such that $\dfrac{f(s)-f(0)}{s} = \int e^{sx}\psi(x)\,dx$, $f(s) = \int e^{xs}\varphi(x)\,dx$

this implies $f(s)-f(0) = \int e^{sx}\left(-\partial_x\psi(x)\right)dx$ so

$\int e^{sx}\left(\varphi(x)+\partial_x\psi(x)\right)dx = f(0)$ implying

$-\partial_x\psi(x) = \varphi(x) + f(0)\delta(x)$. What's important is the homogeneous function character, something about the link between the singularities x^n $n\in\mathbb{N}$ and s^{-n-1}.

First step seems: Find $\psi_n(x)$ such that $\psi_0 = \varphi$, $\hat{\psi}_1 = \dfrac{\hat{\varphi}(s)-\hat{\varphi}(0)}{s}$, $\hat{\psi}_2 = \dfrac{\hat{\varphi}-\hat{\varphi}(0)-\hat{\varphi}'(0)s}{s^2}$, etc. Suppose $\varphi(x)$ supp on $\mathbb{R}_{\geq 0}$.

$$\psi_n(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!}\varphi(x')\,dx' \qquad n\geq 1, \quad x>0$$

$$= \int_x^\infty \varphi(x')\,dx' \qquad n=1$$

$$= \varphi(x) \qquad n=0.$$

Is there another way to write this: If $\varphi(x') = \delta(x'-b)$ where $b>0$

Then
$$\psi_0(x) = \delta(b-x)$$
$$\psi_1(x) = \int_x^\infty \delta(x'-b)\,dx' = H(b-x) = \begin{cases} 1 & x<b \\ 0 & x>b \end{cases}$$
$$\psi_n(x) = \begin{cases} \dfrac{(b-x)^{n-1}}{(n-1)!}H(b-x) & \\ 0 & x>b \end{cases}$$

So if $\varphi(x) = \delta(x-b)$ with $b>0$ say

then
$$\psi_n(x) = H(b-x)\frac{(b-x)^{n-1}}{(n-1)!} = \begin{cases} \dfrac{(b-x)^{n-1}}{(n-1)!} & \text{for } x<b \\ 0 & x>b. \end{cases}$$

b

Somehow you are again wrestling with Taylor remainder

$\psi_n(x) = \frac{(b-x)^{n-1}}{(n-1)!} H(b-x)$

So what can you do?

$e^{ys^{-1}} e^{bs}$

$$e^{ys^{-1}} e^{bs} = e^{bs} + \frac{y}{s} e^{bs} + \frac{y^2}{2!}\frac{e^{bs}}{s^2} + \frac{y^3}{3!}\frac{e^{bs}}{s^3}$$

horizontal part is

$$e^{bs} + y\,\frac{e^{bs}-1}{s} + \frac{y^2}{2!}\,\frac{e^{bs}-1-bs}{s^2} +$$

vertical part is

$$y\,\frac{1}{s} + \frac{y^2}{2!}\,\frac{1+bs}{s^2} + \frac{y^3}{3!}\,\frac{1+bs+\frac{1}{2}b^2s^2}{s^3} + \cdots$$

this should have an expression in terms of Bessel functions. Note it depends only on bs and by

$$e^{by\left(\frac{1}{bs}\right) + bs}$$

$e^{ys^{-1}+sx} = e^{\frac{xy}{xs} + xs}$ so $e^{ys^{-1}+sx}$ depends only on two variables instead of 3. Take $y=1$ to simplify, also to connect maybe with vertically discrete case. put $-n = j-i$ with $n \geq 1$.

$$e^{s^{-1}} e^{xs} = \sum_{i,j \geq 0} \frac{s^{j-i}\,x^j}{i!\,j!} = \sum_{n \geq 1} s^{-n} \underbrace{\sum_{j \geq 0} \frac{x^j}{j!\,(n+j)!}}_{J_n(x)} + \sum_{n \geq 0} s^n$$

Neg

Thus the singular part is given by

$$\left(e^{s^{-1}}\int e^{xs}\varphi(x)\,dx\right)_- = \sum_{n\geq 1}\frac{1}{s^n}\int J_n(x)\varphi(x)\,dx$$

This was the easy part analogous to

$$\left(\frac{\hat\varphi(s)}{s}\right)_- = \frac{\varphi(0)}{s} \qquad \left(\frac{\hat\varphi(s)}{s^2}\right)_- = \frac{\hat\varphi(0)+\hat\varphi'(0)s}{s^2}$$

The vertical component of $e^{s^{-1}}\hat\varphi$ is thus

$$\int\left(\sum_{n\geq 1}\frac{1}{s^n}J_n(x)\right)\varphi(x)\,dx$$

~~But more more effort is needed.~~

$$e^{s^{-1}+xs} - \sum_{n\geq 1}\frac{1}{s^n}J_n(x) \qquad \text{entire fn. of } s$$

$$e^{s^{-1}}\int e^{xs}\varphi(x)\,dx - \int\sum_{n\geq 1}\frac{1}{s^n}J_n(x)\varphi(x)\,dx \qquad \text{entire fn of } s$$

Let's first try to actually construct the regularized version of $e^{ys^{-1}}\hat\varphi(s)$, which has the form

$$\hat\varphi(s) + y\,\frac{\hat\varphi(s)-\hat\varphi(0)}{s} + \frac{y^2}{2!}\,\frac{\hat\varphi(s)-\hat\varphi(0)-\hat\varphi'(0)s}{s^2} + \cdots$$

$$= \hat\psi_0 + y\hat\psi_1 + \frac{y^2}{2!}\hat\psi_2 + \cdots$$

where $\psi_n(x)$ can be specified by

$$(-\partial_x)^n\psi_n(x) = \varphi(x) \qquad x\geq 0$$

$$\psi_n(x) = 0 \qquad x \gg 0.$$

$$\int_0^\infty e^{sx}\left((-\partial_x)\psi_n\right)dx = \int_0^\infty e^{xs}\psi_{n-1}(x)\,dx = \hat{\psi}_{n-1}(s)$$

$$\shortparallel$$

$$\underbrace{\left[e^{sx}(-\partial_x\psi_n)\right]_0^\infty}_{\psi_{n-1}(0)} + \int_0^\infty s e^{sx}\psi_n(x)\,dx$$

$$s\hat{\psi}_n = \hat{\psi}_{n-1}(s) - \psi_{n-1}(0) \qquad \therefore \hat{\psi}_{n-1}(0) = \psi_{n-1}(0)$$

So ~~then~~ you have this formal series

$$\sum_{n\geq 0}\frac{y^n}{n!}\hat{\psi}_n(s) = \int_0^\infty e^{sx}\sum_{n\geq 0}\frac{y^n}{n!}\psi_n(x)\,dx$$

$$= \sum_{n\geq 0}\frac{y^n}{n!}\int_0^\infty e^{sx}\psi_n(x)\,dx$$

$$= \int_0^\infty e^{sx}\sum_{n\geq 0}\frac{y^n}{n!}(-\partial_x)^{-n}\varphi(x)\,dx$$

This is too hard, however you have an explicit formula for

$$\psi_n(x) = \int_x^\infty \frac{(t-x)^{n-1}}{(n-1)!}\varphi(t)\,dt \qquad n\geq 1.$$

take $\varphi(t) = \delta(t-b)$. Then

$$\psi_n^b(x) = \frac{(b-x)^{n-1}}{(n-1)!} \qquad \text{for } x<b, \quad n\geq 1$$

$$\underbrace{\int_0^b e^{sx} \sum_{n\geq 0} \frac{y^n}{n!} \frac{(b-x)^{n-1}}{(n-1)!}\, dx}$$

$$\| $$

Take $y = 1$

$$\sum_{n\geq 1} \frac{y^n}{n!} \underbrace{\int_0^b e^{sx} \frac{(b-x)^{n-1}}{(n-1)!} dx}_{\int_0^b e^{s(b-x)} \frac{x^{n-1}}{(n-1)!} dx}$$

$$\sum_{n\geq 0} \frac{y^n}{n!} \hat{\varphi}_n(s) = \int_0^\infty e^{sx} \sum_{n\geq 0} \frac{y^n}{n!} \varphi_n(x)\, dx$$

$$= \int_0^\infty e^{sx} \underbrace{\varphi(x)}_{\delta(b-x)} dx + \sum_{n\geq 1} \frac{y^n}{n!} \int_0^b e^{sx} \frac{(b-x)^{n-1}}{(n-1)!} dx$$

$$= e^{bs} + y \underbrace{\int_0^b e^{sx} dx}_{\frac{e^{bs}-1}{s}} + \frac{y^2}{2!} \underbrace{\int_0^b e^{xs} \frac{(b-x)}{1!} dx}_{\frac{e^{bs}-1-bs}{s^2}}$$

$$\int_0^b e^{sx} \frac{(b-x)^n}{n!} dx = \left[\frac{e^{sx}}{s} \frac{(b-x)^n}{n!} \right]_0^b + \int_0^b \frac{e^{sx}}{s} \frac{(b-x)^{n-1}}{(n-1)!} dx$$

$$\hat{\varphi}_{n+1} = -\frac{1}{s} \frac{b^n}{n!} + \frac{\hat{\varphi}_n}{s} = \frac{\hat{\varphi}_n - \frac{b^n}{n!}}{s}$$

$$\int_0^b e^{bs} e^{-xs} \frac{x^{n-1}}{(n-1)!} dx$$

Taylor thm.

$$f(x) - f(0) = \int_0^1 dt\; \partial_t f(tx) = \int_0^1 x f'(tx)\, dt$$

$$= \Big[x f'(tx) (1-t)(-1)\Big]_0^1 + \int_0^1 x^2 f''(tx)(1-t)\,dt$$

$$= \Big[x^2 f''(tx)\frac{(1-t)^2}{2}(-1)\Big]_0^1 + \int_0^1 x^3 f^{(3)}(tx)\frac{(1-t)^2}{2!}\,dt$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \underbrace{\int_0^1 x^3 f^{(3)}(tx)\frac{(1-t)^2}{2!}\,dt}$$

$$\int_0^x f^{(3)}(x-y)\frac{y^2}{2!}\,dy = \int_0^x f^{(3)}(y)\frac{(x-y)^2}{2!}\,dy$$

$f(s) = e^{bs}$ $\qquad \int_0^1 s^3 b^3 e^{tb}\frac{(1-t)^2}{2!}\,dt$

$$= \int_0^1 s^3 e^{ts}\frac{(b-bt)^2}{2!}\,d(bt) \quad ?$$

$$= \int_0^b s^3$$

$$f(s) = e^{bs} = 1 + bs + \frac{b^2}{2!}s^2 + \int_0^1 s^3 b^3 e^{bts}\frac{(1-t)^2}{2!}\,dt$$

$$R = \int_0^b s^3 e^{xs}\frac{(b-x)^2}{2!}\,dx$$

Taylor $\qquad e^{bs} = \sum_{j=0}^{n} \frac{(bs)^j}{j!} + \int_0^1 (bs)^{n+1} e^{t(bs)}\frac{(1-t)^n}{n!}\,dt$

$$f(x) = f(0) + \int_0^1 \partial_t f(tx)\,dt$$
$$x f'(tx)\,dt$$

$$= f(0) + \left[x f'(tx)(1-t)(-1)\right]_0^1 + \int_0^1 x^2 f''(tx)(1-t)\,dt$$

$$= f(0) + x f'(0) + \left[x^2 f^{(2)}(tx)\frac{(1-t)^2}{2!}(-1)\right]_0^1 + \int_0^1 x^3 f^{(3)}(tx)\frac{(1-t)^2}{2!}\,dt$$

~~[illegible]~~ Keep on trying to do this

$e^{ys^{-1}}$ $f(s)$ repeat

Repeat the calculation until it becomes transparent

$$e^{ys^{-1}} e^{xs} = \sum_{n\geq 1} \frac{1}{s^n} \sum_{\substack{n=i-j \\ i,j\geq 0}} \frac{y^i x^j}{i!\,j!} + \sum_{n\geq 0} s^n \sum_{\substack{n=j-i \\ i,j\geq 0}} \frac{y^i x^j}{i!\,j!}$$

$i = n+j$ $\qquad$ $j = n+i$

$$= \sum_{n\geq 1} \frac{y^n}{s^n} \underbrace{\sum_{j\geq 0} \frac{(xy)^j}{j!\,(n+j)!}}_{J_n(xy)} + \sum_{n\geq 0} s^n x^n \underbrace{\sum_{i\geq 0} \frac{(xy)^i}{i!\,(i+n)!}}_{J_n(xy)}$$

$$e^{ys^{-1}} \int e^{xs} \varphi(x)\,dx = \underbrace{\sum_{n\geq 1} \frac{y^n}{s^n} \int J_n(xy)\varphi(x)\,dx}_{\text{vertical}} + \underbrace{\sum_{n\geq 0} s^n \int x^n J_n(xy)\, \varphi(x)\,dx}_{\text{horiz.}}$$

What else? Sing. part should be

$$\sum_{n\geq 1} \frac{y^n}{n!} \; \frac{f(0) + f'(0)s + \cdots + f^{(n-1)}(0)\frac{s^{n-1}}{(n-1)!}}{s^n}$$

$f(s) = e^{bs}$

$\varphi(x) = \delta(x-b)$

$$\sum_{n\geq 1} \frac{y^n}{n!} e^{bs} \; \frac{1 + bs + \frac{(bs)^2}{2!} + \cdots + \frac{(bs)^{n-1}}{(n-1)!}}{s^n}$$

Look again: $f(s) = \int e^{xs} \varphi(x)\,dx$ in $\mathbb{R}_{\geq 0}$
where φ has compact support. Then

$$\frac{f(s)-f(0)}{s} = \int e^{xs} \psi_1(x)\,dx$$

$$\frac{f(s)-f(0)-f'(0)s}{s^2} = \int e^{xs} \psi_2(x)\,dx$$

where $\psi_n = (-\partial_x)^{-n} \varphi$, $\psi_n(x) = 0$ $x \gg 0$

$$\psi_n(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!} \varphi(x')\,dx'$$

$n \geq 2$

$-\partial_x \psi_n = \psi_{n-1}$

$-\partial_x \psi_1 = \psi_0 = \varphi$

what do you learn from this?

Problem: Split $e^{ys^{-1}} \int e^{xs} \varphi(x)\,dx$ into horizontal & vertical components.

Special case: dividing by s.

$$\frac{\hat\varphi(s)}{s} = \frac{\hat\varphi(s)-\hat\varphi(0)}{s} + \frac{\hat\varphi(0)}{s}$$

$$\left(e^{ys^{-1}} \hat\varphi(s)\right)_{\text{vert}} = \left(\hat\varphi(s) + y\frac{\hat\varphi(s)}{s} + \frac{y^2}{2!}\frac{\hat\varphi(s)}{s^2} + \cdots\right)_{\text{vert}}$$

$$= y\frac{\hat\varphi(0)}{s} + \frac{y^2}{2!}\frac{\hat\varphi(0)+\hat\varphi'(0)s}{s^2} + \frac{y^3}{3!}\frac{\hat\varphi(0)+\hat\varphi'(0)s+\frac{\hat\varphi''(0)s^2}{2}}{s^3}$$

$$\left(\int e^{ys^{-1}+xs} \varphi(x)\,dx\right)_{\text{ver}} = \sum_{n\geq 1} \frac{y^n}{s^n} \int J_n(xy)\,\varphi(x)\,dx$$

$$\lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \qquad \begin{aligned}(k\lambda-1)v^1 &= h v^2\\ (k\mu-1)v^2 &= \bar{h} v^1\end{aligned}$$

$$\|$$

$$z^m \left(\frac{z-k}{kz-1}\right)^n \begin{pmatrix} \frac{h}{kz-1} \\ 1 \end{pmatrix} \qquad z^\varepsilon = e^{\varepsilon s} \qquad m = \frac{x}{\varepsilon}, \quad n = \frac{y}{\varepsilon}$$

$$(e^{\varepsilon s})^{\frac{x}{\varepsilon}} \left(\frac{e^{\varepsilon s} - k_\varepsilon}{k_\varepsilon e^{\varepsilon s} - 1}\right)^{\frac{y}{\varepsilon}} \left(\frac{c\varepsilon}{k_\varepsilon e^{\varepsilon s} - 1}\right)$$

$$k_\varepsilon = 1 - \frac{1}{2}|c|^2\varepsilon^2$$

$$e^{\varepsilon s} = 1 + \varepsilon s$$

$$e^{xs} \qquad \frac{c}{s} \qquad \frac{c}{\ }$$

$$\left(\frac{1+\varepsilon s + \frac{(\varepsilon s)^2}{2} - 1 + \frac{1}{2}|c|^2\varepsilon^2}{\left(1 - \frac{1}{2}c^2\varepsilon^2\right)\left(1+\varepsilon s + \frac{\varepsilon^2 s^2}{2}\right) - 1}\right)^{y/\varepsilon} = \left(\frac{\varepsilon s + \frac{(\varepsilon s)^2 + \varepsilon^2|c|^2}{2}}{\varepsilon s + \frac{\varepsilon^2 s^2}{2} - \frac{\varepsilon^2|c|^2}{2}}\right)^{y/\varepsilon}$$

$$= \left(\frac{1 + \frac{\varepsilon s + \varepsilon\frac{|c|^2}{s}}{2}}{1 + \frac{\varepsilon s - \varepsilon\frac{|c|^2}{s}}{2}}\right)^{y/\varepsilon} \sim \left(\frac{1 + \varepsilon\frac{|c|^2}{2s}}{1 - \varepsilon\frac{|c|^2}{2s}}\right)^{y/\varepsilon}$$

$$\longrightarrow e^{\frac{y|c|^2}{s}}$$

$$\left(\frac{1 - k_\varepsilon e^{-\varepsilon s}}{k_\varepsilon - e^{-\varepsilon s}}\right)^{y/\varepsilon}$$

$$= \left(\frac{1 - \left(1 - \frac{|c|^2}{2}\varepsilon^2\right)\left(1 - \varepsilon s + \frac{\varepsilon^2 s^2}{2}\right)}{1 - \frac{|c|^2}{2}\varepsilon^2 - \left(1 - \varepsilon s + \frac{\varepsilon^2 s^2}{2}\right)}\right)^{y/\varepsilon} =$$

$$\cancel{\frac{k_\varepsilon^{-1} e^{\varepsilon s} - 1}{k_\varepsilon e^{\varepsilon s} - 1}} \qquad \frac{e^{\varepsilon s} - k_\varepsilon}{e^{\varepsilon s} - k_\varepsilon^{-1}} = \left(\frac{\varepsilon s + \frac{\varepsilon^2 s^2}{2} + \frac{1}{2}|c|^2\varepsilon^2}{\varepsilon s + \frac{\varepsilon^2 s^2}{2} - \frac{1}{2}|c|^2\varepsilon^2}\right)^{1/\varepsilon}$$

$$\cancel{= 1 + \frac{\varepsilon s}{2} - \frac{1}{2}|c|^2}$$

$$= \frac{1 + \left(\frac{s}{2} + \frac{1}{2}\frac{|c|^2}{s}\right)\varepsilon}{1 + \left(\frac{s}{2} - \frac{1}{2}\frac{|c|^2}{s}\right)\varepsilon}$$

$$\left(z^\varepsilon\right)^{1/\varepsilon}\left(\frac{z^\varepsilon - k_\varepsilon}{k_\varepsilon z^\varepsilon - 1}\right)^n \begin{pmatrix} \frac{b\sqrt{\varepsilon}}{k_\varepsilon z^\varepsilon - 1} \\ 1 \end{pmatrix}$$

$$h_\varepsilon = b\sqrt{\varepsilon}$$
$$k_\varepsilon = \left(1 - |b|^2\varepsilon\right)^{1/2}$$
$$= 1 - a\varepsilon$$

$$z^\varepsilon = e^{i\varepsilon s}$$

$$(k_\varepsilon \lambda^\varepsilon - 1)v^1 = b\sqrt{\varepsilon}\, v_\varepsilon^2$$
$$(k\mu - 1)v_\varepsilon^2 = \bar{b}\sqrt{\varepsilon}\, v^1$$

Here's the question to ask. Is the singular part of $e^{ys^{-1}}e^{xs}$ in the vertical space

e.g. given $\hat\varphi(s)$ is it true that What!!!!!!

$$\frac{\hat\varphi(0)}{s} + y\frac{\hat\varphi(0) + s\hat\varphi'(s)}{s^2} + \frac{y^2}{2!}\frac{\hat\varphi(0) + \hat\varphi'(0)s + \hat\varphi''(0)\frac{s^2}{2!}}{s^3}$$

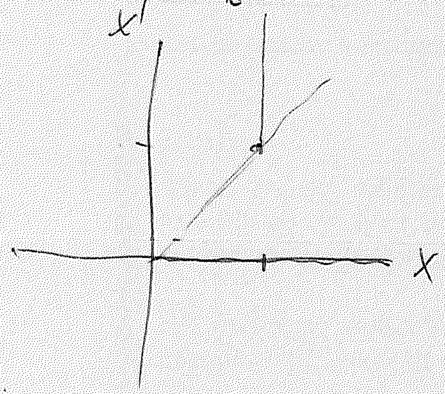
has the form $\int e^{ys^{-1}}\xi(y)\,dy$ where $\xi(y)$ has compact support. $\varphi = \delta(b - x)$ $b > 0$.

$e^{ys^{-1} + bx}$

Repeat. $\varphi_0(x) = \varphi(x)$ given compact supp

$$\varphi_n(x) = (-\partial_x)^{-n}\varphi(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!}\varphi(x')\,dx'$$

$$\int_0^\infty e^{xs}\varphi_n(x)\,dx = \int_0^\infty e^{xs}\left(\int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!}\varphi(x')\,dx'\right)dx$$



$$= \int_0^\infty \underbrace{\left(\int_0^{x'} e^{xs}\frac{(x'-x)^{n-1}}{(n-1)!}\,dx\right)}_{\frac{e^{x's}-1-x's-\cdots-\frac{x'^{n-1}}{(n-1)!}s^{n-1}}{s^n}}\varphi(x')\,dx'$$

there's nothing new here. ~~[illegible]~~
Maybe you should go back to Bessel.

$$f(s) = \int e^{xs}\varphi(x)\,dx$$

$$\varphi_n(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!}\varphi(x')\,dx' \qquad -\partial_x\varphi_n = \varphi_{n-1}$$

$$\varphi_n(x) = 0 \quad x \gg 0$$

$$\varphi_n(0) = \int_0^\infty \frac{x'^{n-1}}{(n-1)!}\varphi(x')\,dx' = \frac{f^{(n-1)}(0)}{(n-1)!}$$

$$s\hat{\varphi}_n(s) = \int \partial_x(e^{xs})\varphi_n(x)\,dx = \underbrace{-\varphi_n(0)}_{-\frac{\hat{\varphi}^{(n-1)}(0)}{(n-1)!}} + \underbrace{\int e^{xs}\varphi_{n-1}(x)\,dx}_{\hat{\varphi}_{n-1}(s)}$$

$$\therefore \hat{\varphi}_n(s) = \frac{\hat{\varphi}_{n-1}(s) - \frac{\hat{\varphi}^{(n-1)}(0)}{(n-1)!}}{s}$$

What is the real question.

Repeat. $\left(e^{ys^{-1}} \int e^{xs} \varphi(x)\,dx \right)_{\text{vert}}$

$$= \sum_{n \geq 1} \frac{y^n}{s^n} \int J_n(xy)\, \varphi(x)\, dx$$

try the other

$$\cancel{\left(e^{ys^{-1}} f(s) \right)_{\text{vert}} = y \frac{f(0)}{s} + \frac{y^2}{2!} \frac{f'(0)}{s^2}}$$

$$\left(e^{ys^{-1}} f(s) \right)_{\text{horz}} = \left(f(s) + y \frac{f(s)}{s} + \frac{y^2}{2!} \frac{f(s)}{s^2} + \cdots \right)_{\text{hor}}$$

$$= f(s) + y \frac{f(s) - f(0)}{s} + \frac{y^2}{2!} \frac{f(s) - f(0) - f'(0)s}{s^2} + \cdots$$

$f(s) = \hat{\varphi}(s)$

$$= \hat{\psi}_0(s) + y \hat{\psi}_1(s) + \frac{y^2}{2!} \hat{\psi}_2(s) + \cdots$$

$$\cancel{\psi_n(x) = \varphi(x) \quad n=0}$$

$$\psi_n(x) = (-\partial_x)^n \varphi(x) \qquad (-\partial_x)^n \psi_n(x) = \varphi(x) \quad x \neq 0$$

$$= 0 \qquad x \gg 0 \qquad \psi_n(x) = 0 \ \text{ or } \ \begin{matrix} x \gg 0 \\ x \ll 0 \end{matrix}$$

$$\psi_n(x) = \int_x^{+\infty} \frac{(x'-x)^{n-1}}{(n-1)!} \varphi(x')\, dx' \qquad x > 0$$

$$= -\int_{-\infty}^{x} \frac{(x'-x)^{n-1}}{(n-1)!} \varphi(x')\, dx' \qquad x < 0$$

$$\psi_n(0^+) - \psi_n(0^-) = \int_{-\infty}^{\infty} \frac{x'^{\,n-1}}{(n-1)!} \varphi(x')\, dx' = \hat{\varphi}^{(n-1)}(0)$$

Take $\varphi(x) = \delta(b-x) \qquad b>0.$

$$\psi_n(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!}\,\delta(b-x')\,dx'$$

$$= \frac{(b-x)^{n-1}}{(n-1)!} \qquad 0<x<b \qquad n\geq 1$$

This is a specific function with the property

$$\hat{\psi}_n(s) = \int_0^b e^{xs}\,\frac{(b-x)^{n-1}}{(n-1)!}\,dx = \frac{e^{bs} - \sum_0^{n-1} \frac{b^j s^j}{j!}}{s^n}$$

I can check this

$$\int_0^b g^{(n)}(x)\,\frac{(b-x)^{n-1}}{(n-1)!}\,dx = g(b) - \sum_{j=0}^{n-1} g^{(j)}(0)\,\frac{b^j}{j!}$$

Now you want to sum the series

$$\sum \psi_n(x)\,\frac{y^n}{n!} \qquad \text{for } \varphi(x) = \delta(b-x)$$

$$\psi_0(x) = \delta(b-x)$$

$$\psi_1(x) = 1 \qquad 0<x<b$$

$$\psi_2(x) = \frac{b-x}{1!} \qquad 0<x<b$$

$$\sum \psi_n(x)\,\frac{y^n}{n!} = \begin{cases} \delta(b-x) + \overbrace{\sum_{n\geq 1} \frac{y^n}{n!}\,\frac{(b-x)^{n-1}}{(n-1)!}}^{y J_1(y(b-x))} & \text{for } 0<x\leq b \\ 0 & \text{for } x>b. \end{cases}$$

So what have you learned

$$\boxed{\left(e^{ys^{-1}} e^{bs}\right)_{hor} = e^{bs} + \int_0^b y\, J_1(y(b-x))\, e^{xs}\, dx}$$

To first order in y

$$= e^{bs} + \int_0^b y\, e^{xs}\, dx$$

$$= e^{bs} + y\,\frac{e^{bs}-1}{s}$$

$$\left(e^{ys^{-1}} e^{bs}\right)_{hor} = e^{bs} + \int_0^b y\, J_1(yx)\, e^{(b-x)s}\, dx$$

$$= e^{bs}\left(1 + \int_0^b y\, J_1(yx)\, e^{-xs}\, dx\right)$$

$$= e^{bs}\left(1 + \int_0^b J_1(yx)\, e^{-yx\frac{s}{y}}\, d(yx)\right)$$

$$= e^{bs}\left(1 + \int_0^{by} e^{-\tilde{x}\frac{s}{y}}\, J_1(\tilde{x})\, d\tilde{x}\right)$$

Go back to $\partial_t \psi = \begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi$, Cauchy problem

$$\psi(x,t) = \exp t\begin{pmatrix} \partial_x & i \\ i & -\partial_x \end{pmatrix}\psi(x,0) \qquad \psi(x,0) = \int e^{ikt}\hat{\psi}_0(k)\,\frac{dk}{2\pi}$$

$$= \int \frac{dk}{2\pi}\, e^{ikx} \exp i\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} t \qquad A_k = \begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix} \quad A_k^2 = (k^2+1)I$$

$$\omega = \sqrt{k^2+1}$$

$$= \int \frac{dk}{2\pi}\, e^{ikx}\left\{ e^{i\omega t}\,\frac{\omega + A_k}{2\omega} + e^{-i\omega t}\,\frac{\omega - A_k}{2\omega}\right\}\hat{\psi}_0(k) \qquad B_\omega = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \quad B_\omega^2 = \omega^2 - 1$$

$$\hat{\psi}_0(\omega)$$

$$\partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}\psi \qquad \psi(x,t) = \int \frac{d\omega}{2\pi}\, e^{i\omega t}\left\{ e^{ikx}\,\frac{k + B_\omega}{2k} + e^{-ikx}\,\frac{k - B_\omega}{2k}\right\}$$

Idea: ~~[illegible]~~ $\|\psi\|^2 = \int \psi^* \psi \, dx$ ind. of t.

$IH(\psi) = \int \psi^* \varepsilon \psi \, dt$ independent of x

The problem — discuss it. You have solutions of the wave expressed in terms of Cauchy data, but you ~~a picture~~ want the ~~[illegible]~~ analytic function picture. So perhaps you want to go between describing $\psi(x,t)$ by a 2 component function $\psi(x,0)$ or $\hat{\psi}_0(k)$, and a corresponding Laurent holom. function of ζ.

Have spectrum described by (ω, k) $\omega^2 = k^2 + 1$.

~~[illegible]~~ modify notation

$$\frac{e^{i(kx-\omega t)}}{2\omega} \begin{pmatrix} \omega - k & -1 \\ -1 & \omega + k \end{pmatrix} + \frac{e^{i(kx+\omega t)}}{+2\omega} \begin{pmatrix} +\omega + k & +1 \\ +1 & +\omega - k \end{pmatrix}$$

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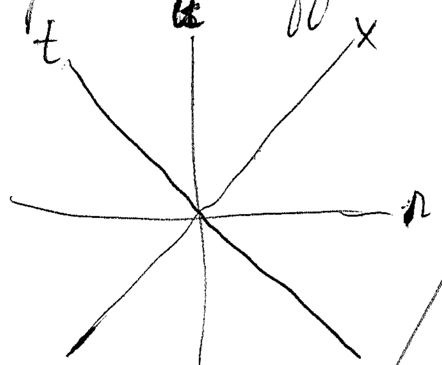
proceed differently.

$$(\partial_t - \partial_x)\psi^1 = i\psi^2 \qquad -\partial_r \psi^1 = i\psi^2$$
$$(\partial_t + \partial_x)\psi^2 = i\psi^1 \qquad \partial_u \psi^2 = i\psi^1$$



$x = r + u$
$t = -r + u$

$$\partial_r = \partial_x - \partial_t$$
$$\partial_u = \partial_x + \partial_t$$

Exp. solns.

$$-\rho V^1 = V^2 \qquad e^{i(r\rho + u\sigma)} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$
$$\sigma V^2 = V^1 \qquad i\sigma = -i\rho^{-1} = (i\rho)^{-1}$$

$$e^{r(i\rho) + u(i\rho)^{-1}} \begin{pmatrix} -i(i\rho)^{-1} \\ 1 \end{pmatrix}$$

$V^1 = -\frac{1}{\rho} = \frac{-i}{i\rho}$

$-\rho^{-1} =$

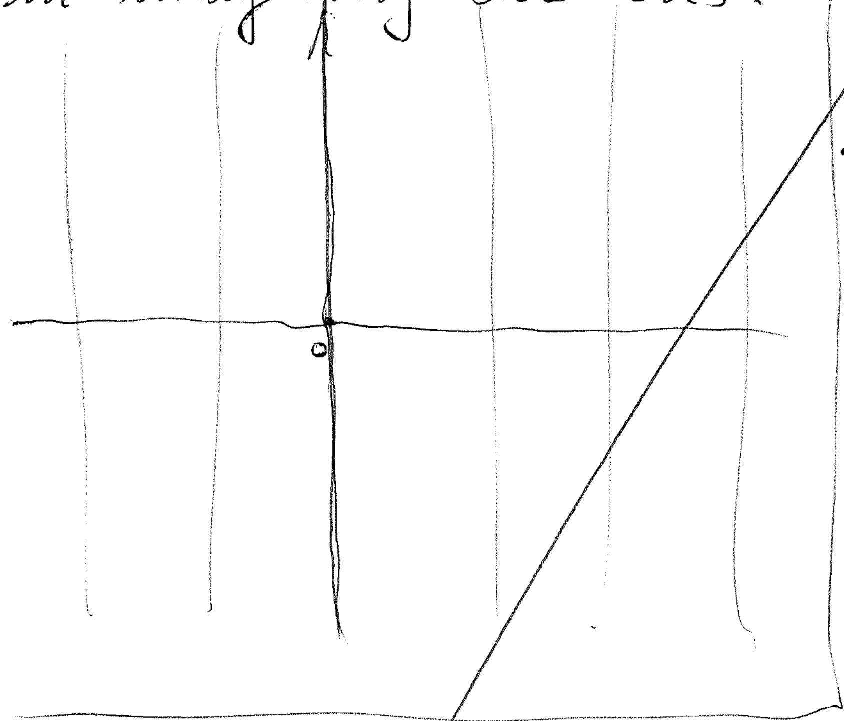
Another attempt.

$$\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}$$

horiz space gen by e^{xs}
vert —— $e^{ys^{-1}}\frac{1}{s}$

$$\partial_x \psi^1 = \psi^2$$
$$\partial_y \psi^2 = \psi^1$$

Grid space should be analytic functions on $\mathbb{C}-\{0\}$ i.e. Laurent series which are somehow adapted to the Fourier integral with x, y real, i.e. you want growth in real directions and things to be oscillatory in imaginary directions.



Let's try to start with this s picture.

$$(k_\varepsilon e^{i\varepsilon\rho}-1)v^1 = c\varepsilon v^2$$
$$(k_\varepsilon e^{i\varepsilon\sigma}-1)v^2 = \bar{c}\varepsilon v^1$$

$$i\rho\, v^1 = c\, v^2$$
$$i\sigma\, v^2 = \bar{c}\, v^1$$

$$= |c|^2$$
$$(i\rho)(i\sigma) = |c|^2.$$

"universal" solution

$$\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}$$

Repeat. $e^{xs+ys^{-1}} = \sum_{i,j\geq 0} \frac{x^i y^j}{i!\,j!} s^{i-j}$

$-n = i-j \quad j = i+n$

$$= \sum_{n\geq 0} s^n \sum_{j\geq 0} \frac{x^{j+n} y^j}{(j+n)!\,j!} + \sum_{n\geq -1} s^{-n} \sum_{i\geq 0} \frac{x^i y^{i+n}}{i!\,(i+n)!}$$

$$= \sum_{n\geq 0} s^n x^n J_n(xy) + \sum_{n\geq -1} s^{-n} y^n J_n(xy)$$

$f = \hat{\varphi}(s) = \int e^{xs} \varphi(x)\,dx$ entire function of s

$$e^{ys^{-1}} \hat{\varphi}(s) = f(s) + y\frac{f(s)}{s} + \frac{y^2}{2!}\frac{f(s)}{s^2} + \cdots$$

~~We get~~

$$\left(e^{ys^{-1}} \hat{\varphi}(s)\right)_{reg} = f(s) + y\frac{f(s)-f(0)}{s} + \frac{y^2}{2!}\frac{f(s)-f(0)-f'(0)s}{s^2} + \cdots$$

~~so what can be done~~. You want to write this as $\sum_{n\geq 0} \frac{y^n}{n!}\hat{\varphi}_n$

$$\hat{\varphi}_n = \frac{f(s)-f(0)-f'(0)s - \cdots - f^{(n-1)}(0)\frac{s^{n-1}}{(n-1)!}}{s^n}$$

$$s^n\hat{\varphi}_n = \int s^n e^{xs} \varphi_n(x)\,dx = \int \partial_x^n(e^{xs})\,\varphi_n(x)\,dx$$

$$= \int e^{xs}(-\partial_x)^n \varphi_n(x)\,~~dx~~$$

$$f - s^n\hat{\varphi}_n = \int e^{xs}\left(\varphi - (-\partial_x)^n \varphi\right)(x)\,dx = \sum_{k=0}^{n-1} a_k s^k \qquad \frac{f^{(k)}(0)}{k!}$$

$$= \int e^{xs} \sum_{k=0}^{n-1} a_k (-\partial_x)^k \delta(x)\,dx$$

There may be something here but in any case you get

$$(-\partial_x)^n \varphi_n(x) = \varphi(x) \qquad x \neq 0$$

$$\varphi_n(x) = 0 \qquad \text{if} \quad x \gg 0,\ x \ll 0$$

$$\psi_n(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!}\,\varphi(x')\,dx' \qquad x>0,\quad n\geq 1$$

$$= -\int_{-\infty}^{x} \frac{(x'-x)^{n-1}}{(n-1)!}\,\varphi(x')\,dx' \qquad x<0,\quad n\geq 1$$

Consider $\varphi(x) = \delta(x-b)$ $\qquad b>0.$

$$\psi_n^b(x) = 0 \qquad x<0$$

$$= \frac{(b-x)^{n-1}}{(n-1)!} \qquad 0<x<b \qquad n\geq 1.$$

$$\sum_{n\geq 0} \frac{y^n}{n!}\,\psi_n^b(x) = \delta(x-b) + \sum_{n\geq 1}\frac{y^n}{n!}\begin{cases}\frac{(b-x)^{n-1}}{(n-1)!} & \text{for } 0<x<b\\ 0 & \text{otherwise}\end{cases}$$

$$= \delta(x-b) + y\,J_1(y(b-x))\,\chi_{(0,b)}(x)$$

$$\left(e^{ys^{-1}+bs}\right)_{hor} = e^{bs} + \int_0^b e^{xs}\,y\,J_1(y(b-x))\,dx$$

$$= e^{bs} + \int_0^b e^{bs-xs}\,y\,J_1(yx)\,dx$$

$$= e^{bs}\left(1+\int_0^{by} e^{-\frac{xs}{y}}\,J_1(x)\,dx\right)$$

$$\left(e^{s^{-1}+bs}\right)_{hor} = e^{bs}\left(1+\int_0^b e^{-xs}\,J_1(x)\,dx\right)$$

$$\left(e^{s^{-1}+bs}\right)_{vert} = e^{s^{-1}+bs} - e^{bs}\left(1+\int_0^b e^{-xs}J_1(x)\,dx\right)$$

$$\frac{f(0)}{s} + \frac{1}{2!}\frac{f(0)+f'(0)s}{s^2} + \frac{1}{3!}\frac{f(0)+f'(0)s+f''(0)\frac{s^2}{2}}{s^3} + \cdots\cdots$$

$$= e^{s^{-1}+bs}\Big|_{\text{sing}} = \frac{1}{s} + \frac{1}{2!}\frac{1+bs}{s^2} + \frac{1}{3!}\frac{1+bs+\frac{(bs)^2}{2!}}{s^3} + \frac{1}{4!}\frac{1+\frac{bs}{1!}+\frac{(bs)^2}{2!}+\frac{(bs)^3}{3!}}{s^4}$$

$$= \sum_{n\geq 1} \frac{1}{s^n} J_n(b) \qquad J_n(b) = \sum_{i\geq 0} \frac{b^i}{i!\,(i+n)!}$$

$$J_n(b) = \int e^{e^{-i\theta}+be^{i\theta}}\, e^{in\theta}\, \frac{d\theta}{2\pi}$$

$$\frac{1}{s}\left(\frac{1}{1!\,0!} + \frac{1}{2!}\frac{1}{1!}b + \frac{1}{3!}\frac{1}{2!}b^2 + \frac{1}{4!}\frac{1}{3!}b^3\right.$$

$$\frac{1}{s^2}\left(\frac{1}{2!\,0!} + \frac{1}{3!\,1!}b + \frac{1}{4!\,2!}b^2\right.$$

Here b is real, positive say

$$\partial_b J_n(b) = \sum_{i\geq 1} \frac{b^{i-1}}{(i-1)!\,(i+n)!} = J_{n+1}(b)$$

$$\sum_{n\geq 1}{}' \frac{1}{s^n}\partial_b^n J_0(b) = \frac{s^{-1}\partial_b}{1-s^{-1}\partial_b} J_0(b)$$

$$= \frac{\partial_b}{s-\partial_b} J_0(b) = \frac{1}{s-\partial_b} J_1(b)$$

$$\frac{1}{s-\partial_b} J_1(b) = e^{bs}\int_0^b e^{-xs} J_1(x)\,dx$$

forgot bdry cond.

$$\partial_b\left(\sum_{n\geq 1} \frac{1}{s^n} J_n(b)\right) = \sum_{n\geq 1}\frac{1}{s^n} J_{n+1}(b)$$

$$s\left(\sum_{n\geq 1}\frac{1}{s^n}J_n(b)\right) = \sum_{n\geq 0}\frac{1}{s^n}J_{n+1}(b)$$

$$(s-\partial_b)\left(\sum_{n\geq 0}\frac{1}{s^n}J_n(b)\right) = J_1(b)$$

$$\therefore \sum_{n\geq 0}\frac{1}{s^n}J_n(b) = e^{bs}\int_{b} e^{-xs} J_1(x)\,dx$$

$$\sum_{n\geq 0}\frac{1}{s^n}J_n(b) = \frac{1}{1-s^{-1}\partial_b}J_0(b)$$

$$= \frac{1}{\ }$$

$$\partial_b\left(\overbrace{\sum_{n\geq 0} s^{-n}J_n(b)}^{F}\right) = \sum_{n\geq 0} s^{-n}J_{n+1}(b)$$

$$= s\sum_{n\geq 0} s^{-n-1}J_{n+1}(b) = s(F-J_0)$$

$$\therefore (\partial_b - s)F = -sJ_0$$

$$F = \int e^{(b-x)s}\, s\, J_0(x)\,dx$$

$$\partial_b \underbrace{\sum_{n\geq 1}\frac{1}{s^n}J_n(b)}_{\Phi} = s\sum_{n\geq 1}\frac{1}{s^{n+1}}J_{n+1}(b) = s\left(\Phi - \frac{1}{s}J_1\right)$$

$$\therefore (\partial_b - s)\Phi(b) = -J_1(b)$$

$$\Phi(b) = -\int_0^b e^{(b-x)s} J_1(x)\,dx$$

$$\left(e^{s^{-1}+bs}\right)_{reg} = \sum_{n\geq 0} s^n J_n(b) = \Phi$$

$$\partial_b \Phi = \frac{1}{s}\sum_{n\geq 0} s^{n+1} J_{n+1}(b) = \frac{1}{s}\left(\Phi - J_0\right)$$

$$\Phi(b) = \int^b e^{s^{-1}(b-x)}$$

$-n = -i+j$ $\qquad n = -i+j \qquad j = i+n$

$$e^{s^{-1}+bs} = \sum_{i,j\geq 0} s^{-i+j}\frac{b^j}{i!\,j!} = \sum_{n\geq 1} s^{-n}\sum_{j\geq 0}\frac{b^j}{(n+j)!\,j!} + \sum_{n\geq 0} s^n b^n \sum_{i\geq 0}\frac{b^i}{i!\,(i+n)!}$$

$$\boxed{e^{s^{-1}+bs} = \sum_{n\geq 1} s^{-n} J_n(b) + \sum_{n\geq 0} s^n b^n J_n(b)}$$

$$\Phi = \sum_{n\geq 1} s^{-n} J_n(b) \qquad \partial_b \Phi = s\sum_{n\geq 1} s^{-n-1} J_{n+1}(b) = s\left(\Phi - s^{-1}J_1(b)\right)$$

So $(\partial_b - s)\Phi = -J_1(b)$. This is a puzzle because the DE equation is regular at $s=0$, so you get ~~a regular soln.~~ the standard part. soln.

$$\Phi(b) = e^{bs}\int_0^b e^{-xs}\left(-J_1(x)\right)dx$$

+ something ind of b times e^{bs}. But ~~this~~ you found that

$$\left(e^{s^{-1}+bs}\right)_{hor} = e^{bs}\left(1+\int_0^b e^{-xs}J_1(x)\,dx\right)$$

Check signs. $\boxed{e^{s^{-1}+bs} = \sum_{n\geq 1} s^{-n} J_n(b) + \sum_{n\geq 0} (bs)^n J_n(b)}$ 195

$$J_n(z) = \sum_{i\geq 0} \frac{z^i}{i!(i+n)!} \qquad \partial_z J_n(z) = J_{n+1}(z)$$

Set $\Phi = \sum_{n\geq 1} s^{-n} J_n(b)$, $\partial_b \Phi = s \sum_{n\geq 1} s^{-n-1} J_{n+1}(b)$

$= s(\Phi - s^{-1} J_1)$. $\quad (\partial_b - s)\Phi = -J_1$, $b \geq 0$

$$\boxed{\Phi = (\text{fn of } s)\, e^{bs} - \int_0^b e^{(b-x)s} J_1(x)\,dx}$$

$$\left(e^{s^{-1}+bs}\right)_{\text{hor}} = e^{bs} + \frac{e^{bs}-1}{s} + \frac{1}{2!}\frac{e^{bs}-1-bs}{s^2} + \cdots$$

$$= e^{bs} + \hat{\varphi}_1 + \frac{1}{2!}\hat{\varphi}_2 + \frac{1}{3!}\hat{\varphi}_3 + \cdots$$

$(-\partial_x)^n \psi_n = \varphi(x) \quad x \neq 0, \quad \psi_n = 0 \quad x \ll 0, \gg 0.$

$$\psi_n(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!}\varphi(x')\,dx' \qquad x > 0 \qquad n \geq 1$$

$$-\int_\infty^x$$

$\varphi(x) = \delta(b-x) \quad b > 0 \quad$ then $\quad \psi_n(x) = \frac{(b-x)^{n-1}}{(n-1)!} \quad 0 < x < b$

$0 \quad x < 0, x > b$

$$\delta(b-x) + \sum_{n\geq 1} \frac{1}{n!}\frac{(b-x)^{n-1}}{(n-1)!} = \delta(b-x) + J_1(b-x) \qquad 0 < x < b$$

$$\left(e^{s^{-1}+bs}\right)_{\text{hor}} = e^{bs} + \int_0^b e^{xs} J_1(b-x)\,dx$$

$$\sum_{n\geq 0} s^n b^n J_n(b) = e^{bs} + \int_0^b e^{(b-x)s} J_1(x)\,dx$$

It seems that

$$\left(e^{s^{-1}+bs}\right)_{hor} \stackrel{def}{=} \sum_{n\geq 0} s^n b^n J_n(b) = e^{bs} + \int_0^b e^{(b-x)s} J_1(x)\,dx$$

and that

$$\Phi \stackrel{def}{=} \sum_{n\geq 1} s^{-n} J_n(b) = \left(\begin{smallmatrix}\text{some fn}\\ \text{of } s\end{smallmatrix}\right) e^{bs} - \int_0^b e^{(b-x)s} J_1(x)\,dx$$

Add the two to get.

$$e^{s^{-1}+bs} = e^{bs} + \underbrace{\left(\begin{smallmatrix}\text{some fn}\\ \text{of } s\end{smallmatrix}\right)}_{e^{s^{-1}}-1} e^{bs}$$

OK things check.

Remaining problem is to write

$$\left(e^{s^{-1}+bs}\right)_{ver} \stackrel{def}{=} \sum_{n\geq 1} \frac{1}{s^n} J_n(b)$$

$$\text{also} = \frac{1}{s} + \frac{1}{2!}\frac{1+bS}{s^2} + \frac{1}{3!}\frac{1+bs+\frac{(bs)^2}{2!}}{s^3} + \cdots$$

in the form $\int e^{ys^{-1}} \varphi(y)\,dy$ with $\varphi(y)$ of compact support

The idea is to use the circle aspect of Laurent series.

$$\oint e^{ys^{-1}+xs} s^{-n} \frac{ds}{2\pi i s} = \text{essentially } J_n(xy).$$

basic idea is arrange

$$ys^{-1} + xs = \sqrt{xy}\cos\theta$$

$$\begin{array}{ccc} W & \xrightarrow{\iota_+} & V \\ {\scriptstyle s}\downarrow & & \downarrow H_x \\ W^+ & \xleftarrow{\pi_+} & V^+ \end{array}$$

You have ~~problem~~ two ~~components~~ vectors $\xi_-, \xi_- \in V$ which are being projected, or split

orth proj $\rightsquigarrow$ Birkhoff decompositions.

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lecture. $$e^{s^{-1}+bs} = \underbrace{\sum_{n\geq 0} s^n b^n J_n(b)}_{\sum_{n\geq 0} s^n \sum_{i\geq 0} \frac{b^{n+i}}{i!(n+i)!}} + \underbrace{\sum_{n\geq 1} s^{-n} J_n(b)}_{\sum_{n\geq 1} s^{-n} \sum_{j\geq 0} \frac{b^j}{j!(n+j)!}}$$

$$\sum_{i,j\geq 0} s^{-i+j} \frac{b^j}{i!\,j!}$$

$n = -i+j \qquad j = n+i$

$-n = -i+j \qquad i = n+j$

$$\left(e^{s^{-1}+bs}\right)_{hor} = \overbrace{e^{bs}}^{\widehat{\delta(b-x)}} + \underbrace{\frac{e^{bs}-1}{s}}_{\hat\psi_1} + \frac{1}{2!}\underbrace{\frac{e^{bs}-1-bs}{s^2}}_{\hat\psi_2} + \frac{1}{3!}\underbrace{\frac{e^{bs}-1-bs-\frac{1}{2!}(bs)^2}{s^3}}_{\hat\psi_3} + \cdots$$

Define $$(-\partial_x)^n \psi_n(x) = \delta(b-x)$$ assume $b>0$.

$$\psi_n(x) = 0 \qquad x > b$$

$n \geq 1$ $$\psi_n(x) = \int_x^\infty \frac{(x'-x)^{n-1}}{(n-1)!}\delta(b-x')\,dx' = \frac{(b-x)^{n-1}}{(n-1)!} \qquad 0<x<b$$

$$\hat\psi_n(s) = \int_0^b e^{xs}\frac{(b-x)^{n-1}}{(n-1)!}\,dx = \int_0^b e^{(b-x)s}\frac{x^{n-1}}{(n-1)!}\,dx$$

$$= \frac{e^{bs}-1-bs-\cdots-\frac{(bs)^{n-1}}{(n-1)!}}{s^n} \qquad n\geq 1$$

$$\left(e^{s^{-1}+bs}\right)_{hor} = e^{bs} + \sum_{n\geq 1} \frac{1}{n!}\widehat{\varphi_n}$$

$$= e^{bs} + \int_0^b \sum_{n\geq 1} \frac{e^{(b-x)s}}{n!} \frac{x^{n-1}}{(n-1)!} dx$$

$$\sum_{n\geq 0} s^n b^n J_n(b) = e^{bs} + \int_0^b e^{(b-x)s} J_1(x)\, dx$$

~~Best~~

$$\sum_{n\geq 0} (s)^n J_n(b) = e^{s} + \int_0^b e^{s - xsb^{-1}} J_1(x)\, dx$$

$$\cancel{\sum_{n\geq 0} s^{-n} J_n(b) = e^{s^{-1}} + \int_0^b e^{s^{-1} - xs^{-1}b^{-1}}}$$

You want $\left(e^{s^{-1}+bs}\right)_{ver} = \sum_{n\geq 1} s^{-n} J_n(b)$

to be written in the form $\int \frac{e^{ys^{-1}}}{s^{-1}} \varphi(y)\, dy$

So you want.

$$\sum_{n\geq 1} s^n J_n(b) = \int s e^{ys} \varphi(y)\, dy$$

$$\sum_{n\geq 0} s^n J_n(b) = e^{s} + \int_0^b e^{(1-xb^{-1})s} J_1(x)\, dx$$

$$\partial_b J_0(b) = J_1(b)$$

$$J_0(b) = 1 + \int_0^b J_1(x)\, dx$$

$$\sum_{n\geq 1} s^n J_n(b) = e^{s} - 1 + \int_0^b \left(e^{(1-xb^{-1})s} - 1\right) J_1(x)\, dx$$

$$f(b) = f(0) + \int_0^b \partial_x f(x)\,dx$$

$$= f(0) + \Big[f'(x)(x-b)\Big]_0^b + \int_0^b f''(x)\frac{(b-x)}{1!}\,dx$$

$$= f(0) + f'(0)b + \underbrace{\Big[f''(x)\frac{(b-x)^2}{2!}(-1)\Big]_0^b}_{f''(0)\frac{b^2}{2!}} + \int_0^b f'''(x)\frac{(b-x)^2}{2!}\,dx$$

$$f(b) = \sum_{j=0}^{n-1}\frac{f^{(j)}(0)}{j!}b^j + \int_0^b f^{(n)}(x)\frac{(b-x)^{n-1}}{(n-1)!}\,dx$$

$$e^{bs} = \sum_{j=0}^{n-1}\frac{(bs)^j}{j!} + \int_0^b s^n e^{xs}\frac{(b-x)^{n-1}}{(n-1)!}\,dx$$

$$\left(e^{s^{-1}}e^{bs}\right)_{hor} = \sum_{n\geq 0}\frac{1}{n!\,s^n}\left(e^{bs} - \sum_{j=0}^{n-1}\frac{(bs)^j}{j!}\right)$$

$$= \sum_{n\geq 0}\frac{1}{n!\,s^n}\sum_{j\geq n}\frac{(bs)^j}{j!} \qquad i+n=j$$

$$= \sum_{n\geq 0}\frac{1}{n!\,s^n}\sum_{i\geq 0}\frac{(bs)^{n+i}}{(n+i)!}$$

$$= \sum_{i\geq 0} s^i b^i \sum_{n\geq 0}\frac{b^n}{n!\,(n+i)!} = \sum_{i\geq 0}(bs)^i J_n(b)$$

$$= e^{bs} + \int_0^b \sum_{n\geq 0}\frac{1}{n!}e^{xs}\frac{(b-x)^{n-1}}{(n-1)!}\,dx$$

$$= e^{bs} + \int_0^b e^{xs} J_1(b-x)\,dx$$

$$\left(e^{s^{-1}+bs}\right)_{\text{reg}} = \sum_{n\geq 1} \frac{1}{n!\,s^n} \sum_{j=0}^{n-1} \frac{(bs)^j}{j!} \qquad \begin{array}{l} i = n-j \\ n = i+j \end{array}$$

$$= \sum_{\substack{i\geq 1 \\ j\geq 0}} \frac{1}{s^i} \frac{b^j}{(i+j)!\,j!} = \sum_{i\geq 1} s^{-i} J_i(b)$$

want

$$\sum_{n\geq 1} \frac{1}{s^n} J_n(b) = \int \frac{e^{ys^{-1}}}{s}\, \varphi(y)\, dy$$

But you know

$$\sum_{n\geq 0} s^n b^n J_n(b) = e^{bs} + \int_0^b e^{xs} J_1(b-x)\, dx - J_0(b)$$

$$\int_0^b s e^{xs} J_0(b-x)\, dx = \int_0^b \partial_x(e^{xs})\, J_0(b-x)\, dx$$

$$= \left[e^{xs} J_0(b-x)\right]_0^b - \int_0^b e^{xs} \underbrace{\partial_x J_0(b-x)}_{-J_1(b-x)}\, dx$$

$$= e^{bs} - J_0(b) + \int_0^b e^{xs} J_1(b-x)\, dx$$

$$\therefore \sum_{n\geq 1} s^{-n} b^n J_n(b) = \int_0^b s^{-1} e^{xs^{-1}} J_0(b-x)\, dx$$

$$\sum_{n\geq 1} s^{-n} J_n(b) =$$

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$$= \int_0^b \frac{s}{b}\, e^{b^{-1}xs^{-1}} J_0(b-x)\, dx = \int_0^1 s\, e^{x's} J_0(b(1-x'))\, dx'$$

$$\left(e^{s^{-1}}e^{bs}\right)_{hor} = \sum_{n\geq 0} s^n b^n J_n(b) = e^{bs} + \int_0^b e^{xs} J_1(b-x)\,dx$$

$$\cancel{\left(e^{s^{-1}}e^{bs}\right)_{ver} = \sum_{n\geq 1} s^{-n} J_n(b) = e^{bs} + \int_0^b e^{xs} J_1(b-x)\,dx - J_0(b)}$$

$$\cancel{= \int_0^b s e^{xs} J_0(b-x)\,dx}$$

$$\int_0^b \underbrace{s e^{xs}}_{\partial_x(e^{xs})} J_0(b-x)\,dx = \left[e^{xs} J_0(b-x)\right]_0^b + \int_0^b e^{xs} J_1(b-x)\,dx$$

$$= e^{bs} - J_0(b) + \int_0^b e^{xs} J_1(b-x)\,dx$$

$$= \sum_{n\geq 1} (sb)^n J_n(b)$$

$$\sum_{n\geq 1} (sb)^n J_n(b) = \int_0^b s e^{xs} J_0(b-x)\,dx$$

$$\sum_{n\geq 1} s^n J_n(b) = \int_0^b s\, e^{\overset{y}{\frac{x}{b}} s} J_0(b-x) \overset{dy}{\frac{dx}{b}}$$

$$= \int_0^1 s e^{ys} J_0(b(1-y))\,dy$$

Last

$$\boxed{\sum_{n\geq 1} s^{-n} J_n(b) = \int_0^1 \frac{e^{ys^{-1}}}{s} J_0(b(1-y))\,dy = \left(e^{s^{-1}}e^{bs}\right)_{ver}}$$

$$\left(e^{as^{-1}+bs}\right)_{hor} = \sum_{n\geq 0} s^n b^n J_n(ab)$$

$$= e^{bs} + \int_0^b e^{xs}\, J_1(a(b-x))\, a\, dx$$

$$\int_0^b s e^{xs}\, J_0(a(b-x))\, dx = \Big[e^{xs} J_0(a(b-x))\Big]_0^b + \int_0^b e^{xs} \underbrace{\partial_x J_0(a(b-x))\, dx}_{J_1(a(b-x))\,(+a)\,dx}$$

$$\boxed{\int_0^b s e^{xs} J_0(a(b-x))\,dx = e^{bs} - J_0(ab) + \int_0^b e^{xs} J_1(a(b-x))\,a\,dx}$$

$$\boxed{= \sum_{n\geq 1} s^n b^n J_n(ab)}$$

$$\boxed{\cancel{\left(e^{as^{-1}+bs}\right)_{ver}} \sum_{n\geq 1} s^{-n} b^n J_n(ab) = \int_0^b \frac{e^{xs^{-1}}}{s} J_0(a(b-x))\,dx}$$

$$\left(e^{as^{-1}+bs}\right)_{hor} = \sum_{n\geq 0} s^n b^n J_n(ab) = e^{bs}\left(1 + \int_0^b e^{-xs} J_1(ax)\,a\,dx\right)$$

$$\left(e^{as^{-1}+bs}\right)_{ver} = \sum_{n\geq 1} s^{-n} a^n J_n(ab) = e^{as^{-1}} \int_0^a \frac{e^{-xs^{-1}}}{s} J_0(bx)\,dx$$

you probably need two more namely

$\left(\frac{1}{s} e^{as^{-1}+bs}\right)_{hor}$ $\quad \left(\frac{1}{s} e^{as^{-1}+bs}\right)_{ver}$

probably $J_0(ab)$ gets shifted.

Riemann Green's fn. ~~Potential scattering~~

Time to understand varying b to $bz^x = be^{i\lambda x}$ and differentiating wrt x.

The basic object before was the group $\mathbb{Z}$ of discrete translations, now you to replace it by $\mathbb{R}$. One encounters problems with the group ring which is non unital. ~~Also different~~ When you get to scattering, you encounter invertible matrices over the group ring.

~~Discrete case (Hilbert's~~ It's important to focus on corresponding objects in the disc. + cont. cases, and to treat them similarly — their properties, behavior should be essentially the same.)

~~Philosophy ... Objects in the~~

Look at inverse scattering, you start with the spectral picture, functions matrices depending on eigenvalue parameter z, λ.

Philosophy gets nowhere, so look at example, where to begin

Go over Bessel stuff one more time.

$$\left(e^{as^{-1}+bs}\right)_{hor} = \sum_{n\geq 0} s^n \sum_{\substack{n=-i+j\\ i,j\geq 0}} \frac{a^i}{i!}\frac{b^j}{j!} \overset{j=n+i}{=} \sum_{n\geq 0} s^n \sum_{i\geq 0} \frac{a^i b^{i+n}}{i!\,(i+n)!}$$

$$= \sum_{n\geq 0} s^n b^n J_n(ab)$$

$$\left(e^{as^{-1}+bs}\right)_{hor} = \frac{e^{bs}}{0!} + \frac{a}{1!}\frac{e^{bs}-1}{s} + \frac{a^2}{2!}\frac{e^{bs}-1-\frac{bs}{1!}}{s^2}$$

$$= \sum_{n\geq 0} \frac{a^n}{n!\,s^n} \sum_{j\geq n} \frac{b^j s^j}{j!}$$

Taylor

$$f(b) = f(0) + \int_0^b \overbrace{}^{-d(b-x)} f'(x)\,dx$$

$$= f(0) + \Big[-f'(x)(b-x)\Big]_0^b + \int_0^b f''(x)\underbrace{(b-x)\,dx}_{-d\left(\frac{(b-x)^2}{2!}\right)}$$

$$= f(0) + f'(0)b + \left[-f''(x)\frac{(b-x)^2}{2!}\right]_0^b + \int_0^b f'''(x)\frac{(b-x)^2}{2!}\,dx$$

$$f(b) - \sum_{j=0}^{n-1} f^{(j)}(0)\frac{b^j}{j!} = \int_0^b f^{(n)}(x)\frac{(b-x)^{n-1}}{(n-1)!}\,dx$$

$$e^{bs} - \sum_{j=0}^{n-1}\frac{(bs)^j}{j!} = \int_0^b s^n e^{sx}\frac{(b-x)^{n-1}}{(n-1)!}\,dx$$

for $n \geq 1$.

$$\left(e^{as^{-1}+bs}\right)_{hor} = \sum_{n\geq 0}\frac{a^n}{n!\,s^n}\left(e^{bs} - \sum_{j=0}^{n-1}\frac{(bs)^j}{j!}\right)$$

$$= \overset{e^{bs}}{\cancel{1}} + \int_0^b e^{sx}\sum_{n\geq 1}\frac{a^n}{n!}\frac{(b-x)^{n-1}}{(n-1)!}\,dx = \overset{e^{bs}+}{\cancel{1}}\int_0^b e^{sx} J_1\big(a(b-x)\big)\,a\,dx$$

$$\left(e^{as^{-1}+bs}\right)_{\text{ve.}} = \sum_{n\geq 1} s^{-n} \sum_{\substack{-i+j=-n \\ i,j\geq 0}} \frac{a^i b^j}{i!\,j!} \qquad n+j=i$$

$$= \sum_{n\geq 1} s^{-n} \sum_{j\geq 0} \frac{a^{n+j} b^j}{(n+j)!\,j!} = \sum_{n\geq 1} s^{-n} a^n J_n(ab)$$

We have established the formula

$$\sum_{n\geq 0} s^n b^n J_n(ab) = e^{bs} + \int_0^b e^{sx} J_1(a(b-x))\, a\, dx$$

$$\Longrightarrow \sum_{n\geq 0} s^n a^n J_n(ab) = e^{as} + \int_0^a e^{sx} \underbrace{J_1(b(a-x))\, b}_{-\partial_x J_0(b(a-x))}\, dx$$

$$= e^{as} + \underbrace{\left[-e^{sx} J_0(b(a-x))\right]_0^a}_{\cancel{-e^{as}} + J_0(ab)} + \int_0^a s e^{sx} J_0(b(a-x))\, dx$$

$$\therefore \sum_{n\geq 1} s^n a^n J_n(ab) = \int_0^a s e^{xs} J_0(b(a-x))\, dx$$

$$\sum_{n\geq 1} s^{-n} a^n J_n(ab) = \int_0^a \frac{e^{ys^{-1}}}{s^{-1}} J_0(b(a-y))\, dy$$

$$\left(e^{as^{-1}+bs}\right)_{ver} = e^{as^{-1}} \int_0^{ab} \frac{e^{-x/bs}}{bs} J_0(x)\,dx$$

$$\left(e^{as^{-1}+bs}\right)_{hor} = e^{bs}\left(1 + \int_0^{ab} e^{-xs/a} J_1(x)\,dx\right)$$

notice that both are unchange by $a \to b, s \to s^{-1}$
except for e^{bs} by itself.

$$\boxed{\begin{aligned} \partial_x \psi^1 &= h(x,y)\psi^2 \\ \partial_y \psi^2 &= \bar{h}(x,y)\psi^1 \end{aligned}}$$

$$\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_y \end{pmatrix} \psi = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix} \psi$$

$h = 0.$

$\partial_x \psi^1 = 0 \Longrightarrow \psi^1$ function of y

$\partial_y \psi^2 = 0 \Longrightarrow \psi^2$ ——— x.

$$\begin{pmatrix} \partial_x & h \\ \bar{h} & \partial_y \end{pmatrix} \psi = f \qquad \text{inhomog.}$$

$$\partial_x H(x) = \delta(x).$$

$\partial_x u = f$ sol. $u(x) = \int^x f(x')\,dx'$

$$= \int \underbrace{H(x-x')}_{1 \text{ when } x > x'} f(x')\,dx'$$

Solve the inhomog equation

$$\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_y \end{pmatrix} \psi = \begin{pmatrix} 0 & h_1 \\ h_2 & 0 \end{pmatrix} \psi$$

$$\partial_x \psi^1 = h_1 \psi^2 \qquad \psi^1(x,y) = \int_0^x h_1(x',y) \psi^2(x',y)\, dx' + \psi^1(0,y)$$

$$\partial_y \psi^2 = h_2 \psi^1 \qquad \psi^2(x,y) = \int_0^y h_2(x,y') \psi^1(x,y')\, dy' + \psi^2(x,0)$$

Use Laplace transf. in both directions.

$$\mathcal{L} = \mathcal{L}_x \mathcal{L}_y \psi = \int_0^\infty e^{-\xi x} dx \int_0^\infty e^{-\eta y} dy\ \psi(x,y).$$

$$\mathcal{L}_x(\partial_x \psi^1) = \int_0^\infty e^{-\xi x} (\partial_x \psi^1)(x,y)\, dx$$

$$= \left[e^{-\xi x} \psi^1 \right]_0^\infty + \xi \int_0^\infty e^{\xi x} \psi^1\, dx$$

$$= -\psi^1(0,y) + \xi (\mathcal{L}_x \psi^1)(\xi, y)$$

$$h\mathcal{L}(\psi^2) = \mathcal{L}(\partial_x \psi^1) = -\mathcal{L}_y \psi^1(0,y) + \xi(\mathcal{L}\psi^1)$$

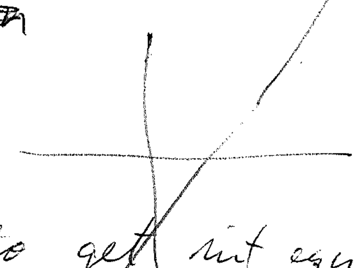
$$h\mathcal{L}(\psi^1) = \mathcal{L}(\partial_y \psi^2) = -\mathcal{L}_x \psi^2(x,0) + \eta(\mathcal{L}\psi^2)$$

$$\begin{pmatrix} 1 & -\eta \\ -\xi & 1 \end{pmatrix} \begin{pmatrix} \mathcal{L}\psi^1 \\ \mathcal{L}\psi^2 \end{pmatrix} = \begin{pmatrix} -\hat{f}^2(\xi) \\ -\hat{f}^1(\eta) \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{L}\psi^1 \\ \mathcal{L}\psi^2 \end{pmatrix} = \frac{1}{1-\xi\eta} \begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix} \begin{pmatrix} -\hat{f}^2(\xi) \\ -\hat{f}^1(\eta) \end{pmatrix}$$

$$= \frac{-1}{1-\xi\eta} \begin{pmatrix} \hat{f}^2(\xi) + \eta \hat{f}^1(\eta) \\ \xi \hat{f}^2(\xi) + \hat{f}^1(\eta) \end{pmatrix}$$

Yesterday found Riemann's Green's function for $\begin{pmatrix}\partial_x & 0\\ 0 & \partial_y\end{pmatrix}\psi = \begin{pmatrix}0 & h\\ \bar{h} & 0\end{pmatrix}\psi$,



$$\partial_x\psi_1 = h\psi_2 \qquad \partial_y\psi_2 = \bar{h}\psi_1$$

namely, you integrate to get int eqn.

$$\psi^1(x,y) = \psi^1(0,y) + \int_0^x (h\psi^2)(x',y)\,dx'$$
$$\psi^2(x,y) = \psi^2(x,0) + \int_0^y (\bar{h}\psi^1)(x,y')\,dy'$$

of Volterra type which you can ~~iterate~~ as usual to obtain unique solution with given $\psi^1(0,y)$, $\psi^2(x,0)$ on the characteristic lines.

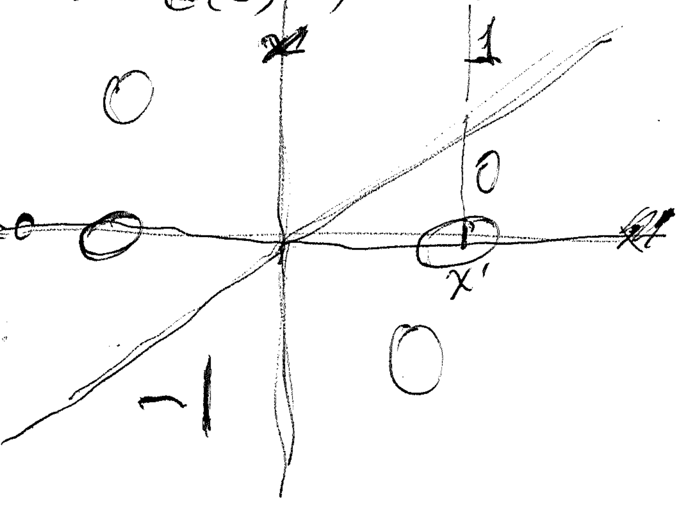
$$\partial_x u = f \qquad u(x) = \int_{-\infty}^{x} f(x')\,dx' = \int_{-\infty}^{\infty} H(x-x')f(x')\,dx'$$

$u(x) = u(0) + \int_0^x f(x')\,dx'$, you seem to want the initial value problem assoc. to LT.

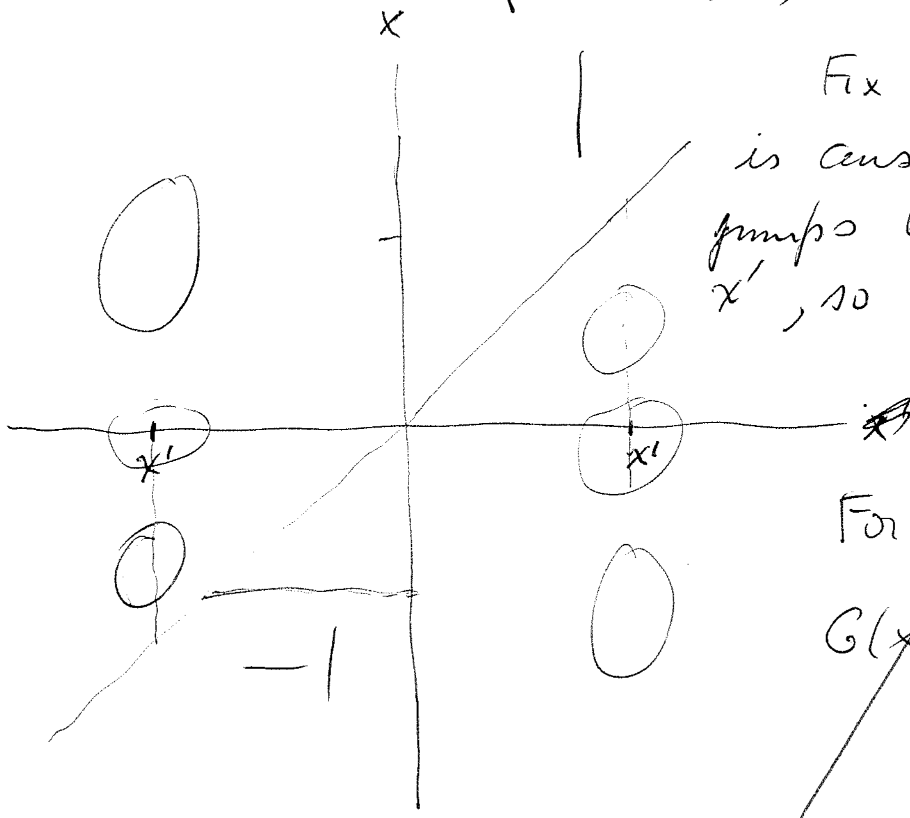
Given $\partial_x u = v$, $u(0) = f$ unique soln. $u = f + \int_0^x v(x')\,dx'$

Point is ∂_x has G.fn.: $\left(\begin{array}{l}\partial_x G(x,x') = \delta(x,x') \\ G(0,x') = 0.\end{array}\right.$

$$G(x,x') = c(x') + H(x-x')$$
$$G(0,x') = c(x') + H(-x') = 0$$



You want $\begin{cases}\partial_x G(x,x') = \delta(x-x') \\ G(0,x') = 0.\end{cases}$



Fix $x'>0$, then $G(x,x')$ is constant in x for $x \neq x'$ jumps by $+1$ as x crosses x', so $G(x,x') = \begin{cases} 0 & x<x' \\ 1 & x>x' \end{cases}$

For $x'<0$,

$$G(x,x') = \begin{cases} 0 & x>x' \\ -1 & x<x' \end{cases}$$

$$u(x) = \int G(x,x') f(x')\,dx' = \int_0^x f(x')\,dx' \qquad x>0.$$

$$= \int_x^0 -f(x')\,dx' = \int_0^x f(x')\,dx' \qquad x<0$$

<u>So G not translation invariant</u>

$$\left(\psi^1 - G_x h \psi^2\right)(x,y) = \psi^1(x,y) - \int_0^x (h\psi^2)(x',y)\,dx' = f^1(y)$$

$$\left(\psi^2 - G_y \bar{h} \psi^1\right)(x,y) = \psi^2(x,y) - \int_0^y (h\psi^1)(x',y')\,dy' \overset{!}{=} f^2(x)$$

---

what about iterating.

to solve

$$\left[\begin{array}{l} \begin{pmatrix}\partial_x & 0\\ 0 & \partial_y\end{pmatrix}\psi = \begin{pmatrix}0 & h\\ \bar{h} & 0\end{pmatrix}\psi \\ \psi^1(0,y) = f^1(y) \\ \psi^2(x,0) = f^2(x) \end{array}\right.$$ given

Let $\varphi = \begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}$ be ∋ $\varphi^1(0,y) = f^1(y)$, $\varphi^2(x,0) = f^2(x)$

~~Then $u = \psi - \varphi$ satisfies~~

Set $\psi = \chi - \varphi$, rather $\chi = \psi + \varphi$,

χ satisfies the 0 bdry condition $\left|\begin{array}{l}\chi^1(0,y)=0\\ \chi^2(x,0)=0\end{array}\right.$

$D = \begin{pmatrix}\partial_x & \\ & \partial_x\end{pmatrix}$ $\quad V = \begin{pmatrix}0 & h\\ \bar{h} & 0\end{pmatrix}$.

$$0 = (D-V)\psi = (D-V)\chi - (D-V)\varphi$$

so you get $\left\{\begin{array}{l} (D-V)\chi = \underbrace{(D-V)\varphi}_{\text{known}} = \beta \\ \chi^1(0,y) = 0 \\ \chi^2(x,0) = 0 \end{array}\right.$

inhomogeneous eqn. 0 bdry cond. Now use that $DG = 1$, G has 0 bdry data

$D\chi = V\chi + \beta$ should have ~~$DG\chi = \chi$~~

$GD\chi = \chi$ since χ has 0 bdry

yes $D(GD\chi - \chi) = 0$ so $GD\chi - \chi = \begin{pmatrix} f(y) \\ g(x) \end{pmatrix}$
imp. since G, χ has 0 on bdry.

$$\chi = GV\chi + G\beta$$

$$(1 - GV)\chi = G(D-V)\varphi = GD\varphi - GV\varphi$$

$$(1-GV)(\chi + \varphi) = \varphi + GD\varphi$$

To solve $(D-V)\psi = 0$ $\quad \psi = \varphi$ on ∂

Let ~~…~~

$\chi = \psi - \varphi$ so $\chi = 0$ on ∂

$$(D-V)\chi = \cancel{(D-V)\psi} - (D-V)\varphi$$

$$\underbrace{GD\chi}_{\chi} - GV\chi = -(GD - GV)\varphi$$

$$(1-GV)\chi = -G(D-V)\varphi$$

$$(1-GV)(\chi + \varphi) = -GD\varphi + \cancel{GV\varphi} + \varphi - \cancel{GV\varphi}$$

$= \varphi$ if φ is actually chosen to be $\begin{matrix} f(y) \\ g(x) \end{matrix}$

whence $\psi = \dfrac{1}{1-GV}\varphi$

Surely you can make this clearer.

Repeat. To solve $(D-V)\psi = 0$
with $\psi = \text{given} \begin{pmatrix} f^1(y) \\ f^2(x) \end{pmatrix}$ on ∂

Set $\varphi(x,y)$ so that $D\varphi = 0$, φ sat. ∂ cond.

Put $\chi = \psi - \varphi$ so $\chi = 0$ on ∂ and $GD\chi = \chi$

$\psi = \chi + \varphi$

$$0 = (D-V)\psi = (D-V)\chi + (\cancel{D}-V)\varphi$$

$$(1-GV)\chi = G(D-V)\chi = GV\varphi$$

$$\chi = \frac{1}{1-GV} GV\varphi$$

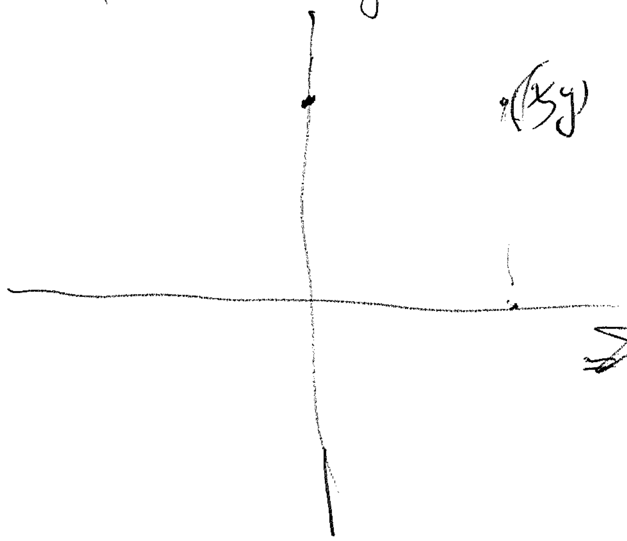
$$\psi = \varphi + \chi = \frac{1}{1-GV}\varphi$$

$$\psi^1(x,y) = f^1(y) + \int^x (h\psi^2)(x', y)\,dx'$$
$$\psi^2(x,y) = f^2(x) + \int_0^y (\bar{h}\psi^1)(x,y')\,dy'$$

$$\psi = \varphi + GV\psi \qquad G = \begin{pmatrix} G_x & 0 \\ 0 & G_y \end{pmatrix}$$

$$D\varphi = 0 \qquad V = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}$$

$\boxed{h = \bar{h} = 1}$ Here goes



$$\partial_x \psi^1 = \psi^2$$
$$\partial_y \psi^2 = \psi^1$$
$$\psi^1(x,y) = f^1(y) + \int_0^x \psi^2(x',y)\,dx'$$
$$\cancel{= f^1(y) + \int_0^x f^2(x)\,dx}$$
$$\psi^2(x,y) = f^2(x) + \int_0^y \psi^1(x,y')\,dy'$$

$$\psi^1(x,y) = f^1(y) + \int_0^x f^2(x')\,dx' + \int_0^x dx' \int_0^y \psi^1(x',y')\,dy'$$
$$\psi^2(x,y) = f^2(x) + \int_0^y f^1(y')\,dy' + \int_0^y dy' \int_0^x \psi^2(x',y')\,dx'$$

L.T. of $\underbrace{\int_0^x f(x)\,dx}_{g(x)}$ $\int_0^\infty s e^{-xs} g(x)\,dx$

$$= \cancel{\left[-e^{-xs} g(x)\right]_0^\infty} + \int e^{-xs} \underbrace{g'(x)}_{f}\,dx$$

$$\mathcal{L}\left\{\int_0^x f(x)\,dx\right\} = \frac{\mathcal{L}(f)}{s}$$

$$\partial_x \psi^1 = \psi^2 \qquad \psi^1(x,y) = f^1(y) + \int_0^x \psi^2(x',y)\,dx'$$
$$\partial_y \psi^2 = \psi^1 \qquad \mathcal{L}\psi^1 = \frac{1}{\xi}\mathcal{L}_y(f^1(y)) + \frac{1}{\xi}\mathcal{L}\psi^2$$

$$\mathcal{L}_x \psi^1(\xi,y) = -\frac{1}{\xi} f^1(y) + \frac{1}{\xi}\mathcal{L}_x \psi^2(\xi,y)$$

$$\boxed{\mathcal{L}\psi^1 = -\frac{1}{\xi}\hat{f}^1(\eta) + \frac{1}{\xi}\mathcal{L}\psi^2}$$

$$\partial_x \psi^1 = \psi^2 \qquad \begin{matrix}\psi^1(0,y) = f^1(y). \\ \psi^2(x,0) = f^2(x)\end{matrix} \qquad \partial_y \psi^2 = \psi^1$$

$$\xi \mathcal{L}_x \psi^1 - f^1(y) = \mathcal{L}_x \psi^2 \qquad \eta \mathcal{L}_y \psi^2 - f^2(x) = \mathcal{L}_y \psi^1$$

$$\xi \widehat{\psi^1} - \hat{f}^1(\eta) = \widehat{\psi^2} \qquad \eta \widehat{\psi^2} - \hat{f}^2(\xi) = \widehat{\psi^1}.$$

$$\hat{\psi}^1 - \eta \hat{\psi}^2 = -\hat{f}^2(\xi) \qquad \begin{pmatrix} 1 & -\eta \\ -\xi & 1 \end{pmatrix} \hat{\psi} = -\begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\eta) \end{pmatrix}$$

$$-\xi \hat{\psi}^1 + \hat{\psi}^2 = -\hat{f}^1(\eta)$$

$$\hat{\psi} = \frac{-1}{1-\xi\eta} \begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix} \begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\eta) \end{pmatrix} = \frac{+1}{\xi\eta - 1} \begin{pmatrix} \hat{f}^2(\xi) + \eta \hat{f}^1(\eta) \\ \xi \hat{f}^2(\xi) + \hat{f}^1(\eta) \end{pmatrix}$$

Simplest case is $f^1(y) = \delta(y)$ 0 $\quad f^2(x) = 0$ $\delta(x)$

$$\hat{\psi}(\xi, \eta) = \frac{1}{\xi\eta - 1} \begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix}$$

$$\int_{-i\infty}^{i\infty} \frac{e^{\eta y}}{2\pi i} \frac{1}{\xi\eta - 1} \begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix} = e^{\xi^{-1} y} \begin{pmatrix} 1 & \frac{\xi^{-1}}{\xi} \\ & \frac{\xi + 1}{\xi} \end{pmatrix}$$

$$\int_{-i\infty}^{+i\infty} \frac{e^{\xi x + \xi^{-1} y}}{2\pi i} \begin{pmatrix} \frac{1}{\xi} & \frac{1}{\xi^2} \\ 1 & \frac{1}{\xi} \end{pmatrix} = \int_{-i\infty}^{i\infty} \frac{d\xi}{2\pi i} \frac{e^{\xi x + \xi^{-1} y}}{\xi} \begin{pmatrix} 1 & \frac{1}{\xi} \\ \xi & 1 \end{pmatrix}$$

$$\mathcal{L}(f') = \int_0^\infty e^{-sx} f'(x)\,dx = \left[e^{-sx} f(x) \right]_0^\infty + s \int_0^\infty e^{-sx} f(x)\,dx$$

$$= -f(0) + s\,\mathcal{L}(f)$$

$$\partial_x \psi^1 = \psi^2 \qquad \mathcal{L}_x \partial_x \psi^1 = -\underbrace{\psi^1(0,y)}_{f^1(y)} + \xi \mathcal{L}_x \psi^1$$

$$\mathcal{L}_y \mathcal{L}_x \psi^2 = -\mathcal{L}_y f^1(y) + \xi \mathcal{L}_y \mathcal{L}_x \psi^1$$

$$\hat{\psi}^2 = -\hat{f}^1(\eta) + \xi \hat{\psi}^1$$

$$\hat{\psi}^1 = -\hat{f}^2(\xi) + \eta \hat{\psi}^2$$

$$\begin{pmatrix} 1 & -\eta \\ -\xi & 1 \end{pmatrix} \begin{pmatrix} \hat{\psi}^1 \\ \hat{\psi}^2 \end{pmatrix} = \begin{pmatrix} -\hat{f}^2(\xi) \\ -\hat{f}^1(\eta) \end{pmatrix}$$

$$\hat{\psi} = \frac{+1}{\xi\eta - 1} \begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix} \begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\eta) \end{pmatrix}$$

Interesting cases

$$\psi^1(0,y) = f^1(y)$$
$$\psi^2(x,0) = f^2(x)$$

are $f^1(y) = 0 \qquad \delta(y)$

$f^2(x) = \delta(x) \qquad 0$

So you need the inverse LT of $\frac{+1}{\xi\eta - 1}\begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix}$

$$\int_{-i\infty}^{i\infty} \frac{d\eta}{2\pi i} e^{y\eta} \frac{1}{\xi\eta - 1} \begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix} = e^{y\xi^{-1}} \frac{1}{\xi} \begin{pmatrix} 1 & \xi^{-1} \\ \xi & 1 \end{pmatrix}$$

$$\boxed{\int_{-i\infty}^{i\infty} \frac{d\xi}{2\pi i} \frac{e^{x\xi + y\xi^{-1}}}{\xi} \begin{pmatrix} 1 & \xi^{-1} \\ \xi & 1 \end{pmatrix} \begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\xi^{-1}) \end{pmatrix} = \psi(x,y)}$$

$y = 0.$

$$\int_{-i\infty}^{i\infty} \frac{d\xi}{2\pi i} \frac{e^{x\xi}}{\xi} \xi \hat{f}^2(\xi) = f^2(x) = \psi^2(x,0) \qquad \psi^1(0,y)$$

$$\int \frac{d\xi}{2\pi i} \frac{e^{y\xi^{-1}}}{\xi} \xi^{-1} \hat{f}^1(\xi^{-1}) = f^2(y) = \psi^1(0,y)$$

It seems that the solution of $\partial_x \psi^1 = \psi^2$, $\partial_y \psi^2 = \psi^1$

such that $\psi^1(0,y) = f^1(y)$, $\psi^2(x,0) = f^2(x)$ is

$$\psi(x,y) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{e^{xs+ys^{-1}}}{s} \begin{pmatrix} 1 & s^{-1} \\ s & 1 \end{pmatrix} \begin{pmatrix} \hat{f}^2(s) \\ \hat{f}^1(s^{-1}) \end{pmatrix}$$

where

$$f^2(x) = \int_{-i\infty}^{i\infty} e^{xs} \hat{f}^2(s) \frac{ds}{2\pi i}$$

$$f^1(y) = \int_{-i\infty}^{i\infty} \frac{e^{ys^{-1}}}{s} \hat{f}^1(s^{-1}) \frac{ds}{2\pi i s}$$

$$= \int_{-i\infty}^{i\infty} \cancel{s}\, e^{ys} \hat{f}^1(s) \frac{ds}{2\pi i \cancel{s}}$$

$f^2(x)$

$f^1(y)$

This seems perfectly correct, but hard to prove

You ~~would~~ want to take $f^1(y) = \delta(y-y')$, $f^2(x) = \delta(x-x')$

$$\hat{f}^1(\eta) = e^{-y'\eta} = e^{-y's^{-1}}$$

$$\hat{f}^2(\xi) = e^{-x'\xi} = e^{-x's}$$

$$K(x,y|x',y') = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} e^{-x's} & s^{-1}e^{-y's^{-1}} \\ se^{-x's} & e^{-y's^{-1}} \end{pmatrix}$$

So the real job is to compute

$$\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} 1 & s^{-1} \\ s & 1 \end{pmatrix}$$

which should involve J_0, J_1 probably $J_0(xy)$ $J_1(xy)$

This is obvious except for the contour.

OK From L.T. theory you expect to get 0 from this integral if either x or $y < 0$, the point being that the contour is $a - i\infty$ to $a + i\infty$ $a >> 0$. But the only singularity is at $s = 0$, so

$$\int_{a-i\infty}^{a+i\infty} \cdots = \oint e^{xs + ys^{-1}} \begin{pmatrix} 1 & s^{-1} \\ s & 1 \end{pmatrix} \frac{ds}{2\pi i s}$$

$$= \begin{pmatrix} J_0(xy) & x J_1(yx) \\ y J_1(xy) & J_0(xy) \end{pmatrix}$$

Check that the columns are indeed solutions fs of $\partial_x \psi^1 = \psi^2 \quad \partial_y \psi^2 = \psi^1$.

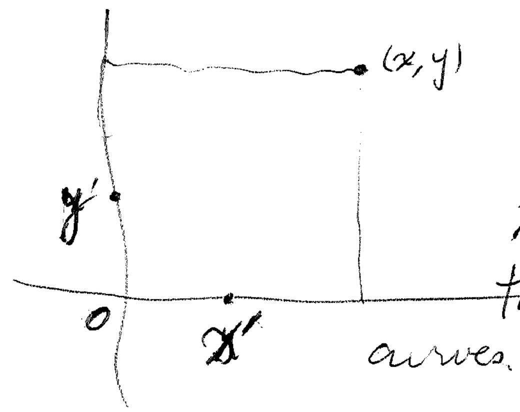
$$\partial_x J_0(xy) = J_1(xy)\, y$$

$$\partial_y \left(y J_1(xy) \right) = \partial_y \left(\sum_{n \geq 0} \frac{y\, x^n y^n}{n!\,(n+1)!} \right) = \sum_{n \geq 0} \frac{x^n (n+1) y^n}{n!\,(n+1)!} = J_0(xy)$$

$$\partial_y J_0(xy) = J_1(xy)\, x$$

$$\partial_x \left(x J_1(yx) \right) = \partial_x \left(\sum_{n \geq 0} \frac{x\, x^n y^n}{n!\,(n+1)!} \right) = \sum_{n \geq 0} \frac{x^n y^n}{n!\, n!} = J_0(xy)$$

But you want Riemann's fundamental solution which is not translation invariant, ~~because~~ at least the way you have done things.



What invariance property might ~~there~~ there be? The kernel $K(x,y \mid x', y')$ the seek is a 2×2 matrix the columns refers to char curves. So think $K(x,y \mid x')$ and $K(x,y \mid y')$

$$K(x,y|x') = \int \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} 1 \\ s \end{pmatrix} e^{-x's}$$

$$= \begin{pmatrix} J_0((x-x')y) \\ yJ_1((x-x')y) \end{pmatrix}$$

$$K(x,y|y') = \int \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix} e^{-y's^{-1}}$$

$$= \begin{pmatrix} xJ_1(x(y-y')) \\ J_0(x(y-y')) \end{pmatrix}$$

~~odd~~

odd Bessel

$$K_s(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} t^s \frac{dt}{t}$$

$$\partial_r K_s(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} (-1)\left(\frac{t+t^{-1}}{2}\right) t^s \frac{dt}{t}$$

$$\boxed{-\partial_r K_s = \frac{1}{2}\left(K_{s+1} + K_{s-1}\right)}$$

$$sK_s = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} \partial_t(t^s)\, dt \qquad (-\partial_t) e^{-r\left(\frac{t+t^{-1}}{2}\right)}$$

$$= \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} (+r)\left(\frac{1-t^{-2}}{2}\right) t\, t^{s} \frac{dt}{t}$$

$$\boxed{\frac{s}{r} K_s = \frac{1}{2}\left(K_{s+1} - K_{s-1}\right)}$$

$$\boxed{\left(-\partial_r + \frac{s}{r}\right) K_s = K_{s+1}} \qquad \boxed{\left(+\partial_r + \frac{s}{r}\right) K_s = -K_{s-1}}$$

$$\left(+\partial_r - \frac{s-\frac{1}{2}}{r}\right)K_{s-\frac{1}{2}} = -K_{s+\frac{1}{2}} \qquad \Big| \qquad \left(\partial_r + \frac{s+\frac{1}{2}}{r}\right)K_{s+\frac{1}{2}} = -K_{s-\frac{1}{2}}$$

$$\underbrace{\left(+\partial_r - \frac{s-\frac{1}{2}}{r}\right)\left(\partial_r + \frac{s+\frac{1}{2}}{r}\right)} K_{s+\frac{1}{2}} = +K_{s+\frac{1}{2}}$$

$$\partial_r^2 + \left(s+\tfrac{1}{2}\right)\underbrace{\left(\partial_r \frac{1}{r}\right)}_{\frac{1}{r}\partial_r - \frac{1}{r^2}} - \left(s-\tfrac{1}{2}\right)\left(\frac{1}{r}\partial_r\right) - \frac{s^2-\frac{1}{4}}{r^2} \qquad ?$$

$$\left(\partial_r - \frac{s}{r}\right)K_s = -K_{s+1} \qquad \left(\partial_r + \frac{s}{r}\right)K_s = -K_{s-1}$$

$$\underbrace{\left(\partial_r - \frac{s-1}{r}\right)\left(\partial_r + \frac{s}{r}\right)} K_s = \left(\partial_r - \frac{s-1}{r}\right)\left(-K_{s-1}\right) = K_s$$

$$\partial_r^2 - \frac{s-1}{r}\partial_r + \underbrace{\partial_r \frac{s}{r}}_{s\frac{1}{r}\partial_r - \frac{1}{r^2}} - \frac{s^2-s}{r^2}$$

$$\partial_r^2 + \left(-(s-1)+s\right)\frac{1}{r}\partial_r + \left(\frac{-s^2+s}{r^2} - \frac{s}{r^2}\right)$$

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{s^2}{r^2}\right)K_s = K_s \qquad r^{-a}\partial_r r^a = \partial_r + \frac{a}{r}$$

$$\underbrace{r^{-a}\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{s^2}{r^2}\right)r^a}\left(r^{-a}K_s\right) = r^{-a}K_s$$

$$\left(\partial_r + \frac{a}{r}\right)^2 + \frac{1}{r}\left(\partial_r + \frac{a}{r}\right) - \frac{s^2}{r^2}$$

$$\partial_r^2 + \cancel{2}\underbrace{\partial_r \frac{a}{r}}_{\frac{a}{r}\partial_r - \frac{a}{r^2}} + \frac{a}{r}\partial_r + \frac{a^2}{r^2} - \frac{s^2}{r^2} = \partial_r^2 + 2\frac{a}{r}\partial_r + \frac{a^2-a-s^2}{r^2}$$

$$K_s(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} t^s \frac{dt}{t}$$

$$\partial_r K_s = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} (-1)\frac{t+t^{-1}}{2}\, t^s \frac{dt}{t} = -\frac{1}{2}\left(K_{s+1}+K_{s-1}\right)$$

$$sK_s = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} \underbrace{st^{s-1}}_{\partial_t(t^s)}\, dt = \int_0^\infty (-\partial_t)\left(e^{-r\left(\frac{t+t^{-1}}{2}\right)}\right) t^{s+1}\frac{dt}{t}$$

$$= \int_0^\infty e^{r\left(\frac{t+t^{-1}}{2}\right)}\, r\left(\frac{t-t^{-1}}{2}\right) t^s \frac{dt}{t} = \frac{r}{2}\left(K_{s+1}-K_{s-1}\right)$$

$$\partial_r K_s = -\frac{1}{2}\left(K_{s+1}+K_{s-1}\right) \qquad \left(\partial_r+\frac{s+1}{r}\right)\left(\partial_r -\right.$$

$$\frac{s}{r} K_s = \frac{1}{2}\left(K_{s+1}-K_{s-1}\right)$$

$$\boxed{\begin{aligned}\left(\partial_r+\frac{s}{r}\right)K_s &= -K_{s-1}\\ \left(\partial_r-\frac{s}{r}\right)K_s &= -K_{s+1}\end{aligned}}$$

$$\left(\partial_r-\frac{s-1}{r}\right)K_{s-1} = -K_s$$

$$\underbrace{r^a\left(\partial_r+\frac{s}{r}\right)r^{-a}}\left(r^aK_s\right) = -r^aK_{s-1}$$

$$\boxed{\left(\partial_r+\frac{s-a}{r}\right)\left(r^aK_s\right) = -\left(r^aK_{s-1}\right)}$$

$$\boxed{\left(\partial_r-\frac{s-1}{r}-\frac{a}{r}\right)\left(r^aK_{s-1}\right) = -r^aK_s} \qquad \cancel{-\frac{2a-1}{r}}$$

$$\left(\partial_r+\frac{s-a}{r}\right)\left(\partial_r-\frac{s+a-1}{r}\right) = \partial_r^2 + \overbrace{\left(\frac{s-a}{r}-\frac{s+a-1}{r}\right)}\partial_r$$

$$+\frac{s+a-1}{r^2} - \frac{(s-a)(s+a-1)}{r^2}$$

$a=0$

$$-s^2+2s-1$$

$$\partial_r^2 + \frac{1}{r}\partial_r + \frac{s-1-s(s-1)}{r^2}$$

$$\left(\partial_r + \frac{s}{r}\right) K_s = -K_{s-1}$$

$$\left(\partial_r - \frac{s}{r}\right) K_s = -K_{s+1}$$

$$\left(\partial_r - \frac{s-1}{r}\right) K_{s-1} = -K_s$$

YUCK

$$\left(\partial_r + \frac{s}{r}\right)\left(\partial_r - \frac{s-1}{r}\right) K_{s-1} = +K_{s-1}$$

$$(r\partial_r + s) K_s = -r K_{s-1}$$

$$K_s = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} t^s \frac{dt}{t} \qquad K_s = K_{-s} \qquad s \log \underset{1+x}{\overset{t}{\shortparallel}}$$

$$f(t) = \frac{t+t^{-1}}{2} \qquad f'(t) = \frac{1-t^{-2}}{2} = 1 \qquad t = +1.$$

$$= 1 + \frac{x^2}{2} \qquad f''(t) = t^{-3} = 1.$$

$$K_s(r) \sim e^{-r} \int_{-\infty}^{\infty} e^{-r\frac{x^2}{2} + sx} = \frac{e^{-r}}{\sqrt{r}}$$

Problem: What are the other Bessels?

$$\partial_x J_n(x) = J_{n+1} \qquad x^n J_n(x) = \sum \frac{x^{j+n}}{j!(j+n)!}$$

$$\partial_x (x^n J_n) = x^{n-1} J_{n-1} \qquad n \geq 1.$$

$$\partial_x J_n = J_{n+1}$$

$$(x\partial_x + n) J_n = J_{n-1} \qquad n \geq 1.$$

$$x\partial_x J_0 = \sum_{j \geq 0} \frac{x^j j}{j!\,j!} = x J_1$$

To make things symmetrical.

$$J_n(x) = \sum_{j\geq 0} \frac{x^j}{j!\,(j+n)!} \qquad n \geq 0$$

$$J_{-n}(x) = \sum_{j\geq n} \frac{x^j}{j!\,(j-n)!} = x^n J_n(x)$$

$$e^{s+ys^{-1}} = \sum_{n\geq 0} s^n x^n J_n(xy) + \sum_{n\geq +1} s^{-n} y^n J_n(xy)$$

<u>You want</u> to set up something for $n = \pm\frac{1}{2}$.

$$e^{s+ys^{-1}} = \sum_{n\geq 0} s^n J_n(y) + \sum_{n\geq 1} s^{-n} \underbrace{y^n J(y)}_{J_{-n}(y)}$$

$$e^{s+ys^{-1}} = \sum_{n\in\mathbb{Z}} s^n J_n(y)$$

$$\partial_y e^{s+ys^{-1}} = \sum_{n\in\mathbb{Z}} s^{n-1} J_n(y) = \sum s^n \partial_y J_n(y)$$

$$\therefore\ \partial_y J_n(y) = J_{n+1} \qquad \forall n.$$

$$e^{xs+s^{-1}} = \sum_{n\geq 0} s^n x^n J_n(x) + \sum_{n\geq 1} s^{-n} J_n(x)$$

$$= \sum_{n\in\mathbb{Z}} s^n J_{-n}(x) \qquad ?$$

~~See~~ now that ~~if~~ $n = \frac{1}{2}$

$$x J_{\frac{1}{2}}\left(\frac{x^2}{4}\right) =$$

$$\sum_{j\geq 0} \frac{x^{2j}}{2\cdots 2j\ 1\cdots 2j-1} = \sum_{j\geq 0} \frac{x^{2j}}{(2j)!}$$

$$= \frac{e^x + e^{-x}}{2}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{1/2}\, \frac{dt}{t}$$

$$= \int_0^\infty e^{-t^2}\, t\, \frac{2dt}{t}$$

$$= \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \sqrt{\pi}$$

First def. $J_n(x) = \begin{cases} J_n(x) & n \geq 0 \\ x^{-n} J_{-n}(x) & n \leq 0. \end{cases}$

$$e^{s+ys^{-1}} = \sum_{n\in\mathbb{Z}} s^n J_n(y) \qquad \partial_x J_n(x) = J_{n+1}(x)$$

$$e^{sy+s^{-1}} = \sum_{n\in\mathbb{Z}} s^n\big(y^n J_n(y)\big)$$

$$\sum s^n \partial_y\big(y^n J_n(y)\big) = \sum s^{n+1}\big(y^n J_n(y)\big)$$

$$\partial_y\big(y^n J_n(y)\big) = y^{n-1} J_{n-1}(y).$$

$$\left.\begin{aligned} \partial_x J_n &= J_{n+1} \\ (x\partial_x + n) J_n &= J_{n-1} \end{aligned}\right) \qquad (x\partial_x + n+1)\underbrace{J_{n+1}}_{\partial_x J_n} = J_n$$

$$\left(x\partial_x^2 + (n+1)\partial_x - 1\right) J_n = 0$$

$$\boxed{\partial_x (x\partial_x + n) J_n = J_n}$$

$$a_i\, i(i+n)\, x^{i-1} = a_{i-1} x^{i-1}! \qquad a_i = \frac{a_{i-1}}{i(i+n)}$$

OKAY

$$\left((r\partial_r)^2 - m^2\right) u = \lambda r^2 u$$

$$a_j (j^2 - m^2) = \lambda a_{j-2}$$

$$a_{m+2k} \underbrace{\left((m+2k)^2 - m^2\right)}_{2k(2m+2k)} = \lambda a_{m+2(k-1)}$$

generalize $J_n(x)=\sum_{j\ge 0}\frac{x^j}{j!\,(j+n)!}$ to

$$J_s(x)=\sum_{j\ge 0}\frac{x^j}{j!\,\Gamma(j+s+1)}$$

[redacted] $\partial_x(x^sJ_s)=\sum\frac{x^{j+s-1}\,(j+s)}{j!\,\Gamma(j+s+1)}$
$= x^{s-1}J_{s-1}$

$$\partial_x J_s(x)=\sum_{j\ge 1}\frac{x^{j-1}}{(j-1)!\,\Gamma(j+s+1)}=\sum_{j\ge 0}\frac{x^j}{j!\,\Gamma(j+1+s+1)}=J_{s+1}(x)$$

$$(x\partial_x+s)J_s(x)=\sum_{j\ge 0}\frac{x^j(j+s)}{j!\,\underbrace{\Gamma(j+s+1)}_{(j+s)\Gamma(j+s)}}=\sum_{j\ge 0}\frac{x^j}{j!\,\Gamma(j+s)}=J_{s-1}(x).$$

whence DE. $\boxed{\partial_x(x\partial_x+s)J_s=J_s}$

$$\boxed{\begin{aligned}\partial_x J_s&=J_{s+1}\\(x\partial_x+s)J_s&=J_{s-1}\\xJ_{s+1}+sJ_s&=J_{s-1}\end{aligned}}$$

and recursion relation $\boxed{\partial_x(x^sJ_s)=x^{s-1}J_{s-1}}$

$J_{-\frac12}$? Recall $\Gamma(\frac12)=\int_0^\infty e^{-t}t^{\frac12}\frac{dt}{t}=\int_0^\infty e^{-t^2}t$ [redacted] $2\frac{dt}{t}=$
$\int_{-\infty}^{\infty}e^{-t^2}dt=\sqrt{\pi}$, so $\Gamma(j+\frac12)=(j-\frac12)(j-\frac32)\cdots($ [redacted] $\frac12)\overbrace{\Gamma(\tfrac12)}^{\sqrt{\pi}}$

$$J_{-\frac12}(x)=\sum_{j\ge 0}\frac{x^j}{(1\cdot 2\cdots j)(\frac12\,\frac32\cdots\frac{2j-1}{2})\sqrt{\pi}}$$

$$=\frac{1}{\sqrt{\pi}}\sum_{j\ge 0}\frac{x^j2^{2j}}{(2j)!}=\frac{1}{\sqrt{\pi}}\sum_{j\ge 0}\frac{(2x^{1/2})^{2j}}{(2j)!}$$

$$\boxed{J_{-\frac12}(x)=\frac{1}{\sqrt{\pi}}\cosh(2x^{1/2})}\qquad\boxed{J_{\frac12}(x)=\frac{1}{\sqrt{\pi}}\frac{\sinh(2x^{1/2})}{x^{1/2}}}$$

$$J_{\frac12}(x)=\frac{1}{x^{1/2}}\sum_{j\ge 0}\frac{x^{j+\frac12}2^{2j+1}}{j!\,1\cdot 3\cdots(2j+1)\sqrt{\pi}}=\frac{1}{\sqrt{\pi}}\frac{\sinh(2x^{1/2})}{x^{1/2}}$$

$\partial_x\cosh(2x^{1/2})=\sinh(2x^{1/2})\,2\tfrac12x^{-1/2}$

$\partial_x x^{1/2}J_{1/2}=\partial_x\sinh(2x^{1/2})/\sqrt{\pi}=\frac{\cosh}{\sqrt{\pi}}(2x^{1/2})\,2\tfrac12x^{-1/2}=x^{-1/2}J_{-\frac12}$

Note that when making the link with polar coordinates one has $x = \left(\frac{r}{2}\right)^2$ or $r = 2\sqrt{x}$.

$$\psi(x,y) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{e^{xs+ys^{-1}}}{s} \begin{pmatrix} 1 & s^{-1} \\ s & 1 \end{pmatrix} \begin{pmatrix} \hat{f}^2(s) \\ \hat{f}^1(s^{-1}) \end{pmatrix}$$

$$\psi^1(0,y) = f^1(y)$$
$$\psi^2(x,0) = f^2(x)$$

take δ fn. $f^1(y) = \delta(y-y')$ $\quad f^2(x) = 0$

$$K(x,y\,|\,y') = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} e^{-y's^{-1}} \\ e^{-y's^{-1}} \end{pmatrix}$$

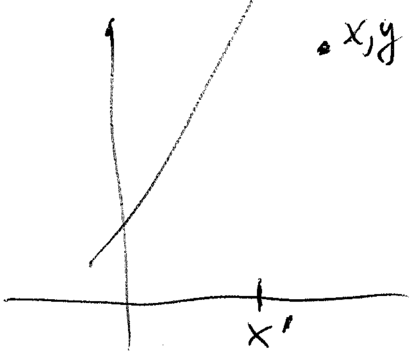
$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i\, s} e^{xs+(y-y')s^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}$$

$$K(x,y\,|\,x') = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} 1 \\ s \end{pmatrix} e^{-x's}$$

$f^1(y) = 0$
$f^2(x) = \delta(x-x')$
$\hat{f}^2(\xi) = e^{-x'\xi}$

$$\boxed{K(x,y\,|\,x') = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{(x-x')s+ys^{-1}} \begin{pmatrix} 1 \\ s \end{pmatrix}}$$

if $x-x'<0$ can push



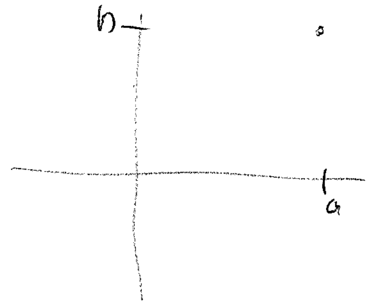
perhaps you should link your study of horizontal-vertical splitting to Riemann fund. solution.

$\lambda^{x,y}_{\mu}\begin{pmatrix}v^1\\ v^2\end{pmatrix} = e^{xs+ys^{-1}}\begin{pmatrix}s^{-1}\\ 1\end{pmatrix}$. You want to ~~[illegible]~~

split

$$e^{as+bs^{-1}}\begin{pmatrix}s^{-1}\\ 1\end{pmatrix}$$ ~~into linear combinations~~

into horizontal + vertical components



$$e^{xs+ys^{-1}} = \sum_{j\geq 0} \frac{e^{xs}}{s^j}\frac{y^j}{j!}$$

$$\left(e^{xs+ys^{-1}}\right)_{hor} = \sum_{n\geq 0}\frac{y^n}{s^n n!}\underbrace{\left(e^{xs}-\sum_{i=0}^{n-1}\frac{(xs)^i}{i!}\right)}_{\int_0^x \underbrace{(\partial_x)^n(e^{xs})}_{s^n}\frac{(x-x')^{n-1}}{(n-1)!}\,dx'}$$

$$= \int_0^x \sum_{n\geq 0}\frac{y^n}{n!}\, e^{x's}\,\frac{(x-x')^{n-1}}{(n-1)!}\,dx'$$

$$= \int_0^x e^{x's}\, y\, J_1\big(y(x-x')\big)\,dx'$$

$$= \int_0^x e^{x's}\, J_1\big(y(x-x')\big)\, y\,dx'$$

$$\left(e^{xs+ys^{-1}} s^{-1}\right)_{hor} = \left(\sum_{n\geq 0} \frac{y^n}{n!} \frac{e^{xs}}{s^{n+1}}\right)_{hor}$$

$$= \sum_{n\geq 0} \frac{y^n}{n!\,\cancel{s^{n+1}}} \int_0^x \cancel{s^{n+1}} e^{x's} \frac{(x-x')^n}{n!}\, dx'$$

$$= \int_0^x e^{x's} J_0\big(y(x-x')\big)\, dx'$$

$$\left(e^{xs+ys^{-1}}\right)_{ver} = \sum_{n\geq 1} \frac{y^n}{n!} \frac{\sum_{j<n} \frac{(xs)^j}{j!}}{s^n}$$

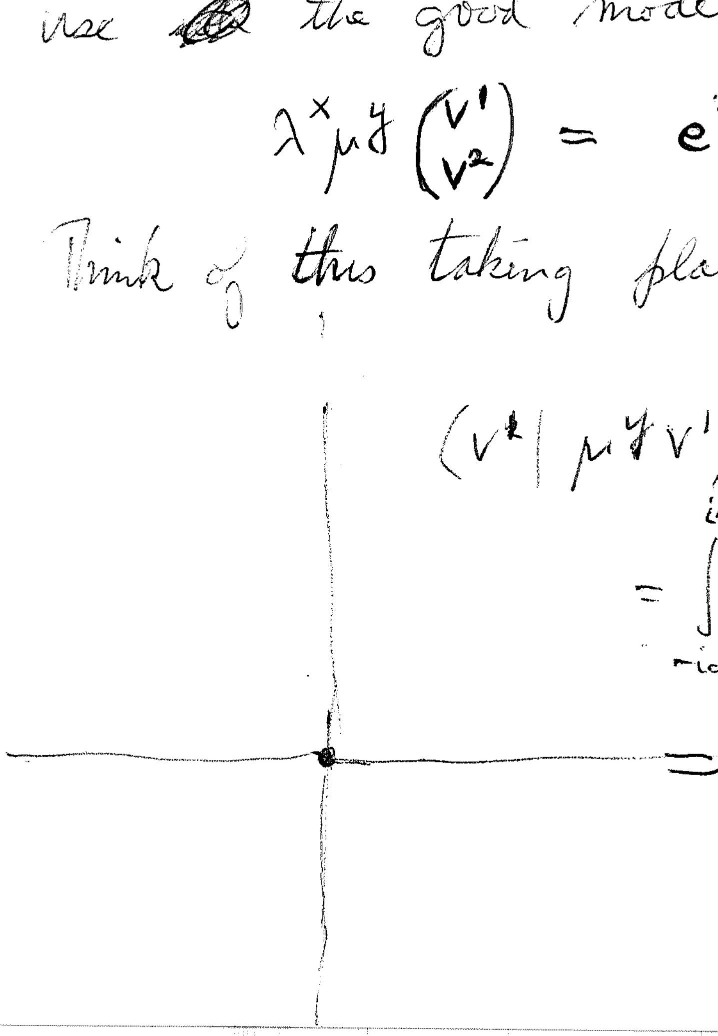
Go back, study ~~all~~ solns of $\partial_x \psi^1 = \psi^2 \quad \partial_y \psi^2 = \psi^1$.
Look at hermitian products. Should be determined by ~~certain~~ certain basic solutions

$$(v^i \,|\, \lambda^x \mu^y v^j) = \psi^j(x,y) \qquad j = 1, 2.$$

use the good model

$$\lambda^x \mu^y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}_{i\infty}$$

Think of this taking place with $\int_{-i\infty}^{i\infty} \overline{f(s)}\, g(s) \frac{ds}{2\pi i}$



$$(v^1 \,|\, \mu^y v^1) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\bar{1}}{s} e^{ys^{-1}} \frac{1}{s}$$

$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} (-1)\, e^{ys^{-1}} \frac{1}{s}$$

$$= \int_{i0^+}^{-i0^+} \frac{ds}{2\pi i \cancel{s}} e^{ys} \cancel{\tfrac{1}{s}}\, s \qquad 0.$$

Notice that $\displaystyle\int_{-\infty}^{\infty} f(t)\,\frac{dt}{t} = \int_{-\infty}^{0^-} f(t)\frac{dt}{t} + \int_{0^+}^{+\infty} f(t)\frac{dt}{t}$

$$= \int_{0^-}^{-\infty} f\left(\tfrac{1}{t}\right)(-1)\frac{dt}{t} + \int_{+\infty}^{0^+} f\left(\tfrac{1}{t}\right)(-1)\frac{dt}{t}$$

$$= \int_{-\infty}^{0^-} f\left(\tfrac{1}{t}\right)\frac{dt}{t} + \int_{0^+}^{+\infty} f\left(\tfrac{1}{t}\right)\frac{dt}{t} = \int_{-\infty}^{\infty} f\left(\tfrac{1}{t}\right)\frac{dt}{t}$$

But $\displaystyle\int_{-i\infty}^{i\infty} f(s)\,\frac{ds}{s} = \int_{-i\infty}^{i0^-} f(s)\frac{ds}{s} + \int_{i0^+}^{+i\infty} f(s)\frac{ds}{s}$

$$= \int_{i0^+}^{+i\infty} f\left(\tfrac{1}{s}\right)(-1)\frac{ds}{s} + \int_{-i\infty}^{-i0^+ = i0^-} f\left(\tfrac{1}{s}\right)(-1)\frac{ds}{s} = -\int_{-i\infty}^{+i\infty} f\left(\tfrac{1}{s}\right)\frac{ds}{s}$$

It possible this is the origin of the -1 in the vertical direction.

Check it. The cycle $\int_{-i\infty}^{i\infty}$ or path on the RS going up the imag. axis is unchanged by $s \mapsto s^{-1}$

$$\partial_x \psi^1 = \psi^2 \qquad \psi^1(0,y) = f^1(y) \qquad \begin{pmatrix} 1 & -\eta \\ -\xi & 1 \end{pmatrix}$$
$$\partial_y \psi^2 = \psi^1 \qquad \psi^2(x,0) = f^2(x)$$

$$\mathcal{L}_y\psi^1 = -f^2(x) + \eta\,\mathcal{L}_y\psi^2 \qquad \hat{\psi}^1 = -\hat{f}^2(\xi) + \eta\,\hat{\psi}^2$$
$$\mathcal{L}_x\psi^2 = \mathcal{L}_x(\partial_x\psi^1) = -f^1(y) + \xi\,\mathcal{L}_x\psi^1 \qquad \hat{\psi}^2 = -\hat{f}^1(\eta) + \xi\,\hat{\psi}^1$$

$$\hat{\psi} = \frac{-1}{1-\xi\eta}\begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix}\begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\eta) \end{pmatrix}$$

$$(\mathcal{L}_x\psi)(\xi, y) = \int_{-i\infty}^{i\infty} \frac{d\eta}{2\pi i}\, e^{y\eta}\frac{1}{\xi\eta - 1}\begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix}\begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\eta) \end{pmatrix} = \frac{e^{y\xi^{-1}}}{\xi}\begin{pmatrix} 1 & \xi^{-1} \\ \xi & 1 \end{pmatrix}\begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\xi^{-1}) \end{pmatrix}$$

$$\psi(x,y) = \int_{-i\infty}^{i\infty} \frac{d\xi}{2\pi i \xi} e^{x\xi + y\xi^{-1}} \begin{pmatrix} 1 & \xi^{-1} \\ \xi & 1 \end{pmatrix} \begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\xi^{-1}) \end{pmatrix}$$

$$\psi^2(x,0) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{xs} \left(s\hat{f}^2(s) + \hat{f}^1(s^{-1}) \right) \overset{\text{if } x>0}{=} f^2(x)$$

$$\psi^1(0,y) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{ys^{-1}} \left(\hat{f}^2(s) + s^{-1}\hat{f}^1(s^{-1}) \right)$$

$$= \int_{-i\infty}^{i\infty} (-) \frac{ds}{2\pi i s} e^{ys} \left(\hat{f}^2(s^{-1}) + s\hat{f}^1(s) \right)$$

$$\int_0^\infty dx \int_0^\infty dy \; e^{-x\xi - y\eta} \; \partial_x \psi^1$$

$$= \int_0^\infty dx \, e^{-x\xi} \, \partial_x (\mathcal{L}_y \psi^1)(x,\eta)$$

$$= -\underline{(\mathcal{L}_y \psi^1)(0,\eta)} + \xi \hat{\psi}^1(\xi,\eta)$$

$$\int_0^\infty dy \, e^{-y\eta} \underbrace{\psi^1(0,y)}_{f^1(y)} = \hat{f}^1(\eta).$$

$\partial_x \psi^1 = \psi^2$ set

$$\hat{\psi}(s,y) = \int_0^\infty e^{-xs} \psi(x,y) \, dx$$

$$\hat{\psi}^2(s,y) = \int_0^\infty e^{-xs} (\partial_x \psi^1)(x,y) \, dx = -f^1(y) + s\hat{\psi}^1(s,y)$$

$$\partial_y \hat{\psi}^2(s,y) = \hat{\psi}^1(s,y)$$

Repeat $\partial_x \psi^1 = \psi^2$, $\partial_y \psi^2 = \psi^1$ | for $x, y \geq 0$
$\psi^1(0,y)$, $\psi^2(x,0)$ are given

$$\hat{\psi}^2 = \overbrace{(\partial_x \psi^1)} = -\psi^1(0,y) + s\hat{\psi}^1 \quad // \quad \hat{\psi}^1 = \frac{\psi^1(0,y) + \hat{\psi}^2(s,y)}{s}$$

$$\partial_y \hat{\psi}^2 = \widehat{\partial_y \psi^2} = \hat{\psi}^1$$

$$(\partial_y - s^{-1})\hat{\psi}^2 = s^{-1}\psi^1(0,y)$$

$$\hat{\psi}^2(s,y) = \int_0^y e^{(y-y')s^{-1}} s^{-1} \psi^1(0,y')\,dy' + \underline{e^{ys^{-1}}\hat{\psi}^2(s,0)}$$

$$\psi^2(x,y) = \int_{-i\infty}^{i\infty} e^{xs}\,\hat{\psi}^2(s,y)\frac{ds}{2\pi i} \qquad e^{ys^{-1}}\int_0^\infty e^{-x's}\psi^2(x',0)\,dx'$$

$J_0(x(y-y'))$ — provided $x>0$,

$$= \int_0^y \left(\int_{-i\infty}^{i\infty} e^{xs+(y-y')s^{-1}} s^{-1}\frac{ds}{2\pi i}\right)\psi^1(0,y')\,dy'$$

$$+ \int_{-i\infty}^{i\infty} e^{xs}\left(\int_0^\infty e^{ys^{-1}-x's}\psi^2(x',0)\,dx'\right)\frac{ds}{2\pi i}$$

$$\int_0^\infty \left(\underbrace{\int_{-i\infty}^{i\infty} e^{(x-x')s+ys^{-1}}\frac{ds}{2\pi i}}_{J_1((x-x')y)\,y}\right)\psi^2(x',0)\,dx'$$

for $x > x'$,
0 otherwise

$$\boxed{\begin{aligned}\psi^2(x,y) &= \int_0^y J_0(x(y-y'))\,\psi^1(0,y')\,dy' \\ &\quad + \int_0^x J_1((x-x')y)\,y\,\psi^2(x',0)\,dx'\end{aligned}}$$

NO

$$\hat{\psi}^1(s,y) = \partial_y \hat{\psi}^2(s,y) = s^{-1}\hat{\psi}^2 + s^{-1}\psi^1(0,y) \quad \frac{e^{ys^{-1}}}{s}\hat{\psi}^2(s,0)$$

$$= \int_0^y e^{(y-y')s^{-1}} s^{-2} \psi^1(0,y')\,dy' + s^{-1}\psi^1(0,y)$$

FT

$x>0$

$$\psi^1(x,y) = \int_0^y \left(\int_{-i\infty}^{i\infty} e^{xs+(y-y')s^{-1}} s^{-2} \frac{ds}{2\pi i} \right) \psi^1(0,y')\,dy' + \psi^1(0,y)$$

$$= \int_0^y J_1(x(y-y'))\,x\,\psi^1(0,y')\,dy' \qquad \mathrm{I}$$

$$\mathcal{L}^{-1}\left(\int_0^\infty \frac{e^{ys^{-1}}}{s} e^{-x's} \psi^2(0,x')\,dx' \right) = \int_0^\infty \left(\int_{-i\infty}^{i\infty} e^{xs} \frac{e^{ys^{-1}-x's}}{s} \frac{ds}{2\pi i} \right) \psi^2(0,x')\,dx'$$

$$= \int_0^x J_0((x-x')y)\,\psi^2(x',0)\,dx'. \qquad \mathrm{II}$$

$$\psi^1(x,y) = \psi^1(0,y) + \int_0^y J_1(x(y-y'))\,x\,\psi^1(0,y')\,dy'$$
$$+ \int_0^x J_0((x-x')y)\,\psi^2(x',0)\,dx'$$

Repeat. $\partial_x \psi^1 = \psi^2 \qquad \partial_y \psi^2 = \psi^1$

$$\partial_y \hat{\psi}^2 = \frac{\hat{\psi}^2 + \psi^1(0,y)}{s}$$

$$\hat{\psi}(s,y) = \int_0^\infty e^{-xs}\psi(x,y)\,dy$$

$$(\partial_y - s^{-1})\hat{\psi}^2 = \frac{\psi^1(0,y)}{s}$$

$$\hat{\psi}^2 = -\psi^1(0,y) + s\hat{\psi}^1$$

$$\hat{\psi}^2(s,y) = \int_0^y e^{(y-y')s^{-1}} \frac{\psi^1(0,y')}{s}\,dy' + e^{ys^{-1}}\hat{\psi}^2(s,0)$$

$$\partial_y \hat{\psi}^2 = \hat{\psi}^1$$

$$\psi^2(x,y) = \int_{-i\infty}^{i\infty} e^{xs} \left\{ \int_0^y e^{(y-y')s^{-1}} \frac{\psi^1(0,y')}{s}\,dy + e^{ys^{-1}}\hat{\psi}^2(s,0) \right\}$$

$$= \int_0^y \underbrace{\left(\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{xs+(y-y')s^{-1}} \right)}_{J_0(x(y-y'))} \psi^1(0,y')\,dy + \underbrace{\int_0^\infty \left(\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{(x-x')s+ys^{-1}} \right) \psi^2(x',0)\,dx'}_{\int_0^x J_1((x-x')y)\,y\,\psi^2(x',0)\,dx'}$$

$$e^{xs+ys^{-1}} = \sum_{n\geq 1} s^{-n} y^n J_n(xy) + \sum_{n\geq 0} s^n x^n J_n(xy)$$

OK

$$\boxed{\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\, e^{(x-x')s+ys^{-1}} = \delta(x-x') + J_1((x-x')y)y}$$

$$\psi^2(x,y) = \int_0^x \left(\delta(x-x') + J_1((x-x')y)\right)y\psi^2(x',0)\,dx' + \int_0^y J_0(x(y-y'))\psi^1(0,y')\,dy'$$

$$\hat{\psi}^1 = \frac{\hat{\psi}^2 + \psi^1(0,y)}{s} = \frac{\psi^1(0,y)}{s} + \int_0^y \frac{e^{(y-y')s^{-1}}}{s}\frac{\psi^1(0,y')}{s}\,dy' + e^{ys^{-1}}\frac{\hat{\psi}^2(s,0)}{s}$$

$$\psi^1(x,y) = \psi^1(0,y) + \int_0^y \left(\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i s}\,\frac{e^{xs+(y-y')s^{-1}}}{s}\right)\psi^1(0,y')\,dy' + \int_0^\infty \left(\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i s}\,e^{xs}e^{ys^{-1}}e^{-x's}\right)\psi^2(x',0)\,dx'$$

$$\psi^1(x,y) = \psi^1(0,y) + \int_0^y J_1(x(y-y'))\,x\,\psi^1(0,y')\,dy' + \int_0^x J_0((x-x')y)\,\psi^2(x',0)\,dx'$$

$$\psi^2(x,y) = \psi^2(x,0) + \int_0^x J_1((x-x')y)\,y\,\psi^2(x',0)\,dx' + \int_0^y J_0(x(y-y'))\,\psi^1(0,y')\,dy'$$

$$\partial_y \psi^2(x,y) = ?$$

$$\partial_y\, y \sum_{j\geq 0} \frac{a^j y^j}{j!\,(j+1)!} = \sum_{j\geq 0} \frac{a^j y^j}{j!\,j!} = J_0(ay)$$

$$\partial_y \big(y J_1(ay) \big) = J_0(ay)$$

$$\therefore\ \partial_y \int_0^x \underbrace{y J_1((x-x')y)}_{J_0((x-x')y)} \psi^2(x',0)\,dx'$$

$$\partial_y \int_0^y J_0(x(y-y'))\, \psi^1(0,y')\,dy'$$

$$= \psi^1(0,y) + \int_0^y J_1(x(y-y'))\, x\, \psi^1(0,y')\,dy'$$

Basic equations are
$$\partial_x J_0(x) = J_1(x)$$
$$\partial_x\, x J_1(x) = J_0(x)$$

$$\partial_x \psi^1(x,y) = \int_0^y \underbrace{\partial_x\big(x J_1(x(y-y'))\big)}_{J_0(x(y-y'))} \psi^1(0,y')\,dy'$$

$$+\ \psi^2(x,0) + \int_0^x \underbrace{\partial_x\{J_0((x-x')y)\}}_{y J_1((x-x')y)} \psi^2(x',0)\,dx'$$

~~try instead~~

$$\partial_x J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x)$$

$$\partial_x\big(x^{+\frac{1}{2}} J_{\frac{1}{2}}(x)\big) = x^{-1/2} J_{-\frac{1}{2}}(x)$$

$$\partial_x\big(\cosh(2x^{1/2})\big) = \sinh(2x^{1/2})\, x^{-1/2}$$

$$\partial_x\big(\sinh(2x^{1/2})\big) = \cosh(2x^{1/2})\, x^{-1/2}$$

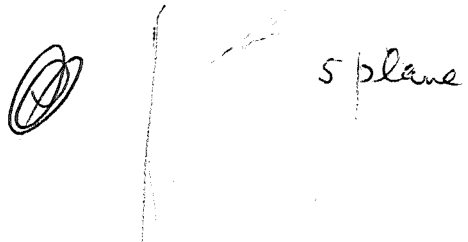
In general
$$\partial_x J_s(x) = J_{s+1}(x)$$
$$\partial_x\big(x^s J_s(x)\big) = x^{s-1} J_{s-1}(x)$$

Some things you perhaps learned during the struggle with the L.T., rather inverse L.T., which should be relevant to determining the hermitian forms.

$$\int_{0^+-i\infty}^{0^++i\infty} \frac{ds}{2\pi i}\, e^{xs+ys^{-1}} = \delta(x) + yJ_1(xy)H(x)$$

$$\int_{0^+-i\infty}^{0^++i\infty} \frac{ds}{2\pi i s}\, e^{xs+ys^{-1}} = J_0(xy)H(x)$$

$$\int_{0^+-i\infty}^{0^++i\infty} \frac{ds}{2\pi i s^2}\, e^{xs+ys^{-1}} = xJ_1(xy)H(x).$$



choice of contour with 0^+ means $=0$ if $x<0$.

So how can you use this? ~~[illegible]~~
<u>One point is</u> that integrating over $s\in i\mathbb{R}$ is wh

---

See if progress can be made on herm. forms. You ~~have~~ now have certain solutions of the DE defined by integrating over the contour $0^+ + i\mathbb{R}$

"Universal" soln is $\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}$

but it has values in the space of entire Laurent series.

Apply $\displaystyle\int_{0^+-i\infty}^{0^++i\infty}\frac{ds}{2\pi i}$ and you get $\psi(x,y) = \begin{pmatrix} J_0(xy)H(x) \\ \delta(x)+yJ_1(xy)H(x) \end{pmatrix}$

Check $\partial_x\big(J_0(xy)H(x)\big) = yJ_1(xy)H(x) + \delta(x)$

$\partial_y\big(\delta(x) + yJ_1(xy)H(x)\big) = J_0(xy)H(x).$

Some other solutions | namely contour S^1!

~~[illegible]~~ e.g.

$$\cancel{\oint \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} s^{n-1} \\ s^n \end{pmatrix}} =$$

$$\cancel{\oint \frac{ds}{2\pi i} e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} s^{n} \\ 1 \; s^{n} \end{pmatrix} = \begin{pmatrix} y^n J_n(xy) \\ y^{n-1} J_{n-1}(xy) \end{pmatrix}}$$

$$e^{xs+ys^{-1}} = \sum_{n \geq 1} s^{-n} y^n J_n(xy) + \sum_{n \geq 0} s^n x^n J_n(xy)$$

$$\oint \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \\ s \\ s^2 \end{pmatrix} = \begin{pmatrix} x J_1(xy) \\ J_0(xy) \\ y J_1(xy) \\ y^2 J_2(xy) \end{pmatrix}$$

This is an interesting class of solutions.
Try putting in $\frac{1}{2}$'s.

$$\partial_y \Big(\begin{matrix} x \, J_{-\frac{1}{2}}(xy) \\ \\ y^{\frac{1}{2}} \, J_{\frac{1}{2}}(xy) \end{matrix}$$

Idea here. To use $\sqrt{s}$.

Instead of ~~[illegible]~~ powers of s use $\frac{1}{2}$ powers $s^{n+\frac{1}{2}}$ $n \in \mathbb{Z}$

$$e^{xs+ys^{-1}} \begin{pmatrix} s^{-3/2} \\ s^{-1/2} \\ s^{1/2} \\ s^{3/2} \\ \vdots \end{pmatrix}$$

Can you integrate these somehow

somehow sections of $\mathcal{O}(-1)$

Review. Yesterday you corrected a mistake – the δ-fn in

$$\int_{0^+-i\infty}^{0^++i\infty} \frac{ds}{2\pi i s}\, e^{xs+y s^{-1}}\, s = \delta(x) + y J_1(xy) H(x)$$

$$1 \qquad J_0(xy) H(x)$$

$$s^{-1} \qquad x J_1(xy) H(x)$$

$$\int_{0^+-i\infty}^{0^++i\infty} \frac{ds}{2\pi i s}\, e^{xs+y s^{-1}} \begin{pmatrix} 1 & s^{-1} \\ s & 1 \end{pmatrix} = \cancel{J_0(xy)H(x)}$$

$$\cancel{\delta(x) + y J_1(yx) H(x)} \quad \cancel{J_0(xy) H(x)}$$

$$\begin{pmatrix} J_0(xy)H(x) & yJ_1(xy)H(x) \\ \delta(x) + yJ_1(xy)H(x) & J_0(xy)H(x) \end{pmatrix}$$

$$\partial_x \psi^1 = \psi^2 \qquad \partial_y \psi^2 = \psi^1 \qquad \hat{\psi}(s,y) = \int_0^\infty e^{-xs} \psi(x,y)\, dx$$

$$\widehat{\psi^2} = \widehat{\partial_x \psi^1} = -\psi^1(0,y) + s\widehat{\psi^1} \qquad \widehat{\psi^1} = \frac{\psi^1(0,y) + \widehat{\psi^2}}{s} = \partial_y(\widehat{\psi^2})$$

$$\widehat{\psi^1} = \widehat{\partial_y \psi^2} = \partial_y(\widehat{\psi^2})$$

$$(\partial_y - s^{-1})\widehat{\psi^2} = s^{-1}\psi^1(0,y)$$

$$\widehat{\psi^2}(s,y) = \int_0^y e^{(y-y')s^{-1}} s^{-1} \psi^1(0,y')\, dy' + e^{ys^{-1}}\, \widehat{\psi^2}(s,0)$$

$$\psi^2(x,y) = \int_0^y \underbrace{\left(\int_{0^+-i\infty}^{0^++i\infty} e^{xs+(y-y')s^{-1}} s^{-1} \frac{ds}{2\pi i} \right)}_{J_0(x(y-y'))H(x)} \psi^1(0,y')\, dy'$$

$$+ \int_0^\infty \left(\int_{-i\infty}^{i\infty} e^{(x-x')s + ys^{-1}} \frac{ds}{2\pi i} \right) \psi^2(x',0)\, dx'$$

$$\hat{\psi}^1 = s^{-1}\hat{\psi}^2 + s^{-1}\psi^1(0,y)$$

$$= \int_0^y e^{(y-y')s^{-1}} s^{-2}\psi^1(0,y')\,dy' + \int_0^\infty e^{ys^{-1}-x's} s^{-1}\psi^2(x',0)\,dx' + s^{-1}\psi^1(0,y)$$

$$\psi^1(x,y) = \psi^1(0,y)\cancel{H(x)} + \int_0^y x J_1(x(y-y'))\cancel{H(x)}\,\psi^1(0,y')\,dy' + \int_0^x J_0((x-x')y)\cancel{H(x-x')}\,\psi^2(x',0)\,dx'$$

$$\psi^2(x,y) = \int_0^y J_0(x(y-y'))\,\psi^1(0,y')\,dy' + \psi^2(x,0) + \int_0^x y J_1((x-x')y)\,\psi^2(x',0)\,dx'$$

$$e^{xs+ys^{-1}}s^{-1} = e^{ys^{-1}}s^{-1} + \int_0^y x J_1(x(y-y'))\,e^{y's^{-1}}s^{-1}\,dy' + \int_0^x J_0((x-x')y)\,e^{x's}\,dx'$$

$$e^{xs+ys^{-1}} = e^{xs} + \int_0^x y J_1((x-x')y)\,e^{x's}\,dx' + \int_0^y J_0(x(y-y'))\,e^{y's^{-1}}s^{-1}\,dy'$$

$$e^{xs+ys^{-1}} = \sum_{n\geq 1} s^{-n} y^n J_n(xy) + \sum_{n\geq 0} s^n x^n J_n(xy)$$

$$e^{xs} + \frac{y}{1!}\frac{e^{xs}-1}{s} + \frac{y^2}{2!}\frac{e^{xs}-1-xs}{s^2} + \cdots$$

$$= e^{xs} + \sum_{n=1}^{\infty} \frac{y^n}{n!} \int_0^x e^{x's} \frac{(x-x')^{n-1}}{(n-1)!}\, dx'$$

$$= e^{xs} + \int_0^x e^{x's}\, y\, J_1((x-x')y)\, dx'$$ ~~anyway.~~

---

inner products. In LT picture you use

$$\int_{0^+-i\infty}^{0^++i\infty} \frac{ds}{2\pi i}\left(\text{---} \right)$$ a linear functional on grid space.

First determine the pos. def. product.

$$\left(\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \middle| \lambda^x \mu^y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right)$$

~~This should amount to~~

First thing to do is to find $\int \varphi(x)\lambda^x\, dx\; v^2 \Big| \int \varphi_1(x)\lambda^x dx\, v^2$

Basically you want $\left(\lambda^x v^2 \middle| \lambda^{x'} v^2 \right) = \delta(x-x')$

$$\int_{-i\infty}^{i\infty} \left(e^{sx}\right)^* \left(e^{sx'}\right) \frac{ds}{2\pi i} = \int_{-\infty}^{\infty} \left(e^{i\rho x}\right)^* \left(e^{i\rho x'}\right) \frac{d\rho}{2\pi}$$

$s = i\rho$

$$= \delta(x-x')$$

Therefore you want $(e^{ipx})^* = e^{-ipx}$

and so $(e^{sx})^* = (e^{ipx})^* = e^{-ipx} = e^{-xs}$

$$\boxed{(e^{xs})^* = e^{-xs}}$$

makes $\lambda^* =$ mult by e^{xs} unitary

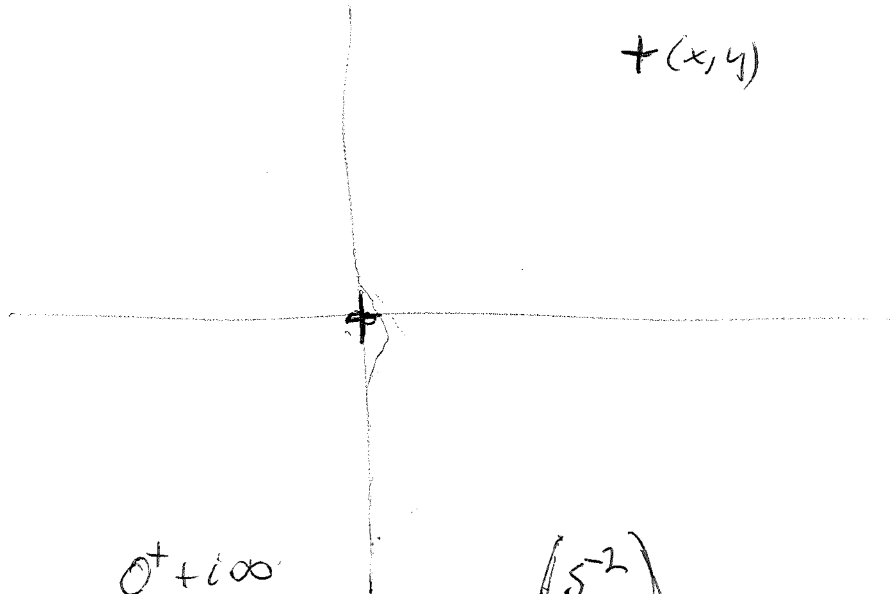
similarly $(e^{ys^{-1}})^* = e^{-ys^{-1}}$. ~~Next go to~~

$$\left(v^2 \,\middle|\, \lambda^* \mu y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right) = \psi(x,y)$$

$$\int_{-i\infty}^{i\infty} e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix} \frac{ds}{2\pi i}$$

the second component is perhap not well defined

Let's take 0^+ contour and see what we get



$$\int_{0^+-i\infty}^{0^++i\infty} e^{xs+ys^{-1}} \begin{pmatrix} s^{-2} \\ s^{-1} \\ 1 \\ s \end{pmatrix} \frac{ds}{2\pi i} = \begin{cases} y J_1(xy) H(x) \\ J_0(xy) H(x) \\ \delta(x) + y J_1(xy) H(x) \end{cases}$$

Try to determine initial values for $(v^1|-)$ using the ~~[illegible]~~ pos. def. product you expect

$$\left(v^1 \middle| \lambda^x \mu^y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}\right) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \left((s^{-1})^* \;\; 1 \right) e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}$$

You expect $(s^{-1})^* = -s^{-1}$

$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} \left(-1 \right) e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}.$$

$$\psi^1(0,y) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s}(-1) e^{ys^{-1}} s^{-1} \qquad \text{put } s \mapsto s^{-1}$$

$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i \cancel{s}} e^{ys} \cancel{s} = \delta(y) \qquad \text{expected for } (v^1|\mu^y v^1)$$

$$\psi^2(x,0) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s}(-1) e^{xs} 1 \qquad \partial_x \psi^2(x,0) = \int \frac{ds}{2\pi i}(-1) e^{xs} = -\delta(x)$$

so $\psi^2(x,0) = -H(x)$ up to a constant.

Actually we expect $v^1 = -\int_0^\infty \lambda^x v^2 \, dx$ in the Hilbert space completion — i.e. $\frac{1}{s} = -\int_0^\infty e^{xs} dx$

Take solution corresp to $\psi^1(0,y) = \delta(y)$, $\psi^2(x,0) = -H(x)$

$$\psi^1(x,y) = \delta(y) + x J_1(xy) H(y) + \int_0^x J_0((x-x')y)(-H(x'))dx'$$

$$\psi^2(x,y) = -H(x) + (-1)\int_0^x y J_1((x-x')y)\,dx' + J_0(xy)H(y)$$

OK

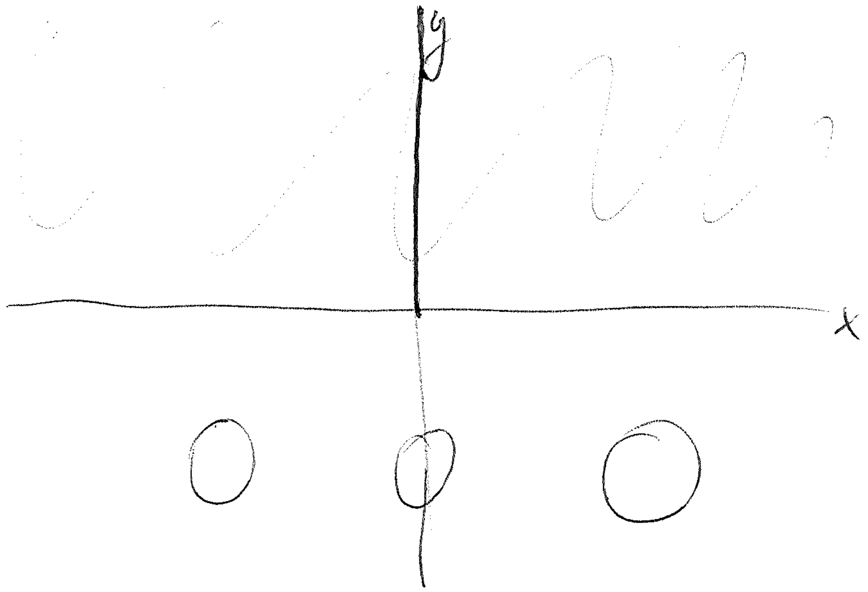
$$\partial_y \int_0^x y J_1((x-x')y)\,dx' = \int_0^x J_0((x-x')y)\,dx'$$

$$\partial_x \int_0^x J_0((x-x')y)\,dx' = 1 + \int_0^x y J_1((x-x')y)\,dx'$$

~~Prob~~ You should separate components

First case $\psi^1(0,y) = \delta(y)$ $\quad \psi(x,y) = \begin{pmatrix} \delta(y) + xJ_1(xy)H(y) \\ J_0(xy)H(y) \end{pmatrix}$

$\psi^2(x,0) = 0$



$$\begin{pmatrix} J_0(xy) \\ yJ_1(xy) \end{pmatrix}$$

2nd case $\psi^1(0,y) = 0$

$\psi^2(x,0) = 1$

$$\psi(x,y) = \begin{pmatrix} \int_0^x J_0((x-x')y)\,dx' \\ 1+\int_0^x yJ_1((x-x')y)\,dx' \end{pmatrix} = \begin{pmatrix} \int_0^x J_0(x'y)\,dx' \\ 1+\int_0^x yJ_1(x'y)\,dx' \end{pmatrix}$$

$$1+\int_0^x yJ_1(x'y)\,dx'$$

$$= 1+\int_0^x \partial_{x'} J_0(x'y)\,dx' = 1+\left[J_0(x'y)\right]_0^x = J_0(xy)$$

$$\int_0^x J_0(x'y)\,dx' = \int_0^x \partial_{x'}\left(x'J_1(x'y)\right)dx' = \left[x'J_1(x'y)\right]_0^x = xJ_1(xy)$$

So in this case $\quad \psi(x,y) = \begin{pmatrix} xJ_1(xy) \\ J_0(xy) \end{pmatrix} \quad \partial_x\psi(x,y) = \begin{pmatrix} J_0(xy) \\ yJ_1(xy) \end{pmatrix}$

<u>Third case:</u> $\psi^1(0,y) = 0$

$\psi^2(x,0) = \delta(x)$.

$$\psi(x,y) = \begin{pmatrix} J_0(xy)H(x) \\ \delta(x) + y J_1(xy) H(x) \end{pmatrix}$$

apply $\int_0^x$

$$\int_0^x \underbrace{J_0(x'y)}_{\partial_{x'}\left(x' J_1(x'y)\right)} H(x')\, dx' = \begin{cases} 0 & x < 0 \\ x J_1(xy) & x > 0 \end{cases} = x J_1(xy) H(x)$$

$$\cancel{\partial_y\left(x J_1(xy) H(x)\right) = xy J_2(xy) H(x).} \quad ?$$

$$\cancel{\psi(x,y) = \begin{pmatrix} x J_1(xy) H(x) \\ xy J_2(xy) H(x) \end{pmatrix}}$$

$$\psi(x,y) = \begin{pmatrix} x J_1(xy) H(x) \\ J_0(xy) H(x) \end{pmatrix}$$

Return to third case:

$\psi^1(0,y) = 0$

$\psi^2(x,0) = \delta(x)$

$$\psi(x,y) = \begin{pmatrix} \int_0^x J_0((x-x')y)\, \delta(x')\, dx' \\ \delta(x) + \int_0^x y J_1((x-x')y)\, \delta(x')\, dx' \end{pmatrix}$$

my interpretation of this was $\psi(x,y) = \begin{pmatrix} J_0(xy) H(x) \\ \delta(x) + y J_1(xy) H(x) \end{pmatrix}$

But $H(x)$ is ambiguous at $x=0$

so what does it mean ~~for~~ that's

$$\psi^1(0,y) = \underbrace{J_0(0)}_{1}\, \underbrace{H(0)}_{?}$$

$$\psi^2(x,0) = \delta(x) \text{ is OK.}$$

Apparently the conclusion is that the problem is not well posed. What problem? the solution with $\psi^1(0,y)=0$, $\psi^2(x,0)=\delta(x)$. (third case above) or $\psi^1(0,y)=\delta(y)$, $\psi^2(x,0)=0$ (first case above).

Special case: Work on the line with ∂_x and its inverse $\partial_x^{-1} = \int_0^x$, i.e. $(\partial_x^{-1}f)(x) = \int_0^x f(x')\,dx'$. This gives solution to IVP. ~~So what does Wh~~

$\partial_x u = f$, $u(0)=0$.

Take $f(x) = \delta(x-a)$. Then you want

$$\partial_x u = \delta(x-a) \qquad u(0)=0.$$

~~means~~ $u(x) = \int_0^x \delta(x'-a)\,dx'$, this u is constant for $x\neq a$ and jumps by $+1$ on passing through a. ~~Whether~~ Seems impossible to specify its value at a, so things become ~~wired~~ confused.

$$\begin{pmatrix} J_0(xy)H(x) \\ \delta(x)+yJ_1(xy)H(x) \end{pmatrix}$$ solution $\psi^1(0,y) = H(0)$?

$\psi^2(x,0) = \delta(x)$.

Problem to compute $(\ |\ )$ on the "grid space", or really to show it is given on the holom. function model by $(f|g) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} (f)^* g$ where $*$ is defined by $(s)^* = -\bar{s}$, thus $s^* = \bar{s}$ for $s \in i\mathbb{R}$?

? we want $(e^{xs})^* = e^{-xs}$ for x real, so that e^{xs} is unitary. The point maybe is that you define a $*$ operator on holom. fns which reduces to conjugate when the function is restricted to $i\mathbb{R}$. Thus $s^* = (i\rho)^* = -i\rho^* = -i\rho = -s$.

Note that
$$\int_{-\infty}^{\infty} \frac{dt}{t} f(t) = \int_{-\infty}^{0^-} f(t)\frac{dt}{t} + \int_{0}^{\infty} f(t)\frac{dt}{t}$$

$$= \int_{0^-}^{-\infty} f(t^{-1})\left(-\frac{dt}{t}\right) + \int_{+\infty}^{0^+} f(t^{-1})\left(-\frac{dt}{t}\right)$$

$$= \int_{-\infty}^{0^-} f(t^{-1})\frac{dt}{t} + \int_{0^+}^{+\infty} f(t^{-1})\frac{dt}{t} = \int_{-\infty}^{\infty} f(t^{-1})\frac{dt}{t}$$

However
$$\int_{-i\infty}^{+i\infty} f(s)\frac{ds}{s} = \left(\int_{-i\infty}^{+i0^-} + \int_{i0^+}^{+i\infty}\right) f(s)\frac{ds}{s}$$

$$= \left(\int_{i0^+}^{+i\infty} + \int_{-i\infty}^{-i0^+}\right) f(s^{-1})\left(-\frac{ds}{s}\right) = -\int_{-i\infty}^{i\infty} f(s^{-1})\frac{ds}{s}$$

true for half line

Def

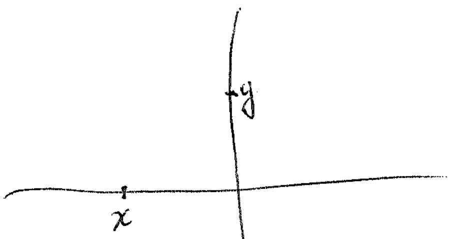
$$(f|g) = \int_{-i\infty}^{i\infty} (f)^* g \,\frac{ds}{2\pi i}$$ and check it works

$$(v^2|\lambda^x v^2) = \int_{-i\infty}^{i\infty} 1\cdot e^{xs}\,\frac{ds}{2\pi i} = \delta(x)$$

$$(v^1|\mu^y v^1) = \int_{-i\infty}^{i\infty} \left(\frac{\cancel{1}}{s}\right)^* e^{ys^{-1}}(s^{-1})\,\frac{ds}{2\pi i} = \int_{-i\infty}^{i\infty} -s^{-1}e^{ys^{-1}}\,\frac{ds}{2\pi i s}$$

$$= \int_{-i\infty}^{i\infty} +\cancel{s}\,e^{ys}\,\frac{ds}{2\pi i\cancel{s}} = \delta(y).$$

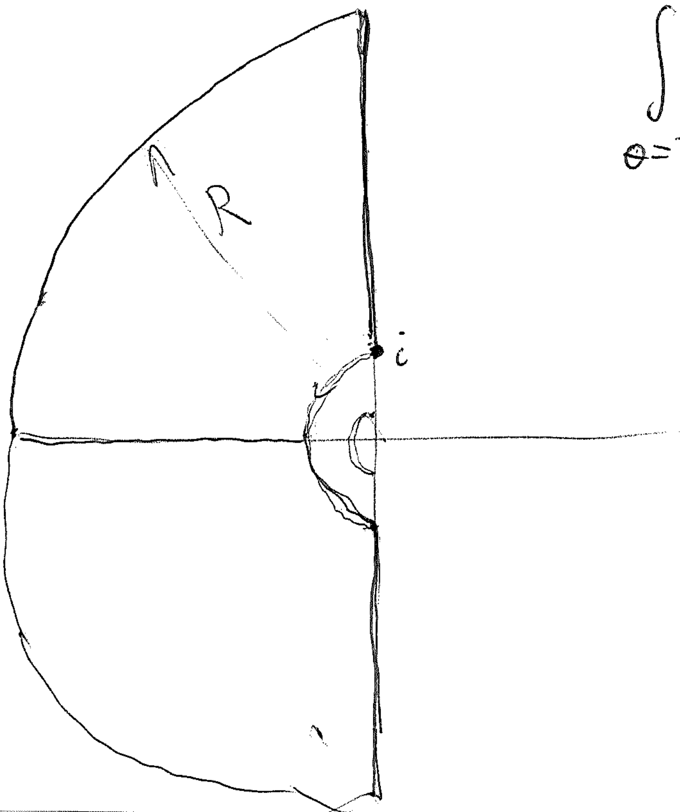
$$(\lambda^x v^2|\mu^y v^1) = \int_{-i\infty}^{i\infty} (e^{xs})^* e^{ys^{-1}} s^{-1}\,\frac{ds}{2\pi i} = \int_{-i\infty}^{i\infty} e^{-xs+ys^{-1}}\,\frac{ds}{2\pi i s}$$

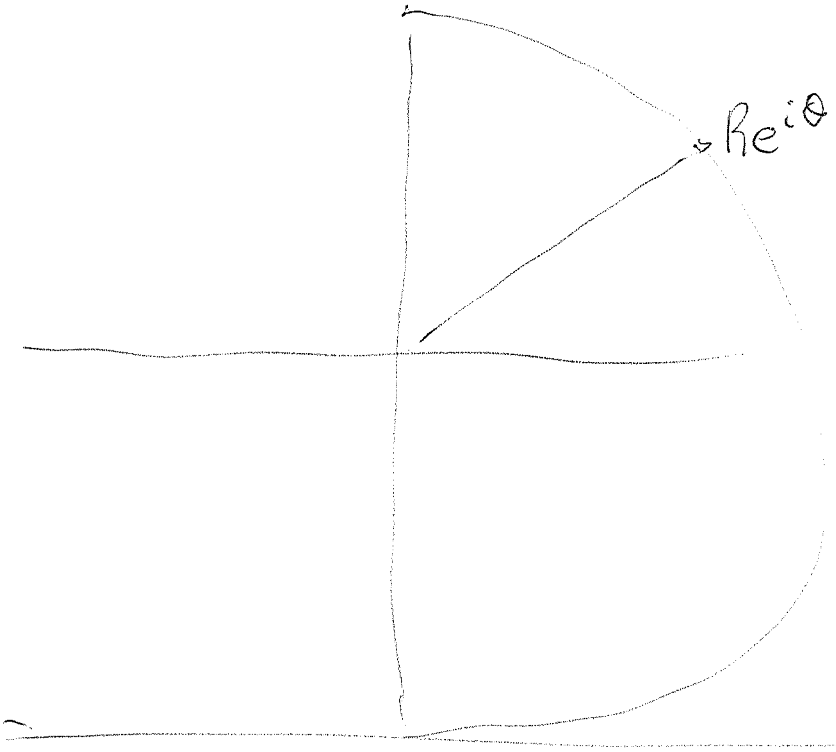


assume $x<0$, $y>0$

Because $x<0$ $e^{(-x)s}$ decays as $\mathrm{Re}(s) \to -\infty$. In the s plane it looks like we can deform the contour from $s =$ i to $+i\infty$

$$\int_{\theta=\pi/2}^{\theta=\pi} \left|e^{(-x)Re^{i\theta}}\right| \frac{d\theta}{2\pi} \qquad e^{(-x)R\underbrace{\cos\theta}_{<0}}$$





$$\left|e^{-xRe^{i\theta}}\right| = e^{-xR\cos\theta}$$

$$\int_0^{\pi/2} e^{-xR\cos\theta}\, R\,d\theta$$

$$\int_0^{\pi/2} e^{-R\sin\theta}\, R\,d\theta$$

$$\int_0^{\varepsilon} e^{-R\theta} R\,d\theta = \left[-e^{-R\theta}\right]_0^{\varepsilon} = 1-e^{-R\varepsilon}$$

group ring of R is non unital - in scattering you encounter invertible matrices over this group ring.

$$K_s(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} t^s \frac{dt}{t}$$

$$\partial_r K_s = -\tfrac{1}{2}\left(K_{s+1} + K_{s-1}\right)$$

$$\frac{s}{r} K_s = \tfrac{1}{2}\left(K_{s+1} - K_{s-1}\right)$$

$$\left(\partial_r + \frac{s}{r}\right) K_s = -K_{s-1}$$

$$\left(\partial_r - \frac{s}{r}\right) K_s = -K_{s+1}$$