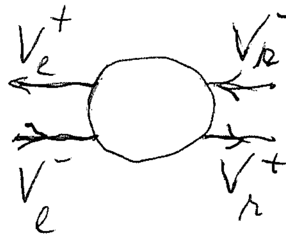
739 Feb 28 Examples

Go back to linear ones.



You want to construct a partial unitary by coupling periodically. You need to identify ~~with z~~ $V_r^- \simeq zV_\ell^+$ and $V_r^+ \simeq zV_\ell^-$

A state will consist then of a sequence

$(\xi_0, \xi_1, \ldots) \in V_\ell^+$?

$\ldots, \xi_0, \xi_1, \ldots$ of elements in V_ℓ^+

~~and~~ together with

$\ldots, \eta_0, \eta_1, \ldots$ of elements in V_ℓ^-

strictly you have $\begin{pmatrix} \sum z^n \xi_n & \in & \bigoplus_{n\geq 0} z^n V_\ell^+ \\ \sum z^n \eta_n & \in & \bigoplus V_\ell^- \end{pmatrix}$

What is the unitary operator? It arises from the coin. $V_\ell^- \oplus V_r^- \simeq V_\ell^+ \oplus V_r^+$ given by the port together with the given isos. $V_\ell^- \simeq V_r^+$, $V_\ell^+ \simeq V_r^-$ except you are ~~not~~ glueing to a translate

$H \quad \ldots z^{-1}V_\ell \oplus V_\ell \oplus zV_\ell \oplus \ldots$

$$V_\ell = V_\ell^+ \oplus V_\ell^- \longrightarrow V_r^+ = zV_\ell^-$$

$$Y = V^- \oplus X \simeq X \oplus V^+$$

Let's begin with a 2-port

$$Y = V^- \oplus bX = aX \oplus V^+$$

$$V^- = V_\ell^- \oplus V_r^- \qquad V^+ = V_\ell^+ \oplus V_r^+$$

form

$$\ell^2(\mathbb{Z}) \otimes Y = \ell^2(\mathbb{Z}) \otimes V^- \oplus \ell^2(\mathbb{Z}) \otimes bX$$
$$= \ell^2(\mathbb{Z}) \otimes V^+ \oplus \ell^2(\mathbb{Z}) \otimes aX$$

$\mathbb{Z}$ acts by translation

The unitary $bX \xrightarrow{\sim} aX$ yield a partial unitary on $\ell^2(\mathbb{Z}) \otimes Y$ commuting w. $\mathbb{Z}$ action. It remains to give unitary c.w. $\mathbb{Z}$ a. $\ell^2(\mathbb{Z}) \otimes V^- \longrightarrow \ell^2(\mathbb{Z}) \otimes V^+$.

It seems I should get straight where the unitary runs, if you are thinking of V^- as incoming and V^+ as outgoing. The ~~model examp~~ case to keep in mind is to couple to half shifts. So we get a unitary on

$$\bigoplus_{n<0} z^n V^- \oplus Y \oplus \bigoplus_{n \geq 0} z^n V^+$$

namely

$$\cdots \oplus z^{-2}V^- \oplus z^{-1}V^- \oplus \underbrace{V^- \oplus bX} \oplus zV^+ \oplus z^2V^+ \oplus \cdots$$
$$\cdots \oplus z^{-2}V^- \oplus z^{-1}V \oplus \overbrace{aX \oplus V^+} \oplus zV^+ \oplus z^2 V^+ \oplus \cdots$$

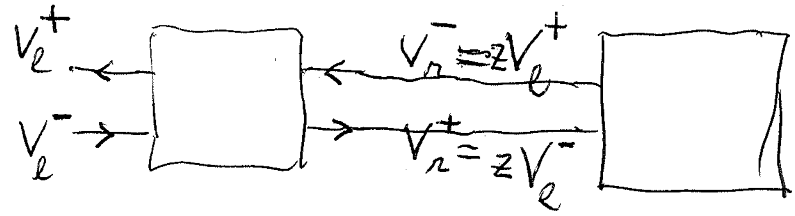
This picture doesn't work. Instead you want to write

$$\oplus\ z^{-1}V^- \oplus aX \oplus V^+ \oplus zV^+$$
$$\searrow \qquad \searrow \qquad \searrow \qquad \searrow \qquad \searrow$$
$$\oplus\ z^{-1}V^- \oplus V^- \oplus bX \oplus zV^+$$

741 Go back to $Y = V^- \oplus bX = aX \oplus V^+$

$V^- = V_\ell^- \oplus V_r^-$ $\qquad$ $V^+ = V_\ell^+ \oplus V_r^+$

$$\ell^2(\mathbb{Z}) \otimes Y = \ell^2(\mathbb{Z}) \otimes V^- \oplus \ell^2(\mathbb{Z}) \otimes bX$$
$$\uparrow 1 \otimes ba^{-1}$$
$$= \ell^2(\mathbb{Z}) \otimes V^+ \oplus \ell^2(\mathbb{Z}) \otimes aX$$



What's the best way to go about this?
You should write down a Hilbert space, namely

$$H = \ell^2(\mathbb{Z}) \otimes Y = (\ell^2(\mathbb{Z}) \otimes V_\ell^-) \oplus (\ell^2(\mathbb{Z}) \otimes V_r^-) \oplus \ell^2(\mathbb{Z}) \otimes bX$$
$$\downarrow z^{-1} \otimes 1 \qquad \downarrow z \otimes 1 \qquad \downarrow 1 \otimes ba^{-1}$$
$$= (\ell^2(\mathbb{Z}) \otimes V_r^+) \oplus (\ell^2(\mathbb{Z}) \otimes V_\ell^+) \oplus \ell^2(\mathbb{Z}) \otimes aX$$

Now you have a constructed a Hilbert space with "fundament domain" Y for the $\mathbb{Z}$ action. You have a $\mathcal{N}(\mathbb{Z})$-Hilbert module fin. gen. free, and a unitary auto of it, whence a unitary matrix over $\mathcal{N}(\mathbb{Z})$, a ~~bounded~~ measurable unitary matrix valued function.

Stop & prepare Tuesday talk.

First result is equiv. of cats. finite $\mathcal{N}(\Gamma)$-Hilbert modules ↔ f.g. generated $\mathcal{N}(\Gamma)$-modules.

2nd result ~~consider~~ $\mathcal{N}(\Gamma)$ is semi-hereditary f.g. submodule of a free module is projective

142. f.p. modules form an ab. category.

PID's. $T(M) \oplus P(M)$.

$\phi(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \end{cases}$

$a = n(\Gamma)$ $\quad a^m \underset{f^*}{\overset{f}{\rightleftarrows}} a^n \quad \phi(f^*f) = e$

$\mathrm{Ker}(f) = \mathrm{Ker}(e)$

$\ell^2(\Gamma)^m \xrightarrow{f} \ell^2(\Gamma)^n$ $\quad$ f bounded op comm. with Γ

$\mathrm{Ker}(f)$ closed Γ-inv. subspace of $\ell^2(\Gamma)^m$

$$\phi(f^*f) = \text{weak (strong?)} \lim_{n\to\infty} (f^*f)^{1/n} = e$$

$\mathrm{Ker}(f) = \mathrm{Ker}(e) = \mathrm{Im}(1-e)$

$\mathrm{Im}(f) = a^m/\mathrm{Ker}(e) = a^m e \oplus a^m(1-e)/ \cong \mathrm{Im}(e)$

So what is going on??

f.p. module

$0 \longrightarrow a^q e \longrightarrow a^p \longrightarrow M \longrightarrow 0$

$0 \longrightarrow \mathrm{Hom}_a(M,a) \longrightarrow a^p \longrightarrow e a^q \longrightarrow \mathrm{Ext}^1_a(M,a) \to 0$

f.g. proj!

$\check{P} = \mathrm{Hom}_a(M,a)$ f.g. proj.

$P = \mathrm{Hom}_a(\mathrm{Hom}_a(M,a),a) \longleftarrow M$

Back to periodic coupling of a 2 port

$$Y = V^- \oplus bX = aX \oplus V^+$$

write this way to describe coupling to a "trans." line

H: $z^{-2}V^- \oplus z^{-1}V^- \oplus aX \oplus V^+ \oplus zV^+ \oplus$

$\oplus z^{-1}V^- \oplus V^- \oplus bX \oplus zV^+ \oplus$

($u = ba^{-1}$)

eigenvector

$$\lambda^{-1}u(\cdots + z^{-2}v_{-2} + z^{-1}v_{-1} \oplus ax \oplus w_0 + zw_1 + \cdots)$$

$$= \lambda^{-1}z^{-1}v_{-2} + \lambda^{-1}v_{-1} + \lambda^{-1}bx + \lambda^{-1}zw_0 + \lambda^{-1}z^2w_1 + \cdots$$

$\lambda^{-1}v_{-2} = v_{-1}$ $\qquad$ $\lambda^{-1}w_0 = w_1$

$\lambda^{-1}v_{-3} = v_{-2}$ $\qquad$ $\lambda^{-1}w_1 = w_2$

$w_n = \lambda^{-n}w_0$
$v_{-n-1} = \lambda^n v_{-1}$

and then $\quad ax + w_0 = \lambda^{-1}bx + \lambda^{-1}v_{-1}$

$(a - \lambda^{-1}b)x = -w_0 + \lambda^{-1}v_{-1}$
$(\lambda a - b)x = -\lambda w_0 + v_{-1}$

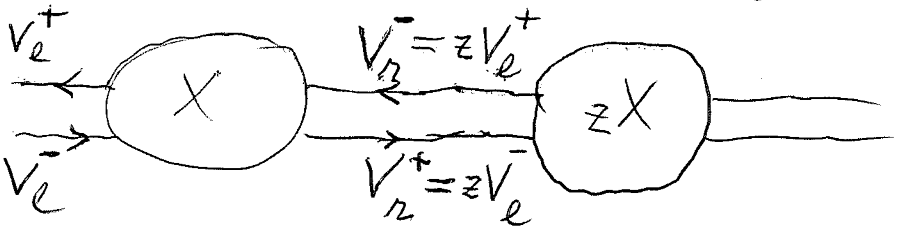
go to

$$Y = V^- \oplus bX = aX \oplus V^+$$

$$\ell^2(\mathbb{Z}) \otimes Y = \ell^2(\mathbb{Z}) \otimes V_\ell^- \oplus \ell^2(\mathbb{Z}) \otimes V_r^- \oplus \ell^2(\mathbb{Z}) \otimes bX$$

$\uparrow 1 \otimes z^{\ell}$ $\qquad$ $\downarrow 1 \otimes z^{r}$ $\qquad$ $\uparrow 1 \otimes ba^{-1}$

$$= \ell^2(\mathbb{Z}) \otimes V_r^+ \oplus \ell^2(\mathbb{Z}) \otimes V_\ell^+ \oplus \ell^2(\mathbb{Z}) \otimes aX$$



i4 You seem to be writing a unitary matrix over $\mathcal{R}(\mathbb{Z})$, which will become a measurable mod null sets unitary matrix function over the circle. Suppose you look for eigenvectors belonging to eigenvalue λ.

Suppose $X = 0$ to simplify. You

$$Y = V_\ell^- \oplus V_r^- = V_\ell^+ \oplus V_r^+$$

and isos. ~~Work in~~

Assume $Y = \mathbb{C}^2$ with $V_\ell^- = \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$ $V_r^- = \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$

$V_r^+ = \mathbb{C}\begin{pmatrix}a\\b\end{pmatrix}$ $V_r^+ = \mathbb{C}\begin{pmatrix}c\\d\end{pmatrix}$

Let $\xi \in L^2(S^1, Y) = L^2(S^1, V_\ell^-) \oplus L^2(S^1, V_r^-)$ be an eigenvector for u $\ni$ $u(\xi) = \lambda \xi$. $\xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$

$$\xi \in L^2(S^1, V_\ell^-) \oplus L^2(S^1 V_r^-)$$

$$\simeq \Big\downarrow \begin{pmatrix} z^{-1} & 0 \\ 0 & z\end{pmatrix}$$

$$L(S^1, V_r^+) \oplus L^2(S^1 V_\ell^+)$$

$$\Big\downarrow \begin{pmatrix} a & c \\ b & d\end{pmatrix}$$

$$L(S^1, V_\ell^-) \oplus L^2(S^1 V_r^-)$$

So
$$u\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix} az^{-1} & cz \\ bz^{-1} & dz \end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \stackrel{?}{=} \lambda \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

$$\lambda^2 - (az^{-1} + dz)\lambda + (ad - bc) = 0$$

essentially the same as before

I think what you need to make this all

745 much cleaner is the eigen

M f.g. $\mathcal{a}$-module $\qquad \mathrm{Hom}_{\mathcal{a}}(M, \mathcal{a}) = M^\vee$

$\mathcal{a}^p \to M \to 0 \quad \Longrightarrow \quad M^\vee \hookrightarrow (\mathcal{a}^p)^\vee = \mathcal{a}^p$

$\therefore$ $M^\vee$ f.g. proj (right)

So can find $M \xrightarrow{(f_i)} \mathcal{a}^s$ $\ni$ $M^\vee = \sum_i f_i \mathcal{a}$

K subm of $\mathcal{a}^n$

K submodule of $\mathcal{a}^n$

Let K be a submodule of $\mathcal{a}^n$

~~If K f.g. then K is f.g. proj.~~

$\bar{K}$ = ann of all $f: \mathcal{a}^n/K \longrightarrow \mathcal{a}$

$K \subset \mathcal{a}^n \rightsquigarrow$ there will be some Γ-subspace of $\ell^2(\Gamma)^n$. Form closure you get ~~a direct factor~~ an idemp on $\ell^2(\Gamma)^n$ whence a corresp. direct summand of

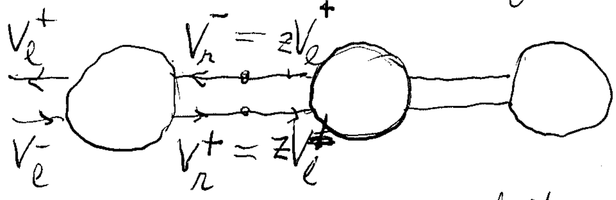
$$0 \longrightarrow P \longrightarrow \mathcal{a}^n \longrightarrow \text{Torsion} \longrightarrow 0$$
$$P \longrightarrow \mathcal{a}^n \times_T Q \longrightarrow Q \longrightarrow 0$$

Consider $0 \to P' \to P \to T \to 0$ $\qquad$ T torsion, P', P f.g. proj.

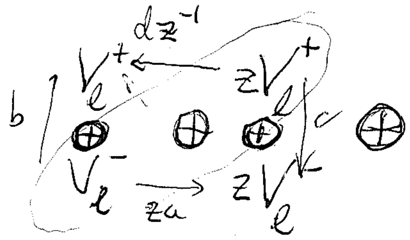
$\ell^2(\Gamma) \otimes_{\mathcal{a}} P' \to \ell^2(\Gamma) \otimes_{\mathcal{a}} P$

745a Mar 1, 98

Consider 2-port self ~~coupled~~ periodically



H =



$= \begin{matrix} L^2(S^1, V_\ell^+) \\ \oplus \\ L^2(S^1, V_\ell^-) \end{matrix}$

What is u on H? The port gives an isom.

$$V_\ell^- \oplus V_r^- \xleftarrow{\begin{pmatrix}1 & 0\\ 0 & z^{-1}\end{pmatrix}} V_\ell^- \oplus zV_\ell^+$$

$\begin{pmatrix} a & c \\ b & d\end{pmatrix}$ $\quad \wr\|$

$$V_r^+ \oplus V_\ell^+ \overset{\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}}{=} zV_\ell^- \oplus V_\ell^+$$

$$\begin{pmatrix} z & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & c \\ b & d\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-1}\end{pmatrix} = \begin{pmatrix} az & c \\ b & dz^{-1}\end{pmatrix}$$

~~Yesterday you did better, namely~~, start with

$$L^2\left(S^1, \begin{matrix} V_\ell^- \\ \oplus \\ V_\ell^+\end{matrix}\right) \supset \begin{pmatrix} V_\ell^- \\ \oplus \\ V_\ell^+\end{pmatrix} \xrightarrow{\begin{pmatrix} az & c \\ b & dz^{-1}\end{pmatrix}} L^2\left(S^1, \begin{matrix} V_\ell^- \\ \oplus \\ V_\ell^+\end{matrix}\right) \qquad \begin{pmatrix} az & cz^{-1} \\ bz & dz^{-1}\end{pmatrix}$$

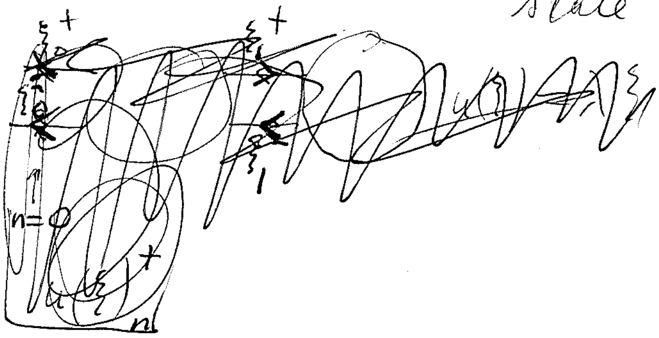
Alt. use $\begin{pmatrix} V_\ell^- \\ V_r^- \end{pmatrix} = \begin{pmatrix} V_\ell^- \\ zV_\ell^+ \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 0 \\ 0 & z\end{pmatrix}\begin{pmatrix} az & c \\ b & dz^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-1}\end{pmatrix} =$

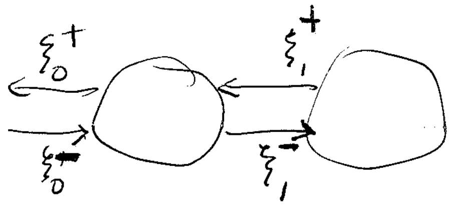
This is clear, but you should concentrate on the eigen~~vector~~ equation. An eigenfunction is a

$$f(z) = \begin{pmatrix} f^-(z) \\ f^+(z)\end{pmatrix} \in L^2\left(S^1, \begin{matrix} V_\ell^- \\ V_\ell^+\end{matrix}\right)$$ such that

$uf = \lambda f.$ $\qquad u\begin{pmatrix} f^- \\ f^+\end{pmatrix} = \begin{matrix} azf^+ \\ bf^- \end{matrix}$

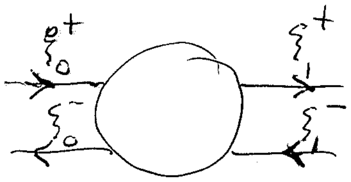
I think I want to use vectors
Try for a language, words, to describe states. State is $\xi = \left(\xi_n^{\pm}\right)$





$$\begin{pmatrix}\xi_0^+\\ \xi_1\end{pmatrix} = \begin{pmatrix} d & b\\ c & a\end{pmatrix}\begin{pmatrix}\xi_1^+\\ \xi_0^-\end{pmatrix}$$

Begin again with a 2 port



$$\begin{pmatrix}\xi_n^+\\ \xi_{n+1}^-\end{pmatrix} = \underbrace{\begin{pmatrix} a & c\\ b & d\end{pmatrix}}_{\in U(2)}\begin{pmatrix}\xi_{n+1}^+\\ \xi_n^-\end{pmatrix}$$

$$z^n\xi_n^+ = z^{-1}a z^{n+1}\xi_{n+1}^+ + c z^n\xi_n^-$$

$\xi(z)$ generating function for $\begin{pmatrix}\xi_n^+\\ \xi_n^-\end{pmatrix}$

$$\xi(z)^+ = z^{-1}a\,\xi(z)^+ + c\,\xi(z)^-$$

$$z^{n+1}\xi_{n+1}^- = b\, z^{n+1}\xi_{n+1}^+ + dz z^n\xi_n^-$$

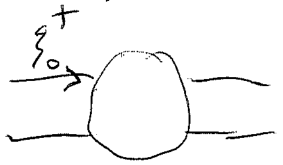
$$\xi(z)^- = b\,\xi(z)^+ + dz\,\xi(z)^-$$

$$\xi = \begin{pmatrix} az^{-1} & c\\ b & dz\end{pmatrix}\xi$$

747 Given a 2-port, you ~~need to construct~~ get a Hilbert space with unitary operator by self coupling periodically. Also it's translation invariant, so the unitary operator lies over $n(\mathbb{Z})$.

Describe the Hilbert space and unitary operator

Let Y be the state space of the 2 port. Y is a hermitian space equipped with distinguished unit vectors:

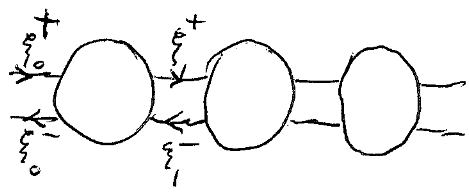


Forget about X for the moment. Basically you have

$$\underset{in}{\mathbb{C}^2} \xrightarrow[\begin{pmatrix} a & b \\ c & d \end{pmatrix}]{\sim} \underset{out}{\mathbb{C}^2} \quad \text{unitary}$$

Why not be very elementary. An element of H is a.

Start again with



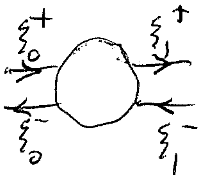
$$\text{state} = \begin{pmatrix} \begin{pmatrix} \xi_n^+ \end{pmatrix}_{n \in \mathbb{Z}} \\ \begin{pmatrix} \xi_n^- \end{pmatrix}_{n \in \mathbb{Z}} \end{pmatrix}.$$

a port gives us [redacted] a unitary correspondence between in and out states. ~~Between in and out states~~

correspondence = subspace of the product. So I am iterating this correspondence. [redacted] Is there a link between iterating a correspondence and the $\mathbb{P}^1$-things you do with a correspondence?

[redacted]

Start again with a 2-port (freq. indep)



This gives a unitary correspondence between in and out states

$$\begin{pmatrix} \xi_0^+ \\ \xi_1^- \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in U(2)} \begin{pmatrix} \xi_1^+ \\ \xi_0^- \end{pmatrix}$$

$$|\xi_0^+|^2 + |\xi_1^-|^2 = |\xi_1^+|^2 + |\xi_0^-|^2$$

When $\exists$ transmission: $|a| > 0$, then

$$|\xi_0^+|^2 - |\xi_0^-|^2 = |\xi_1^+|^2 - |\xi_1^-|^2$$

$$\begin{pmatrix} \xi_0^+ \\ \xi_0^- \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in U(1,1)} \begin{pmatrix} \xi_1^+ \\ \xi_1^- \end{pmatrix}$$

$$\xi_1^- = c\,\xi_1^+ + d\,\xi_0^- \qquad \xi_0^- = d^{-1}\xi_1^- - d^{-1}c\,\xi_1^+$$

$$\xi_0^+ = a\,\xi_1^+ + b\left(-d^{-1}c\,\xi_1^+ + d^{-1}\xi_1^-\right) \qquad = -d^{-1}c\,\xi_1^+ + d^{-1}\xi_1^-$$

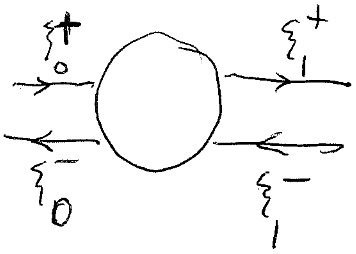
$$= (a - bd^{-1}c)\,\xi_1^+ + bd^{-1}\xi_1^-$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a - bd^{-1}c & bd^{-1} \\ -d^{-1}c & d^{-1} \end{pmatrix}$$

unit. case $\quad c = -\bar{b}, \; d = \bar{a}$

$$= \begin{pmatrix} a + \frac{|b|^2}{\bar{a}} & \frac{b}{\bar{a}} \\ + \frac{\bar{b}}{\bar{a}} & \frac{1}{\bar{a}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\bar{a}} & \frac{b}{\bar{a}} \\ \frac{\bar{b}}{\bar{a}} & \frac{1}{\bar{a}} \end{pmatrix}$$

749



$$\begin{pmatrix} \xi_0^+ \\ \xi_1^- \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in U(2)} \begin{pmatrix} \xi_1^+ \\ \xi_0^- \end{pmatrix}$$

You have a unitary correspondence between in and out states, pseudo-unitary correspondence between left and right states.

To iterate you bring in an identification of left & right states. Please try carefully to put things together.

Suppose

$$V^{in} \oplus bX = aX \oplus V^{out}$$

First thm. Hilbert $\mathcal{N}(\Gamma)$-modules $\quad \in \ell^2(\Gamma)^m$

$$\ell^2(\Gamma)^n \underset{f^*}{\overset{f}{\rightleftarrows}} \ell^2(\Gamma)^m$$

$$\begin{pmatrix} 0 & f^* \\ f & 0 \end{pmatrix} \simeq \Big($$

Go back to your 2 port. General case

$$Y = V^{in} \oplus bX = aX \oplus V^{out}$$

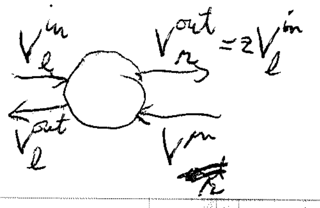
$aX \oplus V^{out}$
$\quad\|$
$V^{in} \oplus bX$

form Hilbert space

$$H = L^2(S^1, Y) = L^2(S^1, V^{in}_l) \oplus L^2(S^1, V^{in}_r) \oplus L^2(S^1, bX)$$

$$\uparrow z \qquad \uparrow z^{-1} \qquad \uparrow ba^{-1}$$

$$= L^2(S^1, V^{out}_r) \oplus L^2(S^1, V^{out}_l) \oplus L^2(S^1, aX)$$



$$V^{out}_r = zV^{in}_l \qquad V^{out}_l = z^{-1}V^{in}_r$$

750 Basically you have a partial unitary which you propose to complete to obtain a unitary. Need suitable notation. Suitable matrix notation.

back to $\mathcal{N}(\Gamma)$.

f.g. Hilbert $\mathcal{N}(\Gamma)$-module = closed invariant subspace of $\ell^2(\Gamma)^{\oplus n}$. $e\,\ell^2(\Gamma)^n$ $\quad e\in M_n(\mathcal{A})$

$e=e^*,\ e^2=e.$

maps given by bounded Γ-equiv. operators.

$$e\,\ell^2(\Gamma)^n \underset{f^*}{\overset{f}{\rightleftarrows}} e'\,\ell^2(\Gamma^{n'})$$

$$\begin{pmatrix} 0 & f^* \\ f & 0 \end{pmatrix} = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}\begin{pmatrix} (f^*f)^{1/2} & 0 \\ 0 & (ff^*)^{1/2} \end{pmatrix}$$

assertions.

semi-hereditary: any f.g submodule of a f.g. proj module is projective.

$\Rightarrow$ f.p. modules form an abelian cat. (closed under ext.

$$\begin{array}{ccccc} \text{fp} & & \text{fp} & & \\ M_1 & \to & M_2 & \to & C \to 0 \end{array}$$

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & F'' & & F'' & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & M' & \to & M \times_{M''} A^n & \to & A^n & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

$K_0(\text{f.p. modules}) \xleftarrow{\sim} K_0(\text{f.g. proj})$

751

$$A^n \twoheadrightarrow M$$

M f.g. $\qquad \mathrm{Hom}_A(M, A) \hookrightarrow A^n$

$$0 \to eA^{n'} \longrightarrow A^n \to M \to 0$$

$$0 \to \mathrm{Hom}_A(M, A) \longrightarrow A^n \longrightarrow eA^{n'}$$

f.g. proj.

$$M \longrightarrow A^n$$

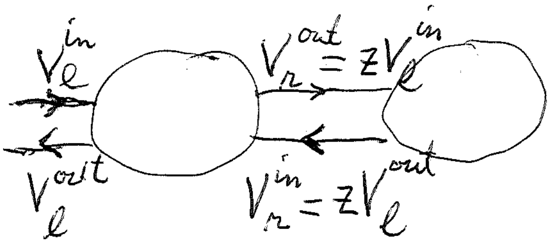
Go back to

$$Y = V^{in} \oplus bX = aX \oplus V^{out}$$

$$L^2(S^1, Y) = L^2(S^1, V^{in}_\ell) \oplus L^2(S^1, V^{in}_r) \oplus L^2(S^1, bX)$$

$$\uparrow z \qquad\qquad \uparrow z^{-1} \qquad\qquad \uparrow ba^{-1}$$

$$= L^2(S^1, V^{out}_r) \oplus L^2(S^1, V^{out}_\ell) \oplus L^2(S^1, aX)$$



This is not clean enough yet to write down a unitary operator. So what

Maybe you should introduce basis elements.

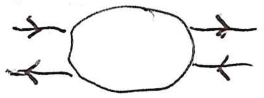
$$u(x_j) = ba^{-1}(x_j)$$

$$u(\xi^{out}_r) = z\xi^{in}_\ell$$

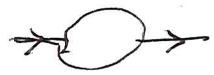
$$u(\xi^{in}_\ell) = ?$$

752 March 2, 98

Consider a 2 port no. X. | $Y = V^- \oplus bX = aX \oplus V^+$ partial unitary as ba^{-1}



my aim is ~~write down~~ find the ~~eigenvalue~~ eigenvector equation

example 1. couple  to trans. lines.

Hilbert space is

$$\oplus\, z^{-2}V^- \oplus z^{-1}V^- \oplus \overbrace{aX \oplus V^+}^{Y} \oplus zV^+ \oplus \cdots = H$$

ba^{-1}

$$\oplus\, z^{-2}V^- \oplus z^{-1}V^- \oplus V^- \oplus bX \oplus zV^+ \oplus \cdots$$

so a state is $\left(\sum_{n<0} z^{+n} v^-_{+n},\ x,\ \sum_{n\geq 0} z^n v_n^+ \right)$

$$\sum_{n<0} z^n v_n^- + y + \sum_{n>0} z^n v_n^+$$

where $y = ax + v_0^+ = v_0^- + bx$

$$u\left(\sum_{n<0} z^n v_n^- + \overset{ax + v_0^+}{\cancel{v_0^- + bx}} + \sum_{n>0} z^n v_n^+ \right)$$

$$= \sum_{n<0} z^n u(v_n^-) + bx + \text{[scribbled out]} + \sum_{n\geq 0} z^n v_n^+$$

$$= \sum z^n \lambda v_n^-$$

753

$$\xi = z^{-2}v^-_{-2} + z^{-1}v^-_{-1} + ax + v^+_0 + z v^+_1 + z^2 v^+_2$$

λ, $u \downarrow$

$$+ z^{-1}v^-_{-2} + v^-_{-1} + bx + z v^+_0 + z^2 v^+_1$$

$u(\xi) = \lambda(\xi)$ says

$$v^-_{-2} = \lambda v^-_{-1} \qquad \boxed{\lambda(ax + v^+_0) = v^-_{-1} + bx} \qquad v^+_0 = \lambda v^+_1$$

$$v^-_{-3} = \lambda v^-_{-2} \qquad\qquad v^+_1 = \lambda v^+_2$$

$$v^-_{-n} = \lambda^{n-1} v^-_{-1} \qquad\qquad v^+_n = \lambda^{-n} v^+_0$$

$$\boxed{(\lambda a - b)x = v^-_{-1} - \lambda v^+_0}$$

$$\begin{array}{ccccccc}
 & & V^+ = \mathrm{Ker}(a^*) & & & & \\
 & & \downarrow & & & & \\
X & \xrightarrow{\lambda a - b} & Y & \longrightarrow & E_\lambda & \longrightarrow & 0 \\
 & \searrow^{\lambda - a^* b} & \downarrow a^* & & & & \\
 & & X & & & &
\end{array}$$

$\lambda - a^*b$ isom for $|\lambda| \geq 1$

so $Y = (\lambda a - b)X \oplus \overbrace{\mathrm{Ker}(a^*)}^{V^+}$

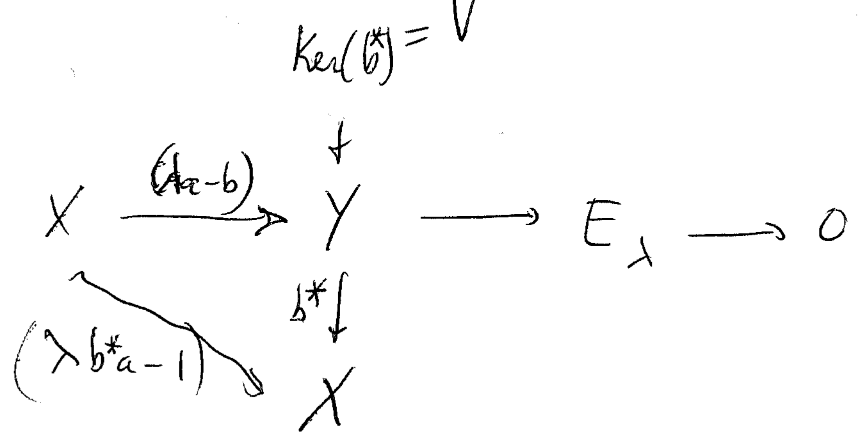
$= (\lambda a - b)X \oplus \underbrace{\mathrm{Ker}(b^*)}_{V^-}$

$$y - \underbrace{(\lambda a - b)(\lambda - a^*b)^{-1}a^*}\, y$$

projects on $(\lambda a - b)X$ kills $\mathrm{Ker}(a^*)$

$$1 - (\lambda a - b)(\lambda - a^*b)^{-1}a^* = 1 - (\lambda a - b)\,a^*(\lambda - ba^*)^{-1}$$

$$= \left[\lambda - ba^* - \lambda aa^* + ba^*\right](\lambda - ba^*)^{-1} = (1 - aa^*)(1 - \lambda^{-1}ba^*)^{-1}$$



$$y - \underbrace{(\lambda a - b)(\lambda b^* a - 1)^{-1} b^*}\, y$$

Kells $\mathrm{Ker}(b^*)$ proj onto $(\lambda a - b)X$

$$1 - (\lambda a - b) b^* (\lambda a b^* - 1)^{-1}$$

$$= \left[(\lambda a b^* - 1) - \lambda a b^* + b b^* \right] (\lambda a b^* - 1)^{-1}$$

$$= (1 - b b^*)(1 - \lambda a b^*)^{-1}$$

$$(\lambda a - b) x = \overbrace{v_{-1}^-}^{\mathrm{Ker}(b^*)} - \lambda \overbrace{v_0^+}^{\mathrm{Ker}(a^*)}$$

$$\underbrace{a^* (\lambda a - b)}_{\lambda - a^* b} x = a^* v_{-1}^-$$

$$x = (\lambda - a^* b)^{-1} a^* v_{-1}^- = a^* (\lambda - b a^*)^{-1} v_{-1}^-$$

$$(\lambda a - b) x = (\lambda a a^* - b a^*)(\lambda - b a^*)^{-1} v_{-1}^-$$

$$\| \quad v_{-1}^- - \lambda v_0^+$$

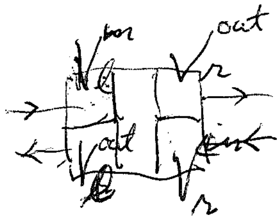
$$v_0^+ = \left(1 - (\lambda a a^* - b a^*)\right)(\lambda - b a^*)^{-1} v_{-1}^-$$

755 $\lambda v_0^+ = [1 - (\lambda a a^* - b a^*)](\lambda - b a^*)^{-1} v_{-1}^-$

$= [\lambda - ba^* - \lambda a a^* + ba^*]$

$$v_0^+ = (1 - aa^*)(\lambda - ba^*)^{-1} v_{-1}^-$$

coupling periodically.



$Y = V^{in} \oplus bX = aX \oplus V^{out}$

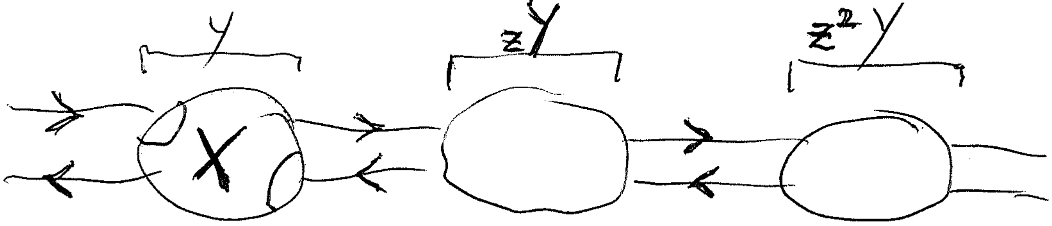
$V^{in} = V_l^{in} \oplus V_r^{in}$ $\qquad$ $V^{out} = V_l^{out} \oplus V_r^{out}$

$H = L^2(S^1, Y) = L^2(S^1, V_l^{in}) \oplus L^2(S^1, V_r^{in}) \oplus L^2(S^1, bX)$

$\qquad\qquad\quad \uparrow z \qquad\qquad z^{-1}\uparrow \qquad\qquad \uparrow ba^{-1}$

$\qquad = L^2(S^1, V_r^{out}) \oplus L^2(S^1, V_l^{out}) \oplus L^2(S^1, aX)$.

think: u should map $aX \longrightarrow bX$ and



$V_l^{in} \oplus bX \oplus V_r^{out}$

Go back and understand a 1-port

$Y = V^{-} \oplus bX = aX \oplus V^{+}$ (in, out)

basic nature of a

Study coupling 2-ports.

$Y_1 = {}'V_l^{in} \oplus {}'V_r^{in} \oplus b'X = a'X \oplus {}'V_l^{out} \oplus {}'V_r^{out}$

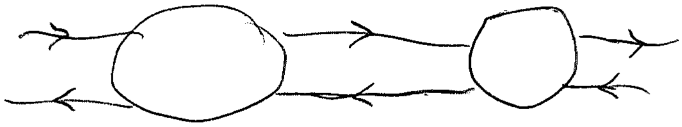
$Y_2 = {}''V_l^{in} \oplus {}''V_r^{in} \oplus b''X = a''X \oplus {}''V_l^{out} \oplus {}''V_r^{out}$

Take $\oplus$, then somehow identify

$'V_r^{in}$ with $''V_l^{out}$

$'V_r^{out}$ with $''V_l^{in}$

Start again. Connect two 2-ports.



$$'Y = 'V_l^{in} \oplus 'V_r^{in} \oplus 'X \simeq 'X \oplus 'V_l^{out} \oplus 'V_r^{out}$$

$$''Y = ''V_l^{in} \oplus ''V_r^{in} \oplus ''X \simeq ''X \oplus ''V_l^{out} \oplus ''V_r^{out}$$

Another viewpoint. Take stable isomorphism

$$'V^{in} \oplus 'X \simeq 'X \oplus 'V^{out}$$

$$''V^{in} \oplus ''X \simeq ''X \oplus ''V^{out}$$

take direct sum and ~~complete~~ extend the ~~iso~~. partial unitary.

$$\begin{matrix} 'V_l^{in} \oplus 'V_r^{in} \oplus 'X \\ ''V_l^{in} \oplus ''V_r^{in} \oplus ''X \end{matrix} \simeq \begin{matrix} 'X \oplus 'V_l^{out} \oplus 'V_r^{out} \\ \oplus \\ ''X \oplus ''V_l^{out} \oplus ''V_r^{out} \end{matrix}$$

notice that the new thing has $X = 'X \oplus ''X \oplus 'V_r^{in} \oplus ''V_l^{in}$

$\simeq$ —— $''V_l^{out} \oplus 'V_r^{out}$

757 Go over preceding. Given first 2-port

$$\boxed{{}^1V_l^{in}} \oplus {}^1X \oplus {}^1V_r^{in} \simeq \boxed{{}^1V_l^{out}} \oplus {}^1X \oplus {}^1V_r^{out}$$

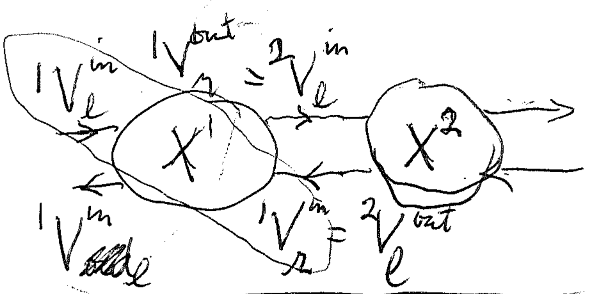
and second

$${}^2V_l^{in} \oplus {}^2X \oplus \boxed{{}^2V_r^{in}} \simeq {}^2V_l^{out} \oplus {}^2X \oplus \boxed{{}^2V_r^{out}}$$

form the direct sum.

$${}^1V_l^{in} \oplus \left({}^1X \oplus {}^1V_r^{in} \oplus {}^2V_l^{in} \oplus {}^2X\right) \oplus {}^2V_r^{in}$$

$$\simeq {}^1V_l^{out} \oplus \left({}^1X \oplus {}^1V_r^{out} \oplus {}^2V_l^{out} \oplus {}^2X\right) \oplus {}^2V_r^{out}$$

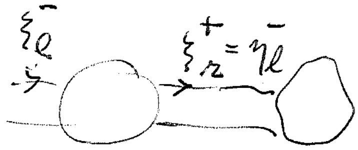


March 3, 1998 Do the above discussion without X's. and primes

$${}^1Y = {}^1V_l^{in} \oplus {}^1V_r^{in} \simeq {}^1V_l^{out} \oplus {}^1V_r^{out}$$

$$\oplus \qquad\qquad \oplus$$

$${}^2Y = {}^2V_l^{in} \oplus {}^2V_r^{in} \simeq {}^2V_l^{out} \oplus {}^2V_r^{out}$$



maybe you should try variables

suppose the port described by 4 variables $\xi_{l\ r}^{-\ +}$

satisfying $|\xi_l^-|^2 + |\xi_r^-|^2 = |\xi_l^+|^2 + |\xi_r^+|^2$

a state of port is $\xi \in \mathbb{C}^4$ $\rightarrow$ $\begin{pmatrix} \xi_r^+ \\ \xi_l^+ \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi_l^- \\ \xi_r^- \end{pmatrix}$

Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525

UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

758 Try coupling these. You get 6 variables

$\xi_{e}^{-}\ \xi_{r}^{+} \qquad \eta_{e}^{-}\ \eta_{r}^{+} \qquad \xi_r^+ = \eta_e^-$

$\xi_r^- = \eta_e^+$

8 variables subject to 6 relations.

torsion mod. M

$$0 \longrightarrow P \longrightarrow \mathcal{A}^n \longrightarrow M \longrightarrow 0$$

$$\ell^2(\Gamma)^n$$

$$\mathcal{A}^m \underset{f^*}{\overset{f}{\rightleftarrows}} \mathcal{A}^n$$

First look at Hilb. space

$$H_0 \underset{f^*}{\overset{f}{\rightleftarrows}} H_1$$

assume f, f^* injective

i.e f injective and $\overline{\mathrm{Im}(f)} = H_1$

$$f = \underbrace{f(f^*f)^{-1/2}}(f^*f)^{+1/2}$$

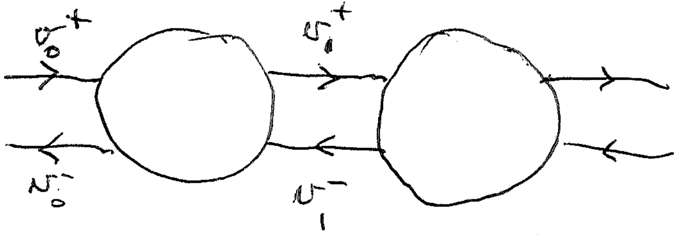
unitary isom ~~[illegible]~~ from H_0 to H_1

$$a = \begin{pmatrix} 0 & f^* \\ f & 0 \end{pmatrix}$$ s.a. on $H_0 \oplus H_1$, anti commutes with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$a = \frac{a}{|a|}|a| \qquad |a| = \begin{pmatrix} (f^*f)^{1/2} & 0 \\ 0 & (ff^*)^{1/2} \end{pmatrix}$$

$\frac{x}{\sqrt{x^2}}$

759



$$H = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \ni \begin{pmatrix} (v_n^+) \\ (v_n^-) \end{pmatrix}$$

$$u(v_n^+) = \alpha v_{n+1}^+ + \beta v_n^-$$
$$u(v_n^-) = \gamma v_n^+ + \delta v_{n-1}^-$$

$$v_z^\pm = \sum_{n\in\mathbb{Z}} z^n v_n^\pm$$

$$u\begin{pmatrix} v_z^+ \\ v_z^- \end{pmatrix} = \begin{pmatrix} \alpha z^{-1} & \beta \\ \gamma & \delta z \end{pmatrix}\begin{pmatrix} v_z^+ \\ v_z^- \end{pmatrix}$$

Something ~~is~~ is nice about this because it leads directly to the desired result. You get directly a unitary operator. You have a state

$$v = \begin{pmatrix} (v_n^+) \\ (v_n^-) \end{pmatrix} \quad \text{and} \quad u(v)_n^+ = \alpha v_{n+1}^+ + \beta v_n^-$$
$$u(v)_n^- =$$

Maybe I have to get the basis vector straight from the general state. Let $\delta_n^\pm$ be the basis vectors. Then

$$u(\delta_n^+) = a\delta_{n+1}^+ + b\delta_n^-$$
$$u(\delta_{n+1}^-) = c\delta_{n+1}^+ + d\delta_n^-$$

Take [redacted] a state [redacted] $\psi = \sum_n \left(\psi_n^+\delta_n^+ + \psi_n^-\delta_n^-\right)$

$$u(\psi) = \sum_n \psi_n^+\left(a\delta_{n+1}^+ + b\delta_n^-\right) + \psi_n^-\left(c\delta_n^+ + d\delta_{n-1}^-\right)$$

$$= \sum_n \left(\psi_{n-1}^+ a + \psi_n^- c\right)\delta_n^+ + \left(\psi_n^+ b + \psi_{n+1}^- d\right)\delta_n^-$$

$$(u\psi)_n^+ = a\psi_{n-1}^+ + c\psi_n^-$$
$$(u\psi)_n^- = b\psi_n^+ + d\psi_{n+1}^-$$

$$\begin{pmatrix} (u\psi)_z^+ \\ (u\psi)_z^- \end{pmatrix} = \begin{pmatrix} az & c \\ b & dz^{-1} \end{pmatrix}\begin{pmatrix} \psi_z^+ \\ \psi_z^- \end{pmatrix}$$

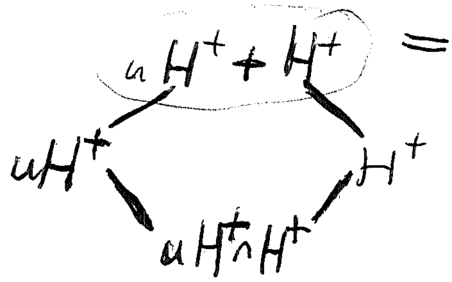
760 Let's make an attempt to understand response from a partial unitary again, specifically the half line partial unitary you've constructed. Use your base $\delta_n^{\pm}$ for $n \geq 0$ with

$$\begin{cases} u(\delta_n^+) = a\delta_{n+1}^+ + b\delta_n^- \\ u(\delta_n^-) = c\delta_n^+ + d\delta_{n-1}^- \end{cases}$$

but $u(\delta_0^-)$ is not defined. u is a unitary on H and we have H^+, we look at uH^+ H^+ and can intersect and sum. $uH^+ \cap H^+$

$$uH^+ + H^+ = H^+ + u(\delta_0^-) = H^+ + c\delta_0^-$$

assumes trans.



What is uH^+?

There should be a partial unitary defined on

$$u^{-1}H^+ \cap H^+ = \{\psi \in H^+ \mid u\psi \in H^+\}.$$

take $\psi = \psi_0^+\delta_0^+ + \psi_1^+\delta_1^+ + \cdots$
$\psi_0^-\delta_0^- +$

$$u(\delta_0^-) = c\delta_0^+ + d\delta_{-1}^-$$

$$u\psi \in \psi_0^- d\delta_{-1}^- + H^+$$

so the domain is $\psi = \sum_{n \geq 0} \psi_n^+\delta_n^+ + \psi_n^-\delta_n^-$ such that $\psi_0^- = 0$.

Maybe you should look at eigenvectors for u. Then you want a ψ such that $u\psi = \lambda\psi$

Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525
Mathematical Institute
24-29 St Giles
Oxford OX1 3LB
Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
UNIVERSITY OF OXFORD

761

$$(u\psi)^+_n = a\psi^+_{n-1} + c\psi^-_n = \lambda\psi^+_n$$
$$(u\psi)^-_n = b\psi^+_n + d\psi^-_{n+1} = \lambda\psi^-_n$$

again you do F.T.

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \qquad \begin{pmatrix} az & c \\ b & dz^{-1} \end{pmatrix}\begin{pmatrix} \psi^+_z \\ \psi^-_z \end{pmatrix} = \lambda \begin{pmatrix} \psi^+_z \\ \psi^-_z \end{pmatrix}$$

$$\lambda^2 - (az + dz^{-1})\lambda + (ad - bc) = 0.$$

$$\lambda^2 - (az + \bar{a}z^{-1})\lambda + 1 = 0$$

$$\lambda = \frac{az + \bar{a}z^{-1} \pm \sqrt{(az+\bar{a}z^{-1})^2 - 4}}{2}$$

$$\begin{aligned}(az+\bar{a}z^{-1})^2 - 4 &= a^2z^2 + 2|a|^2 + \bar{a}^2z^{-2} - 4 \\ &= a^2z^2 - 2|a|^2 + \bar{a}^2z^{-2} - 4 + 4|a|^2 \\ &= (az - \bar{a}z^{-1})^2 - 4(1-|a|^2)\end{aligned}$$

Try to set up recursion. What is your idea? ~~You know~~ Let ψ satisfy $u\psi = \lambda\psi$, want ~~that~~ to use translation invariance. You expect the space of eigenfunctions ~~to be nothe~~ with eigenvalue λ to be 1-dim'l, and if so then ψ should be an eigenfunction for translation. This will give some simple equations which are probably what you already have.

Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525
UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

762 Check this.

$$a\psi^+_{n-1} + c\psi^-_n = \lambda\psi^+_n$$
$$b\psi^+_n + d\psi^-_{n+1} = \lambda\psi^-_n$$

now put ~~is~~ $\psi^\pm_{n-1} = z\psi^\pm_n$ and you get

$$\begin{pmatrix} az & c \\ b & dz^{-1} \end{pmatrix}\begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix} = \lambda \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix} \qquad \forall n.$$

which is the same equation as above. What is the response function – something like ψ^-_0/ψ^+_0

$$\lambda - (az + \bar{a}z^{-1}) + \lambda^{-1} = 0$$

$$\boxed{\lambda + \lambda^{-1} = az + \bar{a}z^{-1}} \qquad a = \bar{a}$$

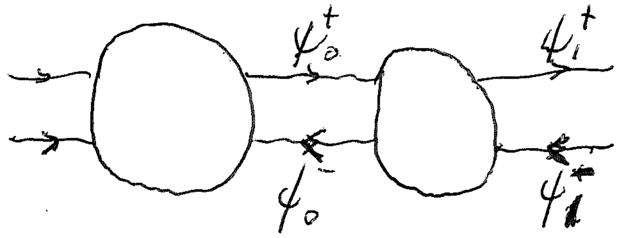
$$z\lambda^2 - (az^2 + \bar{a})\lambda + z = 0 \qquad \text{plane cubic curve.}$$

$$\boxed{\lambda + \lambda^{-1} = a(z + z^{-1})}$$

Can you show

March 4, 1998 ~~[illegible]~~ to Review

$$\psi = \sum \psi^+_n \delta^+_n + \psi^-_n \delta^-_n$$



$$u(\delta^+_0) = a\delta^+_1 + b\delta^-_0$$
$$u(\delta^-_1) = c\delta^+_1 + d\delta^-_0$$

Professor D. Quillen, FRS
Wayneflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525

UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

763

$$\psi = \sum \psi_n^+ \delta_n^+ + \psi_n^- \delta_n^-$$

$$u(\delta_n^+) = a\delta_{n+1}^+ + b\delta_n^-$$

$$u(\delta_{n+1}^-) = c\delta_{n+1}^+ + d\delta_n^-$$

$$u(\psi) = \sum \psi_n^+ (a\delta_{n+1}^+ + b\delta_n^-) + \psi_{n+1}^- (c\delta_{n+1}^+ + d\delta_n^-)$$

$$= \sum (\psi_n^+ a + \psi_{n+1}^- c)\delta_{n+1}^+ + (\psi_n^+ b + \psi_{n+1}^- d)\delta_n^-$$

$$(u\psi)_{n+1}^+ = \boxed{\psi_n^+ a + \psi_{n+1}^- c = \lambda \psi_{n+1}^+}$$

$$(u\psi)_n^- = \boxed{\psi_n^+ b + \psi_{n+1}^- d = \lambda \psi_n^-}$$

$$\begin{pmatrix} \psi_z^+ & \psi_z^- \end{pmatrix} \begin{pmatrix} za & c \\ b & z^{-1}d \end{pmatrix} = \lambda \begin{pmatrix} \psi_z^+ & \psi_z^- \end{pmatrix}$$

$$\begin{pmatrix} za & b \\ c & z^{-1}d \end{pmatrix} \begin{pmatrix} \psi_z^+ \\ \psi_z^- \end{pmatrix} = \lambda \begin{pmatrix} \psi_z^+ \\ \psi_z^- \end{pmatrix}$$

but if you assume ψ is a translation eigenfunction $\psi_{n-1}^{\pm} = \xi \psi_n^{\pm}$, then get

$$a\xi \psi_n^+ + c\psi_n^- = \lambda \psi_n^+$$

$$b\psi_n^+ + d\xi^{-1}\psi_n^- = \lambda \psi_n^-$$

$$\begin{pmatrix} a\xi & c \\ b & d\xi^{-1} \end{pmatrix} \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix} = \lambda \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix}$$

What you want to do now is to discuss the eigenvector equation $u(\psi) = \lambda\psi$:

$$a\psi_n^+ + c\psi_{n+1}^- = \lambda\psi_{n+1}^+$$
$$b\psi_n^+ + d\psi_{n+1}^- = \lambda\psi_n^-$$

~~Compare ratios ψ_n^+/ψ_n^-~~ Suppose

$$S = \frac{\psi_n^-}{\psi_n^+} = \frac{\psi_{n+1}^-}{\psi_{n+1}^+}$$

$$a\psi_n^+ + cS\psi_{n+1}^+ = \lambda\psi_{n+1}^+$$
$$b\psi_n^+ + dS\psi_{n+1}^+ = \lambda S\psi_n^+$$

$$a\psi_n^+ = (\lambda - cS)\psi_{n+1}^+$$

~~[illegible]~~

$$(\lambda S - b)\psi_n^+ = dS\psi_{n+1}^+$$

$$\frac{a}{\lambda S - b} = \frac{\lambda - cS}{dS} = \frac{\lambda S^{-1} - c}{d}$$

$$adS = \lambda^2 S - cS^2 - b\lambda + bcS$$

$$ad = (\lambda S - b)(\lambda S^{-1} - c)$$

$$-bc + ad = \lambda^2 - b\lambda S^{-1} - c\lambda S$$
$$= \lambda^2 - (bS^{-1} + cS)\lambda$$

765 Take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$

$$\lambda^2 - (bS^{-1} - \bar{b}S)\lambda - 1 = 0$$

Recalculate $S\psi^+_{n+1}$

$$a\psi^+_n + c\psi^-_{n+1} = \lambda\psi^+_{n+1}$$

$$b\psi^+_n + d\psi^-_{n+1} = \lambda\psi^-_n$$

$$S\psi^+_{n+1} \qquad S\psi^+_n$$

$$a\psi^+_n = (\lambda - cS)\psi^+_{n+1}$$

$$(b - \lambda S)\psi^+_n + dS\psi_{n+1} = 0.$$

$$(\lambda S - b)\psi^+_n = dS\psi^+_{n+1}$$

$$\frac{\lambda S - b}{a} = \frac{dS}{\lambda - cS}$$

$$(\lambda S - b)(\lambda - cS) = adS$$

$$\lambda^2 S - b\lambda - c\lambda S^2 + bcS = adS$$

$$-\lambda^2 S + b\lambda + c\lambda S^2 + S = 0$$

~~$(b\lambda)S^2 + (1-\lambda^2)S + b\lambda$~~

$$(c\lambda)S^2 + (1 - \lambda^2)S + b\lambda = 0$$

$$S = \frac{-(1-\lambda^2) \pm \sqrt{(1-\lambda^2)^2 - 4bc\lambda^2}}{2c\lambda}$$

$$\lambda^2 - cS\lambda - b\lambda S^{-1} = ad - bc = 1$$

$$c\lambda S + b\lambda S^{-1} + 1 - \lambda^2 = 0$$

$$S + \frac{b}{c}S^{-1} + \frac{1-\lambda^2}{c\lambda}$$

$$S^2 + 2\left(\frac{1-\lambda^2}{2c\lambda}\right)S + \frac{b}{c} = 0$$

766

$$(\lambda S - b)(1 - cS) = adS$$

$$-\lambda^2 S + b\lambda + c\lambda S^2 + S = 0$$

$$c\lambda S^2 + (1-\lambda^2) S + b\lambda = 0$$

$$S^2 + 2\left(\frac{1-\lambda^2}{2c\lambda}\right) S + \frac{b}{c} = 0$$

abs. val. = 1.

$$S = -\left(\frac{1-\lambda^2}{2c\lambda}\right) \pm \sqrt{\left(\frac{1-\lambda^2}{2c\lambda}\right)^2 - \frac{b}{c}}$$

So the two roots have product of abs. value = 1. So either both are on the unit circle or one is in and the other is out.

Let's to reconcile this calculation with the previous calculation of simultaneous eigenvectors for translation and u.

$$a\psi_n^+ + c\psi_{n+1}^- = \lambda\psi_{n+1}^+ \qquad \psi_{n-1}^\pm = z\psi_n^\pm$$

$$b\psi_n^+ + d\psi_{n+1}^- = \lambda\psi_n^-$$

$$a\psi_n^+ + cz^{-1}\psi_n^- = \lambda z^{-1}\psi_n^+$$

$$b\psi_n^+ + dz^{-1}\psi_n^- = \lambda\psi_n^-$$

$$\boxed{\begin{aligned} az\psi_n^+ + c\psi_n^- &= \lambda\psi_n^+ \\ b\psi_n^+ + dz^{-1}\psi_n^- &= \lambda\psi_n^- \end{aligned}}$$

$$S = \frac{\psi_n^-}{\psi_n^+}$$

767

$$\begin{cases} az + cS = \lambda \\ b + dz^{-1}S = \lambda S \end{cases} \qquad bS^{-1} + dz^{-1} = \lambda$$

leads to $(\lambda - cS)(\lambda - bS^{-1}) = (az)(dz^{-1}) = ad$

So its the same as above

You would like to know that S is unitary when $|\lambda| = 1$. Suppose $b = c$ is real.

Then ~~$S = i\alpha \pm \sqrt{1-\alpha^2}$~~ $\qquad \alpha = \dfrac{\lambda^2 - 1}{i2c\lambda} = \dfrac{1}{i2c}(\lambda - \lambda^{-1})$

$|\lambda| = 1 \implies \dfrac{\lambda - \lambda^{-1}}{2i}$ is real

Suppose $b = -c \in \mathbb{R}$ so $|b| = |c| < 1$.

$$S = \alpha \pm \sqrt{\alpha^2 + 1} \qquad \alpha = \frac{\lambda - \lambda^{-1}}{2c}$$

for $|\lambda| = 1$, $\alpha \in i\mathbb{R}$. Tak

Next try for $\mathbb{Z}/2$ symmetry. Go back to

$$u(\delta_n^+) = a\delta_{n+1}^+ + b\delta_n^-$$
$$u(\delta_{n+1}^-) = -\bar{b}\delta_{n+1}^+ + \bar{a}\delta_n^- \qquad |a|^2 + |b|^2 = 1$$

You want reflection to be a symmetry. First ask for $\delta_0^+ \mapsto \delta_1^-$, $\delta_0^- \mapsto \delta_1^+$ to commute with u

$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^- \qquad u(\delta_1^-) = a\delta_0^- + b\delta_1^+$$
$$u(\delta_1^-) = -\bar{b}\delta_1^+ + \bar{a}\delta_0^- \qquad u(\delta_0^+) = -\bar{b}\delta_0^- + \bar{a}\delta_1^+$$

this implies $a = \bar{a}$, $b = -\bar{b}$ i.e. a real, b purely imaginary. Next let

$\delta_n^+ \longmapsto \delta_{-n}^-$ $\qquad u(\delta_{-n}^-) = a\delta_{-n-1}^- + b\delta_{-n}^+$

$\delta_n^- \longmapsto \delta_{-n}^+$ $\qquad u(\delta_{-n-1}^+) = -\bar{b}\delta_{-n-1}^- + \bar{a}\delta_{-n}^+$

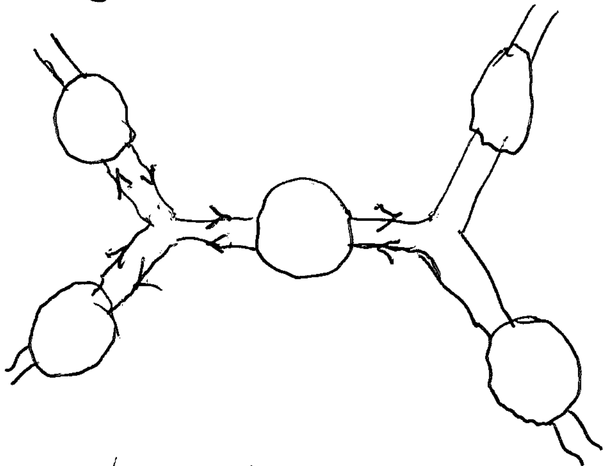
Put $k = -n-1$.

$$u(\delta_k^+) = -\bar{b}\delta_k^+ + \bar{a}\delta_{k+1}^- = \bar{a}\delta_{k+1}^- - \bar{b}\delta_k^+$$

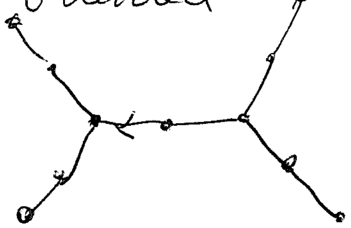
$$u(\delta_{k+1}^+) = a\delta_k^+ + b\delta_{k+1}^- = b\delta_{k+1}^- + a\delta_k^+$$

So it's OKAY.

Now try the tree for $\Gamma = PSL_2(\mathbb{Z})$.



You need to write down your Hilb space. There should be a basis for each connection, i.e. for each oriented oriented edge in the tree

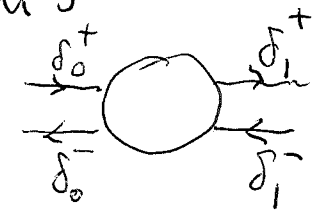


a Γ simplex in this tree has an obvious orientation -

It will take much concentration to get this straight.

769 March 5

Review



$u(\delta_0^+) = a\delta_1^+ + b\delta_0^-$

$u(\delta_1^-) = c\delta_1^+ + d\delta_0^-$

behavior of basis vectors

$$u(\psi_0^+\delta_0^+ + \psi_1^-\delta_1^-) = \underbrace{(\psi_0^+ \ \psi_1^-)\begin{pmatrix}a & b\\ c & d\end{pmatrix}}\begin{pmatrix}\delta_1^+\\ \delta_0^-\end{pmatrix}$$

$$(\psi_0^+ \ \psi_1^-)u =$$

reflection $$\begin{pmatrix}\delta_0^+\\ \delta_1^-\end{pmatrix} \begin{matrix}\longmapsto\\ \longmapsto\end{matrix} \begin{pmatrix}\delta_1^-\\ \delta_0^+\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\delta_0^+\\ \delta_1^-\end{pmatrix}$$

$$u\begin{pmatrix}\delta_0^+\\ \delta_1^-\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\delta_1^+\\ \delta_0^-\end{pmatrix} \qquad \therefore \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}d & c\\ b & a\end{pmatrix}$$

If $\det = 1$. Then $\begin{pmatrix}\bar a & \bar c\\ \bar b & \bar d\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$ $\qquad d = \bar a$, $\bar b = -c$

What is the relation between $\dfrac{\psi_0^-}{\psi_0^+}$ and $\dfrac{\psi_1^-}{\psi_1^+}$?

Although this is probably unnecessary, you might concentrate better if you worked more on the response of a partial unitary, to get the theory cleaner in your mind. Consider then $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ where X, Y are Hilbert spaces, a & b isometries $a^*a = b^*b = 1$ on X, assume $(aX)^\perp$ & $(bX)^\perp$ have the same dimension, say $\dim = 1$. Then choose a unitary iso $(aX)^\perp \simeq (bX)^\perp$ you get a unitary operator u on Y given by ba^{-1} on aX and this unitary on $(aX)^\perp$. Conversely given Y, u and a closed subspace W, you get a partial unitary $W \underset{b = u|_W}{\overset{a = \text{inc.}}{\rightrightarrows}} Y$. What is next?

770

Review response

$$\mathrm{Ker}(b^*) \to Y$$

$$X \xrightarrow{a\lambda - b} Y \xrightarrow{b^*} X, \qquad b^* a\lambda - 1 : X \to X$$

$$(a\lambda - b), 1 : X \oplus \mathrm{Ker}(b^*) \xrightarrow{\sim} X \quad \text{for } |\lambda| < 1.$$

$$\left((a\lambda - b), 1\right) : X \oplus \mathrm{Ker}(a^*) \xrightarrow{\sim} Y \qquad |\lambda| > 1.$$

What you should do is to extend the partial unitary to a unitary u and then work with $(\lambda - u)^{-1}$. [redacted]

Start with (H, u) and a cyclic vector ξ_0 of norm **1**. $X = (\mathbb{C}\xi_0)^\perp$

$$H = \mathbb{C}\xi_0 \oplus X = \mathbb{C}u(\xi_0) \oplus uX$$

Go over the eigenvector equation where a port is connected to incoming & outgoing transmission lines.

$$z^{-1}V^- \oplus aX \oplus V^+ \oplus zV^+$$

$$z^{-1}V^- \oplus V^- \oplus bX \oplus zV^+$$

$$\lambda\xi = \cdots + z^{-1}\lambda v_{-1}^- + a\lambda x + \lambda v_0^+ + z\lambda v_1^+ + \cdot$$

$$u(\xi) = \qquad v_{-1}^- + bx + zv_0^+$$

$$a\lambda x + \lambda v_0^+ = v_{-1}^- + bx$$

$$\boxed{(a\lambda - b)x = -\lambda v_0^+ + v_{-1}^-}$$

$$v_0^+ \in \mathrm{Ker}(a^*), \qquad v_{-1}^- \in \mathrm{Ker}(b^*)$$

Try to figure out what you want.

a = inclusion of $X = (\mathbb{C}\xi_0)^\perp$ in Y

b = restriction of u to X.

$$(\lambda - u)\,x = -\lambda \overset{v_0^+}{\xi_0} + \overset{v_{-1}^-}{\overbrace{\tfrac{\alpha}{\lambda}\, u(\xi_0)}} \qquad \alpha \in \mathbb{C}.$$

Go back to your original calculation of the scattering, namely, the operator which takes $v_{-1}^- \in V^- = \mathrm{Ker}(b^*)$, solves the eigenvector equation for x and v_0^+, and sends v_{-1}^- to v_0^+.

$$(\lambda - a^*b)\,x = a^*(a\lambda - b)\,x$$
$$= a^*(-\lambda v_0^+ + v_{-1}^-) \qquad \text{NO}$$

$$b^*(a\lambda - b)\,x = b^*(-\lambda v_0^+ + v_{-1}^-)$$

$$(+\lambda b^* a - 1)\,x = -\lambda\, b^* v_0^+$$

$$x = (1 - \lambda b^* a)^{-1} \lambda\, b^* v_0^+$$
$$= b^*(1 - \lambda a b^*)^{-1} \lambda v_0^+$$

$$\underset{-\lambda v_0^+ + v_{-1}^-}{(a\lambda - b)\,x} = (a\lambda - b)\, b^*(1 - \lambda a b^*)^{-1} \lambda v_0^+ \quad ?$$

Start again

$$(a\lambda - b)\,x = \underset{\text{killed by } a^*}{-\lambda v_0^+} + \underset{\text{killed by } b^*}{v_{-1}^-}$$

$$(\lambda - a^* b)\,x = a^* v_{-1}^-$$

772

$$x = (\lambda - a^* b)^{-1} a^* v_{-1}^-$$
$$= a^* (\lambda - b a^*)^{-1} v_{-1}^-$$

$$-\lambda v_0^+ + v_{-1}^- = (a\lambda - b) x = (a\lambda - b) a^* (\lambda - b a^*)^{-1} v_{-1}^-$$

$$\lambda v_0^+ = \left(1 - (a\lambda - b) a^* (\lambda - b a^*)^{-1}\right) v_{-1}^-$$

$$= (\underbrace{\lambda - ba^* - (a\lambda - b) a^*}_{\lambda(1 - aa^*)}) (\lambda - b a^*)^{-1} v_{-1}^-$$

$$\boxed{v_0^+ = (1 - aa^*) \underbrace{(\lambda - b a^*)^{-1}}_{\text{defined \& analytic for } |\lambda| > 1.} v_{-1}^-}$$

In good cases can hope to analytically continue to $|\lambda| = 1$.

Where am I? You have a port

$$aX \oplus \operatorname{Ker}(a^*) = \operatorname{Ker}(b^*) \oplus bX$$

with $\operatorname{Ker}(a^*) = V^+$, $\operatorname{Ker}(b^*) = V^-$,

$$\lambda a x + \lambda v_0^+ = v_{-1}^- + bx$$

eigenvector ~~eigenvalue~~ equation is ~~$(a\lambda - b)x = -\lambda v_0^+ + v_{-1}^-$~~

$$(a\lambda - b) x = \underbrace{-\lambda v_0^+}_{\text{killed by } a^*} + \underbrace{v_{-1}^-}_{\text{killed by } b^*}$$

$$(b^* a \lambda - 1) x = -\lambda b^* v_0^+$$

$$x = (1 - \lambda b^* a)^{-1} \lambda b^* v_0^+ = \lambda b^* (1 - \lambda a b^*)^{-1} v_0^+$$

$$(a\lambda - b) x = (a\lambda - b) \lambda b^* (1 - \lambda a b^*)^{-1} v_0^+$$
$$= -\lambda v_0^+ + v_{-1}^-$$

$$v_{-1}^{-} = \lambda\left(1 + (a\lambda - b)\, b^*(1-\lambda a b^*)^{-1}\right) v_0^{+}$$

$$= \lambda\left(1 - \lambda a b^* + (a\lambda - b)\, b^*\right)(1-\lambda a b^*)^{-1} v_0^{+}$$

$$\boxed{v_{-1}^{-} = \lambda(1-bb^*)(1-\lambda a b^*)^{-1} v_0^{+}}$$

defined and analytic for $|\lambda|<1$.

~~Now go b~~

$$|\lambda|\left(\|x\|^2 + \|v_0^{+}\|^2\right) = \|v_{-1}^{-}\|^2 + \|x\|^2$$

~~to understand the unit~~

What to do next. You are missing a link between $(\lambda - ba^*)^{-1}$ and $\frac{1}{\lambda - u}$.

The determinant. Given the partial unitary, say finite dimensional, consider the function which associates to each extension of it to a unitary the characteristic poly.

Suppose given u and the unit vector ξ_1, and remaining orth basis vectors $\xi_2, \dots, \xi_n$. Then the partial unitary given by rest. u to $\xi_1^{\perp} = \mathbb{C}\xi_2 \oplus \cdots \oplus \mathbb{C}\xi_n$ is given by ~~consists of~~ the columns 2nd–nth of the matrix for u

774 and possible extension of this partial unitary are given by

$$\begin{pmatrix} \zeta u_{11} & u_{12} & \cdots & u_{1n} \\ \zeta u_{12} & \vdots & & \vdots \\ \vdots & & & \\ \zeta u_{n1} & u_{n2} & \cdots & u_{nn} \end{pmatrix}$$

$$= u \begin{pmatrix} \zeta & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = u \underbrace{\left(\zeta \, pr_{\mathbb{C}\xi_1} \oplus pr_{(\mathbb{C}\xi_1)^\perp} \right)}_{\Theta_\zeta}$$

Char poly is $\det(\lambda - u\Theta_\zeta) = \det(\lambda\Theta_{\zeta^{-1}} - u)\,\zeta$

What might you look for? Vary ζ

$$d \log \det(\lambda - u\Theta_\zeta) = tr\left\{ \frac{1}{\lambda - u\Theta_\zeta} d(\lambda - u\Theta_\zeta) \right\}$$

$$= -tr\left\{ \frac{1}{\lambda - u\Theta_\zeta} u \, d\zeta \, pr_{\mathbb{C}\xi_1} \right\}$$

$$= (-d\zeta) \underbrace{\left\langle \xi_1, \frac{1}{\lambda - u\Theta_\zeta} u \xi_1 \right\rangle}$$

$$\left\langle \xi_1, u \frac{1}{\lambda - \Theta_\zeta u} \xi_1 \right\rangle$$

$$= \frac{\langle \xi_1, u\xi_1 \rangle}{\lambda} + \frac{\langle \xi_1, u\Theta_\zeta u \xi_1 \rangle}{\lambda^2} + $$

So basically you need info about $\frac{1}{\lambda - u\Theta_\zeta}$

rank 1 pert.

$$u\Theta_\zeta = u \, pr_{\mathbb{C}\xi_1^\perp} + u\zeta \, pr_{\mathbb{C}\xi_1}$$ YES

775 What is $u\, pr_{(\mathbb{C}\xi_1^\perp)}$

$$\underbrace{u\, pr_{(\mathbb{C}\xi_1^\perp)}}_{aa^*}$$

$$u\Theta_\zeta = u\, pr_{\mathbb{C}\xi_1^\perp} + u\zeta\, pr_{\mathbb{C}\xi_1}$$

$$= u\left(aa^* + \zeta(1-aa^*)\right)$$

$$= ba^* + \zeta u(1-aa^*)$$

$$\frac{1}{\lambda - u\Theta_\zeta} = \frac{1}{\lambda - ba^* - \zeta \underbrace{u(1-aa^*)}_{u\xi_1\langle\xi_1|}}$$

$$\left\langle \xi_1, \left(\frac{1}{\lambda-ba^*} + \frac{1}{\lambda-ba^*}\zeta u(\xi_1)\langle\xi_1|\frac{1}{\lambda-ba^*} + \cdots\right)u(\xi_1)\right\rangle$$

$$\left\langle \xi_1, \frac{1}{\lambda-ba^*}u(\xi_1)\right\rangle + \zeta\left\langle \xi_1, \frac{1}{\lambda-ba^*}u(\xi_1)\right\rangle\left\langle \xi_1, \frac{1}{\lambda-ba^*}u(\xi_1)\right\rangle$$

looks like a geometric series

$$t + \zeta t^2 + \cdots = \frac{t}{1-\zeta t}$$

where $t = \left\langle \xi_1, \frac{1}{\lambda-ba^*}u(\xi_1)\right\rangle$

Start again

Let u be a unitary op on Y

let ξ_1 be a unit vector, $\xi_1, \xi_2, \ldots, \xi_n$ orth basis

$aX = (\mathbb{C}\xi_1)^\perp$ $\qquad bX = u(aX)$

$$\Theta_\zeta = \zeta \, \overline{pr_{\mathbb{C}\xi_1}} + pr_{(\mathbb{C}\xi_1)^\perp}$$

(annotations: aa^* above, $a\chi$ below)

$$\Theta_\zeta = \zeta(1-aa^*) + aa^*$$

$$u\Theta_\zeta = uaa^* + \zeta u(1-aa^*)$$

$$= ba^* + \zeta u(\xi_1)\xi_1^*$$

$$(\lambda - u\Theta_\zeta)^{-1} = (\lambda - ba^* - \zeta u(\xi_1)\xi_1^*)^{-1} = (G_0 - V)^{-1}$$

$$= \sum_{n \geq 0} (G_0 V)^n G_0$$

$$d \log \det(\lambda - u\Theta_\zeta) = d \operatorname{tr} \log(\lambda - u\Theta_\zeta)$$

$$= \operatorname{tr}\{(\lambda - u\Theta_\zeta)^{-1}(-u \, d\Theta_\zeta)\}$$

$$= (-d\zeta) \operatorname{tr}\{(\lambda - u\Theta_\zeta)^{-1} u(\xi_1)\xi_1^*\}$$

$$-\frac{d}{d\zeta} \log\det(\lambda - u\Theta_\zeta) = \langle \xi_1, (\lambda - u\Theta_\zeta)^{-1} u(\xi_1) \rangle$$

$$= \left\langle \xi_1, \sum_{n\geq 0} \left(\frac{1}{\lambda - ba^*} \zeta u(\xi_1)\xi_1^* \right)^n \frac{1}{\lambda - ba^*} u(\xi_1) \right\rangle$$

$$= \left\langle \xi_1, \frac{1}{\lambda - ba^*} u(\xi_1) \right\rangle$$

$$+ \left\langle \xi_1, \frac{1}{\lambda - ba^*} \zeta u(\xi_1) \right\rangle \left\langle \xi_1, \frac{1}{\lambda - ba^*} u(\xi_1) \right\rangle$$

777 Key number

$\mathbb{C}\xi_1 = (aX)^\perp$

$\mathbb{C}u(\xi_1) = u(aX)^\perp$
$= (bX)^\perp$

$\langle \xi_1, \frac{1}{\lambda - ba^*} u(\xi_1) \rangle$

$\mathbb{C}\xi_1 = (1-aa^*)Y$

$(1-aa^*)\frac{1}{\lambda - ba^*}(1-bb^*)V_{-1}^{\sim}$

So your have the scattering operator up to a scalar of modulus 1. I'm puzzled because I seem to have something like

$$-\zeta \frac{d}{d\zeta} \log \det(\lambda - u\theta_\zeta) = \frac{\zeta S}{1-\zeta S} = \frac{\zeta}{S^{-1}-\zeta}$$

This is roughly what I want, namely a link between the spectrum of u and $\lambda \ni S(\lambda)$ = a fixed point on S^1.

Try again. Given Y basis $\xi_1, \dots, \xi_n$

$aX = (\mathbb{C}\xi_1)^\perp$, u unitary operator on Y, $bX = uaX$.

$\theta_\zeta = \zeta(1-aa^*) + aa^* = \begin{matrix} 1 \text{ on } aX \\ \zeta \text{ on } (aX)^\perp = \mathbb{C}\xi_1 \end{matrix}$

$$d\left(\log\det(\lambda - u\theta_\zeta)\right) = \operatorname{tr} \frac{1}{\lambda - u\theta_\zeta} d\lambda$$

$u\theta_\zeta = \zeta u(1-aa^*) + ba^*$

$(\lambda - u\theta_\zeta)^{-1} = (\lambda - ba^* - \zeta u(\xi)\xi^*)^{-1}$

$= G_0 + G_0 V G_0 + \dots$ where $G_0 = (\lambda - ba^*)^{-1}$

$V = \zeta u(\xi)\xi^*$

778 $$G_0 V = \frac{1}{\lambda - ba^*} \int u(\xi)\, \xi^*$$

look carefully $\lambda - u\Theta_\zeta = \lambda - ba^* - u\zeta(\xi_1 \otimes \xi_1^*)$

$$\Theta_\zeta = \underbrace{\zeta(1-aa^*)}_{\zeta \text{ on } (aX)^\perp} + \underbrace{aa^*}_{1 \text{ on } aX} = \zeta\, \xi_1 \otimes \xi_1^* + aa^*$$

$(aX)^\perp = \mathbb{C}\xi_1$

$$u\Theta_\zeta = u\zeta(\xi_1 \otimes \xi_1^*) + ba^*$$

Ask when $\det(\lambda - ba^* - \underbrace{u\zeta(\xi_1 \otimes \xi_1^*)}_{\zeta\, u(\xi_1)\otimes \xi_1^*}) = 0$

These zeroes are roughly same as poles of

$$\operatorname{tr} \frac{1}{\lambda - ba^* - \zeta u(\xi_1) \otimes \xi_1^*}$$

$$= \operatorname{tr}\left(\frac{1}{\lambda - ba^*} + \frac{1}{\lambda - ba^*} \zeta u(\xi_1) \otimes \xi_1 \frac{1}{\lambda - ba^*} + \cdots \right)$$

Yes it seems like you want to differentiate wrt ζ.

$$\operatorname{tr} \frac{1}{\lambda - ba^* - \zeta u(\xi_1)\otimes \xi_1^*} (-u(\xi_1) \otimes \xi_1^*)$$

evaluate at $\zeta = 1$.

$$\operatorname{tr} \frac{1}{\lambda - ba^*} u(\xi_1) \otimes \xi_1^* = \left\langle \xi_1, \frac{1}{\lambda - ba^*} u(\xi_1) \right\rangle$$

$$\operatorname{tr} \left(\quad \right)^2 = \left\langle \xi_1, \frac{1}{\lambda - ba^*} u(\xi_1) ~~\otimes \xi_1^*~~ \frac{1}{\lambda - ba^*} u(\xi_1) \right\rangle$$

Now $\xi_1 \in \operatorname{Ker}(a^*) = \operatorname{Im}(1 - aa^*)$. Remember that

$$\underbrace{v_0^+}_{\in \operatorname{Ker}(a^*) = \mathbb{C}\xi_1} = (1 - aa^*)(\lambda - ba^*)^{-1} \underbrace{v_{-1}^-}_{\in \operatorname{Ker}(b^*) = \mathbb{C}u(\xi)}$$ gives the scattering

779 ~~this says~~ Lot

$$t = \left\langle \xi_1, (1-aa^*) \frac{1}{\lambda - ba^*} u(\xi_1) \right\rangle$$

Then $S_\lambda(u(\xi_1)) = t\,\xi_1$

$$-\operatorname{tr}\left(\frac{1}{\lambda - u} u(\xi_1) \otimes \xi_1^*\right) = t + t^2 + \cdots = \frac{t}{1-t}$$

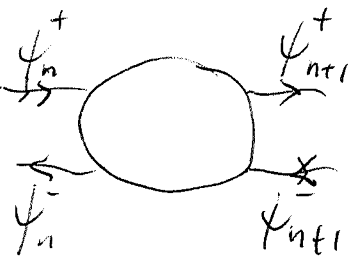
If you can analytically continue to $|\lambda| = 1$ this says that u has eigenv. $\lambda \iff t$ has eigenv. 1.

$$\operatorname{tr}\left(\frac{1}{\lambda - u} u(\xi_1) \otimes \xi_1^*\right) = \left\langle \xi_1, \frac{1}{\lambda - u} u(\xi_1) \right\rangle$$

$$\|$$

$$\frac{t}{1-t} \quad \text{where} \quad t = \left\langle \xi_1, \frac{1}{\lambda - ba^*} u(\xi_1) \right\rangle$$

not very clear although there is some logic to it.



assume $\psi = \sum \psi_n^+ \delta_n^+ + \psi_n^- \delta_n^-$

set $u(\psi) = \lambda \psi$

then
$$\lambda \psi_{n+1}^+ = a\psi_n^+ + c\psi_{n+1}^-$$
and
$$\lambda \psi_n^- = b\psi_n^+ + d\psi_{n+1}^-$$

suppose $\psi_n^- = S\psi_n^+$

$$\lambda \psi_{n+1}^+ = a\psi_n^+ + cS\psi_{n+1}^+$$
$$\lambda S \psi_n^+ = b\psi_n^+ + dS\psi_{n+1}^+$$

$$(\lambda - cS)\psi_{n+1}^+ = a\psi_n^+$$
$$(\lambda S - b)\psi_n^+ = dS\psi_{n+1}^+$$
$$\frac{\lambda - cS}{a} = \frac{\psi_n^+}{\psi_{n+1}^+} = \frac{dS}{\lambda S - b}$$

$$(\lambda - cS)(\lambda S - b) = adS$$
$$\lambda^2 S - c\lambda S^2 - b\lambda + (bc - ad)S = 0 \quad (bc-ad = -1)$$
$$c\lambda S^2 + (1 - \lambda^2)S + b\lambda = 0$$

780

$$a\psi_n^+ = \lambda\psi_{n+1}^+ - c\,\psi_{n+1}^-$$

$$\psi_n^+ = \frac{\lambda}{a}\psi_{n+1}^+ - \frac{c}{a}\psi_{n+1}^-$$

$$\lambda\psi_n^- = b\left(\frac{\lambda}{a}\psi_{n+1}^+ - \frac{c}{a}\psi_{n+1}^-\right) + d\,\psi_{n+1}^-$$

$$= \lambda\frac{b}{a}\psi_{n+1}^+ + \left(d - \frac{bc}{a}\right)\psi_{n+1}^-$$

$$\psi_n^- = \frac{b}{a}\psi_{n+1}^+ + \frac{ad-bc}{\lambda a}\psi_{n+1}^-$$

$$\begin{pmatrix}\psi_n^+\\ \psi_n^-\end{pmatrix} = \begin{pmatrix}\frac{\lambda}{a} & -\frac{c}{a}\\ \frac{b}{a} & \frac{1}{\lambda a}\end{pmatrix}\begin{pmatrix}\psi_{n+1}^+\\ \psi_{n+1}^-\end{pmatrix}$$

$$\begin{pmatrix}\psi_n^-\\ \psi_m^+\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda a} & \frac{b}{a}\\ -\frac{c}{a} & \frac{\lambda}{a}\end{pmatrix}\begin{pmatrix}\psi_{n+1}^-\\ \psi_{n+1}^+\end{pmatrix}$$

$$S = \frac{\lambda^{-1}S + b}{-cS + \lambda}$$

$$-cS^2 + \lambda S = \lambda^{-1}S - \bar{c}$$

$$-cS^2 + (+\lambda - \lambda^{-1})S + \bar{c} = 0$$

$$cS^2 + (\lambda^{-1} - \lambda)S - \bar{c} = 0$$

$\lambda = e^{-i\theta}$

$$S^2 + 2\left(\frac{\lambda^{-1}-\lambda}{2ict}\right)S + 1 = 0$$

$$S^2 + 2t^{-1}\sin\theta\, S + 1 = 0$$

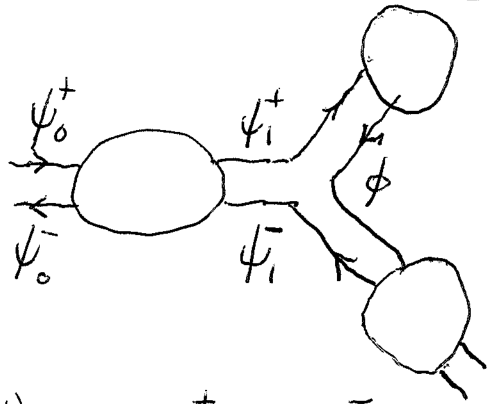
So for $|\cos\theta| \leq |t|$ $|S| = 1$ and for $|\sin\theta| \geq |t|$ ~~~~ two roots are real, ~~~~ product is 1, $S(\lambda)$ is smaller root.

if $c \in i\mathbb{R}$
$|t| < 1$

781 March 6, 1998

Can we construct the eigenvector as a series?

Let's go on to $\Gamma = PSL_2(\mathbb{Z})$. power series in the reflection coeff.



$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^-$$
$$u(\delta_1^-) = -\bar{b}\delta_1^+ + \bar{a}\delta_0^-$$

$u($

[illegible]

Take $u(\psi) = \lambda\psi$

$$\lambda\psi_1^+ = u(\psi)_1^+ = a\psi_0^+ - \bar{b}\psi_1^-$$
$$\lambda\psi_0^- = u(\psi)_0^- = b\psi_0^+ + \bar{a}\psi_1^-$$

Assume $\phi = S\psi_1^+ \qquad \psi_1^- = S\phi \qquad \Rightarrow \boxed{\psi_1^- = S^2\psi_1^+}$

also $\psi_0^- = S\psi_0^+$

$$\lambda\psi_1^+ = a\psi_0^+ - \bar{b}S^2\psi_1^+ \qquad (\lambda + \bar{b}S^2)\psi_1^+ = a\psi_0^+$$
$$\lambda S\psi_0^+ = b\psi_0^+ + \bar{a}S^2\psi_1^+ \qquad (\lambda S - b)\psi_0^+ = \bar{a}S^2\psi_1^+$$

$$\frac{\lambda + \bar{b}S^2}{\bar{a}S^2} = \frac{a}{\lambda S - b}$$

$$\lambda^2 S - b\lambda + \bar{b}\lambda S^3 - |b|^2 S^2 = |a|^2 S^2$$

$$\boxed{\bar{b}\lambda S^3 - S^2 + \lambda^2 S - b\lambda = 0}$$

But $b = it \qquad |t| < 1.$

$$it\lambda S^3 - S^2 + \lambda^2 S + it\lambda = 0$$

$$\boxed{it S^3 - \lambda^{-1}S^2 + \lambda S + it = 0}$$

This should define S as a power series in λ or λ^{-1}.

go back to $\quad cS^2 + (\lambda^{-1}-\lambda)S + \bar{c} = 0$

$c = +it \quad itS^2 + (\lambda^{-1}-\lambda)S + it = 0$

$$S^2 + 2\left(\frac{\lambda^{-1}-\lambda}{2it}\right)S + 1 = 0$$

$$S = -\left(\frac{\lambda^{-1}-\lambda}{2it}\right) \pm \sqrt{\left(\frac{\lambda^{-1}-\lambda}{2it}\right)^2 - 1}$$

I want to find a power series in λ or λ^{-1}.

As $\lambda \to 0$

$$it(1+S^2)\lambda + (1-\lambda^2)S = 0$$

$$S = -\frac{\lambda it}{1-\lambda^2}(1+S^2)$$

You should be able to solve this by iteration to get a power series in λ such that $S(0)=0$.

$$\lambda^{-1}S = -\frac{it}{1-\lambda^2}\left(1+\lambda^2(\lambda^{-1}S)^2\right)$$

$$u = -\frac{it}{1-\lambda^2}(1+\lambda^2 u^2)$$

leads to a power series in λ^2

$$it\, S^3 - \lambda^{-1} S^2 + \lambda S + it = 0$$

seems to be quartic 4th degree plane curve. symmetry

$$S \mapsto \omega S \qquad \lambda \mapsto \omega^{-1}\lambda \qquad \omega^3 = 1$$

$$it(\lambda S)^3 - (\lambda S)^2 + \lambda^3(\lambda S) + it\lambda^3 = 0$$

$$it\, u^3 - u^2 + \lambda^3 u + it\lambda^3 = 0$$

so u should be a function of λ^3.

$$it\, u^3 - u^2 + z u + itz = 0$$

definitely cubic equation, in fact, seems rational.

---

go back to the first case.

$$\begin{pmatrix} \psi_n^- \\ \psi_n^+ \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{\lambda a} & \frac{b}{a} \\ -\frac{c}{a} & \frac{\lambda}{a} \end{pmatrix} \begin{pmatrix} \psi_{n+1}^- \\ \psi_{n+1}^+ \end{pmatrix}$$

$$S = \frac{(ad-bc)\lambda^{-1} S + b}{-cS + \lambda}$$

$$-cS^2 + \lambda S = \overbrace{(ad-bc)}^{\Delta}\lambda^{-1} S + b$$

$$cS^2 + (\Delta\lambda^{-1} - \lambda) S + b = 0$$

$$c(\lambda S)^2 + (\Delta - \lambda^2)(\lambda S) + b\lambda^2 = 0$$

784 Set $u = \lambda s$.

$$c u^2 + (\Delta - \lambda^2) u + b\lambda^2 = 0$$

$$\therefore \quad c u^2 + \Delta u + (-u + b)\lambda^2 = 0$$

$$\lambda^2 = \frac{c u^2 + \Delta u}{b - u}$$

Other poss.

$$c s^2 + (\Delta \lambda^{-1} - \lambda) s + b = 0$$

$$c(\lambda^{-1}s)^2 + (\Delta\lambda^{-2} - 1)\lambda^{-1}s + b\lambda^{-2} = 0$$

$$c(\lambda^{-1}s)^2 - (\lambda^{-1}s) + \lambda^{-2}(\Delta(\lambda^{-1}s) + b) = 0$$

$$\lambda^2 + \frac{\Delta(\lambda^{-1}s) + b}{c(\lambda^{-1}s)^2 - (\lambda^{-1}s)} = 0$$

$$\lambda^2 + \frac{\Delta v + b}{c v^2 - v} = 0$$

quadratic extn. of $\mathbb{C}[v, v^{-1}]$.

$$\lambda^2 + \frac{\Delta v^{-1} + b}{c v - 1} \quad ?$$

$u = \lambda s$

$$\frac{it\, u^3 - u^2}{u + it} + \lambda^3 = 0.$$

785 Consider again

$$cS^2 + (\Delta\lambda^{-1} - \lambda)S + b = 0$$

$|b|<1$

Assume $\Delta = 1$, $b = -\bar{c}$. Then there should be an analytic function $S(\lambda)$ for $|\lambda|<1$ satisfying this equation, unique.

$$(\lambda^{-1}-\lambda)S + b + cS^2 = 0$$

$$(\lambda-\lambda^{-1})S = b + cS^2$$

$$(\lambda^2-1)S = \lambda(b+cS^2)$$

$$S = \frac{-\lambda}{1-\lambda^2}(b+cS^2)$$

It should be possible to iterate this equation starting with $S_0 = 0$. Then $S_1 = \frac{-\lambda}{1-\lambda^2}b$, etc.

Why formal convergence. Rewrite equation as

$$\lambda^{-1}S = \frac{-1}{1-\lambda^2}\left(b + c\lambda^2(\lambda^{-1}S)^2\right)$$

If S_n is the n-th approx and $\lambda^{-1}S = \lambda^{-1}S_{n+1} + O(\lambda^{2k})$

$$\lambda^{-1}S_{n+1} = \frac{-1}{1-\lambda^2}\left(b + c\lambda^2(\lambda^{-1}S_n)^2\right)$$

Then

$$= \frac{-1}{1-\lambda^2}\left(b + c\lambda^2\left[(\lambda^{-1}S + O(\lambda^{2k})\right]^2\right)$$

$$= \frac{-1}{1-\lambda^2}\left(b + c\lambda^2(\lambda^{-1}S)^2 + c\lambda^2 O(\lambda^{2k})\right)$$

$$= \lambda^{-1}S + O(\lambda^{2k+2})$$

Implicit fn. thm. ⇒ analytic convergence near 0.

Find $\lambda^{-1}S = -b + \lambda^2(\text{power series in } \lambda^2)$

$$\lambda^{-1}S = -(1+\lambda^2)(b + c\lambda^2(-b)^2)$$
$$= -b + \lambda^2(-b - cb^2)$$

So S is an odd function of λ.

Put $u = \lambda^{-1}S = u(\lambda^2)$.

$$\lambda u = \frac{-\lambda}{1-\lambda^2}(b + c\lambda^2 u^2)$$
$$(1-\lambda^2)u = -b - c\lambda^2 u^2$$
$$u + b = \lambda^2 u - c\lambda^2 u^2 = \lambda^2(u - cu^2)$$

$$\therefore \boxed{\frac{u+b}{u-cu^2} = \lambda^2}$$

Now supposedly $|S(\lambda)| \leq 1$ for $|\lambda| < 1$, hence $|u(\lambda)| \leq 1$ by Schwarz somebody lemma.

$$\frac{u+b}{-cu+1} = \lambda^2 u$$

$$\begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}(u) = \lambda^2 u \qquad u = \begin{pmatrix} 1 & -b \\ c & 1 \end{pmatrix}(\lambda^2 u)$$

$b = it$
$c = -\bar{b} = b$
$|c| < 1$.

$$\frac{1 + bu^{-1}}{1 - cu} = \lambda^2 u$$

look at $u \longmapsto \lambda^2$

787 Apparently you have the equation

$$\frac{1+bu^{-1}}{1+\bar{b}u} = \lambda^2 u$$

this is an equation relating $u(\lambda) = \lambda^{-1}S(\lambda)$ and λ^2.

$$1 + bu^{-1} = \lambda^2 u + \bar{b}u^2 \quad ?$$

Start again.

$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^-$$

$$u(\delta_1^-) = c\delta_1^+ + d\delta_0^-$$

$$u(\psi_0^+\delta_0^+ + \psi_1^-\delta_1^-) = \psi_0^+(a\delta_1^+ + b\delta_0^-) + \psi_1^-(c\delta_1^+ + d\delta_0^-)$$

$$\lambda\psi_1^+\delta_1^+ + \lambda\psi_0^-\delta_0^-$$

$$\lambda\psi_1^+ = a\psi_0^+ + c\psi_1^-$$

$$\lambda\psi_0^- = b\psi_0^+ + d\psi_1^-$$

$$\lambda\psi_1^+ = a\psi_0^+ + cS\psi_1^+$$

$$\lambda S\psi_0^+ = b\psi_0^+ + dS\psi_1^+$$

$$(\lambda - cS)\psi_1^+ = a\psi_0^+$$

$$dS\psi_1^+ = (-b+\lambda S)\psi_0^+$$

$$\frac{\lambda - cS}{dS} = \frac{a}{-b+\lambda S}$$

$$(\lambda - cS)(\lambda S - b) = adS$$

$$\lambda^2 S - c\lambda S^2 - b\lambda + bcS$$

$$-c\lambda S^2 + (\lambda^2 - 1)S - b\lambda = 0$$

788

$$c\lambda S^2 + (1-\lambda^2)S + b\lambda = 0$$

$$cS^2 + (\lambda^{-1}-\lambda)S + b = 0$$

$$c\lambda^2 u^2 + (\lambda^{-1}-\lambda)\lambda u + b = 0$$

$$c\lambda^2 u^2 + (1-\lambda^2)u + b = 0$$

$$\lambda^2(cu^2 - u) + u + b = 0$$

$$\lambda^2 = \frac{u+b}{u-cu^2}$$

$$\lambda^2 u = \frac{u+b}{1-cu} = \begin{pmatrix} 1 & b \\ -c & 1 \end{pmatrix}(u)$$

$$u = \frac{-(1-\lambda^2) \pm \sqrt{(1-\lambda^2)^2 - 4bc\lambda^2}}{2c\lambda^2}$$

---

transfer matrix $\begin{pmatrix} \psi_n^- \\ \psi_n^+ \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ -\frac{c}{a} & \frac{\lambda}{a} \end{pmatrix} \begin{pmatrix} \psi_{n+1}^- \\ \psi_{n+1}^+ \end{pmatrix}$

$$S = \frac{\lambda^{-1}S + b}{-cS + \lambda} \qquad \lambda^2(\lambda^{-1}S) = \frac{(\lambda^{-1}S) + b}{-c(\lambda^{-1}S) + 1}$$

so it checks.

789 So the equation of interest is

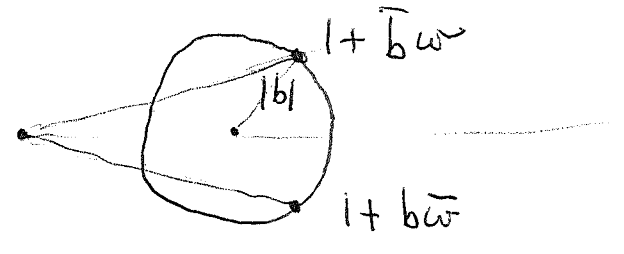
$$z w = \frac{w + b}{\bar{b} w + 1}$$

What should be true? There is a unique analytic function $w = w(z)$ defined for $|z| < 1$ which satisfies this equation, and moreover $|w| < 1$ for $|z| < 1$. It will be easy to look at ~~the function~~ z as a function of w. We know that $|w| = 1 \implies |z| = 1$, z is a degree 2 rational function of w.

$$z = \frac{1 + b w^{-1}}{1 + \bar{b} w} \qquad \text{recall } |b| < 1$$

clearly $|w| = 1 \implies |1 + b w^{-1}| = |1 + b\bar{w}| = |1 + \bar{b} w|$
$\geq 1 - |b|$.

so the possible z's form an arc on the circle



Let's move on ~~the~~ to the Γ-graph.

$$S = \frac{\lambda^{-1} S^2 + b}{\bar{b} S^2 + \lambda}$$

~~$S(0) = 0$ clear~~

~~$\lambda w = \frac{\lambda w^2 + b}{\bar{b} \lambda^2 w^2 + \lambda}$~~

$$\lambda S = \frac{S^2 + b\lambda}{\bar{b} S^2 + \lambda}$$

OK

$$0 = \frac{S(0)^2}{\bar{b} S(0)^2}$$

790

$$S = \begin{pmatrix} \lambda^{-1} & b \\ \bar{b} & \lambda \end{pmatrix} (S^2)$$

$$S = \frac{\lambda^{-1} S^2 + b}{\bar{b} S^2 + \lambda}$$

$$\bar{b} S^3 + \lambda S = \lambda^{-1} S^2 + b$$

$$\bar{b} \lambda S^3 - S^2 + \lambda^2 S - b\lambda = 0$$

$\lambda = 0 \implies S = 0.$ Put $S = \lambda w$

$$\bar{b} \lambda^4 w^3 - \lambda^2 w^2 + \lambda^3 w - b\lambda = 0$$

$$\bar{b} \lambda^3 w^3 - \lambda w^2 + \lambda^2 w - b = 0$$

So we have a contradiction, probably we should be using λ^{-1} instead of λ. So try

$$S = \frac{\lambda S^2 + b}{\bar{b} S^2 + \lambda^{-1}}$$

$$\bar{b} S^3 - \lambda S^2 + \lambda^{-1} S - b = 0.$$

$$\bar{b} \lambda S^3 - \lambda^2 S^2 + S - b\lambda = 0 \qquad \therefore S(0) = 0.$$

put $S = \lambda w$

$$\bar{b} \lambda^4 w^3 - \lambda^4 w^2 + \lambda w - b\lambda = 0$$

791

$$\bar{b}\lambda^3 w^3 - \lambda^3 w^2 + w - b = 0$$

$$\lambda^3\left(\bar{b} w^3 - w^2\right) = b - w$$

$$\lambda^3 = \frac{w-b}{w^2 - \bar{b} w^3} = \frac{1}{w^3}\,\frac{w-b}{w^{-1}-\bar{b}} = \frac{1}{w}\,\frac{1-bw^{-1}}{1-\bar{b}w}$$

so you put $z = \lambda^3$

$$z = \frac{1}{w}\,\frac{1-bw^{-1}}{1-\bar{b}w}$$

$w \mapsto z$ for $\mathbb{P}^1$ to $\mathbb{P}^1$ has degree 3.

Recall w is essentially the scattering matrix. For discrete spectrum you want $|w| = 1$ and $|\lambda| < 1$? ?

Notice that $w \mapsto z$ maps S^1 to S^1 & is a degree -1 map.

I am ultimately interested in the spectrum of the unitary operator. Since I am looking at half space this means I am looking

Go back to the equations

$$\lambda\psi_1^+ = a\psi_0^+ + c\psi_1^-$$

$$\lambda\psi_0^- = b\psi_0^+ + d\psi_1^-$$

and assume $S(\lambda)$ such that $\psi_0^- = S\psi_0^+$, $\psi_1^- = S\psi_1^+$

$$\psi_0^+ = -\frac{c}{a}\psi_1^- + \frac{\lambda}{a}\psi_1^+$$

$$\psi_0^- = \frac{b}{\lambda}\left(-\frac{c}{a}\psi_1^- + \frac{\lambda}{a}\psi_1^+\right) + \frac{ad}{\lambda a}\psi_1^- = \frac{1}{\lambda a}\psi_1^- + \frac{b}{a}\psi_1^+$$

792

$$\begin{pmatrix}\psi_0^-\\ \psi_0^+\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda a} & \frac{b}{a}\\ -\frac{c}{a} & \frac{\lambda}{a}\end{pmatrix}\begin{pmatrix}\psi_1^-\\ \psi_1^+\end{pmatrix}$$

$$S = \frac{\lambda^{-1}S + b}{\underbrace{-c}_{\bar{b}}S + \lambda} = \frac{\lambda^{-1}S + b}{\lambda(1+\bar{b}(\lambda^{-1}S))}$$

$w = \lambda^{-1}S.$ $\quad \lambda w = \dfrac{w+b}{\bar{b}\lambda w + \lambda}$ $\quad \therefore \boxed{\lambda^2 = \dfrac{1+bw^{-1}}{1+\bar{b}w}}$

So $|w| = 1 \implies |\lambda| = 1.$

You want eigenvectors for the transfer matrix i.e. fixpts of $S \longmapsto \dfrac{\lambda^{-1}S+b}{\bar{b}S+\lambda}$

$$\zeta^2 - \left(\frac{1}{\lambda a} + \frac{\lambda}{a}\right)\zeta - \underbrace{\left(\frac{1}{a^2} - \frac{b\bar{b}}{a^2}\right)}_{\frac{a\bar{a}}{a^2} = \frac{\bar{a}}{a}}$$

assume $a > 0$
$0 < a < 1$

$$\zeta = \frac{\left(\frac{1}{\lambda a} + \frac{\lambda}{a}\right) \pm \sqrt{\left(\frac{1}{\lambda a} + \frac{\lambda}{a}\right)^2 - 4}}{2}$$

$$\frac{\lambda + \lambda^{-1}}{a} = 2$$

$$\lambda^2 - 2a\lambda + 1 = 0$$

$$\lambda = \underbrace{a \pm \sqrt{a^2 - 1}}_{\in S^1}$$

---

transfer matrix is

$$\begin{pmatrix}\frac{1}{\lambda a} & \frac{b}{a}\\ \frac{\bar{b}}{a} & \frac{\lambda}{a}\end{pmatrix} \in SU(1,1)$$

$0 < a < 1$
$|a|^2 + |b|^2 = 1$

eigenvalues are roots of

$$\zeta^2 - \left(\frac{1}{\lambda a} + \frac{\lambda}{a}\right)\zeta + 1 = 0$$

$$\zeta = -\frac{\lambda + \lambda^{-1}}{2a} \pm \sqrt{\left(\frac{\lambda+\lambda^{-1}}{2a}\right)^2 - 1}$$

product of two eigenvs is 1.

793 What is the relation between λ, $S(\lambda)$, and ~~[illegible]~~ ξ? What do I know ~~[illegible]~~?

~~[illegible]~~ Let T = transfer matrix

Then
$$T\begin{pmatrix} S \\ 1 \end{pmatrix} = \xi \begin{pmatrix} S \\ 1 \end{pmatrix}$$
$$\frac{1}{\lambda a} S + \frac{b}{a} = \xi S$$
$$\frac{\bar{b}}{a} S + \frac{\lambda}{a} = \xi$$

There is a solution of the eigenvalue eqns. namely

$$\psi_0^- = S \qquad \psi_1^- = \xi S \qquad \psi_2^- = \xi^2 S$$
$$\psi_0^+ = 1 \qquad \psi_1^+ = \xi \qquad \psi_2^+ = \xi^2$$

so we need one root ξ to have $|\ | < 1$ for an ℓ^2 solution to ξ. ξ, λ are different but related by $a\left(\frac{\xi + \xi^{-1}}{2}\right) = \frac{\lambda + \lambda^{-1}}{2}$. ~~[illegible]~~

~~[illegible]~~ I know that if $\lambda = e^{i\theta}$ and $|\cos\theta| > a$ then one root ξ has $|\ | < 1$, so we have an ℓ^2 solution to the right. I think also that if $|\lambda| < 1$, then ~~some~~ one root ξ is $|\ | < 1$. In fact there should be a power series $\xi(\lambda)$ converging for $|\lambda| < 1$.

Transfer matrix. $T = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar{b}}{a} & \frac{\lambda}{a} \end{pmatrix}$ $0 < a < 1$. The eigenvector equation for u is ~~[illegible]~~ $\begin{pmatrix} \psi_n^- \\ \psi_n^+ \end{pmatrix} = T \begin{pmatrix} \psi_{n+1}^- \\ \psi_{n+1}^+ \end{pmatrix}$

794 Suppose $0<|\lambda|<1$. Then we expect a unique solution (up to a scalar factor) of the eigenvector equation which decays as $n \to +\infty$. Then $\psi_n = \begin{pmatrix} \psi_n^- \\ \psi_n^+ \end{pmatrix}$ and $\psi_{n+1} = \begin{pmatrix} \psi_{n+1}^- \\ \psi_{n+1} \end{pmatrix}$ are proportional to $\begin{pmatrix} s \\ 1 \end{pmatrix}$, say $\begin{pmatrix} s \\ 1 \end{pmatrix} = \psi_n$ $\quad c\psi_n = \psi_{n+1}$. Then

$\psi_{n+1} = c\psi_n = cT(\psi_{n+1})$ so c^{-1} is an eigenvalue for T and all ψ_n are eigenvector. Since ψ decays $|c|<1$. So $\zeta = c^{-1}$ has $|\zeta| > 1$.

$$\zeta \psi_{n+1} = T(\psi_{n+1}) = \psi_n$$

Go back to

$$\psi_0 = T(\psi_1) = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar{b}}{a} & \frac{\lambda}{a} \end{pmatrix} \psi_1 = \zeta \psi_1$$

$$\zeta = \frac{\lambda+\lambda^{-1}}{2a} \pm \sqrt{\left(\frac{\lambda+\lambda^{-1}}{2a}\right)^2 - 1}$$

$$2a\lambda\zeta = 1+\lambda^2 \pm \sqrt{(1+\lambda^2)^2 - 4a^2\lambda^2}$$

$$\frac{2a\lambda}{1+\lambda^2}$$

$$x + \sqrt{x^2-1}$$

$$\frac{2a\lambda\zeta}{1+\lambda^2} = -1 \mp \sqrt{1 - \frac{4a^2\lambda^2}{(1+\lambda^2)^2}}$$

795

$$\begin{pmatrix} \psi_0^- \\ \psi_0^+ \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ +\frac{\bar{b}}{a} & \frac{\lambda}{a} \end{pmatrix}}_{T} \begin{pmatrix} \psi_1^- \\ \psi_1^+ \end{pmatrix}$$

$a = \sqrt{1-|b|^2}$

$\psi_0 = T(\psi_1)$ For $|\lambda| < 1$, $\lambda \neq 0$ $\exists!$ up to scalar decaying to the right eigenvector for u with eigenvalue λ. Thus $\psi_0 = \begin{pmatrix} S \\ 1 \end{pmatrix}$ $\psi_1 = c\begin{pmatrix} S \\ 1 \end{pmatrix} = c\psi_0$

$\psi_1 = c\psi_0 = cT(\psi_1)$

$\therefore T(\psi_1) = c^{-1}\psi_1$ so $c^{-1} = \xi$, want $|c| < 1$ for decay $\Rightarrow |\xi| > 1$.

$$\xi^2 - 2\left(\frac{\lambda + \lambda^{-1}}{2a}\right)\xi + 1 = 0.$$

$$\xi = \frac{\lambda+\lambda^{-1}}{2a} \pm \sqrt{\left(\frac{\lambda+\lambda^{-1}}{2a}\right)^2 - 1}$$

$$2a\lambda\xi = 1+\lambda^2 \pm \sqrt{(1+\lambda^2)^2 - 4a^2\lambda^2}$$

$$\frac{2a\lambda\xi}{1+\lambda^2} = 1 + \sqrt{1 - \frac{4a^2\lambda^2}{(1+\lambda^2)^2}}$$

$$= 1 + 1 + \frac{1}{2}\left(-\frac{4a^2\lambda^2}{(1+\lambda^2)^2}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\quad\right)^2 + \cdots$$

$$(\lambda\xi)^2 - 2\left(\frac{1+\lambda^2}{2a}\right)\lambda\xi + \lambda^2 = 0$$

what do you want to know?

Can you show from $a\left(\frac{\xi+\xi^{-1}}{2}\right) = \frac{\lambda+\lambda^{-1}}{2}$

796 March 8, 98. ~~What are~~

transfer matrix: $\psi_0 = \underbrace{\begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar b}{a} & \frac{\lambda}{a} \end{pmatrix}}_{T} \psi_1$ $\qquad a = \sqrt{1-|b|^2}$

When $(\psi_0, \psi_1, \cdots)$ is a decaying eigenvector for u

we have $\psi_0 = \text{const}\begin{pmatrix} \zeta \\ 1 \end{pmatrix}$ $\qquad \psi_n = T^{-n}\psi_0 = \zeta^{+n}\psi_0$

where ζ is the root of

$$\zeta^2 - \left(\frac{\lambda+\lambda^{-1}}{a}\right)\zeta + 1 = 0$$

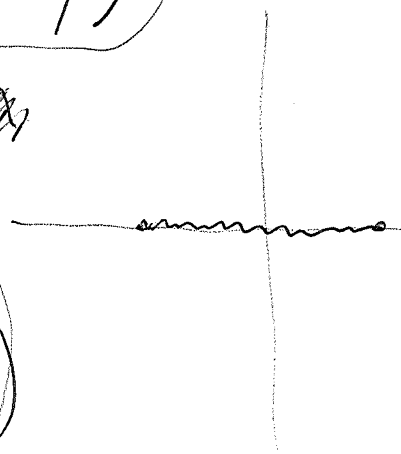
which is analytic at $\lambda = 0$.

$$\zeta = \frac{\lambda+\lambda^{-1}}{2a} - \sqrt{\left(\frac{\lambda+\lambda^{-1}}{2a}\right)^2 - 1}$$

Put $\lambda\eta = \zeta$, $\qquad \lambda^2\eta^2 - \left(\frac{\lambda+\lambda^{-1}}{a}\right)\lambda\eta + 1 = 0$

$$\frac{1+\lambda^2}{a}\eta = 1 + \lambda^2\eta^2$$

$$\eta = \frac{a}{1+\lambda^2}(1+\lambda^2\eta^2)$$



$$x - \sqrt{x^2-1}$$

$$= x\left(1 - (1-x^{-2})^{1/2}\right)$$

$$= x\left(1 - \left\{1 - \tfrac{1}{2}x^{-2} + \frac{\frac{1}{2}(-\frac{1}{2})}{2}x^{-4} + \cdots\right\}\right)$$

$$= \tfrac{1}{2}x^{-1} + \tfrac{1}{8}x^{-3} + \cdots$$

convergence depends on $|x^{-1}| < 1$. $\qquad x = \frac{\lambda+\lambda^{-1}}{2a}$

~~which is never true for λ near 0.~~

ok for λ near 0

$$\zeta = \frac{1}{2}\frac{2a}{\lambda+\lambda^{-1}} + \frac{1}{8}\left(\frac{2a}{\lambda+\lambda^{-1}}\right)^2 + \cdots$$

$$\zeta = \frac{a\lambda}{1+\lambda^2} + \frac{1}{2}\left(\frac{a\lambda}{1+\lambda^2}\right)^2 + \cdots$$

$$S = T(S) = \frac{\lambda^{-1}S + b}{\bar{b}S + \lambda}$$

$$T\begin{pmatrix} S \\ 1 \end{pmatrix} = \rho^{-1}\begin{pmatrix} S \\ 1 \end{pmatrix}$$

$$\lambda^{-1}S + b = \rho^{-1}S$$
$$\bar{b}S + \lambda = \rho^{-1}$$

Maybe this is too hard. The reason for introducing ρ is to get the sign of the decay straight. Maybe you should go back to the case of a partial unitary.

The problem is to see when there is discrete spectrum. The idea here is that this will happen when the resolvents $(\lambda - ba^*)^{-1}$, $(1-\lambda ab^*)^{-1}$ can be analytically continued to part of the unit circle. Look at $S(\lambda)$ for a 1-port. This is always defined for $|\lambda| < 1$ and corresponds to a decaying eigenfunction. Now suppose you can analytically continue to a nbd of a point λ_0, $|\lambda_0| = 1$ and that $|S(\lambda)| = 1$ for $|\lambda| = 1$ and λ close to λ_0. It should be enough that $|S(\lambda_0)| = 1$. Then the value $S(\lambda_0)$ should give a boundary condition completing the partial unitary to a unitary having the discrete eigenvalue λ_0.

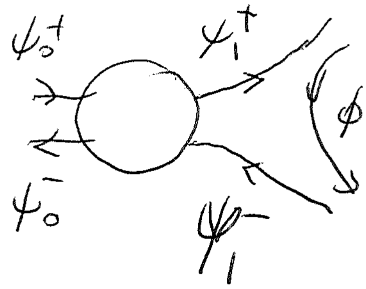
Let's work with the equation $S = T(S^2)$

$$S = \frac{\lambda^{-1}S^2 + b}{\bar{b}S^2 + \lambda} \qquad \bar{b}S^3 - \lambda^{-1}S^2 + \lambda S - b = 0$$

$S(\lambda)$ analytic for $|\lambda| < 1$

so $S(0) = 0$ Put $S = \lambda w$

$$\bar{b}\lambda^3 w^3 - \lambda w^2 + \lambda^2 S - b = 0$$ Contradiction



$$\psi_0^- = S\psi_0^+$$
$$\phi = S\psi_1^+$$
$$\psi_1^- = S\phi = S^2\psi_1^+$$

$$S = \frac{\psi_0^-}{\psi_0^+} = \frac{\frac{1}{\lambda a}\frac{\psi_1^-}{\psi_1^+} + \frac{b}{a}}{\frac{\bar b}{a}\frac{\psi_1^-}{\psi_1^+} + \frac{\lambda}{a}} = \frac{\lambda^{-1}S^2 + b}{\bar b S^2 + \lambda}$$

So apparently S is analytic for $|\lambda| > 1$. ?

~~What~~ What sort of transf is

$$z \longmapsto \frac{\lambda^{-1}z + b}{\bar b z + \lambda} \qquad \begin{pmatrix} \lambda^{-1} & b \\ \bar b & \lambda \end{pmatrix}\frac{1}{c}$$

Why not interchange λ and λ^{-1}.

$$S = \frac{\lambda S^2 + b}{\bar b S^2 + \lambda^{-1}}$$

$$\bar b S^3 - \lambda S^2 + \lambda^{-1}S - b = 0 \qquad \Rightarrow S(0) = 0$$
$$S = \lambda w$$

$$\bar b \lambda^3 w^3 - \lambda^3 w^2 + w - b = 0$$

$$\lambda^3 = \frac{w - b}{w^2(1 - \bar b w)} = \frac{1}{w}\frac{1 - bw^{-1}}{1 - \bar b w}$$

Interchange λ and λ^{-1}.

799 Recap. There should be $S(\lambda) = \lambda w(\lambda^3)$ analytic for $|\lambda| < 1$ at least satisfying ?

~~[struck out]~~ Consider

$$w \longmapsto z = \frac{1}{w}\frac{1-bw^{-1}}{1-\bar{b}w} = \frac{1}{w^2}\frac{w-b}{1-\bar{b}w}$$

This is a degree 3 map from the Riemann sphere to itself. I am interested in this map for $|z^{1/3}w| \leq 1$. It seems that $S(\lambda) = \lambda w(\lambda^3)$ is analytic for $|\lambda| < 1$. Actually what happens?

$$(w^2 - \bar{b}w^3)z = w - b$$

$$w = b + z(w^2 - \bar{b}w^3)$$

You can iterate this equation for small $|z|$. to get an analytic function $w(z) = b +$

$z(b^2)(1-|b|^2) +$

$w_0 = b$

$w_1 = w_0^2 - \bar{b}w_0^3$

note that w

Look carefully at $w \mapsto z = \frac{1}{w}\frac{1-bw^{-1}}{1-\bar{b}w}$ for $|w| = 1$.

is this a diffeomorphism of S^1.

$$\frac{dz}{z} = \left\{-\frac{1}{w} + \frac{1}{1-bw^{-1}}\frac{+bw^{-2}}{w} - \frac{1}{1-\bar{b}w}\frac{(-\bar{b})}{w}\right\}dw$$

$$\frac{dz}{z} = \left\{-1 + \frac{b}{(1-bw^{-1})w^2} + \frac{\bar{b}}{(1-\bar{b}w)}\right\}\frac{dw}{w}$$

$$\frac{d}{dw}\log(1-bw^{-1}) = \frac{1}{1-bw^{-1}}\left(-b(-1)w^{-2}\right) = \frac{b}{w^2 - bw}$$

800

$$-1 + \frac{b}{w^2 - bw} + \frac{\bar b}{1-\bar b w}$$

$$\frac{b}{w(w-b)} = \frac{-1}{w} + \frac{1}{w-b}$$

$$\frac{dz}{z} = \left\{ -1 - \frac{1}{w} + \frac{1}{w-b} + \frac{\bar b}{1-\bar b w} \right\} \frac{dw}{w}$$

$$z = \frac{1}{w} \frac{1-bw^{-1}}{1-\bar b w}$$

$$\log z = -\log w + \log(1-bw^{-1}) - \log(1-\bar b w)$$

$$d \log z = -\frac{dw}{w} + \frac{1}{1-bw^{-1}}(-b(-1)w^{-2}dw) - \frac{1}{1-\bar b w}(-\bar b)dw$$

$$= \left\{ -1 + \frac{b}{w-b} + \frac{\bar b}{w^{-1}-\bar b} \right\} \frac{dw}{w}$$

$$= \left\{ -1 + \frac{bw^{-1} + bw - 2|b|^2}{(w-b)(w^{-1}-\bar b)} \right\} \frac{dw}{w}$$

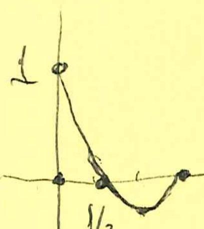
$$= \frac{-1 + bw^{-1} + \bar b w - |b|^2 + bw^{-1} + \bar b w - 2|b|^2}{4 \, \mathrm{Re}\,(be^{-i\theta})}$$

$$\frac{dz}{z} = \frac{-1 - 3|b|^2 + 2\bar b w + 2bw^{-1}}{(w-b)(w^{-1}-\bar b)} \frac{dw}{w}$$

$$4|b| < +1 + 3|b|^2$$

$\frac{8}{3} \overset{?}{<} \frac{3}{3} + 3 \cdot \frac{4}{9}$ $= \frac{7}{3}$ NO.

$$3|b|^2 - 4|b| + 1 =$$



$6|b| - 4$

$|b| = \frac{2}{3}$

$$3 \cdot \frac{4}{9} - 4 \cdot \frac{2}{3} + 1 = \frac{4}{3} - \frac{8}{3} + 1$$

801.

$$\frac{dz}{z} = \left\{ \frac{-1-3|b|^2 + 2\bar{b}w + 2bw^{-1}}{(w-b)(w^{-1}-\bar{b})} \right\} \frac{dw}{w}$$

Does this vanish? Recall $b = it$

$$2it(\underbrace{w^{-1}-w}_{\in i\mathbb{R}}) - 1 - 3|b|^2 = 0 \quad \Rightarrow \quad w = e^{i\theta}$$

It looks like there are 2 singular points on S^1.

Let's check this $z = \frac{1}{w}\frac{1-bw^{-1}}{1-\bar{b}w} = \frac{1}{w^2}\frac{w-b}{1-\bar{b}w}$ $\quad w=0 \rightarrow z=\infty$.

$$\frac{dz}{z} = \left\{ -\frac{1}{w} + \frac{1}{1-bw^{-1}}(-b)(-1)w^{-2} - \frac{1}{1-\bar{b}w}(-\bar{b}) \right\} dw$$

$$= \left\{ -1 + \frac{b}{w-b} + \frac{\bar{b}}{w^{-1}-\bar{b}} \right\} \frac{dw}{w}$$

$bw^{-1} - |b|^2 \quad + \bar{b}w - |b|^2$

$$= \frac{-1 + bw^{-1} + \bar{b}w - |b|^2 + b(w^{-1}-\bar{b}) + \bar{b}(w-b)}{(w-b)(w^{-1}-\bar{b})} \frac{dw}{w}$$

$$= \frac{-1-3|b|^2 + 2bw^{-1} + 2\bar{b}w}{(w-b)(w^{-1}-\bar{b})} \frac{dw}{w}$$

$$\frac{dz}{dw} = \frac{(-1-3|b|^2 + 2bw^{-1} + 2\bar{b}w)}{(w-b)(1-\bar{b}w)} \frac{1}{w^2} \frac{(1-bw^{-1})w}{1-\bar{b}w}$$

$$\boxed{\frac{dz}{dw} = \frac{-1-3|b|^2 + 2bw^{-1} + 2\bar{b}w}{(1-\bar{b}w)^2 w^2}}$$

we have by implicit function theorem an anal. function $w(z) = b + b^2(1-|b|^2)z + \ldots$ for $|z|$ small (hopefully $|z| < 1$) and $|w(z)| < 1$

An interesting question is what is the range of $w(z)$ for $|z|<1$. Note that ~~w(z)~~ $w(z)$ cannot be zero. Ask about the critical points. You have a Riemann surface defined by the correspondence between z and w. Actually z is a rational function of w of degree 3, so there's a map $w \mapsto$ ~~z~~ from the R.S. to itself, and you can ask about the ramification. So the formula for z tells us everything when $w \neq 0, \frac{1}{\bar b}$. The ramification points are w satisfying

$$2bw^{-1} + 2\bar b w = 1+3|b|^2$$

$$2\bar b w^2 - (1+3|b|^2)w + 2b = 0$$

$$w = \frac{1+3|b|^2 \pm \sqrt{(1+3|b|^2)^2 - 16|b|^2}}{4\bar b}$$

Let's put in $b = i|b|$. The two roots have product $\frac{2b}{2\bar b} = -1$, so they will be on the unit circle when $(1+3|b|^2)^2 - 16|b|^2 \leq 0$

$$1 + 6|b|^2 + 9|b|^4 - 16|b|^2 \leq 0$$

$$1 - 10|b|^2 + 9|b|^4 \leq 0$$

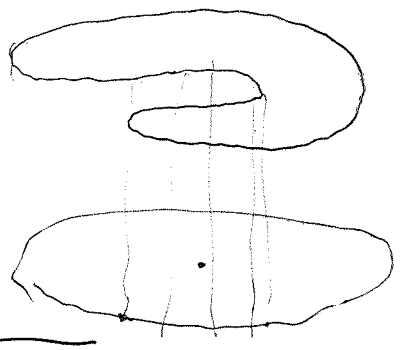
$$(1-9|b|^2)\underbrace{(1-|b|^2)}_{>0} \leq 0$$

need. $|b| \geq \frac{1}{3}$

803 Now before I ~~looking~~ at the map $w \mapsto z$ from $\{|w| = 1\}$ to $\{|z| = 1\}$. and ~~[struck]~~ found this is a diffeom ~~[struck]~~ for $|b| < \frac{1}{3}$.

So for $|b| > \frac{1}{3}$ say you will find some ~~ramification~~ points on the z circle having 3 points on the w circle

graph



z circle

Review what you learned about

$$Y = aX \oplus \mathbb{C}\xi_1 = bX \oplus \mathbb{C}u(\xi_1)$$

$$\mathbb{C}\xi_1 = V^+ \qquad \mathbb{C}u(\xi_1) = V^-$$

The aim is to relate $(\lambda - u)^{-1}$ and $(\lambda - ba^*)^{-1}$

$$u = u(aa^* \oplus \underbrace{(1-aa^*)}_{\xi_1 < \xi_1|}) = ba^* + u\overbrace{(1-aa^*)}^{\pi}$$

$$(\lambda - u)^{-1} = (\lambda - ba^* - u\pi)^{-1}$$

$$= (\lambda - ba^*)^{-1} + (\lambda - ba^*)^{-1} u\pi (\lambda - ba^*)^{-1}$$

$$\pi(\lambda - u)^{-1} u\pi = \underbrace{\pi(\lambda - ba^*)^{-1}}_{S} u\pi + (\underbrace{\pi(\lambda - ba^*)^{-1}}_{S} u\pi)^2 + \cdots$$

?

Two ingredients. $S(\lambda) : V^- \longrightarrow V^+$ for $|\lambda| > 1$

$$S(\lambda)(v_{-1}^-) = \pi(\lambda - ba^*)^{-1} v_{-1}^-$$

other ingredient is the isom. $u : V^+ \xrightarrow{\sim} V^-$, call this β for Boundary condition. Then we have something like $\pi(\lambda-u)^{-1}\beta = S\beta + (S\beta)^2 + \cdots$

$$= \frac{S\beta}{1-S\beta}$$

Maybe $(\lambda-u)^{-1}u = (u^{-1}\lambda-1)^{-1}$ via analytic cont.

i.e. $(\lambda-u)^{-1}u = \lambda^{-1}(1-\lambda^{-1}u)^{-1}u$

$$= \lambda^{-1}\sum_{n\geq 0}\lambda^{-n}u^{n+1} = \sum_{n>0}\lambda^{-n+1}u^{n+1} \qquad |\lambda|>1.$$

$$(u^{-1}\lambda-1)^{-1} = -\sum_{n\geq 0}u^{-n}\lambda^{n} = -\sum_{n\leq 0}\lambda^{-n}u^{n} \qquad |\lambda|<1$$

$$(\lambda-u)^{-1}u = (\lambda-ba^*-u\pi)^{-1}(ba^*+u\pi)$$

$$= (\lambda-ba^*)^{-1}(ba^*+u\pi)$$

$$+ (\lambda-ba^*)^{-1}u\pi(\lambda-ba^*)^{-1}(ba^*+u\pi)$$

March 10, 1998 Go back to $b\in i\mathbb{R}$ $0<\frac{b}{i}<1.$

$a=\sqrt{1-|b|^2}$

$$\begin{pmatrix}\varphi_0^+\\ \varphi_0^-\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda a} & \frac{b}{a}\\ \frac{\bar b}{a} & \frac{\lambda}{a}\end{pmatrix}\begin{pmatrix}\varphi_1^+\\ \varphi_1^-\end{pmatrix}$$

$$S = \frac{\lambda^{-1}S^2+b}{\bar b S^2+\lambda} \qquad \bar b S^3+\lambda S = \lambda^{-1}S^2+b$$

S should be analytic for $|\lambda|<1$ or for $|\lambda|>1$

If S analytic at 0 then $S(0)=0$ and

$\bar b S^3$, λS, $\lambda^{-1}S^2$ all vanish at 0, a contradiction

$\therefore S$ analytic for $|\lambda|>1$.

Change $\lambda\mapsto\lambda^{-1}$ to get

$$\bar b S^3+\lambda^{-1}S = \lambda S^2+b$$

$\Longrightarrow S(0)=0 \Longrightarrow S=\lambda\omega(\lambda)$ where ω analytic

$$\bar b \lambda^3 w^3 + w = \lambda^3 w^2 + b$$

$$\lambda^3 (\bar b w^3 - w^2) = b - w$$

$$\lambda^3 = \frac{w-b}{w^2 - \bar b w^3} = \frac{1}{w^2} \frac{w-b}{1-\bar b w}$$

Then w will be an analytic function of $z = \lambda^3$ for $|z| < 1$. Also we should have by max. principle $|w(z)| < 1$ for $|\lambda| < 1$, i.e. $|z| < 1$.

So now we wish to study this map

$$w \longmapsto z = \frac{w-b}{w^2(1-\bar b w)} \qquad \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

having degree 3. Following Riemann its a 3 sheet covering of the z-plane. It should be etale on the image of $z \mapsto w(z)$. Calculate the ramification

$$\frac{dz}{z} = -2\frac{dw}{w} + \frac{dw}{w-b} - \frac{1}{1-\bar b w} d(1-\bar b w)$$

$$\frac{dz}{dw} = \frac{1}{w^2}\frac{w-b}{1-\bar b w}\left\{ \frac{-2}{w} + \underbrace{\frac{1}{w-b} + \frac{\bar b}{1-\bar b w}}_{\frac{1 - \cancel{\bar b w} + \cancel{\bar b w} - |b|^2}{(w-b)(1-\bar b w)}} \right\}$$

$$= \frac{1}{w^2}\frac{\cancel{w-b}}{1-\bar b w}\left\{ \frac{-2(-\bar b w^2 + (1+|b|^2)w - b) + w(1-|b|^2)}{w(\cancel{w-b})(1-\bar b w)} \right\}$$

Learn how important your privacy is to us.
© 1997 Intuit Inc. Trademark Notices By accessing and using this page you agree to the Terms of Service.
Microsoft Internet Explorer
Help & Feedback • Site Map • Site Index NEW • Glossary • Quicken Store • Quicken Product Support

806

$$\frac{dz}{dw} = \frac{1}{w^3(1-\bar{b}w)^2}\left\{2\bar{b}w^{-2} + (-1-3|b|^2)w + 2b\right\}$$

ramification points are

$$w = \frac{(1+3|b|^2) \pm \sqrt{(1+3|b|^2)^2 - 16|b|^2}}{4\bar{b}}$$

The product of the 2 roots is $\frac{2b}{2\bar{b}} = \frac{b}{\bar{b}} = -1$

$$1 + 6|b|^2 + 9|b|^4 - 16|b|^2 = 1 - 10|b|^2 + 9|b|^4$$
$$= (1-9|b|^2)(\underbrace{1-|b|^2}_{>0})$$

So if $|b| = \frac{1}{3}$ then $w = \frac{1+3|b|^2}{4\bar{b}} = \frac{1+\frac{3}{9}}{4\left(\frac{-1}{3}i\right)} = \frac{1}{-i} = i$

is the only ramification point

$$2\underset{-\frac{i}{3}}{\bar{b}}\,w^2 - \frac{4}{3}w + 2\left(\frac{i}{3}\right) = -\frac{2i}{3}\left(w^2 - 2iw - 1\right)$$
$$= -\frac{2i}{3}(w-i)^2$$

How many ram. points.

so $\frac{dz}{dw}$ has a ~~triple~~ double zero at $w=i$

what happens at w. Also it looks like

z vanishes to order 2 at $w=\infty$

Volume, Advancers, Decliners and Unchanged on NYSE, NASDAQ & AMEX
Dow Jones Industrial Average 30
Companies reaching a 52-Week Low
Companies reaching a 52-Week High
Percent Losers
Percent Gainers
Most Active Issues
Major Market Indexes
Indexes
NEWS & QUOTES

807 Look at z as a function of $\frac{1}{w}$

$$z = w^{-2}\,\frac{1-bw^{-1}}{w^{-1}-\bar b}$$

$$\frac{dz}{d(w^{-1})} = \frac{dz}{-w^{-2}\,dw} =$$

$$=(-w^2)\,\frac{1}{w^{-3}\,(w^{-1}-\bar b)^2}\left(2\bar b + (-1-3|b|^2)w^{-1} + 2bw^{-2}\right)$$

$$= \frac{(-1)\,w^{-1}}{(w^{-1}-\bar b)^2}\left(2\bar b + \cdots \frac{1}{w} \quad \frac{1}{w^2}\right)$$

~~$ax^3 + bx^2 + cx + d$~~ $3x$

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$$\begin{array}{r|l} & \frac{1}{2}x - \frac{b}{4a} \\ 2ax+b & ax^2 + bx + c \\ & ax^2 + \frac{1}{2}bx \\ \hline & \frac{1}{2}bx + c \\ & -2ax\frac{b}{4a} - \frac{b^2}{4a} \\ \hline & c - \frac{b^2}{4a} \end{array}$$

I/B/E/S

Datek Online · DLJdirect · Charles Schwab · Ameritrade® · SURETRADE.COM

Quicken.com Home • Investments • Retirement • Home & Mortgage
Banking & Borrowing • Saving & Spending • Insurance • Taxes • People & Chat

Help & Feedback • Site Map • Site Index NEW • Glossary • Quicken Store • Quicken Product Support

Microsoft Internet Explorer

© 1997 Intuit Inc. Trademark Notices By accessing and using this page you agree to the Terms of Service.
Learn how important your privacy is to us.

80 8

$$f(x) = x^3 + ax^2 + bx + c$$

$$f'(x) = 3x^2 + 2ax + b$$

$$\frac{1}{3}x + \frac{1}{9}a$$

$$3x^2 + 2ax + b \;\overline{)\; x^3 + ax^2 + bx + c}$$

$$x^3 + \frac{2}{3}ax^2 + \frac{1}{3}bx$$

$$\frac{1}{3}ax^2 + \frac{2}{3}bx + c$$

$$\frac{1}{3}ax^2 + \frac{2a^2}{9}x + \frac{1}{9}ab$$

$$\underbrace{\left(\frac{2b}{3} - \frac{2a^2}{9}\right)}_{\alpha}x + \underbrace{\left(c - \frac{ab}{9}\right)}_{\beta}$$

$$\frac{3}{\alpha}x + \frac{1}{\alpha}\left(2a - \frac{3\beta}{\alpha}\right)$$

$$\alpha x + \beta = \;\overline{)\; 3x^2 + 2ax + b}$$

$$3x^2 + \frac{3\beta}{\alpha}x$$

$$\left(2a - \frac{3\beta}{\alpha}\right)x + b$$

$$\left(2a - \frac{3\beta}{\alpha}\right)x + \frac{1}{\alpha}\left(2a - \frac{3\beta}{\alpha}\right)\beta$$

$$b - \frac{2a\alpha - 3\beta}{\alpha^2}\beta$$

$$\alpha^2 b - 2a\alpha\beta + 3\beta^2$$ ~~substitute~~ $$\alpha = 6b - 6a^2$$

$$-2(54ab - 54a^3 - 6a^2b + 6a^4)$$ $$\beta = 9c - ab$$

$$(6b - 6a^2)^2 b - 2a(6b - 6a^2)(9 - a)b + 3(9 - a)^2 b^2$$

6 6 6 3 3 4 4

$$36b^3 - 72a^2b^2 + 36a^4b - 108ab + 108a^3 + 12a^2b - 12a^4$$

$$+ 243c^2 - 54ac^2 + 3a^2b^2$$

6 7 8

$|a| = 1$
$|b| = 2$
$|c| = 3$

809

$f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

$$\begin{array}{r} \frac{1}{3}x + \frac{1}{9}a \\ 3x^2 + 2ax + b \,\big)\, x^3 + ax^2 + bx + c \\ x^3 + \frac{2a}{3}x^2 + \frac{1}{3}bx \\ \hline \frac{1}{3}ax^2 + \frac{2}{3}bx + c \\ \frac{1}{3}ax^2 + \frac{2}{9}a^2x + \frac{1}{9}ab \\ \hline \left(\frac{2}{3}b - \frac{2}{9}a^2\right)x + \left(c - \frac{1}{9}ab\right) \end{array}$$

$\alpha \qquad \beta$

$$\begin{array}{r} \frac{3}{\alpha}x + \frac{1}{\alpha}\left(2a - \frac{3\beta}{\alpha}\right) \\ \alpha x + \beta \,\big)\, 3x^2 + 2ax + b \\ 3x^2 + \frac{3\beta}{\alpha}x \\ \hline \left(2a - \frac{3\beta}{\alpha}\right)x + b \\ \left(2a - \frac{3\beta}{\alpha}\right)x + \frac{1}{\alpha}\left(2a - \frac{3\beta}{\alpha}\right)\beta \\ \hline b - \frac{1}{\alpha}\left(2a - \frac{3\beta}{\alpha}\right)\beta \end{array}$$

$|\alpha| = 2 \qquad |\beta| = 3$

$\alpha^2 b - 2a\alpha\beta + 3\beta^2$

$\left(\frac{6b - 2a^2}{9}\right)^2 b - 2a\left(\ \right) + 3\left(\frac{9c - ab}{9}\right)^2$

$\left(\frac{6b - 2a^2}{9}\right)\left(\frac{9c - ab}{9}\right)$

$(6b - 2a^2)^2 b - 2a(6b - 2a^2)(9c - ab) + 3(9c - ab)^2$

$36b^3 - 24a^2b^2 + 4a^4b - ~~162ac~~ + ~~18a^2b~~ + 243c^2 - 54abc + 3a^2b^2$

810 $\Delta = \alpha^2 b - 2a\alpha\beta + 3\beta^2$

$|a|=1 \quad |b|=2 \quad |c|=3$

$\alpha = 6b - 2a^2 \quad |\alpha| = 2$

$\beta = 9c - ab \quad |\beta| = 3$

$(6b-2a^2)^2 b - 2a(6b-2a^2)(9c-ab) + 3(9c-ab)^2$

$36b^3 - 24a^2b^2 + \cancel{4a^4b} - 108abc + 12a^2b^2 + 36a^3c - \cancel{4a^4b}$

$+ 243c^2 - 54abc + 3a^2b^2$

$= 36b^3 + 243c^2 + a^2b^2(\underbrace{-24+12+3}_{-9}) + 36a^3c + abc(-162)$

$$\boxed{4b^3 + 27c^2 - a^2b^2 + 4a^3c - 18abc}$$

If $c=0$ get $4b^3 - a^2b^2 = (4b-a^2)b^2$

$a = b = 3 \quad c = 1.$

$4\cdot 27 + 27 - 81 + 4\cdot 27 - 18\cdot 9$

$9\cdot 27 \qquad -3\cdot 81$

So we now have the disc. of a cubic

$z = \frac{1}{w^2}\frac{w-b}{1-\bar{b}w}$

$\boxed{w = i \Rightarrow z = \frac{1}{-1}\frac{i - i|b|}{1+i|b|i} = -i}$

$zw^2(1-\bar{b}w) = w - b$

$-\bar{b}zw^3 + zw^2 - w + b = 0$

$\underbrace{\left(-\frac{\bar{b}}{b}z\right)}_{c} w^3 + \underbrace{\left(\frac{z}{b}\right)}_{b} w^2 + \underbrace{\left(-\frac{1}{b}\right)}_{a} w + 1 = 0$

$4\left(\frac{z}{b}\right)^3 + 27\left(-\frac{\bar{b}}{b}z\right)^2 - \left(-\frac{1}{b}\right)^2\left(\frac{z}{b}\right)^2 + 4\left(-\frac{1}{b}\right)^3\left(-\frac{\bar{b}}{b}z\right) - 18\left(-\frac{1}{b}\right)\left(\frac{z}{b}\right)\left(-\frac{\bar{b}z}{b}\right)$

811

$$w^3 + \left(-\frac{1}{\bar b}\right)w^2 + \left(\frac{1}{\bar b z}\right)w + \left(-\frac{b}{\bar b z}\right) = 0$$

$$4\left(\frac{1}{\bar b z}\right)^3 + 27\left(-\frac{b}{\bar b z}\right)^2 - \left(\frac{-1}{\bar b}\right)^2\left(\frac{1}{\bar b z}\right)^2$$
$$+ 4\left(-\frac{1}{\bar b}\right)^3\left(-\frac{b}{\bar b z}\right) - 18\left(-\frac{1}{\bar b}\right)\left(\frac{1}{\bar b z}\right)\left(-\frac{b}{\bar b z}\right)$$

$$4 + 27\left(-\frac{b}{\bar b z}\right)^2(\bar b z)^3 - \frac{1}{\bar b^2}\bar b z$$
$$+4\left(\frac{1}{\bar b^3}\right)\left(+\frac{b}{\bar b z}\right)(\bar b z)^3 - 18\frac{b}{\bar b^3 z^2}\bar b^3 z^3$$

$$4 + 27 b^2 \bar b z - \frac{z}{\bar b}$$
$$+4\frac{b}{\bar b}z^2 - 18bz$$

$$4\frac{b}{\bar b}z^2 + \left(27b|b|^2 - \frac{1}{\bar b} - 18b\right)z + 4$$

Try $b = \frac{i}{3}$ $\qquad \frac{b}{\bar b} = -1$

$$-4z^2 + \left(i - \frac{3}{-i} - 6i\right)z + 4$$
$$(1 - 3 - 6)i$$

$$4z^2 + 8iz - 4 = 0$$

$$z^2 + 2iz - 1 = 0$$

$$(z+i)^2.$$

812

$$z = \frac{1}{w^2}\frac{w-b}{1-\bar{b}w}$$

note that $w = i$

and also $b = i|b|$, says $\quad z = \frac{1}{-1}\frac{i - i|b|}{1+(i|b|)i} = (-1)i\,\frac{1-|b|}{1-|b|}$

$= -i$

Let's compute ramif again

$$z = \frac{1}{w^2}\frac{w-b}{1-\bar{b}w} \qquad w^{-2}\frac{1-bw^{-1}}{w^{-1}-\bar{b}} \quad \text{at } w=\infty$$

$$\frac{dz}{dw} = -\frac{2}{w^3}\frac{w-b}{1-\bar{b}w} + \frac{1}{w^2}\frac{1}{1-\bar{b}w} \pm \frac{1}{w^2}\frac{w-b}{(1-\bar{b}w)^2}(-\bar{b})$$

$$= \frac{1}{w^3(1-\bar{b}w)^2}\left\{-2(w-b)(1-\bar{b}w) + w(1-\bar{b}w) + w(w-b)\bar{b}\right\}$$

$$\left\{-2w+2b+2\bar{b}w^2-2|b|^2w + w - \bar{b}w^2 + w^2\bar{b} - w|b|^2\right\}$$

$$= \frac{1}{w^3(1-\bar{b}w)^2}\left\{2\bar{b}w^2 + w\left(-2-2|b|^2+1-|b|^2\right) + 2b\right\}$$

$$= \frac{1}{w^3(1-\bar{b}w)^2}\left\{2\bar{b}w^2 + \left(-1-3|b|^2\right)w + 2b\right\}$$

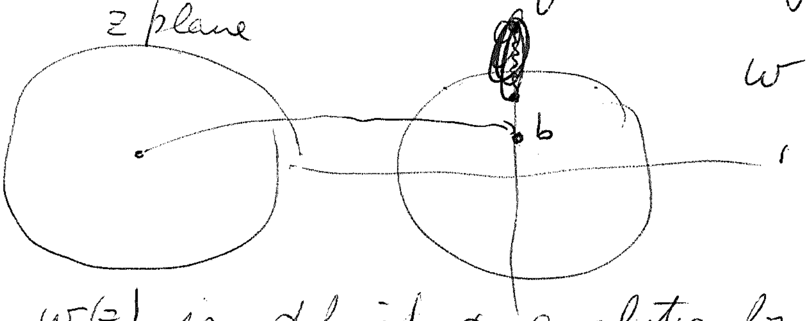
ramif. points ~~$w = \frac{1+3|b|^2}{4\bar{b}} \pm$~~

$$w = \frac{1+3|b|^2 \pm\sqrt{(1+3|b|^2)^2 - 16|b|^2}}{4\bar{b}}$$

disc. is $1+6|b|^2+9|b|^4-16|b|^2 = 1-10|b|^2+9|b|^4$

$= \underbrace{(1-|b|^2)}_{>0}(1-9|b|^2)$

so for $0<|b|<\frac{1}{3}$, numerator will be real. There will be a ramification point on the imag. axis



w plane

prod of roots is $\frac{b}{\bar{b}} = -1$

You believe that $w(z)$ is defined & analytic for $|z|<1$, and that $|w(z)|<1$ there

813 $\frac{dz}{z} = \frac{dw}{w^3(1-\bar b w)^2}\left\{2\bar b w^2 + \cdots\right\}\frac{w^2(1-\bar b w)}{w-b}$

$$\frac{dz}{z} = \underbrace{\left(\frac{2\bar b w^2 + (-1-3|b|^2)w + 2b}{(w-b)(1-\bar b w)}\right)}\frac{dw}{w}$$

you should be able to see that is real and mostly negative

What would you really like to know? I have this $w(z)$ defined for $|z| \leq 1$.

2 ramification points. Als

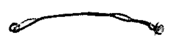
Basically things are a mess.

March 11, 1998 $z = \frac{1}{w^2}\frac{w-b}{1-\bar b w} = \frac{1}{w}\frac{1-bw^{-1}}{1-\bar b w}$

$$\{|z|<1\} \xrightarrow{w(z)} \{|w|<1\}$$

$$z=0 \longmapsto w = b = i|b|$$

try to extend w to the circle $|z|=1$ by radial limits. This should work easily except at ramification points. Suppose $|b| > \frac{1}{3}$. Then there's are an arc of $|w|=1$



where all

Idea: Suppose $|b| < \frac{1}{3}$. ~~Then all $w \ni |w|=1$ are unramified~~ Then have nice diffeom between $|w|=1$ and $|z|=1$. For each $z \in S^1$, consider the set of $w \in S^2 \ni w \longmapsto z$. This set has 3 elements at most and as long as we don't encounter the two ramification points on $i\mathbb{R}_{>0}$, or the point $w = \infty$ we should have two other curves lying over $|z|=1$. One should

814 radial limits of $w(z)$ as $z \to$ boundary $|z|=1$ radially. I know that $w(z)$ maps $|z|<1$ into $|w|<1$. It should be possible to extend this map from $|z|=1$ [redacted] into $|w| \leq 1$ by taking radial limits. [redacted] This should give a curve hopefully a simple closed curve sitting inside $|w|=1$ mapped diffeom. to $|z|=1$

To justify this picture I need to calculate the z-values of the ram. pts. This should mean the zeroes of the discriminant

$$z = \frac{1}{w^2}\,\frac{w-b}{1-\bar{b}w}$$

$$z(w^2 - \bar{b}w^3) = w - b$$

$$(\bar{b}z)w^3 - (z)w^2 + w - b = 0$$

$$w^3 + \left(\frac{-1}{\bar{b}}\right)w^2 + \left(\frac{1}{\bar{b}z}\right)w + \left(\frac{-b}{\bar{b}z}\right) = 0$$

$$4\left(\frac{1}{\bar{b}z}\right)^3 + 27\left(\frac{-b}{\bar{b}z}\right)^2 - \left(\frac{-1}{\bar{b}}\right)^2\left(\frac{1}{\bar{b}z}\right)^2 + 4\left(-\frac{1}{\bar{b}}\right)^3\left(\frac{-b}{\bar{b}z}\right) - 18\left(\frac{-1}{\bar{b}}\right)\left(\frac{1}{\bar{b}z}\right)\left(\frac{-b}{\bar{b}z}\right)$$

$$\frac{1}{(\bar{b}z)^3}\left\{4 + 27(+b)^2\bar{b}z - \left(\frac{1}{\bar{b}}\right)z + 4\left(\frac{b}{\bar{b}}\right)z^2 - 18bz\right\}$$

815

$$4\left(\frac{b}{\bar b}\right)z^2 + \left(27b|b|^2 - \frac{1}{\bar b} - 18b\right)z + 4$$

$$4\left(\frac{b}{\bar b}\right)z^2 + \left(27|b|^4 - 18|b|^2 - 1\right)\frac{z}{\bar b} + 4$$

roots of this give the z values of the ram. points at least for $z \neq 0, \infty$.

~~Check~~

$$z^2 + \left(\frac{27|b|^4 - 18|b|^2 - 1}{4\bar b}\right)z + \frac{\bar b}{b} = 0$$

$$\boxed{z^2 + \left(\frac{27|b|^4 - 18|b|^2 - 1}{4b}\right)z + \frac{\bar b}{b} = 0}$$

check this. If $|b| > \frac{1}{3}$ we know the ram. pts have $|w| = 1$ so $|z| = 1$. ∴ disc. < 0.

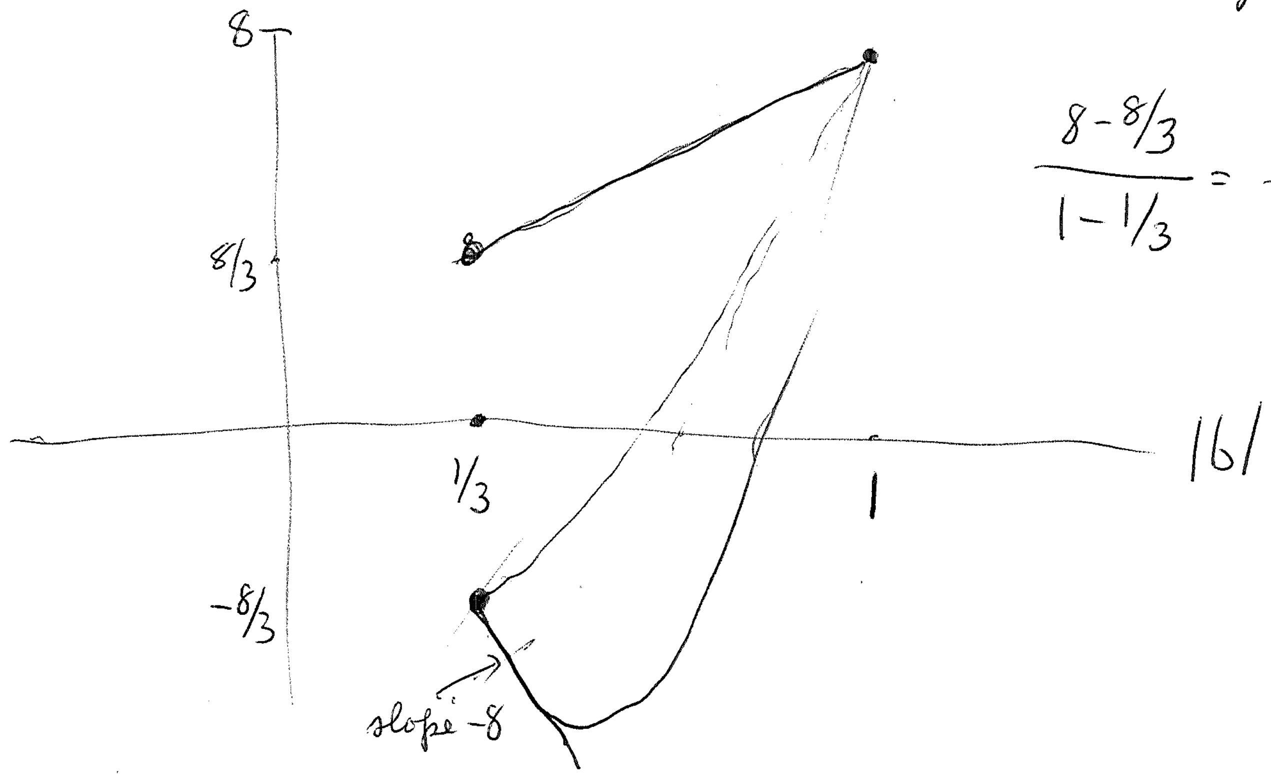
$$\underbrace{\left(\frac{27|b|^4 - 18|b|^2 - 1}{4i|b|}\right)^2}_{\text{purely imag}} - 4\underbrace{\left(\frac{\bar b}{b}\right)}_{-1} > 0 \quad ?$$

$$\left(\frac{27|b|^4 - 18|b|^2 - 1}{4|b|}\right)^2 < 4$$

$$\left|\left(27|b|^4 - 18|b|^2 - 1\right)\right| < 8|b|$$

$|b| = \frac{1}{3}$ $\qquad \frac{1}{3} - 2 - 1 \qquad\qquad 8\,\frac{1}{3}$

816 So it looks OKAY. Namely



$$\frac{8-8/3}{1-1/3} = \frac{8\cdot 2/3}{2/3} = 8$$

$$f(|b|) = 27|b|^4 - 18|b|^2 - 1$$

$$f'(x) = 108x^3 - 36x = (108x^2 - 36)x$$

$$= (3x^2 - 1)\,x\,36$$

$$\left(\frac{1}{3} - 1\right)\frac{1}{3}\,36 = -\frac{2}{9}\,36 = -8$$

$$f''(x) = 108\cdot 3\cdot\frac{1}{9} - 36\cdot\frac{1}{3}$$

$$= 36 - 12 > 0.$$

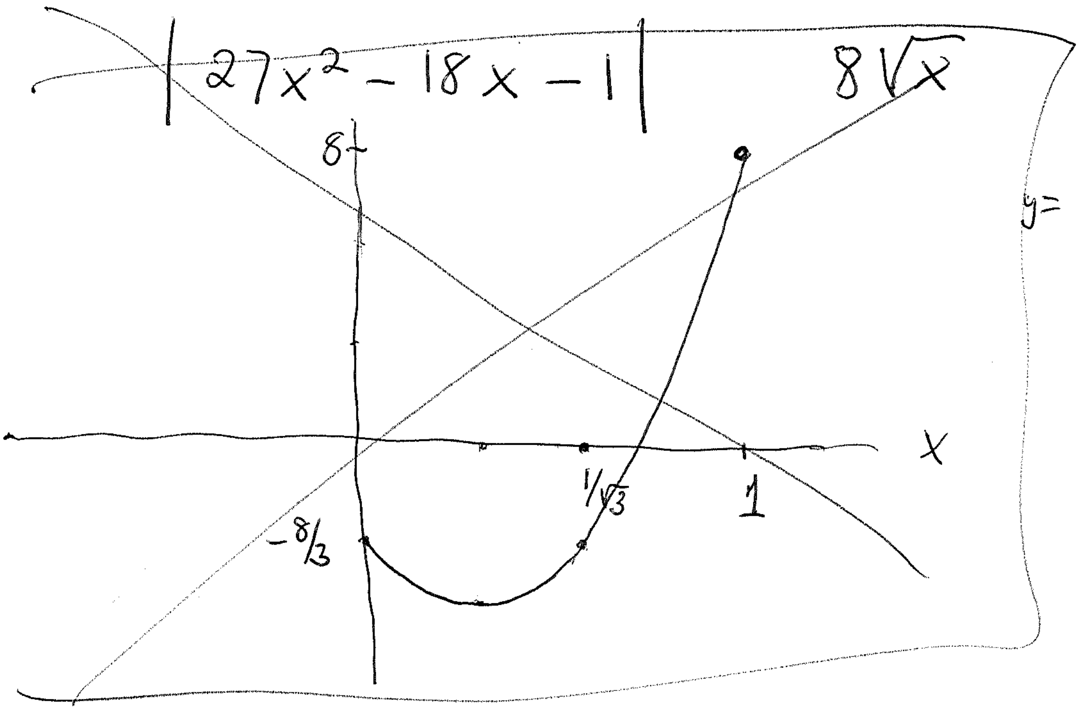
So it checks.

Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525

UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

8/18 $\quad 0 < |b| < \frac{1}{3}$ $\quad\quad 54x - 18 = 0$

$0 < x < \frac{1}{9}$ $\quad\quad x = \frac{1}{3}$

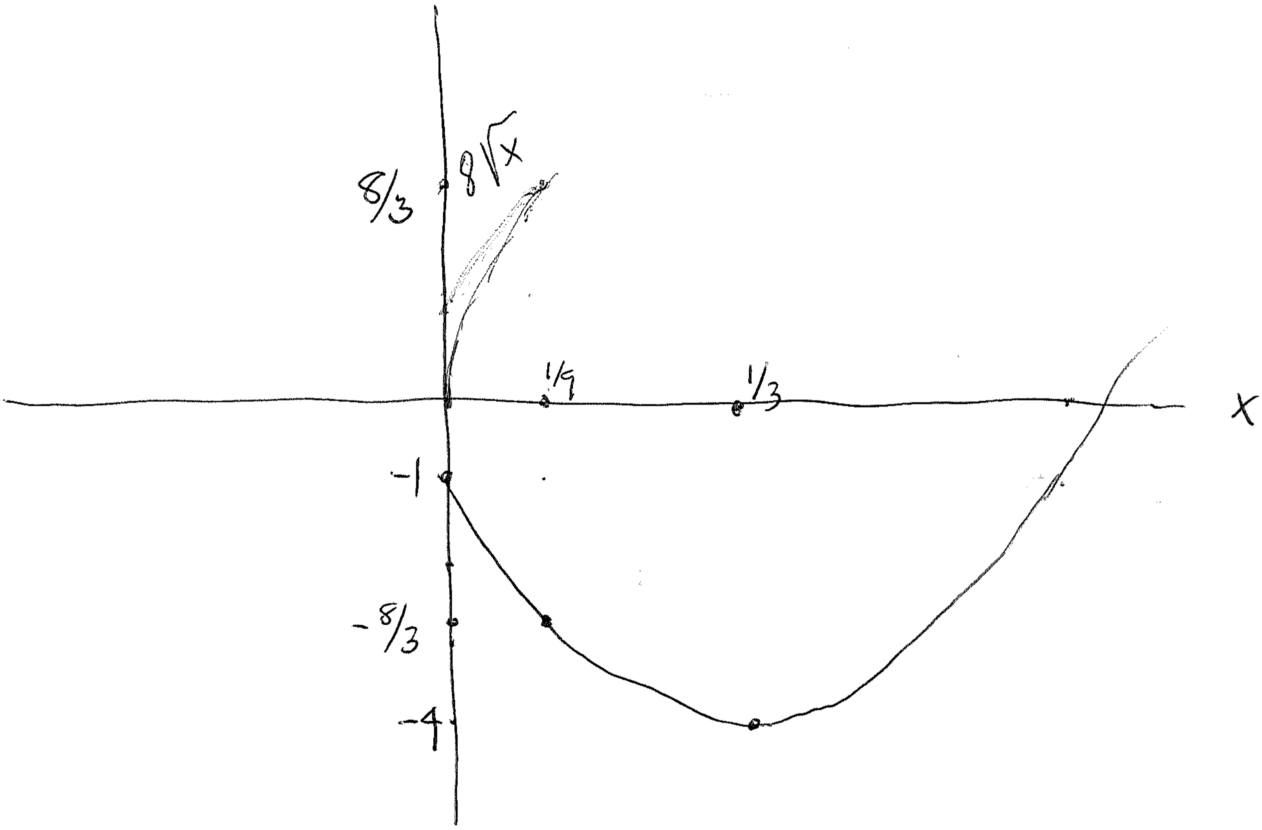


$x = \frac{1}{9}$

$y = 27x^2 - 18x - 1 = -8/3$

$x = 1/3,\ y = -4$

$0 < x = |b|^2 < \frac{1}{9}$



UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525

818

$$z = \frac{1}{w^2}\frac{w-b}{1-\bar{b}w} \qquad b = i|b| \qquad w = iv \qquad z =$$

$$z = \frac{1}{(iv)^2}\frac{iv - i|b|}{1 + i|b|iv} = \frac{-i}{v^2}\frac{v-|b|}{1-|b|v}$$

so you propose the change $iz = z'$ $w = iw'$ and then drop 's.

$$z = \frac{1}{w^2}\frac{w-b}{1-bw} \qquad 0<b<1. \qquad z=1 \text{ if } w=1.$$

w real $\Rightarrow$ z is real.

$$\frac{dz}{dw} = \frac{-2}{w^3}\frac{w-b}{1-bw} + \frac{1}{w^2(1-bw)} + \frac{1}{w^2}\frac{w-b}{(1-bw)^2}(+b)$$

$$= \frac{1}{w^3(1-bw)^2}\left\{-2(w-b)(1-bw) + w(1-bw) + w(w-b)b\right\}$$

$$-2w + 2b + 2bw^2 - 2b^2w + w - bw^2 + bw^2 - wb^2$$

$$2b + w(-1-3b^2) + 2bw^2$$

$$= \frac{2b}{w^3(1-bw)}\left(w^2 - \frac{1+3b^2}{2b}w + 1\right)$$

$$w = +\frac{1+3b^2}{4b} \pm \sqrt{\left(\frac{1+3b^2}{4b}\right)^2 - 1}$$

$$\frac{1+3b^2}{4b} = 1 \qquad 1+3b^2 = 4b$$

$$3b^2 - 4b + 1 = 0$$

$$(3b-1)\underbrace{(b-1)}_{<0} = 0$$

if $\frac{1}{3}<b<1$ then disc <0 so roots on the unit circle. If $0<b<\frac{1}{3}$ then disc >0, so roots >0 with product $= 1$.

819

$$\frac{(1+3t^2)^2-(4t)^2}{(4t)^2} = \frac{(3t^2-4t+1)(3t^2+4t+1)}{(4t)^2}$$

$$= \frac{(3t-1)(t-1)(3t+1)(t+1)}{(4t)^2}$$

$$= \frac{(9t^2-1)(t^2-1)}{(4t)^2}$$

$$w = \frac{1+3t^2 \pm \sqrt{(9t^2-1)(t^2-1)}}{4t}$$

$$2t w^2 - (1+3t^2) w + 2t = 0$$

$$w = \frac{1+3t^2 \pm \sqrt{(1+3t^2)^2 - 16t^2}}{4t}$$

---

let $z = 1$ and solve

$$1 = \frac{1}{w^2}\,\frac{w-t}{1-tw}$$

$$w^2 - tw^3 = w - t$$

$$tw^3 - w^2 + w - t = 0$$

$$w - 1 \,\overline{)\, tw^3 - w^2 + w - t} \quad \to \quad tw^2 + (t-1)w + t$$

$$tw^3 - tw^2$$

$$(t-1)w^2 + w$$

$$(t-1)w^2 - (t-1)w$$

$$tw - t$$

$$w^2 + \left(1-\frac{1}{t}\right)w + 1 = 0$$

$$w = \frac{-\left(1-\frac{1}{t}\right) \pm \sqrt{\left(1-\frac{1}{t}\right)^2 - 4}}{2}$$

check ram. $1 - \frac{1}{t} = \pm 2$

$\frac{1}{t} = 1 \mp 2 = \begin{matrix}-1\\3\end{matrix}$ $\quad t = \frac{1}{3}$

so for $t < \frac{1}{3}$ $\quad \frac{1}{t} > 3$ $\quad 1 - \frac{1}{t} < -2$

$(\quad)^2 > 4$

two roots real. So for $z = 1$ and $t < \frac{1}{3}$ three w values are real.

820 Go back and study a partial unitary. See if progress can be made on completing it to a unitary.

$$Y = aX \oplus V^+ \qquad \lambda(ax + v_0^+)$$
$$= V^- \oplus bX \qquad \quad \| $$
$$\qquad\qquad\qquad (v_{-1}^- + bx)$$

solution is

for $|\lambda| > 1.$ $\quad v_0^+ = (1-aa^*)(\lambda - ba^*)^{-1} v_{-1}^-$

$|\lambda| < 1$ $\quad v_{-1}^- = (1-bb^*)(1-\lambda ab^*)^{-1} \lambda v_0^+$

$$|\lambda|^2 (\|x\|^2 + \|v_0^+\|^2) = (\|v_{-1}^-\|^2 + \|x\|^2)$$

So if we can take the limit as $|\lambda| \to 1$, i.e.

$$S(\lambda) = (1-aa^*)(\lambda - ba^*)^{-1} : V^- \longrightarrow V^+$$

This is analytic for $|\lambda| > 1$.

~~There is no ...~~ ~~...~~ I think you have to understand the periodic 2 port in more detail.

$$\begin{pmatrix} \varphi_0^- \\ \varphi_0^+ \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar{b}}{a} & \frac{\lambda}{a} \end{pmatrix} \begin{pmatrix} \psi_1^- \\ \psi_1^+ \end{pmatrix}$$

$$S = \frac{\lambda^{-1} S + b}{\bar{b} S + \lambda}$$

$$\bar{b} S^2 + S(\lambda - \lambda^{-1}) - b = 0 \qquad \text{if } S(\lambda) \text{ anal.}$$

for $|\lambda| < 1$, then $S(0) = 0 \qquad S = \lambda w$

$$\bar{b} \lambda^2 w^2 + w(\lambda^2 - 1) - b = 0$$

821 If $S(\lambda)$ anal for $|\lambda|>1$, then $S(\infty)=0$ and $S=\lambda^{-1}w$ get.

$$\bar b\lambda^{-2}w^2+w(1-\lambda^{-2})+b=0$$

so these are ~~[illegible]~~ similar.

first.

$$\lambda^2(\bar b w^2+w)=w-b$$

$$\lambda^2=\frac{w+b}{w(1+\bar b w)}$$

$$z=\frac{1}{w}\frac{w+b}{1+\bar b w}=\frac{1}{iw'}\frac{iw'+i|b|}{1-i|b|iw'}$$

shift to ~~$iw' \pm i|b|$~~

$$=\frac{1}{w'}\frac{w'+|b|}{1+|b|w'}$$

drop 's.

$$z=\frac{1}{w}\frac{w+t}{1+tw} \qquad 0<t<1.$$

$$\frac{dz}{dw}=-\frac{1}{w^2}\frac{w+t}{1+tw}+\left(\frac{1}{w}\frac{1}{1+tw}\right)+\frac{1}{w}\frac{w+t}{(1+tw)^2}(-1)t$$

$$=\frac{1}{w^2(1+tw)^2}\left\{-(w+t)(1+tw)+w(1+tw)-w(w+t)t\right\}$$

$$-w-t-tw^2-t^2w+w+tw^2-w^2t-wt^2$$

$$-t-tw^2-2t^2w$$

$$=\frac{-t}{w^2(1+tw^2)}\left(w^2+2tw+1\right)$$

$$w^2+2tw+1=0$$

$$w=-t\pm\sqrt{t^2-1}$$ always on $|w|=1$.

822 Go back to

$z\psi_n = \psi_{n+1}$

$$\begin{pmatrix}\psi_n^-\\ \psi_n^+\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda a} & \frac{b}{a}\\ \frac{\bar{b}}{a} & \frac{\lambda}{a}\end{pmatrix}\begin{pmatrix}\psi_{n+1}^-\\ \psi_{n+1}^+\end{pmatrix}$$

$$\begin{pmatrix}\psi_0^-\\ \psi_0^+\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda a} & \frac{b}{a}\\ \frac{\bar{b}}{a} & \frac{\lambda}{a}\end{pmatrix}\begin{pmatrix}z\psi_0^-\\ z\psi_0^+\end{pmatrix}$$

$$\lambda\begin{pmatrix}\psi_0^+\\ \psi_1^-\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\psi_1^+\\ \psi_0^-\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}z^{-1}\psi_0^+\\ z\psi_1^-\end{pmatrix}$$

$$= \begin{pmatrix}az^{-1} & cz\\ bz^{-1} & dz\end{pmatrix}\begin{pmatrix}\psi_0^+\\ \psi_1^-\end{pmatrix}$$

$$\lambda^2 - (az^{-1}+dz)\lambda + 1 = 0$$

$$\lambda + \lambda^{-1} = a(z+z^{-1})$$

$$z^2 - \left(\frac{\lambda+\lambda^{-1}}{a}\right)z + 1 = 0$$

z eigenvalue of T

$$\begin{vmatrix}\frac{1}{\lambda a} - z & \frac{b}{a}\\ \frac{\bar{b}}{a} & \frac{\lambda}{a} - z\end{vmatrix} = 0$$

eigenvector is $\begin{pmatrix}-\frac{b}{a}\\ \frac{1}{\lambda a} - z\end{pmatrix}$ or $\begin{pmatrix}\frac{\lambda}{a} - z\\ -\frac{\bar{b}}{a}\end{pmatrix} = \begin{pmatrix}\lambda - az\\ -\bar{b}\end{pmatrix}$

$$S = \frac{\lambda - az}{-\bar{b}}$$

823

$$S = \frac{\lambda^{-1}S + b}{\bar{b}S + \lambda} \qquad \bar{b}S^2 + (\lambda - \lambda^{-1})S - b = 0$$

$$-itS + (\lambda - \lambda^{-1}) - itS^{-1}$$

$$S + S^{-1} = \frac{\lambda - \lambda^{-1}}{it}$$

March 12, 1998

$$T = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar{b}}{a} & \frac{\lambda}{a} \end{pmatrix} \qquad T\begin{pmatrix} S \\ 1 \end{pmatrix} = z\begin{pmatrix} S \\ 1 \end{pmatrix} \qquad a := \sqrt{1-|b|^2}$$

So the point is maybe going to be that where $|S(\lambda)| < 1$ and $|\lambda| \neq 1$, there will be continuous spectrum?

$$\frac{1}{\lambda a}S + \frac{b}{a} [redacted] = zS \qquad z^2 - \left(\frac{\lambda + \lambda^{-1}}{a}\right)z + 1 = 0$$

$$\frac{\bar{b}}{a}S + \frac{\lambda}{a} = [redacted] z$$

S is a function of λ.

$$\lambda^{-1}S + b = \bar{b}S^2 + \lambda S$$

$$-\bar{b}S^2 + (\lambda^{-1} - \lambda)S + b = 0 \qquad b = it, \quad -\bar{b} = it$$

$$S^2 - \frac{\lambda - \lambda^{-1}}{it}S + 1 = 0$$

$$S = \frac{\lambda - \lambda^{-1}}{2it} \pm \sqrt{\left(\frac{\lambda - \lambda^{-1}}{2it}\right)^2 - 1}$$

there should be a branch analytic at $\lambda = 0$

$$\lambda S = \frac{\lambda^2 - 1}{2it} \pm \sqrt{\left(\frac{\lambda^2 - 1}{2it}\right)^2 + \lambda^2}$$

824

$$\lambda^2 w^2 - \frac{\lambda^2 - 1}{it} w + 1 = 0$$

~~Go back to the equation~~

$\S^2$ We have $\frac{S(\lambda)}{\lambda} = w(\lambda^2)$

where $w(u)$ satisfies

$$u w^2 - \frac{u-1}{it} w + 1 = 0 \qquad \left(\frac{\lambda^2-1}{it}\right)^2 - 4\lambda^2$$

$$u\left(w^2 - \frac{1}{it} w\right) + \frac{1}{it} w + 1 = 0$$

$$u = \frac{1}{\frac{1}{it} w - w^2}\left(\frac{1}{it} w + 1\right)$$

$$= \frac{1}{w} \frac{w + it}{1 - itw} = \frac{1}{iw'} \frac{iw' + it}{1 - it\,iw'}$$

$$u = \frac{1}{w'} \frac{w' + t}{1 + tw'}$$

drop 's.

$$u = \frac{1}{w} \frac{w+t}{1+tw} = \frac{1+tw^{-1}}{1+tw}$$

$$\frac{du}{dw} = \frac{-1}{w^2} \frac{w+t}{1+tw} + \frac{1}{w} \frac{1}{1+tw} - \frac{1}{w} \frac{w+t}{(1+tw)^2} t$$

$$= \frac{1}{w^2(1+tw)^2}\left(-(w+t)(1+tw) + \cancel{w(1+tw)} - tw(w+t)\right)$$

$$= \frac{+1}{w^2(1+tw)^2}\left(-t - t^2 w - tw^2 - t^2 w \right.$$
$$\left. - tw^2 + (-2t^2)w + t\right)$$

$$\frac{du}{dw} = \frac{-t}{w^2(1+tw)^2}\left(w^2 + 2tw + 1\right)$$

ram. $-t \pm \sqrt{t^2 - 1}$ always on $|w| = 1$.

825

$$u(\omega + t\omega^2) = \omega + t$$

$$(ut)\omega^2 + (u-1)\omega + (-t) = 0$$

$$\text{disc} = (u-1)^2 - 4(ut)(-t) = u^2 - 2u + 1 + (4t^2)u$$

$$= u^2 + (4t^2 - 2)u + 1$$

$$u = -(2t^2-1) \pm \sqrt{(2t^2-1)^2 - 1}$$

$$-1 < 2t^2 - 1 < 1 \quad \text{for } 0<t<1.$$

So now we know that there's no ramification over the disk. $|\lambda^2| < 1$, so $S = i\lambda\omega(\lambda^2)$ is analytic for $|\lambda| < 1$.

Now we want to find the image of $|\lambda| \leq 1$ under S. Given $u = \lambda^2$ we have two values for ω:

$$(ut)\omega^2 + (u-1)\omega + (-t) = 0$$

At this point you might go back to S.

$$\lambda^2 t \left(\frac{S}{i\lambda}\right)^2 + (\lambda^2 - 1)\frac{S}{i\lambda} + (-t) = 0$$

$$-tS^2 + \frac{\lambda - \lambda^{-1}}{i} S - t = 0$$

$$S^2 - \left(\frac{\lambda - \lambda^{-1}}{it}\right) S + 1 = 0$$

Put in $\lambda = e^{i\theta}$ and you get

$$S^2 - \left(\frac{2}{t}\sin\theta\right) S + 1 = 0$$

$$S = \frac{1}{t}\sin\theta \pm \sqrt{\frac{\sin^2\theta}{t^2} - 1}$$

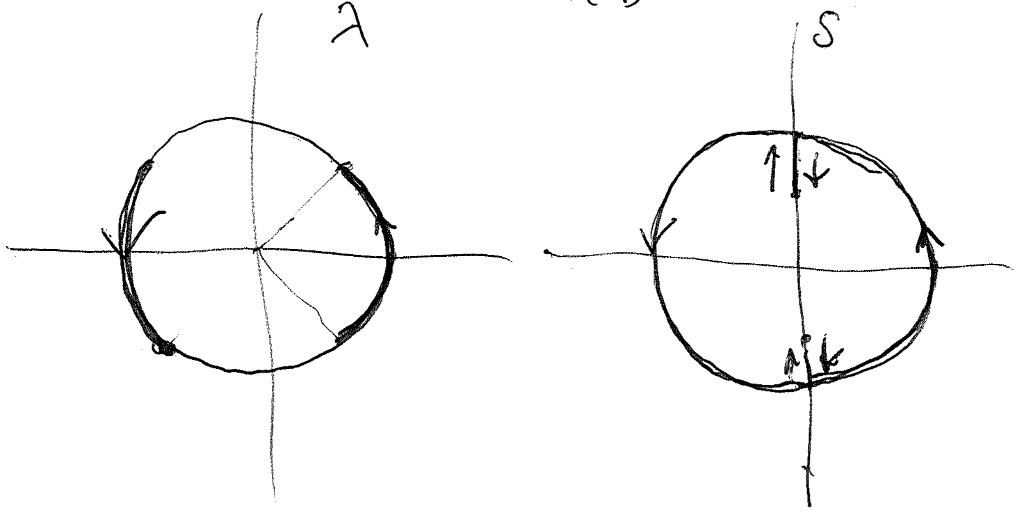
on the unit circle except where $|\sin\theta| > t$

826 So what do you find?

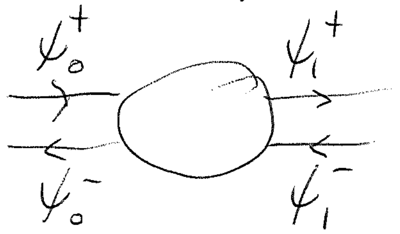
~~[illegible]~~

$$\lambda = e^{i\theta} \longrightarrow S^2 - 2\frac{\sin\theta}{t} S + 1 = 0$$

$$S(\lambda) = e^{\pm \alpha} \qquad \cos\alpha = \frac{\sin\theta}{t}$$



Go over formulas.



$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^- \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$$

$$u(\delta_1^-) = c\delta_1^+ + d\delta_0^-$$

symmetry under $\delta_0^+ \longleftrightarrow \delta_1^- \quad \delta_1^- \longleftrightarrow \delta_0^+$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ b & a \end{pmatrix}$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \qquad \bar{a}b + \bar{b}a = 0 \qquad \det 1 \quad \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{1-t^2} & it \\ it & \sqrt{1-t^2} \end{pmatrix}$$

eigenvector equation $\lambda \begin{pmatrix} \psi_1^+ \\ \psi_0^- \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \psi_0^+ \\ \psi_1^- \end{pmatrix}$

$$\lambda\psi_1^+ = a\psi_0^+ + c\psi_1^- \qquad \lambda\psi_0^- = b\psi_0^+ + d\psi_1^-$$

$$\psi_0^+ = \frac{\lambda}{a}\psi_1^+ - \frac{c}{a}\psi_1^-$$

$$\psi_0^- = \frac{b}{\lambda}\left(\frac{\lambda}{a}\psi_1^+ - \frac{c}{a}\psi_1^-\right) + d\psi_1^- = \frac{b}{a}\psi_1^+ + \left(d - \frac{bc}{a}\right)\frac{1}{\lambda}\psi_1^-$$

827

$$\begin{pmatrix}\psi_0^- \\ \psi_0^+\end{pmatrix} = \underbrace{\begin{pmatrix}\frac{\Delta}{\lambda a} & \frac{b'}{a} \\ -\frac{c}{a} & \frac{\lambda}{a}\end{pmatrix}}_{T}\begin{pmatrix}\psi_1^- \\ \psi_1^+\end{pmatrix}$$

this is the eigenvector eqn written in transfer matrix form.

want a solution of

$$\psi_n = T\psi_{n+1} \qquad \text{for all } n.$$

solution is $\psi_n = T^{-n}\psi_0$ ψ_0 arb.

If you want ℓ^2 solution as $n \to \infty$ you need $T\psi_0 = z\psi_0$ with $|z| > 1$.

Let tentatively $\psi_0 = \begin{pmatrix} S \\ 1\end{pmatrix}$ be an eigenvector

$\ni$ $T\begin{pmatrix} S \\ 1\end{pmatrix} = z\begin{pmatrix} S \\ 1\end{pmatrix}$ $|z| > 1$

Digress back to general theory. H Hilbert space, u unitary, ξ_1 cyclic vector, $X = (\mathbb{C}\xi_1)^\perp$, $a : X \hookrightarrow H$ inclusion, $b : X \to H$ restriction of u to X.

$$V = aX \oplus V^+ = V^- \oplus bX$$

$V^+ = \mathbb{C}\xi_1$, $V^- = \mathbb{C}u(\xi_1)$

eigenvector equation for the partial isom. is

~~$\lambda(ax + \xi_1^+) = \ldots + bx$~~

$$\lambda(ax + v_0^+) = v_{-1}^- + bx$$

$$\boxed{(\lambda a - b)x = v_{-1}^- - \lambda v_0^+}$$

the eigenvector equation for u is this equation together with the condition that

$$u(v_0^+) = v_{-1}^-$$

828 The eigenvector equation for the port can be solved off the unit circle:

$$\begin{cases} x = (\lambda - a^* b)^{-1} a^* v_{-1}^- \\ v_0^+ = (1 - a a^*)(\lambda - b a^*)^{-1} v_{-1}^- \end{cases} \qquad |\lambda| > 1$$

$$\begin{cases} x = (1 - \lambda b^* a)^{-1} \lambda b^* v_0^+ \\ v_{-1}^- = \lambda (1 - b b^*)(1 - \lambda a b^*)^{-1} v_0^+ \end{cases} \qquad |\lambda| < 1.$$

You need examples. $\mathcal{H} = L^2(S^1, d\mu)$ since the rep is cyclic. Finite-dimensional measure. When X is infinite dimensional - stuff can go into X and get lost. That's what $|S(\lambda)| < 1$ for $|\lambda| = 1$ means.

$$aX \oplus V^+ = V^- \oplus bX$$

$$|\lambda|^2 \left(\|x\|^2 + \|v_0^+\|^2 \right) = \|v_{-1}^+\|^2 + \|x\|^2$$

Ω

[illegible]

Try to finish the **periodic picture**. [redacted]

$a = \sqrt{1 - |b|^2}$

$$\begin{pmatrix} \psi_0^- \\ \psi_0^+ \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{b}{a} & \frac{\lambda}{a} \end{pmatrix}}_{T} \begin{pmatrix} \psi_1^- \\ \psi_1^+ \end{pmatrix}$$

$$\zeta^2 - \frac{\lambda + \lambda^{-1}}{a} \zeta + 1 = 0$$

$$\zeta + \zeta^{-1} = \frac{\lambda + \lambda^{-1}}{a}$$

$$\psi_n = T^{-n} \psi_0$$

$$\psi_n = T^n \psi_0$$

basically for each $\lambda \ni |\lambda| < 1$ there should be a unique $\zeta \ni |\zeta| > 1$.

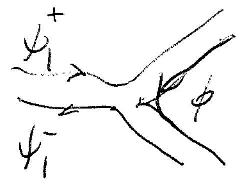
Point is that the two zeta values are inverse, hence if conj of abs. value $\Rightarrow \lambda + \lambda^{-1} \in \mathbb{R}$

829 I have an interesting paradox. How do I make progress. Begin with

$b = it$
$a = \sqrt{1-t^2}$

$$\begin{pmatrix} \psi_0^- \\ \psi_0^+ \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{b}{a} & \frac{\lambda}{a} \end{pmatrix} \begin{pmatrix} \psi_1^- \\ \psi_1^+ \end{pmatrix}$$

$\psi_0^- = S\psi_0^+$ domain



Y $\delta_n^\pm$ $n \geq 0$ orth basis.

$$u(\delta_n^+) = a\delta_{n+1}^+ + b\delta_n^- \qquad n \geq 0$$
$$u(\delta_n^-) = b\delta_n^+ + a\delta_{n-1}^- \qquad n \geq 1$$

domain aX spanned by δ_n^+ $n \geq 0$, δ_n^- $n \geq 1$

range bX ——— δ_n^+ $n \geq 1$, δ_n^- $n \geq 0$

~~Now you need Eigenvalue equations are~~

Eigenvector equations

$$a\psi_n^+ + c\psi_{n+1}^- = \lambda\psi_{n+1}^+ \qquad n \geq 0.$$
$$b\psi_n^+ + d\psi_{n+1}^- = \lambda\psi_n^-$$

$\sum_{n \geq 0} \psi_{n+1} z^{-n} = z\left(\sum_{n \geq -1} \psi_{n+1} z^{-n+1} - \psi_0\right)$

use F.T. $\hat{\psi}(z) = \sum_{n \geq 0} \psi_n^+ z^{-n}$

$$z^{-1}a\hat{\psi}^+ + cz(\hat{\psi}^- - \psi_0^-) = \lambda z(\hat{\psi}^+ - \psi_0^+)$$
$$b\hat{\psi}^+ + dz(\hat{\psi}^- - \psi_0^-) = \lambda\hat{\psi}^-$$

$$\begin{pmatrix} z^{-1}a - \lambda & c \\ b & dz - \lambda \end{pmatrix} \begin{pmatrix} \hat{\psi}^+ \\ \hat{\psi}^- \end{pmatrix} = \begin{pmatrix} c\psi_0^- - \lambda\psi_0^+ \\ dz\psi_0^- \end{pmatrix}$$

830

$$\begin{vmatrix} z^{-1}a-\lambda & b \\ b & zd-\lambda \end{vmatrix} = \lambda^2 - (z^{-1}a + zd)\lambda + ad-bc$$

$$= \lambda^2 - a(z+z^{-1})\lambda + 1$$

~~So suppose you find~~ This is an ~~invertible 2~~ unimodular 2x2 matrix, so the solution is

$$\hat{\psi} = \frac{1}{\lambda^2 - a(z+z^{-1})\lambda + 1} \begin{pmatrix} za-\lambda & -b \\ -b & z^{-1}a-\lambda \end{pmatrix} \begin{pmatrix} b\psi_0^- - \lambda\psi_0^+ \\ za\psi_0^- \end{pmatrix}$$

Laplace inversion formula is

$$\psi_n = \frac{1}{2\pi i} \oint \hat{\psi}(z)\, z^{+n-1}\, dz$$

Contour: $\hat{\psi} = \sum_{n \geq 0} \psi_n z^{-n}$ analytic for $|z|$ large

Contour is taken over a large circle, and then shrunk to [redacted] zero picking up the singularities along the way. Here singularities are two roots of $\lambda^2 - a(z+z^{-1})\lambda + 1$ in the variable z

$a = \sqrt{1-t^2}$ $b = it$

$$z + z^{-1} = \frac{\lambda + \lambda^{-1}}{a}$$

$$z^2 - \left(\frac{\lambda+\lambda^{-1}}{a}\right) z + 1 = 0$$

$$z_i = \frac{\lambda+\lambda^{-1}}{2a} \pm \sqrt{\left(\frac{\lambda+\lambda^{-1}}{2a}\right)^2 - 1} \qquad i = 1, 2.$$

These are the two roots, so

$$\psi_n = z_1^{+n-1}(?) + z_2^{+n-1}(?)$$

note that the z_i with larger $|z_i|$ appears first.