January 18, 1999

I propose to make notes on my lectures about operators and related analytic functions.

Let X be a Hilbert space, let c be a contraction operator on X, which means c satisfies the following equivalent conditions:

1) $\|c\| \leq 1$ 1)$'$ $\|c^*\| \leq 1$

2) $(x, c^*cx) \leq (x,x)$ $\forall x$ 2)$'$ $(x, cc^*x) \leq (x,x)$ $\forall x$

3) $1 - c^*c \geq 0$ 3)$'$ $1 - cc^* \geq 0$.

Prop 1. If c is a contraction on X, then there exists a triple (E, u, j) with E a Hilbert space, u a unitary operator on E and $j : X \to E$ an isometry satisfying

$$j^* u^n j = \begin{cases} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0 \end{cases} \qquad E = \overline{\sum_{n\in\mathbb{Z}} u^n jX}$$

Moreover (E, u, j) is unique up to canonical isom.

Note the above condition for $n \leq 0$ is equivalent to the condition for $n \geq 0$ by adjointness. Also for $n=0$ it says $j^*j = 1$, i.e. $\|jx\| = \|x\|$ $\forall x$, and j is an isometry.

We Construct E by completing the space $\bigoplus_{n\in\mathbb{Z}} z^n X$ (alg direct sum) of Laurent poly functions on $|z|=1$ with values in X.

We define a hermitian form on Laurent polynomials so that $(z^k x_k, z^l x_l)$ equals the desired inner product in E under $z^k x_k \mapsto u^k j x_k$. Thus

$$(z^k x_k, z^l x_l) = (u^k j x_k)^* (u^l j x_l) = x_k^* j^* u^{-k+l} j x_l$$

$$= x_k^* \rho_{-k+l} x_l, \quad \text{where} \quad \rho_n = j^* u^n j = \begin{cases} c^n & n \geq 0 \\ c^{*-n} & n \leq 0. \end{cases}$$

We have to prove positivity of this inner product; then we can complete the space of Laurent polys to obtain the desired (E, u, j). By replacing c by rc with $0<r<1$ and letting $r \uparrow 1$ we can reduce to the case $\|c\| < 1$. Then the Fourier series with coefficients ρ_{-n}:

$$\sum_{n\in\mathbb{Z}} z^{-n}\rho_n = \sum_{n\geq 0} z^{-n}c^n + \sum_{n\geq 1} z^n c^{*n} = \frac{1}{1-z^{-1}c} + \frac{zc^*}{1-zc^*}$$

$$= \frac{1}{1-z^{-1}c}\left(1-zc^* + (1-z^{-1}c)zc^*\right)\frac{1}{1-zc^*} = \frac{1}{1-z^{-1}c}(1-cc^*)\frac{1}{1-zc^*}$$

also

$$= \frac{1}{1-zc^*}\left(1-zc^* + zc^*(1-z^{-1}c)\right)\frac{1}{1-z^{-1}c} = \frac{1}{1-zc^*}(1-c^*c)\frac{1}{1-z^{-1}c}$$

~~[illegible]~~ converges. We have for $x(z) = \sum z^k x_k$ a Laurent polynomial

$$(x(z), x(z)) = \sum_{k,\ell} x_k^* \rho_{-k+\ell} x_\ell = \sum_{k,\ell} x_k^* \int z^{-k+\ell}\rho \frac{d\theta}{2\pi} x_\ell$$

$$= \int x(z)^* \rho\, x(z) \frac{d\theta}{2\pi} = \int \|T(z)x(z)\|^2 \frac{d\theta}{2\pi} \geq 0$$

where $T(z) = (1-c^*c)^{1/2}\frac{1}{1-z^{-1}c}$ or $(1-cc^*)^{1/2}\frac{1}{1-zc^*}$.

Call (E, u, j) the standard dilation of c.

Next, another description. Motivation: Consider $\overline{jX + ujX} \subset E$. This is the completion of the space of pairs $(x_0, x_1) \in X^{\oplus 2}$, equiv. polys $x_0 + zx_1$, for the norm $\|jx_0 + ujx_1\|^2 = \|x_0\|^2 + (jx_0, ujx_1) + (ujx_1, jx_0) + \|x_1\|^2$

$$= \|x_0\|^2 + (x_0, cx_1) + (cx_1, x_0) + \|cx_1\|^2 + \|x_1\|^2 - \|cx_1\|^2$$

$$= \|x_0 + cx_1\|^2 + \|x_1\|^2 - \|cx_1\|^2.$$

Def $\quad \nu_+ x = ujx - jcx = (uj - jc)x$

$\quad\quad\; \nu_- x = u^{-1}jx - jc^*x = (u^{-1}j - jc^*)x$

[redacted] Prop 2. E admits an orthogonal Hilbert space direct sum decomposition

$$E = \bigoplus_{n\le 0}^{(2)} u^n V_- \oplus jX \oplus \bigoplus_{n\ge 0}^{(2)} u^n V_+$$

where $V_\pm = \overline{\nu_\pm X}$.

Proof. Formulas: $\boxed{j^* u^n \nu_+ = 0 \quad n \ge 0,}$

$\boxed{\nu_+^* u^n \nu_+ = 0 \quad n \ne 0.}$ Also $j^* u^n \nu_- = 0 \quad n \le 0,$

$\nu_-^* u^n \nu_- = 0 \quad n \ne 0,$ and $\nu_-^* u^n \nu_+ = 0 \quad n \ge 0$

$$j^* u^n \nu_+ = j^* u^n (uj - jc) = j^* u^{n+1} j - j^* u^n j c = c^{n+1} - c^n c = 0$$

$$\nu_+^* u^n \nu_+ = (uj - jc)^* u^n \nu_+ = (\underbrace{j^* u^{-1} - c^* j^*}_{u^{n-1}}) u^n \nu_+ = 0 \quad n-1 \ge 0$$

$$\nu_-^* u^n \nu_+ = (u^{-1}j - jc^*)^* u^n \nu_+ = j^* u^{n+1} \nu_+ - c j^* u^n \nu_+ = 0.$$

Also should have $\boxed{\nu_+^* \nu_+ = 1 - c^*c, \quad \nu_-^* \nu_- = 1 - cc^*.}$

$$\nu_+^* \nu_+ = (uj - jc)^*(uj - jc) = j^* u^{-1}(uj - jc) = j^*j - \underbrace{j^* u^{-1} j}_{c^*} c = 1 - c^*c.$$

Check direct sum contains jX and is stable under u, u^{-1}. [redacted]

$ujx = jcx + \nu_+ x \implies ujX \subset jX \oplus V_+.$

$u\nu_- x = u(u^{-1}j - jc^* x) = jx - ujc^*x \implies uV_- \subset jX + ujX \subset jX \oplus V_+$

<u>January 19, 1989</u>

Recall the situation: (X, c) dilation (E, u, j), $V_\pm = \overline{\nu_\pm X}$. We know the subspaces $u^n V_+$, $n \in \mathbb{Z}$ are mutually orthogonal, hence $\bigoplus^{(2)}_{n\in\mathbb{Z}} u^n V_+$ is a closed subspace of E which can be identified with $L^2(S^1, V_+)$ via $\sum u^n v_{+,n} \longleftrightarrow \sum z^n v_{+,n}$. Let

$$j_+ : L^2(S^1, V_+) \simeq \bigoplus^{(2)}_{n\in\mathbb{Z}} u^n V_+ \hookrightarrow E$$

be the inclusion. Its adjoint $j_+^* : E \longrightarrow L^2(S^1, V_+)$ is called the <u>outgoing representation</u>. Similarly the adjoint of

$$j_- : L^2(S^1, V_-) \simeq \bigoplus^{(2)}_{n\in\mathbb{Z}} u^n V_- \hookrightarrow E$$

gives the <u>incoming representation</u>.

<u>Prop. 3.</u> $\quad j_+^* j x = \nu_+\left(\frac{1}{z-c} x\right), \quad j_-^* j x = \nu_-\left(\frac{1}{z^{-1}-c^*} x\right).$

First the picture:

$$\begin{array}{ccccccccccccc}
L^2(S^1, V_+) & = & \cdots \oplus & u^{-2}V_+ & \oplus & u^{-1}V_+ & \oplus & & & V_+ & \oplus & uV_+ & \oplus \cdots \\
\downarrow j_+ & & & & \cap & & & & & \| & & \| & \\
E & = & \cdots \oplus & u^{-1}V_- & \oplus & V_- & \oplus & jX & \oplus & V_+ & \oplus & uV_+ & \oplus \cdots \\
\uparrow j_- & & & \| & & \| & & & & \cup & & & \\
L^2(S^1, V_-) & = & \cdots \oplus & u^{-1}V_- & \oplus & V_- & \oplus & & & uV_- & \oplus & u^2V_+ & \oplus \cdots
\end{array}$$

Let $\xi \in E$. $j_+^*\xi$ is the projection of ξ onto $\bigoplus^{(2)}_{\mathbb{Z}} u^n V_+$; the components of $j_+^*\xi$ for $n \geq 0$ can be obtained by projecting ξ onto $V_+ \oplus uV_+ \oplus \cdots$. To obtain the components for $n \geq -N$ one can take $u^N\xi$ project onto $V_+ \oplus uV_+ \oplus \cdots$ and apply u^{-N}. Carry this out for $\xi = jx$.

$$u j x = j c x + \nu_+ x$$
$$u^2 j x = j c^2 x + \nu_+ c x + u \nu_+ x$$
$$u^3 j x = j c^3 x + \nu_+ c^2 x + u \nu_+ c x + u^2 \nu_+ x$$
$$* \qquad u^{n+1} j x = j c^{n+1} x + \sum_{k=0}^{n} u^{n-k} \nu_+ c^k x$$
$$j x = u^{-1-n} j c^{n+1} x + \underbrace{\sum_{k=0}^{n} u^{-1-k} \nu_+ c^k x}$$

gives the components $j_+^* j x$ in degrees $\geqslant -n-1$.

$$\therefore \quad j_+^* j x = \sum_{k=0}^{\infty} u^{-1-k} \nu_+ c^k x \longleftrightarrow \sum_{k=0}^{\infty} \nu_+ z^{-1-k} c^k x$$
$$= \nu_+ \frac{1}{1 - z^{-1} c} z^{-1} x = \nu_+ \left(\frac{1}{z-c} x \right).$$

From $*$ above we get

$$\|x\|^2 = \|c^{n+1} x\|^2 + \sum_{k=0}^{n} \|\nu_+ c^k x\|^2$$

which is also clear since $\|\nu_+ c^k x\|^2 = (c^k x, (1 - c^* c) c^k x)$ $= \|c^k x\|^2 - \|c^{k+1} x\|^2$. This shows that

$$\|x\|^2 = \lim_{n \to \infty} \|c^n x\|^2 + \|j_+^* j x\|^2$$

<u>Prop. 4</u>. One has $\|x\|^2 = \|j_+^* j x\|^2$, i.e. the outgoing representation preserves the norm on jX, iff $\lim_{n \to \infty} \|c^n x\| = 0$, $\forall x \in X$. In this case j_+, j_+^* give a unitary isom between E and $L^2(S^1, V_+)$ and conversely. Similarly $\lim_{n\to\infty} \|c^{*n} x\| = 0 \iff \|x\| = \|j_-^* j x\| \ \forall x \iff j_-, j_-^*$ give unitary isomorphism $L^2(S^1, V_-) \simeq E$.

The equiv. between $\|x\| = \|j_+^* j_+ x\|$, $\forall x$ and $\|c^n x\| \to 0$ $\forall x$ is clear. This holds exactly when $jX \subset j_+ L^2(S^1, V_+)$, otherwise the norm on jX would decrease on projecting onto $j_+ L^2(S^1, V_+)$. Then $E = j_+ L^2(S^1, V_+)$ since E is generated by jX, u, u^{-1}.

<u>January 25, 1999</u>

Recall that the Reimann sphere $\mathbb{C} \cup \{\infty\}$ can be viewed as the space $\mathbb{C}P^1$ of complex lines in $\mathbb{C}^2$, the line $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\mathbb{C}$ corresponding to $\frac{z_1}{z_2} = z$ in the Reimann sphere. This leads to the action of $GL_2(\mathbb{C})$ by fractional linear transformation on the RS. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \dfrac{az+b}{cz+d}$. Let $U(1,1)$ be the subgroup of $GL_2\mathbb{C}$ preserving the hermitian form $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = |z_1|^2 - |z_2|^2$, equivalently, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$ when $g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note $U(1,1)$ contains the diagonal matrices having both entries of modulus 1, so $U(1,1) = SU(1,1) \cdot \{e^{i\phi} I\}$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$, then $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g^{-1}$, i.e.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar a & \bar c \\ \bar b & \bar d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \bar a & -\bar c \\ -\bar b & \bar d \end{pmatrix} \text{ must } = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \text{ i.e.}$$

$a = \bar d$, $b = \bar c$, so $SU(1,1) = \left\{ \begin{pmatrix} \bar d & \bar c \\ c & d \end{pmatrix} \middle| \, |d|^2 - |c|^2 = 1 \right\} =$

$\left\{ \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \right\} \cdot \left\{ \begin{pmatrix} d & \bar c \\ c & d \end{pmatrix} \middle| \begin{matrix} d > 0 \\ d^2 - |c|^2 = 1 \end{matrix} \right\}$. Also $\begin{pmatrix} d & \bar c \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix} \dfrac{1}{\sqrt{1-|h|^2}}$

where $h = \dfrac{\bar c}{d}$ $\left(\text{and } 1 - |h|^2 = 1 - \dfrac{|c|^2}{d^2} = \dfrac{1}{d^2} \text{ so } d = \dfrac{1}{\sqrt{1-|h|^2}}\right)$.

[I've left out $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \Rightarrow \overline{\det g}\,(-1) \det g = -1$ $\Rightarrow |\det g|^2 = 1 \Rightarrow |\det g| = 1$, so one has $SU(1,1) \hookrightarrow U(1,1) \twoheadrightarrow S^1$ exact. Note that $U(1,1)$ is connected, unlike $GL_2(\mathbb{R})$.]

Since $U(1,1)$ preserves $|z_1|^2 - |z_2|^2$, the corresponding fractional linear transfs. preserve the unit circle $|z|^2 - 1 = 0$, the unit disk D: $|z|^2 - 1 < 0$, and the complementary disk $|z|^2 - 1 > 0$.

Recall that D is a model for the hyperbolic plane, where geodesics are circles $\perp$ to ∂D. The group of orientation preserving isometries is $U(1,1)/\text{diag} = SU(1,1)/\{\pm 1\}$. Claim the hyperbolic arclength is

$$ds = \frac{|\delta z|}{1 - |z|^2}$$

δ used to avoid conflict with d in g.

This agrees with the Euclidean length at $z=0$ and is invariant under the $U(1,1)$ action

$$w = \frac{az+b}{cz+d} \qquad \delta w = \frac{(cz+d)a\,\delta z - (az+b)c\,\delta z}{(cz+d)^2} = (ad-bc)\frac{\delta z}{(cz+d)^2}$$

$$1-|w|^2 = \frac{|cz+d|^2 - |\bar{d}z+\bar{c}|^2}{|cz+d|^2} = \frac{|cz|^2+|d|^2+\cancel{cz\bar{d}}+\cancel{\bar{c}\bar{z}d} - |dz|^2-|c|^2-\cancel{\bar{d}zc}-\cancel{d\bar{z}\bar{c}}}{|cz+d|^2}$$

$$= \left(|d|^2-|c|^2\right)\frac{1-|z|^2}{|cz+d|^2} \qquad \boxed{\therefore \frac{|\delta w|}{1-|w|^2} = \frac{|\delta z|}{1-|z|^2}}$$

Schur expansion for $f \in H^\infty$ $\|f\|_\infty \leq 1$

Let $h_0 = f(0)$, so $|h_0| \leq 1$. If f constant, then $f(z) = h_0$ is the order 0 case of the Schur expansion. If f is not constant, then $|h_0| < 1$ and $g = \begin{pmatrix} 1 & -h_0 \\ -\bar{h}_0 & 1 \end{pmatrix} f = \frac{f-h_0}{1-\bar{h}_0 f}$ is a non constant map $D \to D$, which vanishes at 0, hence $f_1 = \frac{g(z)}{z}$ is also in H^∞ of norm ≤ 1, and one has $f = \begin{pmatrix} 1 & +h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} f_1$. Now continue this process for f_1, i.e. set $h_1 = f_1(0)$ so that $|h_1| \leq 1$. If f_1 constant, then $h_1 \neq 0$ otherwise $g = zf_1$ would be constant. We have the Schur exp. of order 1.

$$f = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} h_1 \qquad \text{with } |h_0| < 1,\ 0 < |h_1| < 1.$$

If f_1 non-constant, then $|h_1| < 1$, and proceeding as above we get

$$f = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} f_2$$

with $|h_0|, |h_1| < 1$ and $f_2 \in H^\infty$, $\|f_2\|_\infty \leq 1$. etc.

Summarize by saying either f has Schur expansion of order n

$$f = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_{n-1} \\ \bar{h}_{n-1} & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}(h_n) \qquad \begin{array}{l} |h_0|, \dots, |h_{n-1}| < 1 \\ 0 < |h_n| \leq 1 \end{array}$$

1 Need to review reflection positivity.
Can you first understand Gaussian case?

so what do I do?

Review program. Given $aX \oplus V^+ = V^- \oplus bX = Y$ of type $\mathcal{O}(n)$, get line bundle $L_z = Y/(az-b)X$, get holom section from v_0^- (unit v_0 in V^+), allowing sections y to yield functions:

$$(az-b)x = -y + \tilde{y}(z)\,v_0^-$$

solve $\quad (1-zb^*a)x = b^*y$

$$\tilde{y}(z) = \Big(v_0^-,\ \underbrace{y + (az-b)\overbrace{(1-zb^*a)^{-1}b^*}^{b^*(1-zab^*)^{-1}}y}_{\left(1-zab^* + (az-b)b^*\right)(1-zab^*)^{-1}y}\Big)$$

$$= (v_0^-, (1-zab^*)^{-1}y).$$

$$\int |\tilde{y}(z)|^2 \frac{d\theta}{2\pi} = \int \left(y, (1-z^{-1}ba^*)^{-1}(1-bb^*)(1-zab^*)^{-1}y\right)\frac{dz}{2\pi i z}$$

$$\frac{1}{z-ba^*}(1-bb^*)\frac{1}{1-zab^*} = ?$$

$$\frac{1}{z-ba^*} + ab^*\frac{1}{1-zab^*} = \frac{1}{z-ba^*}\left(1-zab^* + (z-ba^*)ab^*\right)\frac{1}{1-zab^*}$$

Do residue calculation. $\frac{1}{1-zab^*}$ analytic for $|z|\le 1$.

so you evaluate. put $z = ba^*$ $\quad 1-ba^*ab^* = 1-bb^*$.

$$\int \left(y, \frac{1}{\cancel{1-}zba^*}y\right)\frac{dz}{2\pi i z}$$ analytic outside $|z|=1$ except at ∞

This yields the isom. embedding $Y \hookrightarrow L^2(S^1)$

There's too much calculation here.

Forgot $g(z) = \det(1-zab^*)$

2 other versions of the calculation. The ~~previous~~ above ~~case~~ can be understood better by ~~attaching~~ ~~extending~~ the canonical extension

$$H = \cdots \oplus u^{-1}V^- \oplus aX \oplus V^+ \oplus uV^+ \oplus \cdots$$
$$\oplus u^{-1}V^- \oplus V^+ \oplus bX \oplus uV^+ \oplus \cdots$$

by attaching incoming + outgoing subspaces.

Other ~~sketches~~ pictures: X, γ γ contraction

~~Then~~ Form $H =$ completion of $\bigoplus_{n\in\mathbb{Z}} u^n X$ $\gamma^* u^n \gamma = \gamma^n$ $n \geq 0$.

~~Note~~ $V^+ = (1-\gamma^*\gamma)^{1/2} X$

$V^- = (1-\gamma\gamma^*)^{1/2} X.$

$$\underbrace{\|x_0 + u x_1\|^2}_{ax_0+bx_1} = \|x_0\|^2 + (x_0, \gamma x_1) + (\gamma x_1, x_0) + \|x_1\|^2$$
$$= \|\underbrace{x_0 + \gamma x_1}_{a^*(x_0+ux_1)}\|^2 + \|(1-\gamma^*\gamma)^{1/2} x_1\|$$

$$\gamma = a^* b$$
$$1 - \gamma^*\gamma = 1 - b^* a a^* b$$
$$= b^* \pi_+ b$$

$$\int \left\| (1-\gamma\gamma^*)^{1/2} \frac{1}{1-z\gamma^*} x \right\|^2 \frac{dz}{2\pi i z} = \int \left(x, \frac{1}{1-\bar{z}^{-1}\gamma} (1-\gamma\gamma^*) \frac{1}{1-z\gamma^*} x \right) \frac{dz}{2\pi i z}$$
$$= \|x\|^2 \qquad \text{residues inside } S^1.$$

Opinion: It looks as if ~~your your problem~~ a proper explanation of these calculations will involve residue calculus. There should be a link between quasi-determinants and residues.

One parameter version. ~~If~~ You want ~~[illegible]~~ to replace the nearly unitary $\overset{\text{contraction}}{a^* b}$ with a nearly hermitian operator with ~~positive~~ ~~$\leq$~~ imaginary part, call it, β.

$$\int_{-\infty}^{\infty} \left\| (?)^{1/2} \frac{1}{\omega - \beta} x \right\|^2 \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(x, \frac{1}{\omega-\beta^*} (?) \frac{1}{\omega-\beta} x \right)$$

close counter in UHP want spectrum of β in LHP

so sing. at $\omega = \beta^*$ in UHP. ~~$? \beta^* \beta$~~ $? = i\underbrace{(\beta - \beta^*)}_{2i\,\mathrm{Im}(\beta)} \geq 0$

3 $X \overset{a}{\underset{b}{\rightrightarrows}} X$ $\quad -i\lambda = \frac{1-z}{1+z} \quad z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda+i}$ maybe λ should be ∞

$$a = i\varepsilon + A \qquad az - b = (i\varepsilon + A)z - (i\varepsilon - A)$$
$$b = i\varepsilon - A \qquad = (iz - i)\varepsilon + (z+1)A$$
$$= \left(i\frac{z-1}{z+1}\varepsilon + A\right)(z+1)$$
$$= (\lambda\varepsilon - A)(-z-1).$$

~~$a^*a = (-i\varepsilon^* + A^*)(i\varepsilon + A)$~~

Assume $X \overset{\varepsilon}{\underset{A}{\rightrightarrows}} Y$ given and Y has scalar product.

$$\|ax\|^2 = \|i\varepsilon x + Ax\|^2 = \|\varepsilon x\|^2 + (Ax, i\varepsilon x) + (i\varepsilon x, Ax) + \|Ax\|^2$$
$$\|bx\|^2 \qquad - \qquad - \qquad -$$

so $\|ax\| = \|bx\| \iff (Ax, \varepsilon x) = (\varepsilon x, Ax) \quad \forall x.$

$ax = 0 \implies ax = bx = 0 \implies \begin{matrix} a+b = 2i\varepsilon \\ a-b = 2A \end{matrix}$ kills x,

Assume that $\varepsilon x = Ax = 0 \implies x = 0$. Then you have a partial unitary.

Want no bound states. Since a, b and ε, A are expressable ~~...~~ in terms of each other its clear that $az - b$ injective $\forall z \in \mathbb{C} \cup \infty \iff \lambda\varepsilon - A$ injective $\forall \lambda \in \mathbb{C} \cup \infty$.

So what can you do? ~~...~~

So far no scalar prod on X, except that the partial unitary requires $\|x\|^2 = \|ax\|^2 = \|\varepsilon x\|^2 + \|Ax\|^2$ so this is the mistake!! How do you correct?

$$a = (i\varepsilon + A)h^{-1/2} \qquad h = \overset{1}{\varepsilon^*\varepsilon} + A^*A$$
$$b = (i\varepsilon - A)h^{-1/2}$$

4 Suppose we start with ~~X, Y Hilb. spaces~~

$$X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y \qquad X, Y \text{ Hilb. spaces}$$

$$\varepsilon^*\varepsilon = 1 \qquad (\varepsilon \text{ is an emb})$$

$$\varepsilon^* A = A^*\varepsilon$$

$$\lambda\varepsilon - A \quad \text{injective } \forall \lambda \in \mathbb{C}.$$

$$\|(i\varepsilon \pm A)x\|^2 = \|x\|^2 \pm \|Ax\|^2 = \|(1+A^*A)^{1/2}x\|^2$$

$$\|(i\varepsilon \pm A)(1+A^*A)^{-1/2}x\|^2 = \|x\|^2$$

$$a = (i\varepsilon + A)(1+A^*A)^{-1/2}$$

$$b = (i\varepsilon - A)(1+A^*A)^{-1/2}$$

$$ab^* = (i\varepsilon + A)\left((1+A^*A)^{-1/2}(1+A^*A)^{-1/2}\right)(-i\varepsilon^* - A^*)$$

$$b^*a = (1+A^*A)^{-1/2}\underbrace{(-i\varepsilon^* - A^*)(i\varepsilon + A)}_{\varepsilon^*\varepsilon - iA^*\varepsilon - i\varepsilon^*A - A^*A}(1+A^*A)^{-1/2}$$

$$= 1 - A^*A - 2i(\varepsilon^*A)$$

does $A^*\varepsilon = \varepsilon^*A$ commute with A^*A NO

$$A^*\varepsilon A^*A =$$

$$b^*a = h^{-1/2}(-i\varepsilon^* - A^*)(i\varepsilon + A)h^{-1/2}$$

$$1 = h^{-1/2}(-i\varepsilon^* + A^*)(i\varepsilon + A)h^{-1/2}$$

$$1 + b^*a = h^{-1/2}(-2i\varepsilon^*)(i\varepsilon + A)h^{-1/2}$$

$$= h^{-1/2}(+2 - 2i\,\varepsilon^*A)h^{-1/2}$$

$$1 - b^*a = h^{-1/2}(2A^*)(i\varepsilon + A)h^{-1/2}$$

5 $(1-b^*a)(1+b^*a)^{-1} = h^{-1/2}(iA^*\varepsilon + A^*A)(1-i\varepsilon^*A)^{-1}h^{+1/2}$?

Let's look at ~~egg eigenvalue~~ eigenvectors

There are two lines in Y naturally presented, namely $\operatorname{Ker}\varepsilon^*$, $\operatorname{Ker}A^*$. These are the orth complements in Y of the subspace $(\lambda\varepsilon - A)X$ for $\lambda = \infty, 0$. Look at

$$\begin{tikzcd}
 & & e_0 \in \operatorname{Ker}(\varepsilon^*) \arrow[d, hook] \arrow[dr] & & \\
0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, "\lambda - \varepsilon^*A"'] & Y \arrow[r] \arrow[d, "\varepsilon^*"] & L_\lambda \arrow[r] & 0 \\
 & & X & &
\end{tikzcd}$$

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e_0$$

$$(\lambda - \varepsilon^*A)x = -\varepsilon^*y$$

$$y \,\bar{\cancel{\oplus}}\, (\lambda\varepsilon - A)(\lambda - \varepsilon^*A)^{-1}(+\varepsilon^*)y = \tilde{y}(\lambda)e_0$$

$$= y - (\lambda\varepsilon - A)\varepsilon^*(\lambda - A\varepsilon^*)^{-1}y$$

$$= \{\lambda - \cancel{A}\varepsilon^* - (\lambda\varepsilon - \cancel{A})\varepsilon^*\}(\lambda - A\varepsilon^*)^{-1}y$$

$$= (1 - \varepsilon\varepsilon^*)(1 - \lambda^{-1}A\varepsilon^*)^{-1}y$$

$$\tilde{y}(\lambda) = (e_0, (1 - \lambda^{-1}A\varepsilon^*)^{-1}y)$$ this has poles on real axis.

You propose to modify ~~Ker ε~~ ε^* a bit

$b^* = h^{-1/2}(-i\varepsilon^* - A^*)$ $\qquad \varepsilon^* - iA^*$.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr] & Y \arrow[r] \arrow[d, "\varepsilon^* - iA^*"] & L_\lambda \arrow[r] & 0 \\ & & X & & \end{tikzcd}$$

$$(\varepsilon^* - iA^*)(\lambda\varepsilon - A) = \lambda - \varepsilon^* A - i\lambda A^*\varepsilon + iA^*A$$
$$= \lambda + iA^*A - (1+i\lambda)A^*\varepsilon$$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, "\lambda A^*\varepsilon - A^*A"'] & Y \arrow[d, "A^*"] \\ & & X \end{tikzcd}$$

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e_0'$$
$$(\lambda A^*\varepsilon - A^*A)x = -A^*y$$
$$x = -(\lambda A^*\varepsilon - A^*A)^{-1}A^*y$$

$$y + (\lambda\varepsilon - A)(\lambda A^*\varepsilon - A^*A)^{-1}A^*y = \tilde{y}(\lambda)e_0'$$

$$y + (\lambda\varepsilon - A)A^*(\lambda\varepsilon A^* - AA^*)^{-1}y \quad \text{NO good}$$

$$\left\{\lambda\varepsilon A^* - AA^* + (\lambda\varepsilon - A)A^*\right\}(\lambda\varepsilon A^* - AA^*)^{-1}y$$

$$(\lambda A^*\varepsilon - A^*A)^{-1}A^* \overset{?}{=} A^*(\lambda\varepsilon A^* - AA^*)^{-1}$$

$$\Uparrow \quad \text{if both invertible}$$

$$A^*(\lambda\varepsilon A^* - AA^*) = (\lambda A^*\varepsilon - A^*A)A^* \quad \text{OK}$$

$(\lambda\varepsilon - A)(\varepsilon^*(\lambda\varepsilon - A))^{-1}\varepsilon^*$ is OK

$(\lambda\varepsilon - A)(A^*(\lambda\varepsilon - A))^{-1}A^*$ is not OK.

7 Curious $1-xy$ invertible $\Leftrightarrow$ $1-yx$ invertible

$$\left(1+y(1-xy)^{-1}x\right)(1-yx) = 1 - yx + y\cancel{(1-xy)^{-1}}\underbrace{x(1-yx)}_{\cancel{(1-xy)}x}$$

$\varepsilon - xy$ invertible $\Leftrightarrow$ $\varepsilon - yx$ is inv.

need $[\varepsilon, X] = 0$

$$\left[\alpha + y(\varepsilon-xy)^{-1}x\right](\varepsilon - yx) = yx + \overset{1}{\alpha}(\varepsilon-yx) = \varepsilon$$

idea was $(\varepsilon - xy)^{-1} = \varepsilon^{-1}(1-xy\varepsilon^{-1})^{-1}$

$$= \varepsilon^{-1} + \varepsilon^{-1}xy\varepsilon^{-1} + \varepsilon^{-1}(xy\varepsilon^{-1})^2 + \cdots$$

$$\left[\varepsilon^{-1} + \varepsilon^{-1}y(\varepsilon - xy)^{-1}x\right](\varepsilon - yx) = 1 - \varepsilon^{-1}yx + \varepsilon^{-1}yx = 1.$$

$$\begin{array}{ccc} X & \xrightarrow{A} & Y \\ & \searrow^{AA^*} & \downarrow A^* \\ & & X \end{array}$$

$(\lambda\varepsilon - A)x = -y + c_0'$

$A^*Ax = A^*y$

$x = (A^*A)^{-1}A^*y$

you can't do
$(A^*A)^{-1}A^*y = A^*(AA^*)^{-1}y$

$y - A(A^*A)^{-1}A^*y$

$$\left(e_0', y + (\lambda\varepsilon - A)\left(A^*(\lambda\varepsilon - A)\right)^{-1}A^*y\right)$$

$$= \left(e_0', y + \lambda\varepsilon(\lambda A^*\varepsilon - A^*A)^{-1}A^*y\right)$$

8 Think. What might you be missing?

~~Your need~~:

$$0 \longrightarrow X \xrightarrow{\lambda\varepsilon - A} Y \longrightarrow L_\lambda \longrightarrow 0$$

Any $y \in Y$ gives a holom. section of $\{L_\lambda\}$. If $y \neq 0$ section vanishes n points, get equiv between $\mathbb{P}(Y)$ and divisors ≥ 0 of degree n. ~~Two divisors~~ Your goal is to represent Y as functions. One way to do this is to select a $\neq 0$ section e_0, then solve

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e_0$$

Why can you solve this? Look at homogeneous equation

$$(\lambda\varepsilon - A)x = ce_0$$

Linear alg. You have a K-module of type $\mathcal{O}(n)$ meaning $\dim Y = n+1$, $\dim X = n$, $a\lambda - b$ injective $\forall$ all λ includ ∞. Why does any line in Y arise from some λ and line in X.

$$\mathbb{P}X \times \mathbb{P}^1 \longrightarrow \mathbb{P}Y$$

So the simple idea seems to be to add this e_0 to get

$$\begin{matrix} X \\ \oplus \\ 1 \end{matrix} \xrightarrow{(\lambda\varepsilon - A \;\; e_0)} Y$$

Inverting this is equiv. to solving the equation

$$y = (A - \lambda\varepsilon)x + ce_0$$

9 Take a J-matrix version.

$$\begin{pmatrix} -\lambda+b_1 & a_1 & & & e_1 \\ a_1 & -\lambda+b_2 & a_2 & & e_2 \\ & a_2 & \ddots & a_{n-1} & \vdots \\ & & a_{n-1} & -\lambda+b_n & \vdots \\ & & & a_n & e_{n+1} \end{pmatrix}$$

Observe the determinant is a poly of degree n assuming $e_{n+1} \neq 0$.

Let's go over what has been learned. You have this K-module $\lambda\varepsilon - A : X \to Y$ of type $O(n)$ say.

$$\begin{tikzcd} & & \mathbb{C}e \arrow[d] \arrow[dr] & & \\ 0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[r] & L_\lambda \arrow[r] & 0 \end{tikzcd}$$

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e$$

$$y = (A - \lambda\varepsilon)x + \tilde{y}(\lambda)e.$$

There is something funny here because the ~~determinant has~~ characteristic poly has degree $\leq n$. It seems that I want $e_{n+1} \neq 0$. i.e. $e \neq 0$ at ∞.

Anyway, so we

$$(i\varepsilon - A)^*(\lambda\varepsilon - A) = (-i\varepsilon^* - A^*)(\lambda\varepsilon - A)$$
$$= -i\lambda + i(\varepsilon^* A) - \lambda(A^*\varepsilon) + A^*A$$
$$= (-i\lambda + A^*A) + (i-\lambda)\varepsilon^* A$$

10 my problem is to find a non-hermitian extension of A

$$A^*(i\varepsilon + A)(1 - i\varepsilon^* A)^{-1}$$

---

given $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ want $z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda + i}{\lambda + i}$

$a = i\varepsilon + A$ $\|ax\|^2 = \|\varepsilon x\|^2 + (i\varepsilon x, Ax) + (Ax, i\varepsilon x) + \|Ax\|^2$

$b = i\varepsilon - A$ $\|bx\|^2$ $-$ $-$

$\cancel{\varepsilon^* A = A^* \varepsilon}$.

so $\cancel{\left(\|ax\|^2 = \|\varepsilon x\|^2 + \|Ax\|^2\right) = \|(\varepsilon^*\varepsilon + A^*A)^{1/2} x\|^2}$

$$\therefore \quad (\varepsilon x, Ax)_Y = (Ax, \varepsilon x)_Y \qquad \forall x.$$

also $\|x\|^2 = \|\varepsilon x\|^2 + \|Ax\|^2$.

what is ε^* ?

$\cancel{(\varepsilon^* x, y)_Y = (x, \varepsilon y)}$

$$(\varepsilon^* y, x)_X = (y, \varepsilon x)_Y$$

$$\|$$

$$(\varepsilon(\varepsilon^* y), \varepsilon x) + ((A\varepsilon^*) y, Ax)$$

What is ε^*?

$$(\varepsilon^* y, x)_X = (y, \varepsilon x)_Y$$

$$\boxed{(a^* y, x) = (y, (i\varepsilon + A)x)_Y}$$

$$a^* a = (-i\varepsilon^* + A^*)(i\varepsilon + A) = \varepsilon^*\varepsilon + A^*A$$

11 You must understand perfectly what a partial hermitian operator is. Usual picture is a densely defined operator $D \longrightarrow H \oplus H$ satisfying an equal to its annihilator condition. How?

$$D = \left\{ \begin{pmatrix} \xi \\ A\xi \end{pmatrix} \;\middle|\; \xi \in \mathcal{D}_A \right\}$$

D is a closed subspace of $H \oplus H$ such that $pr_1 : D \longrightarrow H$ is injective and has ~~closed~~ dense image.

From C^* module theory ~~you~~ want assume existence of adjoint. it is not..

$$\mathcal{D}_T \Rightarrow \Gamma_T \subset H \oplus H \qquad \mathcal{D}$$

$$(\xi_1, T\xi_2) - (T^*\xi_1, \xi_2) = 0$$

$$\begin{pmatrix} \xi_1 \\ +T^*\xi_1 \end{pmatrix} \perp \begin{pmatrix} +T\xi_2 \\ -\xi_2 \end{pmatrix} \qquad \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \xi_2 \\ T\xi_2 \end{pmatrix}$$

general case. Take $H_1 \oplus H_2$ $\qquad \Gamma_{T^*} \perp \underbrace{\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}}_{J} \Gamma_T$

The good case is when $\Gamma_{T^*} \oplus \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \Gamma_T = \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix}$

i.e. $\begin{pmatrix} 1 & -T \\ T^* & 1 \end{pmatrix} \qquad \begin{matrix} D_{T^*} \\ \oplus \\ D_T \end{matrix} \longrightarrow \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix}$

shift to $\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = 1 + \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$

12 partial unitary = subspace of $H \oplus H$, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ isotropic for $\|\xi_1\|^2 - \|\xi_2\|^2$ hermitian form.

partial hermitian = subspace Γ of $H \oplus H$ isotropic for hermitian form $\xi \mapsto \operatorname{Im}(\xi_1, \xi_2)$ and such that $p_1 : \Gamma \to H$ is injective.

Check 2nd description. Polarize $\xi \overset{\operatorname{Im}}{\longmapsto} (\xi_1, \xi_2)$

$$\frac{1}{4}\sum_{k=0}^{4} i^{-k} Q(\xi + i^k \eta)$$

$$= \frac{1}{4}\sum_{k=0}^{4} i^{-k} \frac{(\xi_1 + i^k\eta_1, \xi_2 + i^k \eta_2) - (\xi_2 + i^k \eta_2, \xi_1 + i^k \eta_1)}{2i}$$

$$= \frac{(\xi_1, \eta_2) - (\xi_2, \eta_1)}{2i} = \frac{1}{2i}\left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right)$$

Suppose $\Gamma = \begin{pmatrix} 1 \\ T \end{pmatrix} W$ $\qquad W = p_1(\Gamma)$

isotropic means

$$0 = \left(\begin{pmatrix} \omega \\ T\omega \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \omega' \\ T\omega' \end{pmatrix} \right)$$

$$= (\omega, T\omega') - (T\omega, \omega')$$

$az - b$

$= a\left(\frac{1+i\lambda}{1-i\lambda}\right) - b$

$= a(1+i\lambda) - b(1-i\lambda) \ldots$

$= a - b + i(a+b)\lambda$

$i\,\frac{a-b}{a+b} - \lambda$

Now do what?

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ i & +i \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ i & +i \end{pmatrix} \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

Remaining step go between unitary + hermitian pictures via C.T.

unitary
$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] & H \\ H & \oplus \end{tikzcd}$$

$$Q(\xi) = \|\xi_1\|^2 - \|\xi_2\|^2$$
$$= \left(\xi, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xi\right)$$

$$az - b = a\left(\frac{1+i\lambda}{1-i\lambda}\right) - b = \frac{a(1+i\lambda) - b(1-i\lambda)}{1-i\lambda} = \frac{(a-b) + i\lambda(a+b)}{\cdot\cdot}$$
$$= \left(i\lambda + \frac{a-b}{a+b}\right)\left(\frac{1}{1-i\lambda}\right)(a+b)$$
$$= \left(\lambda - i\,\frac{a-b}{a+b}\right)\left(\frac{i}{1-i\lambda}\right)(a+b)$$

$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{tikzcd} X \arrow[r, "\begin{pmatrix} a \\ b \end{pmatrix}"] \arrow[d] & H \oplus H \arrow[r, "\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}"] & H \oplus H \\ {} \end{tikzcd}$$

Check.
$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} x = \begin{pmatrix} i(a-b)x \\ (a+b)x \end{pmatrix}$$

$$\cancel{\|\xi_1\|^2 - \|\xi_2\|^2 = \|(a-b)x\|^2 - \|(a+b)x\|^2}$$
$$\cancel{= \|ax\|^2 - (ax, bx) - (bx, ax) + \|bx\|^2}$$
$$\cancel{- \|ax\|^2 + (\quad) + (\quad) - \|bx\|^2}$$
$$\cancel{= -2}$$

$$\mathrm{Im}(\xi_1, \xi_2) = \mathrm{Im}\big(i(a-b)x, (a+b)x\big)$$
$$= \pm\mathrm{Re}\big((a-b)x, (a+b)x\big)$$
$$= \|ax\|^2 - \|bx\|^2$$

$$\varepsilon = a+b$$
$$A = i(a-b)$$

14 The good setting seems to be an isotropic subspace X of $Y \oplus Y$ for the hermitian form $\|\xi_1\|^2 - \|\xi_2\|^2$, and then you want to intersect this with graph of mult by z. In other words you have the basic sequence

$$0 \to \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}(1) \longrightarrow 0 \quad \text{over } \mathbb{P}^1$$

tensored with Y. You combine a hermitian scalar product on Y with the symplectic? structure on $H^0(\mathcal{O}(1))$. Not symplectic but rather $U(1,1)$ structure to get the pseudo scalar product on $Y \otimes H^0(\mathcal{O}(1))$. Then what? X isotropic, find induced v.b. on X°/X. LC circuits are similar, maybe identically except it's a real version.

~~Partial unitary on H = subspace of $H \oplus H$ isotropic for hermitian form $\|\xi_1\|^2 - \|\xi_2\|^2$~~

~~Partial hermitian op on H = subspace of $H \oplus H$ projecting nonsingularly on first factor, i.e. pr_1 injective which isotropic for skew hermitian form $\left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right)$~~

~~$= (\xi_1, \eta_2) - (\xi_2, \eta_1)$. This should be determined by purely imag. function $(\xi_1, \xi_2) - (\xi_2, \xi_1) = 2i\,\mathrm{Im}(\xi_1, \xi_2)$~~

~~$\Gamma \subset H \oplus H$ isotropic subspace such that $\forall (\xi_1, \xi_2) \in \Gamma$ $(\xi_1, \xi_2) \in \mathbb{R}$.~~

~~Thus $\Gamma = \Gamma_A = \left\{ \begin{pmatrix} \xi \\ A\xi \end{pmatrix} \middle| \xi \in \mathcal{D}_A \right\}$. $(\xi, A\xi)$ real~~

15 Review. You have achieved some understanding of partial hermitian operators. A hermitian operator A on H can be identified with a subspace Γ of $H \oplus H$ which is maximal isotropic for the hermitian form $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \longmapsto \mathrm{Im}(\xi_1, \xi_2)$. Corresp herm. bilinear form is found by polarization

proj $p_1 : \Gamma \xrightarrow{\sim} H$

$$\frac{1}{4}\sum_{k=0}^{3} i^{-k} Q(\xi + i^k \eta) = \frac{1}{4}\sum_{k=0}^{3} i^{-k} \frac{(\xi_1 + i^k\eta_1, \xi_2 + i^k\eta_2) - (\xi_2 + i^k\eta_2, \xi_1 + i^k\eta_1)}{2i}$$

$$= \frac{(\xi_1, \eta_2) - (\xi_2, \eta_1)}{2i} = \frac{1}{2i}\left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right)$$

$\frac{1}{i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is hermitian.

Isotropic means $\left(\begin{pmatrix} \xi \\ A\xi \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ A\eta \end{pmatrix} \right)$

$= (\xi, A\eta) - (A\xi, \eta)$ vanishes,

A partial hermitian is subspace $X \subset \begin{matrix} H \\ \oplus \\ H \end{matrix}$ isotropic for same herm. form and such that $p_1 : X \hookrightarrow H$

So the study of partial hermitian operators should reduce to partial unitaries

Now time to sort out previous problems, where you could handle partial unitaries but not partial hermitians. C.T. $z = \frac{1-(-i\lambda)}{1+(-i\lambda)} = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda-i}$

$$i(1-i\lambda)(az-b) \equiv i\big(a(1+i\lambda) - b(1-i\lambda)\big) = i\big(a-b + i\lambda(a+b)\big)$$

$$= -\left(\lambda - i\,\frac{a-b}{a+b}\right)(a+b).$$

$\varepsilon = a+b$

$A = i(a-b)$

$$X \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} \begin{matrix} H \\ \oplus \\ H \end{matrix} \xrightarrow{\frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}} \begin{matrix} H \\ \oplus \\ H \end{matrix}$$

$$\left(\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right) \frac{1}{2}$$

$$\frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

Notice that $\begin{pmatrix} a \\ b \end{pmatrix} : X \to \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ also $\begin{pmatrix} \varepsilon \\ A \end{pmatrix}$ do not use scalar prod on X. So you should equip X with the scala product $\|x\|^2 = \|(a+b)x\|^2$ to arrange that $\varepsilon^*\varepsilon = 1$.

Try to discuss the general theory of the eigenvector equation etc.

$$X \subset X^0 \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \qquad X^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \left(\begin{pmatrix} ax \\ bx \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = 0 \right\}$$

$$X^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| a^* y_1 = b^* y_2 \right\}. \qquad (ax, y_1) = (bx, y_2)$$

$$a^* y_1 = b^* y_2 .$$

e.g. $\begin{pmatrix} V^+ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ V^- \end{pmatrix},$ eigenvector equation

$$W^0 = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix} X}_{W} \oplus \begin{pmatrix} V^+ \\ V^- \end{pmatrix}$$

$(z\ \ -1)$ has kernel $\left\{ \begin{pmatrix} \xi \\ z\xi \end{pmatrix} \right\} = L_z$

$$X \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \xrightarrow{(z\ \ -1)} Y$$

$\dim Y \oplus Y = 2n+2$
$\dim W = n$
$\dim W^0 = n+2$

So you assume $\forall z$ $\quad az - b$ inj, this means no bound states. $W^0 \cap L_z$ should be a line in $W^0/W = V^+ \oplus V^-$. The line is clear namely the correspondence between V^+, V^- given by the eigenvalue equation

$$(az - b)x = -v^+ + v^-$$

So simple.

17 What happens in the electrical setting. Somehow $U(n,n)$ becomes $Sp(2n,\mathbb{R})$.

Go back to partial hermitian setting and find what you did wrongly.

$$X \xrightarrow{\lambda\varepsilon - A} Y$$

Wait, better idea. Is it possible to derive the formula $\tilde{y}(z) = (e_0^-, (1-za^*b)^{-1}y)$ without the tricks used before? This is the solution of $(az-b)x = -y + \tilde{y}(z)e_0^-$. You want maybe to ~~look~~ work in the ~~category~~ double $Y \oplus Y$

In $Y\oplus Y$ you have $\Gamma_z = \begin{pmatrix} 1 \\ z \end{pmatrix} Y$, $W = \begin{pmatrix} a \\ b \end{pmatrix} X$

$W + \Gamma_z$ codim 1 in $Y \oplus Y$

$$\underset{n}{\begin{pmatrix} ax \\ bx \end{pmatrix}} + \underset{n+1}{\begin{pmatrix} y \\ zy \end{pmatrix}} + \underset{1}{\begin{pmatrix} 0 \\ ce_0^- \end{pmatrix}} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

$$\begin{pmatrix} z & -1 \end{pmatrix} \cdot \qquad (az-b)x - ce_0^- = y$$

ax

bx No.

Consider partial herm. case

$$\underbrace{\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X}_{W}, \quad \underbrace{\begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y}_{\Gamma_\lambda}, \qquad W^\circ$$

you want elements e of Y whose sections are nonvanishing in the UHP

$$e \in (\varepsilon\lambda - A)x \Longrightarrow \lambda \text{ lower half plane?}$$

18 partial hermitian ops. $W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

isotropic ~~for~~: $\left(\begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon x' \\ Ax' \end{pmatrix} \right) = (\varepsilon x, Ax') - (Ax, \varepsilon x') = 0$

Note this doesn't depend on any inner product on X.

What is: W° ? $= \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \ (y_1, Ax') - (y_2, \varepsilon x') = 0 \quad \forall x' \right.$

For example. $\begin{pmatrix} 0 \\ y_2 \end{pmatrix}$ with $(y_2, \varepsilon X) = 0$. $\begin{pmatrix} (AX)^\perp \\ (\varepsilon X)^\perp \end{pmatrix}$

Is it true that $\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \oplus \begin{matrix} (AX)^\perp \\ \oplus \\ (\varepsilon X)^\perp \end{matrix} = W^\circ$? NO.

Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W^\circ$ i.e. $(y_1, Ax) = (y_2, \varepsilon x) \quad \forall x$.

We have $Y = \varepsilon X + (\varepsilon X)^\perp$, so $y_1 = \varepsilon x + y_1'$ $\quad y_1' \in \varepsilon X^\perp$

~~So reasons~~

$$\begin{pmatrix} \varepsilon \\ A \end{pmatrix} x = \begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} \in \begin{matrix} (AX)^\perp \\ \oplus \\ (\varepsilon X)^\perp \end{matrix} \qquad \begin{matrix} A^* \varepsilon x = 0 \\ \varepsilon^* A x = 0. \\ \text{possible} \end{matrix}$$

so it is possible for $Ax \in (\varepsilon X)^\perp$. Anyway continue.

You want something say involving $\lambda = \pm i$

$\varepsilon = a + b \qquad i\varepsilon - A = 2ib$

$A = i(a - b) \qquad i\varepsilon + A = 2ia$

$$(i\varepsilon - A)^* (\lambda \varepsilon - A) = (-i\varepsilon^* - A^*)(\lambda \varepsilon - A)$$

$$= -i\lambda \varepsilon^* \varepsilon + \underbrace{i\varepsilon^* A - A^* \lambda \varepsilon}_{(i - \lambda) \varepsilon^* A} + \underbrace{A^* A}_{1 - \varepsilon^* \varepsilon}$$

$$= 1 + (-1 - i\lambda) \varepsilon^* \varepsilon + (i - \lambda) \varepsilon^* A$$

$$= 1 + (\lambda - i)(-i\varepsilon^* \varepsilon - \varepsilon^* A)$$

19 Your problem again. You know that $(az-b)(x)=0 \implies x=0$ for $|z|\neq 1$.

because $\quad z\,a x = b x \implies \underset{|z|\,\|x\|}{\|zax\|} = \underset{\|x\|}{\|bx\|}$

Try $\quad (\lambda\varepsilon - A)x = 0 \qquad \lambda\varepsilon x = Ax$

$$\underset{(\lambda\varepsilon x,\ \varepsilon x)}{(Ax,\ \varepsilon x)} = \underset{(\varepsilon x,\ \lambda\varepsilon x)}{(\varepsilon x,\ Ax)} \implies (\lambda-\bar\lambda)\|\varepsilon x\|^2 = 0.$$

$$(i\varepsilon - A)^*(\lambda\varepsilon - A) = 1 + (\lambda - i)(-i\varepsilon^*\varepsilon - \varepsilon^* A)$$

provided $\varepsilon^*\varepsilon + A^*A = 1$

$$(\lambda\varepsilon - A)x = -y + \tilde{g}(\lambda)\, e_0^- \qquad (i\varepsilon - A)^* e_0^- = 0$$

$$y + (\lambda\varepsilon - A)\Big[1 + (\lambda - i)(-i\varepsilon^*\varepsilon - \overbrace{\varepsilon^* A}^{A^*\varepsilon})\Big]^{-1}(i\varepsilon - A)^*(-y)$$

$$y + (\lambda\varepsilon - A)\Big[1 + (\lambda - i)(\underbrace{-i\varepsilon^* - A^*}_{(i\varepsilon - A)^*})\varepsilon\Big]^{-1}(i\varepsilon - A)^*(-y)$$

$$y - (\lambda\varepsilon - A)(i\varepsilon - A)^*\Big[1 + (\lambda - i)\,\varepsilon(i\varepsilon - A)^*\Big]^{-1} y$$

$$\Big[1 + (\lambda - i)\varepsilon(i\varepsilon - A)^* - (\lambda\varepsilon - A)(i\varepsilon - A)^*\Big]\Big[1 + (\lambda - i)\varepsilon(i\varepsilon - A)^*\Big]^{-1} y$$

$$\underbrace{(\cancel{\lambda\varepsilon} - i\varepsilon - \cancel{\lambda\varepsilon} + A)(i\varepsilon - A)^*}_{(-i\varepsilon + A)(-i\varepsilon^* - A^*)}$$

proj onto V^-

$$\boxed{\tilde{y}(\lambda) = \Big(e_0^-,\ \Big[1 + (\lambda - i)\,\varepsilon(i\varepsilon - A)^*\Big]^{-1} y\Big)}$$

$\boxed{\text{OKAY}}$

20 Can you find an interpretation of trick using the double? Basic trick.

$$
\begin{tikzcd}
X \arrow[r, "az-b"] \arrow[dr, "1-zb^*a"'] & Y \arrow[d, "-b^*"] \\
 & X
\end{tikzcd}
$$

$$(az-b)x = -y + \tilde{y}(z)e_0^-$$

$$(1-zb^*a)x = b^*y$$

$$x = (1-zb^*a)^{-1}b^*y = b^*(1-zab^*)^{-1}y$$

$$\tilde{y}(z)e_0^- = y + (az-b)b^*(1-zab^*)^{-1}y$$

$$= (1-\cancel{zab^*} + \cancel{azb^*} - bb^*)(1-zab^*)^{-1}y$$

$$\boxed{\tilde{y}(z) = (e_0^-, (1-zab^*)^{-1}y)}$$

here you have used $(1-zb^*a)^{-1} \exists \iff (1-zab^*)^{-1} \exists$.

Goes back to $X \underset{q}{\overset{p}{\rightleftarrows}} Y$

$$
\begin{tikzcd}
 & \cancel{X} \arrow[d, "\binom{q}{1}"] \arrow[dr, "1-pq"] & \\
X \arrow[r, "\binom{1}{p}"] \arrow[dr, "1-qp"'] & X \oplus Y \arrow[r, "(-p\ 1)"] \arrow[d, "(1\ -q)"] & Y \\
 & X &
\end{tikzcd}
$$

2×2

$$(1-pq)^{-1} = 1 + p(1-qp)^{-1}q$$

21 Review. Consider $Y \oplus Y$ with hermitian form $\xi \mapsto \|\xi_1\|^2 - \|\xi_2\|^2$. Then an isotropic subspace W has the form $\begin{pmatrix} a \\ b \end{pmatrix} X$ where $\|ax\| = \|bx\|$ for all x. The annihilator is
make this $\|x\|$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, (ax, y_1) = (bx, y_2) \quad \forall x \right\}$$

$$(x, a^* y_1 - b^* y_2) = 0$$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, a^* y_1 = b^* y_2 \right\}.$$

Suppose $a^* y_1 = b^* y_2$. ~~Then $a^* a a^* y_1$~~ $\quad \in \begin{pmatrix} a a^* y_1 \\ b b^* y_2 \end{pmatrix}$

write
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (1 - aa^*) y_1 + aa^* y_1 \\ (1 - bb^*) y_2 + bb^* y_2 \end{pmatrix} = \begin{pmatrix} (1 - aa^*) y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (1 - bb^*) y_2 \end{pmatrix}$$

So $W^0 = W \oplus \begin{pmatrix} V^+ \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ V^- \end{pmatrix}$. $\quad \Gamma_z = \begin{pmatrix} 1 \\ z \end{pmatrix} Y$

Γ_z isotropic for $|z| = 1$. What can we do?

$Y \oplus Y$ contains $W, \Gamma_z, \begin{pmatrix} V^+ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ V^- \end{pmatrix}$

Generically
$$Y \oplus Y = W + \begin{pmatrix} V^+ \\ 0 \end{pmatrix} + \Gamma_z$$
$$= W + \begin{pmatrix} 0 \\ V^- \end{pmatrix} + \Gamma_z$$

Fundamental problem. Consider p.herm. ~~situation~~

$$0 \longrightarrow X \xrightarrow{\lambda\varepsilon - A} Y \longrightarrow E_\lambda \longrightarrow 0$$

Find an element $e_0 \in Y$ which trivializes E over the UHP including the real axis. zeroes of section e_0 are in LHP should be eigenvalues of some **extension** of A. This is tricky because there are $n = \dim X$ zeroes of e_0. Yes!

 Problem: Consider p. herm. situation

$$0 \longrightarrow X \xrightarrow{\lambda\varepsilon - A} Y \longrightarrow E_\lambda \longrightarrow 0$$

To find $e_0 \in Y$ such that the corresponding section of E_0 has its zeroes in the LHP. These zeroes should be the eigenvalues of some ~~point of~~ compression of A to X. Example. Adjust the scalar product on X so that $\varepsilon^*\varepsilon = 1$. ~~Then~~

Better: Let $\varepsilon^* : Y \to X$ ~~be~~ $\ni$ $\varepsilon^*\varepsilon = 1$.

~~Then the section~~ and e_0 generates $\mathrm{Ker}(\varepsilon^*)$.

Then section $\neq 0$ when $\lambda - \varepsilon^* A$ nonsing.

usual calculation will work. Check this $\eta\varepsilon = 1$.

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e_0 \qquad \eta(e_0) = 0$$

$$(\lambda - \eta A)x = -\eta y$$

$$\lambda \neq 0$$

$$x = -(\lambda - \eta A)^{-1}\eta y = -\eta(\lambda - A\eta)^{-1}y$$

$$y - (\lambda\varepsilon - A)\eta(\lambda - A\eta)^{-1}y = [\lambda - \cancel{A\eta} - (\lambda\varepsilon - \cancel{A})\eta](\lambda - A\eta)^{-1}y$$

$$= \lambda(1 - \varepsilon\eta)(\lambda - A\eta)^{-1}y$$

$$\tilde{y}(\lambda) = (e_0, (1 - \lambda^{-1}A\eta)^{-1}y)$$

Observe that ηA and $A\eta$ have the same spectrum $\neq 0$. Now explain ~~this~~ your trick calculation in quasi-det terms.

Use ε' instead of η. Idea you start with

$$X \xrightarrow{\lambda\varepsilon - A} Y \ni e$$

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "e"] & \\ X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, "\lambda p\varepsilon - pA"'] & Y \arrow[r] \arrow[d, "p"] & E_\lambda \\ & Y/\mathbb{C}e & \end{tikzcd}$$

You want $p\varepsilon : X \to Y/\mathbb{C}e$ to be an isom., then define

$\varepsilon' = (p\varepsilon)^{-1} p : Y \to X$

So it seems that we have one splitting

$Y = \varepsilon X \oplus \mathbb{C}e$

and have arranged ~~another splitting~~

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "e"] & \\ X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, "\lambda - \varepsilon' A"'] & Y \arrow[r] \arrow[d, "\varepsilon'"] & E_\lambda \\ & X & \end{tikzcd}$$

How do I relate $\lambda - \varepsilon' A$ to $\lambda - A\varepsilon'$?

You form $X \oplus Y$

It seems maybe worthwhile using $Y \oplus Y$

where one factor is $\mathbb{C}e \oplus \varepsilon X$.

$$\begin{tikzcd} & Y \arrow[d] & \\ X \arrow[r, "\begin{pmatrix} 1 \\ \lambda\varepsilon - A \end{pmatrix}"] & X \oplus Y \arrow[r] \arrow[d] & Y \\ & X & \end{tikzcd}$$

$$(1 - A\varepsilon')^{-1} = 1 + A(1 - \varepsilon' A)^{-1}\varepsilon'$$

$$\begin{tikzcd}
 & Y \arrow[d, "\begin{pmatrix}\varepsilon' \\ 1\end{pmatrix}"] & \\
X \arrow[r, "\begin{pmatrix}1 \\ A\end{pmatrix}"] & X \oplus Y \arrow[r, "(-A \;\; 1)"] \arrow[d, "(1 \;\; -\varepsilon')"] & Y \\
 & X &
\end{tikzcd}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc}$$

$$\begin{tikzcd}
X \arrow[r, "\begin{pmatrix}1 \\ A\end{pmatrix}"] & X \oplus Y \arrow[r, "(-A \;\; 1)"] \arrow[d, equal] & Y \\
X & X \oplus Y \arrow[l, "(1 \;\; -\varepsilon')"'] & Y \arrow[l, "\begin{pmatrix}\varepsilon' \\ 1\end{pmatrix}"']
\end{tikzcd}$$

Main statement is that any matrix coefficient of the inverse matrix is the inverse of the quasi-determinant

$$\boxed{\langle 1 | \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} | 1 \rangle = \frac{d}{ad-bc} = (a - bd^{-1}c)^{-1}}$$

Is there a way to fit

$$(1-pq)^{-1} = \underset{a}{1} + \underset{b}{p}\,\underset{d}{(1-qp)^{-1}}\,\underset{-c}{q} \qquad \begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix}$$

into this ~~state~~ picture

$$\begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -q \end{pmatrix} (1 \;\; p)$$

Is $\begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix}$ invertible? det $= 1$ in comm. case

$$\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -p \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} \underline{1-pq} & -p \\ q & 1 \end{pmatrix}$$

$$\boxed{(A^{-1})_{ij} = (ij \text{ quasi-det})^{-1}}$$

Your problem.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "e"] & \\ X \arrow[r, "\varepsilon - \lambda^{-1}A"] \arrow[dr] & Y \arrow[r] \arrow[d, "\varepsilon'"] & E_\lambda \\ & X & \end{tikzcd}$$

You need to get $(e, \underline{(1 - \lambda^{-1}A\varepsilon')^{-1}}y)$

this requires

$$X \underset{\varepsilon'}{\overset{\lambda^{-1}A}{\rightleftarrows}} Y$$

Ingredients — Abstract calculation

$$\begin{tikzcd} & V \arrow[d, shift left, "i"] \\ X \arrow[r, "b-a"] & Y \arrow[u, shift left, "\iota^*"] \arrow[d, shift left, "b^*"] \\ & X \arrow[u, shift left, "b"] \end{tikzcd}$$

To solve

$$(b-a)x = -y + iv$$

$$(1-b^*a)x = -b^*y$$

$$x = -(1-b^*a)^{-1}b^*y = -b^*(1-ab^*)^{-1}y$$

$$y+(b-a)x = y-(b-a)b^*(1-ab^*)^{-1}y$$

$$= (1-ab^* - (b-a)b^*)(1-ab^*)^{-1}y$$

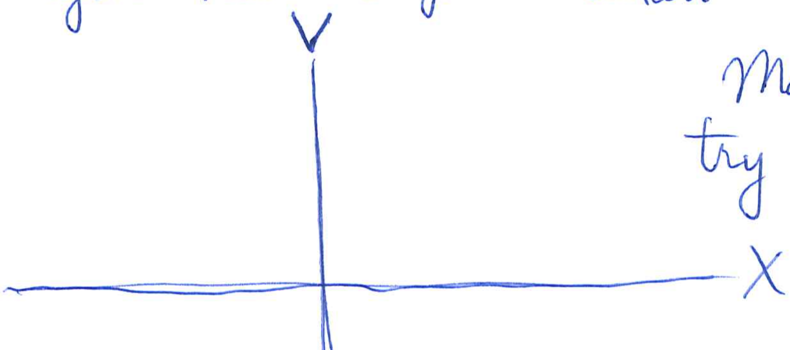
$$iv = \underbrace{(1-bb^*)}_{\iota\iota^* v}(1-ab^*)^{-1}y$$

$$\boxed{v = \iota^*(1-ab^*)^{-1}y}$$

This is a perturbation calculation which should be fairly general. You begin with the splitting

$$Y = bX \oplus \iota V \qquad \begin{tikzcd} X \oplus V \arrow[r, shift left, "(b\ \iota)"] & Y \arrow[l, shift left, "\begin{pmatrix} b^* \\ \iota^* \end{pmatrix}"] \end{tikzcd}$$

and you have a perturbation $b-a$ of b.



Maybe you should try

$$\begin{tikzcd} \begin{pmatrix} X \\ V \end{pmatrix} \arrow[r, shift left, "(b-a\ \ i)"] & Y \arrow[l, shift left, "\begin{pmatrix} b^* \\ \iota^* \end{pmatrix}"] \end{tikzcd}$$

$$(b-a\ \ i)\begin{pmatrix} b^* \\ \iota^* \end{pmatrix} = 1 - ab^*$$

27 You start with the isomorphism and inverse

$$Y \xrightarrow{\begin{pmatrix} b^* \\ i^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V \end{matrix} \xrightarrow{(b \quad i)} Y$$

and you have the pert. of b:

$$Y \xrightarrow{\begin{pmatrix} b^* \\ i^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V \end{matrix} \xrightarrow{(b-a \quad i)} Y \qquad (b-a \quad i)\begin{pmatrix} b^* \\ i^* \end{pmatrix} = 1 - ab^*$$

so you get $(b-a \quad i)^{-1} = \begin{pmatrix} b^* \\ i^* \end{pmatrix}(1-ab^*)^{-1}$

---

back to p. herm. $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ split $Y \underset{(\varepsilon \ \ j)}{\overset{\begin{pmatrix} \varepsilon^* \\ j^* \end{pmatrix}}{\rightleftarrows}} \begin{matrix} X \\ \oplus \\ V \end{matrix}$

$$Y \xrightarrow{\begin{pmatrix} \varepsilon^* \\ j^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V \end{matrix} \xrightarrow{(\lambda\varepsilon - A \quad j)} Y \qquad (\lambda\varepsilon - A \quad j)\begin{pmatrix} \varepsilon^* \\ j^* \end{pmatrix} = \lambda - A\varepsilon^*$$

we need to choose $\varepsilon^* : Y \longrightarrow X$ so that $\varepsilon^*\varepsilon = 1$

such a choice can be altered by an element of X

These ε^* form an affine space of dim n. So what is de Branges choice?

$$\begin{pmatrix} b_1 & a_1 & \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 \\ & & a_3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \end{pmatrix}}_{\text{choice for } \varepsilon^*} \begin{pmatrix} b_1 & a_1 & \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 \\ 0 & 0 & a_3 \end{pmatrix}$$

$$A\varepsilon^* = \begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 \\ & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \qquad \begin{pmatrix} b_1c_1 + a_1c_2 \\ a_1c_1 + b_2c_2 + a_2c_3 \\ a_2c_2 + b_3c_3 \\ a_3c_3 \end{pmatrix}$$

$$\begin{pmatrix} b_1 & a_1 & b_1c_1 + a_1c_2 \\ a_1 & b_2 & a_1c_1 + b_2c_2 \\ & a_2 & a_2c_2 \end{pmatrix} = A\varepsilon^*$$

You should look at $\varepsilon^* A$

$$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{pmatrix}\begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \\ & a_2 \end{pmatrix} = \begin{pmatrix} b_1 & a_1 + c_1a_2 \\ a_1 & b_2 + c_2a_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & c_1 \\ & 1 & & c_2 \\ & & 1 & c_3 \end{pmatrix}\begin{pmatrix} b_1 & a_1 & \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 \\ & & a_3 \end{pmatrix} = \begin{pmatrix} b_1 & a_1 & c_1a_3 \\ a_1 & b_2 & a_2 + c_2a_3 \\ 0 & a_2 & b_3 + c_3a_3 \end{pmatrix}$$

It looks as if you want $c_1 = c_2 = 0$ $c_3 = i$

Then $\varepsilon^* A$.

Now work out the details: Start with $\varepsilon, A : X \to Y$

[redacted] you want to find $\varepsilon' : Y \to X$ $\varepsilon'\varepsilon = 1$.

I want a natural choice of ε'.

If we choose scalar prod on X so that $\varepsilon^*\varepsilon = 1$, i.e. ε is an isometry. Then we can

$1 - \varepsilon\varepsilon^* = \iota\iota^*$ $\varepsilon' = \varepsilon^* +$

29 It looks like I'm not getting ~~a calculation~~ something straightforward. What do you want? I need an ε' such that $\varepsilon'\varepsilon = 1$, equivalently a ~~line~~ complementary to εX, εX corresp. to $\lambda = \infty$.

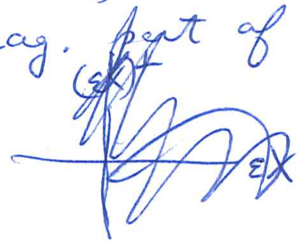
What do you want? You ~~give~~ need a ~~line~~ $\neq 0$ elt e in Y which provide~~s~~ ~~the~~ a section of the line bundle. You want $e \notin \varepsilon X$ so that $\mathbb{C}e \oplus \varepsilon X = Y$ whence $1 = ee' + \varepsilon\varepsilon'$ $\quad \varepsilon'\varepsilon = 1_X$. So you want a line whose assoc. section ~~is non~~ vanishes only in the LHP. Then get $\varepsilon'\varepsilon = 1$.

so $$Y \xrightarrow{\begin{pmatrix}\varepsilon' \\ e'\end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{(\varepsilon - \lambda^{-1}A \quad e)} Y \qquad \begin{aligned} &(\varepsilon - \lambda^{-1}A)\varepsilon' + ee' \\ &= 1 - \lambda^{-1}A\varepsilon' \end{aligned}$$

The choice is the element e, because εX is fixed. So you ask for a natural ~~element~~ line outside εX. The orthogonal complement, but this yields $\varepsilon^* A$ on X which is hermitian, ~~we~~ we want something with a negative imag. part of rank 1.

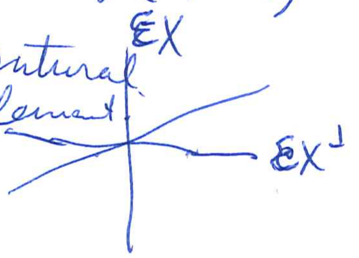
$\varepsilon' = \varepsilon^* + f \qquad f\varepsilon = 0$

$f : Y/\varepsilon X \to X$



$$\varepsilon' A = \varepsilon^* A + fA$$

So $f(Y/\varepsilon X)$ is a line in X. Is there ~~a~~ a natural element



So you have to find a line in X.

 So things look very interesting indeed. Let's review carefully. You have a partial hermitian operator $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ of $\mathcal{O}(n)$ type. Y is a Hilbert space of dim $n+1$, X has dim n, $\lambda\varepsilon - A$ is injective $\forall\lambda$ including ∞. Partial herm. means $\begin{pmatrix}\varepsilon\\A\end{pmatrix}X \subset Y \oplus Y$ is isotropic wrt $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

i.e. $$\left(\begin{pmatrix}\varepsilon\\A\end{pmatrix}(x'), \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\varepsilon\\A\end{pmatrix}(x) \right) = 0 \quad \forall x, x'.$$

$$(\varepsilon x', Ax) = (Ax', \varepsilon x)$$

~~So~~ ~~[illegible]~~ You have line bundle $E_\lambda = Y/(\lambda\varepsilon - A)X$ over $\mathbb{P}^1$, you ~~[illegible]~~ wish to find a natural section ~~vanishing~~ only in the LHP. Such a section amounts to an element of $e \in Y$ $e \notin \varepsilon X$.

not vanishing at ∞

Get splitting $$Y \xrightarrow{\begin{pmatrix}\varepsilon'\\e'\end{pmatrix}} \begin{matrix}X\\ \oplus\\ \mathbb{C}\end{matrix} \xrightarrow{(\varepsilon \ \ e)} Y \qquad 1 = \varepsilon\varepsilon' + ee'$$ etc.

$$\begin{pmatrix}\varepsilon'\\e'\end{pmatrix}(\varepsilon, e) = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Then to solve $$(\lambda\varepsilon - A)x + ce = y$$

Apply ε' $$(\lambda - \varepsilon'A)x = \varepsilon' y$$ can be always ~~[illegible]~~ solved when $\det(\lambda - \varepsilon'A) \neq 0$.

So the ~~issue~~ issue becomes to find a natural choice of e or ε'. ~~Have~~ Have the orthogonal complement of εX. $\mathrm{Ker}(\varepsilon^*)$ once ~~[illegible]~~ scalar product on X chosen so that ε is an isometry - If you take $\varepsilon' = \varepsilon^*$ get $\det(\lambda - \varepsilon^* A) = 0$ which has real roots as $\varepsilon^* A$ is hermitian.

 ~~What to do???~~

What to do? Possibilites { Look at $\left(\binom{\varepsilon}{A}X\right)^0$ in $\begin{smallmatrix}Y\\ \oplus\\ Y\end{smallmatrix}$
Use vanishing at ∞ filtration of Y }

Try 2nd. The idea is to seek ε' in the form $\varepsilon^* + f$ where $f: Y/\varepsilon X \longrightarrow X$. This means you need to find a line in X if $f \neq 0$. So far I have singled out $\lambda = \infty$. From K-module theory you do get natural complementary : ~~filtrations~~ flags by looking order of vanishing at 0 and ∞. $\lambda = i$ might play a special role.

I know from K-module theory ~~then~~ that we can identify Y with $\mathbb{C} + \mathbb{C}\lambda + \cdots + \mathbb{C}\lambda^n$, X degree $< n$ $\varepsilon =$ inc, $A \doteq \lambda$, in an essentially unique way (non-zero scalar). The $\mathbb{C}$ is sections vanishing to order n at $\lambda = \infty$. This gives degree filtration on polys. Notice that translation doesn't change this. YES

Have J matrix picture of the $\boxed{\text{partial (herm.) op.}}$

Let's go on to W^0/W. W is an isotropic subspace of $Y \oplus Y$, ~~so you need to~~ To extend to an isot subspace should be the same as giving a ~~self adj~~ hermitian extension of the partial herm. op. But there is a complex projective line of n dim subspaces containing W and contained in W^0. Is there a natural point? One

~~That~~ You discussed before extending the J matrix

There's should be some condition. You discussed possible $\varepsilon' A$, mainly possible ε' (n dims)

The Cayley transform should take W^0/W into $\begin{smallmatrix}V^+\\ \oplus\\ V^-\end{smallmatrix}$

Hopefully there are extensions of the partial ~~hermitian~~ herm. to a nearly hermitian op $A\varepsilon'$ on Y.

32 Oct 2, 98 Review. p. herm. $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$

$$W = \begin{pmatrix}\varepsilon\\A\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix} \qquad W^0 = \left\{ \begin{pmatrix}y_1\\y_2\end{pmatrix} \;\middle|\; \left(\begin{pmatrix}y_1\\y_2\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\varepsilon x\\Ax\end{pmatrix}\right) = 0 \right.$$

$$\left. \iff (y_1, Ax) = (y_2, \varepsilon x) \right\} \quad \forall x$$

~~Obtain This is projective line~~ W^0/W is 2-diml with hermitian form type $1, -1$. Let $W \subset V \subset W^0$ If $p_1 : V \xrightarrow{\sim} Y$ then V is the graph of an extension of [redacted] (ε, A) to Y. $p_1 W = \varepsilon X$ so use gen.^{e} for $(\varepsilon X)^\perp$. Take $y_1 = e \in (\varepsilon X)^\perp$ $(e, Ax) = (y_2, \varepsilon x)$. There's a unique $y_2 \in \varepsilon X$ with the appropriate property and any multiple of e ~~add~~ can be added to y_2. So ~~you~~ get an affine line of possible V. ~~So the description~~ In the J matrix description

$$\begin{bmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & b_3 & \end{bmatrix}$$

any element of $\mathbb{C}$.
Anyway what else?

Review. ~~As~~ Given $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ $\qquad W = \begin{pmatrix}\varepsilon\\A\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$

W^0 consists of $\begin{pmatrix}y_1\\y_2\end{pmatrix}$ $\quad (y_1, Ax) = (y_2, \varepsilon X) \quad \forall x$

Add to W $\begin{pmatrix}e\\ \varepsilon \cancel{x_2}\end{pmatrix}$ where e spans $(\varepsilon X)^\perp$
and $\varepsilon \cancel{x_2}$ satisfies $(e, Ax) = (\cancel{\varepsilon x_2}, \varepsilon x) \;\forall x$

y_2 determined up to an ~~element~~ a multiple of e.

$y_2 = \varepsilon x_2 + ce$. This defines extn. $\tilde{A}$ of $A\varepsilon^{-1}$

$(e, \tilde{A} e) = 0$

33 Review. $W = \binom{\varepsilon}{A} X$ $\quad W^\circ = \left\{ \binom{y_1}{y_2} \middle| (y_1, Ax) = (y_2, \varepsilon x) \quad \forall x \in X \right\}$

$p_1 : W \xrightarrow{\sim} \varepsilon X$, e unit vector $\perp$ εX, take $y_1 = e$

$\exists!\, x_0 \ni (e, Ax) = \underbrace{(\varepsilon x_0, \varepsilon x)}_{\substack{\| \\ (x_0, x)}} \quad \forall x$ $\quad$ if $\varepsilon^* \varepsilon = 1$.

Then get ~~a hermitian~~ $\binom{e}{\varepsilon x_0} \in W^\circ$. Can describe all extensions of A to a hermitian op $\tilde{A}$ on Y, by

$\tilde{A}\varepsilon = A$, $\tilde{A}e = \varepsilon x_0 + ce$ $\quad c \in \mathbb{R}$. You need $\binom{e}{\varepsilon x_0 + ce}$ to be isotropic: $\operatorname{Im}(e, \varepsilon x_0 + ce) = \operatorname{Im}(c) = 0$.

I want nearly hermitian operator with neg. Imag part. Probably want $c = -i$ so that $(e, \tilde{A}e) = -i$ up to a positive scalar. ~~But you have~~

Go back to C.T. to ~~figure out~~ find what corresponds to $z = 0$, $\lambda = i$

$W = \binom{a}{b} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\qquad W^\circ = W \oplus \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$ $\qquad V^+ = \operatorname{Ker} a^*$

$$az - b = a\frac{1+i\lambda}{1-i\lambda} - b \sim a(1+i\lambda) - b(1-i\lambda)$$
$$-i\underbrace{(a-b)}_{A} + \lambda \underbrace{(a+b)}_{\varepsilon}$$

b arises from $z = 0$, $\lambda = +i$

$$V^- = \big((i\varepsilon - A)X\big)^\perp$$

~~If you want~~ Go back to $W = \binom{\varepsilon}{A} X$ and

$$W^\circ = \binom{\varepsilon}{A} X + \mathbb{C}\binom{e}{\varepsilon x_0} + \mathbb{C}\binom{0}{e}$$

~~The remaining~~ Remaining problem: to connect $\big((i\varepsilon - A)X\big)^\perp$ with the J-matrix

$$\det \lambda - \underbrace{\begin{pmatrix} b_1 & a_1 & & & \\ a_1 & \ddots & \ddots & & \\ & \ddots & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n & a_n \\ & & & a_n & i \end{pmatrix}}_{A} = (\lambda - i) d_n - a_n^2 d_{n-1}$$

$$\det\begin{pmatrix} \lambda-b_1 & -a_1 & & \\ -a_1 & \ddots & & \\ & & \ddots & -a_{n-1} \\ & & -a_{n-1} & \lambda-b_n \end{pmatrix}$$

?? This doesn't look so promising.

You have this way to produce $\tilde{A}$ extending $A\varepsilon^{-1}$ such that $\tilde{A} - \tilde{A}^*$ rank 1 purely imag

You now have clear picture of nearly hermitian extensions of $A\varepsilon^{-1}$.

---

puzzle. What happens. Review.

have p. herm. $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ of type $\mathcal{O}(n)$

$W = \binom{\varepsilon}{A} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ isotropic for

$$\left(\xi, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\xi'\right) = (\xi_1, \xi_2') - (\xi_2, \xi_1')$$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, (y_1, Ax) = (y_2, \varepsilon x) \quad \forall x \in X \right\} \supset W$$

e unit vector in $(\varepsilon X)^\perp$, define x_0 by

$$\underbrace{(e, Ax)}_{\text{linear func. on } X} = (\varepsilon x_0, \varepsilon x)$$

Then $\begin{pmatrix} e \\ \varepsilon x_0 \end{pmatrix} \in W^0$

$$W^0 = \binom{\varepsilon}{A} X \oplus \mathbb{C}\begin{pmatrix} e \\ \varepsilon x_0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ e \end{pmatrix}$$

~~Extending the partial herm $A\varepsilon^{-1}$ to $\tilde{A}$ amounts to $\binom{\varepsilon}{A}X \subset \Gamma_{\tilde{A}}$~~

Consider subspaces: $\overset{n}{W} \subset \overset{n+1}{V} \subset \overset{n+2}{W^0} \ni V = \Gamma_{\tilde{A}}$

$\tilde{A} : Y \to Y$ extends $A\varepsilon$. $\tilde{A}\left| \begin{matrix} \varepsilon x & = & Ax \\ e & = & \varepsilon x_0 + ce \end{matrix} \right.$

$$\left(\varepsilon x+e,\ \tilde{A}(\varepsilon x+e)\right) = \left(\varepsilon x+e,\ Ax+\varepsilon x_0+ce\right)$$

$$= \underbrace{(\varepsilon x, Ax)}_{\text{real}} + (\varepsilon x, \varepsilon x_0) + \underbrace{(e, Ax)}_{(\varepsilon x_0,\ \varepsilon x)} + (e,e)c$$

Im part = Im c.

So $\tilde{A}$ is nearly real hermitian.

J-matrix picture of $\tilde{A}$.

$$\begin{bmatrix} b_1 & a_1 & & & & \\ a_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & a_{n-1} & & \\ & & a_{n-1} & b_n & a_n \\ & & & a_n & c \end{bmatrix}$$

Now ~~one~~ you might use $\tilde{A}$ to construct an isometric embedding

$$\tilde{y}(\lambda) = \left(e,\ (\lambda-\tilde{A})^{-1}y\right)$$

of Y into $L^2(R)$.

Look at poles: $\det(\lambda-\tilde{A}) = 0$

$$\det(\lambda-\underbrace{\tilde{A}}_{M_{n+1}}) = (\lambda-\underbrace{c}_{b_{n+1}})\det(\lambda-\underbrace{\varepsilon^* A}_{M_n}) - a_n^2 \det(\lambda - M_{n-1})$$

$$d_{n+1} = (\lambda - b_{n+1})d_n - a_n^2 d_{n-1}$$

$$\frac{d_{n+1}}{d_n} = \lambda - b_{n+1} - \frac{a_n^2}{\left(\frac{d_n}{d_{n-1}}\right)}$$

~~Something is wrong.~~

~~I am puzzling~~ The preceding is reasonable but it does not yield the relation you want, you expect from de Branges + scattering. ~~The point~~

The important point involves choosing a ~~[illegible]~~ ~~a line in Y whose~~ section of the line bundle.

~~Idea:~~ Let's try to use the line orthogonal to $(i\varepsilon - A)X$ in Y.

Go back to J-matrix & try to combine the $\iota\varepsilon$ with the imag part of c.

$$\begin{matrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \\ & & & a_n \end{matrix}$$

What do you need. ~~[illegible]~~

$$Y \xrightarrow{\binom{\varepsilon'}{e'}} X \oplus \mathbb{C} \xrightarrow{(\lambda\varepsilon - A \;\; e)} Y$$

Present understanding. ~~Let~~

$$(\varepsilon \;\; e)\begin{pmatrix}\varepsilon' \\ e'\end{pmatrix} = \varepsilon\varepsilon' + ee' = 1.$$

$$(\lambda\varepsilon - A \;\; e)\begin{pmatrix}\varepsilon' \\ e'\end{pmatrix} = \lambda - A\varepsilon' \qquad \begin{pmatrix}\varepsilon' \\ e'\end{pmatrix}(\varepsilon \;\; e) = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$$

~~[illegible]~~ Can you calculate ε', e, e' for the line $(\iota\varepsilon - A)X^{\perp}$?

$$A = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \\ 0 & a_2 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A^* = \begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \end{pmatrix} \qquad \varepsilon^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\iota\varepsilon^* + A^* = \begin{pmatrix} b_1 + i & a_1 & 0 \\ a_1 & b_2 + i & a_2 \end{pmatrix} \qquad (\iota\varepsilon^* + A^*)\varepsilon = \begin{pmatrix} b_1 + i & a_1 \\ a_1 & b_2 + i \end{pmatrix}$$

$$(\iota\varepsilon^* + A^*)\varepsilon = i + A^*\varepsilon \qquad \text{invertible}$$

$$\varepsilon' = (i + A^*\varepsilon)^{-1}(\iota\varepsilon^* + A^*) = \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \;\; \underbrace{\begin{pmatrix} b_1 + i & a_1 \\ a_1 & b_2 + i \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ a_2 \end{pmatrix}}_{\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}}\right)$$

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$$\mathrm{Ker}(\varepsilon') = \begin{pmatrix} -\xi_1 \\ -\xi_2 \\ 1 \end{pmatrix}$$

$$(i\varepsilon^* + A^*)\varepsilon = i + A^*\varepsilon$$
$$= i + \varepsilon^* A$$
$$= \varepsilon^*(i\varepsilon + A)$$

$$\underbrace{\begin{pmatrix} 1 & 0 & \xi_1 \\ 0 & 1 & \xi_2 \\ 0 & 0 & 1 \end{pmatrix}}_{\begin{pmatrix} \varepsilon' \\ e' \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & 0 & -\xi_1 \\ 0 & 1 & -\xi_2 \\ 0 & 0 & 1 \end{pmatrix}}_{(\varepsilon \quad e)} = \mathrm{Id}$$

$A\varepsilon'$ has kernel $\mathbb{C}e$

What is $\varepsilon' A$.

$$\begin{pmatrix} 1 & 0 & \xi_1 \\ 0 & 1 & \xi_2 \end{pmatrix} \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \\ & a_2 \end{pmatrix} = \begin{pmatrix} b_1 & a_1 + \xi_1 a_2 \\ a_1 & b_2 + \xi_2 a_2 \end{pmatrix}$$

Try again.

$$A = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & \ddots & b_n \\ & & & a_n \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ & 0 & \\ & & 1 \\ & & 0 \end{pmatrix}$$

$$i\varepsilon^* + A^* = \begin{pmatrix} b_1+i & a_1 & & & 0 \\ a_1 & b_2+i & a_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ 0 & & & a_{n-1} & b_n+i & a_n \end{pmatrix}$$

So $\varepsilon' = (\iota + \varepsilon^* A)^{-1}(\iota\varepsilon^* + A^*)$

$$= \begin{pmatrix} I & \xi \end{pmatrix}$$

$$(\iota+\varepsilon^* A)\xi = \begin{pmatrix}0\\0\\0\\a_n\end{pmatrix}$$

$$\varepsilon' = \varepsilon^* + (\iota+\varepsilon^* A)^{-1}x_0\langle e_{n+1}|$$

$$(e_{n+1}, Ax) = (x_0, x)$$

$$\varepsilon' A = \varepsilon^* A + (\iota+\varepsilon^* A)^{-1}x_0\langle e_{n+1}|A$$

The point somehow

$$A\varepsilon' = A\varepsilon^* + \underbrace{A(\iota+\varepsilon^* A)^{-1}}\, x_0 (e_{n+1}|$$

$$(\iota + A\varepsilon^*)^{-1} A x_0 (e_{n+1}|$$

Review the logic. You have a holom. line bundle E over $\mathbb{P}^1$ with fibres $Y/(\lambda\varepsilon - A)X = E_\lambda$. Any $y \in Y$ gives a holom. section which has n zeroes if ~~specific~~ $y \neq 0$. Get divisors of degree n on $\mathbb{P}^1$ equivalent to lines in $\mathbb{P}^1$.

If $f \notin \varepsilon X$, then $Y = \mathbb{C}f \oplus \varepsilon X$

we get

$$Y \xrightarrow[\sim]{\binom{\varepsilon'}{f'}} X \oplus \mathbb{C} \xrightarrow[\sim]{(\varepsilon\ \ f)} Y$$

$$Y \xrightarrow{\binom{\varepsilon'}{f'}} X \oplus \mathbb{C} \xrightarrow{(\lambda\varepsilon - A\ \ f)} Y \qquad (\lambda\varepsilon - A)\varepsilon' + ff' = \lambda - A\varepsilon'$$

divisor is roots of $\frac{1}{\lambda}\det(\lambda - A\varepsilon') = \det(\lambda - \varepsilon' A)$

In simpler terms

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, "ff"] & & \\
X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, "\lambda - \varepsilon' A"'] & Y \arrow[r] \arrow[d, "\varepsilon'"] & E_\lambda \arrow[r] & 0 \\
 & X & &
\end{tikzcd}$$

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~~What is the point?~~

Different approach. Calculate $\ker(\lambda\varepsilon^* - A^*)$

$$\begin{pmatrix} b_1-\lambda & a_1 & & & \\ a_1 & b_2-\lambda & \ddots & & \\ & \ddots & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n-\lambda & a_n \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ \\ u_{n+1}\end{pmatrix} = 0$$

$$(b_1-\lambda)u_1 + a_1u_2 = 0$$
$$a_1u_1 + (b_2-\lambda)u_2 + a_2u_3 = 0$$

solve starting from $u_1 = 1$.

$$u_2 = \frac{\lambda - b_1}{a_1} \qquad u_3 = \frac{(\lambda-b_2)u_2 - a_1u_1}{a_2}$$

$$d_j = \det(\lambda - M_j) \qquad M_j = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{j-1} \\ & & a_{j-1} & b_j\end{pmatrix}$$

$$d_{j+1} = (\lambda - b_{j+1})d_j - a_j^2 d_{j-1}$$

$$d_{j+1} = (\lambda - b_{j+1})d_j - a_j^2 d_{j-1} \qquad j \geq 0.$$

$$u_{j+1} = \frac{d_j}{a_1\cdots a_j} \qquad d_j = (\lambda - b_j)d_{j-1} - a_{j-1}^2 d_{j-2}$$

$$\underline{a_ju_{j+1}} = (\lambda - b_j)u_j - a_{j-1}u_{j-1}$$

$$\frac{d_j}{a_1\cdots a_{j-1}} = (\lambda - b_j)\frac{d_{j-1}}{a_1\cdots a_{j-1}} - a_{j-1}\frac{d_{j-2}}{a_1\cdots a_{j-2}}$$

$$\begin{pmatrix} b_1-\lambda & & \\ & \ddots & \\ & & b_n-\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_n \end{pmatrix} u_{n+1} = 0$$

$$\begin{pmatrix} 0 \\ \vdots \\ a_n u_{n+1} \end{pmatrix} = (\lambda - M_n)\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

J-matrix

$$A = \begin{bmatrix} b_1 & a_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \\ & & & a_n \end{bmatrix} \qquad \mathrm{Ker}(\lambda \varepsilon^* - A^*).$$

$$\begin{bmatrix} b_1-\lambda & a_1 & & & \\ a_1 & b_2-\lambda & & & \\ & & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n-\lambda & a_n \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_{n+1} \end{bmatrix} = 0$$

$$(b_1-\lambda)u_1 + a_1 u_2 = 0 \qquad a_1 u_2 = (\lambda - b_1) u_1$$

$$a_1 u_1 + (b_2-\lambda) u_2 + a_2 u_3 = 0 \qquad a_2 u_3 = (\lambda - b_2) u_2 - a_1 u_1$$

$$\frac{a_1 u_2}{u_1} = \lambda - b_1 \qquad \frac{a_2 u_3}{u_2} = \lambda - b_2 - a_1^2 \Big/ \frac{a_1 u_2}{u_1}$$

$$a_{j-1} u_{j-1} + (b_j - \lambda) u_j + a_j u_{j+1} = 0$$

$$a_j u_{j+1} = (\lambda - b_j) u_j - a_{j-1} u_{j-1}$$

$$\begin{pmatrix} a_j u_{j+1} \\ u_j \end{pmatrix} = \begin{pmatrix} \frac{\lambda - b_j}{a_{j-1}} & -a_{j-1} \\ a_{j-1}^{-1} & 0 \end{pmatrix}\begin{pmatrix} a_{j-1} u_j \\ u_{j-1} \end{pmatrix}$$

So the recursion formula is

$$\begin{pmatrix} a_j u_{j+1} \\ u_j \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\lambda - b_j}{a_{j-1}} & -a_{j-1} \\ \frac{1}{a_{j-1}} & 0 \end{pmatrix}}_{\det = 1} \begin{pmatrix} a_{j-1} u_j \\ u_{j-1} \end{pmatrix} \qquad \begin{array}{l} u_1 = 1 \\ u_0 = 0 \end{array}$$

$$\begin{pmatrix} a_1 u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} \lambda - b_1 \\ \\ \end{pmatrix}$$

$$\begin{pmatrix} u_{j+1} \\ u_j \end{pmatrix} = \begin{pmatrix} \frac{\lambda - b_j}{a_j} & -\frac{a_{j-1}}{a_j} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_j \\ u_{j-1} \end{pmatrix}$$

So inductively u_{j+1} is a poly $\dfrac{\lambda^j}{a_j \cdots a_1} = \dfrac{\det(\lambda - M_j)}{a_j \cdots a_1}$

~~So never get down to~~

$$(M_n - \lambda) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -a_n u_{n+1} \end{pmatrix}$$

$$\boxed{(\lambda - M_n) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n-1 \\ a_n u_{n+1} \end{pmatrix}}$$

$\mathrm{Ker}(\lambda E^* - A^*)$ gen. by $\begin{pmatrix} u_1 \\ \vdots \\ u_{n+1} \end{pmatrix}$

Here's the question:

Go back to the idea, the important idea, which is the zeroes of the section of $\mathcal{O}(n)$ given by a generator of $\operatorname{Ker}(i\varepsilon^* - A^*)$.

$$\begin{tikzcd} X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr] & Y \arrow[d, "\iota\varepsilon^* - A^*"] \\ & X \end{tikzcd}$$

$$\begin{aligned}(\iota\varepsilon^* - A^*)(\lambda\varepsilon - A) &= i\lambda - \lambda A^*\varepsilon - i\varepsilon^* A + A^*A \\ &= i\lambda + A^*A - (\lambda + i)\varepsilon^* A \\ &= 1 + A^*A + \underbrace{i\lambda - 1}_{i(\lambda+i)} - (\lambda+i)\varepsilon^* A \\ &= 1 + A^*A + (\lambda+i)(i - \varepsilon^* A)\end{aligned}$$

This is no help. Perhaps A^*A is not good. Better might be to have $(\varepsilon^* A)^2 = A^*\varepsilon\varepsilon^* A = A^*A - A^*\underset{e^*e}{\underbrace{\pi}}A$

Somehow this is too hard

$$(\iota\varepsilon^* - A^*)(-i\varepsilon - A) = 1 + A^*A$$

Adopt de Branges approach. Namely consider the Hilbert space with the orthonormal basis given by the sequence of polynomials $u_1, u_2, \dots, u_n$

$$(\iota\varepsilon^* - A^*)(\lambda\varepsilon - A) = \lambda(\iota - A^*\varepsilon) - \underbrace{(\iota\varepsilon^* - A^*)A}_{iA^*\varepsilon - A^*A \,=\, A^*(\iota\varepsilon - A)}$$

$$\lambda - \underbrace{(i - A^*\varepsilon)^{-1}A^*}_{A^*(i - \varepsilon A^*)^{-1}}(\iota\varepsilon - A)$$

$$(\iota\varepsilon^* - A^*)(\lambda\varepsilon - A) = \lambda(\iota\varepsilon^* - A^*)\varepsilon - (\iota\varepsilon^* - A^*)A$$

So we get

$$\left((\iota\varepsilon^* - A^*)\varepsilon\right)^{-1}(\iota\varepsilon^* - A^*)A$$

whose spectrum we want. If you change A by $c\varepsilon$, this operator changes by c.

Go back to the de Branges approach where you have $\overset{u_1}{p_1}, p_2, \ldots, \overset{u_{n+1}}{p_{n+1}}$ polynomials in λ recursion relations. Form

$$\sum_{j=1}^{n} p_j(\lambda)\,p_j(\mu)$$

I think I begin to understand now. You know about extending $A\varepsilon^{-1}$ to a nearly hermitean operator which will then give an L^2 representation. This must be what de Branges does. ~~You know about~~ Work this picture out and correlate with point evaluators.

Recursion relations

$$\lambda p_j = a_j p_{j+1} + b_j p_j + a_{j-1} p_{j-1}$$

$$\sum_{j=1}^{n} \lambda p_j(\lambda)p_j(\mu) = \sum_{j=1}^{n} a_j p_{j+1}(\lambda)p_j(\mu) + \cancel{b_j p_j(\lambda)p_j(\mu)} + \sum_{j=0}^{n-1} a_j p_j(\lambda)p_{j+1}(\mu)$$

$$-\sum_{j=1}^{n} p_j(\lambda)\,\mu p_j(\mu) - \sum_{j=1}^{n} p_j(\lambda)a_j p_{j+1}(\mu) + \cancel{p_j(\lambda)b_j p_j(\mu)} + \sum_{j=0}^{n-1} p_{j+1}(\lambda)a_j p_j(\mu)$$

$$= a_n\left(p_{n+1}(\lambda)p_n(\mu) - p_n(\lambda)p_{n+1}(\mu)\right)$$

$$\sum_{j=1}^{n} p_j(\lambda)p_j(\mu) = a_n\left(\frac{p_{n+1}(\lambda)p_n(\mu) - p_n(\lambda)p_{n+1}(\mu)}{\lambda - \mu}\right)$$

44 Try carefully.

Go back to $W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (y_1, Ax) = (y_2, \varepsilon x), \forall x \in X \right\}$$

~~define $x_0 \in X$ so that~~ y

Let e_{n+1} be a unit vector $\in (\varepsilon X)^\perp$ $\qquad x_n = a_n e_n$

Let $x_n \in X \ni \quad (e_{n+1}, Ax) = (\varepsilon x_n, x)$

Then $W^0 = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \oplus \mathbb{C}\begin{pmatrix} e_{n+1} \\ x_n \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix}$

In terms of the J-matrix.

$$\underbrace{\left[\begin{array}{ccccc|c} b_1 & a_1 & & & 0 & 0 \\ a_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & a_{n-1} & 0 \\ 0 & & & a_{n-1} & b_n & a_n \\ & & & & a_n & c \end{array}\right]}_{A}$$

Now I think can use this $\tilde{A}$ to ~~give a~~ get an isometric embedding

Review:

$(\lambda - \alpha) x = -y + \cdot$

Idea:

$$(e, (\lambda - \alpha)^{-1} y) = \tilde{y}(\lambda)$$

$$\int_{-\infty}^{\infty} |\tilde{y}(\lambda)|^2 \frac{d\lambda}{2\pi} = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left((\lambda-\alpha)^{-1} y, e\right)\left(e, (\lambda-\alpha)^{-1} y\right)$$

$$= \int \frac{d\lambda}{2\pi} \left(y, \underbrace{(\lambda - \alpha^*)^{-1} \cdot |e\rangle\langle e| (\lambda-\alpha)^{-1}}\, y\right)$$

residue

$= \frac{2\pi i}{2\pi}\left(y, |e\rangle\langle e| (\alpha^* - \alpha)^{-1} y\right)$

~~$\frac{2\pi i}{2\pi}(\alpha^* - \alpha)^{-1} |e\rangle\langle e|$~~

$i\, |e\rangle\langle e| = 6(\alpha^* - \alpha)$

Now the requirement is $-i(\alpha^* - \alpha) = |e\rangle\langle e|$

α^* spec. in UHP $\qquad -2\,\frac{\alpha - \alpha^*}{2i} = -2\,\mathrm{Im}(\alpha)$

45 Idea now is to let the imginary part go to ∞.

$$\lambda-\alpha = \begin{bmatrix} \lambda-b_1 & & \\ & \ddots & \\ & \lambda-b_n & -a_n \\ & -a_n & \lambda-c \end{bmatrix}$$

$2\,\mathrm{Im}\,\alpha = \mathrm{Im}\,c.$

$-i(\alpha^*-\alpha) = -i(\bar{c}-c) = -2\,\mathrm{Im}\,c$

So what you try to do then is to

~~Review perturb~~

$$\lambda-\alpha = \left(\begin{array}{cccc|c} \lambda-b_1 & -a_1 & & & \\ -a_1 & \lambda-b_2 & & & 0 \\ & -a_2 & \ddots & & \\ & & & \lambda-b_n & -a_n \\ \hline & 0 & & -a_n & \lambda-c \end{array}\right) \qquad \alpha = \left(\begin{array}{c|c} M_n & \\ & a_n \\ \hline a_n & c \end{array}\right)$$

$$(\lambda-\alpha)^{-1} = \left(\begin{array}{c|c} \lambda-M_n & g \\ \hline g^* & \lambda-c \end{array}\right)^{-1} = \left(\begin{array}{c|c} g(\lambda-c)^{-1}g^* & \\ \hline & \end{array}\right)$$

$$\lambda-M_n - g\frac{1}{\lambda-c}g^* \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Big/ ad-bc$$

$$\left(\lambda-M_n - g\frac{1}{\lambda-c}g^*\right)^{-1}$$

arrange $c \to \infty$ $\quad \dfrac{ad-bc}{-c} = b - ac^{-1}d$

But actually you want $\langle e|(\lambda-\alpha)^{-1}$

So what?

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \text{assume } d \text{ invertible.}$$

$$\begin{pmatrix} \lambda - M & g \\ g^* & \lambda - c \end{pmatrix}^{-1} \qquad \begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a - bd^{-1}c & 0 \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix} = \begin{pmatrix} a-bd^{-1}c & 0 \\ 0 & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix}\begin{pmatrix} (a-bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (a-bd^{-1}c)^{-1} & 0 \\ -d^{-1}c(a-bd^{-1}c)^{-1} & d^{-1} \end{pmatrix}\begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (a-bd^{-1}c)^{-1} \\ \underline{-d^{-1}c(a-bd^{-1}c)^{-1}} \end{pmatrix}$$

$$-d^{-1}c(a-bd^{-1}c)^{-1} = -\frac{1}{\lambda - c} g^* \left(\lambda - M - g \frac{1}{\lambda - c} g^*\right)^{-1}$$

Let's try for a new direction.

Consider

$$\begin{tikzcd} & \tilde{V} \arrow[d] & \\ 0 \arrow[r] & X \arrow[r, "az-b"] & Y \arrow[r] \arrow[d, "b^*"] & {} \\ & & X \end{tikzcd}$$

$$Y \xrightarrow{\begin{pmatrix} b^* \\ e_0^* \end{pmatrix}} X \oplus \mathbb{C} \xrightarrow{(b-az \quad e_0)} Y$$

$$(b-az \quad e_0)\begin{pmatrix} b^* \\ e_0^* \end{pmatrix} = 1 - zab^*$$

$$(b-az)x + \tilde{y}(z)e_0 = y$$

has solution

$$\begin{pmatrix} x \\ c \end{pmatrix} = \begin{pmatrix} b^* \\ e_0^* \end{pmatrix}(1-zab^*)^{-1} y$$

47 and you get an isom. embedding.

$$\tilde{y}(z) = e_0^* (1 - zab^*)^{-1} y$$

$$\int \frac{d\theta}{2\pi} \left| \tilde{y}(z) \right|^2 = \int \frac{d\theta}{2\pi i} \left(y, \frac{1}{1-\bar{z}ba^*} e_0 e_0^* \frac{1}{1-zab^*} y \right)$$

$$= \left(y, e_0 e_0^* \frac{1}{1 - ba^* ab^*} y \right)$$ more arguments needed.

<u>Principle:</u> The element of Y you use to trivialize the line bundle over the UHP ~~determine~~ determines the ~~zeros~~ poles. So if you want the results

Take an LC circuit, form the corresp J-matrix. Calculate the ~~~~.

~~Take a partial unit rotation~~

Put into words the problem. Take a J-matrix determine its response.

Take a partial unitary $\quad aX \oplus V^+ = bX \oplus V^-$

The response function is a map $S(z) : V^- \to V^+$, it really depends upon the line V^-.

~~Take a partial hermitian~~

Given $\quad W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ $\quad$ Also have

$(az - b)x \qquad (z \;\; -1) : \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \longrightarrow Y$

So consider $\quad \overset{n}{W = \begin{pmatrix} a \\ b \end{pmatrix} X} \qquad \overset{n+1}{\begin{pmatrix} 1 \\ z \end{pmatrix} Y} \qquad \overset{n+2}{W^0} = \overset{n}{W} \oplus \begin{matrix} \overset{1}{V^+} \\ \oplus \\ V^- \end{matrix}$

$W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \hookrightarrow W^0 / W^{\perp}$

dim 1

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$$W^\circ \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y = \operatorname{Ker}\left\{ W^\circ \xrightarrow{(z\ \ -1)} Y \right\}$$

$$= \left\{ \begin{pmatrix} ax+v^+ \\ bx+v^- \end{pmatrix} \middle| \ z(ax+v^+) = bx+v^- \right\}$$

$$(az-b)\,x = -zv^+ + v^-$$

$$v^- = (1-bb^*)(1-zab^*)^{-1} z v^+$$

Move on to herm. setting.

$$W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \qquad \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y$$

max. isot. ~~isotropic~~ for λ real

$$W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \hookrightarrow W^\circ / W \qquad \text{line.}$$

Example: Suppose

$$A = \begin{pmatrix} 0 & a_1 & \\ a_1 & 0 & a_2 \\ & a_2 & 0 \\ & & a_3 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & & 0 \\ 0 & 1 & \\ 0 & 0 & 1 \\ & & 0 \end{pmatrix}$$

$$W^\circ = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \oplus \mathbb{C} \begin{pmatrix} e_4 \\ a_3 e_3 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 \\ e_4 \end{pmatrix}$$

$$W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y = W^\circ \cap \operatorname{Ker}(\lambda\ \ -1).$$

$$\big\downarrow (\lambda\ \ -1)$$

$$(\lambda\varepsilon - A)X + \mathbb{C}(\lambda e_4 - a_3 e_3) + \mathbb{C} e_4$$

$$(\lambda - \tilde{A})Y$$

$$\begin{pmatrix} \varepsilon \\ A \end{pmatrix} x + \begin{pmatrix} e_4 \\ a_3 e_3 \end{pmatrix} x_4 + \begin{pmatrix} 0 \\ e_4 \end{pmatrix} \in \operatorname{Ker}(\lambda\ \ -1)$$

$$(\lambda\varepsilon - A)x + (\lambda e_4 - a_3 e_3)x_4 = e_4$$

$$\left[\lambda - \begin{pmatrix} \cdot & a_1 & & \\ a_1 & & a_2 & \\ & a_2 & & a_3 \\ & & a_3 & \end{pmatrix}\right] \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

49 We agreed that solution of

$$\lambda u_1 - a_1 u_2 = 0 \qquad a_1 u_2 = \lambda u_1$$

$$-a_1 u_1 + \lambda u_2 - a_2 u_3 = 0 \qquad a_2 u_3 = \lambda u_2 - a_1 u_1$$

$$-a_2 u_2 + \lambda u_3 - a_3 u_4 = 0 \qquad a_3 u_4 = \lambda u_3 - a_2 u_2$$

$$-a_3 u_3 + \lambda u_4 = 1 \qquad \underset{a_4 u_5}{1} = \lambda u_4 - a_3 u_3$$

$$\left[\lambda - \left(\tilde{A}\right)\right]\begin{pmatrix} u_1 \\ \vdots \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_4 u_5 \end{pmatrix}$$

we take $x_1 = 1 = u_1$ Then find

$x_2 = u_2$

$x_3 = u_3$

$x_4 = u_4$

satisfies $(\lambda E - A)\begin{pmatrix} u_1 \\ \vdots \\ u_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ -a_3 u_4 \\ \lambda u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_4 u_5 \end{pmatrix}$

so we have ~~the element~~ the element

$$\begin{pmatrix} x \\ \cancel{\tilde{A}x} \\ Ax \end{pmatrix} + \begin{pmatrix} 0 \\ e_4 (a_4 u_5) \end{pmatrix} \in W^0 \cap \mathrm{Ker}(\lambda - 1)$$

$$\begin{pmatrix} u_4 e_4 \\ a_3 u_4 e_3 \end{pmatrix} + \begin{pmatrix} 0 \\ (a_4 u_5) e_4 \end{pmatrix}$$

seems to consist of the ~~ ~~ components u_4, $a_4 u_5$

50 Oct 5 Missing point. CT does not commute with taking response function?

~~Eg~~ Consider $W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$, form $W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y$ and ~~[illegible]~~ its image $Z_z \subset W^0/W$. On the other hand can apply $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ to get

$$W' = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \quad \text{and find } W'^{0'} \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y$$

and its image $Z'_\lambda \subset W'^{0'}/W'$. These are all related by the unitary transformation

~~These are all related by the unitary~~ So there is an isom $W^0/W \xrightarrow{\sim} W'^{0'}/W'$

$$\begin{matrix} W^0/W & \xrightarrow{\sim} & W'^{0'}/W' \\ \text{IS} & & \text{IS} \\ V^+ \oplus V^- & & \mathbb{C}\begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix} \end{matrix}$$

$$\begin{pmatrix} \varepsilon \\ A \end{pmatrix} x + \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \qquad \begin{matrix} y_2 = Ae_1 \quad Ae_n \\ y_1 = e_1, \dots, e_n \end{matrix}$$

$$W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} x \qquad W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \ (y_1, Ax) = (y_2, \varepsilon x) \quad \forall x \right\}$$

$$y_1 = e_{n+1} \qquad y_2 = a_n e_n + c\, e_{n+1}$$

$$W^0 \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \ni \begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} + \overset{x_{n+1}}{c} \begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix}$$

$$\lambda(\varepsilon x + e_{n+1} x_{n+1}) = Ax + e_n a_n x_{n+1} + e_{n+1} c$$

~~$(\lambda\varepsilon - A)x + (e_{n+1})\lambda x_{n+1}$~~

$$\left[\lambda - \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & \ddots & \ddots & & \\ & \ddots & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n & a_n \\ & & & a_n & 0 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ c \end{pmatrix}$$

5| You know the vector $\begin{matrix} u_1 \\ | \\ u_{n+1} \end{matrix}$ of orth polys satisfies $(\lambda - M_{n+1})\, u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_{n+1} u_{n+2} \end{pmatrix}$ Take all $b_k^{*} = 0$

u_λ

$\therefore$ get $\begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} u_{n+1} + \begin{pmatrix} 0 \\ a_{n+1} e_{n+1} \end{pmatrix} u_{n+2}$

$W^0 \cap \binom{1}{\lambda} Y$ ~~spanned~~ by $\begin{pmatrix} \varepsilon \\ A \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} u_{n+1} + \begin{pmatrix} 0 \\ a_{n+1} e_{n+1} \end{pmatrix} u_{n+2}$

apply $\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$ or $\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$ to get

$$\begin{pmatrix} i\varepsilon + A \\ i\varepsilon - A \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} i e_{n+1} + a_n e_n \\ i e_{n+1} - a_n e_n \end{pmatrix} u_{n+1} + \begin{pmatrix} e_{n+1} \\ -e_{n+1} \end{pmatrix} a_{n+1} u_{n+2}$$

spanning the corresponding line in the ~~uu~~ Z picture
We need the image of this ~~[illegible]~~
in $\begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$. Looks messy.

~~Apparently what happens is that~~

Try the other direction. Take V^+ which should be ~~[illegible]~~ $\mathrm{Ker}(-i\varepsilon^* - A^*)$ spanned by u^{-i}

~~[illegible]~~ $$(\lambda - \tilde{A})\, u^\lambda = e_{n+1} a_{n+1} u^\lambda_{n+2}$$

$$\therefore \quad (\lambda \varepsilon^* - \varepsilon^* \tilde{A})\, u^\lambda = 0.$$

~~Find~~

52 $\begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} 1 \\ z \end{pmatrix} Y$ $\quad \left(\begin{pmatrix} a \\ b \end{pmatrix} X + \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix} \right) \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y$

$$\begin{pmatrix} \xi \\ z\xi \end{pmatrix} = \begin{pmatrix} ax + v^+ \\ bx + v^- \end{pmatrix} \qquad (az-b)x = -zv^+ + v^-$$

Suppose $\mathrm{Im}\left\{ W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \to W^0/W = \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix} \right\}$ is $\begin{pmatrix} 1 \\ \phi(z) \end{pmatrix} V^+$

~~i.e.~~ $\cancel{(az-b)x = z\ldots}$ ~~$v^- = \phi(z) v^+$. Then~~

$$\cancel{(az-b)x = -zv^+ + \phi(z)v^+}$$
$$\cancel{= (\phi(z) - z)v^+}$$

$$\cancel{(z - a^*b)x}$$

$$z^{-1} S(z)^{-1} v^-$$
$$= z^{-1}(1-a^*a)(1-z^{-1}ba^*)^{-1} v^-$$

$$(az-b)x = -zv^+ + v^-$$
$$(1 - zb^*a)x = zb^*v^+$$
$$x = zb^*(1-zab^*)^{-1} v^+$$
$$v^- = z\left(\overset{(1-zab^*)}{\cancel{1}} + (az-b)b^* (1-zab^*)^{-1} \right) v^+$$

$$v^- = z \underbrace{(1-bb^*)(1-zab^*)^{-1} v^+}_{S(z)v^+} \quad \text{where } S(z) : V^+ \to V^-$$

$\therefore$ L_z line in $\begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$ is $\begin{pmatrix} v^+ \\ zS(z)v^+ \end{pmatrix}$

$L_z = \mathrm{Im}\left(W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \to W^0/W = \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix} \right)$. Note

$L_0 = \begin{matrix} V^+ \\ \oplus \\ 0 \end{matrix} \qquad L_\infty = \begin{matrix} 0 \\ \oplus \\ V^- \end{matrix}$

53 The logic here is that $z \mapsto L_z$ is ~~a~~ regular map from z Riemann spheres to $\mathbb{P}_1(W^\circ/W)$, i.e. rational functions of z after one chooses some sort of coords on W°/W. ~~These~~ $\exists$ a herm. form on W°/W, i.e. a ~~real~~ unit circle in $\mathbb{P}_1(W^\circ/W)$ which is ~~[illegible]~~ the image of $|z|=1$. ~~[illegible]~~

$$W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \qquad W^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (y_1, Ax) = (y_2, \varepsilon x) \right\}$$

$$= W \oplus \mathbb{C}\begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix}$$

$$W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \ni \begin{pmatrix} \varepsilon x + \{e_{n+1}\} x_{n+1} + 0 \\ Ax + \{e_n a_n\} x_{n+1} + e_{n+1} c \end{pmatrix}$$

Then $x_1, \dots, x_{n+1}$ satisfy.

$$\lambda x = \tilde{A} x + e_{n+1} c$$

and we have the solution $x_i = u_i^\lambda \qquad i = 1, \dots, n+1$

$$\left[\lambda - \begin{pmatrix} 0 & a_1 & & & \\ a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_n \\ & & & a_n & 0 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ \vdots \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n+1} u_{n+2}^\lambda \end{pmatrix}$$

$$L_\lambda = \underline{\text{Image of } W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \text{ in } W^\circ/W} \simeq \begin{pmatrix} e_{n+1} \\ e_n a_n \end{pmatrix} \mathbb{C} \oplus \begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix} \mathbb{C}$$

line $\boxed{\begin{pmatrix} e_{n+1} \\ e_n a_n \end{pmatrix} u_{n+1}^\lambda + \begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix} a_{n+1} u_{n+2}^\lambda}$

54 Try other approaches. What is the link between $\mathrm{Ker}(\lambda\varepsilon^* - A^*)$ and $W^\circ \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y$?

$W = \begin{pmatrix}\varepsilon\\ A\end{pmatrix} X$ $\qquad \begin{pmatrix}y\\ \lambda y\end{pmatrix} \in W^\circ \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y$

i.e. $\quad (y, Ax) = (\lambda y, \varepsilon x) \quad \forall x$

or $\quad ((A^* - \lambda\varepsilon^*)y, x) = 0 \quad \forall x.$

Thus $\quad W^\circ \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y = \left\{ \begin{pmatrix}y\\ \lambda y\end{pmatrix} \,\middle|\, (\lambda\varepsilon^* - A^*)y = 0 \right\}.$

$$u^\lambda = \begin{pmatrix} u_1^\lambda \\ \vdots \\ u_{n+1}^\lambda \end{pmatrix} \qquad (\lambda - \tilde{A})u^\lambda = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n+1} u_{n+2}^\lambda \end{pmatrix}$$

So $\forall \lambda$ we have $\begin{pmatrix}u^\lambda\\ \lambda u^\lambda\end{pmatrix} \in W^\circ$, ~~can~~ can ask about hermitian pairing $\left(\begin{pmatrix}u^\lambda\\ \lambda u^\lambda\end{pmatrix}, \begin{pmatrix} & 1\\ -1 & \end{pmatrix} \begin{pmatrix}u^\mu\\ \mu u^\mu\end{pmatrix} \right)$

$$= \mu(u^\lambda, u^\mu) - \bar{\lambda}(u^\lambda, u^\mu) = (\mu - \bar{\lambda})(u^\lambda, u^\mu).$$

$$- (u^\lambda, \tilde{A} u^\mu) + (\tilde{A} u^\lambda, u^\mu)$$

$$= (u^\lambda, (\mu - \tilde{A}) u^\mu) - ((\lambda - \tilde{A})u^\lambda, u^\mu)$$

$$= u_{n+1}^{\bar\lambda} a_{n+1} u_{n+2}^\mu - a_{n+1} u_{n+2}^{\bar\lambda} u_{n+1}^\mu$$

$$\boxed{= a_{n+1} \begin{vmatrix} u_{n+1}^{\bar\lambda} & u_{n+1}^\mu \\ u_{n+2}^{\bar\lambda} & u_{n+2}^\mu \end{vmatrix} = (\mu - \bar{\lambda})(u^\lambda, u^\mu)}$$

$$\begin{tikzcd}
 & & \mathbb{C}v_0^+ \arrow[d] & & \\
0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] \arrow[d, "i\varepsilon + A"'] & Y \arrow[r] \arrow[d, "i\varepsilon^* + A^*"] & L_\lambda \arrow[r] & 0 \\
 & Y \arrow[r, "\lambda\varepsilon^* - A^*"] & X & &
\end{tikzcd}$$

$$z = \frac{-\lambda + i}{\lambda + i}$$

$$(az - b)X = (a(-\lambda+i) - b(\lambda+i))X$$
$$= (\lambda(-a-b) + i(a-b))X$$
$$= (\lambda(a+b) - i(a-b))X$$

$$(i\varepsilon^* + A^*)(\lambda\varepsilon - A) = \lambda(A^*\varepsilon) - i(\varepsilon^* A) + i\lambda\varepsilon^*\varepsilon - A^*A$$

$$\underbrace{\underbrace{(i\varepsilon^*\varepsilon + A^*\varepsilon)}_{(i\varepsilon^* + A^*)\varepsilon}}_{\varepsilon^*(i\varepsilon + A)}\lambda - \underbrace{(i\varepsilon^* + A^*)A}_{A^*(i\varepsilon + A)}$$

seems that $\quad (\mu\varepsilon^* + A^*)(\lambda\varepsilon + A) = (\lambda\varepsilon^* + A^*)(\mu\varepsilon + A)$

Start again. You want to know when $(i\varepsilon^* + A^*)(\lambda\varepsilon - A)$ is singular, i.e.

$$\| $$
$$(\lambda\varepsilon^* + A^*)(i\varepsilon + A)$$

$$\cancel{ab^* x = \bar{z} x}$$
$$\cancel{\iff b^* x = \bar{z} a^* x \text{ and } (1 - aa^*)x = 0}$$
$$\cancel{\iff (\bar{z} a^* - b^*)x = 0 \text{ and } x \in V^+}$$
$$\cancel{\iff x \perp (az - b)X \text{ and } x \perp aX}$$

$$z b^* a x = x \qquad (az - b)x = -y + cv^-$$

$$z(bx', ax) = (x', x) \qquad \forall x' \qquad \text{no meaning.}$$

56 ~~OK~~. OK so what happens?

Let's try again. Consider a partial unitary

$Y = aX \oplus V^+ = V^- \oplus bX$ $\qquad a^*a = b^*b = 1.$

equiv. a subspace $\begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ isotropic wrt $\|y_1\|^2 - \|y_2\|^2$.

Assume of type $O(\infty)$, $a z - b$ always inj.

$$Y \xrightarrow{\begin{pmatrix} b^* \\ e^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{\begin{pmatrix} b & e \end{pmatrix}} Y$$

inverse isos. e unit v. sp V^-

perturbation

$$Y \xrightarrow{\begin{pmatrix} b^* \\ e^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{\begin{pmatrix} b-az & e \end{pmatrix}} Y$$

$$\begin{pmatrix} b-az & e \end{pmatrix} \begin{pmatrix} b^* \\ e^* \end{pmatrix} = \underbrace{bb^* + ee^*}_{1} - zab^*$$

$\forall y \; \exists!$ $\qquad (b-az)x + \tilde{y}(z)e = y$

answer:

$$\begin{pmatrix} x \\ \tilde{y}(z) \end{pmatrix} = \begin{pmatrix} b^* \\ e^* \end{pmatrix} (1-zab^*)^{-1} y = \begin{pmatrix} (1-zb^*a)^{-1} b^* y \\ e^*(1-zab^*)^{-1} y \end{pmatrix}$$

Next point is that $y \longmapsto e^*(1-zab^*)^{-1} y$ is isom. embed of Y into $L^2(S^1)$.

$$\int |\tilde{y}(z)|^2 \frac{d\theta}{2\pi} = \int \left((1-z\,ab^*)^{-1} y, e \right) \left(e, (1-zab^*)^{-1} y \right)$$

$$= \int \left(y, \frac{1}{\cancel{z}-\cancel{q}ba^*} e e^* \frac{1}{\cancel{1}-zab^*} y \right) \frac{dz}{2\pi i \cancel{z}}$$

$$= \|y\|^2.$$

57 You want the same thing in the hermitian picture. So you consider

$$\begin{tikzcd} & \bullet \arrow[d, "\varepsilon"] \\ X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[d, "\varepsilon^*"] \\ & \end{tikzcd}$$

$$\lambda\varepsilon - A = \lambda(a+b) - i(a-b)$$
$$= (\lambda - i)a + (\lambda + i)b$$
$$= -(-\lambda+i)a + (\lambda+i)b$$
$$\sim b - \left(\frac{-\lambda+i}{\lambda+i}\right)a$$

$$\begin{tikzcd} X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[d, "+i\varepsilon^* + A^*"] \\ & X \end{tikzcd}$$

$$\varepsilon = a+b$$
$$A = i(a-b)$$
$$i\varepsilon - A = 2ib.$$

$$(i\varepsilon^* + A^*)(\lambda\varepsilon - A) = i\lambda\, \varepsilon^*\varepsilon + (\lambda - i)\varepsilon^*A - A^*A$$

$$= i\lambda\, \varepsilon^*\varepsilon + (\lambda - i)\varepsilon^*A - (1 - \varepsilon^*\varepsilon)$$

$$\left(\underbrace{(i\varepsilon^* + A^*)\varepsilon}\right)\lambda - (i\varepsilon^* + A^*)A$$

invertible because you can suppose $\varepsilon^*\varepsilon = 1$.
then you have $i + \underbrace{A^*\varepsilon}_{\text{herm.}}$ So it should be true that $\alpha + i\beta$ is invertible where $\alpha = \alpha^* > 0$ and $\beta = \beta^*$, namely $(\alpha + i\beta)x = 0$

$\Rightarrow (x, \alpha x) + i(x, \beta x) = 0 \quad \therefore (x, \alpha x) = 0 \Rightarrow x = 0.$

$$(i\varepsilon^* + A^*)\varepsilon = \varepsilon^*(i\varepsilon + A)$$
$$(i\varepsilon^* + A^*)A = A^*(i\varepsilon + A)$$

so we have $\underbrace{\left((i\varepsilon^* + A^*)\varepsilon\right)^{-1}\left((i\varepsilon^* + A^*)A\right)}$

$$i\varepsilon + A = i(a+b) + i(a-b) = 2ia$$
$$i\varepsilon^* + A^* = i(a^*+b^*) - i(a^*-b^*) = 2ib^*$$

$$(i\varepsilon^* + A^*)\varepsilon = 2ib^*(a+b) = 2i(1+b^*a)$$
$$(i\varepsilon^*+A^*)A = 2ib^*(+i)(a-b) = -2(b^*a-1)$$
$$= 2(1-b^*a)$$

$$\left[(i\varepsilon^*+A^*)\varepsilon\right]^{-1}\left[(i\varepsilon^*+A^*)A\right] = \left(2i(1+b^*a)\right)^{-1}(2)(1-b^*a)$$

$$= \frac{1}{i}\,\frac{1-b^*a}{1+b^*a}$$

$$\frac{1}{i}\,\frac{1-z^{-1}}{1+z^{-1}} = \frac{1}{i}\,\frac{z-1}{z+1} = i\,\frac{1-z}{1+z} = \lambda.$$

Apparently $\boxed{(i\varepsilon^*+A^*)\varepsilon \quad \text{and} \quad (i\varepsilon^*+A^*)A}$ commute.

~~So~~ In this setting we should have an extra condition ~~[illegible]~~ relating $\varepsilon^*\varepsilon$ and A^*A.

$$\varepsilon^*\varepsilon A^*\varepsilon$$

~~$(i)(i\varepsilon^*+A^*)\varepsilon = 2(1+b^*a)$~~

$$(i\varepsilon^*+A^*)\varepsilon = 2i(1+b^*a)$$
$$(-\varepsilon^*+iA^*)A = 2i(1-b^*a)$$

$$i\,\varepsilon^*\varepsilon + iA^*A = 4i$$

$$\varepsilon^*\varepsilon + A^*A = 4$$

$$A^*\varepsilon A^*\varepsilon$$
$$= A^*\varepsilon\varepsilon^*A$$
$$=$$

$$(a^*+b^*)(a+b) + (a^*-b^*)(a-b)$$

$$= a^*a + b^*a + a^*b + b^*b = 4.$$
$$a^*a - b^*a - a^*b + b^*b$$

If you assume

$$(i\varepsilon^*+A^*)\varepsilon = \varepsilon^*(i\varepsilon+A)$$
$$\parallel$$
$$A^*\varepsilon + i(4-A^*A) = \text{const} + \underbrace{A^*(\varepsilon-iA)}$$

$$\varepsilon^*A - iA^*A = (\varepsilon^*-iA^*)A$$