omega^2 + |h|^2 = 0$$

$$\omega = \pm\sqrt{k^2 + |h|^2}$$

So what do you want to accomplish?

You have a skew adjoint operator

$$\begin{pmatrix}\partial_x & h\\ -\bar h & -\partial_x\end{pmatrix} \text{ acting on } \begin{pmatrix}L^2\\ L^2\end{pmatrix} \qquad L^2 = L^2(\mathbb{R},dx).$$

const. coeffs., so can be understood by F.T.

$L^2(\mathbb{R},dx) \sim L^2(\mathbb{R},\frac{dk}{2\pi})$, and it becomes mult by $\begin{pmatrix} ik & h\\ -\bar h & -ik\end{pmatrix}$

time evolution in the F.T. picture is $\exp t\begin{pmatrix} ik & h\\ -\bar h & -ik\end{pmatrix}$

The frequencies are the eigenvalues:

$$\begin{vmatrix} ik-i\omega & h\\ -\bar h & -ik-i\omega\end{vmatrix} = k^2-\omega^2+|h|^2 = 0 \qquad \omega = \pm\sqrt{k^2+|h|^2}$$

What do you want to understand, accomplish? ~~Past~~

Mainly a $\boxed{\text{space-time picture}}$, not a frequency ~~momentum~~ wave vector picture. picture of what?

You have a wave equation, the solutions should be a Hilbert space in the energy norm.

momentarily digress to look at disc. DE with constant h.

$$\begin{pmatrix}p_n\\ q_n\end{pmatrix} = \frac{1}{k}\begin{pmatrix} z & h\\ \bar h z & 1\end{pmatrix}\begin{pmatrix}p_{n-1}\\ q_{n-1}\end{pmatrix} \qquad k=\sqrt{1-|h|^2}$$

look for solutions

$$\begin{pmatrix}p_n\\ q_n\end{pmatrix} = \lambda^n \Psi$$

$$\frac{z}{k^2} - \frac{|h|^2 z}{k^2} = z$$

$$\lambda^n \Psi = \frac{1}{k}\begin{pmatrix} z & h\\ \bar h z & 1\end{pmatrix}\cancel{\lambda^{n-1}}\,\Psi$$

whence

$$0 = \begin{vmatrix} \frac{z}{k}-\lambda & \frac{h}{k}\\ \frac{\bar h z}{k} & \frac{1}{k}-\lambda\end{vmatrix} = \lambda^2 - \frac{1+z}{k}\lambda + z$$

$$\left(\frac{\lambda}{z^{1/2}}\right)^2 - \frac{z^{-1/2}+z^{1/2}}{k}\left(\frac{\lambda}{z^{1/2}}\right) + 1 = 0$$

$$\mu+\mu^{-1} = \frac{1}{k}\left(\sqrt{z}+\sqrt{z}^{-1}\right)$$

wave equation $\partial_t \psi = \begin{pmatrix} \partial_x & h \\ -\bar{h} & -\partial_x \end{pmatrix} \psi$, h const

If $\psi = e^{i(\omega t + \xi x)} \Psi$ $\qquad (c\omega)\Psi = \begin{pmatrix} i\xi & h \\ -\bar{h} & -i\xi \end{pmatrix} \Psi$

$$0 = \begin{vmatrix} i\xi - i\omega & h \\ -\bar{h} & -i\xi - i\omega \end{vmatrix} = \xi^2 - \omega^2 + |h|^2 \implies \omega = \pm\sqrt{\xi^2 + |h|^2}$$

like KLEIN GORDON

$$\omega \Psi = \begin{pmatrix} \xi & -ih \\ i\bar{h} & -\xi \end{pmatrix} \Psi .$$

Change

So many things confusing.
need a good viewpoint.
Start again with the wave eqn. $\partial_t \psi = \begin{pmatrix} \partial_x & h \\ -\bar{h} & -\partial_x \end{pmatrix} \psi$
Global solutions $\simeq$ Cauchy data at $t=0$.

$$\partial_t \psi = X\psi$$

$$\int_0^\infty e^{-st} \partial_t \psi \, dt = X \mathcal{L}\psi \qquad \boxed{\mathcal{L}\psi = (s-X)^{-1}\psi_0}$$

$$\parallel$$

$$\left[e^{-st}\psi\right]_0^\infty + s\int_0^\infty e^{-st}\psi \, dt = -\psi_0 + s\mathcal{L}\psi \qquad \boxed{\psi = e^{tX}\psi_0}$$

You need Green's fn.
G fn. should give desired ~~language~~ object.

Green's fn idea. ~~Obsolete~~ ~~Go back to discrete~~

~~Case~~ Suppose $\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h_x \\ \bar{h}_x & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$ $\qquad \partial_x(u^x) = u^x ik$

$$\partial_x \begin{pmatrix} u^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} u^{-x}(-ik)p_x + u^{-x}\underbrace{\partial_x p_x}_{ikp_x + h_x q_x} \\ \partial_x q_x \quad \bar{h}_x p_x \end{pmatrix} = \begin{pmatrix} u^{-x} h_x q_x \\ \bar{h}_x p_x \end{pmatrix} = \begin{pmatrix} 0 & h_x u^{-x} \\ \bar{h}_x u^x & 0 \end{pmatrix} \begin{pmatrix} u^{-x} p_x \\ q_x \end{pmatrix}$$

$\partial_x \begin{pmatrix} u^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 0 & h_x u^{-x} \\ \bar{h}_x u^x & 0 \end{pmatrix} \begin{pmatrix} u^{-x} p_x \\ q_x \end{pmatrix}$ WAIT, you want to avoid the scattering situation. e.g. take $h_x = im$ $m > 0$.

---

got past obstruction. You

$\partial_t \psi = \begin{pmatrix} \partial_x & im \\ +im & -\partial_x \end{pmatrix} \psi$ $i\omega \Psi = \begin{pmatrix} ik & im \\ +im & -ik \end{pmatrix} \Psi$

$\psi = e^{i(\omega t + kx)} \Psi$ $\omega \Psi = \begin{pmatrix} k & m \\ m & -k \end{pmatrix} \Psi$.

$$\begin{vmatrix} k-\omega & m \\ m & -k-\omega \end{vmatrix} = -k^2 + \omega^2 - m^2 = 0$$

$$\omega = \pm \sqrt{k^2 + m^2}$$

$$\Psi = \begin{pmatrix} m \\ \omega - k \end{pmatrix} \quad \begin{pmatrix} \omega + k \\ m \end{pmatrix}$$

eigenvectors $\begin{pmatrix} m \\ \omega - k \end{pmatrix}$, $\begin{pmatrix} \omega + k \\ m \end{pmatrix}$ for eigenv. ω

$\begin{pmatrix} m \\ -\omega - k \end{pmatrix}$, $\begin{pmatrix} -\omega + k \\ m \end{pmatrix}$ ——— $-\omega$

basic solution (

---

back to solving $\partial_t \psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \psi$. Problem is given $\psi(x,t')$ at $t' = 0$ find ψ at all (x,t). So you want the kernel. $K(x,t; x,0)$. This should be easy via F.T. ~~Thus you want~~ Case $m=0$.

$\partial_t \psi^1 = \partial_x \psi^1$ $\psi^1 = f(x+t)$ general soln.

$\partial_t \psi^2 = -\partial_x \psi^2$ $\psi^2 = g(x-t)$

$$\overset{\#}{\phi}(x,t) = \int K(x,t;x',0)\,\phi(x',0)\,dx'$$

$$\left\{\begin{matrix} f(x+t) \\ g(x-t) \end{matrix}\right\} \begin{matrix} = \int \delta(x+t-x')\,f(x')\,dx' \\ = \int \delta(x-t-x')\,g(x')\,dx' \end{matrix}$$

$$K(x,t;x',t') = \begin{pmatrix} \delta(x+t-x'-t') & 0 \\ 0 & \delta(x-t-x'+t') \end{pmatrix}$$

$$\|$$

$$\tilde{K}(x-x',t-t') = \begin{pmatrix} \delta((x-x')+(t-t')) & 0 \\ 0 & \delta((x-x')-(t-t')) \end{pmatrix}$$

Next try to handle m. Actually you only need to calculate $\tilde{K}(x,t)$ which you expect to begin with $\begin{pmatrix} \delta(x+t) & 0 \\ 0 & \delta(x-t) \end{pmatrix}$

Basic property is $\partial_t \tilde{K} - \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \tilde{K} = \delta(t)\,\delta(x)$

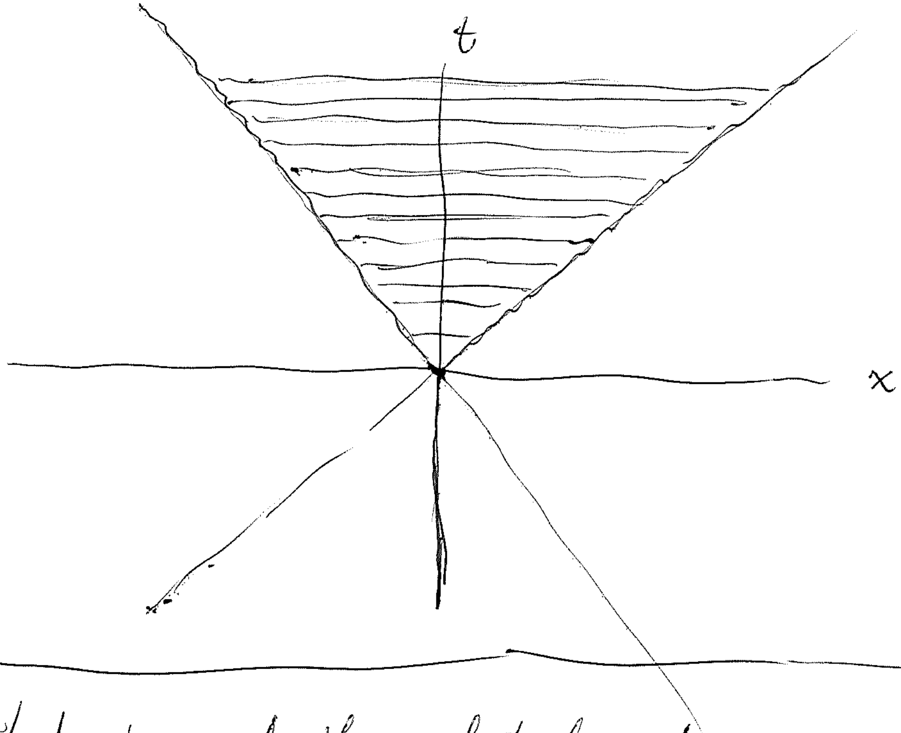
with the boundary condition that $\tilde{K}(x,t) = 0$ for $t<0$. (this means you've solving the ~~IVP at t=0~~ IVP ~~with~~ for $t \geq 0$ with Initial data at $t=0$.

So [illegible] do F.T.

$$\left(i\omega - \begin{pmatrix} ik & im \\ im & -ik \end{pmatrix}\right) \hat{\tilde{K}} = 1$$

Put $E = \tilde{K}$

At this point you need to decide how to proceed. [illegible] ~~So see~~ So draw a picture of $E(x,t)$, the fund. soln. "forward"



Yesterday at the whiteboard you had the following thoughts: about the wave equation on (x,t) plane

$$\partial_t \psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \psi$$ with const. coeffs.

Various interpns of ψ:

distributional function of x,t

dist fn. of x with values in a module with op ∂_t

——— x,t ——— ops ∂_x, ∂_t

Here ∂_t module means a repn. of the group of time translations

You want an analog of the "general soln" or "universal soln." of a Dirac eqn in the discrete case

You seek a solution with values in a suitable top. v.s. repn. M of the translation group, which specializes to any particular solution. Specialize means e.g. taking a quotient of the module M where translations act via a character.

Solving the wave eqn. via F.T. (or L.T.) ought to give some ideas about M. ~~Another~~

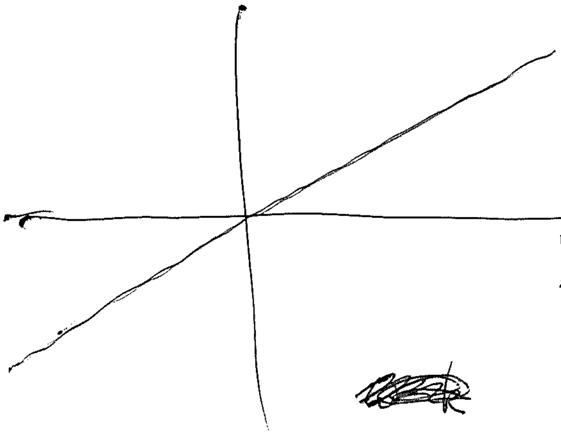
Another idea is that M ~~as~~ is naturally in duality with the vector space of soln. $\phi(x,t)$ with values in $\mathbb{C}^2$.

$\partial_t\psi = \begin{pmatrix}\partial_x & im\\ im & -\partial_x\end{pmatrix}\psi$ use F.T. in x.

$\partial_t\hat\psi = i\begin{pmatrix}k & m\\ m & -k\end{pmatrix}\hat\psi$ $\begin{pmatrix}k & m\\ m & -k\end{pmatrix}\begin{pmatrix}k & m\\ m & -k\end{pmatrix} = \begin{pmatrix}k^2+m^2 & 0\\ 0 & k^2+m^2\end{pmatrix}$

involution $\frac{1}{\sqrt{k^2+m^2}}\begin{pmatrix}k & m\\ m & -k\end{pmatrix}$ is an orthogonal involution

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\theta/2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}^{-1}$$



So what do you want?
to solve the IVP i.e.

$$\hat\psi(k) \longmapsto e^{i\begin{pmatrix}k & m\\ m & -k\end{pmatrix}t}\hat\psi(k)$$

$$\begin{vmatrix}k-\omega & m\\ m & -k-\omega\end{vmatrix} = -k^2+\omega^2-m^2 = 0$$

$$\omega = \pm\sqrt{k^2+m^2}$$

eigenvector $\begin{pmatrix}\omega+k\\ m\end{pmatrix}, \begin{pmatrix}-\omega+k\\ m\end{pmatrix}$

$$\begin{pmatrix}k & m\\ m & -k\end{pmatrix}\begin{pmatrix}\omega+k & -\omega+k\\ m & m\end{pmatrix} = \begin{pmatrix}k\omega+\underbrace{k^2+m^2}_{\omega^2} & -k\omega+\underbrace{k^2+m^2}_{\omega^2}\\ m\omega & -m\omega\end{pmatrix}$$

$$= \begin{pmatrix}\omega+k & -\omega+k\\ m & m\end{pmatrix}\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix}$$

So have a basis $\begin{pmatrix}\omega+k\\ m\end{pmatrix}$ for the eigenspace of $\mathbb{C}^2$ of $\begin{pmatrix}k & m\\ m & -k\end{pmatrix}$ and eigenvalue ω. To solve the IVP let $\psi(x,0)$ be given, expand

$$\psi(x,0) = \int\frac{dk}{2\pi}e^{ikx}\hat\psi(k,0)$$

You want $E(x,t)$ to satisfy

$$\left(\partial_t - \begin{pmatrix}\partial_x & im\\ im & -\partial_x\end{pmatrix}\right)\psi = \delta(x)\,\delta(t)$$

$$\psi = \int\frac{dk}{2\pi}\,e^{ikx}\,\hat{\psi}(k)$$

$$\left(\partial_t - i\begin{pmatrix}k & m\\ m & -k\end{pmatrix}\right)\hat{\psi}(k,t) = \delta(t)$$

Do F.T. in t

$$\left(\omega I - \begin{pmatrix}k & m\\ m & -k\end{pmatrix}\right)\hat{\psi}(k,\omega) = \frac{1}{i} \qquad ?$$

Green's fn.

Be straightforward, begin with wave eqn.

$\partial_t\psi = \begin{pmatrix}\partial_x & im\\ im & -\partial_x\end{pmatrix}\psi$ Interpret as a flow on space of $\begin{pmatrix}\psi^1(x)\\ \psi^2(x)\end{pmatrix}$, one par. gp of unitary autos having inf. gen $\begin{pmatrix}\partial_x & im\\ im & -\partial_x\end{pmatrix}$, on Hilb. space $L^2(\mathbb{R},dx)\oplus L^2(\mathbb{R},dx)$.

Consider global solutions – acted on by space-time translations

Go back to module philosophy. Look for, seek $\overset{\text{universal}}{\text{gen}}$ solution of wave equation.

from whiteboard 01/01/90.

wave eqn. $\partial_t \psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \psi$ $\quad \psi = \begin{pmatrix} \psi^1(x,t) \\ \psi^2(x,t) \end{pmatrix}$

Seek Green's fn. (equivalent to fund. soln. $E(x,t)$ since const coeff

$$\left(\partial_t - \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \right) E(x,t) = \delta(x)\delta(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

need boundary conditions to specify E.

In general for $\partial_t - iA$ both $e^{iAt}H(t)$ and $-e^{iAt}H(-t)$, ($H(t)$ = Heaviside), are Green's fns, forward + backward in time resp. But, if A operates on V and $V = V_+ \oplus V_-$ is an invariant splitting, then you can take one type on V_+ and the other on V_-.

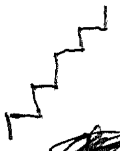
Note for $m = 0$ the forward E is $\begin{pmatrix} \delta(x+t)H(t) & 0 \\ 0 & +\delta(x-t)H(+t) \end{pmatrix}$

FROM 599

What remains to be understood about the partial D.E.

$$\partial_r u = imv$$
$$\partial_s v = imu$$

you want the Hilbert space structure, Krein space also.

Also look at discrete case. recall your idea in walking that you have a bigger symmetry group than just time translations $(m,n) \longmapsto (m-1, n+1)$ and space translations, thinking of space as the staircase , thus get $(m,n) \longmapsto (m+1, n+1)$. Index is 2. In any case can analyze the Hilbert + Krein space arising from a grid with constant h's.

Problem. ~~Hilbert space~~ In the discrete case it is obvious that the Hilbert space is a cyclic rep. of the translation group mentioned, cyclic vector p_0 since

$\boxed{\cdot p_0}$

BACK TO 600

$$\begin{pmatrix}\partial_t & 0\\ 0 & \partial_t\end{pmatrix}\psi = \begin{pmatrix}\partial_x & im\\ im & -\partial_x\end{pmatrix}\psi$$

$$\begin{pmatrix}\partial_t-\partial_x & 0\\ 0 & \partial_t+\partial_x\end{pmatrix}\psi = \begin{pmatrix}0 & im\\ im & 0\end{pmatrix}\psi$$

$$x = r-s$$
$$t = r+s$$

$$\partial_r\,\psi(\overset{x}{\cancel{r-s}},\overset{t}{\cancel{r+s}}) = \psi_x + \psi_t$$
$$\partial_s\,\psi(\overset{x}{\cancel{r-s}},\overset{t}{\cancel{r+s}}) = -\psi_x + \psi_t$$

$$\begin{pmatrix}\partial_r & 0\\ 0 & \partial_s\end{pmatrix}\psi = i\begin{pmatrix}0 & m\\ m & 0\end{pmatrix}\psi$$

Suppose $\psi = \begin{pmatrix}u\\ v\end{pmatrix}$. Then

$\partial_r u = imv$ $\qquad\qquad \partial^2_{rs} u = -m^2 u$

$\partial_s v = imu$

go over to F.T. dual variables ρ,σ to r,s

$i\rho\hat{u} = im\hat{v}$ $\qquad \rho\hat{u} = m\hat{v}$ $\qquad \boxed{\rho\sigma = m^2}$

$i\sigma\hat{v} = im\hat{u}$ $\qquad \sigma\hat{v} = m\hat{u}$

The spectrum i.e. adm. exponentials $e^{i(\rho r+\sigma s)}$ are where $\rho\sigma = m^2$. Two components: where $\rho\sigma > 0$ $\quad \rho,\sigma < 0$. Take $m=1$.

So solve $\hat{v} = \rho \hat{u}$ so

$$\begin{pmatrix} u \\ v \end{pmatrix} = \int_0^\infty d\rho\, e^{i(\rho r + \rho^{-1} s)} \begin{pmatrix} 1 \\ \rho \end{pmatrix} \hat{u}(\rho)$$

$$+ \int_{-\infty}^{0} d\rho\, e^{i(\rho r + \rho^{-1} s)} \begin{pmatrix} 1 \\ \rho \end{pmatrix} \hat{u}(\rho)$$

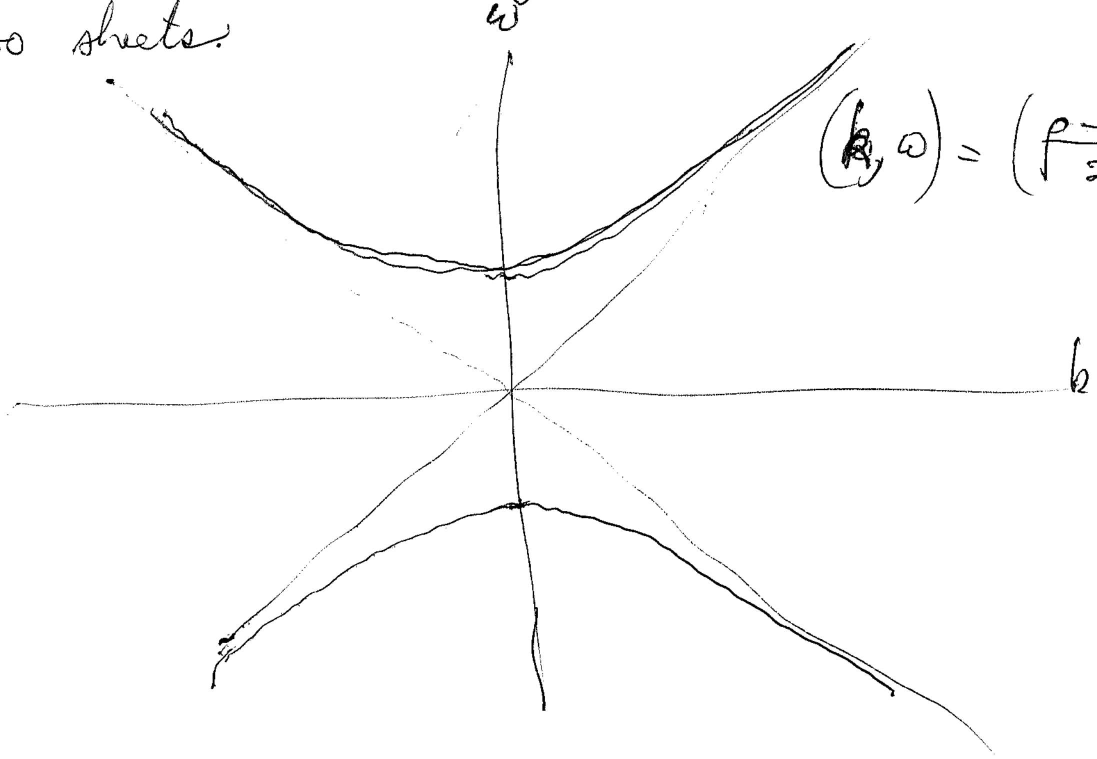
You can put $r = \frac{x+t}{2}$ $s = \frac{t-x}{2}$

$$e^{\cancel{2}i\left(\underbrace{\rho\left(\frac{x+t}{2}\right)} + \underline{\rho^{-1}\left(\frac{t-x}{2}\right)} \right)}$$

$$\underbrace{\left(\frac{\rho-\rho^{-1}}{2}\right)}_{k} x + \underbrace{\left(\frac{\rho+\rho^{-1}}{2}\right)}_{\omega = \pm\sqrt{k^2+m^2}} t \qquad k^2+1 = \omega^2$$

So you are certainly ~~[illegible]~~ parametrizing the two sheets.

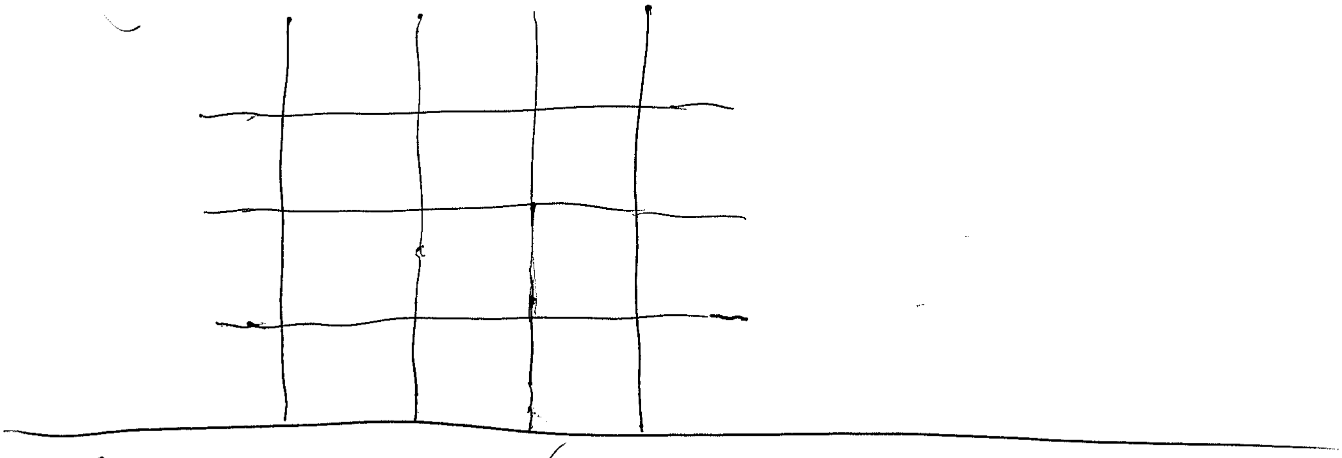
$$(k, \omega) = \left(\frac{\rho-\rho^{-1}}{2}, \frac{\rho+\rho^{-1}}{2} \right)$$



$$\frac{\rho - \frac{m^2}{\rho}}{2} \qquad \frac{\rho + \frac{m^2}{\rho}}{2}$$

GOBACK TO 597

so there should be measure on the dual of the translation group.



start again $\begin{pmatrix}\partial_t-\partial_x & -im\\ -im & \partial_t+\partial_x\end{pmatrix}\psi = 0$

$$x = r-s$$
$$t = r+s$$

$$\partial_r\psi = \partial_x\psi\left(\frac{\partial x}{\partial r}\right) + \partial_t\psi\left(\frac{\partial t}{\partial r}\right)$$
$$= (\partial_x + \partial_t)\psi$$
$$\partial_s\psi = \partial_x\psi\left(\frac{\partial x}{\partial s}\right) + \partial_t\psi\left(\frac{\partial t}{\partial s}\right)$$
$$= (-\partial_x + \partial_t)\psi$$

$$\begin{pmatrix}\partial_s & -im\\ -im & \partial_r\end{pmatrix}\psi = 0 \qquad \psi = \begin{pmatrix}u\\ v\end{pmatrix}$$

$$\frac{1}{i}\begin{pmatrix}\partial_s u\\ \partial_r v\end{pmatrix} = \begin{pmatrix}mv\\ mu\end{pmatrix}$$

$$\frac{1}{i}\partial_s u = mv \qquad \sigma\hat{u} = m\hat{v}$$
$$\frac{1}{i}\partial_r v = mu \qquad \rho\hat{v} = m\hat{u}$$

$$\boxed{\sigma\rho = m^2.}$$

You have a wave equation

$$\partial_t \psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \psi$$ ~~whence a Hilbert space~~

w. constant coefficients. You want to construct a module $\mathcal{M}$ over $\mathcal{A}$ = group ring of translations, ~~having~~ a solution $\psi_{univ}(x,t)$ with values in $\mathcal{M}$ of the wave equation such that ψ is "universal". First make sense of solution with values in V; ~~pair~~ 2 vector $\psi^1(x,t), \psi^2(x,t) \in V$. Then use F.T. so $\psi(x,t)$ becomes $\hat{\psi}(k,\omega)$

$$\underbrace{\begin{pmatrix} \omega-k & -m \\ -m & \omega+k \end{pmatrix}} \hat{\psi}(k,\omega) = 0.$$

$\mathcal{A}$ functions of (k,ω)

so $\mathcal{M}$ is a quotient of $\mathcal{A}\psi^1_{univ} \oplus \mathcal{A}\psi^2_{univ}$ by the ~~rel~~ relations ^, ie:

$$(\omega-k)\hat{\psi}^1 = m\hat{\psi}^2$$
$$(\omega+k)\hat{\psi}^2 = m\hat{\psi}_1$$

$\omega^2 = k^2+m^2$

two components $\omega = \pm\sqrt{k^2+m^2}$

$k \in \mathbb{R}$

So it seems that $\mathcal{M}$ is cyclic $\simeq \mathcal{A}/(\omega^2-k^2-m^2)\mathcal{A}$

~~So it seems that $\mathcal{M}$ is cyclic.~~

Change of variable to characteristic coords

~~$x = r+s$~~
~~$t = -r+s$~~

$$\partial_r f(r+s, -r+s) = \partial_x f - \partial_t f$$
$$\partial_s f(\quad) = \partial_x f + \partial_t f$$

$x = r+s$
$t = r-s$

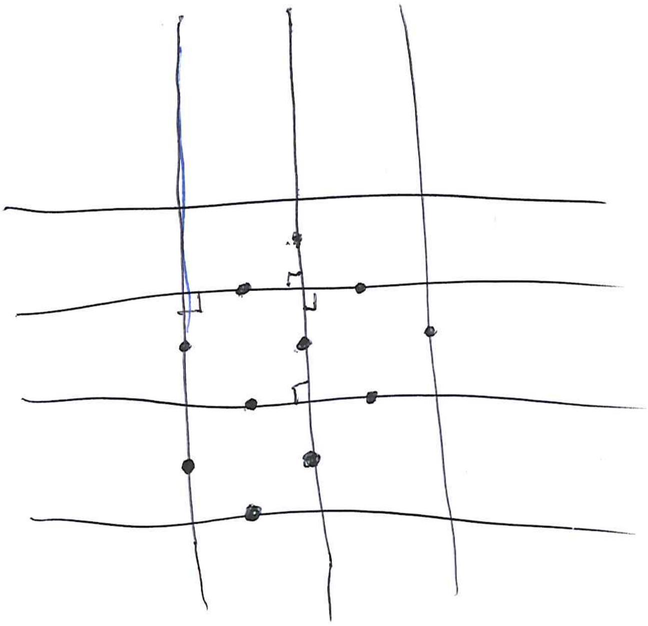
$$\partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h \\ \bar{h} & -ik \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

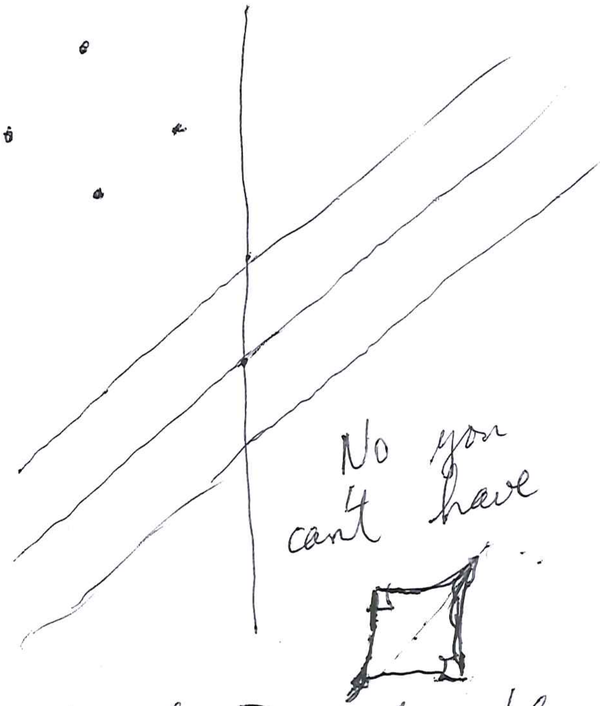
this x is not the x of space time

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

minimize signs: $$(\partial_t - \partial_x)\psi^1 = im\psi^2$$
$$(\partial_t + \partial_x)\psi^2 = im\psi^1$$

Look at discrete case where the notation should be clear. ?

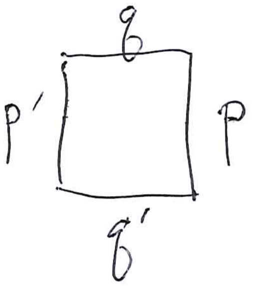




No you can't have

in the Euclidean plane

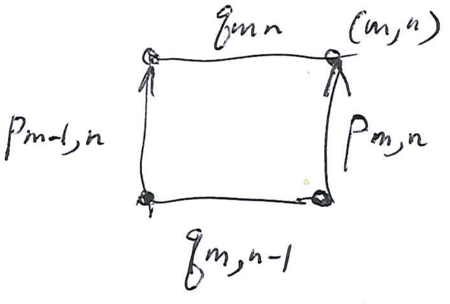
but what about non-Euclidean



$$\begin{pmatrix}p\\ q\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}p'\\ q'\end{pmatrix} \qquad k=\sqrt{1-|h|^2}$$

construct a Hilbert space via the grid

origin



translational symmetry.

$(m,n) \longmapsto (m+1,n)$ or $(m,n+1)$

$$\begin{pmatrix}p_{mn}\\ q_{mn}\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}p_{m-1,n}\\ q_{m,n-1}\end{pmatrix}$$

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χ character

Suppose $$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \underbrace{\chi(m,n)}_{z^m \omega^n}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \chi(m,n) = \begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \underbrace{\begin{pmatrix} p_{m-1,n} \\ q_{m,n-1} \end{pmatrix}}_{\begin{pmatrix} u\,\chi(m-1,n) \\ v\,\chi(m,n-1) \end{pmatrix}}$$

$$\begin{pmatrix} u\,\chi(m-1,n) \\ v\,\chi(m,n-1) \end{pmatrix} = \begin{pmatrix} u z^{-1} \\ v \omega^{-1} \end{pmatrix} \chi(m,n)$$

$$\therefore \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} z^{-1} & 0 \\ 0 & \omega^{-1} \end{pmatrix}}_{\frac{1}{k}\begin{pmatrix} z^{-1} & h\omega^{-1} \\ \bar{h}z^{-1} & \omega^{-1} \end{pmatrix}} \begin{pmatrix} u \\ v \end{pmatrix}$$

spectrum.
$$\left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} z^{-1} & 0 \\ 0 & \omega^{-1} \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} z & 0 \\ 0 & \omega \end{pmatrix} - \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} kz-1 & -h \\ -\bar{h} & k\omega-1 \end{vmatrix} = 0$$

$$\boxed{z\omega + 1 = \frac{\omega + z}{k}}$$

$$(kz-1)(k\omega-1) = |h|^2$$

$$(1-|h|^2)z\omega - k\omega - kz + 1 = |h|^2$$

$$k^2 z\omega - k\omega - kz + k^2 = 0$$

quadratic curve in the $\mathbb{C}\times\mathbb{C}$ plane but you are interested in $z, w \in S^1$.

Can also write $$(zw)^{1/2}+(zw)^{-1/2} = \frac{1}{k}\left(\left(\frac{w}{z}\right)^{1/2}+\left(\frac{w}{z}\right)^{-1/2}\right)$$

which is then the standard $\cos(\theta) = \frac{1}{k}\cos(\phi)$ which I think ~~[illegible]~~ involves elliptic functions, k maybe is the excentricity. But you extracted a square root to get this. So may the quadric curve is the quotient of the elliptic by an obvious auto of degree 2 such as -1. What are singularities of $f = zw+1 - \left(\frac{1}{k}\right)(w+z) = 0$

$$\partial_z f = w - \frac{1}{k} \qquad \partial_w f = z - \frac{1}{k}$$

get a sing. point at $(w,z) = \left(\frac{1}{k}, \frac{1}{k}\right)$, ~~[illegible]~~ ~~[illegible]~~ this is ~~[illegible]~~ where the derivatives vanish, but this point is not on the curve.

$$\left(\frac{1}{k^2}+1-\frac{2}{k^2}\right) = 1-\frac{1}{k^2}\neq$$

$$w + \frac{1}{z} - \frac{1}{k}\left(\frac{w}{z}+1\right) = 0$$

$$f = w + u - \frac{1}{k}(wu+1) = 0$$

$$\partial_w f = 1 - \frac{1}{k}u \qquad \partial_u f = 1 - \frac{1}{k}w$$

vanish at ~~[illegible]~~ $(w,u) = (k,k)$.

~~[illegible]~~ $$f(k,k) = \cancel{k}+k - \frac{1}{k}\left(\cancel{k^2}+1\right) = k - \frac{1}{k}$$

If $w \neq 0$ put $v = \frac{1}{w}$

$$f(z,v) = z\frac{1}{v} + 1 - \left(\frac{1}{k}\right)\left(\frac{1}{v}+z\right)$$

$$\partial_z f = \frac{1}{v} - \frac{1}{k} \qquad \partial_v f = -\frac{z}{v^2} - \frac{1}{k}\left(-\frac{1}{v^2}\right) = \frac{-z+\frac{1}{k}}{v^2}$$

$$v = k \quad z = \frac{1}{k}$$

$$f\left(\tfrac{1}{k}, k\right) = \tfrac{1}{k}k + 1 - \tfrac{1}{k}\left(\tfrac{1}{k} + \tfrac{1}{k}\right) = 2 - \tfrac{2}{k^2} \neq 0.$$

Other way to check. is to form.

$$z = \frac{z_1}{z_0} \qquad w = \frac{z_2}{z_0}$$

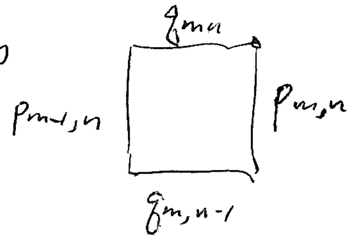
$$zw + 1 - \left(\tfrac{1}{k}\right)(z + w) = \frac{z_1 z_2}{z_0^2} + 1 - \left(\tfrac{1}{k}\right)\left(\frac{z_1}{z_0} + \frac{z_2}{z_0}\right)$$

$$z_1 z_2 + z_0^2 + \left(\tfrac{-1}{k}\right)(z_0 z_1 + z_0 z_2) = 0$$

$$\begin{pmatrix} 1 & -\frac{1}{2k} & -\frac{1}{2k} \\ -\frac{1}{2k} & 0 & \frac{1}{2} \\ -\frac{1}{2k} & \frac{1}{2} & 0 \end{pmatrix} = \left(+\tfrac{1}{2k}\right)\left(\tfrac{1}{2}\right)\left(+\tfrac{1}{2k}\right) + \left(+\tfrac{1}{2k}\right)\left(+\tfrac{1}{2k}\right)\left(\tfrac{1}{2}\right) - 1\ \tfrac{1}{2}\ \tfrac{1}{2}$$

$$= \frac{1}{\cancel{4}k^2} - \frac{1}{4} \neq 0 \qquad k = \pm 1.$$

Review have Hilbert ~~with~~ grid vectors action of translations, U V unitaries



$$U p_{m,n} = p_{m+1,n} \qquad V p_{m,n} = p_{m,n+1}$$

$$U q_{m,n} = q_{m+1,n} \qquad V q_{m,n} = q_{m,n+1}.$$

In the discrete case you set $A = \mathbb{C}[\overset{U^{-1}}{U}, V]^{V^{-1}}$

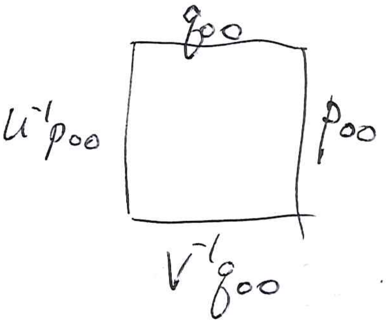
get an A module M with elements $p_{m,n}$, $q_{m,n}$

$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} U^{-1} p_{m,n} \\ V^{-1} p_{mn} \end{pmatrix}$$

Relations

$$\begin{pmatrix} p_{00} \\ q_{00} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} U^{-1} & hV \\ \bar{h}U^{-1} & V^{-1} \end{pmatrix}^{-1}\begin{pmatrix} p_{00} \\ q_{00} \end{pmatrix}$$

$$\begin{pmatrix} U^{-1}p_{0,0} \\ V^{-1}q_{0,0} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix}\begin{pmatrix} p_{00} \\ q_{00} \end{pmatrix}$$



$$\begin{pmatrix} U^{-1}p_{00} \\ q_{00} \end{pmatrix} = \begin{pmatrix} k & \bar{h} \\ \bar{h} & k \end{pmatrix}\begin{pmatrix} p_{00} \\ V^{-1}q_{00} \end{pmatrix}$$

---

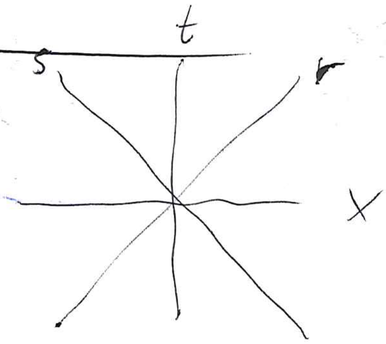
better is $\begin{pmatrix} p_{00} \\ q_{00} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} U^{-1}p_{00} \\ V^{-1}q_{00} \end{pmatrix}$ | $\begin{pmatrix} p_{00} \\ V^{-1}q_{00} \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} U^{-1}p_{00} \\ q_{00} \end{pmatrix}$

$$p_{00} = kU^{-1}p_{00} + hq_{00}$$

$$\therefore \quad q_{0,0} = \frac{1}{h}\left(p_{00} - kU^{-1}p_{00}\right)$$

Problem: First Start with $\partial_t\psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix}\psi$

pass to F.T. $\omega\hat{\psi} = \begin{pmatrix} k & m \\ m & -k \end{pmatrix}\hat{\psi}$



~~2nd introduce char coords.~~

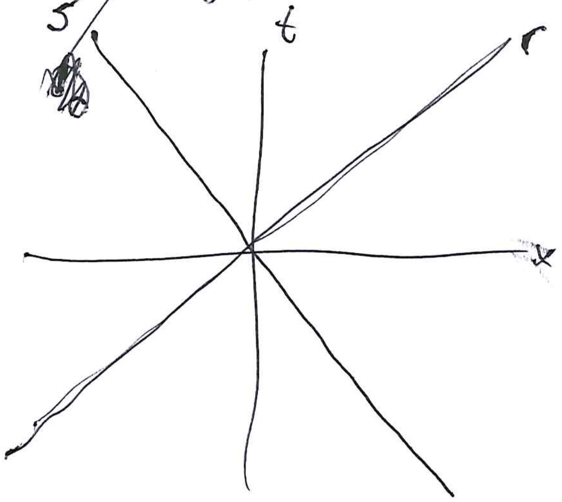
~~$x = r+s$~~

~~$t = r-s$~~

~~$\partial_r f(x,t) = \partial_x f + \partial_t f$~~

~~$\partial_s f = \partial_x f - \partial_t f$~~

~~$\frac{x+t}{2} = r$~~ ~~$\frac{x-t}{2} = s$~~



$x = r-s$

$t = r+s$

$\partial_r = \partial_x + \partial_t$

$\partial_s = -\partial_x + \partial_t$

$$\begin{pmatrix} \underset{\partial_s}{\boxed{\partial_t - \partial_x}} & 0 \\ 0 & \underset{\partial_r}{\boxed{\partial_t + \partial_x}} \end{pmatrix}\psi = \begin{pmatrix} 0 & im \\ im & 0 \end{pmatrix}\psi$$

$\partial_s u = imv$ $\qquad \sigma\hat{u} = m\hat{v}$

$\partial_r v = imu$ $\qquad \rho\hat{v} = m\hat{u}$

You have to run s ↘ if you want the r axis to be ascending staircase

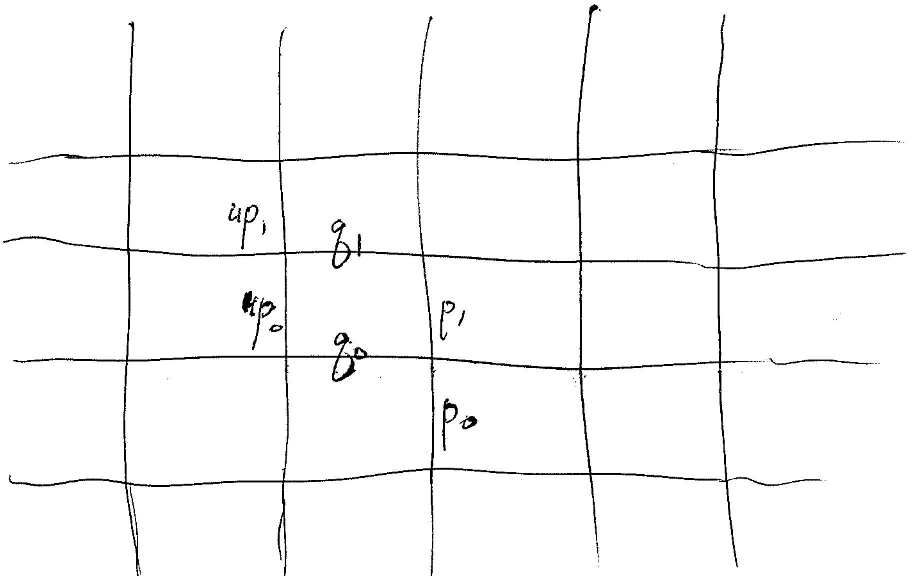
What's the problem? understanding characteristic – philosophy is singularities propagate along them. You don't recognize this from the discrete picture.

So discrete case first – constant h.

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$ Const. coeff.

you've already made F.T. in time.

Question: How do you recognize charateristics for $p(D)u = 0$. Look at highest order term $p_m(D)$, & zeroes of $p_m(\xi)$. Thus you look at large ξ, not something meaningful in the discrete case. However you can always look at $p_m(\xi) = 0$. OKAY. So look at



grid of unit vectors in a Hilb. space.

action of translation. | Look at spectrum of this representation

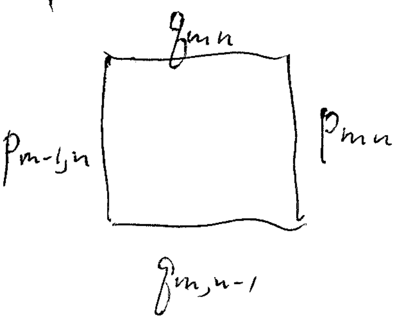
You might make sense of characteristics by factorizing the symbol, i.e. split spectrums into 2 pieces

idea is that $a\partial_x^2 + 2b\partial_x\partial_y + c\partial_y^2$ of non deg & non elliptic factors into ~~linear ones~~ first order ones.

from my viewpoint, so what?

Review

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{\sqrt{\ }} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \sqrt{\ } = \sqrt{1-|h|^2} = k$$



$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{\sqrt{\ }} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} p_{m-1,n} \\ q_{m,n-1} \end{pmatrix}$$

Suppose $\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \lambda^m \mu^n \begin{pmatrix} p_{00} \\ q_{00} \end{pmatrix}$

$$\frac{1}{k} \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} \begin{pmatrix} p_{00} \\ q_{00} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} p_{00} \\ \mu^{-1} q_{00} \end{pmatrix} \quad \Bigg| \quad \frac{1}{k} \begin{pmatrix} \lambda & -h\lambda \\ -\bar{h}\mu & \mu \end{pmatrix} \begin{pmatrix} p_{00} \\ q_{00} \end{pmatrix} = \begin{pmatrix} k p_{00} \\ k q_{00} \end{pmatrix}$$

$$(\lambda - k) p_{00} = h\lambda q_{00}$$
$$(\mu - k) q_{00} = \bar{h}\mu p_{00}$$

$$(\lambda - k)(\mu - k) = (1-k^2)\lambda\mu$$

$$k^2 \lambda\mu - k(\lambda+\mu) + k^2 = \cancel{(1-k^2)\lambda\mu}\ 0$$

$$\boxed{1 + \lambda\mu - \frac{1}{k}(\lambda+\mu) = 0} \qquad \frac{1}{k}$$

$$\frac{(\lambda\mu)^{1/2} + (\lambda\mu)^{-1/2}}{2} = \frac{1}{2}\left(\left(\frac{\lambda}{\mu}\right)^{1/2} + \left(\frac{\cancel{\lambda}}{\mu}\right)^{-1/2}\right)$$

$$\cos(\theta) = \frac{1}{k}\cos(\phi)$$

Try to focus on the problem. You have

$$\partial_t \psi = \begin{pmatrix} \partial_x & im \\ im & -\partial_x \end{pmatrix} \psi \qquad \text{Characteristics.}$$

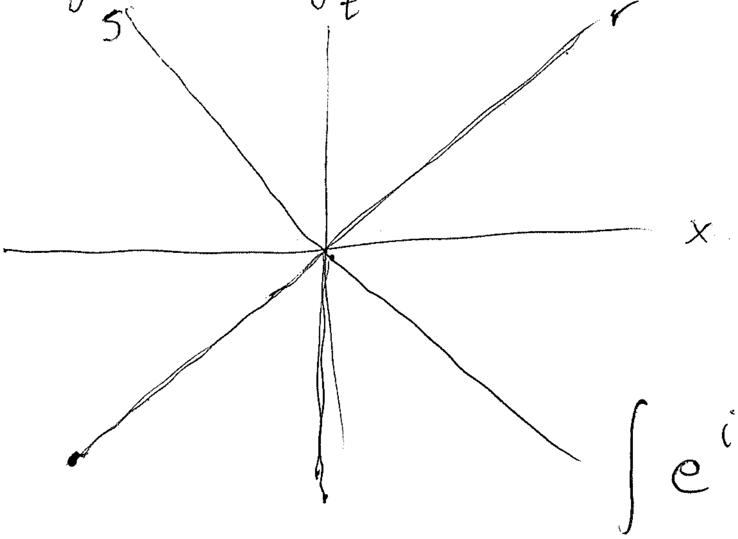
$$\omega \hat\psi = \begin{pmatrix} k & m \\ m & -k \end{pmatrix} \hat\psi \ , \quad 0 = \begin{vmatrix} \omega - k & -m \\ -m & \omega + k \end{vmatrix} = \omega^2 - k^2 - m^2$$

leading term $\omega^2 - k^2 = 0$ $\quad \omega = \pm k.$

you change coordinates

$$\partial_r = \partial_x + \partial_t \qquad x = r - s$$

$$\partial_s = -\partial_x + \partial_t \qquad t = r + s$$



$$\partial_s \psi^1 = im\psi^2 \qquad \sigma\hat\psi^1 = m\hat\psi^2$$

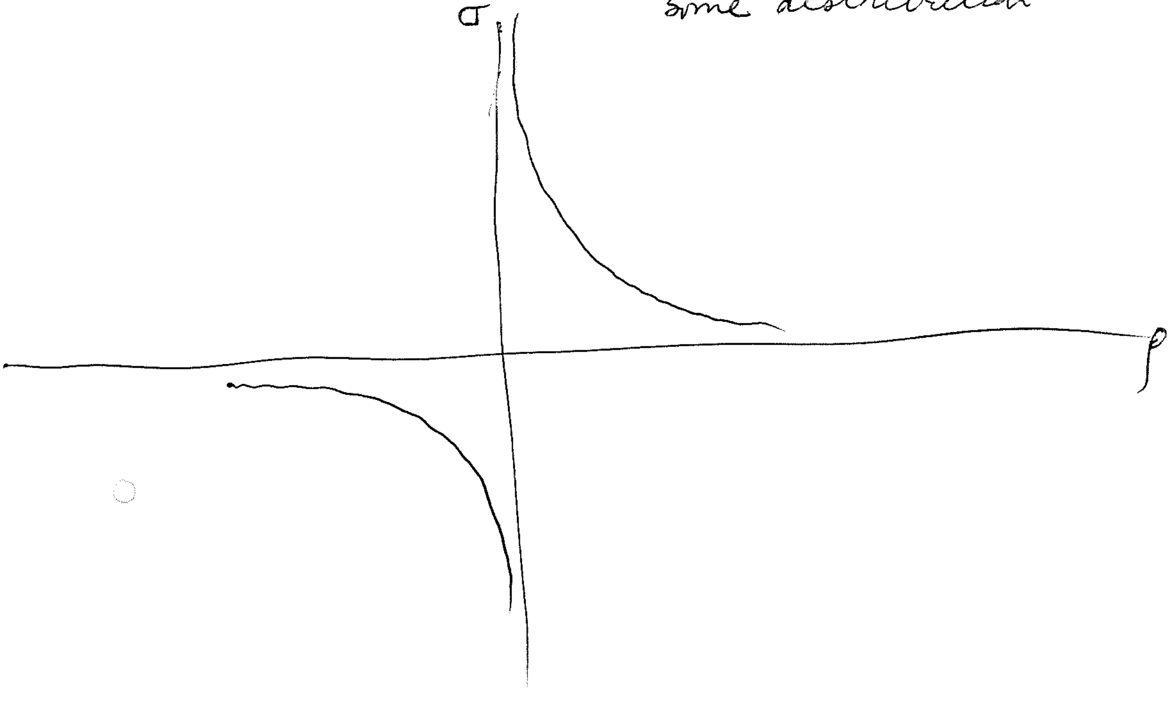
$$\partial_r \psi^2 = im\psi^1 \qquad \rho\hat\psi^2 = m\hat\psi^1$$

$$\int e^{i(r\rho + s\sigma)} \hat\psi(\rho, \sigma) = \psi(r, s)$$

Take $m = 1$, ~~[illegible]~~ get curve $\boxed{\sigma\rho = 1}$

~~[illegible]~~ general solution is

$$\psi = \int_{-\infty}^{\infty} e^{i(\rho r + \rho^{-1} s)} \begin{pmatrix} 1 \\ \rho \end{pmatrix} \underbrace{u(\rho) \frac{d\rho}{\rho}}_{\text{some distribution}}$$



Viewpoint: In the real projective plane, you have a circle ~~cutting the~~ intersecting the real proj line at ∞.

$$\rho = \frac{z_1}{z_0} \qquad \sigma = \frac{z_2}{z_0} \qquad 1 = \rho\sigma = \frac{z_1 z_2}{z_0^2}$$

$$z_1 z_2 - z_0^2 \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \text{non deg.}$$

$$\left(\frac{z_1}{z_0}\right)\left(\frac{z_2}{z_0}\right) = 1$$

$$\frac{z_2}{z_1} = \left(\frac{z_0}{z_1}\right)^2 \qquad \frac{z_1}{z_2} = \left(\frac{z_0}{z_2}\right)^2$$

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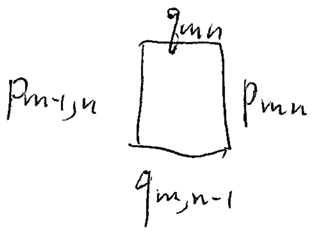
discrete case



$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p_{m-1,n} \\ q_{m,n-1} \end{pmatrix}$$

$$\parallel \qquad\qquad \parallel.$$

$$\lambda^m \mu^n \begin{pmatrix} u \\ v \end{pmatrix} \qquad\qquad \begin{pmatrix} \lambda^{-1} u \\ \mu^{-1} v \end{pmatrix} \lambda^m \mu^n$$

$$\left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1} & \\ & \mu^{-1} \end{pmatrix} \right| = 0$$

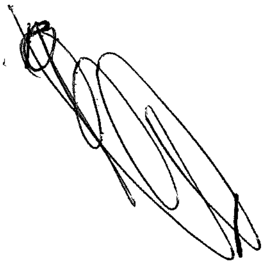
$$\left| \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} - \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \right| = 0 = \begin{vmatrix} \lambda - \frac{1}{k} & -\frac{h}{k} \\ -\frac{\bar{h}}{k} & \mu - \frac{1}{k} \end{vmatrix}$$

$$0 = \left(\lambda - \frac{1}{k}\right)\left(\mu - \frac{1}{k}\right) - \frac{|h|^2}{k^2} = \lambda\mu - \frac{1}{k}(\lambda + \mu) + 1$$

$\lambda = \frac{z_1}{z_0} \quad \mu = \frac{z_2}{z_0}$

$$\frac{z_1 z_2}{z_0^2} - \frac{1}{k}\left(\frac{z_1 + z_2}{z_0}\right) + 1$$

$$z_1 z_2 - \frac{1}{k}\left(z_0 z_1 + z_0 z_2\right) + z_0^2$$



$$\left|\begin{pmatrix} 1 & -\frac{1}{2k} & -\frac{1}{2k} \\ -\frac{1}{2k} & 0 & \frac{1}{2} \\ -\frac{1}{2k} & \frac{1}{2} & 0 \end{pmatrix}\right| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} + \frac{1}{2k}\begin{vmatrix} -\frac{1}{2k} & \frac{1}{2} \\ -\frac{1}{2k} & 0 \end{vmatrix} - \frac{1}{2k}\begin{vmatrix} -\frac{1}{2k} & 0 \\ -\frac{1}{2k} & \frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} + \frac{1}{8k^2} + \frac{1}{8k^2} = \frac{1}{4}\left(\frac{1}{k^2} - 1\right)$$

non singular $\Big\}$ conic section / quadric in $\mathbb{P}^2$. ~~never cubic~~

Quadric surface in $\mathbb{P}^2$ given by a quad form

$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$ affine

$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 = 0$ homog.

Assume $\begin{vmatrix} A & B \\ B & C \end{vmatrix} \neq 0$, then can translate to ~~map~~ make

$D, E = 0$ can factor: $(x - \lambda_1 y)(x - \lambda_2 y) = \text{const.}$

parabola? $x^2 - y = 0$ $\qquad$ $x^2 - yz = 0.$

certainly non-degenerate

$$\begin{pmatrix} 1 & & \\ & 0 & -\frac{1}{2} \\ & -\frac{1}{2} & 0 \end{pmatrix}$$

How does the analysis work?

non deg. quad form over $\mathbb{C}$ in 3 vbls. always ~~then~~ can be put in the form $\begin{pmatrix} 1 & & \\ & 0 & -\frac{1}{2} \\ & -\frac{1}{2} & 0 \end{pmatrix}$

Go back to $\partial_s\psi' = i\psi^2$, $\partial_r\psi' = i\psi^2$ which gives $\sigma u = v$, $\rho v = u$ $\sigma\rho = 1.$

$$\psi = \int e^{i(\rho r+\sigma s)}\begin{pmatrix}1\\ \sigma\end{pmatrix}$$ some distribution supported on the spectral curve

In the discrete case

$$\begin{pmatrix}p_{00}\\ q_{00}\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \mu^{-1}\end{pmatrix}\Big(p_{00}$$

$$0 = \left|\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} - \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\right| = \begin{vmatrix}\lambda-\frac{1}{k} & -\frac{h}{k}\\ -\frac{\bar h}{k} & \mu-\frac{1}{k}\end{vmatrix} = \left(\lambda-\frac{1}{k}\right)\left(\mu-\frac{1}{k}\right) - \frac{|h|^2}{k^2}$$

$$= \lambda\mu - \frac{1}{k}(\lambda+\mu)+1$$

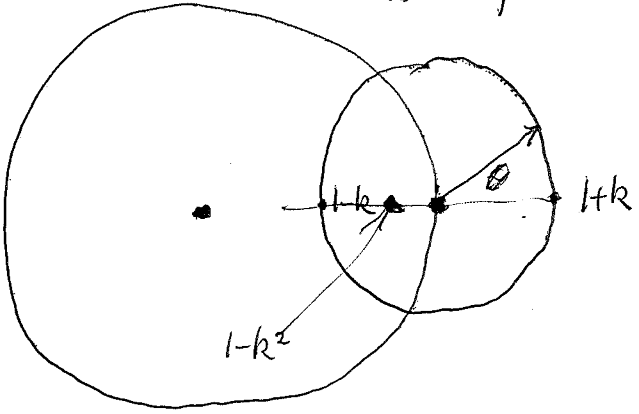
nondegenerate

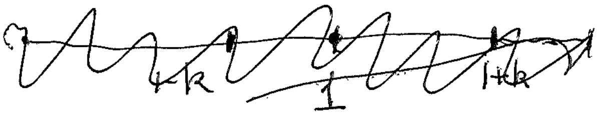
$$(k\lambda-1)(k\mu-1) = |h|^2 = 1-k^2$$

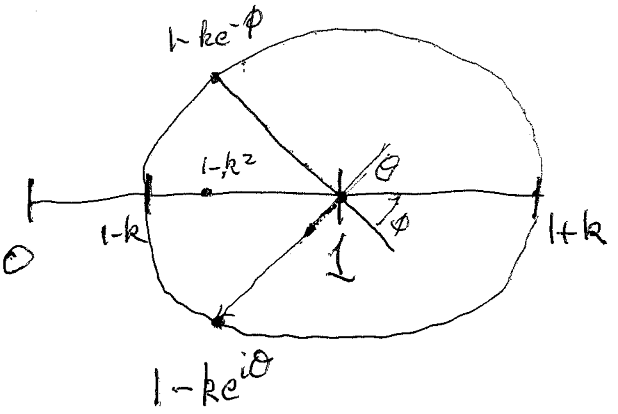
$$(1-k\lambda)(1-k\mu) = 1-k^2 \;(= h\bar h) \qquad 0<k<1.$$

want $\lambda,\mu\in S^1$. Obvious points at $(\lambda,\mu) = (1,-1)$, $(-1,1)$

Question: What is the locus of $(\lambda,\mu)\in S^1\times S^1$ satisf. $(1-k\underset{e^{i\theta}}{\lambda})(1-k\underset{e^{i\phi}}{\mu}) = 1-k^2$







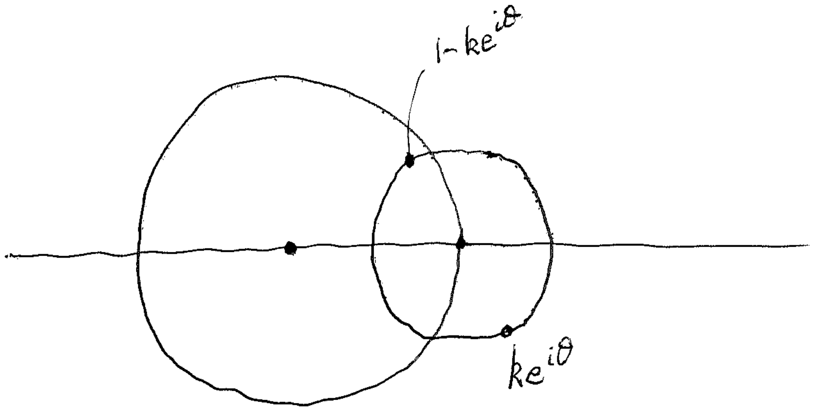
Question: If $|\lambda| = 1$. Is it true that $|\mu| = 1$?

$$1-k\mu = \frac{1-k^2}{1-k\lambda}$$

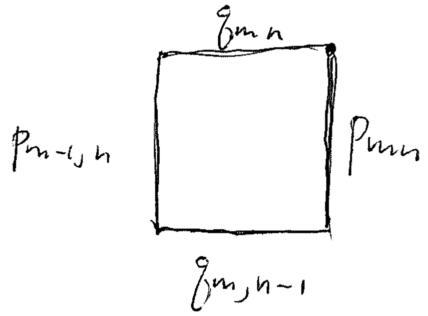
$$k\mu = 1-\frac{1-k^2}{1-k\lambda} = \frac{\cancel{1}-k\lambda-\cancel{1}+k^2}{1-k\lambda}$$

$$\mu = \frac{k-\lambda}{1-k\lambda}$$

$$(1-k\lambda)(1-k\mu) = |h|^2 = 1-k^2$$ 6/13



write up



$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p_{m-1,n} \\ q_{m,n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \lambda^m \mu^n \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1}u' \\ \mu^{-1}v' \end{pmatrix}$$

$$\begin{pmatrix} k\lambda u \\ k\mu v \end{pmatrix} = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$\lambda u = u'$,
$\mu v = v'$

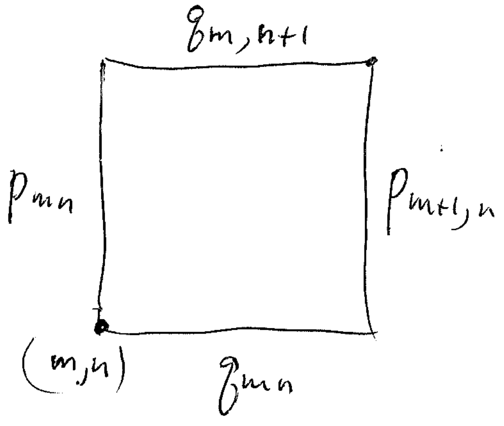
$$(k\lambda-1)u \overset{?}{=} hv$$
$$(k\mu-1)v = \bar{h}v$$

consistency

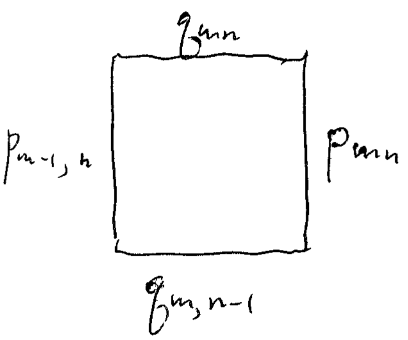
$$(k\lambda-1)(k\mu-1) = |h|^2 = 1-k^2$$

general soln.

$$\psi_{mn} = \int \lambda^m \mu^n \psi_{0,0}$$



$$\begin{pmatrix} \lambda p_{00} \\ \mu q_{00} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p_{00} \\ q_{00} \end{pmatrix}$$



$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p_{m-1,n} \\ q_{m,n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \lambda^{m}\mu^{n}\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix}$$

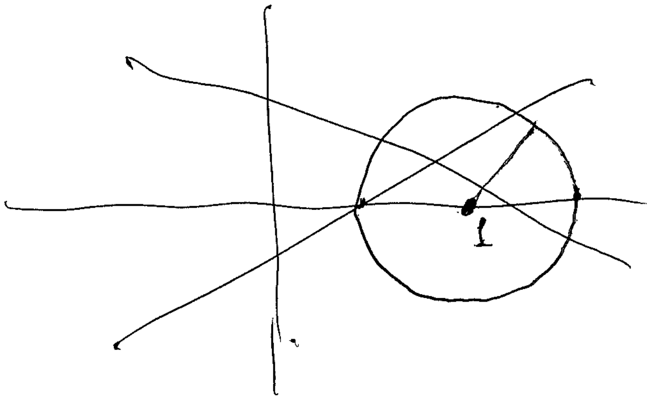
$$k\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \cancel{\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix}} = \frac{\cancel{1}}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \overset{u}{\cancel{\lambda^{-1}}\hat{p}} \\ v\,\cancel{\mu^{-1}}\hat{q} \end{pmatrix}$$

$u = \lambda^{-1}\hat{p}$

$v = \mu^{-1}\hat{q}$

$$\boxed{\begin{aligned} (k\lambda-1)u &= hv \\ (k\mu-1)v &= \bar{h}u \end{aligned}}$$

$$(k\lambda-1)(k\mu-1) = |h|^2 = 1-k^2$$



$$\cancel{(1-k\mu) = \frac{1-k^2}{1-k\lambda}}$$

$$\mu = \begin{pmatrix} 1 & -k \\ k & -1 \end{pmatrix}$$

$$\cancel{+\mu = \frac{1}{k}\left(1+\frac{1-k^2}{k\lambda-1}\right) = +\frac{1}{k}\,\frac{k\lambda\cancel{-1}+\cancel{1}-k^2}{k\lambda-1} = +} \frac{\lambda-k}{k\lambda-1}$$

$$\cancel{\frac{-\lambda+k}{k\lambda-1} = \begin{pmatrix} -1 & k \\ k & -1 \end{pmatrix}}$$

$$k\mu-1 = \frac{1-k^2}{k\lambda-1}$$

$$\mu = \frac{1}{k}\left(1+\frac{1-k^2}{k\lambda-1}\right) = \frac{\lambda-k}{k\lambda-1}$$

$$= \frac{-\lambda+k}{-k\lambda+1} = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}(-\lambda)$$

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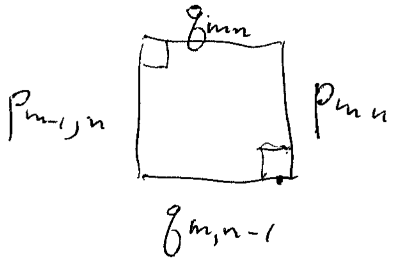
~~The~~ points remaining - your Hilbert space is isomorphic to $L^2(S^1)$, perhaps in a nice way. details?
At the moment what do you have?

Begin again.



$$p_{mn} = k p_{m-1,n} + h q_{mn}$$
$$q_{m,n-1} = -\bar{h} p_{m-1,n} + k q_{m,n}$$

$$p_{mn} = \lambda^m \mu^n \hat{p}$$
$$q_{mn} = \lambda^m \mu^n \hat{q}$$

~~$\mu u = k\lambda u + h v$~~

~~$\mu^{-1} v = -\bar{h}\lambda^{-1} u + k v$~~

$$\boxed{\begin{aligned}\hat{p} &= k\lambda^{-1}\hat{p} + h\hat{q}\\ \mu^{-1}\hat{q} &= -\bar{h}\lambda^{-1}\hat{p} + k\hat{q}\end{aligned}}$$

$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p_{m-1,n} \\ q_{m,n-1} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1}\hat{p} \\ \mu^{-1}\hat{q} \end{pmatrix}}$$

$$(1-k\lambda^{-1})\hat{p} = h\hat{q}$$
$$(k-\mu^{-1})\hat{q} = \bar{h}\lambda^{-1}\hat{p}$$

$$\boxed{\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}}$$

$$\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1}\hat{p} \\ \mu^{-1}\hat{q} \end{pmatrix}$$

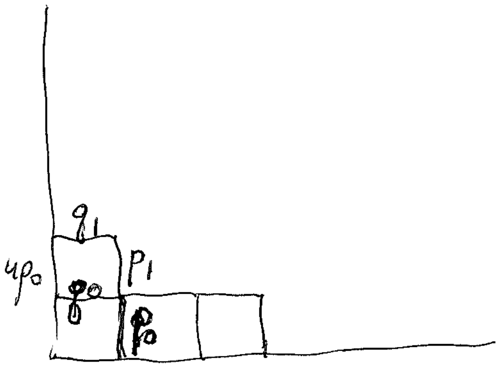
$$\boxed{\begin{pmatrix} \hat{p} \\ \mu^{-1}\hat{q} \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} \lambda^{-1}\hat{p} \\ \hat{q} \end{pmatrix}}$$

$$(\lambda-k)u = h\mu v$$
$$(k\mu-1)v = \bar{h}u$$

$$\Longleftarrow \quad \boxed{\begin{pmatrix} \lambda u \\ v \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} u \\ \mu v \end{pmatrix}}$$

? $\dfrac{\lambda-k}{\mu} = k\lambda - 1$ $\qquad \mu = \dfrac{\lambda-k}{k\lambda-1}$ YES.

You need to work out the Hilb space and Krein space structure. Question: How are they linked. It should be simple in the constant coeff case, except you may not be able to use scattering ideas – asymptotics for (m,n) large.



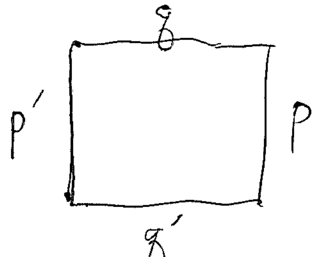
Review what you did before.

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\underbrace{\begin{pmatrix} 1 & h_n u^{-n} \\ \bar{h}_n u^{n} & 1 \end{pmatrix}}\begin{pmatrix} u^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$S^1 \longrightarrow SU(1,1)$$

means that M module over $A = \mathbb{C}[u,u^{-1}]$ general soln. has $Wr : \Lambda^2_A M \Rightarrow A$ and conjugation σ defined in terms of the bases $\begin{matrix} u^{-n}p_n \\ q_n \end{matrix}$ for any n.

Problem: Indef. form in general case



$$\begin{pmatrix} p \\ q \end{pmatrix} = \overbrace{\frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}}^{SU(1,1)}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \underbrace{\begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}}_{\in U(2) \text{ equal diag entries}}\begin{pmatrix} p' \\ q \end{pmatrix}$$

~~Start with~~ Krein structure in the cont. case

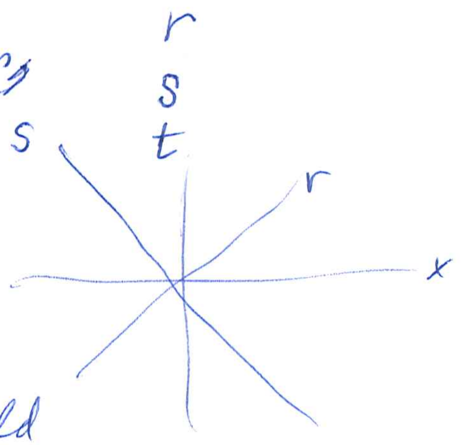
$$\partial_x \begin{pmatrix} u^{-x}p_x \\ q_x \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & h_x u^{-x} \\ \bar{h}_x u^{x} & 0 \end{pmatrix}}_{S^1 \longrightarrow \text{Lie } su(1,1)} \begin{pmatrix} u^{-x}p_x \\ q_x \end{pmatrix}$$

alternative $$\partial_t \psi = \begin{pmatrix} \partial_x & h_x \\ -\bar{h}_x & -\partial_x \end{pmatrix} \psi$$

Idea: If you choose char. coords,

$\partial_r = \partial_x + \partial_t$ $\quad x = r-s$

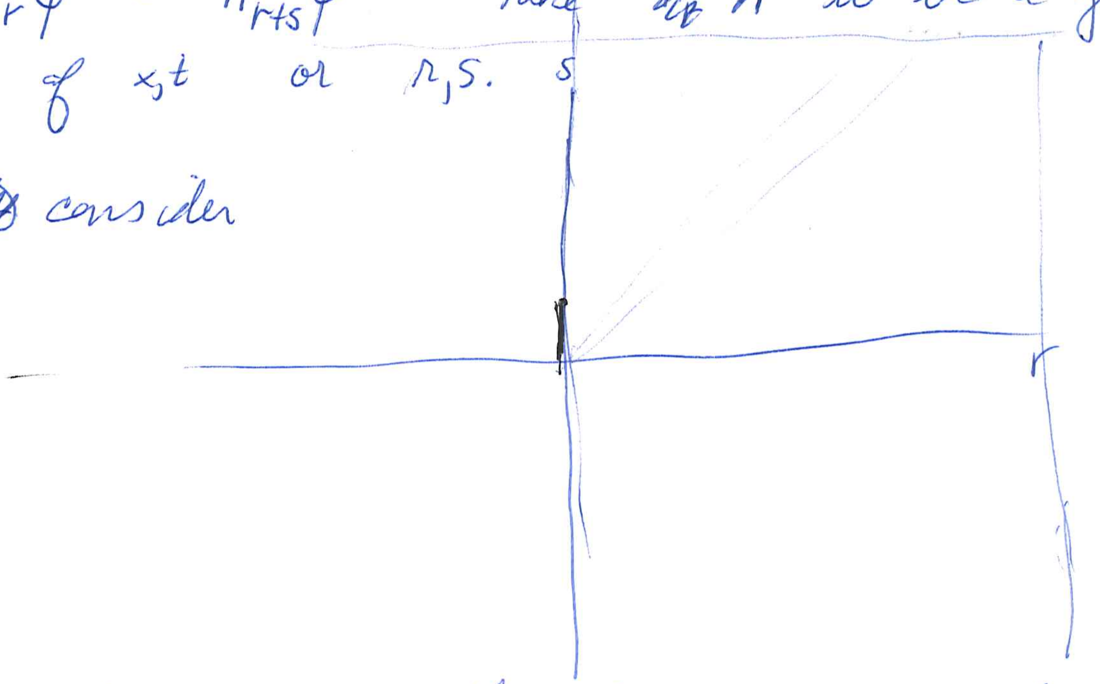
$\partial_s = -\partial_x + \partial_t$ $\quad t = r+s$



$\partial_s \psi^1 = h_{r+s}\psi^2$

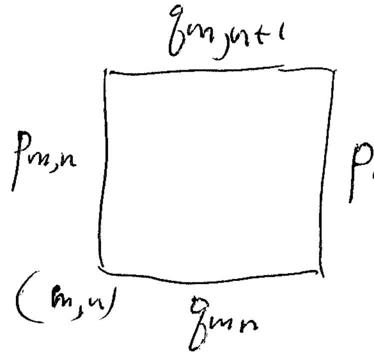
$\partial_r \psi^2 = -\bar{h}_{r+s}\psi^1$ so you should take h to be a general function of x,t or r,s.

So consider



so you have now the chance to do the continuous version.

Go over the discrete case - general grid - and then pass to continuous limit. What is the end result? ~~I see a problem maybe~~



$p_{mn} = \lambda^m \mu^n u$

$q_{mn} = \lambda^m \mu^n v$

$$\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\boxed{\frac{k\lambda - 1}{h} u = v \quad \Big| \quad \frac{k\mu - 1}{\bar{h}} = u}$$ transformed equations

spectral curve: $(k\lambda - 1)(k\mu - 1) = 1 - k^2$ or $\mu = \frac{1}{k}\left(1 + \frac{1-k^2}{k\lambda - 1}\right)$

$= \frac{\lambda - k}{k\lambda - 1} = \frac{-\lambda + k}{1 - k\lambda}$

or $\mu = \underbrace{\begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}}_{\text{in } SU(1,1) \text{ up to scaler factor}} (-\lambda)$

So $\lambda \mapsto \mu$ is a diffeo of S^1 and the spectral curve is S^1.

Let $\mathcal{M}$ be the vector space generated by the edges of the grid subject to the relations from the squares. This is a module over $\mathcal{A} = \mathbb{C}[\lambda, \lambda^{-1}, \mu, \mu^{-1}]$ gen by u, v sat. relations as above.

Now
$\lambda \neq 0, \infty \iff \mu \neq k, k^{-1}$
$\underbrace{\mu \neq 0, \infty}_{\text{needed for } \lambda^m \mu^n} \iff \lambda \neq k, k^{-1}$

Suppose $\mathcal{M}$, $\mathcal{A}$ localized so that $\lambda - k, \mu - k, \lambda - k^{-1}, \mu - k^{-1}$ become invertible, result $\mathcal{M}', \mathcal{A}'$. It seems $\mathcal{M}'$ is a free module with gen. u over $\mathbb{C}[\lambda, \lambda^{-1}, (\lambda - k)^{-1}, (\lambda - k^{-1})^{-1}]$,

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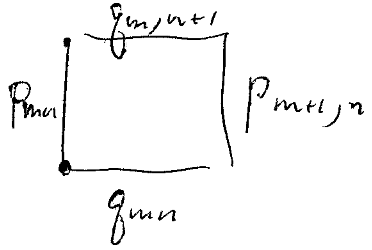
Problem: Describe positive definite + Krein inner products for the constant coeff grid. The F.T. picture gives a representation of $\mathcal{M}$ as functions on a circle. The function 1 corresp to $p_{00} \in \mathcal{M}$, so you need the inner products $(p_{00} | p_{mn})$ i.e. the moments $\int \lambda^m \mu^n \, d\rho$ where $d\rho$ is a pos. measure ~~on the λ circle~~ supported on the spectral curve.

Note $d\rho$ has to be ~~[illegible]~~ $\frac{d\lambda}{2\pi i\lambda}$. The point is that ~~u is a cyclic vector~~ the sequence $p_{m,0} \longleftrightarrow \lambda^m$ is orthonormal



$$\begin{pmatrix} p_{mn} \\ q_{mn}\end{pmatrix} = \lambda^m\mu^n \begin{pmatrix} u\\ v\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & -h\\ -\bar h & 1\end{pmatrix}\begin{pmatrix}\lambda^{m+1}\mu^n u\\ \lambda^m\mu^{n+1} v\end{pmatrix}$$

$$\boxed{\begin{pmatrix}u\\v\end{pmatrix} = \frac1k \begin{pmatrix}1&-h\\-\bar h&1\end{pmatrix}\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix}}$$

$$\boxed{\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \frac1k\begin{pmatrix}1&h\\ \bar h&1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}}$$

$$(k\lambda-1)u = hv$$
$$(k\mu-1)v = \bar h u$$

~~[illegible]~~ Consider M = the v.s. gen. by edges with relations from $\square$'s. M module over $A = \mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$. $= \mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]$, $u = p_{0,0}$, $v = q_{0,0}$, $\lambda^m\mu^n u = p_{mn}$ so u,v generate M as A-module. Also get $(k\lambda-1)(k\mu-1) = 1-k^2$ on M. So ~~in fact becomes~~ $(k\lambda-1)$ $(k\mu-1)$ are invertible on M, and M becomes a module over $\mathbb{C}[\lambda,\mu]/\big((k\lambda-1)(k\mu-1) = 1-k^2\big)$. Get something in the first quadrant $\boxed{\text{GOOD}}$. Clear that λ,μ injective on this first quadrant thing, so you should get M as localizing this cyclic sub-module of M over $\mathbb{C}[\lambda,\mu]$.

Conjecture is that the ~~first~~ quadrant $\mathbb{C}[\lambda,\mu]$-submodule is free of rank 1 over $\mathbb{C}[\lambda,\mu]$. Write it $\mathbb{C}[k\lambda-1, k\mu-1]$

$\mathbb{C}[X,Y]/(XY-1) =$