You need ~~some~~ better control. Let's begin with a ~~positive~~ Laurent polynomial fn. b, form $1+|b|^2$ which is a positive Laurent poly function, then find d ~~~~ poly in $\mathbb{C}[z]$ with $d(0)>0$, roots of d outside S^1, and $|d|^2 = 1+|b|^2$ on S^1.

$1+|b|^2$ Laurent poly real valued on S^1. So its roots are closed under $\lambda \mapsto \bar{\lambda}^{-1}$. ~~it is~~ ~~a finite~~

If $f = 1 + bb^*$ $\qquad b^*(z) = \overline{b(\bar{z}^{-1})}$

$$f(\lambda) = 0 \implies f(\bar{\lambda}^{-1}) = 0 \qquad \text{UFD.}$$

$$(z-\lambda)(z^{-1}-\bar{\lambda}) = z^{-1}(z-\lambda)(1-\bar{\lambda}z)$$

You would like to understand the construction of d, which should be simpler than the ~~construction on~~ splitting business. Do inverse scattering: Point is to start with a Laurent poly b arbitrary. Form

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^2 & d^l \end{pmatrix}$$

---

need big review.

Inverse transform. Given b a function on the circle, say Laurent poly, then $1+|b|^2$ can be uniquely written $1+|b|^2 = |d|^2$ where d is a ~~Laurent~~ poly in z, invertible on unit disk zeroes outside S^1, and $d(0) > 0$. ~~Then~~ Shifting $b \longmapsto z^n b$ does not affect d. so can assume b a poly in z. Then $\beta = \frac{b}{d}$ is analytic in D sat $|\beta(z)| < 1$ so it has a Schur expansion. ~~[illegible]~~ Since β is a ~~Keep on trying~~ rational function of z, this exp. ~~[illegible] Krein straight~~ should be finite.

Krein business. Given b (L. poly) you ~~consider~~ construct d and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ~~and~~ all over $\mathbb{C}[z,z^{-1}] = A$ and you ~~get~~ ~~some~~ form $M = \xi'_- A + \xi'_+ A = \xi_+ A + \xi_- A$. A-module with ~~[illegible]~~ indef herm. form type (1,1) over A. and volume form over A.

$$\underset{\sim}{K}(\xi'_- f + \xi'_+ g) = |f|^2 - |g|^2 = \underset{\sim}{K}(\xi_+ f + \xi_- g)$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\underset{\sim}{K}(\xi'_- f + \xi_- g) = \underset{\sim}{K}\left(\xi'_- f + (\xi'_- c + \xi'_+ d) g\right)$$

$$= \underset{\sim}{K}\left(\xi'_-(f+cg) + \xi'_+ dg\right) = |f+cg|^2 - |dg|^2$$

$$= \begin{pmatrix} f \\ g \end{pmatrix}^* \underbrace{\begin{pmatrix} 1 & c \\ \bar{c} & -1 \end{pmatrix}} \begin{pmatrix} f \\ g \end{pmatrix}$$

eigenvalues: $\lambda^2 + (-1 - |c|^2) = \lambda^2 - |d|^2 = 0$

$\lambda = \pm |d|$.

$$\begin{pmatrix} 1-\lambda & c \\ b & -1-\lambda \end{pmatrix}\begin{pmatrix} 1+\lambda \\ b \end{pmatrix} = \begin{pmatrix} \overset{1-|a|^2+|b|^2}{1-\lambda^2+bc} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & c \\ b & -1 \end{pmatrix}\begin{pmatrix} \lambda+1 & c \\ b & \lambda-1 \end{pmatrix}}_{\begin{pmatrix} \lambda+1+bc & \cancel{c}+c\lambda-\cancel{c} \\ b\lambda+\cancel{b}-\cancel{b} & bc-\lambda+1 \end{pmatrix}} = \begin{pmatrix} \lambda+1 & c \\ b & \lambda-1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

$$\lambda^2 = 1+bc$$

$$\begin{pmatrix} 1-\lambda & c \\ b & -(1+\lambda) \end{pmatrix}\begin{pmatrix} 1+\lambda \\ b \end{pmatrix} = \begin{pmatrix} 1-\lambda^2+bc \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & c \\ b & -1 \end{pmatrix}\begin{pmatrix} 1+\lambda & 1-\lambda \\ b & b \end{pmatrix} = \begin{pmatrix} 1+\lambda+cb & 1-\lambda+cb \\ \cancel{b}+b\lambda & \cancel{b}-b\lambda-\cancel{b} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^2+\lambda & \lambda^2-\lambda \\ b\lambda & -b\lambda \end{pmatrix} = \begin{pmatrix} 1+\lambda & 1-\lambda \\ \underbrace{b \quad\quad b}_{\text{orth}} \end{pmatrix}\begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$$

to make unit vectors. div. by SQRT $\left|1+\lambda\right|^2 + |b|^2 = 1+2\lambda+\lambda^2+bc$

$$= 2\lambda + 2\lambda^2$$

$$= 2\lambda(1+\lambda)$$

Do you learn anything? Recall that you have this ~~space~~ module $M = \xi'_- A + \xi'_+ A$ equipped with pos. def herm. form $\|\xi'_- f + \xi'_+ g\|^2 = |f|^2 + |g|^2$ and indef herm. form $K(\xi'_- f + \xi'_+ g) = \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & c \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$

~~So what goes~~ Diagonalize the Krein form

$$\begin{pmatrix} 1 & c \\ b & -1 \end{pmatrix} \begin{pmatrix} 1+|d| & 1-|d| \\ b & b \end{pmatrix} = \begin{pmatrix} \cancel{1}+|d| - \cancel{1} + |d|^2 & \cancel{1} - |d| - \cancel{1} + |d|^2 \\ b|d| & -b|d| \end{pmatrix}$$

$$= \begin{pmatrix} 1+|d| & 1-|d| \\ b & b \end{pmatrix} \begin{pmatrix} |d| & 0 \\ 0 & -|d| \end{pmatrix}$$

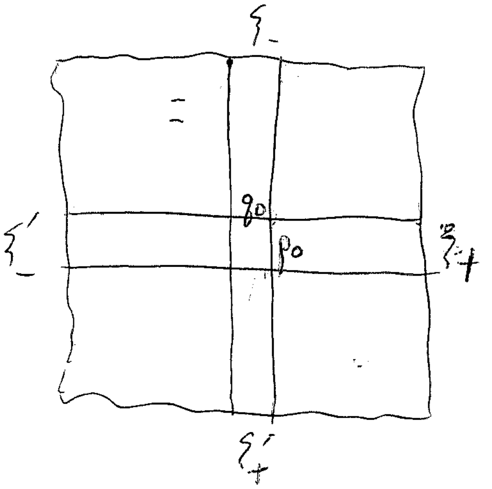
$$(1 \pm |d|)^2 + bc = \cancel{1} \pm 2|d| + |d|^2 - \cancel{1} + |d|^2$$

$$= 2|d|(\pm 1 + |d|) = 2|d|(|d| \pm 1)$$

$$\begin{pmatrix} \frac{|d|+1}{\cancel{2}} & \frac{|d|-1}{\cancel{2}} \\ b & -b \end{pmatrix} \qquad \frac{|d|+1}{\sqrt{2}\sqrt{|d|}\sqrt{|d|+1}}$$

$$\begin{pmatrix} \frac{|d|+1}{\sqrt{2|d|}\sqrt{|d|+1}} & \frac{|d|-1}{\sqrt{2|d|}\sqrt{|d|-1}} \\ \frac{b}{\sqrt{2|d|}\sqrt{|d|+1}} & \frac{-b}{\sqrt{2|d|}\sqrt{|d|-1}} \end{pmatrix}$$

$$\det = \frac{-2b|d|}{2|d|\sqrt{|d|^2-1}} = \frac{-b}{\sqrt{|b|^2}} = -\frac{b}{|b|}$$



~~splitting~~

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

splitting $E = (H_+\xi_+ + H_-\xi_-) \oplus (H_-\xi'_- + H_+\xi'_+)$

splitting $E = (H_+\xi'_- + H_+\xi_-) \oplus (H_-\xi_+ + H_-\xi'_+)$

OKAY. look at latter

$$(H_+ \;\; H_+) \overset{?}{\oplus} (H_- \;\; H_-)\begin{pmatrix} & \\ & \end{pmatrix} = (L^2 \;\; L^2)$$

$$(\xi'_- \;\; \xi_-)\begin{pmatrix}H_+\\ H_+\end{pmatrix} \overset{?}{\oplus} (\xi_+ \;\; \xi'_+)\begin{pmatrix}H_-\\ H_-\end{pmatrix}$$

$$\begin{pmatrix}H_+\\ H_+\end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}H_-\\ H_-\end{pmatrix} = (L^2 \;\; L^2)$$

$$\begin{pmatrix} Id_- & -\pi_- c \\ \pi_- b & Id_-\end{pmatrix} : \begin{pmatrix}H_-\\ H_-\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}H_-\\ H_-\end{pmatrix}$$

Look at the question of whether

$$p_0 \in (H_+\xi'_- + H_+\xi_-) \cap (H_-\xi_+ + H_-\xi'_+)$$

Question: Consider E with Krein form and the subspace $H_+\xi'_- + H_+\xi_-$. What does it mean for the Krein form to be nondegenerate on this subsp

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E'' \arrow[r] & 0 \\ & E'^{*} & E^{*} \arrow[l] & & \end{tikzcd}$$

What is the Krein form on $L^2\xi'_- + L^2\xi_-$

$$\underset{\sim}{K}(f\xi'_- + g\xi_-) = \underset{\sim}{K}\left(f\xi'_- + g(c\xi'_- + d\xi'_+)\right)$$

$$= \underset{\sim}{K}\left((f+gc)\xi'_- + gd\xi'_+\right) = |f+gc|^2 - |gd|^2$$

$$= |f|^2 + \bar{f}gc + \bar{g}\bar{c}f + |g|^2\underbrace{(|c|^2 - |d|^2)}_{-1} = \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{c} \\ c & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

We want to restrict ~~this~~ (integrated) form to $H_+\xi'_- + H_+\xi_-$

Given $f, g \in L^2$ to find $f_+, g_+ \in H_+$ such that

$$K(f\xi'_- + g\xi_-,\ f_0\xi'_- + g_0\xi_-) = K(f_+\xi'_- + g_+\xi_-,\ f_0\xi'_- + g_0\xi'_-)$$

for all $f_0, g_0 \in H_+$. This says simply that

$$f\xi'_- + g\xi_- \in f_+\xi'_- + g_+\xi_- + (H_+\xi'_- + H_+\xi_-)^\perp$$

$$\Uparrow$$

$$(L^2\xi'_- + L^2\xi_-) = (H_+\xi'_- + H_+\xi_-) + (H_-\xi'_- + H_-\xi_-)$$

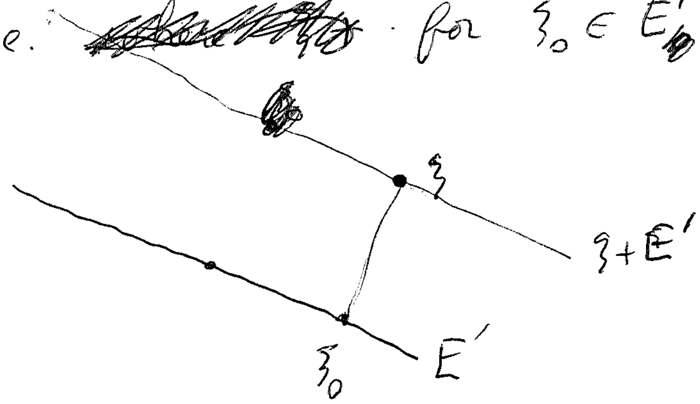
I want to continue this from yesterday. The idea: to see whether the indefinite herm. form yields the splitting. You have the indef form

$$\underset{\sim}{K}(f\xi'_- + g\xi_-) = \underset{\sim}{K}\left(f\xi'_- + g(c\xi'_- + d\xi'_+)\right)$$

$$= \underset{\sim}{K}\left((f+gc)\xi'_- + gd\,\xi'_+\right) = |f+gc|^2 - |dg|^2$$

So $\underset{\sim}{K}(f\xi'_- + g\xi_-) = \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$

You have the subspace $H_+\xi'_- + H_+\xi_-$ and the orthogonal space $H_-\xi_+ + H_-\xi'_+$. If you are in a Hilbert space situation you argue by minimizing, using convexity + completeness that the Hilbert space is the sum of a closed subspace and its orthogonal complement. In the indef case you can look for a stationary point. ~~Precisely~~ More precisely. Suppose $E' \subset E$ and $\xi \in E$, you look for stationary point of K on the coset $\xi + E'$, i.e. for $\xi_0 \in E'$ such that $\xi - \xi_0 \perp E'$.



Try to use non degeneracy of K.

$$\begin{tikzcd} & & & \xi + E' \\ 0 \arrow[r] & E' \arrow[r] \arrow[d, "K|E'"] & E \arrow[r] \arrow[d, "K"] & E/E' \arrow[r] \arrow[u] & 0 \\ & E'^* & E^* \arrow[l] & & \end{tikzcd}$$

Hope: $K|E'$ is an isomorphism. This might be true because the ~~underlying Hilbert spaces~~ underlying topological vector spaces of E, E' are reflexive, better can be ~~endowed~~ made into Hilbert spaces. The non degeneracy ~~might~~ should follow from positivity.

Look at K on E'

First of all $E = L^2\xi'_- + L^2\xi_-$ with $K(f\xi'_- + g\xi_-) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$ and $E' = H_+\xi'_- + H_+\xi_-$, so that E/E' cbiw $H_-\xi'_- + H_-\xi_-$, which is orthogonal to E'. So you only have to show K nondeg. (strongly) on E', i.e. that $E' \xrightarrow{K} E'^*$ is an isom.

So you look at the pairing K on E'.

$$K(f\xi'_- + g\xi_-) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

In terms of the Hilbert space $\begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$ this is the hermitian form assoc. to the hermitian operator of mult. by $\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}$, and non degenerate follows from factoring:

$$\begin{pmatrix} f \\ g \end{pmatrix} \longmapsto \begin{pmatrix} f \\ -g \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} 1 & \bar b \\ b & 1 \end{pmatrix}}_{1+X} \begin{pmatrix} f \\ -g \end{pmatrix} \qquad X \text{ skew adj.}$$

Restricting form from $\begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$ to $\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$ you get the operator

$$\begin{pmatrix} \mathrm{Id}_+ & \pi_+ \bar b \varepsilon_+ \\ \pi_+ b \varepsilon_+ & -\mathrm{Id}_+ \end{pmatrix}$$

The PRINCIPLE is that all the splitting results follow from nondegeneracy, which is established using Hilbert space inner product.

<u>Repeat</u>: $E = L^2\xi_+ \oplus L^2\xi_-$ w. pos. def. form:

$$\|f\xi_+ + g\xi_-\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

Check.

$$f\xi_+ + g\xi_- = f\left(\tfrac{1}{d}\xi'_- + \tfrac{b}{d}\xi_-\right) + g\xi_-$$

$$= \left(f\tfrac{1}{d}\right)\xi'_- + \left(f\tfrac{b}{d} + g\right)\xi_-$$

$$\|f\xi_+ + g\xi_-\|^2 = \int \left|f \overset{\delta}{\boxed{\tfrac{1}{d}}}\right|^2 + \left|f \overset{\beta}{\boxed{\tfrac{b}{d}}} + g\right|^2$$

$$= \int |f|^2 \underbrace{(|\delta|^2 + |\beta|^2)}_{1} + \bar f \bar\beta g + \bar g f \beta + |g|^2$$

$$= \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

to get splitting of E into $E' = H_+\xi_+ + H_-\xi_-$ and its orthogonal complement you need only the non degeneracy of this form on E'. The only point is that $\|\beta\|_\infty < 1$, since E' is not a closed subspace of E otherwise. Then get $\|f\xi_+ + g\xi_-\|^2 \geqslant \varepsilon(\|f\|^2 + \|g\|^2)$. Now you know that

$$\begin{array}{ccccc} H_+\xi_+ + H_-\xi_- & \hookrightarrow & L^2\xi_+ + L^2\xi_- & \twoheadrightarrow & H_-\xi_+ + H_+\xi_- \\ \| & & \| & & \| \\ E' & \hookrightarrow & E & \twoheadrightarrow & E/E' \end{array}$$

You should calculate $(E')^\perp$. $f\xi_+ + g\xi_- \in E'^\perp$

$$\iff \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f_+ \\ g_- \end{pmatrix} = 0 \qquad \forall\, f_+ \in H_+,\ g_- \in H_-$$

$$\cancel{\left(\bar f + \beta \bar g \quad \bar f \beta \right)} \qquad (\bar f_+ \ \ \bar g_-)$$

$$\iff \int \begin{pmatrix} f_+ \\ g_- \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \int \begin{pmatrix} f_+ \\ g_- \end{pmatrix}^* \begin{pmatrix} f + \bar\beta g \\ \beta f + g \end{pmatrix} = 0$$

i.e. $\begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \in \begin{pmatrix} H_- \\ H_+ \end{pmatrix}$ apply $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

i.e. $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \in \begin{pmatrix} H_- \\ H_+ \end{pmatrix}$

$$\begin{pmatrix} f \\ g \end{pmatrix} \in \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} H_- \\ H_+ \end{pmatrix}$$

Start again: $E' = H_+\xi_+ + H_-\xi_- \subset E = L^2\xi_+ + L^2\xi_-$

equipped with herm. form $(g\xi_- | f\xi_+) = \bar{g}\beta f$

$$\|f\xi_+ + g\xi_-\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$E'^\perp = \left\{ f\xi_+ + g\xi_- \;\middle|\; \int \begin{pmatrix} f_+ \\ g_- \end{pmatrix}^* \underbrace{\begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}}_{\begin{pmatrix} f+\bar\beta g \\ \beta f + g \end{pmatrix}} = 0 \quad \forall \begin{pmatrix} f_+ \\ g_- \end{pmatrix} \in \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \right\}$$

$$E'^\perp = \left\{ f\xi_+ + g\xi_- \;\middle|\; \begin{pmatrix} f \\ g \end{pmatrix} \in \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \ni \begin{pmatrix} f+\bar\beta g \\ \beta f + g \end{pmatrix} \in \begin{pmatrix} H_- \\ H_+ \end{pmatrix} \right\}$$

analyze the condition

$$\begin{pmatrix} f + \frac{c}{a} g \\ \frac{b}{d} f + g \end{pmatrix} \in \begin{pmatrix} H_- \\ H_+ \end{pmatrix} \iff \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \in \begin{pmatrix} H_- \\ H_+ \end{pmatrix}$$

$$\iff (f \;\; g)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (H_- \;\; H_+)$$

$$\iff (f \;\; g)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \in H_-\xi'_- + H_+\xi'_+$$

$$\iff f\xi_+ + g\xi_- \in H_-\xi'_- + H_+\xi'_+$$

There should be a better way to ~~do~~ understand this calculation. The role of a, d is funny. unclear

$$E'' \xrightarrow{\sim} E/E'$$

$$\begin{tikzcd}
E' = \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \arrow[r, hook, "\varepsilon"] \arrow[d, "\simeq"'] & E = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \arrow[r, two heads, "\pi"] \arrow[d, "\begin{pmatrix} 1 & \frac{c}{a} \\ \frac{b}{d} & 1 \end{pmatrix}"] & \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = E'' \\
\begin{pmatrix} H_+ \\ H_- \end{pmatrix} & \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \arrow[l, "\pi"'] & \begin{pmatrix} H_- \\ H_+ \end{pmatrix} \arrow[l, hook', "\varepsilon"']
\end{tikzcd}$$

$$\begin{pmatrix} 1 & \frac{c}{a} \\ \frac{b}{d} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\frac{c}{a} \\ -\frac{b}{d} & 1 \end{pmatrix} \underbrace{\frac{1}{1-\frac{bc}{ad}}}_{} = \begin{pmatrix} ad & -cd \\ -ab & ad \end{pmatrix}$$

with ad written over the factor: $ad = \frac{1}{1-\frac{bc}{ad}}$

$$= \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

$$\text{So } (E')^{\perp} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} H_- \\ H_+ \end{pmatrix}$$

$$\simeq \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} H_- \\ H_+ \end{pmatrix} \xrightarrow{\sim} \underbrace{\begin{pmatrix} \xi_+ & \xi_- \end{pmatrix} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}}_{\begin{pmatrix} \xi_-' & \xi_+' \end{pmatrix}} \begin{pmatrix} H_- \\ H_+ \end{pmatrix}$$

This looks pretty clear, namely you have interpreted orthog. projection relative to $\| \, \|^2$ on E as yielding the desired splitting.

indefinite case $E' = H_+\xi_-'^{*} + H_+\xi_- \subset E = L^2\xi_-' + L^2\xi_-$

with $K(f\xi_-' + g\xi_-) = \int \begin{pmatrix}f\\g\end{pmatrix}^{*}\begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}$

$$\begin{tikzcd}
E' \arrow[r, hook] \arrow[d, phantom, "\parallel"] & E \arrow[r, two heads] \arrow[d, phantom, "\parallel"] & E'' \xrightarrow{\sim} E/E' \arrow[d, phantom, "\parallel"] \\
\begin{pmatrix}H_+\\H_+\end{pmatrix} \arrow[r, hook, "\varepsilon_+"] \arrow[d, "\begin{pmatrix}1_+ & \pi_+\bar b\\ \pi_+ b & -1_+\end{pmatrix}", "\simeq"'] & \begin{pmatrix}L^2\\L^2\end{pmatrix} \arrow[r, "\pi_-"] \arrow[d, "\begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}"] & \begin{pmatrix}H_-\\H_-\end{pmatrix} \\
\begin{pmatrix}H_+\\H_+\end{pmatrix} & \begin{pmatrix}L^2\\L^2\end{pmatrix} \arrow[l, "\pi_+"'] & \begin{pmatrix}H_-\\H_-\end{pmatrix} \arrow[l, hook', "\varepsilon_-"']
\end{tikzcd}$$

this is an isom because $1+X$, $X^*=-X$ is always invertible on Hilbert space

So the hermitian form restricted to $\begin{pmatrix}H_+\\H_+\end{pmatrix}$ is (strongly) nondegenerate. Conclude $E = E' \oplus$ orth comp of E' rel K, also that this orth. comp.

$$E'^{\perp K} = \begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}^{-1}\begin{pmatrix}H_-\\H_-\end{pmatrix}$$

$$\begin{pmatrix}1 & c\\ b & -1\end{pmatrix}^{-1} = \begin{pmatrix}-1 & -c\\ -b & 1\end{pmatrix}\frac{1}{-1-bc} = \begin{pmatrix}1 & c\\ b & -1\end{pmatrix}\frac{1}{ad} = \begin{pmatrix}\frac{1}{d} & \frac{c}{d}\\ \frac{b}{d} & -\frac{1}{d}\end{pmatrix}\frac{1}{a}$$

$$= \begin{pmatrix}\frac{1}{d} & \frac{c}{d}\\ \frac{b}{d} & -\frac{1}{d}\end{pmatrix}\begin{pmatrix}H_-\\H_-\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & -\frac{c}{d}\\ \frac{b}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}H_-\\H_-\end{pmatrix}$$

$$E'^{\perp K} = (\xi_-' \ \ \xi_-)\begin{pmatrix}\frac{1}{d} & -\frac{c}{d}\\ \frac{b}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}H_-\\H_-\end{pmatrix} = (\xi_+ \ \ \xi_+')\begin{pmatrix}H_-\\H_-\end{pmatrix}$$

So you understand a bit better.

Now attack the bigger setting where E appears as max isotropic subspace of a rank 4 Krein space. Problem is natural half spaces. Ideal situation?

~~You have to~~ You ~~are~~ should be almost finished here.

Form $L^2\xi_+ \oplus L^2\xi_- \oplus L^2\xi'_- \oplus L^2\xi'_+$

Use $+1 \quad -1 \quad +1 \quad -1$ self adj

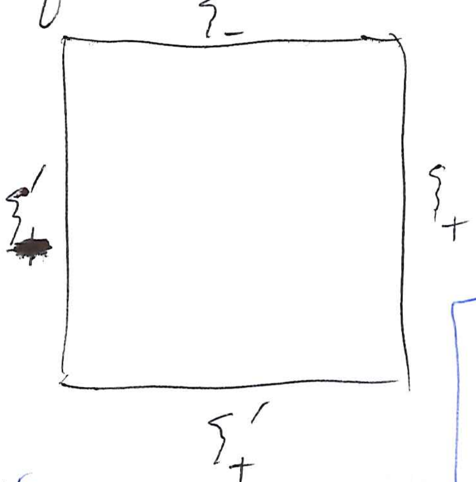
to define K, then E sits inside as an isotropic subspace as the graph of

$$T : \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

or as the graph of

$$S : \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

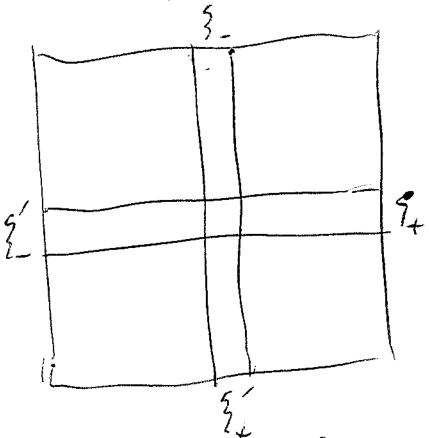
How are you supposed to think about this? Square describing E. Possible viewpt is that ~~you have~~ E appears in various ways as a correspondence



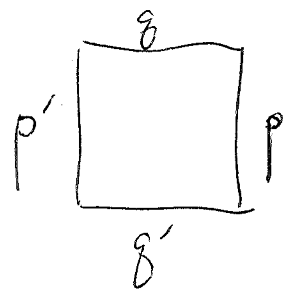
Some ideas. ~~You form~~ Certain things are fixed, such as $\mathcal{L} = L^2\xi_+ + L^2\xi_- + L^2\xi'_- + L^2\xi'_+$ and the indefinite herm. form. The isotropic subspace E depends on $T, S, (h_n)$ so it can vary. ~~Suppose~~ You maybe ~~really~~ want splittings of E to arise from something fixed in $\mathcal{L}$, maybe each of $\xi_\pm$, $\xi'_\pm$ times some $H_\pm$

So the guess is that some fixed constructed ~~...~~ using the splitting $L^2 = H_+ \oplus H_-$ will be in

get back into the spirit of things. Given $E = L^2\xi_+ + L^2\xi_- =$ etc. (there are 4 bases corresp. to the corners of the square)



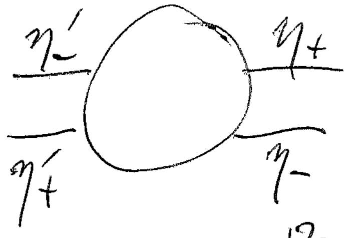
Go back to



Somehow you must distinguish vectors and covectors. ~~How much~~ How much of the pattern desired is known. Basic should be the idea of ~~an~~ a "Lagrangian" subspace of a Krein space. Think maybe of a box with 4 terminals and the hermitian form as related to power, say energy out at right − energy in at the left. ~~So how can this work?~~

~~Say~~ The states of the box are determined by the numbers at the terminals.

You should start with $W \subset \mathbb{C}^4$ so there are 4 linear functionals on W say $\eta_\pm$ on the right and $\eta'_\mp$ on the left.



~~...~~ The η's are the dual bases for obvious basis on $\mathbb{C}^4$.

Power ~~$|\eta_+(w)|^2 \geq |\eta_-(w)$~~

$$|\eta_+ w|^2 - |\eta_- w|^2 - |\eta'_- w|^2 + |\eta'_+ w|^2$$

2 diml subspace W of $\mathbb{C}^4$

$\eta_+(\xi)$ So now you know where the difficulty lies. $\psi : \psi_n = \begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix}$

You want to analyze solutions of DE at given $z \in S$! Have limits

$$\lim_{n \to \pm\infty} \begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix}$$

Have a 2diml space W_z of solutions and these limits give 4 linear functionals on W_z.

Repeat. Consider the DE in the form

$$\psi_n = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\psi_{n-1} \qquad \psi_n = \begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix}$$

Solutions for a 2diml space W_z. If $\psi \in W_z$ then $\lim_{n\to\infty}$ or $n \to -\infty$ ψ_n exists, thereby giving 4 lin. fnls on W_z.

Correlate: ψ is a linear fnl on $M/(z-u)M$ and $\lim_{n\to\infty}\psi_n = \psi\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}$. This space of solns. W_z is ∴ dual to my E. So

$W_z = (M/(z-u)M)^*$, $\xi_\pm$ $\xi'_\mp \in M$

Continue W_z is a 2diml space with 4 linear functionals $\xi_\pm$ $\xi'_\mp$ (arising from these elements of M). Since $\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$

~~[illegible]~~ Go back to $\mathcal{M}$. $\mathcal{M}$ has 4 generators $\xi_\pm, \xi'_\mp$ over $\mathcal{a}$ so it's naturally a ~~quotient~~ of $\mathcal{a}^4$. Corresp to $W_z \subset \mathbb{C}^4$. But how do you make sense of W_z being isotropic?

Puzzle: ~~Maximal isot. subspace.~~

Let W be a max. isot. subspace of a Krein space V.

$W = W^\perp$. $\qquad 0 \longrightarrow W \subset V \longrightarrow V/W \longrightarrow 0$

$V = \begin{pmatrix} Y \\ Y \end{pmatrix}$ with herm. form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Picture of V namely a Hilbert space with $\mathbb{Z}/2$ grading.

$$W \xhookrightarrow{\binom{a}{b}} \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, w^* (a^* \; b^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \;\; \forall w \right\}$$

$$= \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, a^* y_1 = b^* y_2 \right\}.$$

in the max. isot. case $W = \Gamma_u$ a unitary on Y.

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r, hook, "\binom{a}{b}"] \arrow[d, "0"] & V \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] & \\ 0 & W \arrow[l] & V \arrow[l, "(a^* \; b^*)"'] & W^0 \arrow[l, hook'] \end{tikzcd}$$

So ~~[illegible]~~ you learn that ?

Review. $\quad M \quad \mathbb{C}[u, u^{-1}]$ mod gen. by gen solns of D.E.

$\qquad\;\; \mathcal{M} \quad \mathcal{a}$

$$\psi \in \underbrace{\left(\mathcal{M} / (z-u)\mathcal{M} \right)^*}_{W_z} \qquad \text{soln for eigenv. } z$$

have $\xi_\pm \quad \xi'_\mp \in \mathcal{M}$

whence $\psi \longmapsto \psi \big|_{\xi_\pm, \, \xi'_\mp}$ $\qquad$ ~~first maps~~ map $W_z \longrightarrow \mathbb{C}^4$

$$\begin{pmatrix} \xi_+ \\ \xi_{*} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \psi\xi_+ \\ \psi\xi_- \end{pmatrix} = \begin{pmatrix} a_z & b_z \\ c_z & d_z \end{pmatrix} \begin{pmatrix} \psi\xi'_- \\ \psi\xi'_+ \end{pmatrix}$$

so W_z isot. for ~~[illegible]~~ indef. form $|\psi\xi_+|^2 \cancel{+} |\psi\xi_-|^2 \cancel{+} \; +$

~~What you need~~ You need to understand something like a (quotient) isotropic space. ~~To ?~~ ~~?~~ For any $\mathcal{Q}$ module P ~~you can form~~ you have $\mathrm{Hom}_{\mathcal{a}}(M, P) =$ soln of DE with values in P. and this embeds in P^4.

Better fix $z \in S^1$ look at $\left(M/(z-u)M\right)^* = W_z$ $\subset \mathbb{C}^4$ as isot. subsp for $|\psi\xi_+|^2 - |\psi\xi_-|^2 - |\psi\xi_-'|^2 + |\psi\xi_+'|^2$

So $\mathbb{C}\xi_+ \oplus \mathbb{C}\xi_- \oplus \mathbb{C}\xi_-' \oplus \mathbb{C}\xi_+' \twoheadrightarrow W_z$. The kernel is generated by the ~~linear relation~~ elements

$$\xi_+ - a\xi_-' - b\xi_+' \qquad\qquad 1 - |a|^2 + |b|^2 = 0$$

$$\xi_- - c\xi_-' - d\xi_+' \qquad\qquad -1 - |c|^2 + |d|^2 = 0$$

so the kernel is isotropic. Good $\begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix}$ $\quad |d|^2 - |c|^2 = 1$, $\quad |d|^2 = 1 + |c|^2$

There seems to be a lemma here, namely ?

~~To begin with~~

~~Look at a Krein for~~

~~Suppose Let~~

Consider a Krein space $\quad V = \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

and a max. isot. subspace $\quad X = \begin{pmatrix} 1 \\ u \end{pmatrix} Y \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

$$X \hookrightarrow \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \twoheadrightarrow V/X$$

Look: A sequilinear pairing $\quad E \times F \xrightarrow{(\xi|\eta)} \mathbb{C}$ is equiv. to a linear map $F \longrightarrow \overline{E}^* = \mathrm{Hom}_{\mathbb{C}}(\overline{E}, \mathbb{C})$?

~~Another~~ $\quad F \times E \xrightarrow{(\eta|\xi)} \mathbb{C} \qquad (c\eta|c'\xi) = \bar c(\eta|\xi)c'$

$F \xrightarrow{\sigma} E^\vee \qquad E$ Hilb $\qquad E^\vee \simeq \overline{E}$

Consider a Krein space; can assume its in the form $\begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix}$ Y Hilb. space where the indef. form given by the s.a. of $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. An iso subspace has the form $\binom{1}{u}Y = X$. What properties does $\begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix}\Big/\binom{1}{u}Y$ have? You are inclined to write

$$\begin{tikzcd}
X \arrow[r] & \begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix}\Big/X \arrow[dll, dashed] \\
X^{\dagger} & \begin{pmatrix}Y\\ \oplus\\ Y\end{pmatrix}^{\dagger} \arrow[l] & X^{0}
\end{tikzcd}$$

Is it true that the ~~quotient~~ $\begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix}\Big/X$ is naturally isom to X.

$$\begin{tikzcd}
X \arrow[r, "\binom{a}{b}"] \arrow[d, "0"] & \begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix} \arrow[r] \arrow[d, "\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}"] & \begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix}\Big/X \\
X & \begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix} \arrow[l, "(a^* \ b^*)"'] & X^{0} \arrow[l]
\end{tikzcd}$$

The correct assertion is that the quot $\begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix}\Big/X$ is naturally isom to $X^{\dagger}$ the conjugate dual.

Focus. I think you have the following situation. $\mathcal{A}$-module $\mathcal{M}$ quotient of ~~$\mathcal{A}$~~

$\mathcal{A}^4 = \mathcal{A}\xi_+ \oplus \mathcal{A}\xi_- \oplus \mathcal{A}\xi'_- \oplus \mathcal{A}\xi'_+$ with the indefinite herm. form.

$$K(f\xi_+ + g\xi_- + j\xi'_- + k\xi'_+) = |f|^2 - |g|^2 - |j|^2 + |k|^2$$

Put another way, ~~then~~ if you use the pos. def. inner product $\ni$ $\xi_\pm$ $\xi'_\mp$ are orthonormal, then K is the hermitian form corresp to the operator $\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix}$.

$\mathcal{M}$ is the quotient of $\mathcal{A}^4$ by the rank 2 submodule generated by the elts.

$$\xi_+ - a\xi'_- - b\xi'_+ \xrightarrow{K} 1 + |a|^2(-1) + |b|^2(+1)$$

$$\xi_- - c\xi'_- - d\xi'_+ \qquad -1 + |c|^2(-1) + |d|^2(+1)$$

$$K(\xi_+) = 1 - |a|^2 + |b|^2 = 0$$

$$K(\xi_-) = -1 - |c|^2 + |d|^2 = 0$$

$$K(\xi_-, \xi_+) = \bar{c}a(-1) + \bar{d}b(1) = -\bar{c}a + \bar{d}b = 0.$$

So now you have ~~another~~ an isotropic submodule $\mathcal{J}$ inside $\mathcal{A}^4$. We then get some relation between $\mathcal{J}$ and $\mathcal{A}^4/\mathcal{J} = \mathcal{M}$.

$$\mathcal{J} \hookrightarrow \mathcal{A}^4 \twoheadrightarrow \mathcal{M}$$

Do simply first

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "0"'] & Z \arrow[r, two heads] \arrow[d] & Z/W \arrow[dll, dashed] \\ W^* & Z^* \arrow[l, two heads] & \end{tikzcd}$$

So you get a conjugate linear map

$$Z/W \longrightarrow W^\vee$$

when W isotropic.

Let Z be a complex vector space equipped equipped with a ~~sesquilinear~~ hermitian form $K(z_1, z_2)$; enough to give $K(z) = K(z,z)$. W subspace of Z isotropic

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "0"] & Z \arrow[d, "\text{c-lin}"', "z \mapsto K(z,-)"] \\
W^\vee & Z^\vee \arrow[l, two heads]
\end{tikzcd}$$

naturally get an induced sesquilinear ~~form~~ pairing

$K(z_1 + W, w) \qquad Z/W \times W \longrightarrow \mathbb{C}$

equiv. a conj linear map $Z/W \longrightarrow W^\vee$. Now if W equipped with a Hilb. space inner prod. get $W \xrightarrow{\sim} W^\vee$ conj. linear isom. So you get a canonical map $Z/W \longrightarrow W$.

Example

$$\begin{tikzcd}
X \arrow[r, "\binom{a}{b}"] \arrow[d, "0"] & Y \oplus Y \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] & Y \oplus Y / \cancel{X} \\
X & Y \oplus Y \arrow[l, "(a^* \ b^*)"'] & X^\circ
\end{tikzcd}$$

Let Z have Krein form $K(z_1, z_2)$ let W be isotropic Then you get $K(z_2 + W, w)$ pairing $\overline{Z/W} \otimes W \to \mathbb{C}$ whence you have $\overline{Z/W} \longrightarrow \overline{W^\vee}$. When W has a Hilb. ot. $\overline{W^\vee} = W$ so you have $\overline{(Z/W)} \to W$.

Thus
$$\begin{tikzcd} W \arrow[r, hook, "ct"] \arrow[d, "0"'] & Z \arrow[r, two heads] \arrow[d, "K"] & Z/W \arrow[dll] \\ W' & Z \arrow[l, "ct^*"'] & \end{tikzcd}$$
K s.a. op corresp to Krein form

So what seems to happen is that when W is isotropic for the indef. form, then in <u>general</u> you get an anti-linear map $Z/W \to \overline{W'}$

~~So let's continue ε at a time~~

Go back to $Z = L^2\xi_+ \oplus L^2\xi_- \oplus L^2\xi'_- \oplus L^2\xi'_+$

$$K(f\xi_+ + g\xi_- + j\xi'_- + k\xi'_+) = \int(|f|^2 - |g|^2 - |j|^2 + |k|^2)$$

~~[illegible]~~ Now

When to start?

$$\eta \hookrightarrow a\xi_+ + a\xi_- + a\xi'_- + a\xi'_+ \twoheadrightarrow m$$
$$\parallel$$
$$a(\xi_+ - a\xi'_- - b\xi'_+) + a(\xi_- - c\xi'_- - d\xi'_+) \qquad \cancel{\bar{a}c + \bar{b}d}$$

So

$$\begin{tikzcd} X \arrow[r, "\binom{a}{b}"] \arrow[d, "a^*a - b^*b = 0"'] \arrow[drr] & Y \oplus Y \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] & {} \\ X & Y \oplus Y \arrow[l, "(a^* \; b^*)"'] & X^\circ \arrow[l] \end{tikzcd}$$

$$X^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} x = 0 \right\}$$

$$y_1^* a x - y_2^* b x \qquad \forall x \in X$$

$$(a^* y_1 - b^* y_2)^* x$$

$$\therefore X^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| a^* y_1 = b^* y_2 \right\}.$$

~~[illegible]~~

Start with X an isotropic subspace of a Krein space. Can assume the Krein space of the form $\begin{matrix} Y_1 \\ \oplus \\ Y_2 \end{matrix}$ where Y_i is a Hilbert space and the Krein form corresp. to ε, X isot. means you have $X \xrightarrow{\binom{a}{b}} \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

$$\begin{tikzcd}
X \arrow[r, "\binom{a}{b}"] & Y_1 \oplus Y_2 \arrow[d, "s"] & \\
X' & (Y_1 \oplus Y_2)' \arrow[l, "(a' \; b')"'] & X^0 \arrow[l]
\end{tikzcd}$$

Point somehow is that when X is an isotropic subspace of Z there is a map $X \hookrightarrow X^0$ leading to "symplectic quotient" X^0/X also a map $Z/X \longrightarrow X'$? How

$$\begin{tikzcd}
X \arrow[r, hook] & Z \arrow[r, two heads] \arrow[d, "\simeq"] & Z/X \\
X' & Z' \arrow[l, two heads] & X^0 \arrow[l, hook']
\end{tikzcd}$$

get induced maps $X \to X'$ $\quad X^0 \longrightarrow Z/X$ in general, so if you split the exact sequences you end up with usual 2×2 matrix. If X isotropic, then you get $0 \subset X \subset X^\perp \subset Z$

Take a filtration

$$\begin{matrix} X & X^{\perp} & X^{\perp}/X \\ X & Z & Z/X \\ 0 & Z/X^{\perp} & Z/X^{\perp} \end{matrix}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X^{\perp} \arrow[r] \arrow[d] & X^{\perp}/X \arrow[d] \\ X \arrow[r] \arrow[dr, "0"'] & Z \arrow[r] \arrow[d] & Z/X \arrow[d] \\ & Z/X^{\perp} \arrow[r, equal] & Z/X^{\perp} \end{tikzcd}$$

Look $Z = A \oplus B \oplus C$

Z has 2 splitting

$$Z = A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

which should give rise to a 2×2 matrix + inverse.

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[d, "\binom{1}{0}"] & C \arrow[d, "\binom{0}{1}"] \\ A \oplus B \arrow[r, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] & B \oplus C \end{tikzcd} \qquad \begin{tikzcd} A & C \arrow[l, "0"'] \\ A \oplus B \arrow[u, "(1\ 0)"] & B \oplus C \arrow[l, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"'] \arrow[u, "(0\ 1)"'] \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r, shift right, "c"'] \arrow[d, shift right, "a"'] & Q \arrow[l, shift right, "\beta"'] \arrow[d, shift right, "\delta"'] \\ R \arrow[r, shift right, "\gamma"'] \arrow[u, shift right, "\alpha"'] & S \arrow[l, shift right, "b"'] \arrow[u, shift right, "d"'] \end{tikzcd} \qquad \begin{tikzcd} P \oplus S \arrow[r, shift left, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"] & R \oplus Q \arrow[l, shift left, "\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}"] \end{tikzcd}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$a\beta + b\delta = 0$$

$$\delta c + \gamma a = 0$$

So what do you learn? X ~~max isotropic~~ Lagrangian in $Z = \begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix}$ Krein.

means $X = X^{\perp} \overset{\text{def}}{=} \{ z \mid K(x,z) = 0 \ \ \forall x \}$

$$\begin{tikzcd}
X \arrow[r, hook] & Z \arrow[r, two heads] \arrow[d, "S"] & Z/X \\
X' & Z' \arrow[l] & X^{0} \arrow[l, hook']
\end{tikzcd}$$

If $X \to Z \to Z' \to X'$ is zero, then get maps

So. $X^{\perp} = \{ z \mid \text{Image of } z \text{ in } Z' \text{ lies in } X^{0} \}$.
Get maps $X \longrightarrow X^{0}$, $Z/X \longrightarrow X'$. In fact $Z/X \longrightarrow Z/X^{\perp} \longrightarrow X'$

$$Z/X^{\perp} = Z'/X^{0}$$

Finite dims. Z $\mathbb{C}$-vector space with Krein form. non deg herm. form of type n,n. Then you can polarize getting a polarized Hilbert space with Krein form given by ε. So $Z = Y_+ \oplus Y_-$. a max isot. subspace has form $X = \binom{1}{u} Y_+$ where $u : Y_+ \xrightarrow{\sim} Y_-$ is unitary.

$$\begin{tikzcd}
X \arrow[r, "\binom{1}{u}"] & \begin{smallmatrix} Y_+ \\ \oplus \\ Y_- \end{smallmatrix} \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] & Z/X \\
X & \begin{smallmatrix} Y_+ \\ \oplus \\ Y_- \end{smallmatrix} \arrow[l, "(1 \ u^{-1})"'] & X^{0} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\binom{1}{u}"] \arrow[d, "?\ \|"] & \begin{smallmatrix} Y_+ \\ \oplus \\ Y_- \end{smallmatrix} \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] & Z/X \arrow[d, "\|\ ?"] \\
X^{0} \arrow[r] & \begin{smallmatrix} Y_+ \\ \oplus \\ Y_- \end{smallmatrix} \arrow[r, "(1 \ u^{-1})"] & X
\end{tikzcd}$$

So it seems that a Lagrangian subspace of a Krein space is also a quotient space naturally.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "j"] \arrow[d, "S"] & Z \arrow[r] \arrow[d, "S"] & Z/X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X^0 \arrow[r] & Z' \arrow[r, "j'"] \arrow[d, phantom, "\|"] & X' \arrow[r] \arrow[d, phantom, "\|"] & 0 \\ & & Z \arrow[r, "j^*"] & X & \end{tikzcd}$$

from Hilb space structures

Z has herm. form $K(\xi_1, \xi_2)$, non degenerate in the sense that $\xi \longmapsto (\xi_1 \mapsto K(\xi_1, \xi))$ from Z to Z' = antilinear maps $f: Z \to \mathbb{C}$ is an isom. Assume K type n,n i.e. assoc. to polarization $Z = Y_+ \oplus Y_-$ $\therefore K(\xi_1, \xi_2) = (\xi_1 | \varepsilon \xi_2)$.
A max. isot. X in Z has form $\binom{1}{u} Y_+$ where $u: Y_+ \xrightarrow{\sim} Y_-$ is unitary.

Start again. Assume Z dim $2n$ equipped with K herm of type n,n. X isotropic of dim n. Let Z' = anti dual of Z, also for X'. Then

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "S"'] & Z \arrow[r] \arrow[d, "S"', "K"] & Z/X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X^0 \arrow[r] & Z' \arrow[r] & X' \arrow[r] & 0 \end{tikzcd}$$

so $Z/X \xrightarrow{\sim} X'$ is canonically the anti-dual of X

Take $Z = \mathbb{C}\xi_+ \oplus \mathbb{C}\xi_- \oplus \mathbb{C}\xi'_- \oplus \mathbb{C}\xi'_+$

$$X = \mathbb{C}(\xi_+ \quad - a\xi'_- + b\xi'_+)$$
$$\mathbb{C}(\xi_- \quad - c\xi'_- - d\xi'_+)$$

Better to take unitary graph.

$$Z = \begin{matrix} Y_- \\ \oplus \\ Y_+ \end{matrix} \qquad X = \begin{pmatrix} 1 \\ u \end{pmatrix} Y$$

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "\begin{pmatrix} 1 \\ -S \end{pmatrix}"] & \begin{matrix} Y_- \\ \oplus \\ Y_+ \end{matrix} \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] & Q & \\
& & \begin{matrix} Y_- \\ \oplus \\ Y_+ \end{matrix} \arrow[r, "(1 \; -S^{-1})"] & Y \arrow[r] & 0
\end{tikzcd}$$

Z is some sort of extension of X by X' the conjugate dual of X. hyperbolic model for Krein space

You also ~~~~ need half spaces, Connes' F. This involves the circle

Z type (n,n) hermitian space

X isot. subspace of dim n.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook] \arrow[d] & Z \arrow[r, two heads] \arrow[d, "\cong"] & Z/X \arrow[r] & 0 \\ 0 \arrow[r] & X^0 \arrow[r, hook] & \overline{Z'} \arrow[r, two heads] & \overline{X'} \arrow[r] & 0 \end{tikzcd}$$

Let Z be a ~~spa~~ complex v.s. with nondeg. herm. form, X ~~and isotropic~~ a subspace, $X^\perp = \{ z \mid K(z,x) = 0 \ \forall x \}$. Ass. X isotropic: $X \subset X^\perp$. Then

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook, "j"] \arrow[d] & Z \arrow[r] \arrow[d, "\wr S"] & Z/X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X^0 \arrow[r] & Z^* \arrow[r, "j^*"] & X^* \arrow[r] & 0 \end{tikzcd}$$

~~so we get~~ where Z^* = conj dual. Since $X^\perp \subset Z$ is the subspace corresp to $X^0 \subset Z^0$ under $Z \xrightarrow{\sim} Z^*$, ~~the~~ we have $X^\perp \xrightarrow{\sim} X^0$. Serpent

$$0 \longrightarrow X^0/X \longrightarrow Z/X \longrightarrow X^* \longrightarrow 0$$

So if X is Lagrangian, then $Z/X \xrightarrow{\sim} X^*$ is a canonical isom. i.e. you have a canonical extension

$$0 \longrightarrow X \longrightarrow Z \longrightarrow X^* \longrightarrow 0$$

of complex vector spaces. Note $X^* = \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$ $= \mathrm{Hom}_{\mathbb{R}}(X, \mathbb{R}) \stackrel{?}{=} \mathrm{Homcont}(X, \mathbb{R}/\mathbb{Z})$ is the Pontryagin dual of X.

If you polarize Z, this means you choose a pos. def. herm. form, so that $K(\ ,\ )$ is represented by an invertible s.a. op K, then [illegible] then adjust the pos. def. form [illegible] so that $K^2 = 1$.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "j"] & Z \arrow[r] \arrow[d, "s\wr K"'] & Z/X \arrow[r] & 0 \\
& X^{\perp} \arrow[r] & Z \arrow[r, "j^*"] & X &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d] & Z \arrow[r] \arrow[d, "\simeq"] & Z/X \arrow[d] \\
X^{0} \arrow[r] \arrow[d, equal] & Z^{*} \arrow[r] \arrow[d, equal] & X^{*} \arrow[d, equal] \\
X^{\perp} \arrow[r] & Z \arrow[r, "j^*"] & X
\end{tikzcd}$$

$\leftarrow$ use Hilbert space structure on Z.

$$\begin{tikzcd}
X \arrow[r, "\binom{1}{u}"] \arrow[d, equal] & Y \oplus Y \arrow[r, two heads] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] & Z/X \\
X \arrow[r, "\binom{1}{-u}"] & Y \oplus Y \arrow[r, "(1^* \;\; u^{-1})"] & X
\end{tikzcd}$$

~~you're tr~~

you want to say something like X appearing both as a subspace and quotient space

In your example ~~[illegible] a, b, c, d.~~

$$\begin{tikzcd}
X \arrow[r, "\binom{a}{b}"] & Y_+ \oplus Y_- \arrow[r, "(a^* \; -b^*)"] & X
\end{tikzcd}$$

exact when $\binom{a}{b}X$ is Lagrangian

Idea yesterday. ~~[illegible]~~

Setting: a = functions on S^1, F = Hilbert transform

(a, F) determines a K homology class of odd degree which can be paired with

a loop $S^1 \to U(n)$

But you noticed that $U(n)$ = Lag subsp of type (n,n) hermitian space. ~~[illegible]~~

Normal method of constructing the pairing is to ~~[illegible]~~ use the loop as a clutching fn. to get a ~~[illegible]~~ vector bundle over $\mathbb{C}P^1$. vb over $\mathbb{C}P^1$.

~~[illegible] So this might work.~~

~~[illegible]~~ You have a construction already - clutching construction that combines the splitting $L^2 = H_+ \oplus H_-$ with a loop $S: S^1 \to U(n)$ or $GL(n)$, to get vb over $\mathbb{P}^1$. Build up orth polys on S^1

Idea: Do $n = 1$. Form a rank 2 a-module $Z = a\xi_+ \oplus a\xi_-$ with herm form $|f\xi_+ + g\xi_-|^2 = |f|^2 - |g|^2$

A loop $S^1 \xrightarrow{S} U(1)$ will give an isotropic ~~subble~~ subbundle $X \subset Z \to X^*$. Next you have F acting on a i.e. $a = a_+ + a_-$. ~~so you get 4 half spaces in Z.~~ ? ~~[illegible]~~ What do you get?

You want to look at the quotient Z/X which will have generators ξ_+ ξ_- related by $S\xi_- = \xi_+$ and to expect to get the filtration $S H_+ \cap z^n H_-$ in the Hilb. space. $L^2\xi_+ \simeq L^2\xi_-$. So all you need to do is to ~~[illegible]~~ extend F on a to F on $Z = a\xi_+ \oplus a\xi_-$ ~~and [illegible]~~ in a fixed way, then descend it somehow to Z/X as $X = X_S$ varies.

New idea: use the degree of ~~the~~ a general loop; it gives the index, and take degree $\to \pm\infty$ to see what to do. Stick to the Birkhoff factorization.

$$0 \longrightarrow X_S \longrightarrow Z \longrightarrow X_S \longrightarrow 0$$

You now have all the ingredients needed.

Hilbert spaces $\quad E = L^2\xi_+ \overline{\overline{\oplus}} L^2\xi_- \qquad S\xi_- = \xi_+$

This should arise from a DE $\quad p_0 = q_0 = 1 \quad$ Then

$\xi_+ = \lim_{n\to\infty} u^{-n} p_n$, $\xi_- = \lim_{n\to\infty} q_n$, Yes. Take the

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad S = \frac{a^2+b^2}{c^2+d^2} = \frac{\bar{g}}{g} \qquad g \text{ invertible in disk.}$$

Program: $\quad E = L^2\xi_+ = L^2\xi_- \qquad S\xi_- = \xi_+$

should appear as a quotient of $\quad Z = L^2\xi_+ \oplus L^2\xi_-$

$$L^2 \longrightarrow L^2\xi_+ \oplus L^2\xi_- \longrightarrow E$$

Think of cohomology of the v.b. and you get something like

$$\longrightarrow H_+\xi_+ \oplus H_-\xi_- \longrightarrow E$$

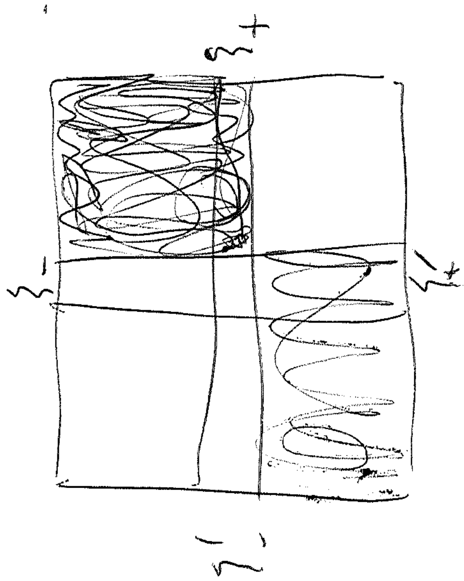
Roughly what happens here is you have a half space inside $L^2\xi_+ \oplus L^2\xi_-$ which map by a Fredholm op. to E

So you learn that somehow all you are doing is to fix a half space

Rank = 1. $E = L^2\xi_+ = L^2\xi_-$ $\quad \xi_+ = g\xi_-$

$g : S^1 \to U(1)$. Put $Z = L^2 \oplus L^2$ so that you have

$$L^2 \xrightarrow{\binom{g}{-1}} \begin{matrix} L^2 \\ \oplus \\ L^2 \end{matrix} \xrightarrow{(\xi_+ \ \xi_-)} E \qquad ?$$



$\cancel{L^2\xi_+ \oplus L^2\xi_- \to L^2(g+1)\xi_-}$

$\cancel{L^2 \xrightarrow{\binom{g}{-1}} \begin{matrix} L^2 \\ \oplus \\ L^2 \end{matrix} \xrightarrow{(\xi_+ \ \xi_-)} E}$

$$L^2 \xrightarrow{\binom{1}{-g}} \begin{matrix} L^2 \\ \oplus \\ L^2 \end{matrix} \xrightarrow{(\xi_+ \ \xi_-)} E$$

$$L^2 \xrightarrow{\binom{\xi_+}{-g\xi_-}} \begin{matrix} L^2\xi_+ \\ \oplus \\ L^2\xi_- \end{matrix} \xrightarrow{+} E$$

So next $\quad \cup$

$$\begin{matrix} H_+\xi_+ \\ \oplus \\ H_-\xi_- \end{matrix} \xrightarrow{+} E = L^2\xi_-$$

Image is $\quad H_+ g + H_-$

Kernel is $\quad \left\{ \binom{f_+}{g_-} \,\middle|\, f_+ g \cancel{+f_-} = 0 \right\} = gH_+ \cap H_-$

$$Z = L^2\xi_+ \oplus L^2\xi'_+ \oplus L^2\xi'_- \oplus L^2\xi_-$$

$$W = L^2\left(\xi_+ - \tfrac{1}{d}\xi'_- - \tfrac{b}{d}\xi_-\right) + L^2\left(\xi'_+ - \left(-\tfrac{c}{d}\right)\xi'_- - \tfrac{1}{d}\xi_-\right)$$

$$K\left(f\xi_+ + g\xi'_+ + j\xi'_- + k\xi'_-\right) = \int\left(|f|^2 + |g|^2 - |j|^2 - |k|^2\right)$$

$$K\left(\xi_+ - \tfrac{1}{d}\xi'_- - \tfrac{b}{d}\xi_-\right) = 1 - \left|\tfrac{1}{d}\right|^2 - \left|\tfrac{b}{d}\right|^2 = 0$$

$$K\left(\xi_+ - \tfrac{1}{d}\xi'_- - \tfrac{b}{d}\xi_-,\ \xi'_+ + \tfrac{c}{d}\xi'_- - \tfrac{1}{d}\xi_-\right)$$

$$= \overline{\left(+\tfrac{1}{d}\right)}(+1)\frac{c}{d} + \overline{\left(+\tfrac{b}{d}\right)}(-1)\left(+\tfrac{1}{d}\right) = \frac{c - \bar{b}}{|d|^2} = 0 .$$

$$Z/W = L^2\xi_+ \oplus L^2\xi'_+ \cong L^2\xi'_- \oplus L^2\xi_-)$$

Half space in Z to consider is

$$H_-\xi_+ \oplus H_-\xi'_+ \oplus H_+\xi'_- \oplus H_+\xi_-$$

which goes to ~~$H_-\xi_+ + H_-\xi'_+ \ldots H_+$~~ $\left(H_-\xi_+ + H_-\xi'_+\right) \overset{?}{\oplus} \left(H_+\xi'_- + H_+\xi_-\right)$

$$\left[\begin{pmatrix} H_- & H_- \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} + \begin{pmatrix} H_+ & H_+ \end{pmatrix}\right]\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

Other half space to consider is

$$H_+\xi_+ \oplus H_+\xi'_+ \oplus H_-\xi'_- \oplus H_-\xi_-$$

which goes to $\left(H_+\xi_+ + H_-\xi_-\right) \overset{?}{\oplus} \left(H_-\xi'_- + H_+\xi'_+\right)$

$$(H_+ \quad H_+)\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} \overset{?}{\oplus} (H_- \quad H_-) = (L^2 \quad L^2)$$

$$\Updownarrow$$

$$(H_- \quad H_-)\begin{pmatrix} \frac{1}{a} & \frac{c}{a} \\ -\frac{b}{a} & \frac{1}{a}\end{pmatrix} \overset{?}{\oplus} (H_+ \quad H_+) = (L^2 \quad L^2)$$

$$(H_- \quad H_-) \oplus (H_+ \quad H_+)\begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d}\end{pmatrix} = (L^2 \quad L^2)$$

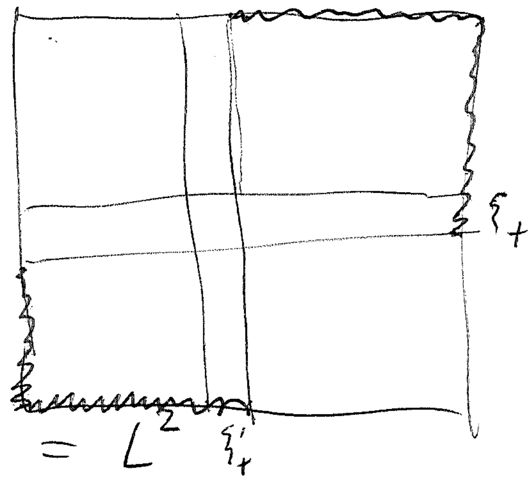
Apply $\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$

$$(H_- \quad H_-) \oplus (H_+ \quad H_+)\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} = (L^2 \quad L^2)$$

$$L^2 = gH_+ \overset{?}{\oplus} H_-$$

Apply $-$: $\quad L^2 = g^{-1}zH_- \overset{?}{\oplus} zH_+$

Apply gz^{-1} $\quad L^2 = H_- + g$



It seems that $\quad gH_+ \overset{?}{\oplus} H_- = L^2$

is not obvious equiv. to $\quad gH_- \overset{?}{\oplus} H_+ = L^2$

Review carefully the splitting results

(1) $(H_+\xi_+ + H_+\xi_+') \oplus (H_-\xi_-' + H_-\xi_-) = E.$

(2) $(H_+\xi_+ + H_-\xi_-) \oplus (H_-\xi_-' + H_+\xi_+') = E$

these seem equivalent

(3) $(H_-\xi_+ + H_-\xi_+') \oplus (H_+\xi_-' + H_+\xi_-) = E$

Compare 1, 2.

(2)
$$\begin{pmatrix} H_+ & H_- \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- & H_+ \end{pmatrix} = \begin{pmatrix} L^2 & L^2 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \oplus \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{c}{a} \\ \frac{b}{d} & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \oplus \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

use $\pi_+ \frac{c}{a} : H_- \to H_+$

$\pi_- \frac{b}{d} : H_+ \to H_-$

contractions.

(1)
$$\begin{pmatrix} H_+ & H_+ \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} + \begin{pmatrix} H_- & H_- \end{pmatrix} = \begin{pmatrix} L^2 & L^2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} + \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} + \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

use $\begin{pmatrix} 1 & -\pi_+ c \\ \pi_+ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ + skew adj

always invertible.

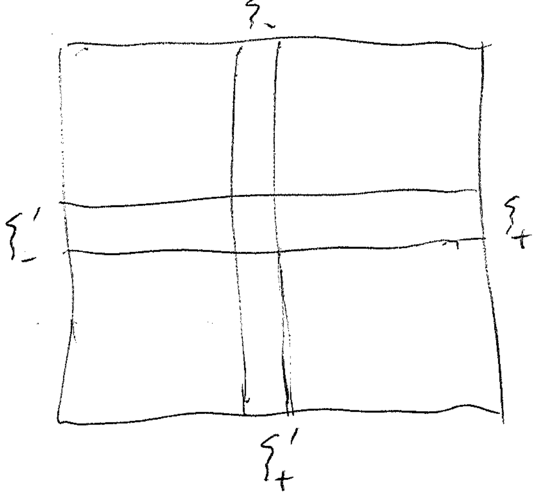
(3)
$$\begin{pmatrix} H_- & H_- \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} + \begin{pmatrix} H_+ & H_+ \end{pmatrix} = \begin{pmatrix} L^2 & L^2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} + \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} + \begin{pmatrix} \cancel{d} H_+ \\ \cancel{d} H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \qquad \begin{pmatrix} 1 & -\pi_- c \\ \pi_- b & 1 \end{pmatrix}$$

everything should fall into place.
Given ~~(…)~~ b a function on S^1.
form $E = L^2\xi_+ + L^2\xi_-$ with

$$K(f\xi_+ + g\xi_-) = \int(|f|^2 - |g|^2) \quad ?$$



$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$

You want to set up ~~the~~

$$K(f\xi'_- + g\xi'_+) = \int(|f|^2 - |g|^2)$$

~~Recall one oversight. A hermitian vector bundle E over X gives a Hilbert module $\Gamma(E)$ over $A = C(X)$.~~
Mistake.

Yesterday you observed that $L^2\xi_+ + L^2\xi_-$ is not a Hilbert module over $C(S^1)$. The Hilbert module is $C(S^1)\xi_+ + C(S^1)\xi_-$ and ~~completing~~ it is complete wrt $\sup_{S^1} |f\xi_+ + g\xi_-|^2 = \sup_{S^1} |f|^2 + |g|^2$. $L^2\xi_+ + L^2\xi_-$ is the completion wrt $\int(|f|^2 + |g|^2)$.

Introduce. $K(f\xi'_- + g\xi'_+) = \int |f|^2 - |g|^2$

$$K(f\xi'_- + g\xi_-) = K(f\xi'_- + g(c\xi'_- + d\xi'_+))$$
$$= K((f+gc)\xi'_- + gd\xi'_+) = \int |f+gc|^2 - |gd|^2$$
$$= \int \begin{pmatrix} f\\ g\end{pmatrix}^* \begin{pmatrix} 1 & \bar b\\ b & -1\end{pmatrix}\begin{pmatrix} f\\ g\end{pmatrix}$$

$$H_+\xi'_- + H_+\xi_- \subset L^2\xi'_{+} + L^2\xi_-$$

Restrict K to this subspace - it should be ~~±~~ nondegenerate. i.e.

$$\begin{pmatrix}H_+\\ H_+\end{pmatrix} \hookrightarrow \begin{pmatrix}L^2\\ L^2\end{pmatrix} \xrightarrow{\begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}} \begin{pmatrix}L^2\\ L^2\end{pmatrix} \xrightarrow{\pi_+} \begin{pmatrix}H_+\\ H_+\end{pmatrix}$$

is the operator.
$$\begin{pmatrix}1 & \pi_+\bar b\\ \pi_+ b & -1\end{pmatrix} = \underbrace{\begin{pmatrix}1 & -\pi_+\bar b\\ \pi_+ b & 1\end{pmatrix}}_{1+\text{skewadj}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

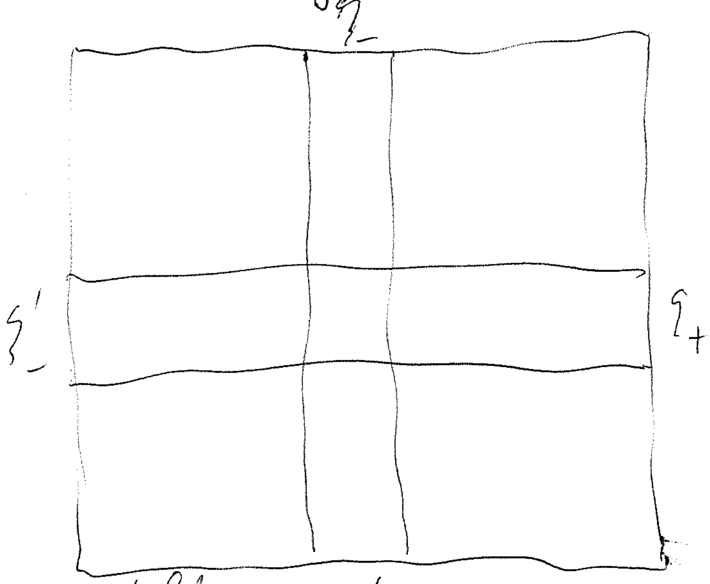
which is ~~one~~ invertible.

$L^2\xi'_- \oplus L^2\xi_-$ Hilb. space orth. direct sum

$$K(f\xi'_- + g\xi_-) = \int \begin{pmatrix}f\\ g\end{pmatrix}^* \begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}$$

~~Let us take some time~~

Recovering:



$L^2\xi'_- \oplus L^2\xi_-$

$$\|f\xi'_- + g\xi_-\|^2 = \|f\|^2 + \|g\|^2$$

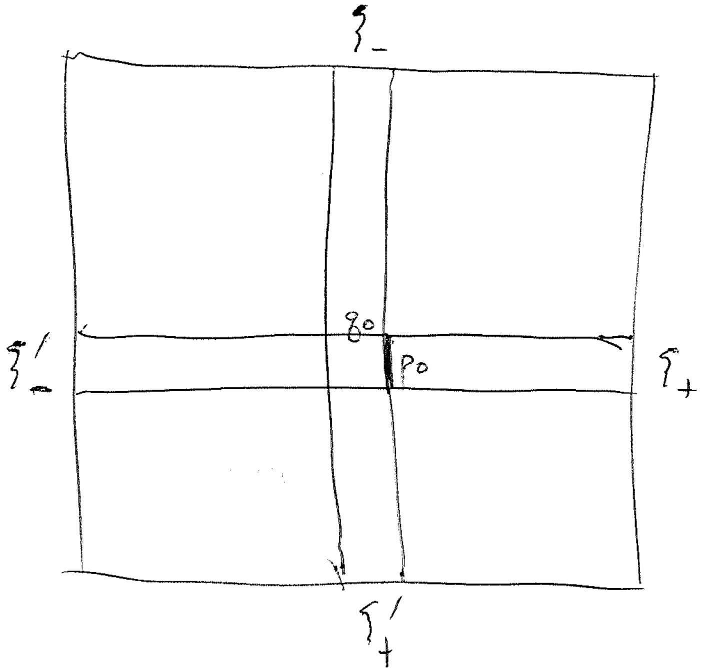
$$K(f\xi'_- + g\xi_-) = \int \begin{pmatrix}f\\ g\end{pmatrix}^* \begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}$$

Recover the potential from b. You have the Krein form and the result that ξ'_+ its restriction to $z^k H_+\xi'_- + H_+\xi_-$ is non degenerate. Check this:

$$\begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}\begin{pmatrix}z^k f\\ g\end{pmatrix}$$

$$\begin{tikzcd}
\begin{pmatrix} z^k H_+ \\ H_+ \end{pmatrix} \arrow[r, hook] & \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \arrow[r, "{\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}}"] & \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \arrow[r, two heads, "{\begin{pmatrix} z^k\pi_+ z^{-k} & 0 \\ 0 & \pi_+ \end{pmatrix}}"] \arrow[dr, "{\begin{pmatrix} \pi_+ z^{-k} & 0 \\ 0 & \pi_+ \end{pmatrix}}"'] & \begin{pmatrix} z^k H_+ \\ H_+ \end{pmatrix} \arrow[d, "{\begin{pmatrix} z^{-k} & 0 \\ 0 & 1 \end{pmatrix}}"] \\
\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \arrow[u, "{\begin{pmatrix} z^k & 0 \\ 0 & 1 \end{pmatrix}}"] & & & \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}
\end{tikzcd}$$

$$\begin{pmatrix} \pi_+ & 0 \\ 0 & \pi_+ \end{pmatrix} \underbrace{\begin{pmatrix} z^{-k} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}\begin{pmatrix} z^k & 0 \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & z^{-k}\bar b \\ b z^k & 1 \end{pmatrix}} \begin{pmatrix} \varepsilon_+ & 0 \\ 0 & \varepsilon_+ \end{pmatrix}$$



Consider

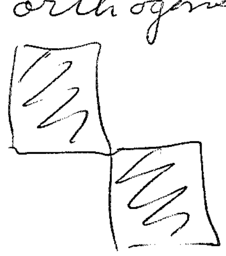
$$H_+\xi'_- + H_+\xi_\infty$$
$$\cup$$
$$zH_+\xi'_- + H_+\xi_-$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{tikzcd}
0 \arrow[r] & zH_+\xi'_- + H_+\xi_- \arrow[r] \arrow[d, "S"] & H_+\xi'_- + H_+\xi_- \arrow[r] \arrow[d, "S"] & \mathbb{C}\xi'_- \arrow[r] & 0 \\
0 \arrow[r] & zH_+\xi'_- + H_+\xi_- & H_+\xi'_- + H_+\xi_- \arrow[l]
\end{tikzcd}$$

$$\exists!\ \tilde{p}_0 \in \left(\xi_-' + zH_+\xi_-' + H_+\xi_-\right)$$

figure out norms! You don't understand the ~~hermitian~~ indef form on p_n. ✱ Because of orthogonal projection methods you do know that regions are $\perp$ for K so that ~~then~~ the hermitian



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$K(p_0) = K\left(a^l\xi_-' + b^l\xi_+'\right) = \int |a^l|^2 - |b^l|^2 = \int 1 = 1$$

$$K(q_0) = K\left(c^l\xi_-' + d^l\xi_+'\right) = \int |c^l|^2 - |d^l|^2 = \int (-1) = -1.$$

$$K(p_0, q_0) = \int \begin{pmatrix} a^l \\ b^l \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c^l \\ d^l \end{pmatrix} \doteq \int \bar{a}^l c^l - \bar{b}^l d^l = 0$$

~~That's not a idea~~ Normalization:

$$p_0 = \frac{d^r}{d}\xi_-' + \frac{b^l}{d}\xi_-$$

$$K(p_0) = \int \begin{pmatrix} \frac{d^r}{d} \\ \frac{b^l}{d} \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\begin{pmatrix} \frac{d^r}{d} \\ \frac{b^l}{d} \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}$$

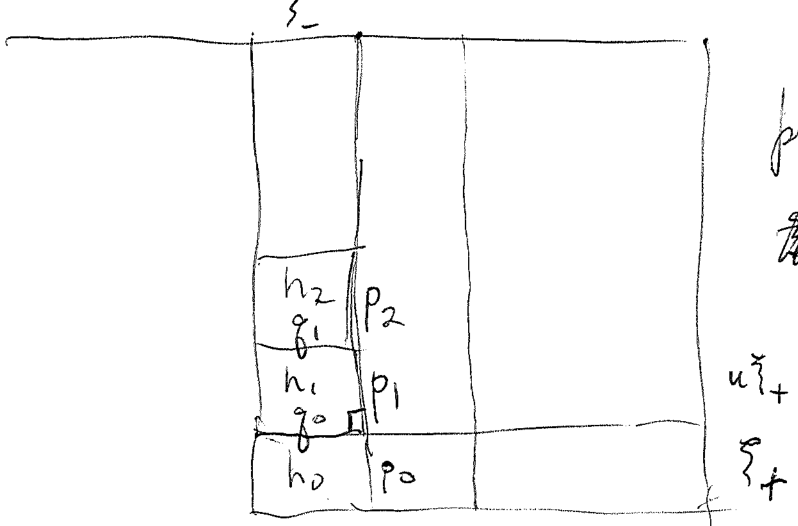
Start again - you have a basic problem with normalizations. First review the pos. def. case.

$$E = L^2\xi_+ + L^2\xi_- \qquad \|f\xi_+ + g\xi_-\|^2 = \int \begin{pmatrix} f\\ g\end{pmatrix}^* \begin{pmatrix}1 & \bar\beta\\ \beta & 1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}$$

since $(g\xi_- \mid \xi_+ f) = \int \bar g \beta f$ why? $\beta = \frac{b}{d}$. $= \bar g \frac{b}{d} f$.

$$\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix} \qquad (g\xi_- \mid \xi_+ f) = \left(g\xi_- \,\middle|\, \left(\tfrac{1}{d}\xi_-' + \tfrac{b}{d}\xi_-\right) f\right)$$

since $\|\beta\|_\infty < 1$, this norm is pos. def., hence you take a filtration and get orthogonal splitting



The problem is that orthog. produces a non unit vector

$\xi_+ =$

$$\begin{pmatrix}p_n\\ q_{n-1}\end{pmatrix} = \begin{pmatrix}k_n & h_n\\ -\bar h_n & k_n\end{pmatrix}\begin{pmatrix}up_{n-1}\\ q_n\end{pmatrix} \qquad \begin{pmatrix}up_{n-1}\\ q_n\end{pmatrix} = \begin{pmatrix}k_n & -h_n\\ \bar h_n & k_n\end{pmatrix}\begin{pmatrix}p_n\\ q_{n-1}\end{pmatrix}$$

$$\begin{pmatrix}p_n\\ q_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar h_n & 1\end{pmatrix}\begin{pmatrix}up_{n-1}\\ q_{n-1}\end{pmatrix}$$

$$q_n = \bar h_n p_n + k_n\left(\bar h_{n-1}p_{n-1} + k_{n-1}\left(\bar h_{n-2}p_{n-2} + k_{n-2}\left(\cdots\right.\right.\right.$$

$$+\ k_n\cdots k_2\,\bar h_1 p_1 + k_n\cdots k_1 q_0$$

$$q_n = \sum_{i=1}^{n} k_n\cdots k_{i+1}\,\bar h_i p_i + k_n\cdots k_1 q_0$$

$$\xi_- = \sum_{i=1}^{\infty} \prod_{i+1}^{\infty} k_j\, \bar h_i p_i + \prod_{1}^{\infty} k_j\, q_0$$

cont case $L^2 = L^2(\mathbb{R})$ variable k

$L^2\xi'_- + L^2\xi_-$

$\underset{\sim}{K}(f\xi'_- + g\xi_-) = \underset{\sim}{K}(f\xi_- + g(c\xi_- + d\xi'_+))$

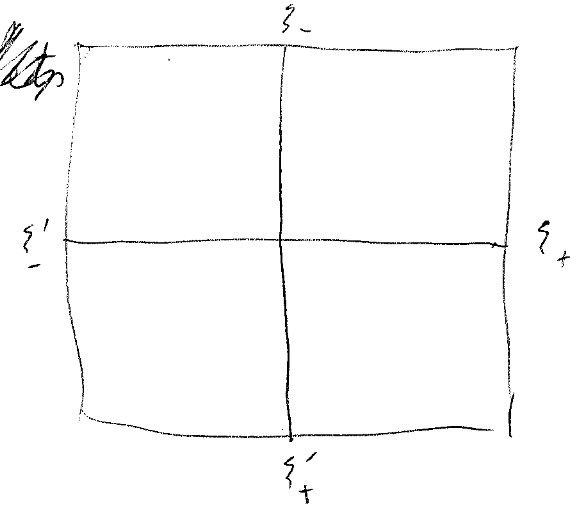
non degenerate on $H_+\xi'_- + H_+\xi_-$

$= \underset{\sim}{K}((f+gc)\xi_- + d\xi'_+)$

$= \int |f+gc|^2 - |d|^2 = \begin{pmatrix}f\\g\end{pmatrix}^* \begin{pmatrix}1 & c\\ b & -1\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}$

or $L^2\xi_+ + L^2\xi_-$

$\|f\xi_+ + g\xi_-\|^2 = \int \begin{pmatrix}f\\g\end{pmatrix}^*\begin{pmatrix}1 & \bar\beta\\ \beta & 1\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}$



what form does orth proj take

$p_0 \in (1+H_+)\xi_+ + H_-\xi_-$

$p_0 \perp (H_+\xi_+ + H_-\xi_-)$

$p_0 = d^\wedge\xi_+ - b^\wedge\xi_-$

$d^\wedge\beta - b^\wedge \in H_+$

$d^\wedge - b^\wedge\bar\beta \in 1+H_-$

$$\begin{pmatrix}1 & +\bar\beta\\ \beta & 1\end{pmatrix}\begin{pmatrix}d^\wedge\\ -b^\wedge\end{pmatrix} \in \begin{pmatrix}1+H_-\\ H_+\end{pmatrix}$$

~~so what~~ these are in $\mathbb{C}+L^2 \xrightarrow{\pi_-} \mathbb{C}+H_-$

Apply $\begin{pmatrix}\pi_+ & 0\\ 0 & \pi_-\end{pmatrix}$

$$\begin{pmatrix}Id & \pi_+\bar\beta\\ \pi_-\beta & Id\end{pmatrix}\begin{pmatrix}d^\wedge\\ -b^\wedge\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

Repeat. $q_0 \in H_+\xi_+ + (1+H_-)\xi_-$, $q_0 \perp H_+\xi_+ + H_-\xi_-$

$q_0 = -c^\wedge\xi_+ + a^\wedge\xi_-$

$-c^\wedge\beta + a^\wedge \in 1+H_+$

$-c^\wedge + a^\wedge\bar\beta \in H_-$

$$\begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} -c^r \\ a^r \end{pmatrix} \in \begin{pmatrix} H_- \\ 1+H_+ \end{pmatrix}$$

$$\begin{pmatrix} Id & \pi_+\bar\beta \\ \pi_-\beta & Id \end{pmatrix}\begin{pmatrix} -c^r \\ a^r \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} Id & \pi_+\bar\beta \\ \pi_-\beta & Id \end{pmatrix}\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Splitting: $E = (H_+\xi_+ + H_-\xi_-) \overset{?}{\oplus} (H_-\xi'_- + H_+\xi'_+)$

$$\begin{pmatrix} H_+ & H_- \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- & H_+ \end{pmatrix} = \begin{pmatrix} L^2 & L^2 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

Apply $\begin{pmatrix} a^{-1} & \\ & d^{-1} \end{pmatrix}$

$$\begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \oplus \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} Id & \pi_+\bar\beta \\ \pi_-\beta & Id \end{pmatrix} : \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} H_+ \\ H_- \end{pmatrix}$$

~~You want to follow~~ Let's try to recover the potential in the continuous case.

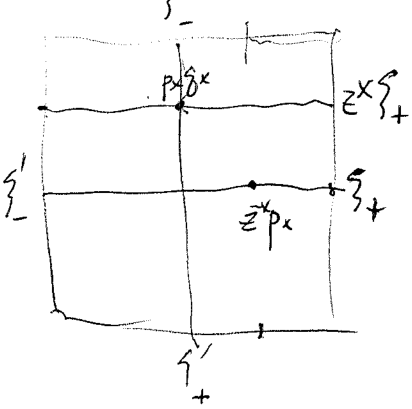
Do in the discrete case first, settle normalizations.

~~settle normal~~ $$\begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

~~Do in the discrete~~

$$\begin{pmatrix} z^{-x}p_x \\ q_x \end{pmatrix} = \begin{array}{cc} 1+H_+ & z^{-x}H_- \\ \end{array} \hspace{-6em} \begin{pmatrix} d^r_x & -b^r_x \\ -c^r_x & a^r_x \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^l_x & b^l_x \\ c^l_x & d^l_x \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{array}{cc} z^xH_+ & 1+H_- \end{array} \qquad\qquad \begin{array}{cc} z^xH_- & 1+H_+ \end{array}$$



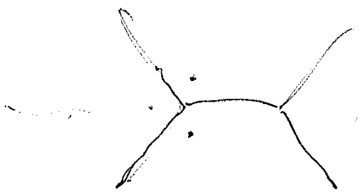
You know how to produce this factorization in the Hilbert space context - positive definite case. Your program is to get to non linear Schroedinger equ.

$$\begin{array}{cc} 1+H_+ & H_- \end{array} \qquad\qquad \begin{array}{cc} 1+H_- & H_+ \end{array}$$

$$\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} d^r_x & -z^xb^r_x \\ -c^r_xz^{-x} & a^r_x \end{pmatrix} \begin{pmatrix} z^x\xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^l_x & z^xb^l_x \\ z^{-x}c^l_x & d^l_x \end{pmatrix} \begin{pmatrix} z^x\xi'_- \\ \xi'_+ \end{pmatrix}$$

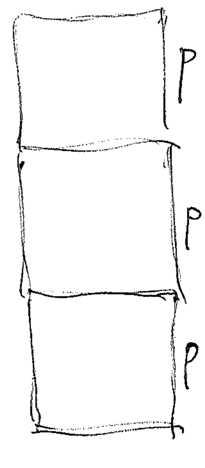
$$\begin{array}{cc} H_+ & 1+H_- \end{array} \qquad\qquad \begin{array}{cc} H_- & 1+H_+ \end{array}$$

$$\begin{pmatrix} z^x\xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & z^xb \\ z^{-x}c & d \end{pmatrix} \begin{pmatrix} z^x\xi'_- \\ \xi'_+ \end{pmatrix}$$

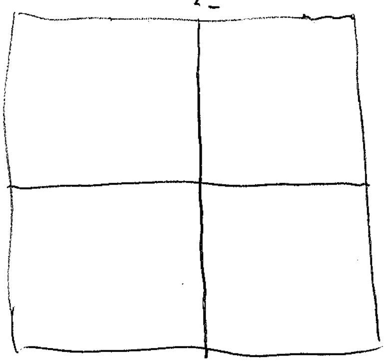


When all the crud is cleaned up you are left with the factorization for $\begin{pmatrix} a & z^xb \\ z^{-x}c & d \end{pmatrix}$. So you have to deal with this factorization with varying parameters.

~~Something simple and~~
discuss unnormalized approach.
mix ξ_+



Continuous case for fun. $E = \xi'_- L^2 + \xi_- L^2$



$$K(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \frac{dk}{2\pi}$$

Restrict ~~[illegible]~~ K to $\xi'_- z^x H_+ + \xi_- H_+$

$$\begin{pmatrix} \pi_x & \\ & \pi_0 \end{pmatrix} \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} \varepsilon_x & 0 \\ 0 & \varepsilon_0 \end{pmatrix} = \begin{pmatrix} Id_x & \pi_x \bar b \varepsilon_0 \\ \pi_0 b \varepsilon_x & -Id_0 \end{pmatrix}$$

This should be invertible on $\begin{pmatrix} z^x H_+ \\ H_+ \end{pmatrix}$ by Hilbert space th.

~~$z^{-x} p_x =$ [illegible] $= \xi_- [illegible] \xi_- g$~~ Set this up properly. Find splitting

Take $x = 0$. Want ~~splitting~~

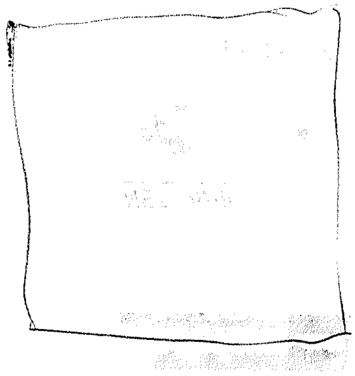
$$E = \left(\xi'_- H_+ + \xi_- H_+\right) \overset{?}{\oplus} \left(\xi_+ H_- + \xi'_+ H_-\right) \qquad ?$$

$$\begin{pmatrix} L^2 \\ L^2 \end{pmatrix} = \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \frac{1}{d} & -\frac{\bar b}{d} \\ +\frac{b}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix} = \begin{pmatrix} \xi'_- & \xi_- \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix}$$

So what? Lets be precise about the factorization

Again we start with b ~~[illegible]~~ a bdd meas. fn. on S' form $E = \xi'_- L^2 + \xi_- L^2$ $\quad K(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$



Can $(\xi'_- L^2)^\perp \neq \begin{pmatrix} f \\ g \end{pmatrix} \Big| f = \bar b g$

$(\xi'_- L^2)^\perp = \{(\xi'_- \bar b - [illegible]\, \xi_- [illegible])\, [illegible]\, g \in L^2 [illegible]$

~~$K(\xi'_- g - \xi_- \bar b g) = \int \begin{pmatrix} g \\ -\bar b g \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ -\bar b \end{pmatrix}$~~

$$(\xi_-' L^2)^\perp = \{(\xi_-' f + \xi_- g) \mid f + \bar b g = 0\}$$

$$= \{(-\xi_-' \bar b + \xi_-) g \mid g \in L^2\}$$

$$K(-\xi_-' \bar b g + \xi_- g) = \int \begin{pmatrix} -\bar b g \\ g \end{pmatrix}^* \underbrace{\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} -\bar b g \\ g \end{pmatrix}}_{\begin{pmatrix} 0 \\ -(1+|b|^2) g \end{pmatrix}}$$

$$= -\int g^* (1+|b|^2) g$$

$$\cancel{S_0 \quad (\xi_-' L^2)^\perp \simeq L^2\left(S^1, (1+|b|^2)\frac{d\theta}{2\pi}\right)}$$

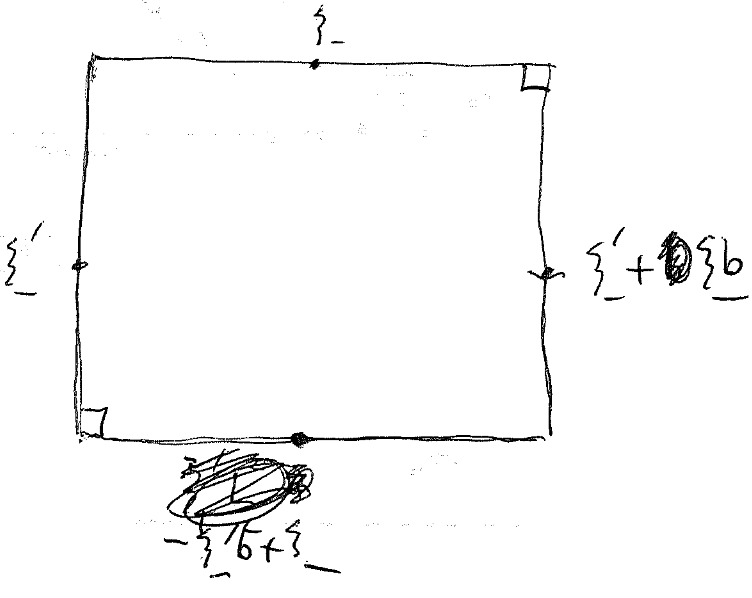
$$(\xi_-' L^2)^\perp = (-\xi_-' \bar b + \xi_-) L^2 \simeq$$

$$(\xi_- L^2)^\perp = \{(\xi_-' + \xi_- b) L^2\}$$

$$\int \begin{pmatrix} f \\ bf \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ bf \end{pmatrix} = \int f^* (1+|b|^2) f$$

compare filtration

$$(\xi_-' + \xi_- b)\, z^{x} H_+$$



Consider $(\xi_-' + \xi_- b)L^2$ $\qquad$ $K(\xi_-' f + \xi_- bf)$

$$= \int \begin{pmatrix} f \\ bf \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ bf \end{pmatrix}$$

$$= \int f^*(1+|b|^2)f$$

Ask for $(\xi_-' + \xi_- b)(1+\phi) \perp (\xi_-' + \xi_- b)H_+$

You want. $\xi_-' + \xi_-' \underset{\substack{\cap \\ H_+}}{\phi} + \xi_- \underset{\substack{\cap \\ H_+}}{\psi} \perp \xi_-' H_+ + \xi_- L^2$

$$\int \begin{pmatrix} H_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} 1+\phi \\ \psi \end{pmatrix} = 0$$

means $\quad 1 + \phi + \bar b \psi \perp H_+$

and $\quad b(1+\phi) = \psi$

$$\boxed{\begin{array}{l} \pi_+(\phi + \bar b \psi) = 0 \\ b(1+\phi) = \psi \end{array}}$$

Can you understand why these eqns can be solved with suitable assumptions on b. $\qquad \phi = -\pi_+(\bar b \psi)$

$$\phi = -\pi_+(\bar b b(1+\phi)) = -\pi_+(\bar b b) - \pi_+ \bar b b \phi$$

$$\underbrace{\phi + \pi_+ \bar b b \phi}_{\pi_+(1+\bar b b)\phi} = -\pi_+ \bar b b$$

$$\pi_+(1+\bar b b)\phi = -\pi_+(\bar b b)$$

to understand you write $\quad H_+ \underset{T^* = \pi_+ \bar b}{\overset{T = b\varepsilon_+}{\rightleftarrows}} L^2$

Then you have $(1 + T^*T)\phi = -T^* b$ whence

$$\phi = -(1+T^*T)^{-1} T^* b = -T^*(1+TT^*)^{-1} b$$

and $\quad \psi = b + b\phi = b - bT^*(1+TT^*)^{-1} b$

So there is something here you don't understand.

Check:
$$\boxed{\xi_-' + \xi_-'\phi + \xi_-\psi \perp (\xi_-' H_+ + \xi_- L^2)}$$

$$\int \begin{pmatrix} H_+ \\ L^2 \end{pmatrix} \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1+\phi \\ \psi \end{pmatrix} = 0 \qquad \begin{aligned} 1+\phi+\bar{b}\psi &\perp H_+ \\ b(1+\phi) &= \psi \end{aligned}$$

maybe you need to analyze the meaning of ~~l~~ when $\xi_-' \notin E = \xi_-' L^2 + \xi_- L^2$. What happens is maybe that ξ_-' should give rise to a linear functional on a dense subspace of E.

$$\int \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \int f + \bar{b} g$$

~~So you~~ You need to understand the condition

$$\int \begin{pmatrix} H_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1+\phi \\ \psi \end{pmatrix} = 0$$

SAVE FOR CONT CASE

$$\int H_+^* \begin{pmatrix} 1 & \bar{b} \end{pmatrix} \begin{pmatrix} 1+\phi \\ b+b\phi \end{pmatrix} = 0$$

$$\underbrace{\qquad\qquad}$$

$$1+\phi + \bar{b}b(1+\phi) = \cancel{(1+\phi)(}$$

$$\boxed{\int H_+^* (1+\bar{b}b)(1+\phi) = 0}$$

how to interpret this? $\qquad \phi \in H_+$

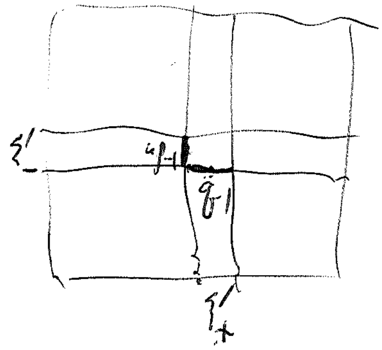
In the discrete case $\xi_-'(1+\phi) + \xi\psi \perp \xi_-' z H_+ + \xi_- L^2$

so $\phi \in z H_+$, so you find that $\cancel{(1+\bar{b}b)(}$?

$$h_0 = (q_0|p_0) = (q_0 \mid \cancel{\xi'_- a^\ell} + \xi'_+ b^\ell) \qquad \begin{array}{cc} zH_- & H_+ \end{array}$$

$$= \left(\cancel{\xi_+ \tfrac{\ell}{a}} + \xi'_+ \frac{\overline{a^\lambda}}{a} \,\Big|\, \xi'_+ b^\ell\right) = \int \frac{\overline{a^\lambda}}{a} b^\ell = \left(\frac{\overline{a^\lambda}}{a} \,\Big|\, b^\ell\right)$$

$$= \overline{\frac{a^\lambda}{a}}(\infty)\, b^\ell(0) = \cancel{\quad} \quad (?)$$



$$\frac{d^\lambda(0)}{\cancel{d(0)}}\cancel{d^\ell(0)}\ \frac{b^\ell(0)}{d^\ell(0)}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix}\begin{pmatrix} u p_{-1} \\ q_{-1} \end{pmatrix}$$

$$\therefore \begin{pmatrix} a_0^\ell & b_0^\ell \\ c_0^\ell & d_0^\ell \end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix} \begin{array}{cc} zH_- & zH_+ \\ \multicolumn{2}{c}{\begin{pmatrix} a_{-1}^\ell & b_{-1}^\ell \\ c_{-1}^\ell & d_{-1}^\ell \end{pmatrix}} \\ H_- & H_+ \end{array}$$

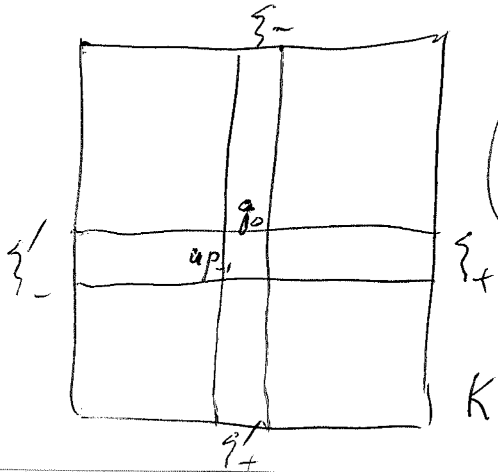
$$\frac{b_0^\ell}{d_0^\ell} = \begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix}\begin{pmatrix} \dfrac{b_{-1}^\ell}{d_{-1}^\ell} \end{pmatrix} \qquad \text{put } z=0$$

get $$\frac{b_0^\ell}{d_0^\ell}(0) = h_0$$ so you get the

formula ~~[struck]~~

so $$\boxed{h_0 = \frac{b^\ell(0)}{d^\ell(0)}}$$

Reconcile approaches so go back to your K-version, and correlate with factorization. Before you had $\xi'_- zH_+$ $\xi_- zH_+$

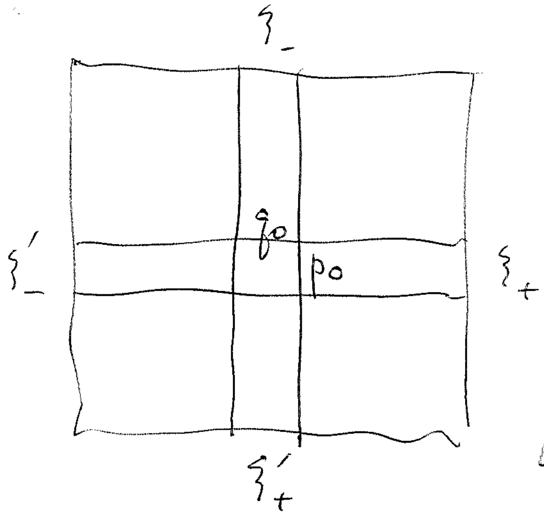


$$\tilde q_0 = \xi'_-(-\phi) + \xi_-(1-\bar\psi)$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_1 \end{pmatrix}\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}\begin{pmatrix} -\phi \\ 1-\psi \end{pmatrix} \qquad \begin{array}{l} \pi_1(-\phi + \bar b - \bar b\psi) = 0 \\ \pi_1(-b\phi - 1 + \psi) = 0 \end{array}$$

$$\pi_1 \bar b = \phi + \pi_1 \bar b \varepsilon \psi \qquad \psi = \pi_1 b \varepsilon \phi$$

$$K(\tilde q_0) = \int \begin{pmatrix} \cancel{-\phi} \\ 1-\cancel{\psi} \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}\begin{pmatrix} -\phi \\ 1-\psi \end{pmatrix} = \int(-b\phi - 1 + \psi)$$



$$\begin{pmatrix} p_0 \\ q_0\end{pmatrix} = \begin{array}{c}\begin{array}{cc} zH_- & H_+\end{array}\\ \begin{pmatrix} a^l & b^l \\ c^l & d^l\end{pmatrix} \\ \begin{array}{cc} zH_- & H_+\end{array}\end{array}\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix} = \begin{array}{c}\begin{array}{cc} H_+ & H_-\end{array}\\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r\end{pmatrix} \\ \begin{array}{cc} zH_+ & zH_-\end{array}\end{array}\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l\end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r\end{pmatrix}\begin{pmatrix}\xi'_+ \\ \xi_+\end{pmatrix}$$

write $\begin{pmatrix} a^l & b^l \\ c^l & d^l\end{pmatrix} = \begin{pmatrix} a^l(0) & 0 \\ 0 & d^l(0)\end{pmatrix} + \begin{pmatrix} \alpha^l & b^l \\ c^l & \delta^l\end{pmatrix}$

$$\cap \quad \begin{pmatrix} H_- & H_+ \\ zH_- & zH_+\end{pmatrix}$$

$$\begin{pmatrix} d^r & -b^r \\ -c^r & a^r\end{pmatrix} = \begin{pmatrix} d^r(0) & 0 \\ 0 & a^r(0)\end{pmatrix} + \begin{pmatrix} \delta^r & -b^r \\ -c^r & \alpha^r\end{pmatrix}$$

$$\cap \quad \begin{pmatrix} zH_+ & H_- \\ zH_+ & H_-\end{pmatrix}$$

$$\begin{pmatrix}\xi_+ \\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi_-\end{pmatrix}$$

$$p_0 = \xi'_- \underset{zH_-}{a^l} + \xi'_+ \underset{H_+}{b^l} = \xi_+ \underset{H_+}{d^r} + \xi_-(\underset{H_-}{-b^r})$$

$$\|p_0\|^2 = (\xi'_- a^l + \xi'_+ b^l \mid \xi_+ d^r + \xi_-(-b^r))$$

$$= \underbrace{(\xi'_- a^l \mid \xi_+ d^r)}_{(\xi'_- a^l \mid \frac{d^r}{d}\xi'_-)} - \underbrace{(\xi'_+ b^l \mid \xi_- b^r)}_{(\xi_- \frac{1}{d} b^l \mid \xi_- b^r)}$$

$$\xi_+ = \frac{1}{d}\xi'_- + \frac{b}{d}\xi_- \qquad \xi'_+ = \cancel{\xi'_-\left(-\frac{c}{d}\right)} + \xi_-\left(\frac{1}{d}\right)$$

$$(\xi'_- a^l \mid \frac{d^r}{d}\xi'_-) \qquad (\xi_- \frac{1}{d} b^l \mid \xi_- b^r) = \int \left(\frac{b^l}{d}\right)^* b^r = 0$$

H_+ H_-

$$(\xi'_- a^l \mid \frac{d^r}{d}\xi'_-) = \int (a^l)^* \frac{d^r}{d} = \int \frac{d^l d^r}{d} = \frac{d^l d^r(0)}{d}.$$

$$\boxed{d(0) = d^l(0)\, d^r(0)}$$

$$h_0 = (q_0|p_0) = \left(\xi'_- c^l + \xi'_+ d^l \,\middle|\, \xi_+ d^r + \xi_-(-b^r)\right)$$

$$= \Big(\xi'_- c^l \,\Big|\, \underbrace{\xi_+ d^r}_{\xi'_- \frac{d^r}{d} + \xi_- \frac{bd^r}{d}}\Big) - \Big(\underbrace{\xi'_+ d^l}_{\xi'_-\left(-\frac{c}{d}\right)d^l + \xi_-\left(\frac{1}{d}\right)d^l} \,\Big|\, \xi_- b^r\Big)$$

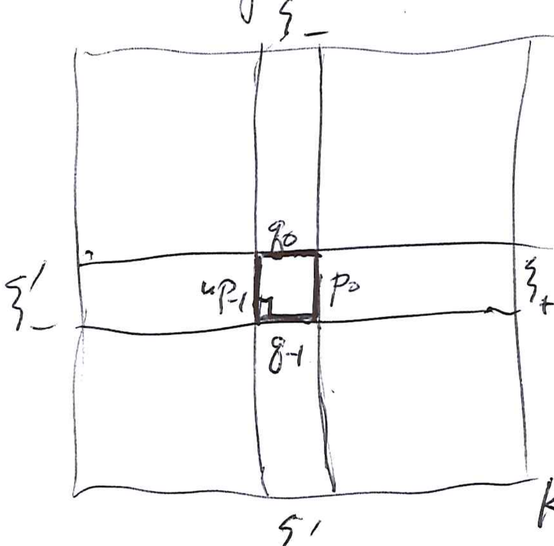
$$= \int \underbrace{(c^l)^*}_{zH_-} \underbrace{\frac{d^r}{d}}_{H_+} - \underbrace{\int \underbrace{\left(\frac{d^l}{d}\right)^*}_{H_+} \underbrace{b^r}_{H_-}}_{0}$$

$$= \overline{c^l(\infty)}\, \frac{d^r(0)}{d(0)} = \frac{b^l(0)}{d^l(0)} = h_0$$ once we know $(c^l)^* = b^l$.

Point is that $d^l(0)$ is under control.

$$d^l(0) = \prod_{n\le 0} \frac{1}{k_n}$$

Run through the ~~same~~ sort of calculation in the K-situation



$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \frac{1}{k_0}\begin{pmatrix}1 & h_0\\ \bar h_0 & 1\end{pmatrix}\begin{pmatrix}up_{-1}\\ q_{-1}\end{pmatrix}$$

$$K(q_{-1}, p_0) = K\left(q_{-1}, \frac{1}{k_0}up_{-1} + \frac{h_0}{k_0}q_{-1}\right) = -\frac{h_0}{k_0}$$

$$\begin{pmatrix}up_{-1}\\ q_{-1}\end{pmatrix} = \frac{1}{k_0}\begin{pmatrix}1 & -h_0\\ -\bar h_0 & 1\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix}$$

$$K(q_0, up_{-1}) \overset{K}{=} \left(q_0, \frac{1}{k_0}p_0 - \frac{h_0}{k_0}q_0\right) = \frac{h_0}{k_0}$$

$$\begin{pmatrix}p_0\\ q_{-1}\end{pmatrix} = \begin{pmatrix}k_0 & h_0\\ -\bar h_0 & k_0\end{pmatrix}\begin{pmatrix}up_{-1}\\ q_0\end{pmatrix} \qquad \begin{pmatrix}p_0\\ q_{-1}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -\frac{\bar h_0}{k_0} & \frac{1}{k_0}\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix}$$

$$\boxed{q_{-1} = -\frac{h_0}{k_0}p_0 + \frac{1}{k_0}q_0}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar h_0 & 1 \end{pmatrix}\begin{pmatrix} u p_{-1} \\ q_{-1} \end{pmatrix} \qquad \begin{pmatrix} u p_{-1} \\ q_{-1} \end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & -h_0 \\ -\bar h_0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$q_{-1} = -\frac{\bar h_0}{k_0} p_0 + \frac{1}{k_0} q_0$$

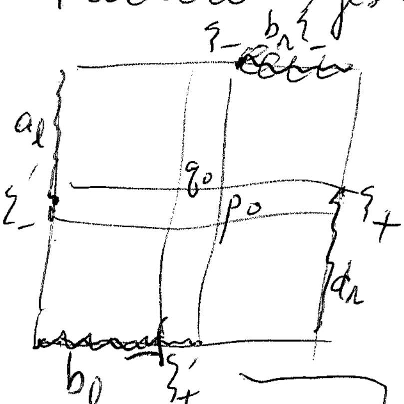
$$K(q_{-1}, p_0) = K\left(-\frac{\bar h_0}{k_0} p_0, p_0\right) = -\frac{\bar h_0}{k_0}$$

$$\begin{pmatrix} p_0 \\ q_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\bar h_0}{k_0} & \frac{1}{k_0} \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\bar h_0}{k_0} & \frac{1}{k_0} \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{\bar h_0}{k_0} & \frac{1}{k_0} \end{pmatrix} \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}}_{\cap \begin{pmatrix} H_+ & H_+ \\ H_+ & H_+ \end{pmatrix}}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \in \begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}$$

Review yesterday calculation,



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\|p_0\|^2 = \left(a^l \xi'_- + b^l \xi'_+ \,\middle|\, d^r \xi_+ - b^r \xi_-\right)$$

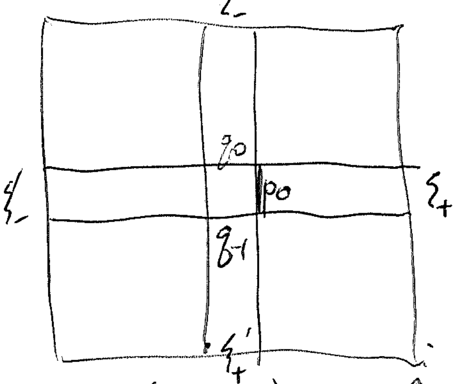
$$= (a^l \xi'_- \,|\, d^r \xi_+) - \cancel{(b^l \xi'_+ \,|\, b^r \xi_+)}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad = \left(a^l \,\middle|\, d^r \frac{1}{d}\right) = \int \frac{d^l d^r}{d} = \frac{d^l(0) d^r(0)}{d(0)} = 1$$

$$(q_0 | p_0) = \left(c^l \xi'_- + d^l \xi'_+ \,\middle|\, d^r \xi_+ - b^r \xi_-\right) = \cancel{(c^l \xi'_- \,|\, b^r \xi_-) + (d^l \xi'_+ \,|\, d^r \xi_+)}$$

$$= (c^l \xi'_- \,|\, d^r \xi_+) - \cancel{(d^l \xi'_+ \,|\, b^r \xi_-)} = \left(c^l \,\middle|\, \frac{d^r}{d}\right) = \int \frac{b^l d^r}{d} = \frac{b^l(0)}{d^l(0)}$$

~~880 ...~~



$$\cancel{\begin{pmatrix} p_0 \\ q_{-1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$ 484

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d}\begin{array}{c} \qquad\quad H_+ \\ \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \\ zH_+ \qquad \end{array}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{array}{c} \qquad\quad H_- \\ \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \\ zH_- \qquad \end{array}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\cancel{0 = K(q_0, p_0) = K\Big(\overbrace{-\frac{c^r}{d}\xi'_- + \frac{d^l}{d}\xi_-}^{0}, \frac{a^l}{a}\xi_+ - \frac{b^r}{a}\xi'_+\Big)}$$

$$K(p_0, p_0) = K\Big(\frac{d^r}{d}\xi'_- + \frac{b^l}{d}\xi_-, \frac{a^l}{a}\xi_+ - \frac{b^r}{a}\xi'_+\Big)$$

$$= K\Big(\frac{d^r}{d}\xi'_-, \frac{a^l}{a}\underbrace{\xi_+}_{a\xi'_- + b\xi'_+}\Big) - K\Big(\frac{b^l}{d}\underbrace{\xi_-}_{c\xi'_- + d\xi'_+}, \frac{b^r}{a}\xi'_+\Big)$$

$$= +\Big(\frac{d^r}{d}\Big|\frac{a^l}{\cancel{a}}\cancel{a}\Big) + \Big(\frac{b^l}{\cancel{d}}\cancel{d}\Big|\frac{b^r}{a}\Big)$$

$$= \int \frac{a^r a^l}{a} + \int \frac{a^l \overbrace{b^r}^{H_-}}{a} = \frac{a^r a^l}{a}(\infty)$$

$$\begin{pmatrix} u\,p_{-1} \\ q_{-1} \end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & -h_0 \\ -\bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\bar{h}_0}{k_0} & \frac{1}{k_0} \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\bar{h}_0}{k_0} & \frac{1}{k_0} \end{pmatrix}\frac{1}{a}\begin{array}{c} \qquad\quad H_- \\ \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \\ zH_- \qquad \end{array}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$= \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ -\frac{\bar{h}_0}{k_0}a^l + \frac{1}{k_0}c^l & \frac{\bar{h}_0}{k_0}b^r + \frac{1}{k_0}a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

~~80 ...~~ but $\in H_-$ so $-\bar{h}_0 a^l(\infty) + c^l(\infty) = 0$

same as $h_0 = \dfrac{b^l(0)}{d^l(0)}$.

Observation. ~~Because p_0, q_0 is~~

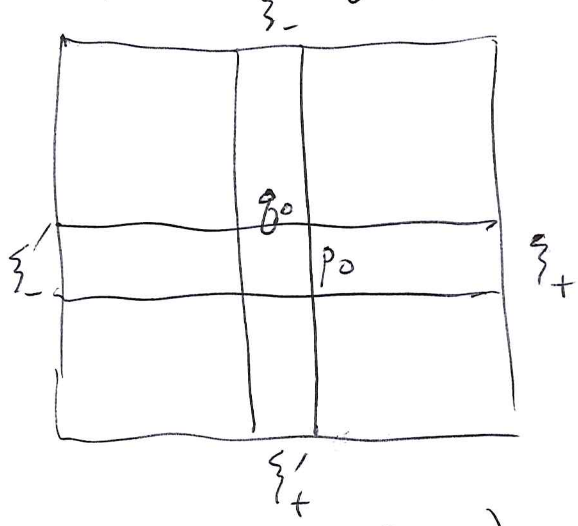
$$p_0 L_2 + q_0 L_2 = \xi'_- L_2 + \xi'_+ L_2$$

is a Krein isom, you know that

$\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$ satisfies $d^l = \overline{a^l}$ and $b^l = \overline{c^l}$

NO.

pos. def. product.



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & \overset{H_+}{b^l} \\ \underset{zH_-}{c^l} & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & \overset{H_-}{-b^r} \\ \underset{zH_+}{-c^r} & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} a^r & \overset{H_-}{b^r} \\ \underset{zH_+}{c^r} & d^r \end{pmatrix} \begin{pmatrix} a^l & \overset{H_+}{b^l} \\ \underset{zH_-}{c^l} & d^l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$d^r(0) d^l(0) = d(0)$$

$$a^r(\infty) a^l(\infty) = a(\infty).$$

$$g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$g^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so $|\det(g)|^2 = 1$. If $\det = 1$.

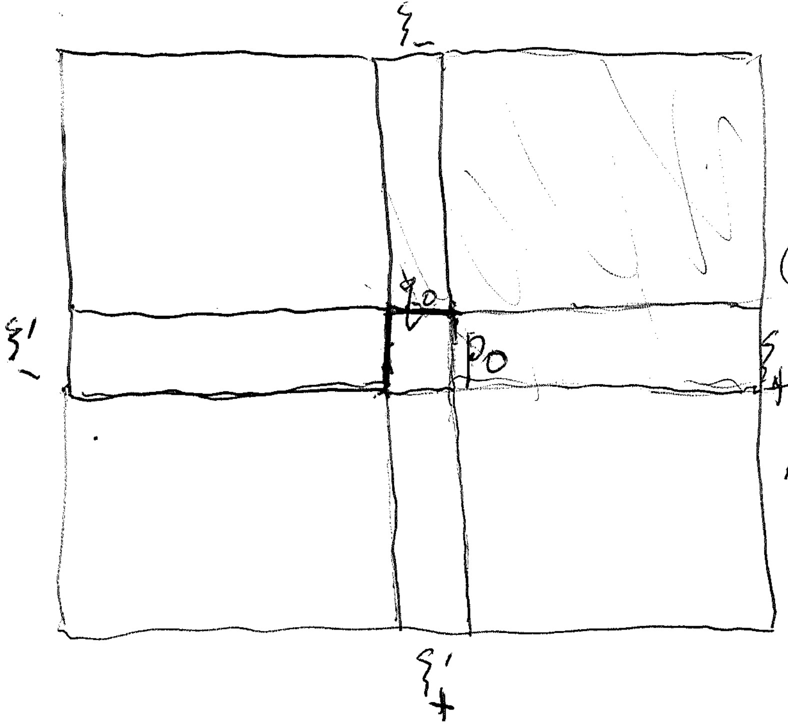
$$(\xi_- g \mid \xi_+ f) = \int \bar{g} \beta f$$

$$\| \xi_+ f + \xi_- g \|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\| (\xi_+ f + \xi_- g)^c \|^2 = \| \xi_+ \bar{g} + \xi_- \bar{f} \|^2 = \int \begin{pmatrix} \bar g \\ \bar f \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \bar g \\ \bar f \end{pmatrix}$$

$$= \int (g \;\; f) \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \bar g \\ \bar f \end{pmatrix} = \int (\bar g \;\; \bar f) \begin{pmatrix} 1 & \beta \\ \bar\beta & 1 \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix}$$

YES



So you have a conjugation on E

$$(\xi_+ f + \xi_- g)^c = \xi_+ \bar{g} + \xi_- \bar{f}$$

preserving the inner product. It carries $\xi_+ zH_+ + \xi_- H_-$ into $\xi_+ zH_+ + \xi_- H_-$

$$(\xi_+ H_+ + \xi_- H_-)^c = (\xi_+ zH_+ + \xi_- zH_-)$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^R & -b^R \\ -c^R & a^R \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{matrix} p_0 \\ q_0 \end{matrix} \text{ unique}$$

such that $\perp (\xi_+ zH_+ + \xi_- H_-)$, and such

$$p_0 \in \xi_+ H_+ + \xi_- H_-$$
$$q_0 \in \xi_+ zH_+ + \xi_- zH_-$$

~~and finally~~ you want $\|p_0\| = \|q_0\| = 1$.

and phase condition $d^R(0) > 0$, $a^R(\infty) > 0$.

Module M over $A = \mathbb{C}[z, z^{-1}]$ generated by general solution of DE, has SU(1,1) ~~structure~~ U(1,1) structure $\cap$ $SL_2(\mathbb{C})$ structure. How to make this precise.

Let V be a 2 diml complex v.s. with hermitian form K of type (1,1). Then

K induces a hermitian form on $\Lambda^2 V$

$$K(v_1 \wedge v_2, v_1 \wedge v_2) = \begin{vmatrix} K(v_1, v_1) & K(v_1, v_2) \\ K(v_2, v_1) & K(v_2, v_2) \end{vmatrix}$$

So take v_1 with $K(v_1) = 1$, $v_2 \in v_1^\perp$ $K(v_2) = -1$. Then $K(v_1 \wedge v_2, v_1 \wedge v_2) = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$

Suppose chosen in $\Lambda^2 V$ an elt $v_1 \wedge v_2$ with $K(v_1 \wedge v_2) = -1$

You claim V gets a real structure

Suppose v_+, v_- and w_+, w_- two bases related by an $SU(1,1)$-matrix $\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_+ \\ w_- \end{pmatrix}$

Real structure

$d = \bar{a}$, $c = \bar{b}$ $ad - bc = 1$.

Then you have a Wronskian pairing $\Lambda^2 V \to \mathbb{C}$. ?

What you need to do is define an M the structures of interest: Wronskian, conjugation, hermitian form K, positive def herm. form (?)

Use the completion $E_\cdot = L^2\xi_+ + L^2\xi_-$ or $L^2\xi_+ + L^2\xi'_+$

It seems that hermitian form K is "over" A ~~...~~ an A-valued hermitian form, like a Hilbert module. What about the pos. def. form?

e.g. take $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$ and write in unitary form $\begin{pmatrix} p_0 \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d^\ell} & \frac{b^\ell}{d^\ell} \\ -\frac{c^\ell}{d^\ell} & \frac{1}{d^\ell} \end{pmatrix} \begin{pmatrix} \xi'_- \\ q_0 \end{pmatrix}$

$$p_0 = \frac{1}{d^\ell}\xi'_- + \frac{b^\ell}{d^\ell} q_0$$

$$(q_0 | p_0) = \left(q_0 \middle| \frac{1}{d^\ell}\xi'_-\right) + \left(q_0 \middle| \frac{b^\ell}{d^\ell} q_0\right) \quad ?$$

You want structure under control.

Look at M = the $A = \mathbb{C}[z, z^{-1}]$ module gen. by $u^{-n}p_n, q_n$ subject to DE relations. Since transition between different n are $SU(1,1)$-matrices, you get an $SU(1,1)$ structure over the circle. What is an $SU(1,1)$ structure on a 2 dim'l vector space V. It consists of ~~...~~ a torsor of "admissible isos." $\mathbb{C}^2 \xrightarrow{\sim} V$, a torsor for the group $SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix} \middle| |a|^2 - |b|^2 = 1 \right\}$.

Conjugation? $\begin{pmatrix} x \\ y \end{pmatrix}^c = \begin{pmatrix} \bar y \\ \bar x \end{pmatrix}$ $\qquad \begin{pmatrix} ax + by \\ \bar b x + \bar a y \end{pmatrix}^c = \begin{pmatrix} b\bar x + a \bar y \\ \bar a \bar x + \bar b \bar y \end{pmatrix}$

$$= \begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix} \begin{pmatrix} \bar y \\ \bar x \end{pmatrix}$$

$\therefore (gv)^c = g v^c$

skew form. $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \wedge \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

$$\begin{pmatrix} \bar y_1 \\ \bar x_1 \end{pmatrix} \wedge \begin{pmatrix} \bar y_2 \\ \bar x_2 \end{pmatrix} = \begin{vmatrix} \bar y_1 & \bar y_2 \\ \bar x_1 & \bar x_2 \end{vmatrix} = -\overline{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \wedge \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}}$$

OKAY because $(e_1 \wedge e_2)^c = e_2 \wedge e_1 = -e_1 \wedge e_2$.

What next. You have recursion relations

$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n u^{-n} \\ \bar h_n u^n & 1 \end{pmatrix} \begin{pmatrix} u^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

If $\sum |h_n| < \infty$ then can obtain

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$(\xi_- L^2)^{\perp^K} = (\xi'_- + \xi_- b) L^2 \simeq \{(\xi'_- + \xi_- b) f \mid \text{norm}$$

$$K(\xi'_- f + \xi_- b f) = \int \begin{pmatrix} f \\ bf \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ bf \end{pmatrix}$$

$$= \int \begin{pmatrix} f \\ bf \end{pmatrix}^* \begin{pmatrix} (1+|b|^2) f \\ 0 \end{pmatrix} = \int |f|^2 + |bf|^2$$

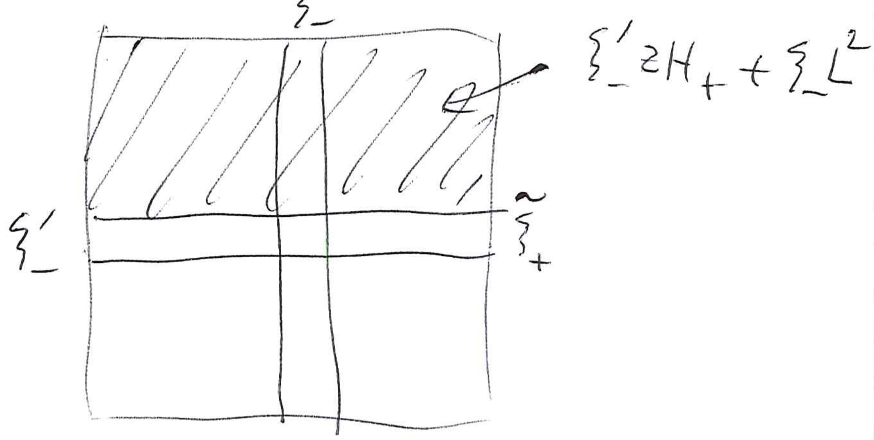
Prediction result: $d\mu = \rho \frac{d\theta}{2\pi}$

$$\tilde g \in 1 + zH_+(d\mu) \subset L^2(d\mu)$$

Problem is: ~~Structure of argum~~ Does $\exists$ link with graph construction, is there any significance to the T, T^*, HTT^* appearing for the K_- orthogonal projection onto the half space $\xi'_- zH_+ + \xi_- L^2$, commutation with graph construction.

Do this again.

$$(\xi_- L^2)^{\perp^K} = (\xi'_- + \xi_- b) L^2$$



$$\tilde\xi_+ = \xi'_-(1-\phi) + \xi_-(-\psi)$$

$$\begin{pmatrix} \pi_1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} 1-\phi \\ -\psi \end{pmatrix} = \begin{pmatrix} \pi_1(1-\phi-\bar b\psi) \\ b - b\phi + \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\phi + \pi_1 \bar b \psi = 0$$
$$b\phi - \psi = +b$$

$$\begin{pmatrix} 1 & T^* \\ \underset{b\varepsilon_1}{T} & -1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\begin{pmatrix} zH_+ \\ L^2 \end{pmatrix} \xrightarrow{\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} Id, & T^* \\ T & -Id \end{pmatrix} \Big\downarrow \qquad \Big\downarrow \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}$$

$$\begin{pmatrix} zH_+ \\ L^2 \end{pmatrix} \xleftarrow{\begin{pmatrix} \pi_1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

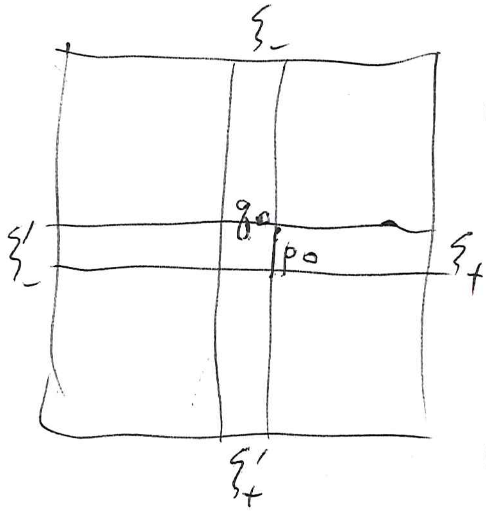
$$\begin{pmatrix} \pi_1 \varepsilon_1 & \pi_1 \bar{b} \\ b\varepsilon_1 & -1 \end{pmatrix}$$

$$\xi'_- : \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \pi_1 & \pi_1 \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

What is important, possibly, is the analogy $\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}$ vs. $\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}$

One thing you need to understand rapidly is why inverse scattering leads to ~~quant~~ quantum groups. Inverse scattering goes from scattering data i.e. an element of the loop group to a potential. From my viewpoint you go from b function on S^1 to a sequence h_n of complex numbers, which ~~[illegible]~~ map is ~~something like~~ a non-linear transform agreeing to first order with the F.T.



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \underset{zH_-}{\overset{H_+}{\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}}} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \underset{zH_+}{\overset{H_-}{\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}}} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d} \underset{zH_+}{\overset{zH_-}{\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}}} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a} \overset{H_-}{\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

~~$\frac{a\ell(0)}{b}\ell(0)$~~ You get a formula for h_0 namely $\frac{b^{\ell}}{d^{\ell}}(0)$. Recall how.

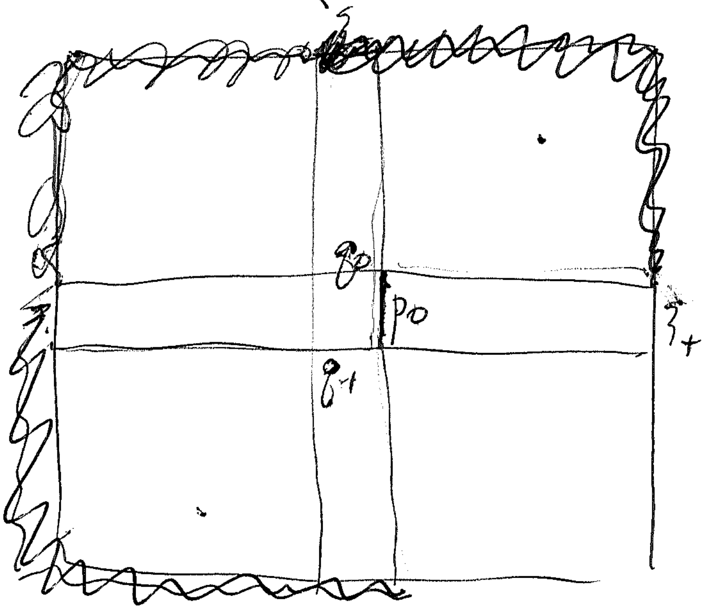
$$(q_0|q_0) = \left(\xi_-' \frac{d^r}{d} + \xi_- \frac{b^\ell}{d} \,\middle|\, \xi_-' \frac{-c^r}{d} + \xi_- \frac{d^\ell}{d} \right)$$

$$\cancel{= \int \underbrace{\left(\frac{d^r}{d}\right)^*}_{H_+} \underbrace{\left(\frac{-c^r}{d}\right)}_{zH_+} + \underbrace{\left(\frac{b^\ell}{d}\right)^*}_{H_+} \left(\frac{d^\ell}{d}\right)}$$

$$\cancel{(q_0|p_0) = \left(\xi_-' \left(\frac{-c^r}{d}\right) + \xi_- \left(\frac{d^\ell}{d}\right) \,\middle|\, \xi_+ \right)}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix}$$

$$(q_0|p_0) = \left(\xi_+(-c^r) + \xi_-(a^r) \,\middle|\, \xi_-' a^\ell + \xi_+' b^\ell \right)$$

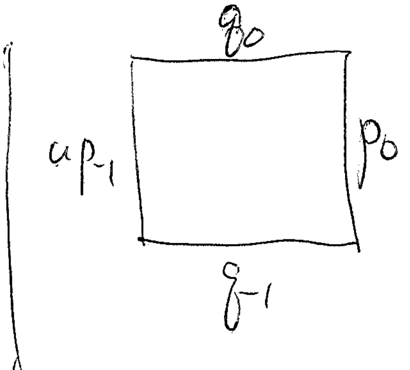


$$\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

$$\left(\xi_- a^r \,\middle|\, \cancel{\xi_-'\left(-\frac{c}{d}\right)} + \frac{1}{d}\xi_- \right) b^\ell$$

$$\left(\xi_- \,\middle|\, \xi_- \frac{b^\ell d^r}{d} \right) = \frac{b^\ell d^r}{d}(0)$$

$$= \frac{b^\ell}{d^\ell}(0)$$



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{k_0} \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} up_{-1} \\ q_{-1} \end{pmatrix}$$

$$q_0 = \frac{\bar{h}_0}{k_0} up_{-1} + \frac{1}{k_0} q_{-1} \qquad \boxed{K(q_0, up_{-1}) = \frac{h_0}{k_0}}$$

$$\begin{pmatrix} up_{-1} \\ q_{-1} \end{pmatrix} = \frac{1}{k_0} \begin{pmatrix} 1 & -h_0 \\ -\bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

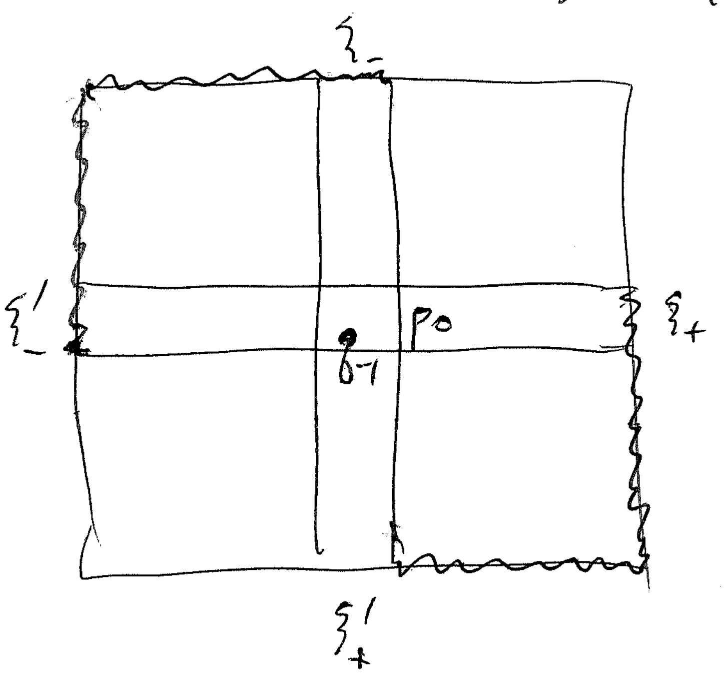
$$q_{-1} = -\frac{\bar{h}_0}{k_0} p_0 + \frac{1}{k_0} q_0$$

$$\boxed{K(q_{-1}, p_0) = -\frac{h_0}{k_0}}$$

$$\begin{pmatrix} p_0 \\ q_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{h_0}{k_0} & \frac{1}{k_0} \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{h_0}{k_0} & \frac{1}{k_0} \end{pmatrix} \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{h_0}{k_0} & \frac{1}{k_0} \end{pmatrix} \frac{1}{a} \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$



$$p_0 = \xi_+\left(\frac{a^l}{a}\right) + \xi'_+\left(\frac{-b^r}{a}\right)$$

$$q_{-1} = \xi'_-\left(-\frac{h_0}{k_0}\frac{d^r}{d} - \frac{c^r}{k_0 d}\right) + \xi_-\left(-\frac{h_0}{k_0}\frac{b^l}{d} + \frac{1}{k_0}\frac{d^l}{d}\right)$$

$$K(q_{-1}, p_0) = K\left(\xi'_-\left(-\frac{h_0 d^r + c^r}{k_0 d}\right), \boxed{\xi_+}\left(\frac{a^l}{a}\right)\right)$$

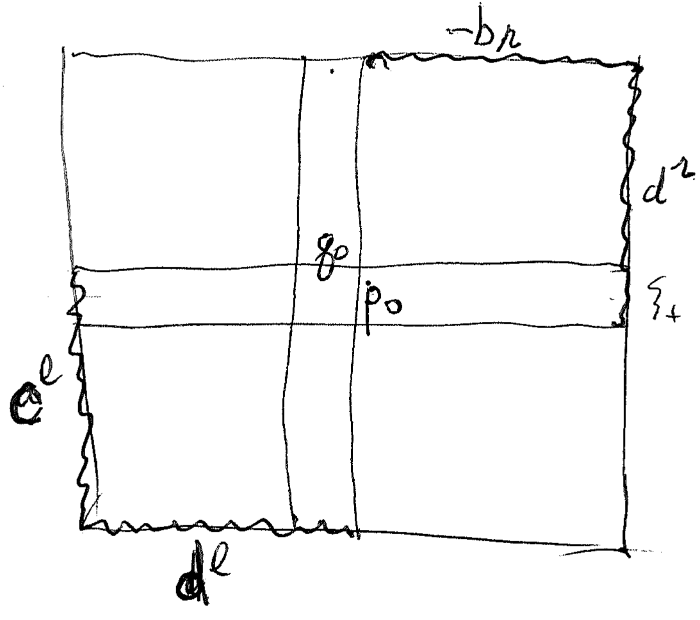
$$\xi'_- a + \xi'_+ b$$

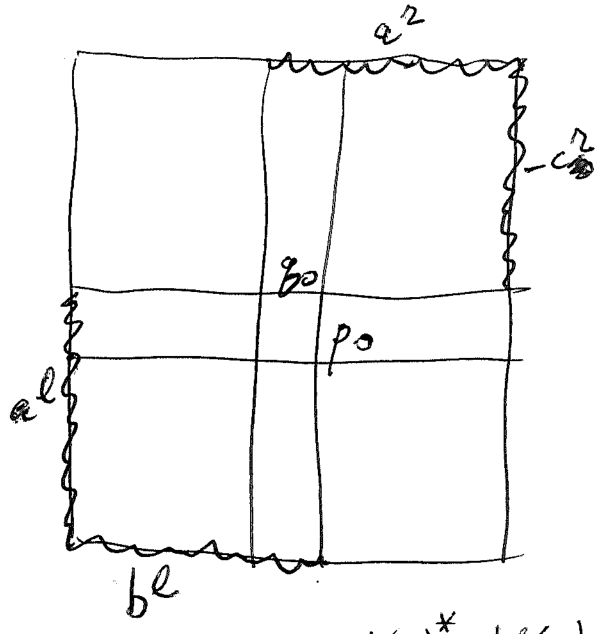
$$= K\left(\xi'_-\left(-\frac{h_0 d^r + c^r}{k_0 d}\right), \ \xi'_- a \frac{a^l}{a}\right)$$

$$= \int \left(-\frac{h_0 d^r + c^r}{k_0 d}\right)^* \left(a^l\right) = \int\left(-\frac{\overset{H_-}{h_0 a^r} + \overset{zH_-}{\cancel{b^r}}}{k_0 a}\right) a^l$$

$$= -\int \frac{h_0}{k_0}\frac{a^r a^l}{a} = -\frac{h_0}{k_0}$$

$(q_0|p_0) = h_0.$

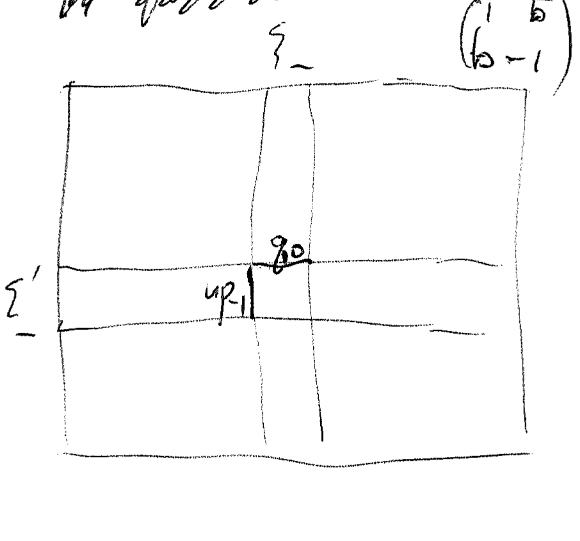




$$\frac{a^r(\infty)^*\, b^l(0)}{d(0)} = \frac{b^l(0)}{d^l(0)}$$

$$(q_0|p_0) = (\xi'_- c^l \,|\, \xi_+ d^r)$$
$$= \left(\xi'_- c^l \,\middle|\, \xi'_- \frac{1}{d} d^r + \cancel{\xi_-}\right) = \frac{b^l d^r}{d}(0) = \frac{b^l(0)}{d^l(0)}.$$

Review what we did before



$up_{-1} = \xi'_-(s-f) + \xi_-(-g)$ $\quad f,g \in zH_+$
$\quad \phi, \psi$

$q_0 = \xi'_-(-\phi) + \xi_-(t-\psi)$ $\quad s,t>0$

$\pi_1(s-f-\bar b g) = 0$ $\quad f + T^* g = 0$

$\pi_1(bs - bf + g) = 0$ $\quad Tf - g = s\pi_1 b$

$\pi_1(-\phi + \bar b t - \bar b \psi) = 0$ $\quad \phi + T^*\psi = +t\pi_1 \bar b$

$\pi_1(-b\phi - \cancel{t} + \psi) = 0$ $\quad T\phi - \psi = 0$

$$K(up_{-1}) = \int \begin{pmatrix} s-f \\ -g\end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1\end{pmatrix}\begin{pmatrix} s-f \\ -g\end{pmatrix} = \int s(s-f-\bar b g)$$
$$= s^2 - \int (s\pi_1 b)^* g = s^2 - \int (Tf-g)^* g = s^2 + \int -f^* T^* g + g^* g$$
$$= s^2 + \|f\|^2 + \|g\|^2 = \underline{1}.$$

$f + T^* g = 0 \qquad g = Tf - \pi_1 b$

$Tf - g = \pi_1 b \qquad f + T^*(Tf - \pi_1 b) = 0 \qquad (1+T^*T) f = T^* \pi_1 b$

$\therefore \underline{f} = (1+T^*T)^{-1} T^* \pi_1 b \qquad T = \pi_1 b \varepsilon_1$

Describe exactly what's being done. You have ~~$\mathcal{E} = \xi'_- L^2 + \xi_- L^2$~~ graph construction (part of C.T.) given $T : H_1 \to H_2$, look at $\Gamma_T \subset H_1 \oplus H_2$ $\qquad \Gamma_T = \begin{pmatrix} 1 \\ T \end{pmatrix} H_1 \quad (\Gamma_T)^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} H_2$

$\begin{pmatrix} \varepsilon \\ A \end{pmatrix}(X) \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ isotropic for $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= i(y_1^* y_2 - y_2^* y_1)$

$= -2\,\mathrm{Im}(y_1^* y_2)$

$\begin{pmatrix} \varepsilon \\ A \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ A \end{pmatrix} = \varepsilon^* A - A^* \varepsilon = 0$

You want $\varepsilon^* \varepsilon = 1$.

So what next? You want to find a good viewpoint. ~~...~~ It should involve the graph construction. Review a little.

Given $T : H \to H'$ ~~map~~ bdd op between Hilbert space, form its graph $\Gamma_T = \begin{pmatrix} 1 \\ T \end{pmatrix} H \subset H \oplus H'$.

$\Gamma_T^\perp = \left\{ \begin{pmatrix} \xi \\ \xi' \end{pmatrix} \middle| \cancel{\begin{pmatrix} H \\ H' \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ T\eta \end{pmatrix} = 0} \right.$ equiv.

$\begin{pmatrix} \xi \\ \xi' \end{pmatrix}^* \begin{pmatrix} 1 \\ T \end{pmatrix} H = 0 \Big\} \qquad \xi + T^* \xi' = 0$

$\therefore (\Gamma_T)^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} H'$.

Put another way – equip $H \oplus H'$ with the hermitian form $K\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \cancel{\begin{pmatrix}\xi\\ \eta\end{pmatrix}^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}}$

$$K\left(\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right) = \frac{1}{i}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^*\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \frac{1}{i}\left(\xi^*\eta - \eta^*\xi\right) = \cancel{2}\,\mathrm{Im}(\xi^*\eta)$$

Then $K\left(\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \begin{pmatrix}1\\ T\end{pmatrix}\xi_1\right) = \cancel{\frac{1}{i}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^*\begin{pmatrix}\xi_1\\ -T\xi_1\end{pmatrix} \quad \frac{1}{i}\left(\xi - T^*\eta\right)^*\xi_1}$

$$\frac{1}{i}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^*\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1\\ T\end{pmatrix}\xi_1 = \frac{1}{i}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^*\begin{pmatrix}T\\ -1\end{pmatrix}\xi_1$$

$$= \frac{1}{i}\left(\xi^*T - \eta^*\right)\xi_1 = \frac{1}{i}\left(T^*\xi - \eta\right)^*\xi_1 = 0 \quad \forall \xi_1 \iff \begin{pmatrix}\xi\\ \eta\end{pmatrix} \in \begin{pmatrix}1\\ T^*\end{pmatrix}H$$

unclear if this means something

~~In any case you have a graph.~~

$$\boxed{(\Gamma_T)^{\perp_K} = \Gamma_{T^*} \quad \text{for this particular herm. form } K.}$$

Go back to
$$\begin{matrix}H\\ \oplus\\ H'\end{matrix} = \begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix}\begin{pmatrix}H\\ H'\end{pmatrix} = \begin{pmatrix}1\\ T\end{pmatrix}H + \begin{pmatrix}-T^*\\ 1\end{pmatrix}H'$$

This is orthogonal splitting. (Kasparov idea where adjoints need not exist comes to mind.)

~~[illegible] Point is clear~~

Describe situation, You have op b on L^2, you form $\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}$ s.a. on $L^2 \oplus L^2$, ~~compress~~ you compress this to $z^m H_+ \oplus z^m H_-$ to get

$$\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix} \quad \text{where} \quad T = \pi_m b \, \varepsilon_m$$

$$= \begin{pmatrix} \pi_n & 0 \\ 0 & \pi_m \end{pmatrix} \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} \varepsilon_n & \\ & \varepsilon_m \end{pmatrix}$$

~~Does the form~~ T is a Toeplitz operator. It's norm ~~doesn't~~ shouldn't change. In fact you get a standard extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

as generalized by Pimsner.

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algebra generated by ε, X $\quad \varepsilon^2 = 1$, $\varepsilon X + X\varepsilon = 0$

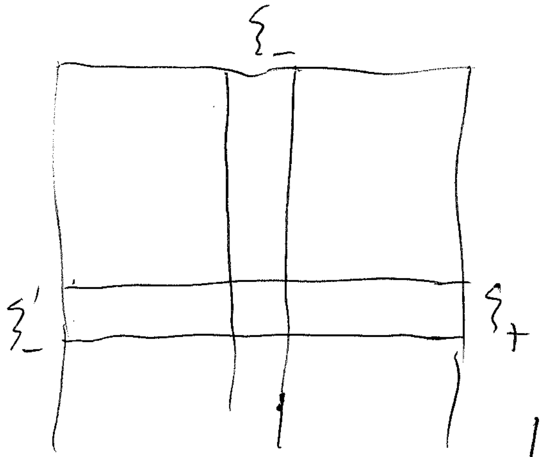
~~has $X \ldots$~~ crossed product ~~of~~ $k[\varepsilon] \otimes k[X]$ where ε anticommutes with X, center $k[X^2]$.

Given a module $M = M_+ \oplus M_-$ $\quad \varepsilon = \pm 1$ on $M_\pm$

$$X = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} \qquad (1+X)\varepsilon = \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}$$

$$(1+X)\varepsilon(1+X)\varepsilon = 1 - X^2 = \begin{pmatrix} 1+T^*T & \\ & 1-TT^* \end{pmatrix} \qquad g^{1/2}\varepsilon$$

$$\frac{(1+X)\varepsilon}{\sqrt{1-X^2}} = \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix} \begin{pmatrix} (1+T^*T)^{-1/2} & 0 \\ 0 & (1+TT^*)^{-1/2} \end{pmatrix}$$



$$K(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

$$\tilde{\xi}_+ = \xi'_-(1-f) + \xi_-(-g) \qquad f \in zH_+ \quad g \in L^2.$$

$$\perp \left(\xi'_- zH_+ + \xi_- L^2 \right)$$

1st method $\perp \xi_- L^2$ means

$$\int \begin{pmatrix} 0 \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1-f \\ -g \end{pmatrix} = 0 \qquad \text{i.e.} \quad b(1-f) + g = 0.$$

so $\tilde{\xi}_+ = \left(\xi'_- + \xi_-(+b)\right)(1-f)$ set $\tilde{f} = 1-f$

then $\tilde{\xi}_+ = (\xi'_- + \xi_- b)\tilde{f}$ is to be $\perp \xi'_- zH_+$

$$\int \begin{pmatrix} zH_+ \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} \tilde{f} \\ b\tilde{f} \end{pmatrix} = 0 \quad \text{means} \quad (1+|b|^2)\tilde{f} \in zH_-$$

This should imply: $(1+|b|^2)\tilde{f}\bar{\tilde{f}} \in zH_- \overline{H_+} = zH_-$

$\therefore (1+|b|^2)|\tilde{f}|^2 \in \mathbb{C}$. etc. What else?

You have this approach based on doing orth the other way. This means:

$$\begin{pmatrix} \pi_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1-f \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

where $T = b\varepsilon_1 : zH_+ \xrightarrow{\varepsilon_1} L^2 \xrightarrow{b} L^2$. Check that the solution of this equation gives the same result.

$$f + T^* g = 0 \qquad Tf - g = b$$

$$f + \pi_1 \bar{b} g = 0 \qquad b - bf + g = 0$$

$\pi_1(\tilde f - \bar b g) = 0$ $\qquad b\tilde f + g = 0$

$\pi_1(\tilde f - \bar b(-b\tilde f)) = 0$ $\qquad \underbrace{\pi_1((1+\bar b b)\tilde f) = 0}$

means exactly that $(1+|b|^2)\tilde f \in zH_-$

So given $b \in L^\infty$ you construct in this way a $\tilde f \in \overset{1+z}{}H_+ \ni (1+|b|^2)\tilde f \in zH_-$. In fact

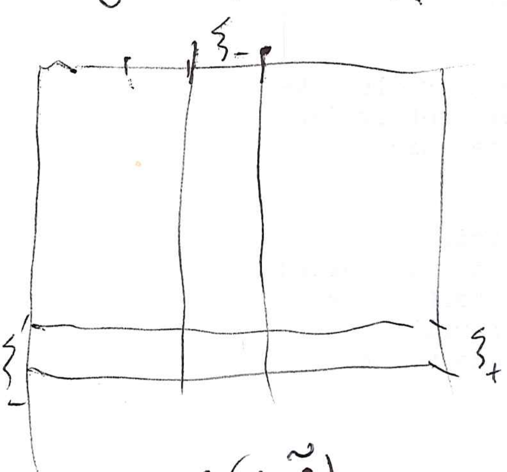
$\tilde f = 1-f \in 1+zH_+$ and $(1+|b|^2)(1-f) \in zH_-$

$1-\bar f \in 1+H_-$ $\qquad (1+|b|^2)(1-f)(1-\bar f) \in zH_- \cap \overline{zH_-} = 0$

why is $1-f$ invertible on $|z| \leq 1$?

Go back to the pos. def. case. Yes.

Given b let $\beta = \frac{b}{\sqrt{1+|b|^2}}$ ~~tentatively~~, then construct δ invertible analytic on disk with $|\delta|^2 = 1-|\beta|^2 = \frac{1}{1+|b|^2}$. This δ ~~should~~ depends only upon $|\beta|$.



$$\|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix}f\\g\end{pmatrix}^* \begin{pmatrix}1 & \bar\beta\\ \beta & 1\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}$$

$$\tilde\xi'_- = \xi_+(\overset{zH_+}{1-f}) + \xi_-(\overset{L^2}{-g})$$

$$\begin{pmatrix}\pi_1 & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \bar\beta\\ \beta & 1\end{pmatrix}\begin{pmatrix}1-f\\ -g\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\beta(\tilde f) = g$ $\qquad \cancel{\tilde f = \pi_1\bar\beta g \quad \pi_1\bar\beta\beta\tilde f}$

$\pi_1(\tilde f) = \pi_1\bar\beta g$ $\qquad \cancel{(1-|\beta|^2)\tilde f = 1}$

$$0 = \pi_1(\tilde f - \bar\beta g) = \pi_1(\tilde f - \bar\beta\beta\tilde f) \qquad (1-|\beta|^2)\tilde f \in zH_-$$

so $\quad f \longmapsto (\xi_-' + \xi_- b) f \quad$ isom. of TVS.

$$L^2 \longrightarrow \underbrace{(\xi_-' + \xi_- b)}_{\eta} L^2 \underset{\text{closed.}}{\subset} E$$

$$L^2 \longrightarrow \eta L^2 \longleftarrow L^2$$

$$f \longmapsto \eta f$$

$$qg \longleftarrow g$$

$$\eta H_+ \ominus \eta z H_+ = \mathbb{C} q$$

$$\eta = c q$$

$L^2(S^1, d\mu) \qquad \tilde{q} = 1 + a_1 z + \cdots + a_n z^n \oplus z^{n+1} H_+$

$H_+ \overset{\tilde{q}}{\supset} zH_+$. Then get

Do it. Hilbert space $\eta \cdot L^2 \overset{\text{closed}}{\subset} E \qquad \|\eta f\|^2 = \int f^*(1+|b|^2) f = \|f\|^2 + \|bf\|^2$

Have map $\quad L^2 \longrightarrow \eta L^2 \quad$ isom. of TVS.

$\qquad f \longmapsto \eta f$

$V = \eta H_+ \subset \eta L^2 \qquad V$ closed stable under z

moreover $\begin{cases} \bigcup u^n V & \text{dense in } \eta L^2 \\ \bigcap u^n V = 0 \end{cases}$

Then get $\quad L^2 \xrightarrow{\sim} \eta L^2 \quad$ isom. of H.S. with u.

$\qquad f \qquad\qquad \eta q f \qquad \eta q$ spans $V \ominus uV$

~~Thus~~ Have Hilb space with z action

$E' = \eta L^2$ have $\eta H_+ \ominus \eta z H_+ \ni \eta g$

Then $\eta z^n g$ orth basis for ηL^2

$\eta z^n g \quad n \geq 0$ ———— ηH_+

$$\begin{array}{ccc} \eta L^2 & \xleftarrow{\sim} & L^2 \\ \eta g & & 1 \\ \cap & & \\ \eta H_+ & & H_+ \end{array}$$

~~$\eta \in \eta H_+$~~ $\eta g z^n, n \geq 0$ orth basis for $\overset{V}{\overset{\|}{\eta H_+}}$

so $\exists f \in H_+$ such that $\eta g f = \eta$

Say it again. You have $\begin{array}{ccc} L^2 & \xrightarrow{\sim} & E' \\ f & \longmapsto & \eta f \end{array}$ iso of TVS$_u$ with

also have $\overset{\text{Hilb}}{}$ iso $\begin{array}{ccc} E' & \longleftarrow & L^2 \\ \eta g \mathfrak{g} & \longleftarrow\!\shortmid & \mathfrak{g} \end{array}$ where $g \in H_+$

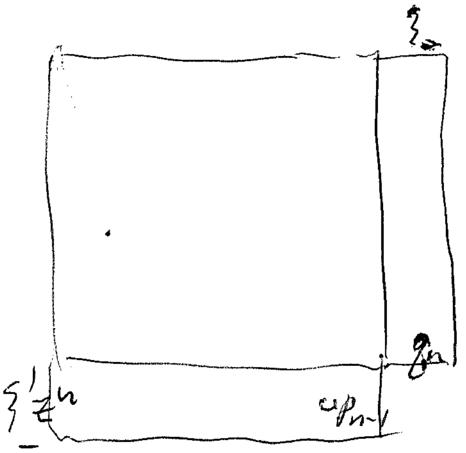
$$\begin{array}{ccccc} L^2 & \xrightarrow{\sim} & E' & \xleftarrow{\sim} & L^2 \\ f & \longmapsto & \eta f & & \\ & & \eta g \mathfrak{g} & \longleftarrow\!\shortmid & \mathfrak{g} \\ g\mathfrak{g} & & \longleftarrow\!\shortmid & & \mathfrak{g} \end{array}$$

So g inv. in C^∞

But also this is an isom of $\begin{array}{ccc} H_+ & \xleftarrow{\sim} & H_+ \\ g\mathfrak{g} & & \mathfrak{g} \end{array}$

$\therefore$ ~~g~~ $g \in H_+^\infty \ni$

Back to K- saturation



$$\widetilde{up}_{n-1} = \underbrace{\xi'_-(z^n - f)}_{z^{n+1}H_+} + \underbrace{\xi_-(-g)}_{zH_+}$$

$$\tilde{g}_n = \xi'_-(-\phi) + \xi_-(1-\psi)$$

$$\begin{pmatrix} \pi_{n+1} & 0 \\ 0 & \pi_1 \end{pmatrix}\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\begin{pmatrix} z^n - f & -\phi \\ -g & 1-\psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}\begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} 0 & \pi_{n+1}\bar{b} \\ \pi_1(bz^n) & 0 \end{pmatrix}$$

$$(1+TT^*)g = -\pi_1(bz^n)$$

$$(1+T^*T)\phi = \pi_{n+1}(\bar{b}) \qquad f + T^*g = 0, \quad T\phi = \psi \qquad \boxed{T = \pi_1 b \varepsilon_{n+1}}$$

$$\xi'_- z^n = \widetilde{up}_{n-1} + \xi'_- f + \xi_- g$$

$$\xi_- = \tilde{g}_n + \xi'_-\phi + \xi_-\psi$$

$$K(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \int g^*(-1-TT^*)g = -\|g\|^2 - \|f\|^2$$

$$K(\xi'_-\phi + \xi_-\psi, \xi'_- f + \xi_- g) = \int \begin{pmatrix} \phi \\ \psi \end{pmatrix}^* \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \int \psi^*(-1-TT^*)g$$

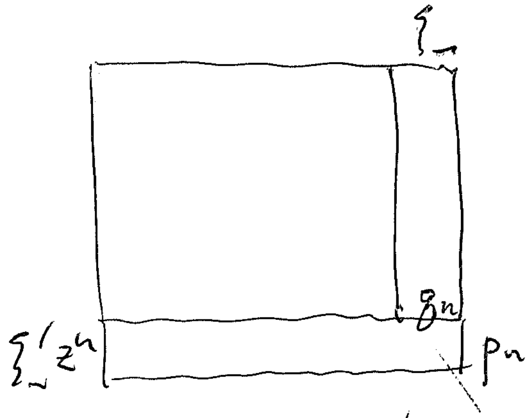
$$K(\xi'_-\phi + \xi_-\psi) = \int \begin{pmatrix} \phi \\ \psi \end{pmatrix}^* \begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \int \phi^*(1+T^*T)\phi$$

$$\int bz^n = K(\xi_-, \xi'_- z^n) = K(\tilde{g}_n, \widetilde{up}_{n-1}) + \underbrace{\int \psi^*(-1-TT^*)g}_{Q}$$

$$Q = \int \phi^* T^* \pi_1(bz^n) = \int (\pi_{n+1}(\bar{b}))^* (1+T^*T)^{-1} T^* \pi_1(bz^n)$$

$$bz^n = \sum_{j \in \mathbb{Z}} b_j z^{n-j} \qquad \pi_1(bz^n) = \sum_{j<n} b_j z^{n-j}$$

$$\bar{b} = \sum \bar{b}_j z^j \qquad \pi_{n+1}(\bar{b}) = \sum_{j>n} \bar{b}_j z^j$$



$$\tilde{p}_n = \underset{z^{n+1}H_+}{\xi'_-(z^n - f)} + \underset{H_+}{\xi_-(-g)}$$

$$\tilde{q}_n = \underset{z^{n+1}H_+}{\xi'_-(-\phi)} + \underset{zH_+}{\xi_-(1-\psi)}$$

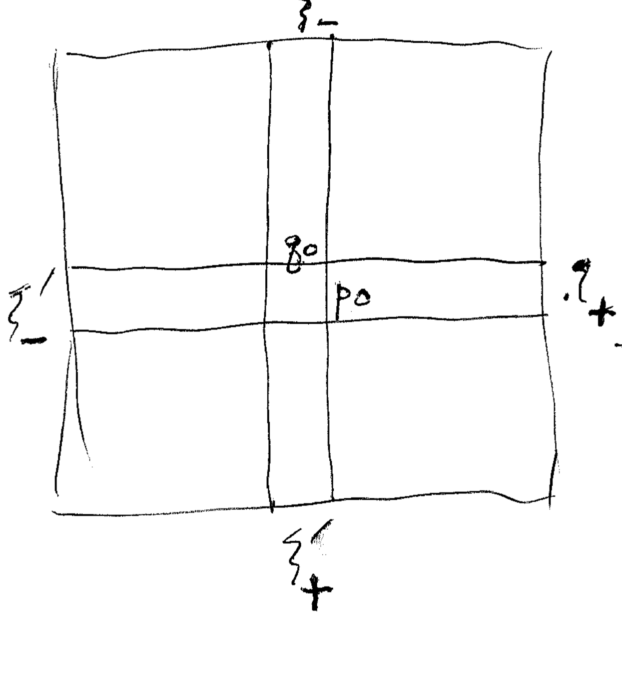
You want to make a serious effort to push this through.

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & \overset{H_+}{b^l} \\ \underset{zH_-}{c^l} & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -\overset{H_-}{b^r} \\ \underset{zH_+}{-c^r} & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

Is there some way to proceed, to obtain the desired estimate using the indef. norm picture

IDEA: fix $n=0$, use conjugation in some way to convert ξ'_-, p_0 to ξ'_+, q_0. Then you are probably ~~[illegible]~~ in a position to handle the fact that $\|\tilde{q}_0\|$, $\|\tilde{p}_0\|$ are complementary.

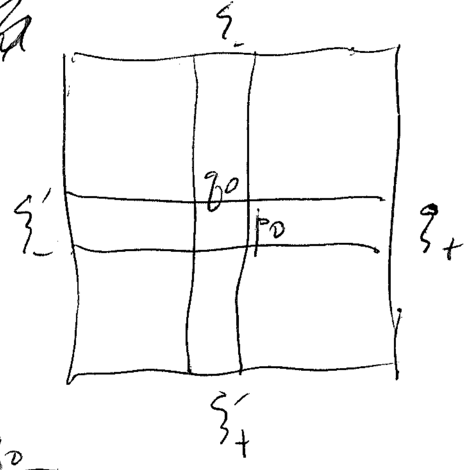


$$E = \xi_+ L^2 + \xi_- L^2$$

$$K(\xi_+ f + \xi_- g) =$$

~~Construct~~ Conjugation depends on a choice of origin.

~~Sketch~~

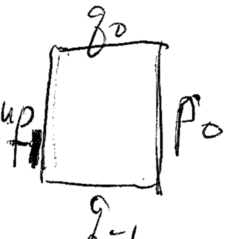


$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$a=\bar{d}$
$c=\bar{b}$
$ad-bc=1$

Wronskian, conjugation

$$(\xi_+ f + \xi_- g)^c = \xi_+ \bar{g} + \xi_- \bar{f}$$



$$\begin{pmatrix}\xi_-\\ \xi_+\end{pmatrix} = \begin{pmatrix}d & c\\ b & a\end{pmatrix}\begin{pmatrix}\xi'^{c}_-\\ \xi'^{c}_+\end{pmatrix}$$

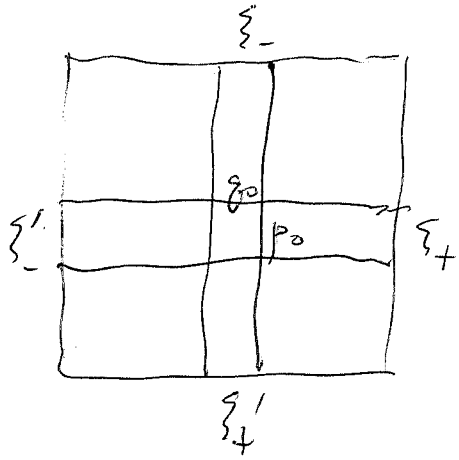
$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}b & a\\ d & c\end{pmatrix}\begin{pmatrix}\xi'^{c}_-\\ \xi'^{c}_+\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'^{c}_+\\ \xi'^{c}_-\end{pmatrix}$$

~~Idea~~. You understand splitting, i.e. take $\xi'_- z^m H_+ + \xi_- z^n H_+$ identify its K-orth comp with $\xi_+ z^m H_- + \xi'_+ z^n H_-$ and E is the direct sum. Why?

$$\begin{pmatrix}\xi'_- & \xi_-\end{pmatrix}\begin{pmatrix}z^m H_+\\ z^n H_+\end{pmatrix} \overset{?}{\oplus} \underbrace{\begin{pmatrix}\xi_+ & \xi'_+\end{pmatrix}}\begin{pmatrix}z^m H_-\\ z^n H_-\end{pmatrix} = (\xi'_-\ \xi_-)\begin{pmatrix}L^2\\ L^2\end{pmatrix}$$

$$\begin{pmatrix}\xi_- & \xi'_-\end{pmatrix}\begin{pmatrix}\frac{1}{d} & -\frac{c}{d}\\ \frac{b}{d} & \frac{1}{d}\end{pmatrix}$$

$$\begin{pmatrix}z^m H_+\\ z^n H_+\end{pmatrix} \overset{?}{\oplus} \begin{pmatrix}\frac{1}{d} & +\frac{\bar{b}}{d}\\ \frac{b}{d} & -\frac{1}{d}\end{pmatrix}\begin{pmatrix}z^m H_-\\ z^n H_-\end{pmatrix} = \begin{pmatrix}L^2\\ L^2\end{pmatrix}$$



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$$\tilde{g}_0 = \xi_-'(-\phi) + \xi_-(1-\psi) \quad \overset{K}{1} \;\; \xi_-' z H_+ + \xi_- z H_+$$

$$\begin{pmatrix} 1 & \pi_1 \bar{b}\varepsilon_1 \\ \pi_1 b \varepsilon_1 & -1 \end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix} = \begin{pmatrix} 0 \\ \pi_1 b\end{pmatrix}$$

$$\xi_- = \tilde{g}_0 + \xi_-'\phi + \xi_-\psi$$

$$K(\xi_-'\phi + \xi_-\psi) = \int \begin{pmatrix}\phi \\ \psi\end{pmatrix}^* \begin{pmatrix} \cancel{1} & \cancel{T^*} \\ T & -1\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix} = \int \psi^*(-1-TT^*)\psi$$

$$= -\|\psi\|^2 - \|\phi\|^2$$

-1
11

$$K(\xi_-) = K(\tilde{g}_0) +$$

$$\therefore\ -K(\tilde{g}_0) = 1 + \|\psi\|^2 + \|\phi\|^2 = \|\tilde{g}_0\|^2 = t^2$$

$$g_0 = \xi_-'\left(\frac{-\phi}{t}\right) + \xi_-\left(\frac{1-\psi}{t}\right) \qquad \phi, \psi \in zH_+$$

$$p_0 = \xi_+'\left(\frac{-\phi^*}{t}\right) + \xi_+\left(\frac{1-\psi^*}{t}\right) \qquad \phi^*, \psi^* \in H_-$$

$$= \begin{pmatrix}\xi_-' & \xi_-\end{pmatrix}\begin{pmatrix}\frac{1}{d} & -\frac{\bar{b}}{d} \\ \frac{b}{d} & \frac{1}{d}\end{pmatrix}\cancel{\begin{pmatrix}\frac{-\phi^*}{t} \\ \frac{1-\psi^*}{t}\end{pmatrix}}\begin{pmatrix}\frac{1-\psi^*}{t} \\ \frac{-\phi^*}{t}\end{pmatrix}$$

$$p_0 = \frac{1}{d}\left(\xi_-' \underbrace{\frac{1-\psi^* + \bar{b}\phi^*}{t}}_{H_+} + \xi_- \underbrace{\frac{b(1-\psi^*) - \phi^*}{t}}_{H_+}\right)$$

$$\begin{pmatrix}\pi_1 & 0 \\ 0 & \pi_1\end{pmatrix}\begin{pmatrix}1 & \bar{b} \\ b & -1\end{pmatrix}\begin{pmatrix}-\phi \\ 1-\psi\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$-\phi + \bar{b} - \bar{b}\psi \in zH_-$$
$$-b\phi - 1 + \psi \in zH_-$$

$$-\phi^* + b(1-\psi^*) \in H_+$$
$$-1 + \psi^* - \bar{b}\phi^* \in H_+$$

$$(g_0|p_0) = \left(\frac{-\phi}{t}\,\middle|\,\frac{1-\psi^*+\bar{b}\phi^*}{td}\right) + \left(\frac{1-\psi}{t}\,\middle|\,\frac{b(1-\psi^*)-\phi^*}{td}\right)$$

$$E = \xi'_- L^2 + \xi_- L^2 = \xi_+ L^2 + \xi'_+ L^2$$

where $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$

i.e. $\begin{pmatrix} \xi_+ & \xi_- \end{pmatrix} = \begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$W_2(\xi_+ f_1 + \xi_- g_1, \xi_+ f_2 + \xi_- g_2) = f_1 g_2 - f_2 g_1 = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

$$W_1\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}^c, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}^c\right) = \begin{vmatrix} \bar g_1 & \bar g_2 \\ \bar f_1 & \bar f_2 \end{vmatrix}$$

$$K(\xi_+ f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$K((\xi_+ f + \xi_- g)^c) = K(\xi_+ \bar g + \xi_- \bar f) = \int \begin{pmatrix} \bar g \\ \bar f \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} \bar g \\ \bar f \end{pmatrix}$$

$$= \int \begin{pmatrix} g \\ f \end{pmatrix}^t \begin{pmatrix} 1 & b \\ \bar b & -1 \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix} = \int \cancel{\begin{pmatrix} f \\ g \end{pmatrix}^t}$$

$$= \int \begin{pmatrix} \bar f \\ g \end{pmatrix}^t \begin{pmatrix} \bar b & -1 \\ 1 & b \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix} = \int \begin{pmatrix} \bar f \\ \bar g \end{pmatrix}^t \begin{pmatrix} -1 & \bar b \\ b & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} ?$$

$$E = \xi'_- L^2 + \xi'_+ L^2$$

$|a|^2 - |b|^2 = 1$
$|c|^2 - |d|^2 = -1$
$\bar a c = \bar b d \quad \frac{c}{d} = \overline{\left(\frac{b}{a}\right)}$

$$K(\xi'_- f + \xi'_+ g) = |f|^2 - |g|^2$$

$$K(\xi_+ f + \xi_- g) = \cancel{K\left(\begin{pmatrix} f \\ g \end{pmatrix}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)} K\left(\begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}\right)$$

$$= |af + cg|^2 - |bf + dg|^2 = \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} \bar a a - \bar b b & \bar a c - \bar b d \\ \bar c a - \bar d b & \bar c c - \bar d d \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\xi_+ \wedge \xi_- = 1 \qquad \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \wedge \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \overset{?}{=} i \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

$$\xi_+^c = \xi_- \implies \xi_+ = \xi_-^c$$

$$(\xi_+ \wedge \xi_-)^c = \xi_- \wedge \xi_+ = -\xi_+ \wedge \xi_-$$

So $\quad Wr(\xi_+, \xi_-) = i$

$$Wr(\xi_+ f_1 + \xi_- g_1, \xi_+ f_2 + \xi_- g_2) = i \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} \qquad \text{YES!!}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix}$$

$$\underset{i}{Wr(\xi_+, \xi_-)} = Wr(a\xi_-' + b\xi_+', c\xi_-' + d\xi_+')$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \underset{i}{Wr(\xi_-', \xi_+')}$$

Next : $\quad \xi_+ = a\xi_-' + b\xi_+'$
$\quad \xi_- = c\xi_-' + d\xi_+'$

$$\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+' \\ \xi_-' \end{pmatrix} = \begin{pmatrix} \xi_-'^c \\ \xi_+'^c \end{pmatrix} = \begin{pmatrix} \bar d & -\bar b \\ -\bar c & \bar a \end{pmatrix} \begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+' \\ \xi_-' \end{pmatrix} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix} \begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix} \qquad \therefore \; a = \bar d, \; c = \bar b$$

$$K(\xi_+ f + \xi_- g) = |f|^2 - |g|^2 \qquad \begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

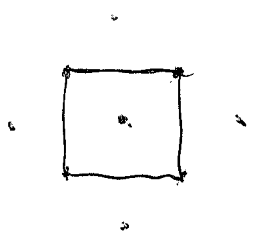
$$\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi_-'\\ \xi_+'\end{pmatrix}$$

$$\cancel{K(\xi_-' f + \xi_- g) = K\big(\tfrac{1}{a}} \qquad \cancel{\begin{pmatrix}\xi_-'\\ \xi_-\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ \frac{c}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_+'\end{pmatrix}}$$

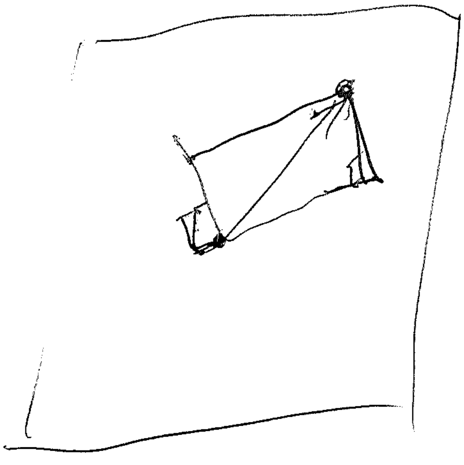
ok

$$K(\xi_-' f + \xi_- g) = K\big((\xi_+ d - \xi_- b) f + \xi_- g\big)$$
$$= K\big(\xi_+ d f + \xi_-(-bf + g)\big)$$
$$= |df|^2 - |-bf+g|^2 = |d|^2|f|^2 - |b|^2|f|^2 + \bar{b}\bar{f}g + \bar{g}bf - |g|^2$$
$$= \begin{pmatrix} f\\ g\end{pmatrix}^* \begin{pmatrix} 1 & \bar{b}\\ b & -1\end{pmatrix}\begin{pmatrix} f\\ g\end{pmatrix}$$

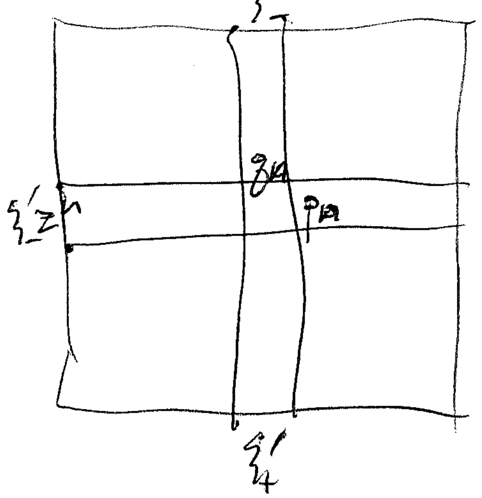
$$Wr\big(\underbrace{\xi_-' f_1 + \xi_- g_1}_{\xi_+(df_1) + \xi_-(-bf_1+g_1)},\ \xi_-' f_2 + \xi_- g_2\big) = \begin{vmatrix} df_1 & df_2\\ -bf_1+g_1 & -bf_2+g_2\end{vmatrix}$$
$$= d\begin{vmatrix} f_1 & f_2\\ g_1 & g_2\end{vmatrix}$$



subdivision



$$E = \xi_-' L^2 + \xi_- L^2 = \xi_+ h^2 + \xi_+' L^2$$



$z^{n+1}H_+$ $\quad H_-$

Check
$$\tilde{p}_n = \xi_+(z^n - \cancel{f}) + \xi_-(-\cancel{g})$$
$$\tilde{q}_n = \xi_+(-\phi) + \xi_-(1-\psi)$$

$$\begin{pmatrix}\pi_{n+1} & 0 \\ 0 & \pi_-\end{pmatrix}\begin{pmatrix}1 & \bar\beta \\ \beta & 1\end{pmatrix}\begin{pmatrix}z^n - f & -\phi \\ -g & 1-\psi\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$\begin{pmatrix}1 & T^* \\ T & 1\end{pmatrix}\begin{pmatrix}f & \phi \\ g & \psi\end{pmatrix} = \begin{pmatrix}0 & \pi_{n+1}(\bar\beta) \\ \pi_-(\beta z^n) & 0\end{pmatrix}$$

$T = \pi_-\beta\varepsilon_{n+1}$
$: z^{n+1}H_+ \to H_-$

$$f + T^*g = 0 \qquad (1-TT^*)g = \pi_-(\beta z^n)$$
$$T\phi + \psi = 0 \qquad (1-T^*T)\phi = \pi_{n+1}(\bar\beta)$$

$$\|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix}f \\ g\end{pmatrix}^* \begin{pmatrix}\cancel{1} & \cancel{T^*} \\ T & 1\end{pmatrix}\begin{pmatrix}f \\ g\end{pmatrix} = \int g^*(1-TT^*)g = \|g\|^2 - \|f\|^2$$

$$\|\xi_+\phi + \xi_-\psi\|^2 = \int \begin{pmatrix}\phi \\ \psi\end{pmatrix}^* \begin{pmatrix}1 & T^* \\ \cancel{T} & \cancel{1}\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix} = \int \phi^*(1-T^*T)\phi = \|\phi\|^2 - \|\psi\|^2$$

$$\|\xi_+ f + \xi_- g\|^2 = \|g\|^2 - \|T^*g\|^2 = -K(\xi_+ f + \xi_- g)$$
$$\|\xi_+\phi + \xi_-\psi\|^2 = \|\phi\|^2 - \|T\phi\|^2 = K(\xi_+\phi + \xi_-\psi)$$

$$\int \beta z^n = (\xi_- \mid \xi_+ z^n) = (\tilde{q}_n \mid \tilde{p}_n) + (\xi_+\phi + \xi_-\psi \mid \xi_+ f + \xi_- g)$$
$$= \|\tilde{q}_n\|\|\tilde{p}_n\| h_n + \int \begin{pmatrix}\phi \\ \psi\end{pmatrix}^* \begin{pmatrix}\cancel{1} & \cancel{T^*} \\ T & 1\end{pmatrix}\begin{pmatrix}f \\ g\end{pmatrix} = \int \psi^*(1-TT^*)g$$

$$\text{error} = \int \psi^*(1-TT^*)g = \int (-T\phi)^*(1-TT^*)g = -\int \phi^* T^*(1-TT^*)g$$

$$\boxed{\text{error} = -\int (\pi_{n+1}(\bar\beta))^*(1-T^*T)^{-1}T^*\pi_-(\beta z^n)}$$

have to express things in terms of β.

$$\beta = \sum_{j\in\mathbb{Z}} \beta_j z^{-j} \qquad \beta z^n = \sum_j \beta_j z^{n-j} \qquad \sum_{j>n} |\beta_j|^2$$

$$\pi_-(\beta z^n) = \sum_{j>n} \beta_j z^{n-j} \qquad \|\pi_-(\beta z^n)\|^2$$

$$\pi_{n+1}(\bar\beta) = \sum_{j>n} \bar\beta_j z^j \qquad \|\pi_{n+1}(\bar\beta)\|^2$$

$$\|\tilde q_n\|^2 = \|\xi_-\|^2 - \|\xi_+\phi + \xi_-\psi\|^2 = 1 - \|\phi\|^2 + \|T^*\phi\|^2$$

$$\|\xi_+\phi + \xi_-\psi\|^2 = \int \phi^* \underbrace{(1-T^*T)\phi}_{\pi_{n+1}(\bar\beta)} = \int (\pi_{n+1}(\bar\beta))^*(1-T^*T)^{-1}\pi_{n+1}(\bar\beta)$$

$$\le \frac{1}{1-\|T\|^2}\sum_{j>n}|\beta_j|^2$$

$$\|\xi_+ f + \xi_- g\|^2 = \int g^*(1-TT^*)g = \int (\pi_-(\beta z^n))^*(1-TT^*)^{-1}\pi_-(\beta z^n)$$

$$\le \frac{1}{1-\|T\|^2}\sum_{j>n}|\beta_j|^2$$

$$|(\xi_+\phi + \xi_-\psi \mid \xi_+ f + \xi_- g)| \le \frac{\|T\|}{1-\|T\|^2}\sum_{j>n}|\beta_j|^2$$

only ε better than $\frac{1}{1-\|T\|^2}$

$$-\int \beta z^n + h_n = h_n(1 - \|\tilde q_n\|\|\tilde p_n\|) + \text{error}.$$

$$\|\tilde p_n\|\,\|\tilde q_n\| = (1-\| \quad \|^2)^{1/2}(1-\| \quad \|^2)^{1/2}$$

$$\ge 1 - \frac{1}{1-\|T\|^2}\sum_{j>n}|\beta_n|^2$$

Work out the details more.

$$\xi_+ z^n = \tilde p_n + (\xi_+ f + \xi_- g)$$

$$\xi_- = \tilde q_n + (\xi_+\phi + \xi_-\psi)$$

$$\|\tilde{p}_n\|^2 = 1 - \|\xi_+ f + \xi_- g\|^2$$

$$\|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} \cancel{1} & \cancel{T^*} \\ T & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \int g^*(1 - TT^*) g$$

$$= \int \left(\pi_-(\beta z^n)\right)^* (1 - TT^*)^{-1} \left(\pi_{n+1}(\bar\beta)\right)$$

$$\leq \frac{1}{1 - \|T\|^2} \sum_{j>n} |\beta_j|^2$$

$$\therefore \quad 1 \geq \|\tilde{p}_n\| \geq \left(1 - \frac{1}{1-\|T\|^2} \sum_{j>n} |\beta_j|^2\right)^{1/2}$$

so

$$1 \geq \|\tilde{p}_n\| \|\tilde{q}_n\| \geq 1 - \frac{1}{1-\|T\|^2} \sum_{j>n} |\beta_j|^2$$

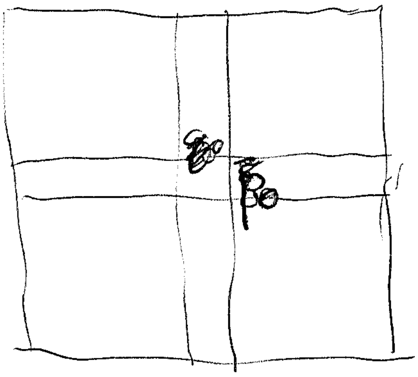
$$\cancel{h_n - \int \beta z^n = h_n - \|\tilde{p}_n\|\|\tilde{q}_n\|}$$

$$\int \beta z^n = \|\tilde{p}_n\|\|\tilde{q}_n\| h_n - \underbrace{\int \left(\pi_{n+1}(\bar\beta)\right)^* (1 - T^*T)^{-1} T^* \left(\pi_-(\beta z^n)\right)}$$

$$|\ \cdot\ | \leq \frac{\|T\|}{1 - \|T\|^2} \sum_{j>n} |\beta_j|^2$$

$$\int \beta z^n - h_n = \underbrace{\left(\|\tilde{p}_n\|\|\tilde{q}_n\| - 1\right) h_n + \text{err}}$$

$$|\ \cdot\ | \leq \Bigg(\underbrace{\frac{1}{1-\|T\|^2} + \frac{\|T\|}{1-\|T\|^2}}_{\frac{1}{1-\|T\|}}\Bigg) \sum_{j>n} |\beta_j|^2$$



$$\cancel{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}} \quad H_- \quad H_+$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

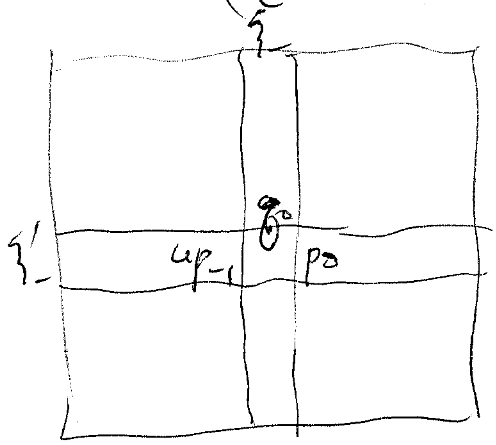
$$\begin{matrix} zH_+ & H_+ & zH_- & H_+ \end{matrix}$$

$$d = c^r b^l + d^r d^l$$

$$d(0) = d^r(0) d^l(0).$$

$$\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}(0) = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}$$

$$\boxed{d(0) = \frac{1}{\prod k_n}}$$



$$\widetilde{up}_{-1} = \xi'_-(1-f) + \xi_-(-g)$$

$$\tilde{q}_0 = \xi'_-(-\phi) + \xi_-(1-\phi)$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} up_{-1} \\ q_{-1} \end{pmatrix}$$

$$\begin{pmatrix} up_{-1} \\ q_{-1} \end{pmatrix} = \frac{1}{k_0}\begin{pmatrix} 1 & -h_0 \\ -\bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$up_{-1} = \frac{1}{k_0}p_0 - \frac{h_0}{k_0} q_0$$

$$K(q_0, up_{-1}) = K\left(q_0, \cancel{\frac{1}{k_0}p_0} - \frac{h_0}{k_0} q_0\right)$$

$$= \left(-\frac{h_0}{k_0}\right) \underbrace{K(q_0, q_0)}_{-1} = \frac{h_0}{k_0}$$

$$K(\xi_- \mid \xi'_-) = K(\tilde{q}_0, \widetilde{up}_{-1}) + \text{error}$$

$$\begin{matrix} \| & & \| \\ \int b & & \|\tilde{q}_0\| \, \|\widetilde{up}_{-1}\| \frac{h_0}{k_0} \end{matrix}$$

So $\cancel{\int \frac{b}{d}}$

$$\int b - \underbrace{d(0)}_{>1} h_0 = \text{error}$$

$$K(\widetilde{up}_{-1}) = \|\widetilde{up}_{-1}\|^2 = 1 + \|f\|^2 + \|g\|^2 > 1$$

~~possible that~~ agrees with $\|\widetilde{up}_{-1}\| = \prod_{j<0} \frac{1}{k_j}$

You want a conceptual way to see that

$\int b z^n \sim d(0) h_n$ is a reasonable approximation

$$d(0) = \prod_{n \in \mathbb{Z}} \frac{1}{k_n} > 1$$

correct to first order I think.

$$\log \frac{1}{k} = -\frac{1}{2} \log(1 - |h|^2) = \frac{|h|^2}{2}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \prod_{+\infty}^{-\infty} \frac{1}{k_n} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}$$

to first order in (h_n) this is

$$\begin{pmatrix} 1 & \sum h_n z^{-n} \\ \sum \bar{h}_n z^n & 1 \end{pmatrix}$$

so $b = \sum h_j z^{-j}$

$$\int b z^n = \int \sum h_j z^{n-j} = h_n$$

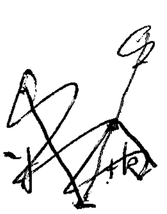
$$\frac{1}{d(0)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \prod_{\infty}^{-\infty} \begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}.$$

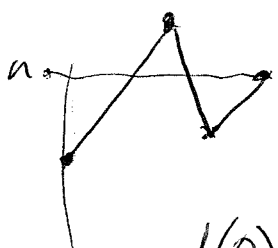
3rd order to $b = \sum_{j > k > l} h_j z^{-j} \bar{h}_k z^{+k} h_l z^{-l}$

2nd order to $d = 1 + \sum_{j > k} \bar{h}_j z^{+j} h_k z^{-k}$

$$\int \sum_{j > k > l} h_j \bar{h}_k h_l z^{-j+k-l+n} = \sum_{\substack{j > k > l \\ -j+k-l+n=0}} h_j \bar{h}_k h_l$$

$$n = j - k + l$$





$$d(0) = 1 + \frac{1}{2} \sum_{\mathbb{Z}} |h_n|^2$$

$$d(0) = \prod \frac{1}{k_n} = \prod \frac{1}{\sqrt{1-|h_n|^2}}$$

$$\log d(0) = \sum -\tfrac{1}{2}\log(1-|h_n|^2)$$

$$= \sum_{n\in\mathbb{Z}} \sum_{k\geq 1} \left(+\tfrac{1}{2}\right)\left(\tfrac{1}{k}|h_n|^{2k}\right)$$

$$= +\tfrac{1}{2}\left(\sum_n |h_n|^2 + \tfrac{1}{2}\sum_n |h_n|^4 + \cdots\right)$$

$$d(0) = 1 + \tfrac{1}{2}\sum_n |h_n|^2 + \text{4th order}$$

Do again

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \prod_{+\infty}^{-\infty} \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar h_n z^n & 1 \end{pmatrix}$$

$$d(0) = \prod \frac{1}{k_n}$$

$$d(0)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \cdots \begin{pmatrix} 1 & h_j z^{-j} \\ \bar h_j z^j & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_k z^{-k} \\ \bar h_k z^k & 1 \end{pmatrix} \cdots$$

$$j > k.$$

$$d(0)^{-1} b = \sum_j h_j z^{-j} + \sum_{j>k>l} h_j z^{-j}\, \bar h_k z^k\, h_l z^{-l} + \text{5th}$$

$$d(0)^{-1} d = 1 + \sum_{j>k} \bar h_j z^j\, h_k z^{-k} + \text{4th}$$

$$\int d(0)^{-1} b z^n = h_n + \sum_{\substack{j>k>l \\ n=j-k+l}} h_j \bar h_k h_l + \text{5th}.$$

Something ~~seems~~ interesting seems to be happening. PUT INTO WORDS ~~anyway it seems that [illegible]~~

~~Put~~ You have

$$d(0) = 1 + \tfrac{1}{2}\sum_{\mathbb{Z}} |h_j|^2 + \text{4th}$$

$$d(0)^{-1} d = 1 + \sum_{j>k} \bar{h}_j h_k z^{j-k} + \text{4th.}$$

so

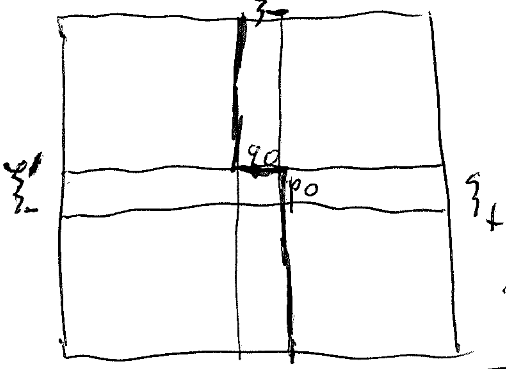
$$d = \underbrace{1 + \tfrac{1}{2}\sum |h_j|^2 + \sum_{j>k}{}' \bar{h}_j h_k z^{j-k}}_{\text{is halfway between}} + \text{4th}$$

$$\sum_{j>k} \bar{h}_j h_k z^{j-k} \quad \text{and} \quad \sum_{j\geq k} \bar{h}_j h_k z^{j-k}$$

Similarly it looks like $\int bz^n$ is half-way between $\int d(0)^{-1} bz^n$ and its modification by introducting the extra terms $j \geq k$, or $k = \ell$

Upshot is that you can't tell at the moment whether $\int d(0)^{-1} bz^n$ is a better approx to h_n than $\int bz^n$. Somehow you feel that because $|h_n| < 1$ and $d(0) > 1$, you expect $d(0) h_n$ to be closer to $\int bz^n$.

One more time



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{array}{c} \quad H_+ \\ \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \\ ZH_- \quad \end{array} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{array}{c} \quad H_- \\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \\ ZH_+ \quad \end{array} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a} \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

To construct p_0, q_0 using K-form.

$$(q_0|p_0) = \cancel{\left(c^l \xi'_- + d^l \xi'_+ \,\middle|\, \frac{1}{d}\left(d^r \xi'_- + b^l \xi_- \right) \right)}$$

$$= \left(\frac{1}{d}\left(-c^r \xi'_- + d^l \xi_-\right) \,\middle|\, \frac{1}{a}\left(a^l \xi_+ - b^r \xi'_+\right) \right) \quad ?$$

Represent $(\ |\ )$ as $K(\ ,B)$ OKAY what next?

$$q_0 = \xi'_-\left(-\frac{c^r}{d}\right) + \xi_-\left(\frac{d^l}{d}\right)$$

$$= \xi_+\left(\frac{c^l}{a}\right) + \xi'_+\left(+\frac{a^r}{a}\right)$$

$$= \begin{pmatrix} \xi'_- & \xi_- \end{pmatrix} \begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \frac{c^l}{a} \\ -\frac{b^a}{a} \end{pmatrix}$$

$$\|q_0\|^2 = \int \frac{|c^r|^2 + |d^l|^2}{|d|^2} \stackrel{\checkmark}{=} \int \frac{|c^l|^2 + |a^r|^2}{|a|^2}$$

~~two~~ ~~conjugation is funny. Before you had~~ $\cancel{p_0^c = q_0}$

$$q_0^c = \xi_+\left(\frac{a^l}{a}\right) + \xi'_+\left(-\frac{b^r}{a}\right) = p_0$$

$$(q_0|p_0) = \int \left(\frac{c^l}{a}\right)^* \left(\frac{a^l}{a}\right) + \left(\frac{a^r}{a}\right)^* \left(-\frac{b^r}{a}\right)$$

---

what is $K(\xi^c, \xi)$

$$K(\xi_+ f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \|f\|^2 - \|g\|^2$$

$$K\left((\xi_+ f_1 + \xi_- g_1)^c, \xi_+ f_2 + \xi_- g_2\right) = K\left(\xi_+ \bar{g}_1 + \xi_- \bar{f}_1, \xi_+ f_2 + \xi_- g_2\right)$$

$$= \int \begin{pmatrix} \bar{g}_1 \\ \bar{f}_1 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \int g_1 f_2 - f_1 g_2$$

So what's important? Begin with

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

defining a $\mathbb{C}[u,u^{-1}]$-module M with gen. p_n, q_n for $n \in \mathbb{Z}$ and above relations, free of rank 2, obvious bases (p_n, q_n) for each $n \in \mathbb{Z}$. Observe that

$$\frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \in SU(1,1) \qquad \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in U(1,-1).$$

Get a hermitian form K on M over A values in A,

$$K(p_n, p_n) = 1 \qquad K(q_n, p_n) = 0$$
$$K(p_n, q_n) = 0 \qquad K(q_n, q_n) = -1.$$

K independent of n.

For any n. ~~[illegible]~~

$K(\xi, \eta)$ well defd hermitian form over A.

Wronskian? Consider $\Lambda^2_A M$, free rank 1 module over A, bases $p_n \wedge q_n$ but

$$p_n \wedge q_n = u\, p_{n-1} \wedge q_{n-1}$$

So you can change your bases to $u^{-n} p_n \wedge q_n = u^{-n+1} p_{n-1} \wedge q_{n-1}$

Important — now you need to understand ~~what~~ how the conjugation varies.

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You are given 2 dim'l V over $\mathbb{C}$, with reduction to $SL(2,\mathbb{R})$, i.e. a conjugation $\sigma : V \to V$ anti-linear, $\sigma^2 = 1$, and $\omega \neq 0$ in $\Lambda^2 V$ such that $\sigma(\omega) = \omega$.

Do intrinsically, and don't mess up the signs.

$V \simeq \mathbb{C}^2$. $0 \neq \omega \in \Lambda^2 \mathbb{C}$ given. K a hermitian form of type 1,1 given. Better is to give an oriented ~~circle~~ circle in PV.

You have to understand this precisely ~~in detail~~

Reflection through the circle, orthogonal complement for the hermitian form.

Idea: A volume element $0 \neq \omega \in \Lambda^2 \mathbb{C}$ gives a duality $l \otimes V/l = \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$, whence $L \otimes (\mathcal{O} \otimes V)/L \simeq \mathcal{O} \otimes \Lambda^2 V = \mathcal{O}$, then a $\neq 0$ tangent vector = map $l \to V/l$ gives a $\neq 0$ quadratic function on l whence a real line in l where this quad fn. is ≥ 0. Thus along the oriented circle you have a real structure on L.

Get formulas straight. Suppose you have $\mathbb{C}^2$ with herm. form $\begin{pmatrix} \bar{s} \\ \bar{t} \end{pmatrix}^{*} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = K\left(\begin{pmatrix} s \\ t \end{pmatrix}\right)$. Obvious to try

$$K(\xi, \eta) = \frac{\xi^c \wedge \eta}{\omega}$$

$$K(\eta, \xi) = \frac{\eta^c \wedge \xi}{\omega} = \left(\frac{\eta \wedge \xi^c}{\omega^c}\right)^c$$

$$= \left(-\frac{\xi^c \wedge \eta}{\omega^c}\right)^c$$

$$\overline{K(\xi,\eta)} = \frac{\xi \wedge \eta^c}{\omega^c}$$ so you need $\omega^c = -\omega$

~~Also~~ Perhaps the way to think is that you are given σ and $0 \neq \omega \in \Lambda^2 V$ such that $\omega^\sigma = -\omega$. Then define

$$K(\xi, \eta) = \frac{\sigma(\xi) \wedge \eta}{\omega}$$

$\mathbb{C}$ linear in η
$\mathbb{C}$ anti " " ξ.

$$\overline{K(\xi, \eta)} = \frac{\xi \wedge \sigma(\eta)}{\sigma(\omega)} = \frac{-\sigma(\eta) \wedge \xi}{-\omega} = K(\eta, \xi).$$

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what to do? Go back to A ~~=~~ $\mathbb{C}[u, u^{-1}]$ module M gen. by $p_n, q_n \in \mathbb{Z}$ relations ~~above~~ standard. Then get $K(\xi, \eta) \in A$ herm. form over A, so that $K(p_0 f + q_0 g) = |f|^2 - |g|^2$.

$$K(p_0 f[redacted] + q_0 g[redacted], p_0 \phi[redacted] + q_0 \psi[redacted]) = \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Three structures

σ conjugation
$\omega \in \Lambda^2 V$ volume
$K(\xi, \eta)$ indf herm. form

$$K(\xi, \eta) = \frac{\sigma(\xi) \wedge \eta}{\omega}$$ is a herm. form provided $\sigma(\omega) = -\omega$

~~Examples~~ Suppose given K and ω. Define $\sigma(\xi)$ by $K(\xi, \eta) = \dfrac{\sigma(\xi) \wedge \eta}{\omega}$ i.e. $\sigma(\xi)$ represents the linear functional $\eta \longmapsto K(\xi, \eta)$. ~~Then~~

$$K(\eta, \xi) = \frac{\sigma(\eta) \wedge \xi}{\omega} =$$

$$\overline{K(\xi, \eta)} = \overline{\frac{\sigma(\xi) \wedge \eta}{\omega}}$$

$$K(\xi, \xi) = \frac{\sigma(\xi) \wedge \xi}{\omega}$$

Start with K choose ξ_+, ξ_- ~~also~~ with $K(\xi_+) = 1$, $K(\xi_+, \xi_-) = 0$, $K(\xi_-) = -1$. Then get volume elt. $\xi_+ \wedge \xi_- \in \Lambda^2 V$. How unique? Another choice related by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$

$\det \in S^1$ First look at $\omega = \xi_+ \wedge \xi_-$

Then $K(\xi_+ f + \xi_- g, \xi_+\phi + \xi_-\psi)$

$$= \bar f\phi - \bar g\psi = \begin{vmatrix} -\bar g & \phi \\ \bar f & \psi \end{vmatrix}$$

seems that $\sigma(\xi_+ f + \xi_- g) = \xi_+(-\bar g) + \xi_- \bar f$

Given K indef on V 2 diml. Choose basis e_+, e_- $K(e_+,e_+) = 1, K(e_+,e_-) = 0, K(e_-,e_-) = -1$. Then

$\sigma(e_+ f + e_- g) = e_+\bar g + e_-\bar f$. What is ω if

$$K(\xi,\eta) = \frac{\sigma(\xi)\wedge\eta}{\omega} \quad ?$$

$$\cancel{K(\sigma(\xi),\eta) = K(e_+\bar g + e_-\bar f, e_+\phi + e_-\psi)}$$

$$\cancel{= \bar g\phi - \bar f\psi = \frac{(e_+\bar g + e_-\bar f)\wedge(e_+\phi + e_-\phi)}{e_+\wedge e_-}}$$

$$K(\sigma\xi,\xi) =$$

Given $K(e_+,e_+) = 1$, $K(e_+,e_-) = 0$, $K(e_-,e_-) = -1$

$$K(e_+ f + e_- g, e_+\phi + e_-\psi) = \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \bar f\phi - \bar g\psi$$

$$\|$$

$$\sigma(e_+ f + e_- g) \wedge (e_+\phi + e_-\psi)$$

$$\|$$

$$\rho\,(e_+\bar g + e_-\bar f)\wedge(e_+\phi + e_-\psi) = \rho(\bar g\psi - \bar f\phi)\, e_+\wedge e_-$$

Here $|\rho| = 1$. If σ is a conjugation so is $\xi\sigma$ where

$$\xi\sigma\,\xi\sigma = \xi\bar\xi\sigma^2$$

$|\rho| = 1$ since

so $\rho = -1$