Puzzle – in going over scratch you observe you have lost the viewpoint that dividing the F.T. $\hat{\varphi}$ by s^n amounts applying the operator $(-\partial_x)^n$ to φ ~~with~~ allowing a discontinuity at 0:

$$\frac{\hat{\varphi}(s)-\hat{\varphi}(0)}{s} = \hat{\psi}(s)$$ where $\psi(x)$ defined

by $(-\partial_x \psi) = \varphi$ $\quad x \neq 0$

$\psi(x) = 0 \quad x \ll 0$ or $\gg 0$.

i.e. $\psi(x) = \int_x^{\infty} \varphi(x')\,dx' \quad x > 0$

$\quad\quad -\int_{-\infty}^{x} \varphi(x')\,dx' \quad x < 0$

$$\left(e^{ys^{-1}}\hat{\varphi}(s)\right)_{hol} = \hat{\varphi}(s) + \frac{y}{1!}\,\frac{\hat{\varphi}(s)-\hat{\varphi}(0)}{s} + \frac{y^2}{2!}\,\frac{\hat{\varphi}(s)-\hat{\varphi}(0)-\hat{\varphi}'(0)s}{s^2} + \cdots$$

$(-\partial_x)^n \psi_n(x) = \varphi(x) \quad x \neq 0$

$$\psi_n(x) = \begin{cases} \int_x^{+\infty} \varphi(x')\,\frac{(x'-x)^{n-1}}{(n-1)!}\,dx' & x > 0 \\ 0 & x < 0 \end{cases}$$

assume φ supp $\subset \mathbb{R}_{\geq 0}$

$-\partial_x \psi_n = \psi_{n-1} \quad n \geq 1.$

$$\hat{\psi}_{n-1}(s) = \int_0^{\infty} (-\partial_x \psi_n) e^{xs}\,dx = \left[-\psi_n e^{xs}\right]_0^{\infty} + \int_0^{\infty} \psi_n\, s e^{xs}\,dx$$

$$\psi_n(0) + s\hat{\psi}_n(s)$$

By ind. $\hat{\varphi}_{n-1}(s) = \dfrac{\hat{\varphi}(s) - \hat{\varphi}(0) - \cdots - \hat{\varphi}^{(n-2)}(0)\frac{s^{n-2}}{(n-2)!}}{s^{n-1}}$

$\therefore \quad \hat{\varphi}_{n-1}(0) = \dfrac{\hat{\varphi}^{(n-1)}(0)}{(n-1)!} = \varphi_n(0)$

$\hat{\varphi}_n(s) = \dfrac{\hat{\varphi}(s) - \qquad - \varphi^{(n-1)}(0)\frac{s^{n-1}}{(n-1)!}}{s^{n}}$

$$\cancel{\left(e^{ys}\hat{\varphi}(s)\right)_{hor} = \hat{\varphi}(s) + \sum_{n\geq 1} \frac{y^n}{n!} \underbrace{\int \varphi(x') \frac{(x'-x)^{n-1}}{(n-1)!} dx'}_{\hat{\varphi}_n(s)}}$$

$$\left(e^{ys}\hat{\varphi}(s)\right)_{hor} = \hat{\varphi}(s) + \frac{y}{1!}\frac{\hat{\varphi}(s)-\hat{\varphi}(0)}{s} +$$

$$= \hat{\varphi}(s) + \sum_{n\geq 1} \frac{y^n}{n!} \underbrace{\hat{\varphi}_n(s)}_{\int_0^\infty e^{xs} \int_0^x \varphi(x') \frac{(x'-x)^{n-1}}{(n-1)!} dx'\, dx}$$

$$\cancel{= \hat{\varphi}(s) + \int_0^x \varphi(x') \sum_{n\geq 1} \frac{y^n}{n!}}$$

$$= \hat{\varphi}(s) + \int_0^\infty e^{xs}\left(\int_0^x \varphi(x') \underbrace{\sum_{n\geq 1} \frac{y^n}{n!} \frac{(x'-x)^{n-1}}{(n-1)!}}_{y J_1(y(x'-x))} dx'\right) dx$$

$$= \varphi(x) + \int_0^x y J_1(y(x'-x))\, \varphi(x)'\, dx' \wedge$$

For tomorrow's lecture calculate inner product. You have grid space described as the sum of horizontal and vertical subspaces.

$E = E_{hor} \oplus E_{ver}$ $\quad$ E_{hor} consists of $\int e^{xs}\varphi(x)\,dx$

φ has compact support, piecewise continuous (approx)

E_{ver} consists of $\int e^{ys^{-1}} s^{-1}\varphi(y)\,dy$

$$\left(\int e^{x's}\varphi_1(x')\,dx' \,\middle|\, \int e^{xs}\varphi_2(x)\,dx\right) = \int \overline{\varphi_1(x)}\,\varphi_2(x)\,dx$$

$$\|$$

$$\int dx'\,dx\ \overline{\varphi_1(x')}\left(e^{x's}\,\middle|\,e^{xs}\right)\varphi_2(x)$$

$s = i\rho$ $\quad$ ρ real

$$\int_{-i\infty}^{i\infty} e^{-x's+xs}\frac{ds}{2\pi i}$$

$$\left(\int e^{y's^{-1}}s^{-1}\varphi_1(y')\,dy' \,\middle|\, \int e^{ys^{-1}}s^{-1}\varphi_2(y)\,dy\right)$$

$$\int\int \overline{\varphi_1(y')}\left(e^{y's^{-1}}s^{-1}\,\middle|\,e^{ys^{-1}}s^{-1}\right)\varphi_2(y)\,dy'\,dy$$

$$\|$$

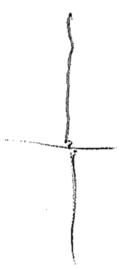
$$\int_{-i\infty}^{i\infty}\left(e^{y's^{-1}}s^{-1}\right)^*\left(e^{ys^{-1}}\right)\frac{ds}{2\pi i s}$$

$$e^{-y's^{-1}}(-s)^{-1}$$

$s \mapsto s^{-1}$

$$(-1)\int_{-i\infty}^{i\infty} e^{(-y'+y)s^{-1}}s^{-1}\frac{ds}{2\pi i s} = \int_{-i\infty}^{i\infty} e^{(-y+y')s}\,\cancel{s}\,\frac{ds}{2\pi i \cancel{s}} = \delta(-y+y')$$

$$\int_{0^+}^{+\infty} f(t)\, \frac{dt}{t} \;\overset{t \mapsto t^{-1}}{=}\; \int_{+\infty}^{0^+} f(t^{-1}) \left(-\frac{dt}{t}\right) = \int_{0^+}^{+\infty} f(t^{-1}) \frac{dt}{t}$$



$$\int_{i0^+}^{+i\infty} f(s) \frac{ds}{2\pi i s} \;\overset{s \mapsto s^{-1}}{=}\; \int_{-i\infty}^{-i0^+} f(s) \left(-\frac{ds}{2\pi i s}\right)$$

$$\int_{-i\infty}^{-i0^+} f(s) \frac{ds}{2\pi i s} = \int_{i0^+}^{+i\infty} f(s) \left(-\frac{ds}{2\pi i s}\right)$$

$$\therefore \int_{-i\infty}^{+i\infty} f(s) \frac{ds}{2\pi i s} = -\int_{-i\infty}^{+i\infty} f(s^{-1}) \frac{ds}{2\pi i s}$$

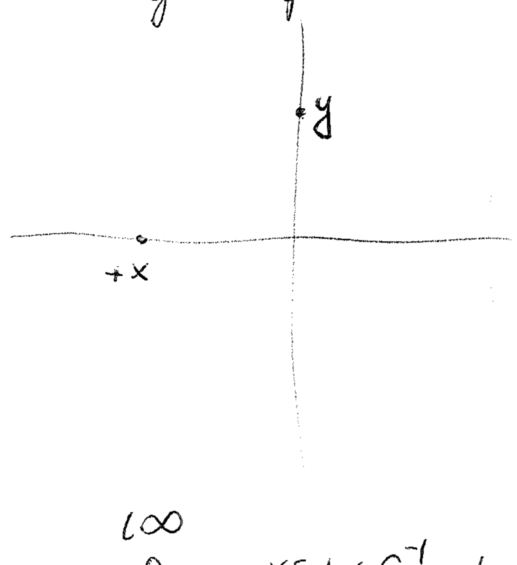
$$\left(\int e^{xs} \varphi(x)\, dx \;\middle|\; \int e^{y s^{-1}_{s^{-1}}} \varphi_1(y)\, dy \right)$$

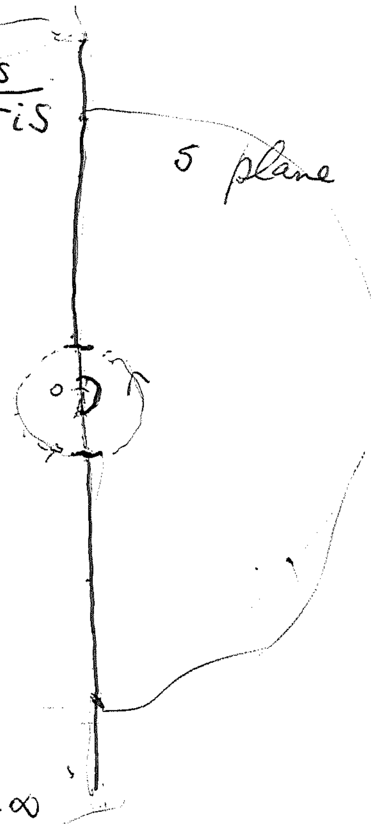
$$= \int \overline{\varphi(x)}\, dx\; \varphi_1(y)\, dy\; \underbrace{\left(e^{xs} \;\middle|\; e^{y s^{-1}_{s^{-1}}} \right)}$$

$y > 0$

$$\int_{-i\infty}^{i\infty} e^{-xs + ys^{-1}} \frac{ds}{2\pi i s}$$

grid ~~plane~~ plain

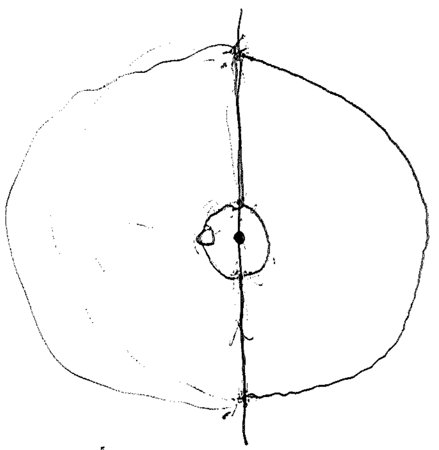


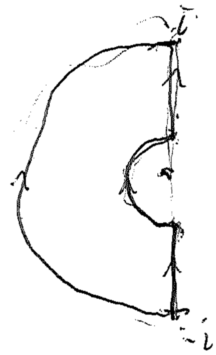


$$\int_{-i\infty}^{i\infty} e^{xs + ys^{-1}} \frac{ds}{2\pi i s} = \int_{-i\infty}^{-i} + \int_{-i}^{i} + \int_{i}^{+i\infty}$$

$x > 0$

$e^{ys^{-1}} s^{-1}$ $\quad y>0$





$e^{ys^{-1}} s^{-1}$

$$\int_{-i}^{i} = \oint$$

provided $e^{ys^{-1}}$ decays as $s \downarrow 0$ means $y<0$.

If $y>0$, then $\int_{-i}^{i} = \oint$

If $x>0$, then $\int_{-i\infty}^{-i} + \int_{i}^{i\infty} + \oint = 0$

Looks like $\boxed{x, y > 0 \Rightarrow \int_{-i\infty}^{i\infty} e^{xs+ys^{-1}} \frac{ds}{2\pi i s} = 0}$

$x<0, y>0.$ $\quad {}_{y>0}\int_{-i}^{i} = \oint$

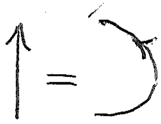
$x<0$ $\quad \int_{-i\infty}^{-i} + \oint + \int_{i}^{i\infty} = 0$

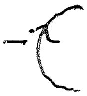
$$\int_{-i\infty}^{-i} + \int_{-i}^{i} + \int_{i}^{i\infty} = \oint - \oint = \oint = -J_0(xy)$$

$x<0,\ y<0$

get 0

$x>0,\ y<0.$



$= -$ 

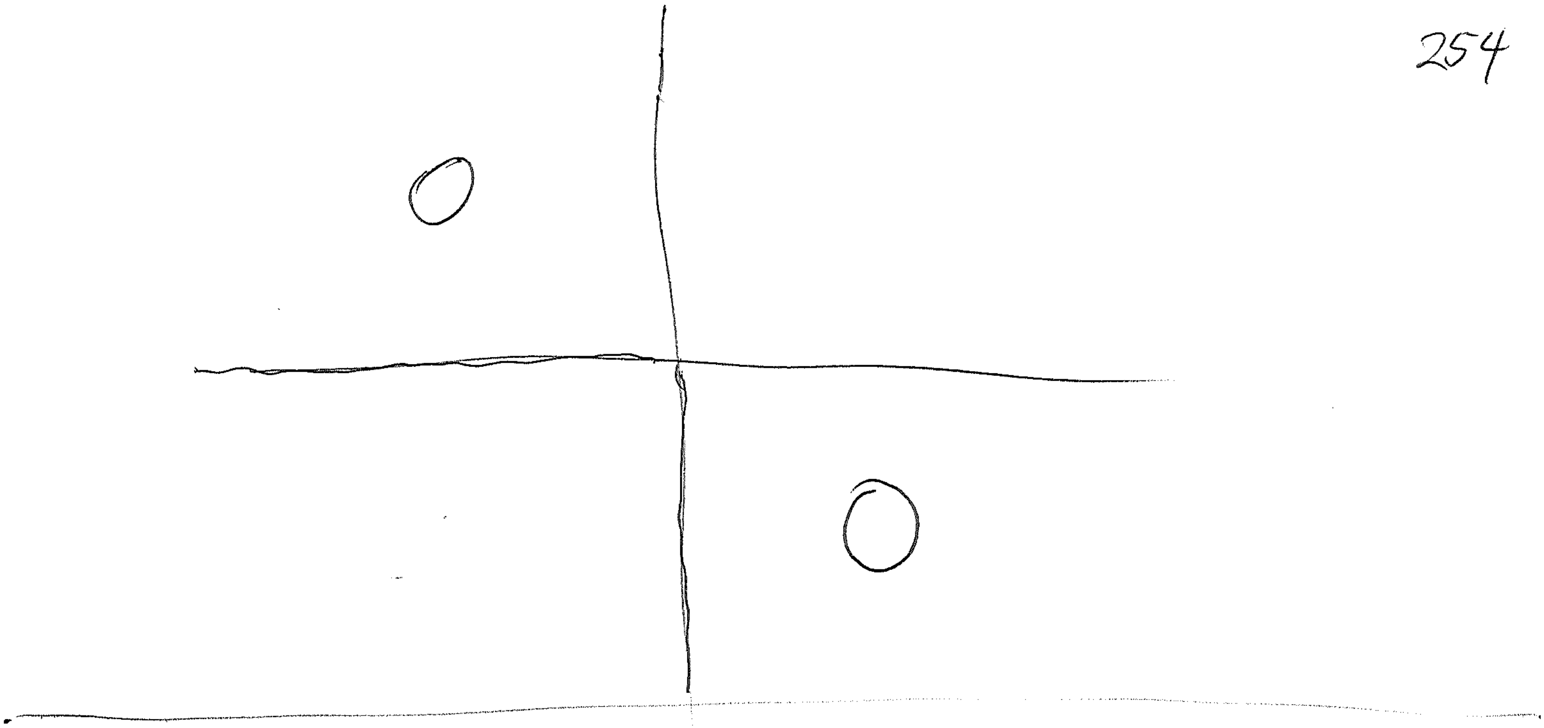
$x>0,\ y<0$ get $\displaystyle\int_{-i\infty}^{i\infty} e^{xs+ys^{-1}}\frac{ds}{2\pi i s} = J_0(xy)$

$-J_0(xy)$	0
0	$J_0(xy)$

$$\left(e^{xs}\,\middle|\,e^{ys^{-1}}s^{-1}\right) = 0 \quad \text{if } x>0,\ y<0 \text{ or } x<0,\ y>0$$

$$= J_0(-xy) \quad \text{if } x<0,\ y<0$$

$$= -J_0(-xy) \quad \text{if } x>0,\ y>0$$



Check a few things.

$$\int_{0^+-i\infty}^{0^++i\infty} = \int_{-i\infty}^{i\infty} + \int_{D}$$

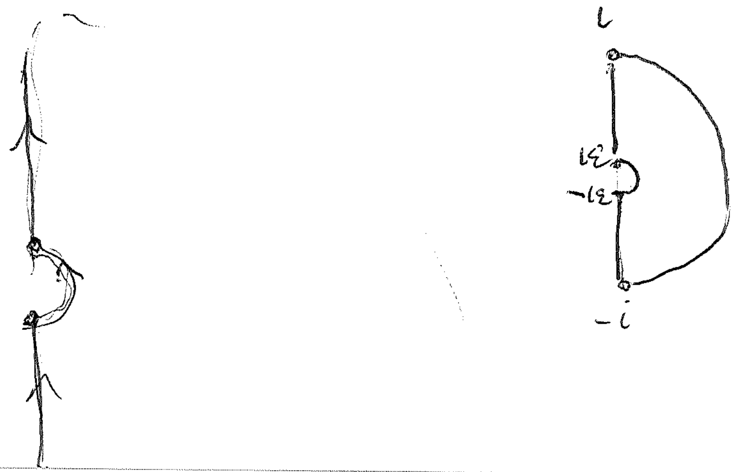
vanishing for $x<0$

$$\int_{D} e^{xs+ys^{-1}} \frac{ds}{2\pi i s} = 0 \quad \text{for} \quad y<0$$

$x<0$

$$\int_{0^+-i\infty}^{0^++i\infty} e^{xs+ys^{-1}} \frac{ds}{2\pi i s} = J_0(xy)\,H(x)$$

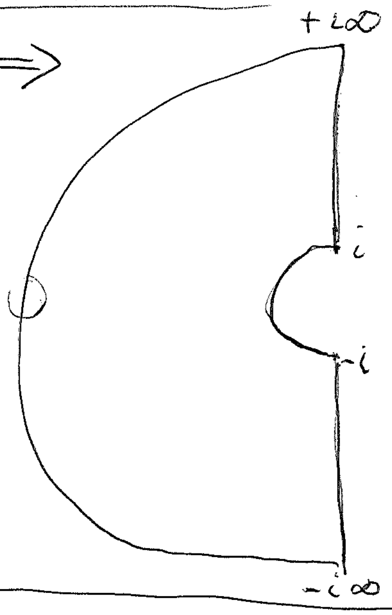
$x<0$



Can I do this better for tomorrow

$$\int e^{xs+ys^{-1}} \frac{ds}{2\pi i s}$$

If $x>0 \Longrightarrow$



$$\int_{-i\infty}^{-i} + \int_{-i}^{i} + \int_{i}^{+i\infty} = 0$$

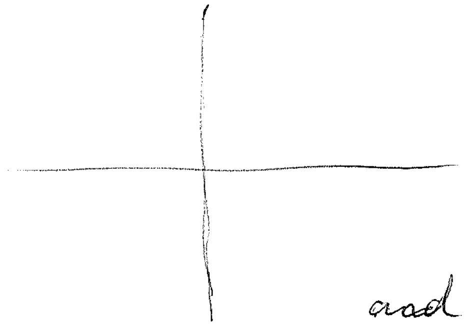
$$\int_{-i\infty}^{-i} + \int_{i}^{i\infty} = \begin{cases} \int_{-i}^{i} & \text{if } x>0 \\ \int_{-i}^{i} & \text{if } x<0 \end{cases}$$

$$\int_{-i}^{i} = \begin{cases} \int_{-i}^{+i} & \text{if } y<0 \\ \int_{-i}^{i} & \text{if } y>0 \end{cases}$$

$$\therefore \int_{-i\infty}^{+i\infty} = 0 \quad \text{if } xy>0$$

$$= J_0(xy) \quad \text{if } x>0,\ y<0$$

$$-J_0(xy) \quad \text{if } x<0,\ y>0$$



(You want to determine the herm. inner product.

You should give a residue formula and check it

Proceed first for (|). $$(f|g) = \int_{-i\infty}^{i\infty} f^* g \frac{ds}{2\pi i}$$

If $f^*(s)$ is analytic, then $f^*(s) = \overline{f(-\bar{s})}$

if $s \in i\mathbb{R}$ $f^*(s) = \overline{f(s)}$.

$$\int_{-\infty}^{\infty} \overline{f(ik)}\, g(ik) \frac{dk}{2\pi}$$

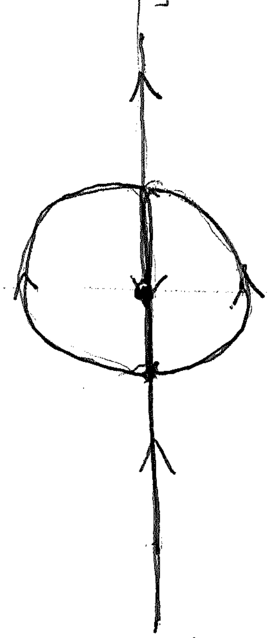
$$\left(\hat{\varphi}_1(s) \middle| \hat{\varphi}_2(s)\right) =$$

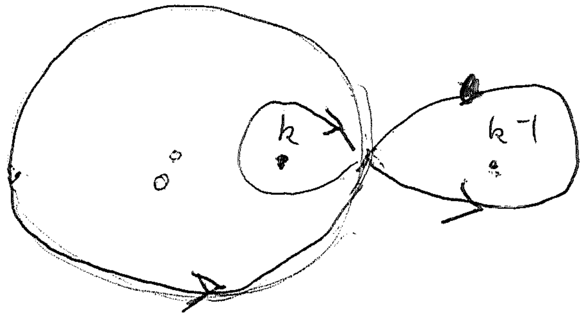
$$\hat{\varphi}_j(s) = \int e^{xs} \varphi_j(x)\, dx$$

$$\hat{\varphi}_j(ik) = \int e^{ikx} \varphi_j(x)\, dx$$

---

what's the analog of $\mathrm{res}_{\{0,k^{-1}\}} = \mathrm{res}_{\{0,k\}} + \mathrm{res}_{\{k^{-1}\}} - \mathrm{res}_{\{k\}}$.



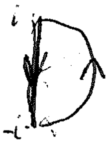


$$\int_{-i\infty}^{-i} + \int_{i}^{\;} - \int_{-i}^{i} + \int_{i}^{\;} + \int_{i}^{+i\infty}$$

$i \curvearrowright -i$ $\qquad$ i, $-i$

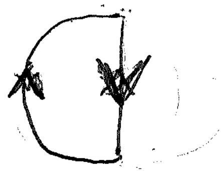
$$= \int_{-i\infty}^{i\infty} + \int + \int$$

$$\int = \begin{cases} 0 & \text{if } y<0 \\ J_0(xy) & \text{if } y>0 \end{cases}$$



$J_0(xy)$	$J_0(xy)$
0	0

$$\int = \begin{cases} -J_0(xy) & \text{if } y<0 \\ 0 & \text{if } y>0 \end{cases}$$



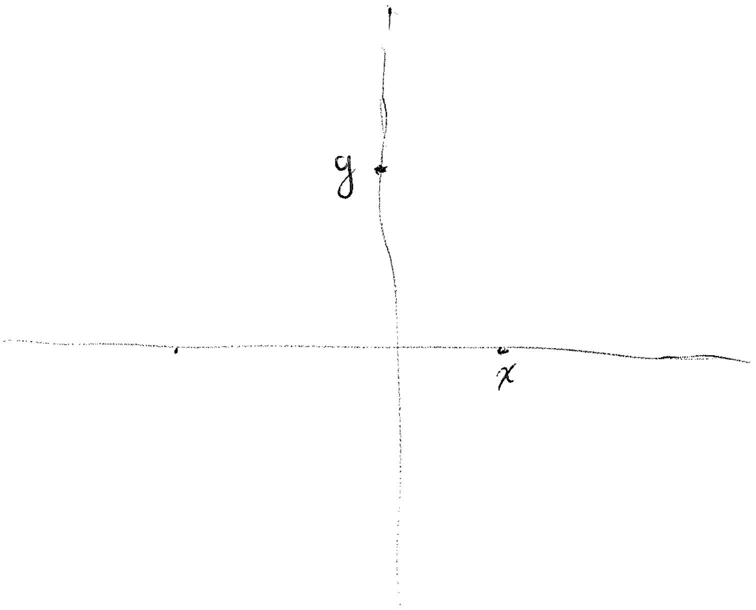
0	0
$-J_0(xy)$	$-J_0(xy)$

$$\int_{-i\infty}^{i\infty} = \begin{cases} 0 & \text{if } xy>0 \\ J_0(xy) & x>0 \quad y<0 \\ -J_0(xy) & x<0 \quad y>0 \end{cases}$$

$-J_0(xy)$	0
0	$J_0(xy)$

$$\text{Sum} = \begin{cases} J_0(xy) & x>0 \quad y>0 \\ 0 & x>0 \quad y<0 \\ 0 & x<0 \quad y>0 \\ -J_0(xy) & x<0 \quad y<0 \end{cases}$$

0	$J_0(xy)$
$-J_0(xy)$	0



$J_0(-xy)$	0
0	$-J_0(-xy)$

inverse scattering, go back to the idea of the fixed vectors ξ'_-, ξ_- being projected orthogonally with respect to the filtration

$\xi'_- z^x H_+ + \xi_- H_+$, decreasing as x increases

First maybe look at the cont. version of orth polys. Start with

$$IH(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

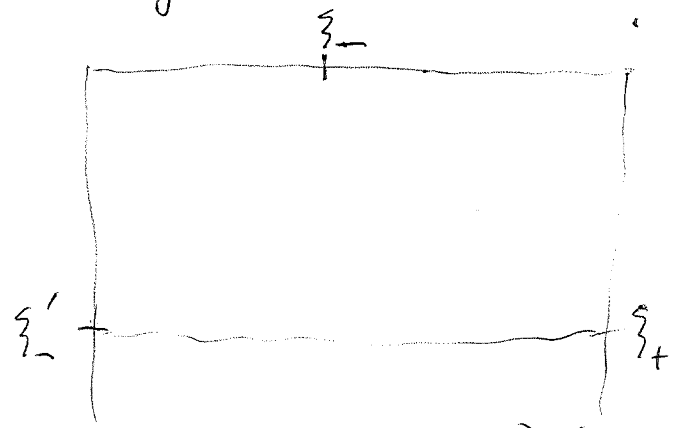
Find $(\xi_- L^2)^\perp$ $\qquad 0 = \int \begin{pmatrix} 0 \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \qquad bf = g$

you want $(\xi'_- z^x H_+ + \xi_- L^2)^\perp$

$$0 = \int \begin{pmatrix} z^x H_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \qquad bf = g$$

so you know that $(\xi'_- - \xi_- b) f \perp \xi_- L^2$

~~next to~~ point will be that you have some vector maybe $\xi'_- - \xi_- b$?



hunt for

$\xi_+ = \xi'_-(1-f) + \xi_-(-g)$

$f \in H_+$, $g \in L^2$.

$\xi_+ \perp \xi'_- H_+ + \xi_- L^2$

$$\int \begin{pmatrix} H_+ \\ L^2 \end{pmatrix} \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1-f \\ -g \end{pmatrix}$$

$\pi_+(1 - f - \bar{b} g) = 0$

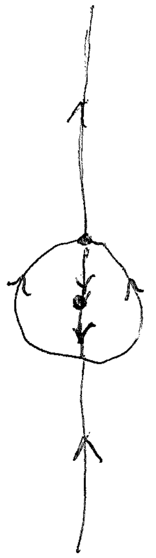
$b(1-f) + g = 0$

$$\begin{pmatrix} \varepsilon_+^* & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}\begin{pmatrix} \varepsilon_+ f \\ g \end{pmatrix} = \begin{pmatrix} \varepsilon_+^* & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\begin{pmatrix} 1 & \varepsilon_+^* \bar b \\ b\varepsilon_+ & -1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\int_{-i\infty}^{i\infty} + \int_{D} + \int_{C} \left(\frac{-1}{s}\right) e^{(-y'+y)s^{-1}} \frac{ds}{2\pi i s}$$

$$\cancel{s}\, e^{(-y'+y)s^{-1}} \frac{ds}{2\pi i \cancel{s}}$$

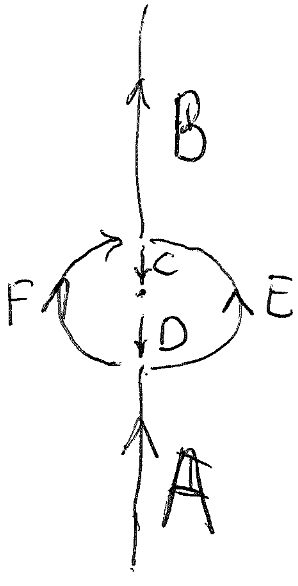


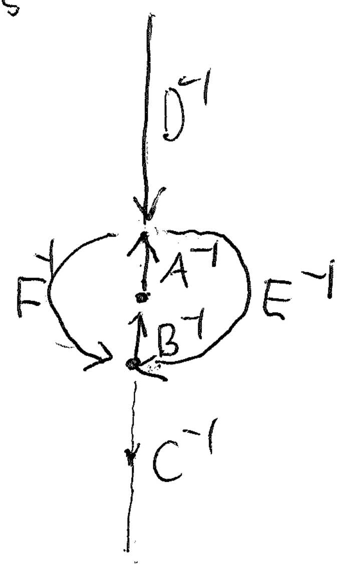
$$\int_{i}^{+i\infty} \rightsquigarrow \int_{-i}^{-i\infty}$$

PAPER

what is $\displaystyle\int_{D} + \int_{C} e^{(-x'+x)s} \frac{ds}{2\pi i}$

$s \mapsto s^{-1}$





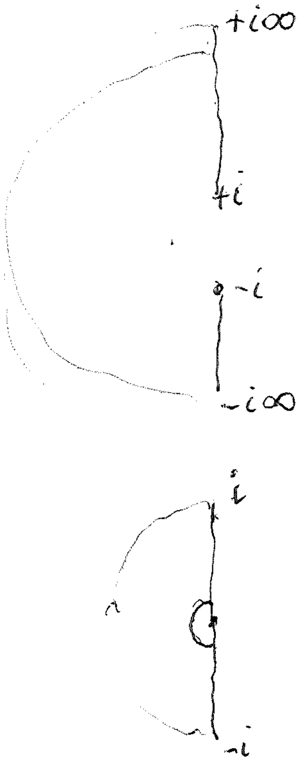
$-i \quad (+\infty \to 1)$

$i(-\infty)$

$(-i)0^+ \qquad (-i)(1)$

$i\infty^+ \qquad i1$

$$\int e^{xs+ys^{-1}}\,\frac{ds}{2\pi i s}$$ over various contours



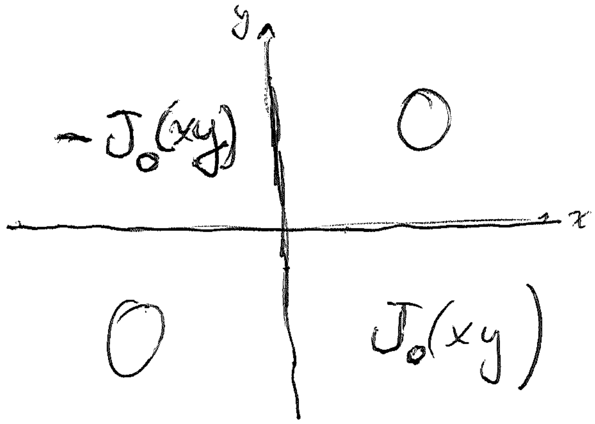
$$x>0 \implies \int_{-i\infty}^{-i} + \int_{i}^{+i\infty} = \int_{-i}^{i}$$

$$x<0 \implies \quad '' \quad = \int$$

$$y>0 \implies \int_{-i}^{i} = \int$$

$$y<0 \implies \int_{-i}^{i} = \int$$

$$\int_{-i\infty}^{i\infty} e^{xs+ys^{-1}}\,\frac{ds}{2\pi i s} =$$



anyway back to inverse scattering.

$\rho \frac{d\theta}{2\pi}$ smooth pos. $\rho > 0$ measure on S^1.

~~[illegible]~~ no, you want $\rho \frac{dk}{2\pi}$ on $\mathbb{R}$.

"sequence" of powers $z^x = e^{ikx}$ $x > 0$.

Look for $p_x = z^x - \int_0^x z^{x'} f(x,x')\,dx'$

$$(z^y | p_x^{\mp}) = \int e^{-iky} e^{ikx} \frac{dk}{2\pi} - \int_0^x \left(\int e^{-iky} e^{ikx'} f(x,x') \right) dx'$$

$$\cancel{(z^y|p_x) = \delta(x-y) - \int_0^x \underbrace{\left(\int \frac{dk}{2\pi} e^{-iky + ikx'}\right)}_{\delta(x'-y)} f(x,x')\,dx'}$$

$$\cancel{= \delta(x-y) - \int_0^x f(x,y)}$$

$$p_x = z^x - \int_0^x f(x,x') z^{x'}\,dx'$$

$$(z^y | p_x) = \int e^{ik(x-y)} \rho(k) \frac{dk}{2\pi} - \int_0^x dx'\, f(x,x') \int e^{ik(x'-y)} \rho(k) \frac{dk}{2\pi}$$

---

given $d\mu = \rho(k) \frac{dk}{2\pi}$ can you find

$$p_x = e^{ikx} + \int_0^x e^{ikx'} g(x')\,dx'$$

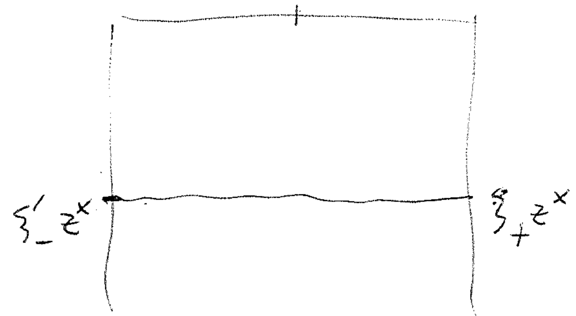
to be $\perp$ to e^{iky} $\forall$ $0 < y < x$.

$$\int e^{-iky} p_x(k) \frac{dk}{2\pi} = \cancel{0} + \int_0^x \underbrace{\left(\int_{-\infty}^{\infty} e^{-iky} e^{ikx'} \rho(k) \frac{dk}{2\pi} \right)}_{\hat{\rho}(x'-y)} g^{\mp}(x')\,dx'$$

$$\underbrace{\int \frac{dk}{2\pi} e^{-iky} e^{ikx} \rho(k) \frac{dk}{2\pi}}_{\hat{\rho}(x-y)}$$

$\therefore$ want $\quad 0 = \hat{\rho}(x-y) + \int_0^x \hat{\rho}(x'-y) g(x')\,dx'$

Set up a simpler problem. For each x you want to project $\xi_-' z^x$ perp to $\xi_-' z^x H_+ + \xi_- L^2$



~~any~~ this problem should be ind of x.

$$\xi_-' z^x(1-f_x) + \xi_-(-g_x)$$

$$= \xi_-' z^x - \left(\xi_-' z^x f_x + \xi_- g_x\right)$$

$$\int \begin{pmatrix} z^x H_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} z^x(1-f_x) \\ -g_x \end{pmatrix} = 0$$

$$\int \begin{pmatrix} H_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} z^{-x} \\ b z^x & -1 \end{pmatrix} \begin{pmatrix} 1-f_x \\ -g_x \end{pmatrix} = 0$$

$$\mathcal{E}_+^*\left(1-f_x - z^{-x}\bar{b} g_x\right) = 0$$

$$b z^x(1-f_x) + g_x = 0$$

$$g_x = -z^x b(1-f_x)$$

$$\mathcal{E}_+^*\left(1-f_x + |b|^2(1-f_x)\right) = 0$$

$$\mathcal{E}_+^*\left[(1+|b|^2)(1-f_x)\right] = 0$$

Return then to projecting $\xi_-' z^x, \xi_-$ onto & $\perp$ to $\xi_-' z^x H_+ + \xi_- H_+$

$$\begin{tikzcd}
zF_{n-1} \arrow[r, dash, "q_n"] \arrow[d, dash, "zp_{n-1}"'] & F_n \arrow[d, dash, "p_n"] \\
zF_{n-2} \arrow[r, dash, "q_{n-1}"'] & F_{n-1}
\end{tikzcd}$$

$$\begin{pmatrix}p_n\\ q_{n-1}\end{pmatrix} = \underbrace{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}}_{\in U(2)}\begin{pmatrix}zp_{n-1}\\ q_n\end{pmatrix}$$

q_{n-1}, q_n have >0 const. terms $\Rightarrow \delta > 0$

$$\begin{pmatrix}p_n\\ q_n\end{pmatrix} = \begin{pmatrix}\frac{\alpha\delta-\beta\gamma}{\delta} & \frac{\beta}{\delta}\\ -\frac{\gamma}{\delta} & \frac{1}{\delta}\end{pmatrix}\begin{pmatrix}zp_{n-1}\\ q_{n-1}\end{pmatrix}$$

p_n, zp_{n-1} have >0 leading coeff $\Rightarrow \frac{\alpha\delta-\beta\gamma}{\delta} > 0$

$\Rightarrow \alpha\delta-\beta\gamma>0$, $|\ \ |=1$. $\therefore \begin{pmatrix}\alpha & \beta\\ \gamma & \underset{>0}{\delta}\end{pmatrix}\in SU(2)$

$$\begin{pmatrix}\bar\alpha & \bar\gamma\\ \bar\beta & \bar\delta=\delta\end{pmatrix} = \begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}$$

$\alpha=\delta$ $\quad \bar\gamma=-\beta$

$$\begin{pmatrix}\delta & \beta\\ -\bar\beta & \delta\end{pmatrix} \qquad \delta = (1+|\beta|^2)^{1/2}$$

Cont. case? measure $d\mu(k)$ on $\mathbb{R}$ say $\rho(k)\frac{dk}{2\pi}$

$F_{<x}$ spanned by $e^{ikx'}$ $0\le x'<x$.
consists of $\int_0^x e^{ikx'}\varphi(x')\,dx'$

there's some mysterious difference between this and $\mathbb{C}e^{ikx}+F_{<x}$. Then $p_x = e^{ikx}+\int_0^x e^{ikx'}\varphi(x')\,dx'$ $\perp F_{<x}$.

$$(e^{iky}|p_x) = (e^{iky}|e^{ikx}) + \int_0^x (e^{iky}|e^{ikx'})\,\varphi(x')\,dx'.$$

where $\left(e^{iky} \middle| e^{ikx'}\right) = \int e^{-iky+ikx'} \rho(k) \frac{dk}{2\pi} = f(u$

$$\underline{0 = f(-y+x) + \int_0^x f(-y+x')\varphi(x')dx'} \qquad \forall\, y<x$$

$$(z^i | z^j) = \int z^{-i+j}\, d\mu = \mu(-i+j)$$

$$\tilde{p}_n = \sum_{j\leq n} c_{nj} z^j \qquad c_{nn} = 1.$$

$$= z^n + \sum_{j<n} c_{nj} z^j$$

$$0 = (z^i | \tilde{p}_n) = \underbrace{(z^i | z^n)} + \sum_{j<n} c_{nj} (z^i | z^j) \qquad i<n$$

$$0 = \mu(-i+n) + \sum_{j<n} c_{nj}\, \mu(-i+j)$$

$$\boxed{0 = \mu(n-i) + \sum_{j<n} c_{nj}\, \mu(j-i) \qquad \forall i<n}$$

$$\cancel{0 = \mu(-i+n) + \sum_{j<n} \mu(-i+j) c_{nj}}$$

You need to look carefully at GS.

Someone gives you a pos. def. matrix

Basic $1, z, z^2, \ldots$ put in unitary symmetry later.

GS ~~...~~ Split invertible matrix into unitary times upper triangular.

What is Gram Schmidt about? ~~Q~~ Involves a filtration and an inner product. Use inner product to split a flag. $0 \subset F_1 \subset F_2$

$$G/B = K/T$$

$B \longleftrightarrow$ Flag
~~K~~ $K \longleftrightarrow$ inner product.
? T disappears

What is continuous analog?
Filtration = self-adjoint op by spectral thm.
flag = mult. one filtration. B upper triangular matrices should becomes kernels

$$\delta(x,x') + k(x,x') \qquad \text{Volterra operator}$$

hermitian ~~quadratic~~ form. $\delta(x,x') + h(x,x')$.
Ultimately you start with $f(x)$ $\quad 0 \le x \le 1$.

OK begin again - Given filtration

$$0 \subset F_1 \subset F_2 \subset \cdots$$

with ~~[illegible]~~ 1 diml quotients. Always the flag is given ~~[illegible]~~ in terms of an initial basis.
$e^{ikx} = z^x$ instead of z^n. In our situation given the "basis" $e^{ikx}, 0 \le x$, the filtration is then

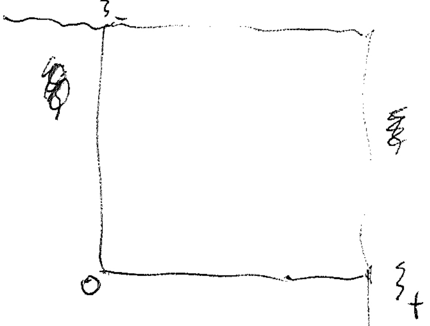
$$F_x = \left\{ \int_0^x e^{ikx'} \varphi(x')\,dx' \right\} \quad \text{with } \varphi \in L^2.$$

In addition ~~you~~ are given $(z^{x'} | z^x) = \int z^{x-x'}\,d\mu$, i.e. a hermitian matrix

---

What assumptions on $d\mu$? appropriate to scattering.

$$\partial_x \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} 0 & z^{-x} h(x) \\ \bar{h}(x) z^x & 0 \end{pmatrix} \begin{pmatrix} z^{-x} p_x \\ q_x \end{pmatrix} \qquad \partial_x \begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h(x) \\ \bar{h}(x) & 0 \end{pmatrix} \begin{pmatrix} p_x \\ q_x \end{pmatrix}$$



$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

$$\xi_+ = S\xi_- \qquad S = \frac{a+b}{c+d}$$

You have a unitary repn of R with 3 cyclic vectors, namely ξ_+, ξ_-, $\xi_0 = p_0 = q_0$. The last gives a repn at L^2 some measure. $(\xi_0 | e^{ikx}\xi_0) = \int e^{ikx} d\mu(k)$

but $\xi_+ = (a+b)\xi_0$ $\qquad \delta(x) = (\xi_+ | e^{ikx}\xi_+) = \underbrace{(\xi_0 | e^{ikx} |a+b|^2 \xi_0)}_{\int e^{ikx} |a+b|^2 d\mu}$

$\xi_- = (c+d)\xi_0$

$$\Rightarrow \quad d\mu = \frac{1}{|a+b|^2}\frac{dk}{2\pi} \quad \text{also} \quad \frac{1}{|c+d|^2}\frac{dk}{2\pi}$$

You want to ~~[illegible]~~ understand the form of the ~~[illegible]~~ inner product: Clear

$$(e^{ikx} | e^{iky}) = \int_{-\infty}^{\infty} e^{ik(-x+y)} \underbrace{d\mu(k)}_{\rho(k)\frac{dk}{2\pi}} \qquad \rho(k) = \frac{1}{|a+b|^2}$$

But about $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ you know it $\in \begin{pmatrix} 1+H_- & H_- \\ H_+ & 1+H_+ \end{pmatrix}$

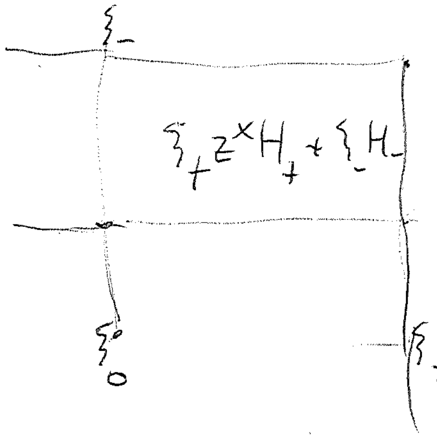
$$\begin{pmatrix} 0 & z^{-x}h(x) \\ h(x)z^{x} & 0 \end{pmatrix} \quad x>0 \qquad \text{so } a+b \in 1+H_- \quad c+d \in 1+H_+$$

$\rho(k)$ You need some Hardy space knowledge but basically ~~[illegible]~~ $\hat{\rho}(x)$ should $\in \delta(x) + k(x)$

In fact $\frac{1}{c+d}$ should be in $1+H_+$, $\frac{1}{a+b} \in 1+H_-$

so $\rho \in 1+$. But you want the inverse direction.

So let's begin with the measure $\rho(k)\frac{dk}{2\pi}$, or do we want the scattering for $S(k)$? The relation is



$$\xi_+ = (a+b)\xi_0 \qquad S\xi_- = \xi_+ \qquad S = \frac{a+b}{c+d}$$
$$\xi_- = (c+d)\xi_0$$

You have two orthogonalization methods, better, two filtrations - from the upper right, or lower left corner.

From upper right the filt is $\xi_+ z^x H_+ + \xi_- H_-$

Need $\|\xi_+ f + \xi_- g\|^2$. Use the unitary isom

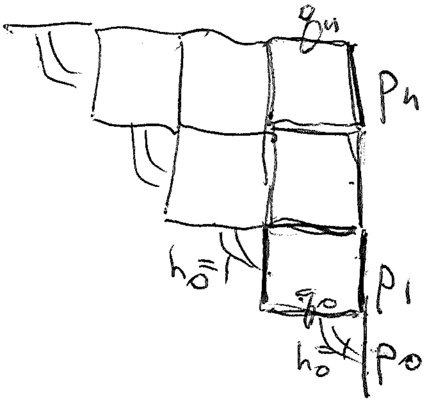
$$E = L^2(\mathbb{R}, \frac{dk}{2\pi}) \qquad \xi_+ f + \xi_- g \longmapsto Sf + g$$
$$\xi_- \longmapsto 1, \quad \xi_+ \longmapsto S$$

$$\|\xi_+ f + \xi_- g\|^2 = \|Sf + g\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

not obviously pos. def, but should be once you restrict to appropriate ~~half~~ spaces.

But you want to understand the filtration from the lower left, which means ? In the discrete case $p_0, \dots, p_n$ same span as $1, z, \dots, z^n$.

So the lower left region is a 



In the cont case you get span of $e^{ikx'}$ for $0 \le x' < x$

$$F_{x} = \left\{ \int_0^x e^{ikx'} \varphi(x') dx' \right\}$$

$$p_x = e^{ikx} + \int_0^x e^{ikx'} \alpha(x,x') dx'$$

$$q_x = 1 + \int_0^x e^{ikx'} \beta(x,x'$$

$$p_x = e^{ikx} + \int_0^x e^{iky} \alpha_x(y) dy$$

$$\left(e^{ikx'} \middle| p_x\right) = \int e^{ik(x-x')} d\mu + \int_0^{x} \left(\int e^{ik(y-x')} d\mu\right) \alpha_x(y)\, dy$$

make good notation

$$p_x = e^{ikx} + \int_0^x \alpha_{xy} e^{iky} dy$$

$$p_x = \int_0^x z^y \alpha_{yx}\, dy + z^x$$

$$\left(z^{x'} \middle| p_x\right) = \boxed{\int_0^x h_{x'y} \alpha_{yx}\, dy + h_{x'x} = 0} \quad \text{for} \quad x' < x$$

You're given h_{xy} hermitian positive. To find α_{yx} triangular: supp $y < x$

$$\sum_{0 \le y < \infty} h_{x'y} \alpha_{yx} + h_{x'x} = 0 \qquad \forall x' < x$$

first straighten out the lin alg.

$(h_{jk}) = (z^j | z^k)$ herm. "non deg".

$$p_k = z^k + \sum_{j<k} z^j t_{jk}$$

$$0 = (z^i | p_k) = h_{ik} + \sum_{j<k} h_{ij} t_{jk} \qquad \forall i < k$$

---

h_{jk} defined for all $j\,k$.

t_{jk} defined for $j \le k$, $t_{kk} = 1$.

$$0 = \sum_{j \le k} h_{ij} t_{jk} \qquad \forall i < k.$$

Pos. def matrix h_{jk} $\quad 1 \le j,k \le n$ $\quad$ ~~n^2 dim $\mathbb{R}$~~

triangular t_{jk} $\quad 1 \le j \le k \le n$ $\quad \frac{n(n-1)}{2}$ dim $\mathbb{C}$

with $t_{jj} = 1$.

It's amazing how hard you find this

You have (h_{jk}) given symmetric

$$(h_{jk}) = (z^j | z^k)$$

t_{jk} for $j<k$ extended so that $t_{kk} = 1$.

$$p_k = \sum_{j<k} z^j t_{jk} + z^k = \sum_{j \le k} z^j t_{jk}$$

$$0 = (z^i | p_k) = \sum_{j<k} h_{ij} t_{jk} + h_{ik}$$

$i < k$

You should now embark upon understanding the analysis beyond orth. projection. First discrete case. ~~that is~~ begin with basis vectors

e_n, $n \geq 0$ ~~[illegible]~~ for a vector space V eg $\mathbb{C}[z]$ with $e_n = z^n$, $n \in \mathbb{N}$, a herm. form on V whence a herm. symm. matrix $h_{mn} = (e_m | e_n)$. Then you want to construct from $(e_n, n \in \mathbb{N}) = (e_0, e_1, \ldots)$ an orth. sequence $\boxed{p_n = \sum_{0 \leq m < n} e_m t_{mn} + e_n}$, this is a new basis, so $(p_{n'} | p_n) = 0$ $n' \neq n$ iff $(e_j | p_n) = 0$ for all $j < n$, i.e.

$$0 = \sum_{m<n} h_{jm} t_{mn} + h_{jn} \qquad \forall j < n.$$

Get one equation for each j, n with $0 \leq j < n$. But you can solve them one at a time for each n. Why. For a given n you have n equations $(\forall j < n)$ and n unknowns (t_{mn}) $(\forall m < n)$. The matrix of coeffs $(h_{jm})_{j,m<n}$ is nondeg by positivity.

Transpose to cont. case and try same arguments. ~~This time you begin with~~ Formally replace e_n by e_x for $x \in \mathbb{R}_{\geq 0}$, concretely suppose $d\mu(k)$ given on $\mathbb{R}$, $e_x = e^{ikx}$, and

$$h_{yx} = (e_x | e_y) = \int e^{-iky + ikx} \, d\mu(k).$$

Actually you work inside $L^2(\mathbb{R}, d\mu)$ with ~~the~~ basis e^{ikx}, $x \geq 0$ for $H^2(\mathbb{R}, d\mu)$. Next put

$$p_x = \int_0^x e_y t_{yx} \, dy + e_x$$

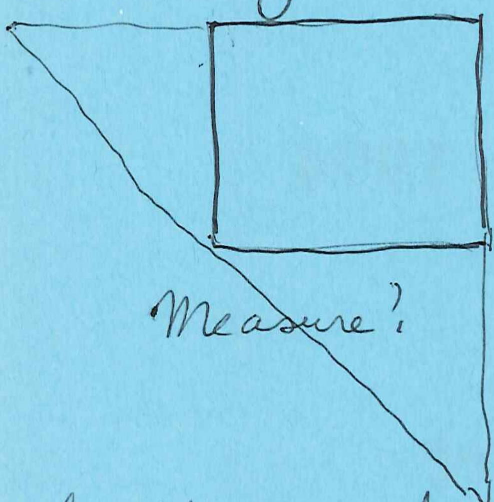
to choose $y \mapsto t_{yx}$ so that $(e_b | p_x) = 0$ $\forall b < x$

$$0 = \int_0^x h_{by}\, t_{yx}\, dy + h_{bx} = 0 \qquad \forall\, b < x$$

You want to solve for each x. t_{yx} is a function on $[0,x)$. Need to formulate ~~undeterminacy~~ nondegeneracy. You have this inner product corresp to "matrix" $h_{yx} = ($

Things are not trivial because e_x is not usually in the $L^2(d\mu)$. What's happening. you have the subspace F_x generated by the e_x or really $\int_0^x e_y \varphi(y)\, dy$ with $\varphi \in C_0^\infty$, and you have a linear functional on this space namely $(e_x | -)$. This issue is to represent this linear fnl by $\int_0^x e_y t_{yx}\, dy$

Next specify the $d\mu$ appropriate to scattering.



Measure?

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \lim_{x \to +\infty} \begin{pmatrix} z^{-x} p_x \\ q^x \end{pmatrix} = \begin{array}{cc} 1+H_- & H_-^* \\ \multicolumn{2}{c}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \\ H_+ & 1+H_+ \end{array} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Look in $\xi_- L^2$ for the elt corresp to $\xi_0 = p_0 = q_0 = e_0$ $\qquad \xi_- = (c+d)e_0$

$e_0 = \xi_- \frac{1}{c+d}$ so the measure should be $d\mu = \frac{1}{|c+d|^2}\frac{dk}{2\pi}$

$$(e_0 | e^{ikx} e_0) = \int e^{ikx}\, d\mu$$

Review: Aiming for cont. version of Szegö when F_x (roughly) spanned by $e_y = e^{iky}$ $0 \le y < x$

and $h_{xy} = (e_x | e_y) = \int e^{-ikx + iky} \, d\mu(k)$. ~~integral~~

$$p_x = \int_0^x e_y \, t_{yx} \, dy + e_x \, .$$

Basically you understand notation but not analysis. You learned zilch. ~~But the rules~~

What you can do: In discrete case you ask for $p_x = \sum_{y<x} e_y t_{yx} + e_x \perp F_x$?

$$p_x \perp F_x \iff \sum_{y<x} (e_{y'} | e_y) \, t_{yx} + (e_{y'} | e_x) = 0 \quad \forall y' < x$$

x fixed - unknowns for each $y < x$
eqn — — $y' < x$

The issue is the invertibility of $(e_{y'} | e_y)$ which results from positivity, better from representing linear fnl on the closed subspace F_x.

~~That~~ Try to proceed with these eqns. ~~means~~ This means ~~various~~ rep. of elements of F_x by

$\int_0^x e_y \, q_{yx} \, dy$ ~~etc~~ with $(e_y | e_x) = \int e^{+ik(-y+x)} d\mu(k)$

What form does $(e_y | e_x)$ have? e.g.

$d\mu(k) = \rho(k) \frac{dk}{2\pi}$. ~~No when You know~~

You have $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix}$ and $\rho(k) = \frac{1}{|c+d|^2}$

important, essential: $\rho(k) \sim 1$ as $|k| \to \infty$.

so that $\tilde{\rho}(x) = \int e^{ikx}\rho(k)\frac{dk}{2\pi} = \delta(x) + \cdots$

You may assume $\hat{\rho} - \delta$ smooth compact support.
Set up equations. ~~[illegible]~~

$$p_x = e^{ikx} + \int_0^x f(x,x')\,e^{ikx'}\,dx'$$

$$(e^{iky}|p_x) = \delta(x-y) + \sigma(x-y) + \int_0^x \big(\delta(x'-y) + \sigma(x'-y)\big) f(x,x')\,dx'$$

$$(e^{iky}|e^{ikx}) = \int e^{-iky+ikx}\rho(k)\frac{dk}{2\pi}$$

$$= \delta(x-y) + \sigma(x-y).$$

so you end up with

$$0 = -\sigma(x-y) + f(x,y) - \int_0^x f(x,x')\,\sigma(x'-y)\,dx' \qquad 0<y<x$$

Here you should consider x fixed, ~~this~~ $\sigma(x'-y)$ is the inner product matrix $f(x,y)$ for $0 \le y \le x$ is unknown

Question, Idea: When is $\delta(x) + \sigma(x)$ positive-definite? non unital versions in the continuous case. Another version - when is $\delta(x-y) + k(x,y)$ a positive matrix?

$$(1+f)(1-\sigma) = 1$$

$$f - f\sigma - \sigma = 0$$

$$f = \sigma + f\sigma$$

It's hard to see the positivity.

What you forgot. ~~[illegible]~~
the underlying topological vector space (Krein space) and filtration ~~[illegible]~~ do not depend on, are independent of, the measure etc.

Different approach. Focus on the splitting

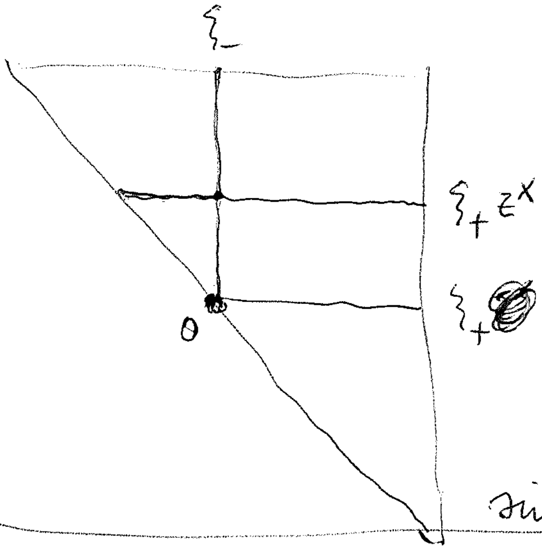
First try $L^2(\mathbb{R}, \rho \frac{dk}{2\pi})$ $\quad \rho(k)$ smooth > 0, $= 1$ far out

Work in $L^2(\mathbb{R}, \frac{dk}{2\pi}) = H_+^2 \oplus H_-^2$

$$\|f\|^2 = \int f^* \rho f \, \frac{dk}{2\pi}$$

$$F_x = H_+ \cap e^{ikx} H_- = \left\{ \int_0^x e^{ikx'} \varphi(x')\,dx' \;\middle|\; \varphi \in L^2([0,x]),\ \mathrm{Supp}\,\varphi \subset [0,x] \right\}$$

to understand, examine $\quad H_+ = F_x \oplus F_x^\perp$



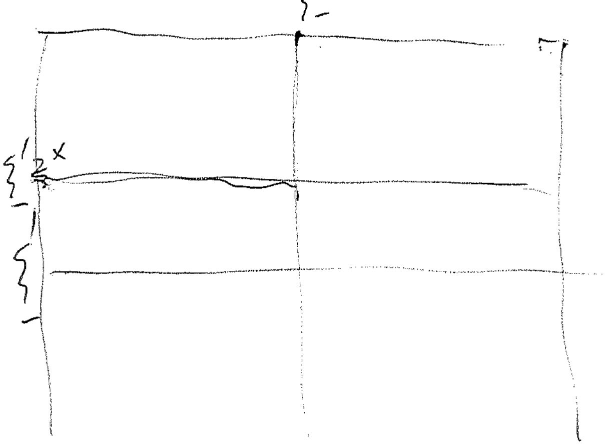
Conj. $F_x^\perp = \xi_+ z^x H_+ + \xi_- H_-$

You expect

$F_x^\perp =$

Let's drop this example, since it's not that simple.

Instead



Instead consider $\xi'_- L^2 \oplus \xi_- L^2$ with trans. $e^{ikx} = z^x$, and

$$IH(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

Focus on the splitting. ~~[illegible]~~
You first want the projection onto F_x; this will be a bounded operator. ~~[illegible]~~ Up to now you look at p_x which involves δ function close but outside F_x. ~~Take~~ But proj defined for everything

~~[illegible]~~ Go back to $d\mu = \rho \frac{dk}{2\pi}$, $\rho > 0$ is now inner product.

Say again: you work in $L^2(\mathbb{R}, \frac{dk}{2\pi})$ with F_x = F.T. of $L^2((0,x), dx)$. Have

$$H_+ = \underbrace{\left(H_+ \cap e^{ikx} H_-\right)}_{F_x} + e^{ikx} H_+$$

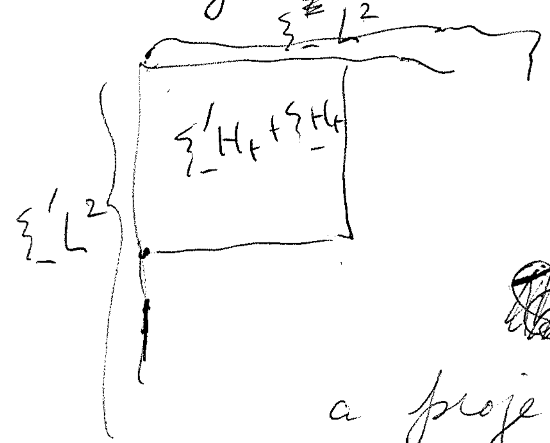
splitting of H_+ into complementary closed subspaces. Have $\int |f|^2 \rho \frac{dk}{2\pi}$ pos. herm. inn. product to get proj op.

$$\begin{tikzcd}
F_x \arrow[d, phantom, "\parallel"] & H_+ & e^{ikx}H_+ \arrow[d, phantom, "\cong"] \\
W \arrow[r, hook, "\varepsilon_x"] \arrow[d, "\varepsilon_x^* \rho \varepsilon_x"] & V \arrow[r] & V/W \\
W & V \arrow[l, "\varepsilon_x^*"'] &
\end{tikzcd}$$

so you get a nice proj. operator by inverting $\varepsilon_x^* \rho \varepsilon_x$. ~~[illegible]~~ Now you want to make this concrete. At some point you must deal with the H.T. How?

Simplest might be the scattering ~~situation~~ ~~[illegible]~~ involving the off diagonal function bz^x depending on x. ~~Fix the Hilb~~ Start with the splitting

Can you avoid p_x, q_x? δ-functions?



$$IH_b(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \frac{dk}{2\pi}$$

Fix half space $\xi'_- H_+ + \xi_- H_+$

~~There~~ For each b you get a projection op. on $\xi'_- L^2 + \xi_- L^2$ or an orth complement wrt IH_b. Given a ~~motion~~ path b_t, you ~~need~~ get a path in the Grass.

This is a very specific problem, b can be as nice as you want. Yes. Set up carefully. Try to understand what ~~ope~~ sort of object you have. For each b a factorization of the scattering matrix S_b. Given b you complete it to ~~$\frac{1}{d}$~~ $\frac{1}{d}\begin{pmatrix} 1 & b \\ -\bar b & 1 \end{pmatrix}$. Your fact. uses formulas.

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

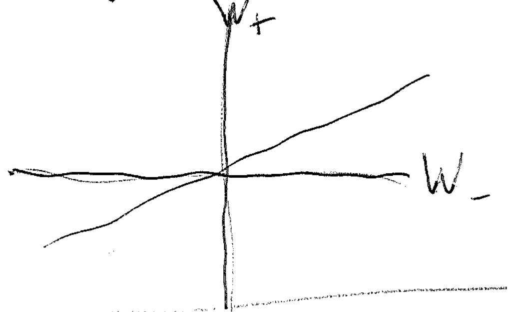
$$= \frac{1}{d} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix} = \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\frac{1}{d} \begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix} \frac{1}{d}$$

You are trying to do something. What?
For each b you get an indefinite hermitian inner product on $L^2 + L^2$

At the moment you have E constructed from b, can be described two copies of $L^2(\mathbb{R}, \frac{dk}{2\pi})$ in two ways, say incoming picture $\{\xi'_- L^2 + \xi_- L^2\} \ni I_b(\xi'_- f + \xi g) = \int \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}$

also the ~~filtration~~ closed subspace W_+ $\xi'_- H_+ + \xi_- H_+$. Restriction of I_b to W non-deg $\Longrightarrow E = W_+ \oplus W_+^{\perp_b}$. You do have already the complement $W_- = \xi'_- H_- + \xi_- H_-$ for W_+ so $W_+^{\perp_b}$ can be described as the graph of a map from W_- to W_+.



What's important?

Yesterday decided I needed to know Toeplitz better. Start with $V = L^2 \oplus L^2$ subspace $W_+ = H_+ \oplus H_+$, complement $W_- = H_- \oplus H_-$, put herm. form $I_b = (-|B-)$, $B = \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}$ on V, then its compr. to W_+ is $A = \begin{pmatrix} 1 & T_b^* \\ T_b & -1 \end{pmatrix}$ $\quad T_b = \varepsilon_+^* b \varepsilon_+$;

which is invertible. Call Then get projection

$V \longrightarrow W_+ \quad \xi \longmapsto \varepsilon_+ A^{-1} \varepsilon_+^* B \xi = \varepsilon_+ (\varepsilon_+^* B \varepsilon_+)^{-1} \varepsilon_+^* B \xi$

operator from $V/W^+ = W^-$ to W^+ whose graph is that to $\varepsilon_+ W_+$ wrt T_b.

So now you find ~~that~~ you need ~~also and~~ some analysis You have defined an operator on the Hilb space level.

Viewpoint: Working with Hardy space, or Hilbert transform, should be closely ~~to~~ related to Laplace transform. e.g.

$$\hat{\varphi}(k) = \int_0^\infty e^{iky} \varphi(y)\, dy$$

$$\psi(x) = \int_0^\infty \left(\int_{-\infty}^{\infty} e^{-ikx} b(k) e^{iky} \frac{dk}{2\pi} \right) \varphi(y)\, dy$$

$$= \int_0^\infty \hat{b}(x-y) \varphi(y)\, dy$$

You look at $\varepsilon_+^* b \varepsilon_+$, Toeplitz op, $b(k)$ is a function of k, the Toeplitz operator is convolution by $\hat{b}$ then rest. to $x \geq 0$. How to get control over the analysis.

Look at $\begin{pmatrix} 1 & T_b^* \\ T_b & -1 \end{pmatrix}$ $\qquad T_b^* = (\varepsilon^* b \varepsilon)^* = \varepsilon^* \bar{b} \varepsilon$

~~Not need to get~~

$$\begin{pmatrix} 1 & T^* \\ T & -1 \end{pmatrix}^2 = \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}$$

automatically invertible

Fight with this. ~~Why~~ How do you know invertibility of $1+T^*T$?

$$\Gamma_T = \begin{pmatrix}1\\T\end{pmatrix}H \subset \begin{pmatrix}H\\H\end{pmatrix}$$

How can you get control of a Toeplitz op, $b(k)$ assumed rapidly decreasing

What do you know?

$$\begin{tikzcd} W_+ \arrow[r, hook, "\varepsilon"] \arrow[d, "A"] & V \arrow[r] \arrow[d, "B"] & W_- \\ W_+ & V \arrow[l, "\varepsilon^*"'] & \end{tikzcd}$$

proj. $\quad v \longmapsto \varepsilon(\varepsilon^* B\varepsilon)^{-1}\varepsilon^* Bv$

$\qquad V \longrightarrow W_+$

you need this operator from W_- to W_+.

$$B^2 = \begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}^2 = 1+|b|^2$$

---

maybe handle things by C.T.

$$L^2\left(\mathbb{R}, \frac{dk}{2\pi}\right) \text{———} L^2\left(S^1, \frac{d\theta}{2\pi}\right)$$

$$H_{\pm} \longleftrightarrow H_{\pm}$$

$$f(k) \qquad\qquad f(z)$$

sections of $\mathcal{O}(-1)$, $s(z) = f(z)\begin{pmatrix}z\\1\end{pmatrix} \subset \mathbb{C}^2$

$$\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}^* f\right)(z) = f\left(\frac{az+b}{cz+d}\right) \sim \begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} \cancel{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}z\\1\end{pmatrix}}$$

$$\left(g^*(s)\right)(z) = g^{-1}s(g\cdot z) \qquad \begin{pmatrix}\frac{az+b}{cz+d}\\ 1\end{pmatrix} f\left(\frac{az+b}{cz+d}\right)$$

$$\mathcal{O}(-1) \hookrightarrow \mathcal{O}\otimes T \longrightarrow Q$$

$$\mathcal{O}\otimes\Lambda^2 T = \Lambda^2(\mathcal{O}\otimes T) = \mathcal{O}(-1)\otimes Q \quad ?$$

Claim $\mathcal{O}(-2) = \mathcal{O}(-1)^{\otimes 2} \simeq \Omega^1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*} f(z)\,dz = f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right) \qquad \frac{(cz+d)a\,dz - (az+b)c\,dz}{(cz+d)^2}$$

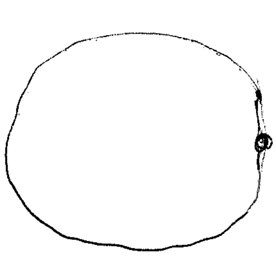
$$= f\left(\frac{az+b}{cz+d}\right)\frac{ad-bc}{(cz+d)^2}\,dz$$

$$\underbrace{f_1(z)\begin{pmatrix} z \\ 1 \end{pmatrix}}_{s_1} \wedge d\left(f_2(z)\begin{pmatrix} z \\ 1 \end{pmatrix}\right) = f_1 f_2 \begin{vmatrix} z & dz \\ 1 & 0 \end{vmatrix} = f_1 f_2(-dz).$$

$$s_1 \wedge ds_2$$

Go over corresp between line and circle pictures.

$i\mathbb{R}$	S^1
s	z



$z = \frac{s-1}{s+1}$

?

$e^s = z$ $\quad -\infty \longleftrightarrow 0$
$\quad +\infty \longleftrightarrow \infty.$?

$$z = e^{ip} = 1+ip = \frac{1+ip/2}{1-ip/2}$$

Basically you want the RH s-plane to corresp to the disk, so that $z = e^{-s}$ $\quad z^x = e^{-xs}$ and the LT $\int_0^\infty e^{-xs}\varphi(x)\,dx$ is a cont. power series

There should be a simple corresp. between

$L^2\left(i\mathbb{R}, \frac{ds}{2\pi i}\right)$ and $L^2\left(S^1, \frac{d\theta}{2\pi}\right)$

but you should review your formulas.

$s(x) = f(x)\begin{pmatrix} x \\ 1 \end{pmatrix} \qquad ds \wedge s = f(x)^2 \begin{vmatrix} dx & x \\ 0 & 1 \end{vmatrix} = f(x)^2 dx$

Thus you get conj $f(x) \longmapsto \overline{f(x)}$ for real str.

next $\quad f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \longmapsto f(z)^2 \begin{vmatrix} dz & z \\ 0 & 1 \end{vmatrix} = f(z)^2 dz = f(z)^2\, iz\, d\theta$

$f(z)\begin{pmatrix} z \\ 1 \end{pmatrix}$ real when $f(z)(iz)^{1/2}$ is real.

$$\begin{array}{ccc} c\begin{pmatrix} z \\ 1 \end{pmatrix} & \longrightarrow & c(iz)^{1/2} \text{ is real} \\ & & \downarrow \\ \bar{c}(iz)^{-1}\begin{pmatrix} z \\ 1 \end{pmatrix} & \longleftarrow & \bar{c}(iz)^{-1/2} \end{array}$$

So $\left[c\begin{pmatrix} z \\ 1 \end{pmatrix}\right]^* = -i\bar{c}\begin{pmatrix} 1 \\ z^{-1} \end{pmatrix}$

$$\left\| f(z)\begin{pmatrix} z \\ 1 \end{pmatrix} \right\|^2 = \int \overline{f(z)}\,(iz)^{-1} f(z)\, dz = \int |f(z)|^2 \frac{dz}{iz}$$

$\boxed{\frac{(1+ix)^2}{(1+ix)(1-ix)}}$

$z = \frac{1+ix}{1-ix} = \frac{(1+ix)^2}{(1-ix)(1+ix)}$ $\quad \boxed{z^{1/2} = \frac{1+ix}{\sqrt{1+x^2}}}$ $\quad \boxed{z^{-1/2} = \frac{1-ix}{\sqrt{1+x^2}}}$

$\longrightarrow$

$f(z)(iz)^{1/2}$

maybe all this should be augmented by half ant grid space.

$$(k_\varepsilon \lambda^\varepsilon - 1) v^1 = h_\varepsilon v^2_\varepsilon \qquad h_\varepsilon = b\sqrt{\varepsilon} \qquad v^2_\varepsilon = v^2\sqrt{\varepsilon}$$

$$(k_\varepsilon \mu - 1) v^2_\varepsilon = \bar{h}_\varepsilon v^1 \qquad \bar{h}_\varepsilon = \bar{b}\sqrt{\varepsilon}$$

$$k_\varepsilon = \sqrt{1-|b|^2\varepsilon} = 1 - a\varepsilon \quad a = \tfrac{1}{2}|b|^2$$

$$(-a+s)\, v^1 = b v^2$$

$$(\mu - 1)\, v^2 = \bar{b} v^1 \qquad \mu = 1 + \frac{|b|^2}{s-a} = \frac{s+a}{s-a}$$

So you end up working with $E^\wedge$ = horizontal space consis of $\int e^{xs}\varphi(x)dx$ $\oplus$ vertical space = rational functions of s vanish at ∞ reg. off $\pm a$.

~~φ~~

Consider Toeplitz operator associated to a smooth function β on the circle $i\mathbb{R}\cup\infty$, or on S^1, so you consider $\beta(s) = \beta(ik)$ multiplying on Hardy space viewed either horizontally as consisting of $\int_0^\infty e^{xs}\varphi(x)dx$, i.e. continuous power series, or vertically as ℓ^2 linear combinations of $\left(\frac{s+a}{s-a}\right)^n \frac{1}{s-a}$, followed by projection back onto Hardy space.

You have an isom, so what do you need? In discrete case you look at

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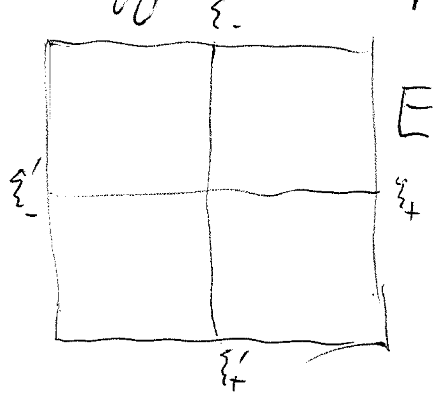
Given $b(\omega)$ on $\mathbb{R}$ you want to understand corresp. Toeplitz op.

Start again to get around this obstruction.
What do you have?

$V = L^2 \oplus C^2$ equipped with $B = \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}$
self-adjoint invertible: $B^2 = 1 + |b|^2$. Have
$W = H_+ \oplus H_+ \overset{\varepsilon}{\subset} V$ compression $A = \varepsilon^* B \varepsilon$
$= \begin{pmatrix} 1 & T_b^* \\ T_b & -1 \end{pmatrix}$ also self adj + invertible.
Get splitting $V = W \oplus \mathrm{Ker}(\varepsilon^* B : V \to W)$.

Point forgotten:
Perhaps you need to use d as in Birkhoff decomp.



$$E = (\xi'_- H_+ + \xi_- H_+) \overset{?}{\oplus} (\xi_+ H_- + \xi'_+ H_-)$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{d} \begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix} = \begin{pmatrix} \xi'_- & \xi_- \end{pmatrix} \begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix} \frac{1}{d}$$

$$E = (\xi'_- H_+ + \xi_- H_+) \overset{?}{\oplus} (\xi_+ H_- + \xi'_+ H_-)$$

$$\begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} = \begin{pmatrix} \xi'_- & \xi_- \end{pmatrix} \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

equivalent to

$$\begin{pmatrix} L^2 \\ L^2 \end{pmatrix} = \begin{pmatrix} H_- \\ H_- \end{pmatrix} \overset{?}{\oplus} \begin{pmatrix} \frac{1}{a} & \frac{\bar{b}}{a} \\ -\frac{b}{a} & \frac{1}{a} \end{pmatrix} \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

$$\begin{pmatrix} L^2 \\ L^2 \end{pmatrix} = \begin{pmatrix} H_- \\ H_- \end{pmatrix} \oplus \begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix} \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}$$

Birkhoff fact.

$$V = \begin{matrix} L^2 \\ \oplus \\ L^2 \end{matrix} \;, \quad V_+ = \begin{matrix} H_+ \\ \oplus \\ H_+ \end{matrix} \;, \quad V_- = \begin{matrix} H_- \\ \oplus \\ H_- \end{matrix}$$

$$B = \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \text{ on } V, \quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B\varepsilon = \begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix} = 1 + X$$

$S = \frac{1}{d}\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix}$ done ~~in~~ wrong basis, so really

S should be $\frac{1}{d}\begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix}$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{d}\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

~~start again~~

$$\xi_+ f + \xi'_+ g = \left(\frac{1}{d}\xi'_- + \frac{b}{d}\xi_-\right)f + \left(-\frac{\bar{b}}{d}\xi'_- + \frac{1}{d}\xi_-\right)g$$

$$= \left(\frac{1}{d}f - \frac{\bar{b}}{d}g\right)\xi'_- + \left(\frac{b}{d}f + \frac{1}{d}g\right)\xi_-$$

$$(f\ g)\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}^t \begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}\frac{1}{d}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$= \begin{pmatrix} \frac{f - gc}{d} & \frac{fb + g}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ & \xi'_+ \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \underbrace{(\xi'_-\ \xi_-)}\frac{1}{d}\begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}.$$

$$\boxed{S = \frac{1}{d}\begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix} = \frac{1}{d}B\varepsilon}$$

$$V = \begin{pmatrix} L^2 \\ \oplus \\ L^2 \end{pmatrix} \qquad V_+ = \begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \qquad V_- = \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\left(V = \left\{ \begin{pmatrix} \xi_-' & \xi_- \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right\} \right) \quad V \text{ equipped with } \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$W_- = \left\{ \begin{pmatrix} \xi_+ & \xi_+' \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix} \right\} \qquad \begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix} \frac{1}{d} \begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ & \xi_+ \end{pmatrix} = \begin{pmatrix} \xi_-' & \xi_- \end{pmatrix} \underbrace{\frac{1}{d} \begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix}}_{B\varepsilon}$$

So what do you find

$$W_- = S\varepsilon V_- = \frac{1}{d} \begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix} V_- = \frac{1}{d} \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} V_-$$

So the formula for $W_- = (V_+)^{\perp_b}$ is quite simple.

Can you check this – maybe simplify:

$$V = V_+ \oplus \frac{1}{d} B V_- \qquad \text{should be obvious}$$

So where are we? You begin with the mult of b on L^2 say $L^2(S^1)$, form $B = \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}$ on $\begin{matrix} L^2 \\ \oplus \\ L^2 \end{matrix}$

Note $B\varepsilon = \begin{pmatrix} 1 & +\bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix}$ = the $1 + X$ in C.T. theory, so it's natural to consider

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} = \frac{1}{\sqrt{1+|b|^2}} \begin{pmatrix} 1 & -\bar{b} \\ b & +1 \end{pmatrix}.$$

Note here that

$$B^2 = (1+|b|^2)I.$$

You are ~~getting~~ confused. The point: there's a link with the Grassmannian stuff. How? $B=\begin{pmatrix}1&\bar b\\ b&-1\end{pmatrix}$ is self adjoint, and it square is $(1+|b|^2)I$, which admits "square roots" d and $\bar d = a$. Thus $\frac{1}{d}B=\frac{1}{d}\begin{pmatrix}1&\bar b\\ b&-1\end{pmatrix}$ is an involution F, so we know $\frac{1}{d}B\varepsilon = \frac{1}{d}\begin{pmatrix}1&-\bar b\\ b&1\end{pmatrix} = S$ is unitary and inverted by ε. But this S is the scattering:

$$(\xi_+ \quad \xi_+') = (\xi_-' \quad \xi_-)\begin{pmatrix}1&-\bar b\\ b&1\end{pmatrix}\frac{1}{d}$$

Recall C.T. formulas.

$$X=\begin{pmatrix}0&-T^*\\ T&0\end{pmatrix} \qquad g=\frac{1+X}{1-X} \qquad F=g\varepsilon$$

$$F\begin{pmatrix}1&-T^*\\ T&1\end{pmatrix} = g\varepsilon\,(1+X) = \frac{1+X}{1-X}(1-X)\varepsilon = (1+X)\varepsilon$$

$$F(1+X) = (1+X)\varepsilon \qquad \therefore \left\{\begin{array}{l} F=+1 \text{ on } \Gamma_T \\ -1 \text{ on } \Gamma_T^{\perp}. \end{array}\right.$$

$$F = g^{1/2}\varepsilon g^{-1/2}, \quad g^{1/2} = \frac{1+X}{\sqrt{1-X^2}}$$

It seems something is strange because above you want $F=\frac{1}{d}B$, $F\varepsilon = \frac{1}{d}B\varepsilon = S$ $\therefore S=g$

But on the other hand, $B\varepsilon = (1+X)\varepsilon = \begin{pmatrix}1&\bar b\\ b&-1\end{pmatrix}$?

~~[illegible]~~ Again. In C.T. theory

$$X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} \qquad g = \frac{1+X}{1-X} \qquad F = g\varepsilon = \begin{matrix} +1 \text{ on } \Gamma_T \\ -1 \text{ on } \Gamma_T^{\perp} \end{matrix}$$

Also $F = g^{1/2}\varepsilon g^{-1/2}$ $\qquad g^{1/2} = \frac{1+X}{\sqrt{1-X^2}}$

In the scatt situation

$$X = \begin{pmatrix} 0 & -\bar b \\ b & 0 \end{pmatrix} \qquad 1 - X^2 = (1+|b|^2)I = |d|^2 I$$

$$\therefore \quad \frac{B}{d} = \frac{1}{d}(1+X) = \frac{1}{d}\begin{pmatrix} 1 & -\bar b \\ b & 1 \end{pmatrix}$$

is a unitary S ~~[illegible]~~

$$\frac{B}{d}\varepsilon = \frac{1}{d}\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}$$

is also unitary ~~[illegible]~~ but not an involution necessarily

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So there's a hint that the Grassmannian is involved. Ideally you want insight into KK cap-product between S and the Hilbert transform F, or ~~splitting~~ $L^2 = H_+ \oplus H_-$.

So repeat $\quad (\xi_+ \;\; \xi_+') = (\xi_-' \;\; \xi_-)\underbrace{\frac{1}{d}\begin{pmatrix} 1 & -\bar b \\ b & 1 \end{pmatrix}}_{S = \frac{1}{d}B\varepsilon}$

Claim that $\quad \overset{V}{\begin{pmatrix} L^2 \\ \oplus \\ L^2 \end{pmatrix}} = \overset{V_+}{\begin{pmatrix} H_+ \\ H_+ \end{pmatrix}} \oplus S \overset{V_-}{\begin{pmatrix} H_- \\ H_- \end{pmatrix}}$

Claim that $\quad V_+ \oplus V_- \xrightarrow{\quad} \begin{matrix} L^2 \\ \oplus \\ L^2 \end{matrix}$

$$\begin{matrix} V_+ \\ \oplus \\ V_- \end{matrix} \xrightarrow{(j_+,\, S j_-)} \qquad j_+^* S j_-$$

$V/_{4}$ ~~Sy...~~

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus S \begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \quad ?$$

equiv. to

$$\begin{pmatrix} H_+ \\ H_+ \end{pmatrix} \oplus \begin{pmatrix} 1 & -\bar b \\ b & 1 \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix} \xrightarrow{\overset{?}{\sim}} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}$$

Do what you can?

$$\xi'_- f + \xi_- g - (\xi'_- \phi + \xi_- \psi) \in (\xi'_- H_+ + \xi_- H_+)^\perp$$

$$\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f - \phi \\ g - \psi \end{pmatrix} = 0 .$$

$$j_+^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} j_+ \begin{pmatrix} \phi \\ \psi \end{pmatrix} = j_+^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

there is geom. here to be worked on. Go back.

$$(\xi'_- H_+ + \xi_- H_+)^\perp \overset{?}{=} \xi_+ H_- + \xi'_+ H_- \quad ?$$

$$= (\xi'_- \quad \xi_-) \underbrace{S \begin{pmatrix} H_- \\ H_- \end{pmatrix}}_{\frac{1}{d}\begin{pmatrix} 1 & -\bar b \\ b & 1 \end{pmatrix}\begin{pmatrix} H_- \\ H_- \end{pmatrix}}$$

$$\mathrm{Ker}\; j_+^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}$$

$$\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \underbrace{\begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \frac{1}{d} \begin{pmatrix} 1 & -\bar b \\ b & 1 \end{pmatrix}}_{d \cdot \quad 1+|b|^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

Converse $\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & -\bar b \\ b & +1 \end{pmatrix} \overbrace{\begin{pmatrix} f \\ -g \end{pmatrix}}^{\in L^2 \oplus L^2} = 0$

$$\frac{1}{d} \begin{pmatrix} 1 & -\bar b \\ b & +1 \end{pmatrix} \begin{pmatrix} f \\ -g \end{pmatrix} \in \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\begin{pmatrix} f \\ -g \end{pmatrix} \in \underbrace{\frac{\bar d}{+1+|b|^2}}_{\frac{1}{d}} \begin{pmatrix} +1 & +\bar b \\ -b & +1 \end{pmatrix} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

Need this in much better shape. You really need the operators under control. You have to get from the ~~oth~~ splittings – the fact that $V = L^2 \oplus L^2$ splits into V_+ and $W = V_+^{\perp_b}$ to an actual factorization of $S = \frac{1}{d}\begin{pmatrix} 1 & -\bar b \\ b & 1 \end{pmatrix}$. S is not in L^2. Everything should be done using operators on L^2, perhaps you should work in the C^* algebra generated by the Hilbert transform and the multiplication operator b.

You begin with $B = \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix}$ on $\mathcal{L}(V)$, $V = L^2 \oplus L^2$, commutes with multiplication by fns. on S^1 or $\mathbb{R}$

You start with an elt $\begin{pmatrix} f \\ g \end{pmatrix}$ of V, $\mapsto e_+ B \begin{pmatrix} f \\ g \end{pmatrix} \longmapsto$

$$(e_+ B e_+)^{-1} e_+ B \begin{pmatrix} f \\ g \end{pmatrix} \in e_+ \begin{pmatrix} L^2 \\ \oplus \\ L^2 \end{pmatrix} = V_+ .$$

$$\longmapsto \begin{pmatrix} f \\ g \end{pmatrix} - (e_+ B e_+)^{-1} e_+ B \begin{pmatrix} f \\ g \end{pmatrix} \in W = V_+^{\perp_b} = \operatorname{Ker}(e_+ B)$$

Let $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \operatorname{Ker}(e_+ B)$: $\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$

$\Longrightarrow \frac{1}{d} \begin{pmatrix} 1 & \bar b \\ b & -1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \begin{pmatrix} H_- \\ H_- \end{pmatrix} \Longrightarrow \frac{\bar d}{1+|b|^2} \begin{pmatrix} +1 & +\bar b \\ +b & -1 \end{pmatrix}$

$$\begin{pmatrix} f \\ g \end{pmatrix} \in V \longmapsto eB\begin{pmatrix} f \\ g \end{pmatrix} \longmapsto \begin{pmatrix} f \\ g \end{pmatrix} - e(eBe)^{-1} eB\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \operatorname{Ker}(eB)$$

$$\int \begin{pmatrix} H_+ \\ H_+ \end{pmatrix}^* B\begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0 \qquad \int \begin{pmatrix} d^{-1}H_+ \\ d^{-1}H_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$$

$$H_+ = \frac{1}{d} H_+ \qquad e\, \frac{1}{d}\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$$

$$\frac{1}{d}\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

Go direct from $eB\begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$ to

$$\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \begin{pmatrix} H_- \\ H_- \end{pmatrix} \Rightarrow \frac{1}{d}\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \left[\frac{1}{d}\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\right]^{-1} \begin{pmatrix} H_- \\ H_- \end{pmatrix}$$

$$S = \frac{1}{d}\begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix} \qquad S^* = \frac{1}{d}\begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix}$$

$$\frac{d}{+1+|b|^2}\begin{pmatrix} +1 & +\bar{b} \\ +b & -1 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} = S\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in S\begin{pmatrix} H_- \\ H_- \end{pmatrix}.$$

So what have we got.

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} f \\ g \end{pmatrix} - e(eBe)^{-1} eB\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

$\longmapsto$

So you have explicit splitting of V into

depending on e and b.