$$p_0 = \frac{d^r}{d}\xi'_- + \frac{b^l}{d}\xi_- = \frac{a^l}{a}\xi_+ - \frac{b^r}{a}\xi'_+$$

$$\frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

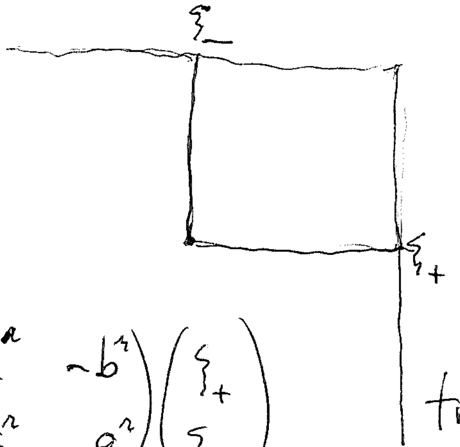
$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}$$

$\underbrace{\phi g_+ = g_-}$? $\qquad\qquad \phi H_+$

want to think of these are left mult operators and there being commuting right mult operators.

Look at rank 1 briefly. Given $\phi(\lambda)$ S^1 valued. Form $L^2\xi_- \simeq L^2\xi_+$ $\quad \xi_+ = \phi\xi_-$

You want



$$p_0 \in (1+H_+)\xi_+ + H_-\xi_-$$

$$q_0 \in H_+\xi_+ + (1+H_-)\xi_-$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^a & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

transfer matrix

here you start with $\left(f\xi_- \mid \xi_+\right) = \int \bar{f}\phi$

The game here is ~~that~~ to start with $\phi\xi_- = \xi_+$

i.e. $\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} & \phi \\ & \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$ degenerate scattering matrix.

to Start with ϕ and to construct p_0, q_0. You are

~~Applying what the shift~~ Basically

$$p_0 = (1+\hat{d})\xi_+ - \hat{b}\xi_-$$

$$0 \underset{x>0}{=} (z^x\xi_+ | p_0) = \hat{d}_x - \underbrace{(z^x\xi_+ | \hat{b}\xi_-)}_{\int z^{-x}\bar{\phi}\, b}$$

$$0 \underset{y<0}{=} (z^y\xi_- | p_0) = \underbrace{(z^y\xi_- | (1+\hat{d})\xi_+)}_{\int z^{-y}(1+\hat{d})\phi} - \underbrace{(z^y\xi_- | b\xi_-)}_{b_y}$$

You should write it.

$$0 \underset{y<0}{=} (z^y\xi_- | p_0) = (z^y | (1+\hat{d})\phi - b)$$

$$0 \underset{x>0}{=} (z^x\xi_+ | p_0) = (z^x | \hat{d} - \bar{\phi} b)$$

Thus get $\quad (1+\hat{d})\phi - b \in 1 + H_+ \qquad \hat{d} \in H_+$

$$\hat{d} - \bar{\phi} b \in H_- \qquad b \in H_-$$

$$\hat{d} = \pi_+ \bar{\phi}\, b$$

$$\pi_-\phi + \pi_-\phi\hat{d} - b = 1 \quad ?$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} d^\ell & -b^\ell \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^\ell & a^\ell \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \begin{pmatrix} \frac{d^\ell}{d} & -\frac{b^\ell}{d} \\ \frac{c^r}{d} & \frac{d^r}{d} \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^\ell & a^\ell \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \begin{pmatrix} -\frac{b^\ell}{d} \\ \frac{d^r}{d} \end{pmatrix} = \begin{pmatrix} b^r \\ a^\ell \end{pmatrix} \in \begin{pmatrix} H_- \\ 1+H_- \end{pmatrix}$$

$$\cap \begin{pmatrix} H_+ \\ 1+H_+ \end{pmatrix}$$

$$p_0 = d\xi_+ - b\xi_-$$

$x>0$ $\quad 0 = (z^x \xi_+ | p_0) = (z^x | d - \bar{\phi} b)$

$y<0$ $\quad 0 = (z^y \xi_- | p_0) = (z^y | d\phi - b)$

$$\hat{d} - \bar{\phi} b \in H_- \implies \hat{d} = \pi_+(\bar{\phi} b)$$

$$d\phi - b \in 1 + H_+ \implies \phi + \hat{d}\phi - b \in 1 + H_+$$

$$\hat{\phi} + \pi_-(\hat{d}\phi) = b$$

$$\hat{\phi} + \pi_-\phi\pi_+\bar{\phi} b = b$$

$$b = (1 - \pi_-\phi\pi_+\bar{\phi})^{-1}\hat{\phi}$$

$$\hat{d} = \pi_+\bar{\phi}(1 - \pi_-\phi\pi_+\bar{\phi})^{-1}\hat{\phi}$$

Begin a giant review. You have to construct the factorization:

In the transfer setting

$$\begin{array}{cccc} & \tilde H_+^* \quad H_- & \tilde H_- \quad H_+ \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = & \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} & \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \\ & H_+ \quad \tilde H_+^* & H_- \quad \tilde H_+ \end{array}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{b}{a} \\ \frac{c}{d} & 1 \end{pmatrix}\begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix} = \begin{array}{cc} \tilde H_- & H_- \\ \multicolumn{2}{c}{\begin{pmatrix} \frac{a^r}{a} & \frac{b^r}{a} \\ \frac{c^r}{d} & \frac{d^r}{d} \end{pmatrix}} \\ H_+ & \tilde H_+ \end{array}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{b}{a} \\ \frac{c}{d} & 1 \end{pmatrix}\begin{array}{c} \tilde H_+ \\ \begin{pmatrix} d^l \\ -c^l \end{pmatrix} \\ H_- \end{array} = \begin{pmatrix} \frac{a^r}{a} \\ \frac{c^r}{d} \end{pmatrix}$$

$$d^l \underbrace{-\frac{b}{a}}_{\bar\gamma} c^l = \frac{a^r}{a} \in \tilde H_- \qquad d^l - \pi_+\left(\frac{b}{a} c^l\right) = 1$$

$$- \pi_-\left(\frac{b}{a} c^l\right) = \frac{a^r}{a} - 1$$

$$\underbrace{\frac{c}{d}}_{-\gamma} d^l - c^l = \frac{c^r}{d} \in H_+ \qquad \pi_-\left(\frac{c}{d} d^l\right) - c^l = 0$$

Put

$$\cancel{\begin{pmatrix} 1 & \pi_+\bar\gamma \\ -\pi_-\gamma & 1 \end{pmatrix}\begin{pmatrix} d^l \\ -c^l \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} 1 & \pi_+\bar\gamma \\ \pi_-\gamma & 1 \end{pmatrix}\begin{pmatrix} -b^l \\ a^l \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$d^l + \frac{b}{a}(-c^l) = \frac{a^r}{a} \qquad d^l + \pi_+ \frac{b}{a}(-c^l) = 1$$

$$\frac{c}{d} d^l + (-c^l) = \frac{c^r}{d} \qquad \pi_- \frac{c}{d} d^l - c_l = 0$$

$$d^l + \pi_+ \frac{b}{a}\left(-\pi_- \frac{c}{d} d^l\right) = 1 \qquad \left(I_{H_+} - \pi_+ \frac{b}{a} \pi_- \frac{c}{d}\right) d^l = 1$$

$$d^{l} = \left(1 - \pi_{+}\frac{b}{a}\,\pi_{-}\frac{c}{d}\right)^{-1} 1 \quad \leftarrow \text{the function } 1$$

$$c^{l} = \pi_{-}\frac{c}{d}\left(\qquad \right)^{-1} 1$$

In matrix form

$$\begin{pmatrix} id_{H_+} & \pi_{+}\frac{b}{a} \\ \pi_{-}\frac{c}{d} & id_{H_-} \end{pmatrix}\begin{pmatrix} d^{l} & -b^{l} \\ -c^{l} & a^{l} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Scattering version

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{array}{cc} \tilde H_{-} & H_{-} \\ \multicolumn{2}{c}{\begin{pmatrix} a^{r} & b^{r} \\ -c^{l} & a^{l} \end{pmatrix}} \\ H_{-} & \tilde H_{-} \end{array} \begin{array}{cc} \tilde H_{+} & H_{+} \\ \multicolumn{2}{c}{\begin{pmatrix} d^{r} & b^{l} \\ -c^{r} & d^{l} \end{pmatrix}} \\ H_{+} & \tilde H_{+} \end{array}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\frac{1}{d}\begin{pmatrix} d^{l} & -b^{l} \\ c^{r} & d^{r} \end{pmatrix} = \begin{pmatrix} a^{r} & b^{r} \\ -c^{l} & a^{l} \end{pmatrix}$$

$$\frac{d^{l}}{d} + b\frac{c^{r}}{d} = a^{r} \quad \in 1 + H_{-}$$

$$-c\frac{d^{l}}{d} + \frac{c^{r}}{d} = -c^{l} \quad \in H_{-}$$

$$\underbrace{\frac{d^{l}}{d} - 1}_{H_+} + b\frac{c^{r}}{d} = a^{r} - 1 \in H_{-} \quad \Longrightarrow \quad \frac{d^{l}}{d} + \pi_{+} b\frac{c^{r}}{d} = 1$$

$$-\pi_{+} c\frac{d^{l}}{d} + \frac{c^{r}}{d} = 0$$

$$\frac{d^{l}}{d} + \pi_{+} b\left(\pi_{+} c\,\frac{d^{l}}{d}\right) = 1 \qquad \left(id_{H_+} + \pi_{+} b\,\pi_{+}\bar b\right)\frac{d^{l}}{d} = 1$$

get $$\frac{d^{l}}{d} = \left(id_{\tilde H_+} + \pi_{+} b\,\pi_{+}\bar b\right)^{-1} 1$$

$$\frac{c^{r}}{d} = \pi_{+}\bar b\left(\qquad \right)^{-1} 1$$

Some interesting stuff is happening here. because you are not apparently working with the scattering matrix, $S = \begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \frac{1}{d}$, but maybe you can rewrite things.

$$\underbrace{\begin{pmatrix} \pi_+ & 0 \\ 0 & \pi_+ \end{pmatrix} \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{\bar{b}}{d} & \frac{1}{d} \end{pmatrix}}_{\begin{pmatrix} \pi_+ \frac{1}{d} & \pi_+ \frac{b}{d} \\ -\pi_+ \frac{\bar{b}}{d} & \pi_+ \frac{1}{d} \end{pmatrix}} \begin{pmatrix} d^\ell \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is an operator on $H_+^{\oplus 2}$

$$\pi_+ \frac{1}{d} d^\ell = \frac{1}{d} d^\ell \qquad \forall d^\ell \in H_+$$

$$\pi_+ \frac{b}{d} d^\ell = \pi_+ (b_- + b_+) \frac{d^\ell}{d} = \pi_+ \left(b_- \frac{d^\ell}{d} \right) + b_+ \frac{d^\ell}{d}$$

So you argue that $\pi_+ S$ on $H_+^{\oplus 2}$ is invertible, because it factors into $\begin{pmatrix} id & \pi_+ b \\ -\pi_+ \bar{b} & id \end{pmatrix} \cdot \frac{1}{d}$ the product of two invertibles

So it's interesting these methods which apparently yield the same result.

$$b \rightsquigarrow d \in 1 + H_+ \qquad |d|^2 = 1 + |b|^2$$

$$\downarrow$$

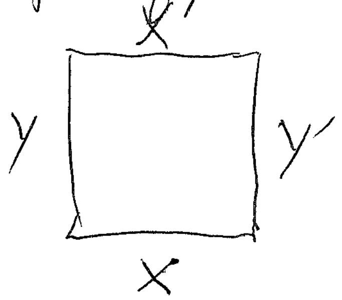
$$X = \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \rightsquigarrow 1 + X = \begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \rightsquigarrow \overset{S \,\shortparallel}{\frac{1+X}{\sqrt{1-X^2}}} = \begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \frac{1}{d}$$

where d involves polar decomposition in the circle setting.

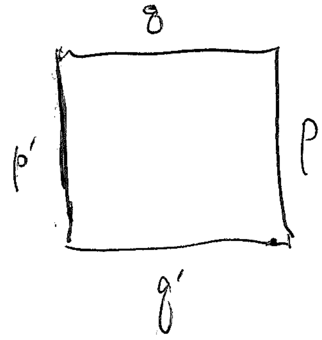
S is a kind of $\frac{1}{2}$ C.T. Then

Situation with pos. square root.

do transfer + scattering in general.



$$\begin{pmatrix} y' \\ x' \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix}$$



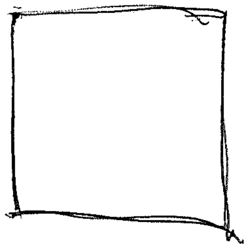
$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix}$$

$$t = \frac{h}{\sqrt{1-|h|^2}}, \quad 1+|t|^2 = \frac{|h|^2}{1-|h|^2} + 1 = \frac{1}{1-|h|^2} = \frac{1}{k^2}$$

$$k = \frac{1}{\sqrt{1+|t|^2}} \qquad t = \frac{h}{k}$$

Somehow in this situation $|h|<1$ "the contraction" corresp. to $t = h(1-|h|^2)^{-1/2}$, $h = kt = t(1+|t|^2)^{-1/2}$



$$\frac{F}{g\varepsilon}\begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix} = g^{\varepsilon}(1+X) = \frac{1+X}{1-X}(1-X)\varepsilon = (1+X)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$F = +1$ on $\operatorname{Im}\begin{pmatrix}1\\ T\end{pmatrix}$.

$ -1$ on $\operatorname{Im}\begin{pmatrix}-T^*\\ 1\end{pmatrix}$

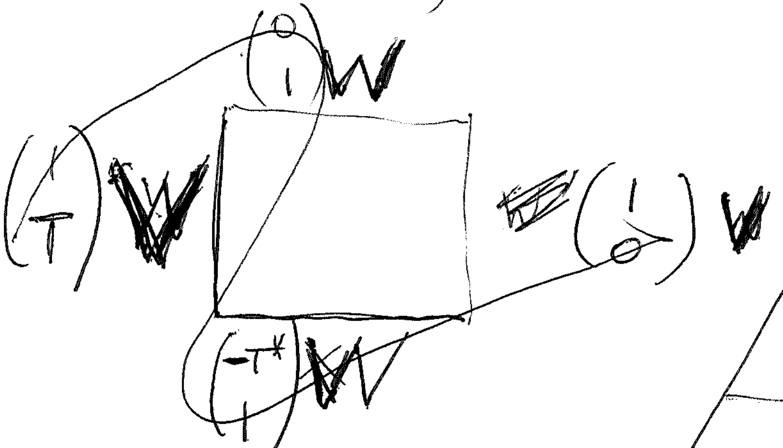
But you want

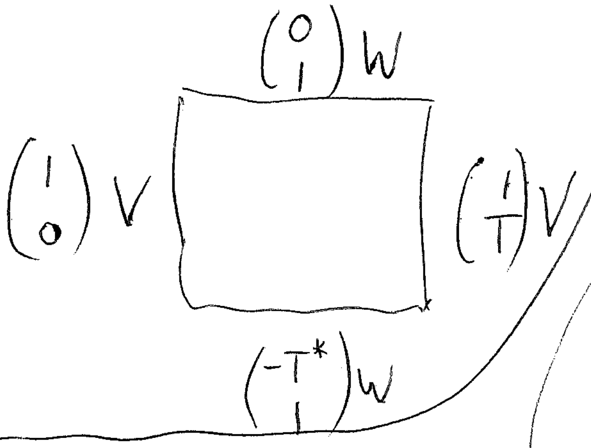
$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix}\Big(\quad \Big)$$

$$g^{1/2} = \begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix}\begin{pmatrix}(1+T^*T)^{-1/2} & 0\\ 0 & (1+TT^*)^{-1/2}\end{pmatrix}$$

Given $T : V \longrightarrow W$, get comp. subspaces

$\begin{pmatrix}1\\ T\end{pmatrix}V$, $\begin{pmatrix}-T^*\\ 1\end{pmatrix}W \subset \begin{matrix}V\\ \oplus\\ W\end{matrix}$





go back to the ~~beginning~~. Review formulas

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ c^r & d^r\end{pmatrix}\begin{pmatrix}a^l & b^l\\ c^l & d^l\end{pmatrix}$$

$$\begin{pmatrix}1 & b\\ -c & 1\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ -c^l & a^l\end{pmatrix}\begin{pmatrix}d^r & b^l\\ -c^r & d^l\end{pmatrix}$$

$$\begin{pmatrix}1 & \frac{b}{a}\\ \frac{c}{d} & 1\end{pmatrix}\begin{pmatrix}d^l & -b^l\\ -c^l & a^l\end{pmatrix} = \begin{pmatrix}\frac{a^r}{a} & \frac{b^r}{a}\\ \frac{c^r}{d} & \frac{d^r}{d}\end{pmatrix}$$

$$\begin{pmatrix}1 & -\frac{b}{d}\\ -\frac{c}{a} & 1\end{pmatrix}\begin{pmatrix}a^r & b^r\\ c^r & d^r\end{pmatrix} = \begin{pmatrix}\frac{d^l}{d} & -\frac{b^l}{d}\\ -\frac{c^l}{a} & \frac{a^l}{a}\end{pmatrix}$$

$$\begin{pmatrix}Id_{\tilde H_+} & \pi_+\frac{b}{a}\\ \pi_-\frac{c}{d} & Id_{\tilde H_-}\end{pmatrix}\begin{pmatrix}d^l & -b^l\\ -c^l & a^l\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}Id_{\tilde H_-} & \pi_-\left(-\frac{b}{d}\right)\\ \pi_+\left(-\frac{c}{a}\right) & Id_{\tilde H_+}\end{pmatrix}\begin{pmatrix}a^r & b^r\\ c^r & d^r\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & b\\ -c & 1\end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^\ell & a^\ell\end{pmatrix}\begin{pmatrix} d^r & b^\ell \\ -c^r & d^\ell\end{pmatrix}$$

$$\begin{pmatrix}1 & b\\ -c & 1\end{pmatrix}\begin{pmatrix} d^\ell & -b^\ell \\ +c^r & d^r\end{pmatrix}\frac{1}{d} = \begin{pmatrix} a^r & b^r\\ -c^\ell & a^\ell\end{pmatrix}$$

$$\begin{pmatrix}Id_{\tilde H_+} & \pi_+ b\\ \pi_+(-\bar b) & Id_{\tilde H_+}\end{pmatrix}\begin{pmatrix}\frac{d^\ell}{d} & -\frac{b^\ell}{d}\\ \frac{c^r}{d} & \frac{d^r}{d}\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Kasparov yoga. $\mathcal{A} = \overset{\text{ring of}}{\text{nice}}$ functions on $\mathrm{Re}(\lambda)=0$. probably vanishing at ∞. Look at solns. of

$$\partial_x \begin{pmatrix} z^{-x}p_x \\ q_x\end{pmatrix} = \begin{pmatrix} 0 & h_x z^{-x} \\ \bar h_x z^x & 0\end{pmatrix}\begin{pmatrix} z^{-x}p_x\\ q_x\end{pmatrix}$$ ~~Assume h decays fast~~

form a module M over $\mathcal{A}$, free of rank 2,

Assume h decays fast, so that propagators exist to $\pm\infty$. Then you get basic elements of M, $\xi_\pm$ $\xi'_\pm$

Let's consider discrete case h fin. support.

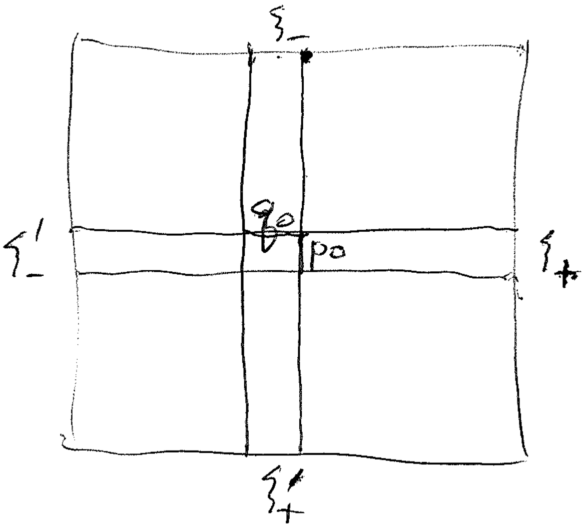
Then $\mathcal{A} = \mathbb{C}[z,z^{-1}]$

$$\begin{pmatrix} z^{-n}p_n \\ q_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n}\\ \bar h_n z^n & 1\end{pmatrix}\begin{pmatrix} z^{-n+1}p_{n-1}\\ q_{n-1}\end{pmatrix}$$

Have usual

$$\begin{pmatrix}p_0\\q_0\end{pmatrix} = \begin{array}{c} \begin{array}{cc} zH_- & H_+ \end{array} \\ \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix} \\ \hline \begin{array}{cc} zH_- & H_+ \end{array} \end{array} \begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{array}{c} \begin{array}{cc} H_+ & H_- \end{array} \\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \\ \hline \begin{array}{cc} zH_+ & zH_- \end{array} \end{array} \begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

$$= \frac{1}{d}\begin{array}{c} \begin{array}{cc} H_+ & H_+ \end{array} \\ \begin{pmatrix} d^r & b^\ell \\ -c^r & d^\ell \end{pmatrix} \\ \begin{array}{cc} zH_+ & H_+ \end{array} \end{array}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix} = \frac{1}{a}\begin{array}{c} \begin{array}{cc} zH_- & H_- \end{array} \\ \begin{pmatrix} a^\ell & -b^r \\ c^\ell & a^r \end{pmatrix} \\ \begin{array}{cc} zH_- & zH_+ \end{array} \end{array}\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix}$$



$$\begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \underbrace{\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\begin{pmatrix}a^r & b^r\\ -c^l & a^l\end{pmatrix}\frac{1}{d}\begin{pmatrix}d^r & b^l\\ -c^r & d^l\end{pmatrix}$$

$$\cap \qquad\qquad \cap$$

$$\begin{pmatrix}zH_- & H_-\\ zH_- & zH_-\end{pmatrix} \qquad \begin{pmatrix}H_+ & H_+\\ zH_+ & H_+\end{pmatrix}$$

What aim? You have this module $\mathcal{M}$ of solutions ~~[illegible]~~ $\psi(n,z)$ of the DE. This is a ~~[illegible]~~ rank 2 module over $\mathcal{A} = \mathbb{C}[z,z^{-1}]$, a right module - try this

Try following: $\mathcal{M}$ = solutions $\psi(n,z)$ of D.E. with $\psi(n,z) \in \mathbb{C}[z,z^{-1}]^{\oplus 2}$ $\forall n$.

$\mathcal{M}$ = $\mathbb{C}[z,z^{-1}]$-module of solutions of the DE with values in $\mathbb{C}[z,z^{-1}]^{\oplus 2}$. Assume (h_n) fin. supp

$\Longrightarrow$ $\begin{pmatrix}z^{-n}p_n\\ q_n\end{pmatrix}$ ind of n for $n \gg 0$ and ——— $n \ll 0$.

Get two isos. $\mathcal{A}^{\oplus 2} \xleftarrow{\sim} \mathcal{M} \xrightarrow{\sim} \mathcal{A}^{\oplus 2}$. These isos. corresp to bases $\{\xi'_-, \xi'_+\}$ $\{\xi_+, \xi_-\}$.

$\mathcal{M}$ is the space of solns of the DE with values in $\mathbb{C}[z,z^{-1}]^{\oplus 2}$ - rank 2 free $\mathcal{A}$-module

$\mathcal{M} \ni \psi = (\psi(n,z))$ $\quad \psi(n,z) \in \mathbb{C}[z,z^{-1}]^{\oplus 2}$ $\forall n$.

$\exists$ Obvious linear functionals, actually $\mathcal{M} \longrightarrow \mathcal{A}$ $\mathcal{A}^0$-linear

Suppose $\psi \in \mathcal{M}$ i.e. $\psi(n,z) \in \mathbb{C}[z,z^{-1}]^{\oplus 2}$ $\forall n$

and $\psi(n,z) = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \psi(n-1,z)$. $\forall n$

Then $$\psi(\infty, z) = \underbrace{\left(\prod_{n=-\infty}^{\infty} \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \quad \text{reverse order} \right)}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \psi(-\infty, z)$$

Take $\psi = \xi'_-$ i.e. $\psi(-\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Then $\xi'_-(\infty) = \begin{pmatrix} a \\ c \end{pmatrix}$ so that

$$\xi'_- = a\xi_+ + c\xi_-$$

Defining properties $\quad \xi_+(\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi_-(\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\quad \xi'_+(-\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \xi'_-(-\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\xi'_-(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xi'_-(-\infty) = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$= (\xi_+ a + \xi_- c)(\infty)$$

$$\xi'_+(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xi'_+(-\infty) = \begin{pmatrix} b \\ d \end{pmatrix}$$

$$= (\xi_+ b + \xi_- d)(\infty)$$

$$(\xi_+ \ \ \xi_-)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\xi'_- \ \ \xi'_+)$$

$$\begin{pmatrix} \xi_+ & \xi_- \end{pmatrix} = \begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

New formulas. Consider a discrete DE with (h_n) finite support. E space of solutions $\psi = \psi(n,z)$ with $\psi(n,z) \in \mathbb{C}[z,z^{-1}]$ $\forall n$. enuf for 1 n.

$$\psi(n,z) = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \psi(n-1,z)$$

Let $\cancel{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} =$

$$\psi(\infty,z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi(-\infty,z)$$

$$= \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \underbrace{\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \psi(-\infty,z)}_{\psi(0,z).}$$

E is a rank 2 free module over $\mathbb{C}[z,z^{-1}]$

Very invariant. You want to explain why its a pre Hilbert space. ~~What's important is that you~~

~~have~~ Repeat discrete DE with (h_n) fin. supp.

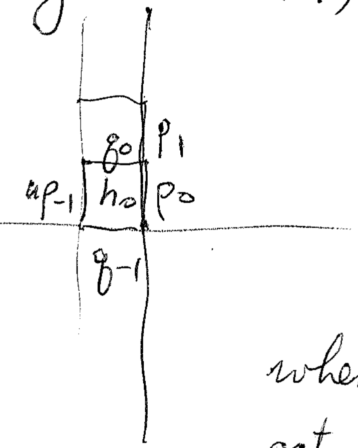
E = solutions of $\psi(n,z) = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \psi(n-1,z)$

with $\psi(n,z) \in \mathbb{C}[z,z^{-1}]^{\oplus 2}$. Then $\psi(n,z)$ const for $n \gg 0$ and for $n \ll 0$. So what you have here is an $\mathcal{O} = \mathbb{C}[z,z^{-1}]$ submodule of $\prod_n \mathbb{C}[z,z^{-1}]^{\oplus 2}$. You want to connect this submodule with your Hilbert space.

Ideas to incorporate: Distributions. Finite vectors inside the Hilbert space

~~Struggle to get the coherent core~~

Your Hilbert space is constructed with a grid of unit vectors, with 2×2 unitaries given by the (h_n). This is a well-defined



Hilb. space with u for any sequence (h_n). Get limits $\lim_{n\to\pm\infty}\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix}$ when h_n is ℓ^2, and (h_n) is ℓ^1 get invertible wave ops.

~~So if you have eigen~~ I have this pre-Hilbert space with u spanned by the grid unit vectors. Free module of rank 2 over $\mathbb{C}[u,u^{-1}]=\mathcal{A}$ basis p_0, q_0. All of your formulas pertain to this pre Hilbert space of finite vectors.

You have this $\mathcal{A}$ module E free over rank 2 given by finite vectors. Then solutions ψ of the DE with values in $\mathcal{A}^{\oplus 2}$ are the same as $\mathcal{A}$ linear maps $E \xrightarrow{\phi} \mathcal{A}$? Check

Given ϕ let $\psi(n,z) = \begin{pmatrix}\phi(u^{-n}p_n)\\ \phi(q_n)\end{pmatrix}$

since $\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & h_n u^{-n}\\ \bar h_n u^n & 1\end{pmatrix}\begin{pmatrix}u^{-n+1}p_{n-1}\\ q_{n-1}\end{pmatrix}$

1173.07
1167.71

you apply $\phi : E \to \mathcal{A}$ $\qquad \phi u = z\phi$

$$\psi(u,z) = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\psi(u-1,z)$$

Maybe E is naturally a $\mathbb{C}[z,z^{-1}]$-module with z acting as u, and then $\mathrm{Hom}_{\mathcal{A}}(E,\mathcal{A})$ is naturally a right module. Let's continue with factorization

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{array}{cc} zH_- & H_- \\ \multicolumn{2}{c}{\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}} \\ zH_+ & H_+ \end{array} \begin{array}{cc} zH_- & H_+ \\ \multicolumn{2}{c}{\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}} \\ zH_- & H_+ \end{array}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{array}{cc} zH_- & H_- \\ \multicolumn{2}{c}{\begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}} \\ zH_- & zH_- \end{array} \begin{array}{cc} H_+ & H_+ \\ \multicolumn{2}{c}{\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}} \\ zH_+ & H_+ \end{array}$$

Two methods of reconstruction

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix}$$

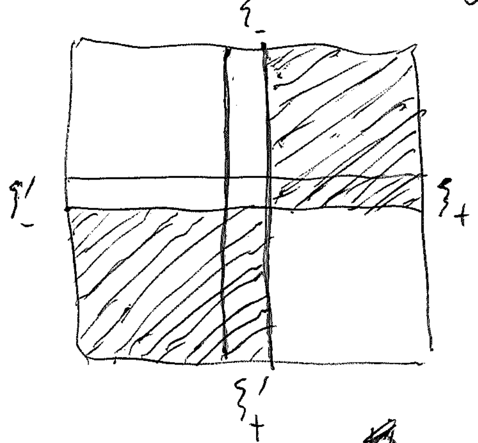
$$\begin{pmatrix} 1 & -\frac{b}{d} \\ -\frac{c}{a} & 1 \end{pmatrix} \begin{array}{cc} zH_- & H_- \\ \multicolumn{2}{c}{\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}} \\ zH_+ & H_+ \end{array} = \begin{pmatrix} \frac{d^l}{d} & \frac{-b^l}{d} \\ \frac{-c^l}{a} & \frac{a^l}{a} \end{pmatrix} \in \begin{pmatrix} H_+ & H_+ \\ zH_- & zH_- \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{Id}_{zH_-} & -\pi_{zH_-}\frac{b}{d} \\ -\pi_{H_+}\frac{c}{a} & \mathrm{Id}_{H_+} \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} \frac{d^l}{d}(0) & -\frac{b^l(0)}{d(0)} \\ -\frac{c^l}{a}(\infty) & \frac{a^l}{a}(\infty) \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} \begin{array}{cc} H_+ & H_+ \\ \multicolumn{2}{c}{\begin{pmatrix} d^l & -b^l \\ c^r & d^r \end{pmatrix}} \\ zH_+ & H_+ \end{array} \frac{1}{d} = \begin{pmatrix} a^r & b^r \\ -c^l & d^l \end{pmatrix} \in \begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}$$

$$\begin{pmatrix} Id_{H_+} & \pi_{H_+} b \\ -\pi_{H_+} c & Id\, H_+ \end{pmatrix}\begin{pmatrix} \frac{d^l}{d} & -\frac{b^l}{d} \\ \frac{c^r}{d} & \frac{d^r}{d}\end{pmatrix} = \begin{pmatrix} a^r(\infty) & 0 \\ -c^l(\infty) & a^l(\infty)\end{pmatrix}$$

Look at splitting. Consider discrete DE.



Given scattering data i.e. $b(\lambda)$ ~~[illegible]~~ to construct splittings

$$E = (H_+\xi_+ \oplus H_-\xi_-) \oplus (H_-\xi'_- \oplus H_+\xi'_+)$$

(this is $\perp$ for Hilb. space structure)

$$H_+\xi_+ + H_-\xi_- = \begin{pmatrix} H_+ & H_- \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} H_+ & H_- \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$\therefore$ want to prove: $\begin{pmatrix} H_+ & H_- \end{pmatrix}$ is comp. to $\begin{pmatrix} H_+ & H_- \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ inside $\begin{pmatrix} L^2 & L^2 \end{pmatrix}$. Equiv.

$$\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \text{ is comp to } \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \text{ in } \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} H_- \\ H_+ \end{pmatrix} \quad \text{———} \quad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} H_- \\ H_+ \end{pmatrix} \quad \text{———}$$

Another splitting

$$E = (H_+\xi'_- \oplus H_+\xi_-) \oplus (H_-\xi_+ \oplus H_-\xi'_+)$$

$$\begin{pmatrix} H_+ & H_+ \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \text{ is comp to } \begin{pmatrix} H_- & H_- \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \text{ is comp to } \begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \quad \text{———} \quad \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \text{ is comp to } \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} 1 & -b \\ c & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \text{ ——— } \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

Since a is an auto of L^2 autom on H_-

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \text{ is comp to } \begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

Since d isom on L^2, H_+

$$\text{"} \quad \text{———} \quad \begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} \text{"} \end{pmatrix}$$

etc. etc.

Q: What is the meaning of

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \text{ is comp to } \begin{pmatrix} H_- \\ H_- \end{pmatrix} \; ?$$

means $\pi_+ : \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \longrightarrow \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$ rest. to $\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$ is an isom. Stronger is for $\pi_+\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} : \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \circlearrowleft$ to be an isom.

Another splitting

$$E = \left(H_-\xi_-' \oplus H_-\xi_-\right) \oplus \left(H_+\xi_+ \oplus H_+\xi_+'\right)$$

$$\begin{pmatrix} H_- & H_- \end{pmatrix} \text{ is comp to } \begin{pmatrix} H_+ & H_+ \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

$$\begin{pmatrix} H_- \\ H_- \end{pmatrix} \text{ ——— } \cancel{\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}\begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}\begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}}$$

$$\begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} = \begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

Prove. ~~Given~~ $\cancel{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ ~~factor it~~

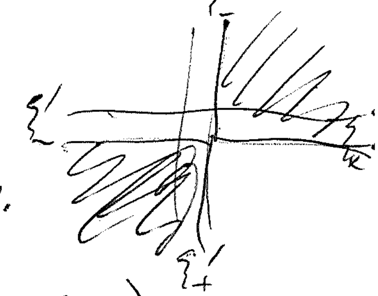
Given $b(z)$ smooth on S^1 find $d(z) \ni$ $|d|^2 = 1 + |b|^2$ and d extends holom. to D, $d(0) > 0$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{d} & b \\ \bar{b} & d \end{pmatrix} \quad + \quad \det = 1.$$

Form $E = \begin{pmatrix} a & a \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} a & a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$

Notice that if a is smooth fns. on S^1, then

$a = a_+ + a_-$ $\quad$ (a_+ powers z^n $n \geq 0$; a_- ——— $n < 0$.)



smoothness means decay of Fourier coeffs.

discuss splitting: $E = (H_+ \xi_+ + H_- \xi_-) \oplus (H_- \xi'_- + H_+ \xi'_+)$

i.e. $\begin{pmatrix} H_- & H_+ \end{pmatrix}$ comp to $\begin{pmatrix} H_+ & H_- \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\begin{pmatrix} H_- \\ H_+ \end{pmatrix}$ ——— $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix}$

$\begin{pmatrix} H_+ \\ H_- \end{pmatrix}$ ——— $\begin{pmatrix} d & b \\ c & a \end{pmatrix}\begin{pmatrix} H_- \\ H_+ \end{pmatrix}$

$\begin{pmatrix} H_+ \\ H_- \end{pmatrix}$ ——— $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} H_- \\ H_+ \end{pmatrix}$ $\quad$ conj by $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

Look at this carefully from receipt

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{array}{c} zH_- \quad H_- \\ \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \\ zH_+ \quad H_+ \end{array} = \begin{array}{c} H_+ \quad H_+ \\ \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix} \\ zH_- \quad zH_- \end{array}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

Confusing

Confusing. To establish the splitting

$$\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\begin{pmatrix}H_-\\H_+\end{pmatrix} \text{ is comp to } \begin{pmatrix}H_+\\H_-\end{pmatrix}$$

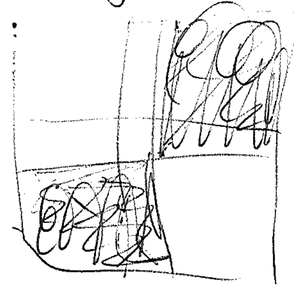
$$\begin{pmatrix}1 & -\frac{b}{d}\\ -\frac{c}{a} & 1\end{pmatrix}\begin{pmatrix}H_-\\H_+\end{pmatrix} \text{ ——— } \begin{pmatrix}H_+\\H_-\end{pmatrix}$$

Enough to show

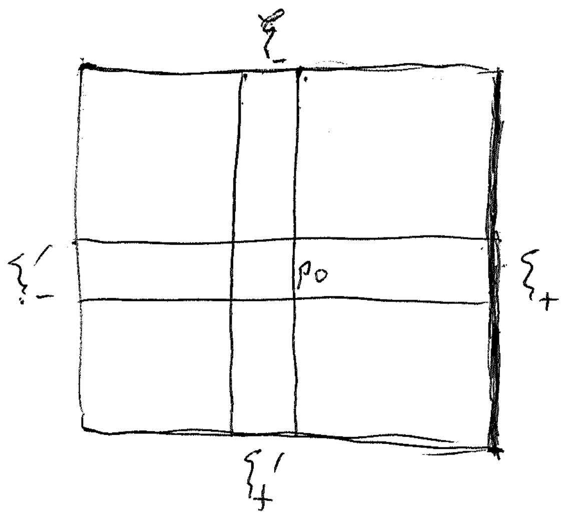
$$\underbrace{\begin{pmatrix}\pi_- & 0\\ 0 & \pi_+\end{pmatrix}\begin{pmatrix}1 & -\beta\\ -\bar\beta & 1\end{pmatrix}}_{\begin{pmatrix}\mathrm{Id}_{H_-} & -\pi_-\beta\\ -\pi_+\bar\beta & \mathrm{Id}_{H_+}\end{pmatrix}} : \begin{pmatrix}H_-\\H_+\end{pmatrix} \longrightarrow \begin{pmatrix}H_-\\H_+\end{pmatrix} \quad \text{is bij}$$

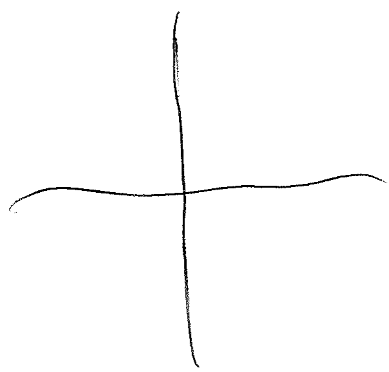
So suppose you know that the splitting ~~[illegible]~~ holds.

$$E = \left(H_+\xi_+ \oplus H_-\xi_-\right) \overset{\perp}{\oplus} \left(H_-\xi'_- \oplus H_+\xi'_+\right)$$



~~Then~~





$$\xi_+ = kp_0 + z f_+ \xi_+ + f_- \xi_-$$

Review splittings:

1) $E = (H_+\xi_+ \oplus H_-\xi_-) \overset{\perp}{\oplus} (H_-\xi_-' \oplus H_+\xi_+')$

Proof. $H_+\xi_+ + H_-\xi_- = (H_+ \ H_-)\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\xi_-' \\ \xi_+\end{pmatrix}$

~~so you~~ So assertion equiv to

$$(H_- \ H_+) \oplus (H_+ \ H_-) \xrightarrow[\left(\text{inc.}, \begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)]{} (L^2 \ L^2)$$

being an isom.

equiv to $(H_+ \ H_-) \xrightarrow{\cdot\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot\begin{pmatrix}\pi_+ & \\ & \pi_-\end{pmatrix}} (L^2/H_- \ \ L^2/H_+)$

$$\begin{pmatrix} H_+ \\ H_-\end{pmatrix} \xrightarrow{\begin{pmatrix} d^{-1} & \\ & a^{-1}\end{pmatrix}} \begin{pmatrix} H_+ \\ H_-\end{pmatrix} \xrightarrow{\begin{pmatrix}\pi_+ & \\ & \pi_-\end{pmatrix}\begin{pmatrix} a & c \\ b & d\end{pmatrix}} \begin{pmatrix} H_+ \\ H_-\end{pmatrix} \qquad \text{NO}$$

$$\begin{pmatrix} H_- \\ H_+\end{pmatrix} \oplus \begin{pmatrix} H_+ \\ H_-\end{pmatrix} \xrightarrow{\left(\text{Inc.}, \begin{pmatrix} a & \cancel{b} \\ b & d\end{pmatrix}\right)} (L^2 \ L^2)$$

$$(H_- \ H_+) \oplus (H_+ \ H_-)\begin{pmatrix} a & b \\ c & d\end{pmatrix} = (L^2 \ L^2) \quad ?$$

$$\begin{pmatrix} H_- \\ H_+\end{pmatrix} \oplus \begin{pmatrix} a & c \\ b & d\end{pmatrix}\begin{pmatrix} H_+ \\ H_-\end{pmatrix} = \quad ''$$

apply $\begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1}\end{pmatrix}$ to both sides

$$\begin{pmatrix} H_- \\ H_+\end{pmatrix} \oplus \begin{pmatrix} 1 & \frac{c}{a} \\ \frac{b}{d} & 1\end{pmatrix}\begin{pmatrix} H_+ \\ H_-\end{pmatrix} = \quad ''$$

$$\begin{pmatrix} H_+ \\ H_-\end{pmatrix} \oplus \begin{pmatrix} 1 & \frac{b}{d} \\ \frac{c}{a} & 1\end{pmatrix}\begin{pmatrix} H_- \\ H_+\end{pmatrix} = \quad ''$$

try iff $\begin{pmatrix} Id_{H_-} & \pi_-\frac{b}{d} \\ \pi_+\frac{c}{a} & Id_{H_+}\end{pmatrix}$ bij. on $\begin{pmatrix} H_- \\ H_+\end{pmatrix}$

Other splitting

$$E = (H_+\xi_-' + H_+\xi_-) \oplus (H_-\xi_+ + H_-\xi_+')$$

$$(H_+ \ H_+) \oplus (H_- \ H_-)\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \overset{?}{\xrightarrow{\sim}} (L^2 \ L^2)$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ +\frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

Apply $\begin{pmatrix} d & \\ & d \end{pmatrix}$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow{\sim} \text{''}$$

Apply $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow{\sim} \cdot$$

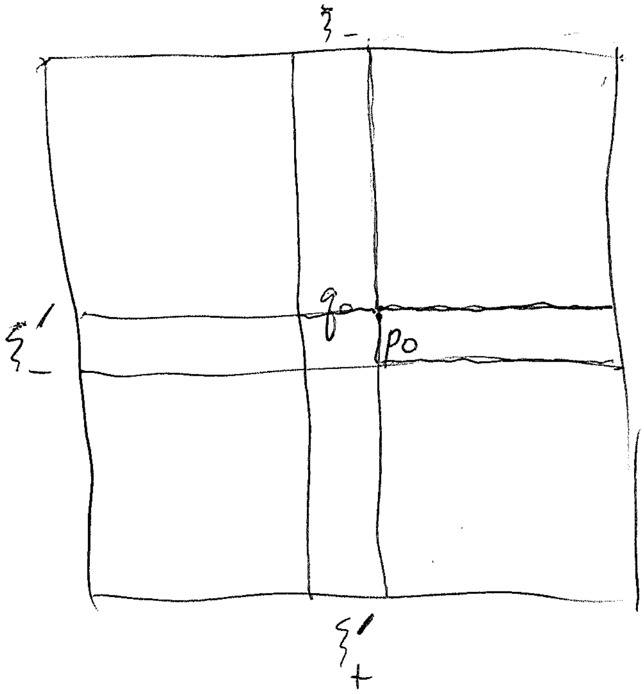
$$\begin{pmatrix} Id_{H_-} & \pi_- b \\ -\pi_- c & Id_{H_-} \end{pmatrix} \text{ inv. on } \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{a} & \frac{c}{a} \\ -\frac{b}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

Apply $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & +b \\ -c & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \ ?$$

Q: What is the meaning of such a splitting? Does it yield factorization?



Does splitting yield ~~a~~ factorization? You ~~?~~ want to factor a matrix like S or the transfer matrix. This means another bases for E over $\mathcal{a}$

~~Also~~ So at the moment we have splittings

$$E = \underbrace{(H_+\xi_+ + H_-\xi_-)} \overset{\perp}{\oplus} (H_-\xi'_- + H_+\xi'_+)$$

$$(H_+ \ H_-)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

So ~~basically~~ you almost understand factoring.

How to proceed? ~~Compare $H_+\xi_+ + H_-\xi_-$ with~~

Take subspaces + codim 1 inclusions

$$\begin{array}{ccc} zH_+\xi_+ + zH_-\xi_- & \supset & zH_+\xi_+ + H_-\xi_- \\ \cap & & \cap \\ H_+\xi_+ + zH_-\xi_- & \supset & H_+\xi_+ + H_-\xi_- \end{array}$$

intersect with the complement for $zH_+\xi_+ + H_-\xi$ which is $zH_-\xi'_- + H_+\xi'_+$, in fact this is the orth complement. Then ~~get~~ you get a 2 diml space isom to $H_+\xi_+ + zH_-\xi_- / zH_+\xi_+ \oplus H_-\xi_- \simeq \mathbb{C}\xi_+ + \mathbb{C}\xi_-$

Then you get
$$\xi_+ = k^1 p_0 + \overset{zH_+}{(\ )}\xi_+ + \overset{H_-}{(\ )}\xi_-$$
$$\xi_- = k^2 q_0 + (\ )\xi_+ + (\ )\xi_-$$

The point (~~at~~ at the moment) is that given β

Yesterday I almost understood factorization, rather how it might follow from splitting. You can prove splitting: Recall

$$E \overset{?}{=} \underbrace{(H_+\xi_+ + H_-\xi_-)}\ \overset{?}{\oplus}\ (H_-\xi_-' + H_+\xi_+')$$

$$\underbrace{(\xi_+\ \ \xi_-)}\begin{pmatrix}H_+\\ H_-\end{pmatrix} \qquad \overset{\shortparallel}{(\xi_-'\ \ \xi_+')}\begin{pmatrix}H_-\\ H_+\end{pmatrix}$$

$$(\xi_-'\ \ \xi_+')\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}H_+\\ H_-\end{pmatrix} \qquad \begin{pmatrix}h & \frac{c}{a}\\ \frac{b}{d} & \#\end{pmatrix}\begin{pmatrix}H_+\\ H_-\end{pmatrix} \overset{?}{\oplus} \begin{pmatrix}H_-\\ H_+\end{pmatrix} = \begin{pmatrix}L^2\\ L^2\end{pmatrix}$$
$$\Updownarrow$$

$$\therefore \begin{pmatrix}Id_{H_+} & \pi_+\bar\beta\\ \pi_-\beta & Id_{H_-}\end{pmatrix} = \begin{pmatrix}\pi_+ & 0\\ 0 & \pi_-\end{pmatrix}\begin{pmatrix}1 & \frac{c}{a}\\ \frac{b}{d} & 1\end{pmatrix} : \begin{pmatrix}H_+\\ H_-\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}H_+\\ H_-\end{pmatrix}$$

This is true because ~~$\|\beta\|$~~ $\|\pi_+\bar\beta\|_\infty, \|\pi_-\beta\|_\infty < 1$

$$E = (\underbrace{H_+\xi_-'} + H_+\xi_-) \overset{?}{\oplus} (H_-\xi_+ + H_-\xi_+')$$

$$(\xi_-'\ \ \xi_-)\begin{pmatrix}H_+\\ H_+\end{pmatrix} \qquad (\xi_+\ \ \xi_+')\begin{pmatrix}H_-\\ H_-\end{pmatrix} = (\xi_-'\ \ \xi_-)\begin{pmatrix}\frac{1}{d} & -\frac{c}{d}\\ \frac{b}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}H_-\\ H_-\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{d} & -\frac{c}{d}\\ \frac{b}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}H_-\\ H_-\end{pmatrix} \overset{?}{\oplus} \begin{pmatrix}H_+\\ H_+\end{pmatrix} = \begin{pmatrix}L^2\\ L^2\end{pmatrix} \qquad \pi_-$$

Apply $\begin{pmatrix}d & 0\\ 0 & d\end{pmatrix}$
$$\begin{pmatrix}1 & -c\\ b & 1\end{pmatrix}\begin{pmatrix}H_-\\ H_-\end{pmatrix} \overset{?}{\oplus} \begin{pmatrix}H_+\\ H_+\end{pmatrix} = \begin{pmatrix}L^2\\ L^2\end{pmatrix}$$

~~Th~~ So splitting is well understood, and the argument works in the continuous case. Next you want factorization of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and S. This is trickier because it's not a Hilb. space argument apparently; ~~so~~ rather you ~~need~~ want operators.

Idea: Can you construct the projection operators associated to a splitting. Yes, clearly. You have

$$\begin{tikzcd}
 & H_-^{\oplus 2} \arrow[d, "(S)"] \arrow[dr, "\sim"] & \\
H_+^{\oplus 2} \arrow[r, "inc"] & L^{2,\oplus 2} \arrow[r, "\pi_-"] & H_-^{\oplus 2}
\end{tikzcd}$$

$$\begin{tikzcd}
 & H_-^{\oplus 2} \arrow[d, "\frac{1}{d}\begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}"] & \\
H_+^{\oplus 2} \arrow[r, hook, "\pi_+^*"] \arrow[d, "s", "d"'] & L^{2,\oplus 2} \arrow[r, "\pi_-"] \arrow[d, "s", "d"'] & H_-^{\oplus 2} \arrow[d, "\pi_- d\, \pi_-^*"] \\
H_+^{\oplus 2} \arrow[r, hook, "\pi_+^*"] & L^{2,\oplus 2} \arrow[r, "\pi_-"] & H_-^{\oplus 2}
\end{tikzcd}$$

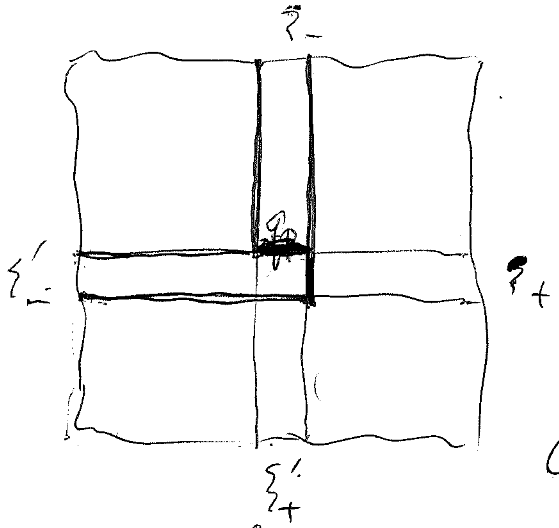
~~You want the proj~~ You ~~should~~ have ~~~~ splitting

$$S(H_-^{\oplus 2}) \oplus H_+^{\oplus 2} \xrightarrow{\sim} L^{2,\oplus 2}$$

and there should be a ~~projection of L^2~~ formula for the inverse isomorphism.

$$\begin{tikzcd}
 & H_-^{\oplus 2} \arrow[d, "S = \frac{1}{d}\begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}"] & & \\
0 \arrow[r] & H_+^{\oplus 2} \arrow[r, "\varepsilon_+"] \arrow[d, "s", "d"'] & L^{2,\oplus 2} \arrow[r, "\pi_-"] \arrow[d, "s", "d"'] & H_-^{\oplus 2} \arrow[r] \arrow[d, "J", "\pi_- d\, \varepsilon_-"'] & 0 \\
0 \arrow[r] & H_+^{\oplus 2} \arrow[r, "\varepsilon_+"] & L^{2,\oplus 2} \arrow[r, "\pi_-"] & H_-^{\oplus 2} \arrow[r] & 0
\end{tikzcd}$$

puzzle



You know how to establish the splitting

$$E = (H_+\xi'_- + H_+\xi_-) \oplus (H_-\xi_+ + H_-\xi'_+)$$

a ~~point~~ vertex in the grid determines such a ~~[illegible]~~ forward and backward light cone splitting, but it is not $\perp$ for the positive definite inner product. Then you have? Because $L^2\xi'_- \perp L^2\xi_-$ you get $\mathbb{C}\xi'_- \oplus \mathbb{C}\xi_-$ for $(H_+\xi'_- + H_+\xi_-) \ominus (zH_+\xi'_- + zH_-\xi_-)$

What are you trying to do? You have an ~~[illegible]~~ incoming subspace $H_+\xi'_- + H_+\xi_-$

Start with the fact you understand:

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} zH_- & H_+ \end{array} \\ \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \end{array} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{array}{c} \begin{array}{cc} zH_- & H_+ \end{array} \\ \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \\ \begin{array}{cc} zH_- & H_+ \end{array} \end{array} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d} \begin{array}{c} \begin{array}{cc} H_+ & H_+ \end{array} \\ \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \\ \begin{array}{cc} zH_+ & H_+ \end{array} \end{array} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}$$

Sim.

$$= \begin{array}{c} \begin{array}{cc} H_+ & H_- \end{array} \\ \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \\ \begin{array}{cc} zH_+ & zH_- \end{array} \end{array} \begin{pmatrix} \xi_+ \\ \xi \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\ }{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$= \frac{1}{a} \begin{array}{c} \begin{array}{cc} zH_- & H_- \end{array} \\ \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \\ \begin{array}{cc} zH_- & zH_- \end{array} \end{array} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\boxed{\begin{pmatrix} d & b \\ -c & a \end{pmatrix} = \begin{array}{c} \begin{array}{cc} zH_- & H_- \end{array} \\ \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \\ \begin{array}{cc} zH_+ & zH_- \end{array} \end{array} \begin{array}{c} \begin{array}{cc} H_+ & H_+ \end{array} \\ \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \\ \begin{array}{cc} zH_+ & H_+ \end{array} \end{array}}$$

So where are you? Aim: factoryation of S.

Answer: You proved: $E = \left(H_+\xi_-' + H_+\xi_-\right) \oplus \left(H_-\xi_+ + H_-\xi_+'\right)$

so $\left(H_+\xi_-' + H_+\xi_-\right) \cap \left(zH_-\xi_+ + zH_-\xi_+'\right)$ is

2 diml.

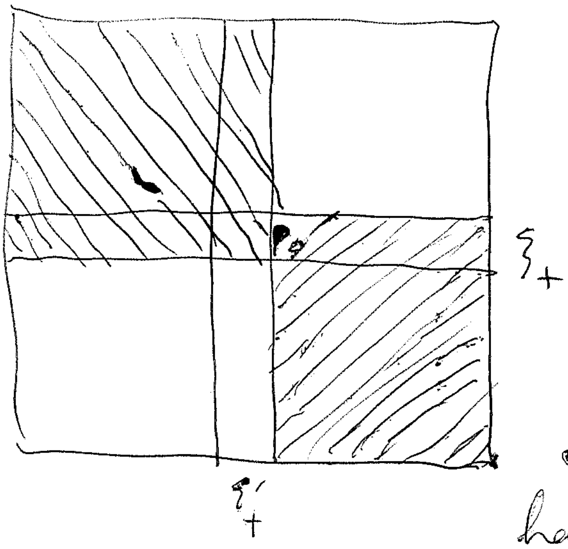
There's something confusing here. When you have

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d}\begin{array}{c} \begin{array}{cc} H_+ & H_+ \end{array} \\ \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \\ \begin{array}{cc} zH_+ & H_+ \end{array} \end{array} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{array}{c} \begin{array}{cc} zH_- & H_- \end{array} \\ \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \\ \begin{array}{cc} zH_- & zH_- \end{array} \end{array} \begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

you see that

$$p_0 \in \left(H_+\xi_-' + H_+\xi_-\right) \cap \left(zH_-\xi_+ + H_-\xi_+'\right)$$



Can you see any orthogonality properties of p_0? No

Try something else. Consider $\left(H_-\xi_+ + H_-\xi_+'\right)$ $\left(H_+\xi_-' + H_+\xi_-\right)$

Is it possible that these half spaces are orthogonal wrt a indefinite hermitian form.

~~Go back to the idea~~

Hardy projection - Hilbert transform.

Idea: Given $b(z)$ form

$$1+X=\begin{pmatrix}1 & -\bar b\\ b & 1\end{pmatrix} \qquad 1-X^2=\begin{pmatrix}1+|b|^2 & 0\\ 0 & 1+|b|^2\end{pmatrix}$$

C.T.

$$1+\pi X=\begin{pmatrix}Id & -\pi_+\bar b\\ \pi_+ b & Id\end{pmatrix} \quad \text{on } H_+^{\oplus 2}$$

What you want to understand is roughly this: Given basic factorization in the equivalent forms

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{array}{c}zH_-\quad H_-\\ \begin{pmatrix}a^r & b^r\\ c^r & d^r\end{pmatrix}\\ zH_+\quad H_+\end{array}\begin{array}{c}zH_-\quad H_+\\ \begin{pmatrix}a^l & b^l\\ c^l & d^l\end{pmatrix}\\ zH_-\quad H_+\end{array}$$

$$\begin{pmatrix}1 & b\\ -c & 1\end{pmatrix}=\begin{array}{c}zH_-\quad H_-\\ \begin{pmatrix}a^r & b^r\\ -c^l & a^l\end{pmatrix}\\ zH_-\quad zH_-\end{array}\begin{array}{c}H_+\quad H_+\\ \begin{pmatrix}d^r & b^l\\ -c^r & d^l\end{pmatrix}\\ zH_+\quad H_+\end{array}$$

that the factorization is unique (up to a few scalar factors). Also want the two factorizations to agree.

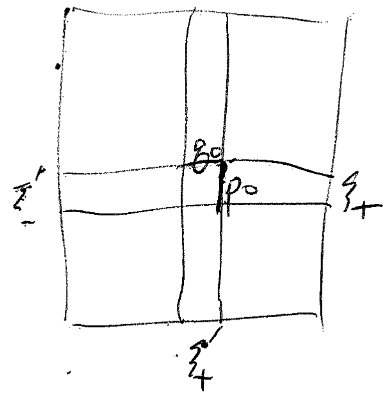
Solution methods:

$$\begin{pmatrix}1 & \frac{b}{a}\\ \frac{c}{d} & 1\end{pmatrix}\begin{pmatrix}+d^l & -b^l\\ -c^l & a^l\end{pmatrix}=\begin{pmatrix}\frac{a^r}{a} & \frac{b^r}{a}\\ \frac{c^r}{d} & \frac{d^r}{d}\end{pmatrix}\in\begin{pmatrix}zH_- & H_-\\ zH_+ & H_+\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}\pi_+ & \\ & \pi_-\end{pmatrix}\begin{pmatrix}1 & \frac{b}{a}\\ \frac{c}{d} & 1\end{pmatrix}}_{\begin{pmatrix}Id_{H_+} & -\pi_+\frac{b}{a}\\ -\pi_-\frac{c}{d} & Id_{H_-}\end{pmatrix}}\begin{pmatrix}d^l & -b^l\\ -c^l & a^l\end{pmatrix}=\begin{pmatrix}\frac{a^r}{a}(\infty) & 0\\ 0 & \frac{d^r}{d}(0)\end{pmatrix}$$

inverting this operator matrix amounts to orthogonal projection.

e.g. $\begin{array}{c} \\ \begin{pmatrix}p_0\\ q_0\end{pmatrix} \\ \\ \end{array} = \begin{array}{cc} zH_- & H_+ \\ \multicolumn{2}{c}{\begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell\end{pmatrix}} \\ zH_- & H_+ \end{array} \begin{array}{c} \\ \begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \\ \\ \end{array}$



$p_0 \in (zH_-)\xi'_- + H_+\xi'_+$

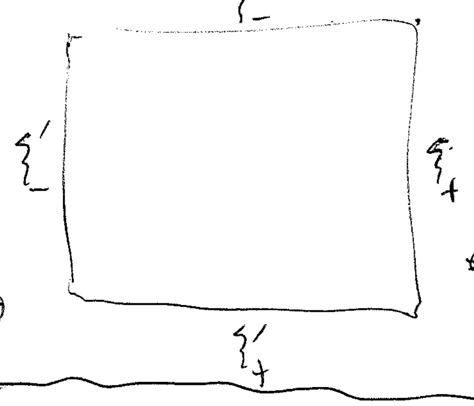
$\ominus \quad H_-\xi'_- + H_+\xi'_+$

Start with $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \bar d = a \quad \bar b = c \quad d \in H_+$

$|d|^2 - |b|^2 = 1.$

$S = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$

The idea is that $E =$



Perhaps you should think of something of rank 2 over the circle. ~~So for~~ ~~each~~

Recall equivalence

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix} \frac{ab-cd}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

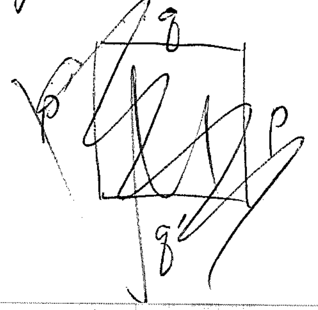
In the scattering situation $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SU(1,1) \Leftrightarrow \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} \in U(2)$ diag $=$

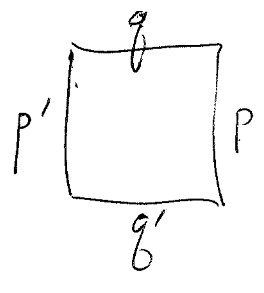
My way to understand this was ~~to~~ consider E as a subspace of a 4 dim'l space. Change notation

Consider $W \subset A^4$ $\qquad A = C(S^1) \quad C^*$ algebra

A^4 column vectors a right A-module $\qquad \xi^*\eta = (\xi_1^* \ \xi_2^* \ \xi_3^* \ \xi_4^*)\begin{pmatrix}\eta_1\\ \vdots \\ \eta_4\end{pmatrix} = \sum \xi_i^* \eta_i$

You want <u>the Krein version</u>



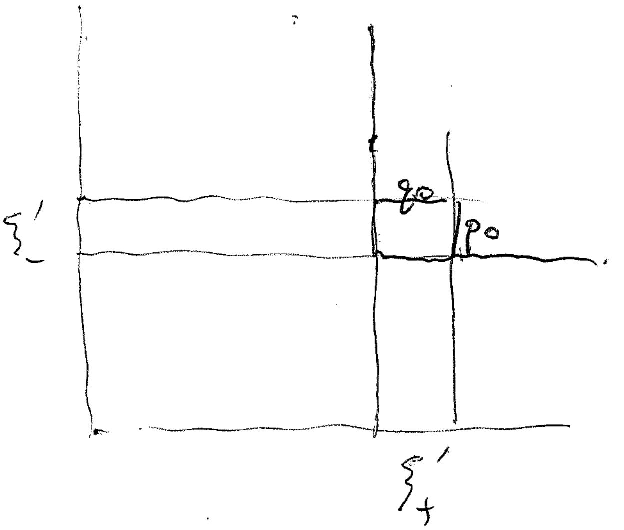


Constraint

Program. You have some linear algebra in a Krein space of type (2,2). Call this K, and you have W max isotropic in K.

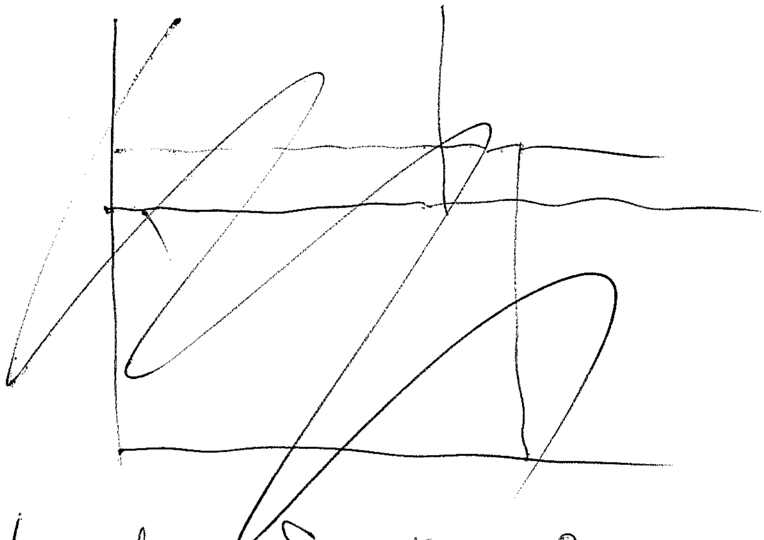
Anyway I digressed to look at partial unitary of type 1,1 – the motivation comes from earlier experience with Krein spaces. If you have such a partial unitary $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\quad Y = X \oplus \mathbb{C}\xi_+ = \mathbb{C}\xi_- \oplus uX$

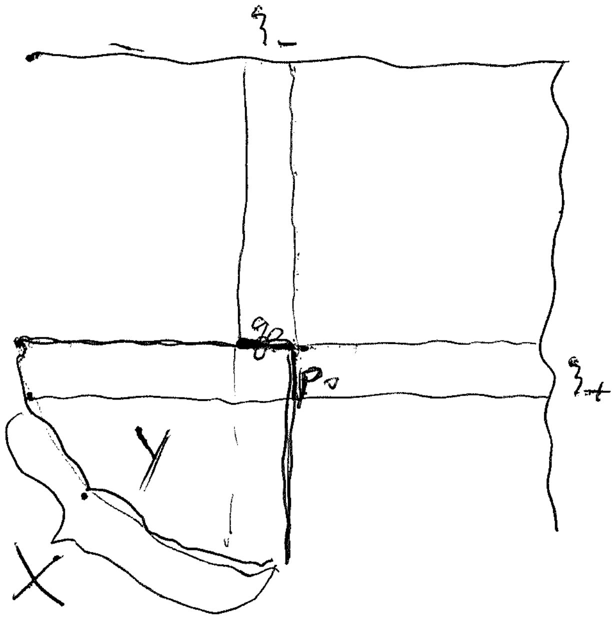
You want this to yield a disc. DE with $h_n = 0 \quad n \le 0$.



$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

This doesn't quite fit it seems.





How does S enter. Point is that you've ~~constructed~~ $L^2\xi_+ + L^2\xi_-$ glued together with β defd by

$$(z^n\xi_- \mid \xi_+) = (z^n \mid \beta) = \beta_n$$

know $\beta_n = 0 \quad n < 0$

OKAY $\quad E = \Gamma(\mathcal{E}) \quad \mathcal{E}_z = \overset{\text{space of}}{\text{solns}}\ \psi(n, z)$.

Define $\xi_\pm, \xi'_\pm \in E$ by

$$\xi_+(\infty, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\xi_-(\infty, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\xi'_-(-\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\xi'_+(-\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\underbrace{\psi(\infty, z) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$$
$$\Downarrow$$
$$\psi = \alpha \xi_+ + \beta \xi_-$$

$$\psi(-\infty, z) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$
$$\Downarrow$$
$$\psi = \gamma \xi'_- + \delta \xi'_+$$

But $\psi(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi(-\infty)$

$$\Downarrow$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

Let $\boxed{\psi(\infty, z) = T(z)\psi(-\infty, z)}$

$$\boxed{\begin{aligned} \xi'_-(-\infty, z) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \xi'_+(-\infty, z) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}}$$

~~so if you~~

Then $\xi'_-(\infty, z) = T(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

$$\xi'_+(\infty, z) = T(z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$\therefore$ with this definition of $\xi_\pm\ \xi'_\pm$ you have

$$\xi'_- = a\xi_+ + c\xi_-$$
$$\xi'_+ = b\xi_+ + d\xi_-$$

So you have the transpose.

Go back to your Krein business. ~~When~~ you have both left + right asymptotics. Instead of dealing with $H_\pm$ maybe you can look at the eigenvector eqn and growth. So you consider solutions of $\psi_n = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\psi_{n-1}$ for $z \in \mathbb{C}^\times$. This is a 2 dim'l vector space. Assume h_n finite supp. Then ψ_n const. for $n >> 0$ (resp $n << 0$), so you have ~~two~~ left and right coordinates for E_z

Start again. Consider the eigenvector equation for a disc. DE with (h_n) fin. supp.: $\psi_n = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\psi_{n-1}$

For each $z \in \mathbb{C}^\times$ you get ~~a~~ 2 dim'l soln. space E_z, moreover ψ_n ~~is~~ const for $n >> 0$ and $n << 0$, so you have natural left + right coordinates for E_z ~~related~~ $\psi(-\infty) \leftarrow \psi \longmapsto \psi(+\infty) \in \mathbb{C}^2$ related by

$$\psi(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi(-\infty).$$

For each z we have E_z a 2 dim'l solution space. The collection of E_z's is a rank 2 ~~free module~~ vector bundle $\mathcal{E}$ over $\mathbb{C}^\times$. Left + right + incoming + outgoing trivializations ~~given are~~ of $\mathcal{E}$. Think of E as the space of sections of this v. bundle. $\xi_\pm, \xi'_\mp$ are the elements of $E = \Gamma(\mathcal{E})$ with

$$\xi_+(\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi_-(\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\xi'_-(-\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi'_+(-\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus $\xi'_-(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\xi'_-(-\infty) = \begin{pmatrix} a \\ c \end{pmatrix}$

E_z consists of $n \mapsto \psi(n,z)$ sats. DE ..

$\mathcal{E}$ ——— $(n,z) \mapsto \psi(n,z)$ ———.

$$\psi(\infty, z) = \underbrace{T(z)}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \psi(-\infty, z) \qquad \text{all } \psi(\cdot, z) \text{ in } E_z \text{ or } \psi(\cdot,\cdot) \text{ in } \mathcal{E}$$

$$\xi'_-(-\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi'_+(-\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle -\infty | \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle -\infty | \begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle \infty | \begin{pmatrix} \xi_+ & \xi_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(z) \langle -\infty | = \langle \infty |$$

$$T(z) = T(z) \langle -\infty | \begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} = \langle \infty | \begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix}$$

If $\begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \xi_+ & \xi_- \end{pmatrix}$

then $\langle \infty |$ $\qquad$ $\langle \infty |$

$$T(z) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Thus

$$\begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} = \begin{pmatrix} \xi_+ & \xi_- \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\xi'_- = \xi_+ a + \xi_- c$$

$$\begin{pmatrix}\xi'_- & \xi'_+\end{pmatrix} = \begin{pmatrix}\xi_+ & \xi_-\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$\boxed{\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}}$$

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

Repeat the calculation:

$$\psi(n,z) = \frac{1}{k_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar{h}_n z^n & 1\end{pmatrix}\psi(n-1,z)$$

$$\psi(\infty,z) = \underbrace{\begin{pmatrix}a(z) & b(z)\\ c(z) & d(z)\end{pmatrix}}_{T(z)}\psi(-\infty,z)$$

$$T(z) = \cdots\cdots \frac{1}{k_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar{h}_n z^n & 1\end{pmatrix}\frac{1}{k_{n-1}}\begin{pmatrix}1 & h_{n-1}z^{-n+1}\\ \bar{h}_{n-1}z^{n-1} & 1\end{pmatrix}\cdots\cdots$$

Then $\Xi = \Gamma(\Xi)$ has dist. solutions. $\xi_\pm$ $\xi'_\mp$

$$\langle\infty|\begin{pmatrix}\xi_+ & \xi_-\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \langle-\infty|\begin{pmatrix}\xi'_- & \xi'_+\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$T(z)\langle-\infty|\begin{pmatrix}\xi'_- & \xi'_+\end{pmatrix} = \langle\infty|\begin{pmatrix}\xi'_- & \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \langle\infty|\begin{pmatrix}\xi'_- & \xi'_+\end{pmatrix}$$

$$\langle\infty|\xi'_- = \begin{pmatrix}a\\ c\end{pmatrix} \qquad \boxed{\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}}$$

$$\langle\infty|\xi'_+ = \begin{pmatrix}b\\ d\end{pmatrix}$$

$\exists$ much confusion eigenvalues

$\mathcal{M}$ module of rank 2 over $\mathcal{A} = \mathbb{C}[u, u^{-1}]$.
$\mathcal{M}$ contains abstract elements $p_n, q_n, \xi_\pm, \xi'_\pm$.
A solution of the DE with eigen value z is
$\psi : \mathcal{M} \longrightarrow \mathbb{C}$ $\quad \psi.(u-z) = 0$

since $u^{-n} p_n = \xi_+ \qquad n >> 0$

$$\psi\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \psi\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

Given (h_n) fin support, you construct $\mathcal{M}$ a free rank 2 module over $\mathcal{A} = \mathbb{C}[u, u^{-1}]$ generated by elements $p_n, q_n \quad n \in \mathbb{Z}$ satisfying

$$\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n u^{-n} \\ \bar{h}_n u^n & 1 \end{pmatrix}\begin{pmatrix} u^{-n+1} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

So $\quad u^{-n} p_n$ constant for $n >> 0$ call it ξ_+
$\quad q_n$ ———— ξ_-
$\quad u^{-n} p_n$ ———— $n << 0$ $\quad \xi'_-$
$\quad q_n$ ———— ξ'_+

Then a solution of $\overset{\text{disc}}{\wedge}$ Dirac eqn. with pot (h_n) eigenvalue $z \in \mathbb{C}^\times$ is a $\psi : \mathcal{M} \longrightarrow \mathbb{C}$ linear functional $\ni$ $\psi(\cancel{z} - u)\cancel{\mathcal{M}} = 0$

By construction $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \underbrace{\begin{pmatrix} a_u & b_u \\ c_u & d_u \end{pmatrix}}\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix}$ 792.43 323

$$= \cdots \frac{1}{k_n}\begin{pmatrix} & \end{pmatrix}\frac{1}{k_{n-1}}\begin{pmatrix} & \end{pmatrix} \cdots$$

To understand earlier mistake. What did you do?
You take a $\psi : M/(z-u)M \longrightarrow \mathbb{C}$

$$\psi(n,z) = \begin{pmatrix} \psi(u^{-n}p_n) \\ \psi(q_n) \end{pmatrix}$$

$$\psi(\infty,z) = \begin{pmatrix} \psi(\xi_+) \\ \psi(\xi_-) \end{pmatrix} \qquad \psi(-\infty,z) = \begin{pmatrix} \psi(\xi_-') \\ \psi(\xi_+') \end{pmatrix}$$

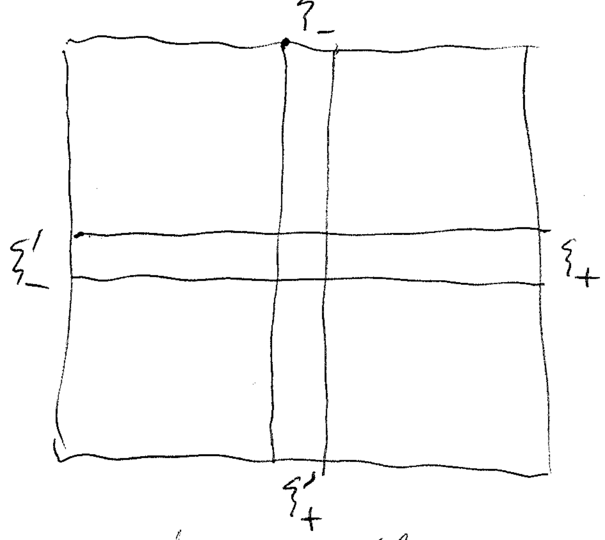
$$\begin{pmatrix} a_z & b_z \\ c_z & d_z \end{pmatrix}\psi(-\infty,z) = \begin{pmatrix} a_z & b_z \\ c_z & d_z \end{pmatrix}\begin{pmatrix} \psi(\xi_-') \\ \psi(\xi_+') \end{pmatrix}$$

$$= \begin{pmatrix} \psi(a_u\xi_-' + b_u\xi_+') \\ \psi(c_u\xi_-' + d_u\xi_+') \end{pmatrix} = \begin{pmatrix} \psi(\xi_+) \\ \psi(\xi_-) \end{pmatrix} = \psi(\infty,z)$$

Think harder.

You have a rank 2 v.b. over $S^2 - \{0,\infty\}$
M over $A = \mathbb{C}[z,z^{-1}]$. Suggested is ~~to try~~ to extend it to a vector bundle over S^2. There's a lot of choice but perhaps things simplify. If you are taking the

Picture of $\mathcal{M}$. $\nwarrow u$



$$M = \left(H_+\xi'_- + H_+\xi_-\right) \overset{?}{\oplus} \left(H_-\xi_+ + H_-\xi'_+\right)$$

$$\begin{pmatrix} \mathcal{A}_+ & \mathcal{A}_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \mathcal{A}_- & \mathcal{A}_- \end{pmatrix}\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{A} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} a_- \\ a_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} a_+ \\ a_+ \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}$$

certainly there's a problem with Laurent polys...

Try to finish today. Life goes on. Yes. Consider zilch. Recall given (h_n) get $\mathcal{M}$ a rank 2 .. $\mathcal{A}$-module generated by p_n, q_n, so that solutions of the DE with eigenvalue z are linear functionals on $\mathcal{M}/(z-u)\mathcal{M}$. $\mathcal{M}$ is a pre Hilbert space and u is a unitary op. So what?

~~Let me try again to go on~~

Point discovered yesterday. Recall that

$b \in \mathcal{A} \implies d \in \mathcal{A}_+$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} z\mathcal{A}_- & \mathcal{A} \\ \mathcal{A} & \mathcal{A}_+ \end{pmatrix}$

but $S = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$ will not be in $\mathcal{A}$ necessarily. $\frac{b}{d}$ is what is ~~So $\mathcal{M}$ is clear.~~ used in constructing the splitting $E = \left(H_+\xi_+ + H_-\xi_-\right) \oplus \left(H_-\xi'_- + H_-\xi'_+\right)$

Recall: $\quad \begin{pmatrix} H_+ & H_- \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- & H_+ \end{pmatrix} = \begin{pmatrix} L^2 & L^2 \end{pmatrix}$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

apply $\begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$ to get $\begin{pmatrix} 1 & \frac{c}{a} \\ \frac{b}{d} & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$

apply $\begin{pmatrix} \pi_+ & \cancel{\pi_-} \\ 0 & \pi_- \end{pmatrix}$ $\quad \begin{pmatrix} Id_{H_+} & \pi_+ \frac{c}{a} \\ \pi_- \frac{b}{d} & Id_{H_-} \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \xrightarrow{\overset{?}{\sim}} \begin{pmatrix} H_+ \\ H_- \end{pmatrix}$ Yes.

Wait: You construct $\mathcal{M}$ over $\mathcal{A} = \mathbb{C}[u,u^{-1}]$

~~get~~ Assume (h_n) fin. supp $\Rightarrow$ ~~(h_n) clear!~~

get $\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \overset{n \ll 0}{=} \begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} \overset{n \gg 0}{=} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$

So that $\mathcal{M} = \mathcal{A}\xi_+ + \mathcal{A}\xi_- = \mathcal{A}\xi'_- + \mathcal{A}\xi'_+ = \mathcal{A}p_n + \mathcal{A}q_n \quad \forall n$

But it's usually not true that $\mathcal{M} = \mathcal{A}\xi_+ \oplus \mathcal{A}\xi'_+$

Start with $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$ $\quad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$

$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$

Consider $E = (H_+\xi'_- + H_+\xi_-) \overset{?}{\oplus} (H_-\xi_+ + H_-\xi'_+)$

$$(H_+ \quad H_+) \overset{?}{\oplus} (H_- \quad H_-)\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = (L^2 \quad L^2)$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

apply d. $\quad \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} = $ "

$$\begin{pmatrix} Id_{H_-} & -\pi_- c \\ \pi_- b & Id_{H_-} \end{pmatrix}$$

Yesterday I understood how splitting leads to factorization How splitting yields factorization. ~~Come on now~~ The idea was ~~~~ a vector bundle of pure slope over $\mathbb{P}_1$. So if you have ~~an S such that $SH_+ \oplus H_- = H$, then~~

$$H_+ \oplus H_- = H \qquad uH_+ \subset H_+ , \; u^{-1}H_- \subset H_-$$

~~~~ and $V = H_+ \cap zH_-$, ~~~~ this isn't quite correct, but OK with finiteness conditions. One thing to do before ~~~~ crap takes over.

Go back to $\mathcal{a} = \mathbb{C}[z,z^{-1}]$ to

$$M = \mathcal{a}\xi_+ \oplus \mathcal{a}\xi_- = \mathcal{a}\xi_-' \oplus \mathcal{a}\xi_+'$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} \qquad \begin{matrix} a = \bar{d} & ad-bc=1 \\ b = \bar{c} & a,b,c,d \in \mathcal{a} \\ d \in \mathcal{a}_+ & \end{matrix}$$

~~so d is a polynomial in z~~ You know that d is a poly in z with roots outside S^1. b can be arbitrary L. poly $|d|^2 = 1 + |b|^2$. So now what happens is you ~~~~ shift to

$$\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

means you ~~to add~~ invert d.

$$M[d^{-1}] = \mathcal{a}[\tfrac{1}{d}]\xi_+ \oplus \mathcal{a}[\tfrac{1}{d}]\xi_+'$$

Splitting $M = (H_-\xi_+ + H_-\xi_+') \overset{?}{\oplus} (H_+\xi_-' \oplus H_+\xi_-)$

$$(H_- \;\; H_-)\begin{pmatrix} \cancel{\frac{1}{d}} & \cancel{\frac{b}{d}} \\ -\cancel{\frac{c}{d}} & \cancel{\frac{1}{d}} \end{pmatrix} \oplus (H_+ \;\; H_+) = (L^2 \;\; L^2)$$

$$\begin{pmatrix} Id & \pi_- c \\ \pi_- b & Id \end{pmatrix} : \begin{pmatrix} H_- \\ H_- \end{pmatrix} \longrightarrow \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$
~~~~

To accomplish the splitting you have to invert the operator $\begin{pmatrix} Id & \\ & Id \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -\pi_- c \\ \pi_- b & 0 \end{pmatrix}}_{X}$ on $\begin{pmatrix} H_- \\ H_- \end{pmatrix}$

$$\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & T^* \\ -T & 1 \end{pmatrix} \begin{pmatrix} (1+T^*T)^{-1} & 0 \\ 0 & (1+TT^*)^{-1} \end{pmatrix}$$

$$(1+X)^{-1} = \frac{1}{1+X}\frac{1-X}{1-X} = \frac{1-X}{1-X^2}$$

$$g = \frac{1+X}{1-X} \qquad \overset{F}{\overbrace{g\varepsilon}}(1+X) = g(1-X)\varepsilon = (1+X)\varepsilon$$

$F = +1$ on $\operatorname{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}$

-1 on $\left(\operatorname{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}\right)^{\perp} = \operatorname{Im}\begin{pmatrix} -T^* \\ 1 \end{pmatrix}$

what's happening here is a mixing of the construction of the scattering matrix $\frac{1}{d}\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix}$ with π_-. ~~They are told~~ NO ~~not yet~~. you haven't ~~then used~~ something like d.

$$\begin{pmatrix} 1 & +b \\ -\bar{b} & 1 \end{pmatrix}^{-1} = \frac{1}{1+X} = \frac{1-X}{1-X^2} = \begin{pmatrix} 1 & +b \\ -\bar{b} & 1 \end{pmatrix} \frac{1}{1+|b|^2}$$

$$\frac{1+X}{1-X} = \frac{(1+X)^2}{1-X^2} = \left(\frac{1+X}{\sqrt{1-X^2}}\right)^2$$

can take suitable sqrt. d of $1+|b|^2$

$$\frac{1}{d}\begin{pmatrix} d & b \\ -\bar{b} & d \end{pmatrix} = \frac{1+X}{\sqrt{1-X^2}} \qquad X = \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \qquad \sqrt{1-X^2} = |d|^2$$

gets nowhere.

So what else is left. $su(1,1)$ side

$$E = (H_+ \xi_+ + H_- \xi_-) \overset{?}{\oplus} (H_- \xi'_- + H_+ \xi'_+)$$

$$(H_+ \;\; H_-)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{?}{\oplus} (H_- \;\; H_+) = (L^2 \;\; L^2)$$

$$\begin{pmatrix} 1 & c/a \\ b/d & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} Id & \pi_+ \frac{c}{a} \\ \pi_- \frac{b}{d} & Id \end{pmatrix} : \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \longrightarrow \begin{pmatrix} H_+ \\ H_- \end{pmatrix}$$

What you might do now is to prove the opposite splitting

$$E = (H_+ \xi_+ + H_+ \xi'_+) \overset{?}{\oplus} (H_- \xi'_- + H_- \xi_-)$$

$$(H_+ \;\; H_+)\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \overset{?}{\oplus} (H_- \;\; H_-) = (L^2 \;\; L^2)?$$

$$\begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\pi_+ c \\ \pi_+ b & 1 \end{pmatrix} : \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\xrightarrow{\sim}} \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{c}{a} \\ +\frac{b}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

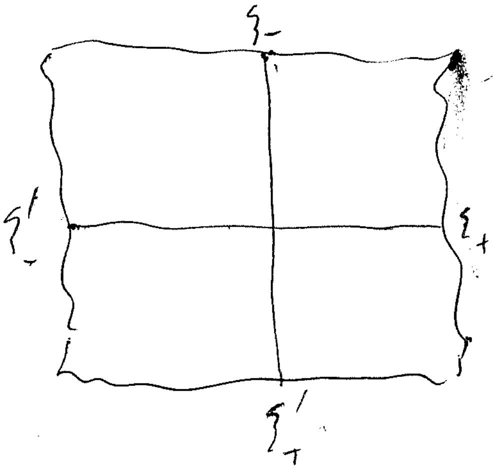
$$E = (H_+ \xi_+ + H_+ \xi'_+) \overset{?}{\oplus} (H_- \xi'_- + H_- \xi_-)$$

$$(H_+ \;\; H_+) \overset{?}{\oplus} (H_- \;\; H_-) \begin{pmatrix} \frac{1}{a} & -\frac{c}{a} \\ +\frac{b}{a} & \frac{1}{a} \end{pmatrix} = (L^2 \;\; L^2)$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \frac{1}{\cancel{a}} & +\frac{b}{\cancel{a}} \\ -\frac{c}{\cancel{a}} & \frac{1}{\cancel{a}} \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} Id & +\pi_- b \\ -\pi_- c & Id \end{pmatrix} : \begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

What are the possible ~~[illegible]~~ splittings



$$E = (H_+ \xi_+ + H_- \xi_-) \oplus (H_- \xi'_- + H_+ \xi'_+)$$

note this is the same as

$$E = (H_+ \xi_+ + H_+ \xi'_+) \oplus (H_- \xi'_- + H_- \xi_-)$$

I think there are 4 reductions.

$$(H_+ \;\; H_-) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{?}{\oplus} (H_- \;\; H_+) \qquad (H_+ \;\; H_+) \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \oplus (H_- \;\; H_-)$$

$$(H_+ \;\; H_-) \oplus (H_- \;\; H_+) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad (H_+ \;\; H_+) \oplus (H_- \;\; H_-) \begin{pmatrix} \frac{1}{a} & -\frac{c}{a} \\ \frac{b}{a} & \frac{1}{a} \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- \\ H_+ \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} H_- \\ H_+ \end{pmatrix} \qquad \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \frac{1}{a} & \frac{b}{a} \\ -\frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

The vertical pairs are obviously equivalent; the horiz pairs should be equivalent by Legendre transform.

Next stage

$$\begin{pmatrix} 1 & \frac{c}{a} \\ \frac{b}{d} & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} + \begin{pmatrix} H_- \\ H_+ \end{pmatrix} \qquad \begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} H_+ \\ H_- \end{pmatrix} + \begin{pmatrix} 1 & -\frac{c}{d} \\ -\frac{b}{a} & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_+ \end{pmatrix} \qquad \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

Finally

$$\begin{pmatrix} Id_+ & \pi_+ \frac{c}{a} \\ \pi_- \frac{b}{d} & Id_- \end{pmatrix} \text{ inv. on } \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \qquad \begin{pmatrix} Id_+ & -\pi_+ c \\ \pi_+ b & Id_- \end{pmatrix} \text{ inv. on } \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

$$\begin{pmatrix} Id_- & -\pi_+ \frac{c}{d} \\ -\pi_- \frac{b}{a} & Id_- \end{pmatrix} \text{ inv on } \begin{pmatrix} H_- \\ H_+ \end{pmatrix} \qquad \begin{pmatrix} Id_- & \pi_- b \\ -\pi_- c & Id_- \end{pmatrix} \text{ inv. on } \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

Next splitting (~~[illegible]~~ forward + backward light cones)

$$E = (H_-\xi_+ + H_-\xi_+') + (H_+\xi_-' + H_+\xi_-)$$

$$(H_- \quad H_-)\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} + (H_+ \quad H_+) \qquad (H_- \; H_-) + (H_+ \; H_+)\begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{d} & -\frac{c}{d} \\ \frac{b}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} + \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \qquad \begin{pmatrix} H_- \\ H_- \end{pmatrix} + \begin{pmatrix} \frac{1}{a} & \frac{c}{a} \\ -\frac{b}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

$$\begin{pmatrix} 1 & -c \\ b & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} + \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \qquad \begin{pmatrix} H_- \\ H_- \end{pmatrix} + \begin{pmatrix} 1 & c \\ -b & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

$$\begin{pmatrix} Id_- & -\pi_- c \\ \pi_- b & Id_- \end{pmatrix} \text{ inv. on } \begin{pmatrix} H_- \\ H_- \end{pmatrix} \qquad \begin{pmatrix} Id_+ & \pi_+ c \\ -\pi_+ b & Id_+ \end{pmatrix} \text{ inv. on } \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

---

u unitary on E $\quad \forall z \notin S^1 \quad (z-u)^{-1} \;\exists$ on E

So review the Green's function, what did you ~~Return to [illegible] Green~~ do? You worked in the continuous case I recall. The Green's function depends on z, which means? You can fix $p_0 \in E$ and ask for $\frac{1}{z-u} p_0$ is defined by geometric series for $|z| \neq 1$. How can we understand this Linear functional vanishing on ~~[illegible]~~

Take new viewpoint. How to proceed?

Go back. Inside of E you have $\xi_\pm \quad \xi'_\mp$

Point. A solution of DE is a linear fnl $E \xrightarrow{\psi} \mathbb{C}$ killing $(z-u)E$ e.g.

Recall earlier analysis. ~~Look at~~ Fix a z and look at solutions ψ of

$$\begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} & \end{pmatrix}\begin{pmatrix} z^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix} \quad \text{with } p_n, q_n \in \mathbb{C}$$

2 dim'l space $\cancel{\psi(n,z) = \begin{pmatrix} z^{-n}p_n(z) \\ q_n(z) \end{pmatrix}}$. Assume fin. supp.

$$\psi(n,z) = \begin{pmatrix} \psi(z^{-n}p_n) \\ \psi(q_n) \end{pmatrix}$$

Then $\psi(\infty) = \begin{pmatrix} \psi(\xi_+) \\ \psi(\xi_-) \end{pmatrix} \qquad \psi(-\infty) = \begin{pmatrix} \psi(\xi'_-) \\ \psi(\xi'_+) \end{pmatrix}$

so $\quad \psi(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)\,\psi(-\infty) \quad$ because

$$\psi(\infty) = \psi\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \psi\begin{pmatrix} a_u & b_u \\ c_u & d_u \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$= \begin{pmatrix} a_z & b_z \\ c_z & d_z \end{pmatrix}\psi\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}.$$

Fix $|z|<1$. Ask for decaying solution on left / on rt.

for $n \ll 0$

$$\psi(\overbrace{u^{-n}p_n}^{\xi'_-}) = z^{-n}\psi(p_n)$$

$$\psi(q_n) = \psi(\xi'_+) \text{ constant} \qquad n \ll 0$$

$$\therefore \begin{pmatrix} \psi(p_n) = z^n\psi(\xi'_-) \\ \psi(q_n) = \psi(\xi'_+) \end{pmatrix} \qquad \begin{matrix} n \ll 0 \\ n \ll 0 \end{matrix}$$

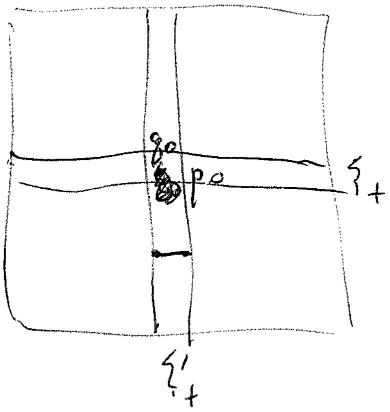
You know that p_n is an orth sequence so ~~ξ~~ if ψ is represented by a vector in E, so

$\psi = (\Phi|$ then $\psi(p_n) \to 0$ as $n \to -\infty$.

Similarly $\psi(\xi'_+) = \psi(q_n)$ $n \ll 0$

$$\psi(u^n \xi'_+) = z^n \psi(\xi'_+) = z^n \psi(q_n) \qquad n < 0$$

$$\therefore \psi(q_n) = z^{-n} \underbrace{\psi(u^n \xi'_+)}_{\downarrow\, 0} \qquad ?$$



$$u^n \xi_+ = p_n \qquad n \gg 0$$

$$z^n \psi(\xi_+) = \psi(p_n) \to 0 \qquad \text{as } n \to \infty$$

$$u^n q_{-n} = u^n \xi'_+ \qquad n \gg 0$$

$$z^n \psi(\xi'_+) = \psi(u^n q_{-n}) \to 0 \qquad \text{as } n \to +\infty$$

looking left.

$$\boxed{\begin{matrix} u^{-n} p_n \Longrightarrow \xi'_- \\ q_n \Longrightarrow \xi'_+ \end{matrix} \quad \text{as } n \to -\infty}$$

$$\psi(\xi'_-) = z^{-n} \underbrace{\psi(p_n)}_{\downarrow\, 0} \qquad n \to -\infty$$

$$\psi(\xi'_+) = \psi(q_n) = z^n \underbrace{\psi(u^{-n} q_n)}_{\downarrow\, 0} \qquad n \to \infty$$

$\psi : E \to \mathbb{C}$ $\qquad \psi(z-u) = 0$.

$\psi(\xi'_-) = ?$

$\psi(\xi'_+) = ?$

$$\psi(\xi'_-) = \psi(u^{-n} p_n) \qquad n \ll 0$$
$$= z^{-n} \psi(p_n)$$
$$\psi(\xi'_+) = \psi(q_n) = z^n \psi(u^{-n} q_n) \qquad n \ll 0$$

$$|\psi(\xi'_-)| \leq |z|^{-n}\|\psi\| \quad n \ll 0$$
$$|\psi(\xi'_+)| \leq |z|^{n}\|\psi\|. \quad n \ll 0$$

If $|z| < 1$, then $\psi(\xi'_+) = 0$

Similarly $\psi(\xi_+) = \psi(u^{-n}p_n) = z^{-n}\psi(p_n) \quad n \gg 0$

$$\psi(\xi_-) = \psi(q_n) = z^n\psi(u^{-n}q_n) \quad n \gg 0$$

$$|\psi(\xi_+)| \leq |z|^{-n}\|\psi\|$$
$$|\psi(\xi_-)| \leq |z|^{n}\|\psi\|$$

If $|z| < 1$, then $\psi(\xi_-) = 0$

Try this again. So it seems that for

$|z| < 1$. $\quad \psi(\xi'_-) = \psi(\xi_-) = 0$

$|z| > 1$. $\quad \psi(\xi_+) = \psi(\xi'_+) = 0$.

This seems clear. Check it again

you have $\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix} \leftarrow \begin{pmatrix}u^{-n}p_n \\ q_n\end{pmatrix} \longrightarrow \begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix}$

So

So given ψ representable

$$\psi(\xi_+) = \lim_{n\to\infty} \underbrace{\psi(u^{-n}p_n)}_{\substack{\| \\ z^{-n}\psi(p_n)}} = 0 \quad \text{if } |z| > 1.$$

$$|z^{-n}\psi(p_n)| \leq |z|^{-n}\|\psi\| \to 0$$

$$\psi(\xi'_+) = \lim_{n\to-\infty}\psi(q_{-n}) = \lim_{n\to-\infty} z^{-n}\psi(u^n q_{-n})$$

$$|z^{-n}\psi(u^n q_{-n})| \leq |z|^{-n}\|\psi\|$$

~~Now let's take care~~

The preceding is probably not rigorous, since you use ψ bounded on E and $\psi(z-u)E = 0$. ??

Idea from yesterday - use the Hilbert space E ~~to understand~~ and unitary operator u to define $(z-u)^{-1}$ for $|z| \neq 1$ then study $(z-u)^{-1}p_0$ and $(z-u)^{-1}q_0$ ~~Did you sort~~ You looked at the Green's fn. before. ~~seems that the~~

Yesterday you looked as a linear functional on E (bdd) ψ. Wait.

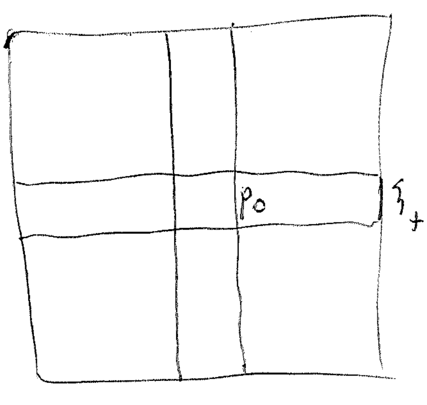
Consider $(z-u)^{-1}p_0$ this is a well defined element of E depending on z. Let $\psi : E/(z-u)E \longrightarrow \mathbb{C}$

You have to distinguish between M the $\mathbb{C}[z,z^{-1}]$-module of "finite" vectors and the Hilbert space completion E of M. If $z \notin S^1$, then $(z-u)$ is invertible on E, but on M, it's injective with image of codim 2.

$$\psi = \left(\frac{1}{\bar{z} - u^*} p_0 \right| \qquad \text{Then}$$

$$\psi\,(z-u)\eta = \left(\frac{1}{\bar{z}-u^*}p_0 \middle| (z-u)\eta\right)$$

$$= \left(p_0 \middle| \cancel{\frac{1}{z-u}(z-u)}\,\eta\right) \qquad \begin{array}{l}\cancel{0 \text{ on space}}\\ = 0 \text{ on } p_0^{\perp}\end{array}$$



$$\psi(\xi_+) = \lim_{n\to\infty} \underbrace{\psi(u^{-n}p_n)}_{\substack{\parallel \\ \bar{z}^{-1}\psi(u^{-n+1}p_n)}} \;?$$

because $\psi(u\eta) = z\psi(\eta)$ for $\eta = u$?

Missing something!

Go back to M in finite supp case

Let ψ be a soln of DE with eigenv. z

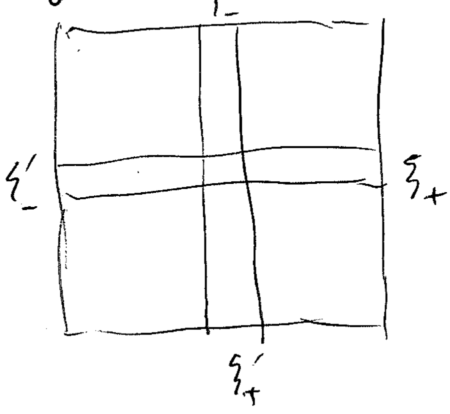
$\psi(n) \in \mathbb{C}^2$ $\qquad \psi(n) = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \psi(n-1)$

Then ψ same as $\phi : M/(z-u)M \longrightarrow \mathbb{C}$

$$\psi(n) = \begin{pmatrix} \phi(u^{-n} p_n) \\ \phi(q_n) \end{pmatrix}. \qquad \psi(\infty) = \begin{pmatrix} \phi(\xi_+) \\ \phi(\xi_-) \end{pmatrix}$$

$$= \phi\left(\begin{pmatrix} a_u & b_u \\ c_u & d_u \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}\right) = \begin{pmatrix} a_z & b_z \\ c_z & d_z \end{pmatrix}\underbrace{\begin{pmatrix} \phi(\xi'_-) \\ \phi(\xi'_+) \end{pmatrix}}_{\psi(-\infty)}.$$

Green's function again. Let's go over the details of the $\mathbb{P}^1$ bundle.



Form $E = L^2\xi'_- \oplus L^2\xi_- = L^2\xi_+ \oplus L^2\xi'_+$

$$E = (H_+\xi'_- + H_+\xi_-) \oplus (H_-\xi_+ + H_-\xi'_+) \qquad \text{Why?}$$

$$(H_+ \;\; H_+) \overset{?}{\oplus} (H_- \;\; H_-)\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = (L^2 \;\; L^2)$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \frac{1}{\cancel{d}} & -\frac{c}{\cancel{d}} \\ \frac{b}{\cancel{d}} & \frac{1}{\cancel{d}} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

need $\begin{pmatrix} Id & -\pi_- c \\ \pi_- b & Id \end{pmatrix}$ invertible on $\begin{pmatrix} H_- \\ H_- \end{pmatrix}$, which follows from Hilbert space theory.

$$\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}^{-1} = \frac{1}{1+X} = (1-X)\frac{1}{1-X^2}$$

What do you want here? I think you want a subring of $L^\infty(S^1)$.

Note that $\frac{1}{1+X} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}^{-1}$ is closely connected to the orthogonal projection onto $\Gamma_T = \binom{1}{T}H$. In fact

$F = g\varepsilon = \frac{1+X}{1-X}\varepsilon$ $\qquad F(1+X) = \frac{1+X}{1-X}\varepsilon(1+X) = (1+X)\varepsilon$

so $F = +1$ on $\binom{1}{T}H$
$\qquad -1$ on $\binom{-T^*}{1}H$ $\qquad$ OKAY.

There is some geometry to understand here, probably a kind of Kasparov product. You are mixing 2 things. Two processes $T \longmapsto \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}^{-1}$, Hardy projection $\pi_- : H \to H_-$

What's essential? The splitting $H = H_+ \oplus H_-$ and the mult. operator b.

Program: Work out details of splitting, factorization, G fn.

A first idea: Start with a transfer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which is an analytic function on the unit circle, hence extends analytically to an annulus. This should happen if h_n decays exponentially. Let A be the ring of analytic functions on the circle, really on a nbd. of the unit circle.

~~You have the module M over $\mathbb{C}[u,u^{-1}]$ and~~

$\Rightarrow$ page 339

a Given b form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ etc.

splitting

$$E = \left(H_+ \xi'_- + H_+ \xi_-\right) \overset{?}{\oplus} \left(H_- \xi_+ + H_- \xi'_+\right)$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{\alpha} & -\frac{c}{\alpha} \\ \frac{b}{\alpha} & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

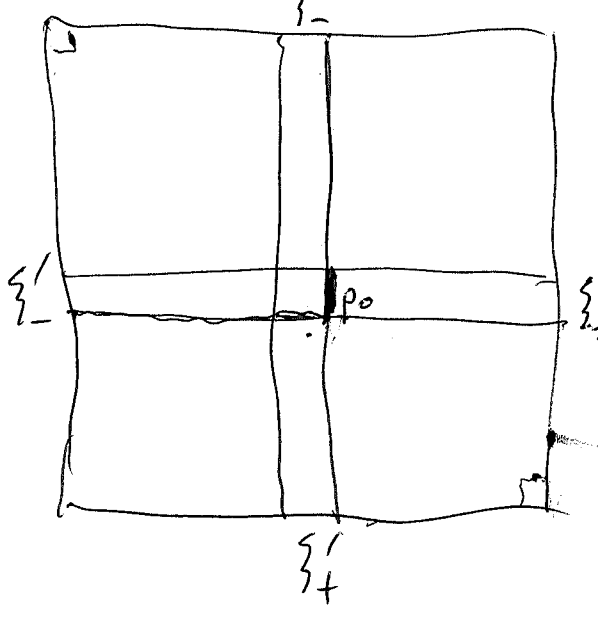
equiv. to $\begin{pmatrix} Id_- & -\pi_- c \\ \pi_- b & Id_- \end{pmatrix}$ invertible on $\begin{pmatrix} H_- \\ H_- \end{pmatrix}$?

true since $\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}$ invertible always

$$\begin{pmatrix} \frac{1}{\alpha} & \frac{c}{\alpha} \\ -\frac{b}{\alpha} & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

equiv to $\begin{pmatrix} Id_+ & \pi_+ c \\ -\pi_+ b & Id_+ \end{pmatrix}$ invertible on $\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$?

Aim now to deduce factorization.



$$V = \left(H_+ \xi'_- + H_+ \xi_-\right) \cap \left(z H_- \xi_+ + z H_- \xi'_+\right)$$

$$\xi_+ = \tilde{p}_0 + \eta$$

$$\tilde{p}_0 \in \left(\underbrace{H_+ \xi'_- + H_+ \xi_-}_{\eta \in}\right) \cap \left(z \underbrace{H_- \xi_+ + H_- \xi'_+}_{\xi_+ \notin}\right)$$

b $\quad p_0 \in H_+ \xi_-' + H_+ \xi_-$

$$p_0 \in zH_- \xi_+ + H_- \xi_+' = \mathbb{C}\xi_+ + H_-\xi_+ + H_-\xi_+'$$

$$\tilde{p}_0 = \xi_+ + f_-\xi_+ + g_-\xi_+'$$

$$\tilde{p}_0 = \qquad f_+\xi_-' + g_+\xi_-$$

$$\begin{pmatrix} a^\ell & +b^r \\ -c^\ell & a^\ell \end{pmatrix} \begin{array}{c} H_+ \quad H_+ \\ \begin{pmatrix} d^r & b^\ell \\ -c^\ell & d^\ell \end{pmatrix} \\ zH_+ \quad H_+ \end{array}$$

$$\cap$$

$$\begin{pmatrix} zH_- & H_- \\ zH_- & zH_- \end{pmatrix}$$

can understand ~~by~~ from

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d^r & b^\ell \\ -c^r & d^\ell \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^\ell & -b^r \\ c^\ell & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix}$$

so $\quad p_0 = \dfrac{d^r}{d}\xi_-' + \dfrac{b^\ell}{d}\xi_-$

$$p_0 = \underset{zH_-}{\frac{a^\ell}{a}}\xi_+ - \underset{H_-}{\frac{b^r}{a}}\xi_+'$$

what steps are needed?

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from 337. $\quad \mathcal{a}$ = ring of analytic fns. on unit circle:
$f(z) = \sum a_n z^n \qquad |a_n| \le C\varepsilon^{|n|}$ some $C > 0$, $0 < \varepsilon < 1$.
Given $b \in \mathcal{a}$ you form transfer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c = \bar{b}$, and $\bar{a} = d$ analytic on closed unit disk. Now form $\mathcal{M} = \mathcal{a}\xi_+ + \mathcal{a}\xi_+' = \mathcal{a}\xi_-' + \mathcal{a}\xi_-$.

No it's not quite right. You want $\mathcal{a}$ to contain functions ~~analytic~~ on S^1 which extend analytically to a fix annulus. Then $\mathcal{a}/(\lambda - u)\mathcal{a} \simeq \mathbb{C}$ for λ in the annulus.

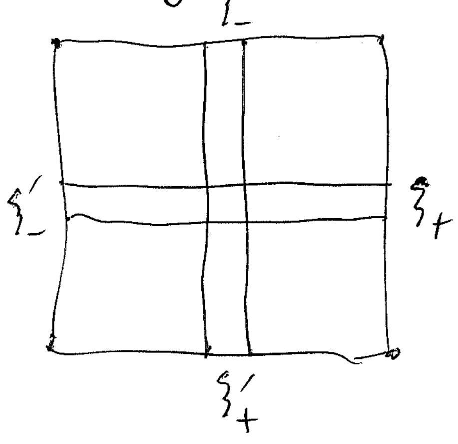
Start again with b holom. in an ~~[illegible]~~ annulus around S^1. Then get d ~~analytic~~ holom (non zero) in disk $> S^1$ sat $|d|^2 = 1 + |b|^2$ on S^1, $d(0) > 0$.

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a = \bar{d} \quad c = \bar{b}$$

$$S = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

unitary on S^1.
holom. in same annulus $> S^1$.

Now form ~~[illegible]~~ let $\mathcal{A}$ be ring of an. functions in ~~[illegible]~~ a small enough annulus about S^1 so that S, T defd over $\mathcal{A}$, form free module M over ~~[illegible]~~ $\mathcal{A}$ ~~[illegible]~~ of rank 2 ~~[illegible]~~ ~~[illegible]~~ generated by $\xi_\pm$ $\xi'_\mp$ related by



$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = T \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = S \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

Given $\lambda \in$ An then $M/(\lambda - u)M$ is 2 diml, and linear functionals on it ~~[illegible]~~ can ultimately be viewed as solution of the D.E. with eigenvalue λ.

$M/(\lambda - u)M$ ~~2~~ diml basis with various bases. What's the basic viewpoint here? By working over an annulus I ~~[illegible]~~ get eigenfunctions. In the Hilbert space situation $E/(\lambda - u)E$ is 0 for $|\lambda| \neq 1$, and for $|\lambda|=1$ $(\lambda-u)E$ is probably not closed. So you don't have eigenvectors.

~~So you have eigenfunctions in a abstract~~ ~~situat~~ You have eigenvectors in the algebraic case $M = \mathbb{C}[u,u^{-1}]$-module with basis p_0, q_0. for all λ. $M = \mathbb{C}[u,u^{-1}]$-module gen. by p_n, q_n satisfying ... $\left(M/(z-u)M\right)^* =$ solutions of D.E. with eigenvalue z.

But the ~~scattering~~ left right transfer requires a limit, and scattering requires inversion of d, ~~some sort of limit~~

$$\mathcal{M} = \mathcal{a}\xi'_- \oplus \mathcal{a}\xi_- = \mathcal{a}\xi_+ \oplus \mathcal{a}\xi'_+$$ ~~But the~~

~~To finish this | left reg First identify~~

By introducing $\mathcal{a}$ analytic fns on an annulus you get eigenfunctions $\psi \in \left(\mathcal{M}/(\lambda-u)\mathcal{M}\right)^*$ for λ in the annulus, 2 dim'l space of eigenfns. i.e. solution of the D.E. $\begin{pmatrix}\psi z^{-n}p_n \\ \psi q_n\end{pmatrix}$ You would like to say this has asymptotics $\begin{pmatrix}\psi\xi'_- \\ \psi\xi'_+\end{pmatrix}$ as $n \to -\infty$, etc. ~~What happens is that limits~~

b_n^l inv. $h_n z^{-n}$

$$\begin{pmatrix}z^{-n}p_n \\ q_n\end{pmatrix} = \begin{pmatrix}a_n^l & b_n^l \\ c_n^l & d_n^l\end{pmatrix}\begin{pmatrix}\xi'_- \\ \xi'_+\end{pmatrix}$$

$$\begin{pmatrix}\cancel{zH_-} & \cancel{z^{-n}H_+} \\ \cancel{z^{n+1}H_-} & \cancel{H_+}\end{pmatrix} \in \begin{pmatrix}zH_- & z^{-n}H_+ \\ z^{n+1}H_- & H_+\end{pmatrix}$$

So
$$\begin{pmatrix} \psi\, z^{-n} p_n \\ \psi\, q_n \end{pmatrix} = \begin{pmatrix} a_n^\ell(\lambda) & b_n^\ell(\lambda) \\ c_n^\ell(\lambda) & d_n^\ell(\lambda) \end{pmatrix} \begin{pmatrix} \psi\, \xi'_- \\ \psi\, \xi'_+ \end{pmatrix}$$

In the fin. supp case
$$\begin{pmatrix} \psi\, z^{-n} p_n \\ \psi\, q_n \end{pmatrix} = \begin{pmatrix} \psi\, \xi'_- \\ \psi\, \xi'_+ \end{pmatrix}. \qquad n \ll 0$$

Thus $\quad \psi(\xi'_-) = \psi(z^{-n} p_n) = \lambda^{-n}\psi(p_n)$

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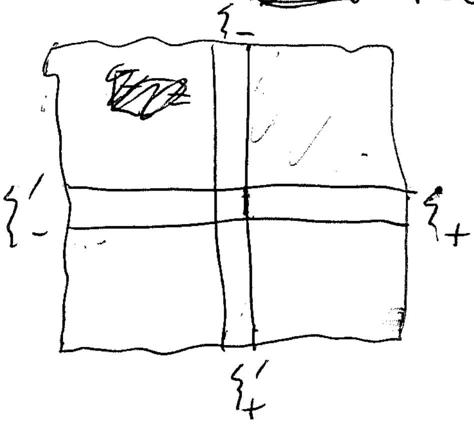
1 hour review fact.



$$E = (H_+\xi'_- + H_+\xi_-) \overset{?}{\oplus} (H_-\xi_+ + H_-\xi'_+)$$

$$(H_+ \ H_+) \overset{?}{\oplus} (H_- \ H_-) \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = (L^2 \ L^2)$$

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \frac{1}{\cancel{d}} & -\frac{b}{\cancel{d}} \\ +\frac{b}{\cancel{d}} & \frac{1}{\cancel{d}} \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

Yes because $\left(\begin{array}{c|c} Id_- & -\pi_- b \\ \hline \pi_- b & Id_- \end{array}\right)$ invertible on $\begin{pmatrix} H_- \\ H_- \end{pmatrix}$

$$p_0^\perp = (H_-\xi'_- + H_+\xi'_+) + (zH_+\xi_+ + H_-\xi_-)$$

~~Take a dive~~ Let's try to understand the construction of V. ~~Everything has been handled in the L^2 theory~~

$$p_0 \in (H_+\xi_+ + H_-\xi_-) \cap (H_+\xi'_+ + zH_-\xi'_-)$$

$$p_0 \in (H_+\xi'_- + H_+\xi_-) \cap (zH_-\xi_+ + H_-\xi'_+)$$

splitting $E = \left(H_+\xi'_- + H_+\xi_-\right) \oplus \left(H_-\xi_+ + H_-\xi'_+\right)$

$p_0 \in \left(H_+\xi'_- + H_+\xi_-\right) \oplus \left(zH_-\xi_+ + H_-\xi'_+\right)$

$$p_0 = f_+\xi'_- + g_+\xi_- = f_-\xi_+ + g_-\xi'_+$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d}\begin{array}{c} \begin{array}{cc} H_+ & H_+ \end{array} \\ \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \\ \begin{array}{cc} zH_+ & H_+ \end{array} \end{array} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{array}{c} \begin{array}{cc} zH_- & H_- \end{array} \\ \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \\ \begin{array}{cc} zH_- & zH_- \end{array} \end{array} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$p_0 = \underset{H_+}{\frac{d^r}{d}}\xi'_- + \underset{H_+}{\frac{b^l}{d}}\xi_- = \underset{zH_-}{\frac{a^l}{a}}\xi_+ - \underset{H_-}{\frac{b^r}{a}}\xi'_+$$

$$q_0 = \underset{zH_+}{\frac{-c^r}{d}}\xi'_- + \underset{H_+}{\frac{d^l}{d}}\xi_- = \underset{zH_-}{\frac{c^l}{a}}\xi_+ + \underset{zH_-}{\frac{a^r}{a}}\xi'_+$$

What do you learn?

Go back to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} d^l & -b^l \\ -c^l & a^l \end{pmatrix}$$

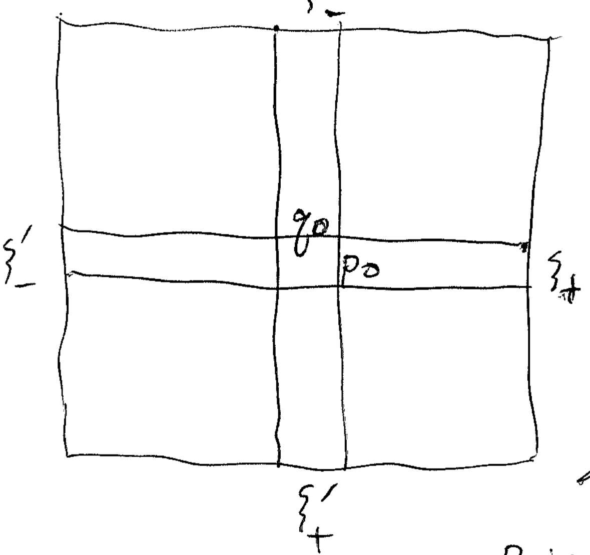
$$\begin{pmatrix} 1 & -\frac{b}{d} \\ -\frac{c}{a} & 1 \end{pmatrix} \begin{array}{c} \begin{array}{cc} zH_- & H_- \end{array} \\ \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \\ \begin{array}{cc} zH_+ & H_+ \end{array} \end{array} = \begin{pmatrix} \frac{d^l}{d} & -\frac{b^l}{d} \\ -\frac{c^l}{a} & \frac{a^l}{a} \end{pmatrix} \in \begin{pmatrix} H_+ & H_+ \\ zH_- & zH_- \end{pmatrix}$$

Apply $\begin{pmatrix} \pi_- & 0 \\ 0 & \pi_+ \end{pmatrix}$ to both sides

$\pi_- : L^2 \longrightarrow L^2/zH_+ = zH_-$ $\qquad$ $\pi_+ : L^2 \longrightarrow L^2/H_- = H_+$

$$\begin{pmatrix} Id_- & -\pi_-\frac{b}{d} \\ -\pi_+\frac{c}{a} & Id_+ \end{pmatrix}\begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} = \begin{pmatrix} \frac{d^l}{d}(0) & -\frac{b^l}{d}(0) \\ -\frac{c^l}{a}(\infty) & \frac{a^l}{a}(\infty) \end{pmatrix}$$

Confusion reigns.



Starting with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ you construct E. (This depends only on $\beta = \frac{b}{d}$ by rule

$$(f\xi_- | g\xi_+) = \int \bar{f} g \beta \quad \text{whence}$$

$$(z^k \xi_- | z^j \xi_+) = \int z^{\overline{k-j}} \beta = \beta_{k-j}$$

~~Recall~~ You have bifiltration

$z^p H_+ \xi_+ + \bar{z}^q H_- \xi_-$ increases with p, q.

Define p_0 by the properties $\|p_0\| = 1$.

$$p_0 \in (H_+\xi_+ + H_-\xi_-) \ominus (zH_+\xi_+ + H_-\xi_-)$$

$$p_0 \in s\xi_+ + zH_+\xi_+ + H_-\xi_- \qquad \text{with } s > 0.$$

Say $p_0 = \sum_{j(\geq 0)} s_j u^j \xi_+ - \sum_{k(<0)} t_k u^k \xi_-$

$$\underset{k<0}{0 =} (u^k\xi_- | p_0) = \sum_j s_j \beta_{k-j} - t_k$$

$$\underset{j\geq 0}{0 \overset{(u^j\xi_+|p_0)}{=}} s_j - \sum_k t_k \underbrace{(u^j\xi_+ | u^k\xi_-)}_{\overline{\beta_{k-j}}}$$

$$s(z) = \sum_{j\geq 0} s_j z^j \in H_+$$
$$t(z) = \sum_{k<0} t_k z^k \in H_-$$

$$\boxed{\begin{matrix} s\beta - t \in H_+ \\ s - t\bar{\beta} \in zH_- \end{matrix}}$$

$$p_0 = \underbrace{d^r}_{s}\xi_+ - \underbrace{b^r}_{t}\xi_-$$

$$\boxed{\begin{array}{ll} d^r\beta - b^r \in H_+ & d^r \in H_+ \\ d^r - b^r\bar\beta \in zH_- & b^r \in H_- \end{array}}$$

$$q_0 = -c^r\xi_+ + a^r\xi_-$$

$$q_0 \in zH_+\xi_+ + zH_-\xi_- \ominus zH_+\xi_+ + H_-\xi_-$$

$$q_0 = -\sum_{j>0} c_j u^j\xi_+ + \sum_{k\le 0} a_k u^k\xi_-$$

$$\underset{k<0}{0=}\left(u^k\xi_- \middle| q_0\right) = -\sum_j c_j \underbrace{\left(u^k\xi_- \middle| u^j\xi_+\right)}_{\beta_{k-j}} + a_k$$

$$\underset{j>0}{0=}\left(u^j\xi_+ \middle| q_0\right) = -c_j + \sum a_k \underbrace{\left(u^j\xi_+ \middle| u^k\xi_-\right)}_{\bar\beta_{k-j}}$$

Theis want
$$\boxed{\begin{array}{ll} c^r \in zH_+ & a^r - c^r\beta \in H_+ \\ a^r \in zH_- & -c^r + a^r\bar\beta \in zH_- \end{array}}$$

$$\begin{pmatrix} 1 & -\beta \\ -\bar\beta & 1 \end{pmatrix}\begin{array}{cc} zH_- & H_- \\ \left(\begin{matrix} a^r & b^r \\ c^r & d^r \end{matrix}\right. & \left.\begin{matrix} \\ \\ \end{matrix}\right) \\ zH_+ & H_+ \end{array} = \begin{array}{cc} H_+ & H_+ \\ \left(\begin{matrix} \frac{d^l}{d} & -\frac{b^l}{d} \\ -\frac{c^l}{a} & \frac{a^l}{a} \end{matrix}\right) & \\ zH_- & zH_- \end{array}$$

~~One reason you are having trouble is that~~

Look at the equs. for $\begin{pmatrix} b^r \\ d^r \end{pmatrix}$

$$b^r - \beta d^r \in H_+ \qquad\qquad b^r = \Pi_-(\beta d^r)$$

$$-\bar\beta b^r + d^r \in zH_-$$

$$-\Pi_+(\bar\beta b^r) + d^r \in \mathbb{C}$$

$$d^r - \Pi_+\bar\beta\,\Pi_-\beta d^r \in \mathbb{C}$$

What I missed

$$\begin{pmatrix}1 & -\beta\\ -\bar\beta & 1\end{pmatrix}\begin{array}{c} \begin{array}{cc} zH_- & H_- \end{array}\\ \begin{pmatrix} a^r & b^r\\ c^r & d^r\end{pmatrix}\\ \begin{array}{cc} zH_+ & H_+ \end{array}\end{array} = \begin{array}{c} \begin{array}{cc} H_+ & H_+ \end{array}\\ \begin{pmatrix} \frac{d^l}{d} & -\frac{b^l}{d}\\ -\frac{c^l}{a} & \frac{a^l}{a}\end{pmatrix}\\ \begin{array}{cc} zH_- & zH_- \end{array}\end{array}$$

$$a^r - \beta c^r \in H_+ \implies \boxed{\pi_- a^r = \pi_-(\beta c^r)}$$

$$-\bar\beta a^r + c^r \in zH_- \qquad -\pi_+(\bar\beta a^r) + c^r \in \mathbb{C}$$

$$b^r - \beta d^r \in H_+ \implies \boxed{b^r = \pi_-(\beta d^r)}$$

$$-\bar\beta b^r + d^r \in zH_- \implies \boxed{-\pi_+(\bar\beta b^r) + d^r \in \mathbb{C}}$$

funny

$a^r \in zH_-$ $c^r \in zH_+$	$a^r - \beta c^r \in H_+$ $-\bar\beta a^r + c^r \in zH_-$

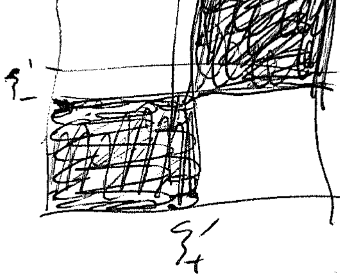
Instead of $\pi_+ = \pi_{H_+}$ $\quad \pi_- = \pi_{H_-}$ work with π_{zH_+}, π_{zH_-}. Then you get,

$$a^r - \pi_{zH_-}(\beta c^r) \in \pi_{zH_-}(H_+) = \mathbb{C}$$

$$c^r - \pi_{zH_+}(\bar\beta a^r) \in \pi_{zH_+}(zH_-) = 0.$$

$$\therefore \quad a^r - \pi_{zH_-}\beta\, \pi_{zH_+}\bar\beta\, a^r \in \mathbb{C}$$

Let's recall the splitting analysis, which is pretty clean



$$E = (H_+\xi_+ + H_-\xi_-) \overset{?}{\oplus} (H_-\xi'_- + H_+\xi'_+)$$

$$(H_+\ H_-)\begin{pmatrix} a & b\\ c & d\end{pmatrix} \overset{?}{\oplus} (H_-\ H_+) = (L^2\ L^2)$$

$$\begin{pmatrix} 1 & \frac{c}{a} \\ \frac{b}{d} & 1 \end{pmatrix}\begin{pmatrix} H_+ \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_- \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

Apply $\begin{pmatrix} \pi_+ & \\ & \pi_- \end{pmatrix}$ to get $\begin{pmatrix} Id_+ & \pi_+\left(\frac{c}{a}\right) \\ \pi_-\left(\frac{b}{d}\right) & Id_- \end{pmatrix}$ inv. on $\begin{pmatrix} H_+ \\ H_- \end{pmatrix}$

So for ~~splitting~~ you need exactly $\begin{pmatrix} 1 & \pi_+\bar{\beta} \\ \pi_-\beta & 1 \end{pmatrix}$ is invertible on $\begin{pmatrix} H_+ \\ H_- \end{pmatrix}$.

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} zH_- & H_+ \\ zH_- & H_+ \end{pmatrix} \qquad \begin{pmatrix} H_+ & H_- \\ zH_+ & zH_- \end{pmatrix}$$

Finally $\frac{1}{z-u}p_0 = \sum_{n\geq 0} z^{-n-1}u^n p_0 \qquad |z|>1$

$$= -\sum_{n<0} z^{-n-1}u^{+n} p_0$$

$$= -\sum_{n>0} z^{n-1}u^{-n} p_0 \qquad |z|<1.$$

$\frac{1}{z-u}p_0 =$ ~~$\frac{1}{z-u}$~~

$$= \frac{1}{zu^{-1}-1}u^{-1}p_0 = -\frac{1}{1-zu^{-1}}u^{-1}p_0 = -\sum_{n\geq 0} z^n u^{-n-1} p_0$$

Next can you calculate this resolvent

Let's start with a review

Given (h_n) decaying fast enough to form

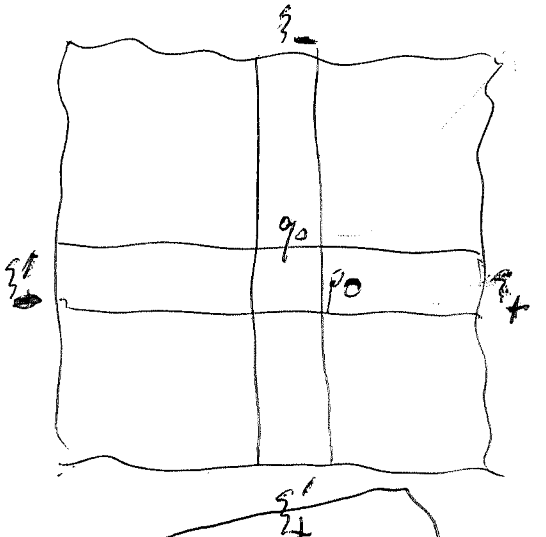
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \cdots \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \frac{1}{k_{n-1}}\begin{pmatrix} & \\ & \end{pmatrix} \cdots$$

you form the NO.

Given (h_n) you form the module M over $\mathbb{C}[z, z^{-1}]$ generated by p_n, q_n satisfying the relations given by the discrete Dirac eqn. M has the "grid" of unit vectors $z^k p_n, z^k q_n$. Scattering situation for h_n decaying sufficiently. Hilbert space completion

$$(\lambda - u)^{-1}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \sum_{n \geq 0} \lambda^{-n-1} u^n \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

can you calculate this in terms of $H_+ \xi_-' + H_+ \xi_-$



Somehow this should follow from

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

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So put $\frac{1}{\lambda - u} p_0 = f_+ \xi_-' + g_+ \xi_-$

$$p_0 = (\lambda - u) f_+ \xi_-' + (\lambda - u) g_+ \xi_-$$

$$p_0 = \frac{d^r}{d} \xi_-' + \frac{b^l}{d} \xi_-$$

So its trivial. If you use $\begin{matrix} \xi_-' \\ \xi_- \end{matrix}$ as bases for E, then

$$p_0 = \begin{pmatrix} \frac{d^r}{d} & \frac{b^l}{d} \end{pmatrix}$$

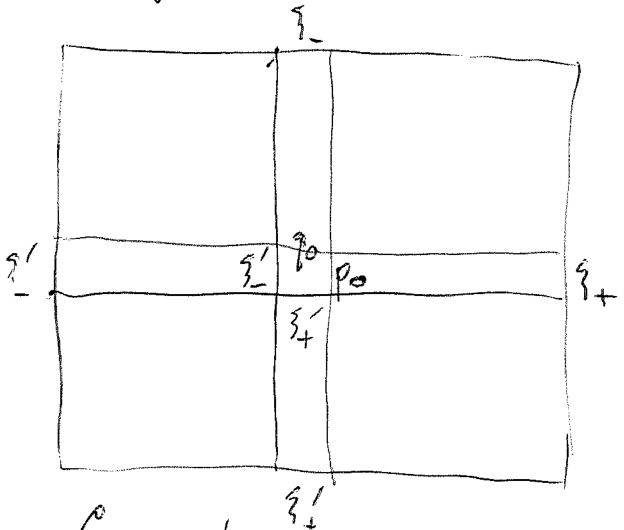
so

$$\frac{1}{\lambda - u} p_0 = \frac{1}{\lambda - z}\begin{pmatrix} \frac{d^r}{d} & \frac{b^l}{d} \end{pmatrix}$$

is analytic on D for $|\lambda| > 1$.

So what next? Go back over the whole inverse business. Start with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form

E) Let's proceed differently, try to connect up with partial unitaries.



Case $h_n = 0$ $\quad n < 0$.

$$Y = H_+\xi_+ + zH_-\xi_-$$
$$X = H_+\xi_+ + H_-\xi$$
$$Y = X \oplus \mathbb{C}\xi'_+ = uX \oplus \mathbb{C}\xi'_-$$

Other case $h_n = 0$ $\quad n > 0$

$p_0 = \xi_+$, $q_0 = \xi_-$.

$$Y = zH_-\xi'_- + H_+\xi'_+$$
$$X = H_-\xi'_- + H_+\xi'_+$$

$Y = X \oplus \mathbb{C}$ [redacted] $\xi_+ = uX + \mathbb{C}$ [redacted] ξ_-

---

Write lots of pages. You have to make progress. ~~[illegible]~~. I'm still trying to understand factorization, ~~of~~ Green's functions. need precise questions.

Recall idea of M = module over $\mathbb{C}[z, z^{-1}]$ gen. by p_n, q_n for $n \in \mathbb{Z}$ satisfying

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \text{[redacted]} \\ \bar{h}_n \text{[redacted]} & 1 \end{pmatrix}\begin{pmatrix} u p_{n-1} \\ q_{n-1} \end{pmatrix}$$

This can be defined in general for any (h_n) $\quad |h_n| < 1$. You know how to put ~~[illegible]~~ a positive def. herm. inner product such that u is unitary. Can complete. On the other hand for each n you get a basis $\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix}$ ~~over~~ for M over $\mathbb{C}[z, z^{-1}]$ such that the transitions are $\frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}$. This ~~[illegible]~~

~~[illegible]~~ should give a ~~[illegible]~~ kind of $SU(1,1)$ structure to M. Volume element preserved

Make this more precise. M is a module over $\mathbb{C}[z,z^{-1}]$ free of rank 2 with ~~various~~ ~~bases~~ distinguished bases ~~[illegible]~~ related to each other by ~~[illegible]~~ $SU(1,1)$ loops. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a,b,c,d \in \mathbb{C}[z,z^{-1}]$ $\quad a = d^*$ $\quad b = c^*$ $\quad ad - bc = 1.$ $\quad d^*d - b^*b = 1.$

$$f^*(z) = \overline{f(\bar{z}^{-1})}$$

If $|z| = 1$ i.e. $z = \bar{z}^{-1}$ then $f^*(z) = \overline{f(z)}$.

Let's try to get some structure here. First we can fix $z \in \mathbb{C}^1$ and look at $M/(z-u)M$. a 2 dim'l fibre. M is algebraically a vector bundle over the circle.

M module over $\mathbb{C}[z,z^{-1}]$, free of rank 2

so $M_\lambda = M/(\lambda - u)M$ 2 dim'l fibre - natural Wronskian(?) structure $SU(1,1)$. Have R.S.

What is the structure on a 2 dim'l space V arising from an $SU(1,1)$ equivalence class of bases. Should be herm. form of type 1,1. On $\mathbb{C}^2$ what can be done with $\xi^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\xi$ and a vol.

$g \in U(1,1)$ means $g^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

or that $g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then ~~$\frac{1}{ad-bc}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$~~ $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}$

So given H indef. herm. form on $V \simeq \mathbb{C}^2$

$H(\eta, \xi) = \eta^* A \xi$ with $A^* = A$.

Given also $\omega : \Lambda^2 V \xrightarrow{\sim} \mathbb{C}$. Pick ξ with $\xi^* A \xi = 1$

then an η with $\xi^* A \eta = 0$ $\eta^* A \eta = -1$

$H(v_1, v_2)$ nondeg. herm. on V 2 diml.

~~.....~~ Review factorization.

V 2diml over $\mathbb{C}$

~~hermitian~~ sesqui-lin form $T(v_1, v_2)$ $\qquad \bar{V} \otimes V \to \mathbb{C}$

herm. $T(v_2, v_1) = \overline{T(v_1, v_2)}$

det. by $T(v,v)$ real smooth, homog. deg 2 inv. under i

A hermitian form on V should induce one on $\Lambda^2 V$

$\sigma^* V \longrightarrow V^\vee$

$$\sigma^*(V \otimes V) = \sigma^* V \otimes \sigma^* V \longrightarrow V^\vee \otimes V^\vee = (V \otimes V)^\vee$$

So you get a hermitian form on $\Lambda^2 V$

+ volume elt. $\therefore$ a number Vol. elt., No

V diml with herm. form T indef. $T(v,v) \gtreqless 0$.

Pick ~~.....~~ e_1 with $T(e_1, e_1) = 1$ Pick e_2 in orth comp. $T(e_1, e_2) = 0$ $T(e_2, e_2) = -1$. Then what ~~..... e_1, e_2 .....~~ What you are doing is to pick $L_+, L_- \subset V$ with $L_+ \perp L_-$ and ~~...~~ herm. form $\pm$ on $L_\pm$.

$$g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \implies -|\det g|^2 = -1. \qquad |\det g| = 1.$$

~~The point~~ ~~An indefinite herm. form~~

Real str.?

Given V 2 diml over $\mathbb{C}$ with hermitian form of type $+,-$ look at $\mathbb{P}V$ pick a pos. line L_+ then $V = L_+ \oplus L_-$ where $L_- = (L_+)^\perp$. So V is 2 diml Krein space. ~~choose~~ Consider ~~the~~ setting up isom. $V \cong \mathbb{C}^2$ usual $\{{}^x\begin{pmatrix}1&0\\0&-1\end{pmatrix}\}$, i.e. picking a basis e_1, e_2 with $(e_1|e_1) = 1$, $(e_2|e_2) = -1$, $(e_1|e_2) = 0$.

So you choose the line L_+, then e_1, e_2 are det. up to S^1. Another choice gives $g \in U(1,1) \xrightarrow{det} S^1$

~~Suppose that you put things~~

$$(v_1 \wedge v_2 | v_3 \wedge v_4) = \begin{vmatrix} (v_1|v_3) & (v_1|v_4) \\ (v_2|v_3) & (v_2|v_4) \end{vmatrix}$$

So $\Lambda^2 V$ should have a ~~strictly~~ negative herm. form.

check: ~~If~~ $g \in U(1,1)$ then $\det(g) \in U(1)$.

The classification

$$SU(1,1) \simeq SL(2,\mathbb{R})$$

I think this should translate to a statement that a 2 diml complex v.s. V equipped with Krein form and volume ray has a real structure. $\Lambda^2 V$ inherits a negative hermitian form, there's a unique volume element of norm -1 lying in the volume ray. The volume ray can be replaced by a volume element of norm -1.

To ~~explain~~ explain why this yields a real structure. The hermitian form on V yields a circle in $\mathbb{P}V = $ R.S. these are the isotropic lines.

~~So what~~ Something subtle is happening it seems. Given

$$V = \mathbb{C}^2 \quad \text{with hermitian form } \xi^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}\xi$$

$$\bar V \longrightarrow V^\vee \qquad\qquad V \longrightarrow V^\vee$$

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \longmapsto \begin{pmatrix}\bar\xi_1 & -\bar\xi_2\end{pmatrix} \qquad\qquad \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \longmapsto \begin{pmatrix}\xi_2 & -\xi_1\end{pmatrix} c$$

So basically ~~you have~~ once you pick a diag. basis.

$$\mathrm{Herm}(\xi,\xi) = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = |\xi_1|^2 - |\xi_2|^2$$

$$\mathrm{Vol}(\xi,\eta) = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \xi_1\eta_2 - \xi_2\eta_1$$

Take,

$$\mu(v'_4 \wedge v_4) = (\phi v'_4 \mid v_4) \qquad\qquad \phi(v_1 c) = \phi(v_1)\bar c$$

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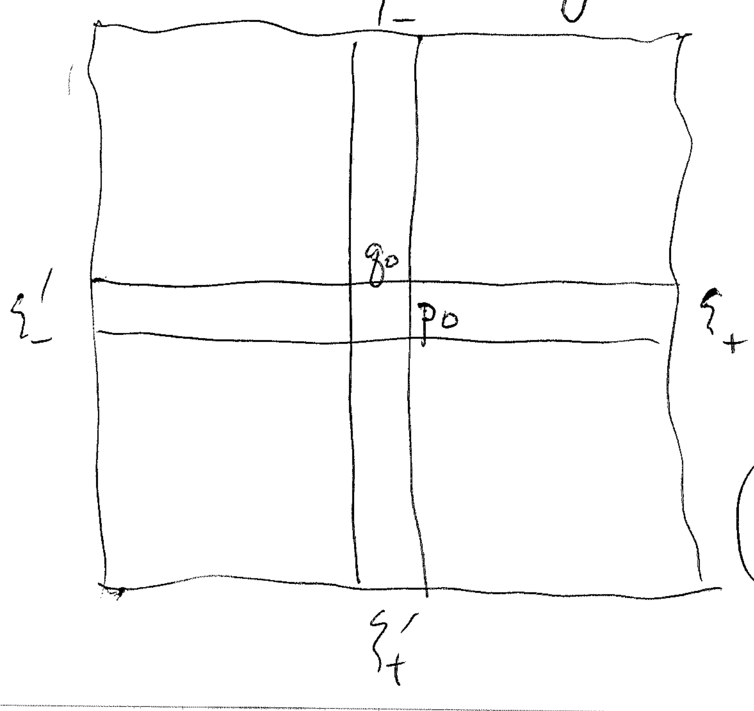
back to splitting and factorization. Try to make links with G fns. Transfer picture



$$p_0 \in (H_+\xi_+ + H_-\xi_-) \cap (zH_-\xi'_- + H_+\xi'_+)$$

$$q_0 \in (zH_+\xi_+ + zH_-\xi_-) \cap (zH_-\xi'_- + H_+\xi'_+)$$

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{array}{cc} zH_- & H_+ \\ \multicolumn{2}{c}{\begin{pmatrix}a^\ell & b^\ell\\ c^\ell & d^\ell\end{pmatrix}} \\ zH_- & H_+\end{array}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} = \begin{array}{cc} H_+ & H_- \\ \multicolumn{2}{c}{\begin{pmatrix}d^r & -b^r\\ -c^r & a^r\end{pmatrix}} \\ zH_+ & zH_-\end{array}\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix}$$

Worry just about p_0

$$p_0 \in (H_+\xi_+ + H_-\xi_-) \cap (zH_-\xi'_- + H_+\xi'_+)$$

$$p_0 = d^r\xi_+ - b^r\xi_- = a^l\xi'_- + b^l\xi'_+$$

$$\cancel{(d^r \quad -b^r)\begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix} = (a^l \quad b^l)}$$

$$\cancel{\begin{array}{ccc} & H_+ & & zH_-\\ \begin{pmatrix}\frac{1}{d} & -\frac{c}{d}\\ \frac{b}{d} & \frac{1}{d}\end{pmatrix} & \begin{pmatrix}d^r\\ -b^r\end{pmatrix} & = & \begin{pmatrix}a^l\\ b^l\end{pmatrix}\\ & H_- & & H_+\end{array}}$$

$$(d^r \quad -b^r)\begin{pmatrix}a & b\\ c & d\end{pmatrix} = (a^l \quad b^l)$$

$$\begin{array}{cccc} & H_+ & & zH_-\\ \begin{pmatrix}a & c\\ b & d\end{pmatrix} & \begin{pmatrix}d^r\\ -b^r\end{pmatrix} & = & \begin{pmatrix}a^l\\ b^l\end{pmatrix}\\ & H_- & & H_+\end{array}$$

$$\begin{array}{cccc} & H_+ & & zH_-\\ \begin{pmatrix}1 & \frac{c}{a}\\ \frac{b}{d} & 1\end{pmatrix} & \begin{pmatrix}d^r\\ -b^r\end{pmatrix} & = & \begin{pmatrix}\frac{a^l}{a}\\ \frac{b^l}{d}\end{pmatrix}\\ & H_- & & H_+\end{array}$$

$$\begin{pmatrix}Id_+ & \pi_+\frac{c}{a}\\ \pi_-\frac{b}{d} & Id_-\end{pmatrix}\begin{pmatrix}d^r\\ -b^r\end{pmatrix} = \begin{pmatrix}\frac{a^l}{a}(\infty)\\ 0\end{pmatrix}$$

$$d^r - \pi_+\frac{c}{a}b^r = \text{const.} \neq 0$$

$$\pi_-\frac{b}{d}d^r - b^r = 0$$

$$\therefore \quad d^r - \pi_+\frac{c}{a}\pi_-\frac{b}{d}d^r = \text{const} \neq 0.$$

$$z^{-1}g_0 \in (H_+\xi_+ + H_-\xi_-) \cap (H_-\xi_-' + z^{-1}H_+\xi_+')$$

$$z^{-1}g_0 = -z^{-1}c^r\xi_+ + z^{-1}a^r\xi_- = z^{-1}c^l\xi_-' + z^{-1}d^l\xi_+'$$

$$\begin{pmatrix}-z^{-1}c^r & z^{-1}a^r\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}z^{-1}c^l & z^{-1}d^l\end{pmatrix}$$

$$\begin{pmatrix}1 & c/a\\ b/d & 1\end{pmatrix}\begin{array}{c}H_+\\ \begin{pmatrix}-z^{-1}c^r\\ z^{-1}a^r\end{pmatrix}\\ H_-\end{array} = \begin{array}{c}H_-\\ \begin{pmatrix}z^{-1}c^l/a\\ z^{-1}d^l/d\end{pmatrix}\\ z^{-1}H_+\end{array}$$

$$\begin{pmatrix}Id_+ & \pi_+\frac{c}{a}\\ \pi_+\frac{b}{d} & Id_+\end{pmatrix}\begin{pmatrix}-z^{-1}c^r\\ z^{-1}a^r\end{pmatrix} = \begin{pmatrix}0\\ z^{-1}\frac{d^l}{d}(0)\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a^r & b^r\\ c^r & d^r\end{pmatrix}\begin{pmatrix}a^l & b^l\\ c^l & d^l\end{pmatrix}$$

$$\begin{pmatrix}1 & b/a\\ \frac{c}{d} & 1\end{pmatrix}\begin{array}{cc}H_+ & H_+\\ \multicolumn{2}{c}{\begin{pmatrix}d^l & -b^l\\ -c^l & a^l\end{pmatrix}}\\ zH_- & zH_-\end{array} = \begin{array}{cc}zH_- & H_-\\ \multicolumn{2}{c}{\begin{pmatrix}a^r/a & b^r/a\\ c^r/d & d^r/d\end{pmatrix}}\\ zH_+ & H_+\end{array}$$

$$\begin{array}{c}\begin{pmatrix}Id_+ & \pi_+\frac{b}{a}\\ \pi_-\frac{c}{d} & Id_-\end{pmatrix}\\ \begin{array}{cc}zH_- & zH_-\end{array}\end{array}\begin{pmatrix}d^l & -b^l\\ -c^l & a^l\end{pmatrix} = \begin{pmatrix}\frac{a^l}{a}(\infty) & 0\\ 0 & \frac{d^r}{d}(0)\end{pmatrix}$$

Is there some way to deal simply with an F i.e. $H = H_+ \oplus H_-$ and Toeplitz operators. Philosophy You have two abelian

~~situations~~ situations over the circle which are roughly equivalent

$$b \longmapsto T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{loop in } SU(1,1)$$

$$\longmapsto S = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \quad \text{———} \quad U(2)$$

abelian ~~means~~ functions on the circle. The construction of T amounts to?

The T situation ends with ~~...~~ Toeplitz ops ~~going from~~ odd

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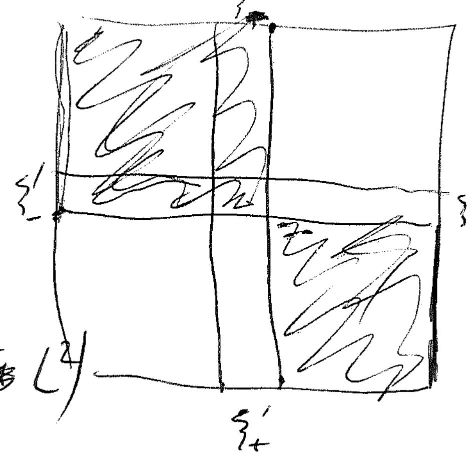
Fit things in a pattern — try.



$$* \quad E = (H_+\xi'_- + H_+\xi_-) \overset{?}{\oplus} (H_-\xi_+ \oplus H_-\xi'_+)$$

$$(H_+ \ \ H_+) \overset{?}{\oplus} (H_- \ \ H_-)\begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} = (L^2 \oplus L^2)$$

$$\begin{pmatrix} \frac{1}{\cancel{d}} & -\frac{c}{\cancel{d}} \\ \frac{b}{\cancel{d}} & \frac{1}{\cancel{d}} \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} = \begin{pmatrix} L^2 \\ \oplus \\ L^2 \end{pmatrix}$$

$$\begin{pmatrix} Id_- & -\pi_- c \\ \pi_- b & Id_- \end{pmatrix} : \begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow[\sim]{\cong ?} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

~~...~~ Question. Go back to M over $\mathbb{C}[z,z^{-1}]$ generated by the general solution the D.E. Assume (h_n) fin. support so that $\xi_\pm, \xi'_\pm \in M$.

M module over $A = \mathbb{C}[z,z^{-1}]$ generated by elements p_n, q_n sat
$$\begin{pmatrix} z^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

(h_n) fin. supp. $\Rightarrow$ $\xi_+ = z^{-n}p_n$ $n >> 0$ $\exists$.

$\underline{\Lambda^2_A M}$ $\quad z^{-n}p_n \wedge q_n = z^{-n+1}p_{n-1} \wedge q_{n-1}$ $\quad \therefore \Lambda^2_A M = A$.

~~So what gives - think it out~~

A is a $*$-alg $\quad \left(\sum a_n z^n\right)^* = \sum \bar{a}_n z^{-n}$

E is pre Hilbert C^*-module over A means a right module with pairing $(\xi'|\xi)$.
$(\xi'a'|\xi a) = a'^*(\xi'|\xi)a$. Can ask for completeness.
There should be a Krein version of this.

Let's see if this can be understood. You know M itself has various bases related by $SU(1,1)$ matrices over A. Right - left?

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{array}{l} a,b,c,d \in A \\ a = d^*, \ b = c^*, \quad ad - bc = 1. \end{array}$$

You need to fit ~~enter~~ into Hilbert transform mode. Basic example of a K-homology class. Fredholm module. F on the left, A on the right.
How to handle this?

$$E = \xi_+ A + \xi_- A \qquad \begin{array}{l} \xi = \xi_+ a_1 + \xi_- a_2 \\ \eta = \xi_+ b_1 + \xi_- b_2 \end{array}$$

$$(\xi|\eta) = (\xi_+ a_1 + \xi_- a_2 | \xi_+ b_1 + \xi_- b_2)$$
$$= a_1^*(\xi_+|\xi_+)b_1 + \cdots \quad = a_1^* b_1 - a_2^* b_2$$

$$\xi^* K \eta = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^* \underbrace{(\xi_+ \ \xi_-)^* \left(\overset{K}{\cancel{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}}} \right) (\xi_+ \ \xi_-)}_{} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} \xi_+^* K \xi_+ & \xi_+^* K \xi_- \\ \xi_-^* K \xi_+ & \xi_-^* K \xi_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

~~So what?~~

$$(\xi_+ \ \ \xi_-) = (\xi_-' \ \ \xi_+') \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$ ~~certainly no problem.~~

What's the link with F.

~~So basically you have~~

Situation. You have A and the ~~[illegible]~~ Hilbert transform F. Functions on S^1 and an elliptic ΨDO. A itself is ~~a ΨDO~~. Hilbert module over itself $E = \xi_+ A + \xi_- A = \xi_-' A + \xi_+' A$

$$(\xi_+ \ \ \xi_-) = (\xi_-' \ \xi_+') \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

So you deal with a vector bundle of rank 2 over S^1 equipped with a $SU(1,1)$ structure Krein + vol in each ~~f~~ fibre — this is all pointwise.

So what else do you look at? ~~OKAY~~

module M over A ~~set~~ with four elements $\xi_\pm \ \xi_\mp'$ related by

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

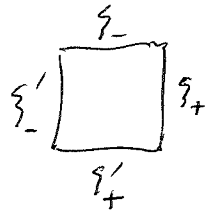
unitary

Review. M module over $A = \mathbb{C}[z,z^{-1}]$ gen. by elements p_n, q_n satisfying DE. Assume (h_n) f.s. or that A is enlarged sufficiently so that ~~the ends~~ $z^{-n}p_n, q_n$ have limits as $n \to +\infty$ and as $n \to -\infty$. Then

$M = \xi_+ A + \xi_- A = \xi'_- A + \xi'_+ A$. $\quad (\xi_+ \ \ \xi_-) = (\xi'_- \ \ \xi'_+)\, T$

$T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. ~~You want to focus on~~

A is a ~~star~~ $*$ alg so has an inner product $a_1^* a_2$. You want to generalize from $\mathbb{C}$ to A. the situation



$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

~~to what you have is~~ You need to adjoin $\frac{1}{d}$ to A.

$\frac{1}{|d|^2} = \frac{1}{1+|b|^2}$ ~~You need to adjoin~~

Over $\mathbb{C}$ you have a 4 diml Krein space with basis $\xi_\pm$, $\xi'_\mp$, ~~and Krein form~~ and an isotropic subspace. Need to review the stuff on partial unitaries and contractions. ~~But the~~

$$Y = aX \oplus \mathbb{C}\xi_+ = bX \oplus \mathbb{C}\xi_-$$

Consider $\begin{pmatrix} Y \\ Y \end{pmatrix}$ with Krein form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Then $W = \begin{pmatrix} a \\ b \end{pmatrix} X$ is isotropic

and $W^\circ = \begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\qquad W^\circ / W$ = Krein space $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$

Spectrum. To get ~~the operator~~ a spectrum you want $W^\circ \subset Z \subset W$

$$(az-b)x = -v_- + v_+$$

$$\begin{tikzcd}
 & & \cancel{X} \arrow[d, "\binom{a}{b}"] \arrow[dr, bend left, "az-b"] & & \\
0 \arrow[r] & Y \arrow[r, "\binom{1}{z}"] & Y \oplus Y \arrow[r, "(z\ -1)"] & Y \arrow[r] & 0
\end{tikzcd}$$

Idea: Choose $W^\circ \subset Z \subset W$. Remember $\mathcal{O}(-1)$

$$\begin{tikzcd}
 & & \mathcal{O}\otimes W \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}(-1)\otimes Y \arrow[r] & \mathcal{O}\otimes (Y \oplus Y) \arrow[r] \arrow[d] & \mathcal{O}(1)\otimes Y \arrow[r] & 0 \\
 & & \mathcal{O}\otimes W^\circ & &
\end{tikzcd}$$

Do your reviewing. Do reviewing.

Given $Y = aX \oplus V_+ = bX \oplus V_-$ $\qquad a : X \hookrightarrow Y$, $b = ua : X \to Y$.

eigenvector eqn.

$ax_1 + v_+$

$$y = x_1 + v_+ = ux_2 + v_-$$

$$uy = ux_1 + uv_+ \qquad \lambda y = \lambda u x_2 + \lambda v_-$$

In general. Given u unitary on Z and $Y \subset Z$, let $X = u^{-1}Y \cap Y \underset{u}{\overset{in}{\rightrightarrows}} Y$

$$Z = Y^\perp \oplus X \oplus V_+ \qquad \cancel{\lambda\xi = \lambda\eta + \lambda x_1 + \lambda v_+}$$

$$= Y^\perp \oplus V_- \oplus uX \qquad u\xi = u\eta + ux_1 + uv_+$$

$\lambda\xi = \lambda(\eta + x_1 + v_+) = \lambda\eta + \lambda u x_2 + \lambda v_-$ $\qquad$ proj onto uX

$= u\eta + ux_1 + uv_+$ $\qquad$ get $x_1 = \lambda x_2$

so end up with $\lambda x_2 + v_+ = u x_2 + v_-$

or $\boxed{(\lambda - u)x = -v_- + v_+}$

Next a p.u. is a pair $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\ni$ $a^*a = 1$, $b^*b = 1$.

same as isotropic subspace $W = \binom{a}{b}X$ in the Krein space $\begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix}$. Have $W^\circ =$

$W \oplus \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\quad W^\circ$: $\begin{pmatrix} a \\ b \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$

$a^* y_1 = b^* y_2$ call this x

then $y_1 = a x + (1 - aa^*) y_1$

$y_2 = b x + (1 - bb^*) y_2$ $\quad \therefore W^\circ = \binom{a}{b} X \oplus \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$.

$\mathrm{Ker}\left\{ W^\circ \subset \begin{smallmatrix} Y \\ \oplus \\ Y \end{smallmatrix} \xrightarrow{(\lambda\ -1)} Y \right\}$ consists of

$\begin{pmatrix} ax + v_+ \\ bx + v_- \end{pmatrix}$ such that $\lambda a x + \lambda v_+ = bx + v_-$

$(\lambda a - b)x = -\lambda v_+ + v_-$

$0 \longrightarrow \mathcal{O}(-1) \otimes Y \subset \mathcal{O} \otimes \begin{pmatrix} Y \\ Y \end{pmatrix} \longrightarrow \mathcal{O}(1) \otimes Y \longrightarrow 0$

What? Somehow W° / W generates the line bundle

Focus on Z: $W \subset Z \subset W^\circ$

Assume. $a\lambda - b : W \hookrightarrow Y \quad \forall \lambda$

so get line bundle $\mathcal{L}_\lambda = Y / (a\lambda - b) X$

Interpret this. In fact you know that this is just solving the eigenvector eqn.

$$(az-b)x = -y + \xi_+ \hat{y}(z)$$

Chk:
$$(z-a^*b)x = -a^*y$$

$$x = (z-a^*b)^{-1}(-a^*y) = -z^{-1}(1-z^{-1}a^*b)(a^*y)$$
$$= -z^{-1}a^*(1-z^{-1}ba^*)^{-1}y$$

$$\xi_+ \hat{y}(z) = y + (az-b)(-\cancel{z^{-1}}a^*)(z-\cancel{z^{-1}}ba^*)^{-1}y$$
$$= \left[z-ba^* + (az-b)(-a^*) \right](z-ba^*)^{-1}y$$
$$= \underbrace{(1-aa^*)}_{\xi_+ \xi_+^*}(1-z^{-1}ba^*)^{-1}y$$

$$\hat{y}(z) = \xi_+^*(1-z^{-1}ba^*)y$$

Still I need a good interpretation.

You are solving $(a\lambda - b)x = -y + \xi_+ \hat{g}(\lambda)$
This should be interpretable in terms of a bdry condition, a resolvent.

$$Z = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix} X}_{W} + \begin{pmatrix} V_+ \\ 0 \end{pmatrix}$$ should be complementary to $l_\lambda \otimes Y = \left\{ \begin{pmatrix} 1 \\ \lambda \end{pmatrix} y \right\}$

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix} y$$
$$\supseteq$$

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, "\wr"'] & \begin{pmatrix} Y \\ Y \end{pmatrix} \arrow[r, dash] \arrow[d, "(\lambda\ -1)"] & {} \\ & Y \end{tikzcd}$$

isom means

$$x, v_+ \longmapsto (\lambda a - b)x + \lambda v_+ = y$$

$$Y = X \oplus V_+ = uX \oplus V_-$$

$$W = \begin{pmatrix} 1 \\ u \end{pmatrix} X \subset \begin{pmatrix} Y \\ \oplus \\ Y \end{pmatrix} \subset$$

$$W^\circ = \begin{pmatrix} 1 \\ u \end{pmatrix} X \oplus \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \qquad T \otimes Y$$

$$W^\circ \cap l_\lambda \otimes Y = \left\{ \begin{pmatrix} x + v_+ \\ ux + v_- \end{pmatrix} \middle| \underbrace{\lambda(x+v_+) = ux + v_-}_{(\lambda - u)x = -\lambda v_+ + v_-} \right\}$$

bdry conditions Yes.

$$1 - c^*c = 1 - ab^*ba^* = 1 - aa^* = \pi_+$$

Aim: Embed Y inside sections

$$\mathcal{O}(-1) \otimes Y \longrightarrow \mathcal{O} \otimes T \otimes Y \longrightarrow \mathcal{O}(1) \otimes Y$$

Recall basic embedding. On Y you have the contraction $c = b^* a^*$ $\qquad u = ba^{-1}$

$$y \longmapsto \underbrace{\sum_{n \geq 0} z^{-n} \underbrace{\pi_+}_{\xi_+^*} c^n y}_{\text{has } l^2 \text{ norm.}} = \underset{\xi_+^*}{\pi_+} \frac{1}{1 - z^{-1}c} y$$

$$\sum_{n \geq 0} \underbrace{\| \xi_+^* c^n y \|^2}_{(y,\, c^{*n} \underbrace{\xi_+ \xi_+^*}_{1 - c^*c} c^n y)} = \|y\|^2 - \lim_{n \to \infty} \|c^n y\|^2$$

somehow embeds Y into functions on the circle with values in V_+. Extending analytically to $|z| > 1$. Yes.

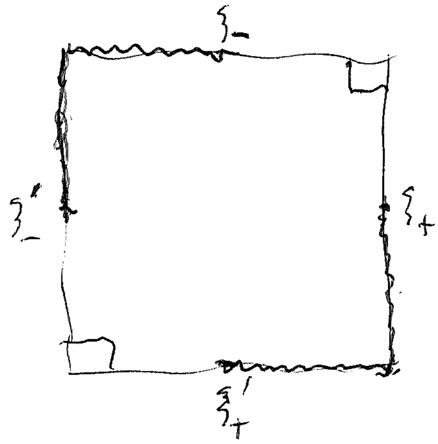
Try hard once more. Go back to M with its $SU(1,1)$ structure and $U(2)$ structure. Integrating should yield a pos. def. inner product in the $U(2)$ case and a Krein product in the $SU(1,1)$ case. Orthogonality holds in the pos. def. herm. inner prod between $H_+\xi_+ + H_-\xi_-$ and $H_-\xi'_- + H_+\xi'_+$. Is there something analogous for the Krein herm. inner product.

You have 8 ~~spaces around~~ $H_\pm \times$ four: $\xi_\pm, \xi'_\mp$

$L^2\xi_+ \perp L^2\xi_-$

$L^2\xi'_- \perp L^2\xi'_+$ for Krein.

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$



$$K(f\xi'_-, g\xi_+) = K(f\xi'_-, g(a\xi'_- + b\xi'_+))$$
$$= \int \bar{f} g a$$

will be zero is $f \in H_+,\ g \in H_-$

$$K(f\xi'_+, g\xi_-) = K(f\xi'_+, g(c\xi'_- + d\xi'_+))$$
$$= -\int \bar{f} g d \quad = 0 \quad \text{if} \quad f \in H_-,\ g \in H_+$$

because $(\xi'_+|\xi'_+) = (\xi_-|\xi_-) = -1$

$(\xi_+|\xi_+) = (\xi'_-|\xi'_-) = +1$

$$\left.\begin{array}{l} H_+\xi'_- \overset{K}{\perp} H_-\xi_+ \\ H_-\xi'_+ \overset{K}{\perp} H_+\xi_- \end{array}\right\} \Rightarrow H_+\xi'_- + H_+\xi_- \overset{K}{\perp} H_-\xi_+ + H_-\xi'_+$$

these are complements so the Krein form is the direct sum of what you ~~xxx~~ have on summand.

$$K\left(f\xi_-' + g\xi_-, f\xi_-' + g\xi_-\right) = ?$$

$$\cancel{\left(g\xi_-, f\xi_-'\right) - \left(g(c\xi_-'\right.}$$

$$K\left(f\xi_-', g\xi_-\right) = K\left(f\xi_-', g(c\xi_-' + d\xi_+')\right)$$
$$= +\int \bar{f}gc$$

$$K\left(g\xi_-, f\xi_-'\right) = K\left(g(c\xi_-' + d\xi_+'), f\xi_-'\right) = \int \overline{gc}f.$$

$$K\left(f\xi_-' + g\xi_-, f\xi_-' + g\xi_-\right) = \int |f|^2 + \bar{f}gc + \overline{gc}f - |g|^2$$

~~If~~ $f = 0$ then get $-\int |g|^2 < 0$

$g = 0$ ——— $\int |f|^2 > 0$

$$\begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix}^t \underbrace{\begin{pmatrix} 1 & c \\ \bar{c} & -1 \end{pmatrix}}_{\begin{pmatrix} 1 & c \\ \bar{c} & |c|^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -|d|^2 \end{pmatrix}} \begin{pmatrix} f \\ g \end{pmatrix}$$

Go over this. You're given $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transfer matrix g and you propose to define a Krein module over $\mathcal{A}$: $\mathcal{M} = \xi_-'\mathcal{A} + \xi_+'\mathcal{A} = \xi_+\mathcal{A} + \xi_-\mathcal{A}$

$$(\xi_+ \ \ \xi_-) = (\xi_-' \ \ \xi_-)\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$K\left(\xi_+ f + \xi_- g\right) = \left(\xi_+ f + \xi_- g\right)^*(K)\left(\xi_+ f + \xi_-^* g\right)$$

$$K(\xi_+ f + \xi_- g) = \cancel{(\xi_+ f + \xi_- g)^* K} \; |f|^2 - |g|^2$$
$$= \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$K(\xi'_- f' + \xi'_+ g') = K\left(\underbrace{\begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{\begin{pmatrix} \xi_+ & \xi_- \end{pmatrix}} \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} \begin{pmatrix} f' \\ g' \end{pmatrix} \right)$$

$$= \begin{pmatrix} f' \\ g' \end{pmatrix}^* \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} \right)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} \begin{pmatrix} f' \\ g' \end{pmatrix} = |f'|^2 - |g'|^2$$

~~Check the local~~ Notice

$$g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \implies |\det(g)|^2 = +1.$$

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$g^{-1} = \frac{1}{\det(g)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

~~Krein form~~ $\mathcal{M} = \xi'_- \mathcal{a} \oplus \xi'_+ \mathcal{a} = \xi_+ \mathcal{a} \oplus \xi_- \mathcal{a}$

$$\underset{\sim}{K}(\xi'_- f + \xi'_+ g) = |f|^2 - |g|^2 \quad \text{local Krein form}$$

$$\underset{\sim}{K}\left(\begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right) = \cancel{\begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} \xi'^*_- K \xi'_+ & \xi'^*_- K \xi \\ \xi'^*_+ K \xi'_+ & \xi'^*_+ K \xi \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}}$$

$$= \begin{pmatrix} \bar{g} & \bar{f} \end{pmatrix} \underbrace{\begin{pmatrix} \xi'^*_- \\ \xi'^*_+ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \xi'_- & \xi'_+ \end{pmatrix}} \begin{pmatrix} f \\ g \end{pmatrix}$$

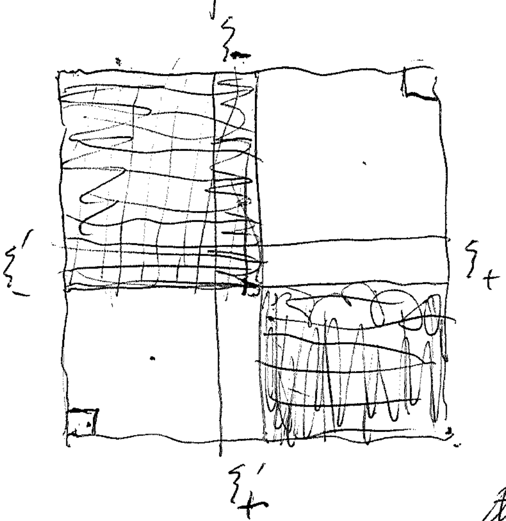
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} |a|^2 - |b|^2 & \bar{a}c - \bar{b}d \\ \bar{b}a - \bar{a}b & \bar{b}c - \bar{d}d \end{pmatrix}$$

$\bar{c}b$

get global Krein form by integration

$$K(\xi_-'f+\xi_+'g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \|f\|^2 - \|g\|^2$$

next point mind like mud.



So you make $\mathcal{M} = \xi_+ a \oplus \xi_- a$
$= \xi_-' a \oplus \xi_+' a$

into a Krein space using ptwise form

$$\begin{pmatrix} \underset{\sim}{K}(\xi_+,\xi_+) & \underset{\sim}{K}(\xi_-,\xi_+) \\ \underset{\sim}{K}(\xi_+,\xi_-) & \underset{\sim}{K}(\xi_-,\xi_-) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The point becomes interesting with incoming & outgoing bases.

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_-' \\ \xi_+' \end{pmatrix}$$

$$\underset{\sim}{K}(\xi_-', \xi_-) = \underset{\sim}{K}(\xi_-', c\xi_-' + d\xi_+') = c$$

$$(\xi_-', \xi_-)^* \underset{\sim}{K}(\xi_-', \xi_-) = \begin{pmatrix} 1 & c \\ \bar{c} & 1 \end{pmatrix}$$

$$K(\xi_-'f, \xi_+ g) = K(\xi_-'f, (a\xi_-' + b\xi_+')g)$$

$$= \int \bar{f} g a \quad = 0 \text{ if } f \in H_+, g \in H_-$$

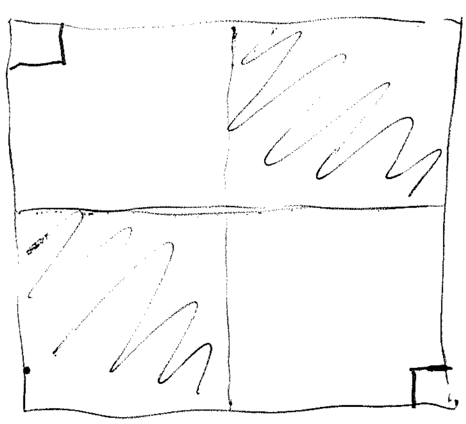
$$K(\xi_+'f, \xi_- g) = K(\xi_+'f, (c\xi_-' + d\xi_+')g)$$

$$= \int \bar{f} d g = 0 \quad \text{if } f \in H_-, g \in H_+$$

$$\xi_-' H_+ + \xi_- H_+ \perp \xi_+ H_- + \xi_+' H_-$$

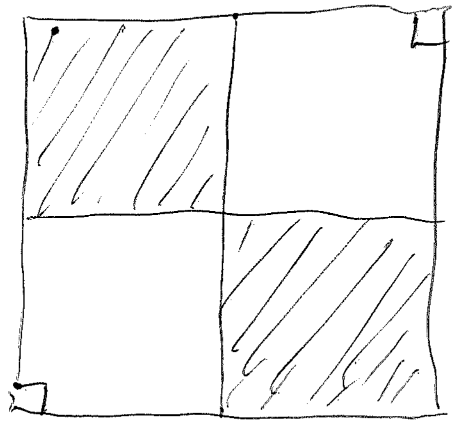
Here's what to do: You have things you ~~did~~ in the context of the Hilbert space E like orthogonal projection. Propose to do the analogs with the indefinite form.

pos def picture



shaded rectangles are $\perp$

indef picture



shaded squares are $\perp$

In the positive def picture you want the half spaces $H_+\xi_+ + H_-\xi_-$ and $H_-\xi'_- + H_+\xi'_+$, so you want to define E using the non orth. bases $\xi_\pm$ (or $\xi'_\mp$) i.e.

$$\| \xi_+ f + \xi_- g \|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

the positivity results from $\|\beta\| < 1$. or

$$\begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar\beta \\ \beta & |\beta|^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1-|\beta|^2 \end{pmatrix}$$

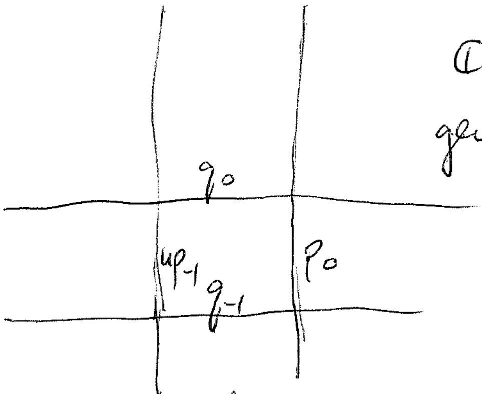
In the indefinite picture you consider the half spaces $H_+\xi'_- + H_+\xi_-$ $\quad H_-\xi_+ \oplus H_-\xi'_+$ so you look at the non orth. (for indef form) ξ'_-, ξ_- (or ξ_+, ξ'_+):

$$K(\xi'_- f + \xi_- g) = K(\xi'_- f + (c\xi'_- + d\xi'_+)g)$$
$$= K(\xi'_-(f+cg) + \xi'_+ d) =$$
$$= \|f+cg\|^2 - \|dg\|^2.$$

---

situation as of yesterday: you seem to have found that M over $\mathbb{C}[z,z^{-1}]$ has both a ~~finite~~ positive definite pairing and a Krein pairing at least globally because of staircase bases.

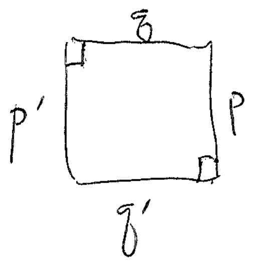
Define M. It's the $\mathbb{C}[z,z^{-1}]$-module with generators p_n, q_n $n \in \mathbb{Z}$ sat



$$\begin{pmatrix}p_n\\ q_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar h_n & 1\end{pmatrix}\begin{pmatrix}z p_{n-1}\\ q_{n-1}\end{pmatrix} \quad \forall n$$

Clearly free of rank 2 with basis p_n, q_n for any n. ~~Krein st~~

IDEA. You have pos. def. inner product and an indefinite inner product, so there should be a ~~self~~ adjoint operator around.



$$K(px+qy)^2 = |x|^2 - |y|^2$$
$$\|px+qy\|^2 = \begin{pmatrix}x\\ y\end{pmatrix}^* \begin{pmatrix}1 & \bar h\\ h & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$
$$\|$$
$$\|px\|^2 + (qy|px) + (px|qy) + \|qy\|^2$$
$$= |x|^2 + \bar y h x + \bar x \bar h y + |y|^2$$

$$\begin{pmatrix}p\\ q\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}p'\\ q'\end{pmatrix}$$
$$\begin{pmatrix}p\\ q'\end{pmatrix} = \begin{pmatrix}k & h\\ -\bar h & k\end{pmatrix}\begin{pmatrix}p'\\ q\end{pmatrix}$$
$$\begin{pmatrix}p'\\ q\end{pmatrix} = \begin{pmatrix}k & -h\\ \bar h & k\end{pmatrix}\begin{pmatrix}p\\ q'\end{pmatrix}$$

(go to 371)

So you look at ~~$M = a\xi'_- + a\xi'_+ = a\xi_+ + a\xi_-$~~

$$M = \xi'_- a \oplus \xi'_+ a = \xi_+ a \oplus \xi_- a \qquad \Big| \qquad (\xi_+\ \ \xi_-) = (\xi'_-\ \ \xi'_+)\begin{pmatrix} a & c\\ b & d\end{pmatrix}$$

$$(\xi'_-\ \ \xi'_+)^* K (\xi'_-\ \ \xi'_+) = \begin{pmatrix} \xi'^*_- K\xi'_- & \xi'^*_- K \xi'_+ \\ \xi'^*_+ K\xi'_- & \xi'^*_+ K\xi'_+\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\begin{pmatrix}f\\g\end{pmatrix}^* (\xi'_-\ \ \xi'_+)^* K (\xi'_-\ \ \xi'_+)\begin{pmatrix}f\\g\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix}$$

$$\xi_- = \xi'_- c + \xi'_+ d$$

$$(\xi'_- f + \xi'_+ g)^* K (\xi'_- f + \xi'_+ g) = \begin{pmatrix}f\\g\end{pmatrix}^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix} = |f|^2 - |g|^2$$

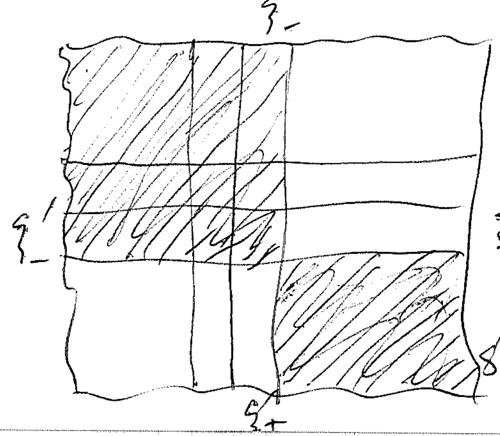
$$(\xi'_-\ \ \xi_-)^* K (\xi'_-\ \ \xi_-) =$$

$$\underbrace{(\xi'_-\ \ \ \xi'_- c + \xi'_+ d)^*}_{\begin{pmatrix}1&0\\ \bar c & \bar d\end{pmatrix}(\xi'_-\ \ \xi'_+)^*} K \underbrace{(\xi'_-\ \ \ \xi'_- c + \xi'_+ d)}_{(\xi'_-\ \ \xi'_+)\begin{pmatrix}1&c\\0&d\end{pmatrix}}$$

$$= \begin{pmatrix}1&0\\ \bar c&\bar d\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&c\\0&d\end{pmatrix} = \begin{pmatrix}1&0\\ \bar c&-\bar d\end{pmatrix}\begin{pmatrix}1&c\\ &d\end{pmatrix} = \begin{pmatrix}1&c\\ \bar c & \underbrace{|c|^2-|d|^2}_{-1}\end{pmatrix}$$

$$= \begin{pmatrix}1&c\\ \bar c&|c|^2\end{pmatrix} + \begin{pmatrix}0&0\\0&-|d|^2\end{pmatrix}$$

~~So you~~



$H_+\xi'_- + H_+\xi_- \qquad H_-\xi_+ + H_-\xi'_+$

It appears that for simple-minded reasons M is the direct sum of the 2 planes $sp(z^n p_0,\ z^n q_0)$

b a lot of checking is needed. But what you claim ~~.....~~ about the global Krein structure ~~should be evident~~ might be clear ~~by~~ from a staircase orthonormal bases.

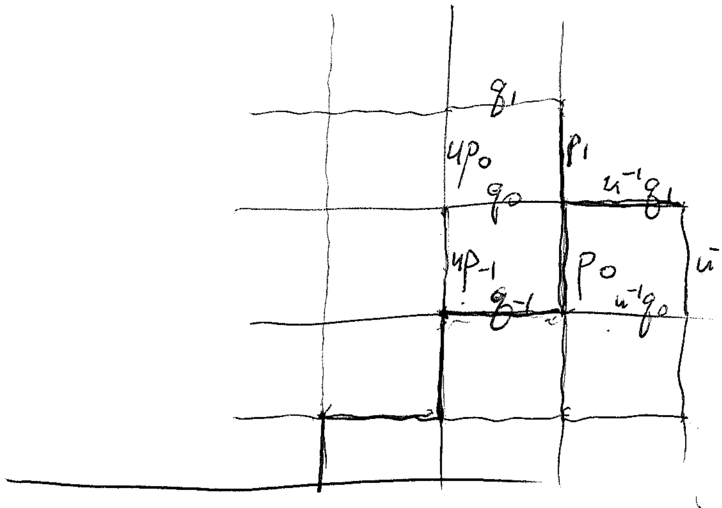
So let's start by checking things carefully.

~~anyway what next~~.

See what can be done.

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

First observation: Form M over $\mathbb{C}[z, z^{-1}]$ and you get this grid.



Make M into a pre Hilbert-space by saying the elements in a any staircase form an orth. basis.

Krein

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}$$

Review ~~.....~~ splitting.

$$E = (H_+ \xi'_- + H_+ \xi_-) \overset{?}{\oplus} (H_- \xi_+ + H_- \xi'_+)$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$= \frac{1}{d}\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

It looks like once you have ~~....~~ arranged $\frac{1}{|d|^2}$ to be in $\mathcal{A}$ that the splitting might be ~~...~~ easy

(go back to 368)

$$\begin{pmatrix} x \\ y \end{pmatrix}^{*} \underbrace{(p\ q)^{*}\ (p\ q)}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} p^* \\ q^* \end{pmatrix}(p\ q) = \begin{pmatrix} 1 & \bar{h} \\ h & 1 \end{pmatrix}$$

transpose because you replace $\begin{pmatrix} p \\ q \end{pmatrix}$ by $(p\ q)$

Then you want K such that

$$\begin{pmatrix} x \\ y \end{pmatrix}^{*}\begin{pmatrix} 1 & \bar{h} \\ h & 1 \end{pmatrix}^{K}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^{*}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Better to work with basis p', q

$$\cancel{K(p'x+qy)} = \cancel{K(\ldots)}$$

$$\cancel{= K(p'x + (p\bar{h} + q'k)y)}$$

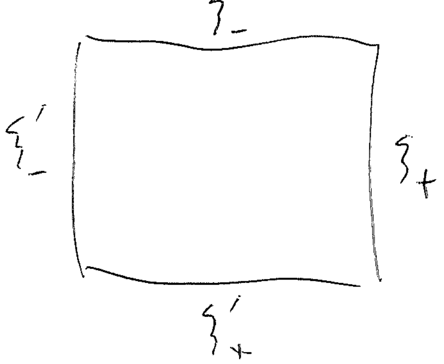
$$\cancel{K(p'x+\ldots)} \quad \cancel{K(xp + yq')} = \cancel{K(xp + y(-hp' + kq))}$$

$$K(xp' + yq) = K\left(xp' + y\left(\tfrac{\bar{h}}{k}p' + \tfrac{1}{k}q'\right)\right)$$

$$= K\left(\left(x + y\tfrac{\bar{h}}{k}\right)p' + y\tfrac{1}{k}q'\right)$$

$$= \left|x + y\tfrac{\bar{h}}{k}\right|^2 - \left|y\tfrac{1}{k}\right|^2$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$



$$\underset{\sim}{K}(\xi'_- f + \xi_- g)$$

$$= \underset{\sim}{K}(\xi'_- f + (\xi'_- c + \xi'_+ d)g)$$

$$= \|f + cg\|^2 - \|dg\|^2$$

So the self adj op is $\begin{pmatrix} 1 & c \\ \bar{c} & -1 \end{pmatrix}$ rel $(\xi'_-\ \xi_-)$