637 Feb 12

Let's go over the picture. You work over the Riemann sphere, parameter s, the disk $Re(s) > 0$, the circle $i\mathbb{R} \cup \infty$. The scattering operator $S(s)$ is analytic ~~on the~~ a nbd of closed disk, unitary on the boundary. It gives rise to a vector bundle F over $\mathbb{C}P^1$ via clutching. The fact that S is analytic inside in disk ~~~~ should say that F is either of ~~positive~~ negative type. ~~The~~ ~~~~ The aim now is to construct the ~~~~ canon. resolution

$$0 \longrightarrow F \longrightarrow \mathcal{O} \otimes (\ ) \longrightarrow \mathcal{O}(1) \otimes (\ ) \longrightarrow 0$$

and to ~~~~ understand the K-module arising.

Your initial approach to scattering ~~is ba~~ uses the unit circle and disk $|z| < 1$. This leads directly to a partial unitary operator. The details have to be written out carefully before you can lecture on them Monday. ~~~~ Question: Is the difference between z and s simply discrete versus continuous? Trans line ~~~~ of unit length (the sort of thing you want for the edges of a graph) yields the variable $e^{s} = e^{-i\omega}$ or $e^{-s} = e^{i\omega}$ such that RHP corresp to $|z| < 1$.

~~~~ Idea: Return to Lap Transform suppose ~~given~~ an incoming wave $(E+I)(x,t) = f(x+t)$, the reflected wave is what? $(Sf)(t-x)$?

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad \text{Assume } f(t) = 0 \text{ for } t \le 0$$

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} \hat{f}(s)\, ds$$
~~~~

Solutions are

$$S = \frac{B}{A}$$

$$\begin{pmatrix} E+I \\ E-I \end{pmatrix}^{\wedge}(x,t) = \begin{pmatrix} A e^{s(x+t)} \\ B e^{s(-x+t)} \end{pmatrix}$$

$$B = SA$$

$$(E+I)(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s(x+t)} \hat{f}(s)\, ds$$

$$(E-I)(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s(-x+t)} (S\hat{f})(s)\, ds$$

All this is in the continuous time setting

I want to formulate the idea of twisting the line bundle and taking sections.

Look at $\hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$. This is analytic for $Re(s) >$ constant (which measures the growth of $f(t)$), so $\hat{f}$ is entire when f has compact support. Actually $f(t) = \delta(t)$ is the important case I guess: $\hat{f}(s) = 1$, then the output is

$$(E-I)(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s(-x+t)} S(s)\, ds$$

($x > t$ before the signal can reach x)

Note that if $-x+t < 0$, then can push the contour to the right to get 0. The singularities of S, i.e. poles, lie in $Re(s) < 0$. These will contribute terms $e^{(-\mu)(-x+t)} = e^{\mu(x-t)}$ with $Re(\mu) > 0$. Typical scattering because for x bounded $(E-I)(x,t)$ decays as $t \to \infty$.

639 suppose the contour $Re(s) = \cancel{x}-b$ $\quad b \geq 0$

$$e^{(-b)(\cancel{x}-x+t)} = e^{b(x-t)} \to 0 \quad \text{as } b\to\infty \text{ if } x-t<0$$
$x<t$.

Should get zero then after all the polar contributions of S. So the only stuff of significance are the ~~poles~~ singularities of S ~~at the~~ which corresp by reflection to the zeros of S in the RHP.

<u>Next</u> go onto discrete theory.

~~Go to discrete~~ theory. $\quad L^2(S^1) = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n \;\middle|\; \sum \|a_n\|^2 < \infty \right\}$

$L^2(S^1, V) \qquad V = \mathbb{C}^n$ with $(v_1, v_2) = v_1^* v_2$

$U_0 =$ mult by z, forward shift.

U perturbation of U_0

$$E_1 \overset{a}{\underset{b}{\rightrightarrows}} E_0$$

$$\cdots \oplus z^{-2}V \oplus z^{-1}V \oplus \mathrm{Im}(a) \oplus V \oplus zV$$

$$\oplus z^{-2}V \oplus z^{-1}V \oplus V \oplus \mathrm{Im}(b) \oplus zV \oplus \cdots$$

So the basic ingredient is a stable unitary isomorphism between two hermitian spaces W_1, W_2 i.e. a unitary $\qquad W_1 \oplus X \simeq X \oplus W_2$

You are unclear about what to do.

Coupling. Recovering from S the partial unitary. This should be straightforward up to some choices. But then there's the coupling question

A first idea is ~~that~~ to consider fixed incoming space, whence a complementary outg. sp.

640 To find exposition. Take 1-port. $H = L^2(S^1)$

$$L^2(S^1) = H^- \oplus H^+ = \bigoplus^{(2)}_{n<0} z^n\mathbb{C} \oplus \bigoplus^{(2)}_{n\geq 0} z^n\mathbb{C}.$$

U_0 = mult by z. U perturbation of U_0.

Maybe the first thing to get straight is outgoing subspaces: $UM \subset M$, $\bigcap U^n L = 0$, $\overline{\bigcup U^n L} = H$.

$V = L \ominus U(L) \Rightarrow H = \bigoplus^{(2)}_{n\in\mathbb{Z}} U^n V$ TFAE

(i) outgoing subspace L

(ii) closed subspace V such that $V \perp U^n(V)$ for all $n \neq 0$, and $\ni$ $\overline{\oplus U^n V} = H$ no U, U^{-1} inv. subspace $\perp$ to V.

(iii) ~~...~~ If V_0 given subspace as in (ii), then third thing is a

$S : S^1 \longrightarrow$ Unitary group(V_0) $\qquad S \in L^\infty(S^1, U(V_0))$

~~...~~ S extends to analytic function on disk

$\Longleftrightarrow$ $L = SH^+ \subset H^+$.

---

examples. $H = L^2(S^1)$. $M_0 = H^2(S^1) = \bigoplus^{(2)}_{n\geq 0} z^n\mathbb{C}$

Take $M_0 \supset M$ $\qquad M_0/M$ f.d.

mult by z gives a contraction of on $M_0 \ominus M$

can find a poly $f(z)$ $\ni$ $f(z)$

$$f(z) = \prod_{i=1}^{d} (z - \lambda_i), \quad |\lambda_i| < 1, \quad M = fM_0$$

$$S(z) = \prod \frac{z - \lambda_i}{1 - \bar{\lambda}_i z} \qquad SM_0 = fM_0 = M$$

641 Start again. You study scattering operators. Focus on discrete time. Basically you have $H = L^2(S^1, \mathbb{C}^n)$ with $U_0 = z$, really different pictures, any two ~~[illegible]~~ are related by an $S : S^1 \longrightarrow$ ~~[illegible]~~ U_n, $S(z)$ bdd meas, rather in L^∞, unitary values. One does topology here so that you want S ~~[illegible]~~ continuous at least. Actually I'm concerned with S rational function of z, U_n valued on $|z| = 1$. So what can you do? ~~[illegible]~~

Use S as a clutching function to get a vector bundle over $\mathbb{CP}^1 = \mathbb{C} \cup \infty$. Because $S(z)$ analytic, regular ~~on~~ nbd of $|z| \leq 1$.

Rice 4:45 ~~What happens?~~

Concentrate. You have clutching function S, which gives rise to v.b. F over $\mathbb{CP}^1$, which has canonical resolution, yielding a K-module. So now you need to straighten this out. A problem is ~~[illegible]~~ the fibre F_z at $z \in \mathbb{CP}^1$.

You need some simple geometry. ~~[illegible]~~ You have various objects to relate, you have ~~duality~~ technical problems with duality. ~~[illegible]~~ Probably you want to work with negative bundles, subbundles of trivial bundles.

Begin with $S(z)$ merom, analytic for $|z| \leq 1$.

~~[illegible]~~ Cech complex for cohomology

$\mathbb{C}[z]$

Γ_+ functions holom on $|z| \le 1$

Γ_- ———————— $|z| \ge 1$ vanishing at ∞.

$\Gamma = \Gamma_- \oplus \Gamma_+$

The negative bundle should have Čech cx.

$$\Gamma_- \oplus S\Gamma_+ \hookrightarrow \Gamma_- \oplus \Gamma_+$$

so $H^0(F) = 0$ $H^1(F) = \Gamma_+/S\Gamma_+$? There are two bundles to keep straight F and $F(\pm 1)$?

$$0 \to F \to \mathcal{O} \otimes \underbrace{H^0(\check{F})^*}_{H^1(F(-2))} \longrightarrow \mathcal{O}(1) \otimes \underbrace{H^1(F(-1))}_{\| \atop H^0(\check{F}(-1))^*} \to 0 \ ?$$

~~[illegible]~~

Ultimately you want to be able to connect the fibres of the v.b. with the K-module. There should be simple alg.

What not start with what you know: namely you have

$$L^2(S^1) = H^- \oplus \quad ?$$

What you have is H^+ and $SH^+ \subset H^+$, and you want the partial unitary. You want to go from SH^+ to the partial unitary.

~~Now SH^+ I think has~~
~~H and so forth~~

If you are thinking of $W \underset{b}{\overset{a}{\rightrightarrows}} V$, then

$W = H^+ \ominus SH^+$ and $V = H^+ \ominus zSH^+$

Basically you have

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$$L^2 = H^- \oplus H^+$$
$$= H^- \oplus H^+ \ominus S \quad ?$$

Go back to $W_1 \oplus X \simeq X \oplus W_2$. Then you have

$$\overbrace{z^{-1}W_1 \oplus}^{H^-} \; X \oplus \overbrace{W_2 \oplus zW_2 \oplus}^{SH^+}$$

$$\begin{array}{c} z \quad\quad 1 \quad\quad z \quad\quad z \end{array}$$

$$z^{-1}W_1 \oplus W_1 \oplus X \oplus \underbrace{zW_2 \oplus z^2 W_2}_{zSH^+}$$

So it seems you first for $X = H^+ \ominus SH^+$
$= H^+ \cap S(H^-)$

So this can be cleaned up with effort.

Basically this happen between H^- and zSH^+

~~[illegible]~~

You have W = top of H^+ = $zH^- \cap H^+$
You also have W_2 = top of SH^+

Feb 13. $H = L^2(S^1, W) \supset z^0 V = W_0 \wedge_0 W$ (basepoint)
another subspace gen. H, $z^n W \perp W$ $n \neq 0$. S
scattering op $\ni$ $SW_0 = W$. Assume rational in z
SH^+ finite codim in H^+. $\therefore S$ analytic $|z| \leq 1$.

~~[illegible]~~

IDEA: For your graph you perhaps need to pass from a $U(3)$ scattering matrix to a $U(1,2)$ matrix of some sort.

644 $\quad H = L^2(S^1; W) \supset H^+ = \bigoplus_{n \geq 0}^{(2)} z^n W \quad , \quad W_0 = z^0 W$

What are you trying to accomplish? A good picture of a vector bundle over $\mathbb{C}P^1$, given a scattering operator S. Start with the Hilbert space $H = L^2(S^1, V)$, unitary op mult by z, the outgoing subspace $H^+ = \bigoplus_{n \geq 0}^{(2)} z^n V$ and another outgoing subspace contained in H^+, which we know can be rep. SH^+, S unitary in $L^\infty(S^1, \mathrm{End}(V))$, S unique up to right mult by a constant unitary. You have outg. spaces

$$\begin{array}{ccc} H^+ & \overset{X}{\supset} & SH^+ \\ V\ \cup & & \cup\ SV \\ zH^+ & \underset{zX}{\supset} & zSH^+ \end{array}$$

From this picture you want to get a partial unitary

Let $\quad X = H^+ \ominus SH^+$

$\qquad V = H^+ \ominus zH^+$

Then $\quad H^+ \ominus zSH^+ = V \oplus zX$

$\qquad\qquad\qquad = X \oplus SV$

Call this subspace Y. So $Y = \hat{V} \oplus z\overset{d}{X} = \overset{d}{X} \oplus S\overset{r}{V}$

$-d$ is the degree of the vector bundle. You want an ~~exact~~ ex seq

$$0 \longrightarrow F \longrightarrow \mathcal{O} \otimes Y \longrightarrow \mathcal{O}(-1) \otimes X \longrightarrow 0$$

roughly, in any case a ~~parameter family of~~ pencil $az + b$ of ~~maps~~ surjections from Y to X.

You have something quite simple, namely a space Y split in two ways, an isomorphism: mult by z, between ~~the~~ summands zX and X, and an isomorphism

Start again. $H = L^2(S^1, V) \supset H^+ = H^2(S^1, V) = \bigoplus\limits_{n \geq 0}^{(2)} z^n V$

Given an outgoing subspace $SH^+ \subset H^+$, S the scattering op. Have outgoing subspace

$$\begin{array}{ccc} H^+ & \overset{X}{\supset} & SH^+ \\ V\ \cup & & \cup\ SV \\ zH^+ & \underset{zX}{\supset} & zSH^+ \end{array}$$

$V = H^+ \ominus zH^+$

$X = H^+ \ominus SH^+$

Let $Y = H^+ \ominus zSH^+$. Then Y has two splittings $Y = V \oplus zX = X \oplus SV$ and the unitary z gives an isomorphism $zX \simeq X$, so that we have a partial unitary operator $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ given by the two embeddings of X into Y, one of which involves the op z. What can we do with a partial unitary, What is the S operator assoc. to a partial unitary, ~~What can we do with~~ Notice that the subspaces V + SV of Y are linked by S, where S is a matrix of analytic functions. Thus SV is the space spanned by the columns of S. Now you have to introduce specializing at a point ζ of S^1.

Start again - make serious attempt to work this out.

Begin with $H = L^2(S^1) \supset H^+ = H^2(S^1)$ and another outgoing space SH^+. Have outgoing subsp.

$$\begin{array}{ccc} H^+ & \overset{X}{\supset} & SH^+ \\ V\ \cup & & \cup \\ zH^+ & \supset & zSH^+ \end{array}$$

 whence ~~putting~~ putting

$$X = H^+ \ominus SH^+$$
$$V = H^+ \ominus zH^+$$
$$Y = H^+ \ominus zSH^+$$

we have $Y = \overset{n}{V} \oplus z\overset{d}{X} = \overset{d}{X} \oplus S\overset{n}{V}$

All takes place in a space of holom. functions on the disk with L^2 boundary values. $V = \mathbb{C}$ the constant functions. $S(V)$ = multiples of $S(z)$. We can evaluate at a point ζ of S^1, getting a linear functional.

S is some function $\displaystyle\prod_{j=1}^{n} \frac{(z-\lambda_1)\cdots(z-\lambda_n)}{(1-\bar\lambda_1 z)\cdots(1-\bar\lambda_n z)}$

$$X = SH^- \cap H^+ = H^+ \cap (SH^+)^\perp$$

somehow consists of analytic $f^+ \in H^+$ such $S^{-1} f^+ \in H^-$

Then $S^{-1} f^+ = f^-$ or $S^{-1} = \dfrac{f^-}{f^+}$ $\quad S = \dfrac{f^+}{f^-}$

$$S f^- = f^+$$

$$\prod \frac{(z-\lambda_j)}{1-\bar\lambda_j z} f^- = f^+$$

$$\prod_j (z-\lambda_j) f^- = \prod (1-\bar\lambda_j z) f^+$$

$$\prod \frac{1}{1-\bar\lambda_j z} f^- = \prod \frac{1}{z-\lambda_j} f^+$$

$$\frac{z-\lambda}{1-\bar\lambda z} f^- = f^+$$

$$f^- = \frac{1}{z-\lambda} = \frac{1}{z} + \frac{\lambda}{z^2} + \frac{\lambda^2}{z^3} + \cdots$$

$$\prod_{j=1}^{d} \frac{1}{z-\lambda_j} \, f^+ \subset H^-$$

provided f^+ is a poly of degree $< d$?

Given $S = \dfrac{z-\lambda}{1-\bar\lambda z}$, what is $SH^- \cap H^+$

note that $\dfrac{1}{1-\bar\lambda z}$ preserves H^+.

$$\frac{1}{z-\lambda} \in H^-$$

So

$$\frac{z^j}{\prod_1^d (z-\lambda_k)} \in H^- \qquad \text{for } 0 \le j < d$$

$$\underbrace{\prod_{k=1}^{d}\left(\frac{z-\lambda_k}{1-\bar\lambda_k z}\right)}_{S} \; \underbrace{\frac{z^j}{\prod_1^d (z-\lambda_k)}}_{f_-} = \underbrace{\frac{z^j}{\prod (1-\bar\lambda_k z)}}_{f_+}$$

So now what? Back to

$$\begin{array}{ccc} & X & \\ H^+ & \supset & SH^+ \\ V\ \cup & & \cup\ SV \\ zH^+ & \supset & zSH^+ \\ & zX & \end{array} \qquad \begin{aligned} Y &= H^+ \ominus zSH^+ \\ &= V \oplus zX = X \oplus SV \end{aligned}$$

continue with this example $\quad \mathbb{C}1 \oplus zX = X \oplus S\mathbb{C}$

where $X = SH^- \cap H^+ = \dfrac{\{\text{polys degree} < d\}}{\prod_{k=1}^{n} (1-\lambda_k z)}$

648 Another idea I've forgotten from the past is that there should be an analogue of this factorization of S in general. Why? The idea is that H^+/SH^+ is a f.d. module over $\mathbb{C}[z]$, so you have the Jordan (normal) form, ~~etc.~~ Back to

$$\mathbb{C}1 \oplus zX = X \oplus S\mathbb{C}$$

$$Y = \mathbb{C}1 \oplus z\frac{\bigoplus_{i=0}^{d-1}\mathbb{C}z^i}{\prod(1-\bar\lambda_k z)} = \frac{\bigoplus_{i=0}^{d-1}\mathbb{C}z^i}{\prod(1-\bar\lambda_k z)} \oplus \mathbb{C}\frac{\prod_1^d(z-\lambda_k)}{\prod_1^d(1-\bar\lambda_k z)}$$

and

$$Y = \frac{\bigoplus_{i=0}^{d} z^i\mathbb{C}}{\prod(1-\bar\lambda_k z)}.$$

It gets clearer

What you need to see is the ~~ottp~~ K-module and the line bundle corresponding to the clutching function S. Can you find ~~...~~ maps $X \rightrightarrows Y$? I would actually like a pencil of surjections $Y \xrightarrow{a\xi+b} X$. ~~You need a pla~~

Let's proceed backwards. Suppose we have a partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$, say X is a subspace of codim 1 in Y and $b: X \longrightarrow Y$ is an isometry. What should S be? Some [redacted] unitary isom between $\mathrm{Ker}(a^*)$ and $\mathrm{Ker}(b^*)$ depending on ξ. Rough idea is

$$\begin{tikzcd}
 & \mathrm{Ker}(a^*) \arrow[d, <->] \arrow[dr, "\sim"] & \\
X \arrow[r, "a\xi+b"] & Y \arrow[r] \arrow[d] & L_\xi \\
 & \mathrm{Ker}(b^*) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

649 It's amazing how little progress on these ideas you have ~~been~~ made. $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ The remarkable fact that a partial unitary leads to a S-operator between the complements. $S(z) : \mathrm{Ker}(a^*) \to \mathrm{Ker}(b^*)$.
Actually S may only be defined up to a constant of absolute value 1.

$$\mathbb{C}1 \oplus \frac{\mathbb{C}z}{1-\bar\lambda z} = \frac{\mathbb{C}}{1-\bar\lambda z} \oplus \mathbb{C}\,\frac{z-\lambda}{1-\bar\lambda z}$$

You could try writing the corresponding invertible $2\times$ matrices passing between these decompositions

$$\begin{tikzcd}[column sep=small]
\oplus \arrow[dr] & z^{-1}\mathrm{Ker}(b^*) \arrow[dr, "z"] & \overbrace{\mathrm{Im}(a)}^{X} \arrow[dr, "ba^{-1}"] & \overbrace{\mathrm{Ker}(a^*)}^{S(V)} \arrow[dr, "z"] & z\,\mathrm{Ker}(a^*) \arrow[dr] & \oplus \\
\oplus & z^{-1}\mathrm{Ker}(b^*) & \underbrace{\mathrm{Ker}(b^*)}_{V} & \underbrace{\mathrm{Im}(b)}_{zX} & z\,\mathrm{Ker}(a^*) & \oplus
\end{tikzcd}$$

Can we look for an eigenfunction wrt z.
You want something ψ satisfying $z\psi = \zeta\psi$.
Go back to $H = H^- \oplus H^+ = H^- \oplus X \oplus \underbrace{SH^+}_{SV \oplus SzH^+}$

~~Do your example~~

Look for an eigenvector wrt z
$z\psi = \zeta\psi$. Consider

Feb 14.

$$H = H^- \oplus \underbrace{X \oplus S(V)}_{\substack{\parallel \\ V \oplus zX}} \oplus S\cancel{z}H^+$$

The eigenvector should be $\sum_{n<0} \zeta^{-n} z^n v \in H^-$

650 You have

$$\psi = + z^{-2}v_{-2} + z^{-1}v_{-1} + \underbrace{x + w_0}_{v_0 + zx'} + zw_1 + z^2 w_2 + \cdots$$

$$\rho^{-1}z\psi = \rho^{-1}z^{-1}v_{-2} + \rho^{-1}v_{-1} + \rho^{-1}zx + \rho^{-1}zw_0 + \rho^{-1}z^2 w_1$$

$\rho^{-1}v_{-2} = v_{-1}$ $\qquad$ $\rho^{-1}w_0 = w_1$

$\rho^{-1}v_{-1} = v_0$ $\qquad$ $\rho^{-1}w_1 = w_2$

$\rho^{-1}x = x'$

Notice all the v_{-n} for $n<0$ are determined by v_0: $v_{-n} = \rho^n v_0$ $n \le 0$. Also $w_n = \rho^{-n} w_0$, $n \geq 0$. Remaining eqns are

$\boxed{\rho^{-1}x = x'}$ $\qquad$ $\boxed{x + w_0 = v_0 + zx'}$

To solve $\quad x + w_0 = v_0 + \rho^{-1}zx$. Simple equation

$$(1 - \rho^{-1}z)x = v_0 - w_0$$

This is an eigenvalue problem associated to a partial unitary. Reformulate

Start with $Y \overset{a}{\supset} X \xrightarrow{b} Y$ set $V = \underbrace{Y \ominus X}_{\mathrm{Ker}(a^*)}$

$W = Y \ominus \mathrm{Im}\, b = \mathrm{Ker}(b^*)$, then given $\rho \in \mathbb{C}$ solve

$$(a - \rho^{-1}b)x = v - w$$

Better is

$$X \xrightarrow{a - \rho^{-1}b} Y \longrightarrow E_\rho$$

Are you assuming no bound states

651 $\quad Y \xleftarrow{a} X \xrightarrow{b} Y$ $\qquad a^*a = 1$, $b^*b = 1$

$$(a - \zeta^{-1} b)x = 0$$

$$ax = \zeta^{-1} bx$$

$$\|x\| = |\zeta^{-1}| \, \|bx\| \quad \Rightarrow \quad |\zeta| = 1.$$

bound states.

So you ~~what~~ can now analyze a partial isometry $Y \xleftarrow{a} X \xrightarrow{b} Y$ $\qquad a^*a = 1$, $b^*b = 1$

IDEA Can you ~~use~~ partial outgoing subspaces to understand a port split into two? ~~[illegible]~~ This idea suggested by bound states. What is an outgoing subspace M for H equipped with unitary u? First $uM \subset M$ and second $\bigcap_{n \geq 0} u^n M = 0$. Then if $V = M \ominus u(M)$ one has an isom.

$$L^2(S^1, V) \simeq \overline{\bigcup_{n \in \mathbb{Z}} u^n M}$$ smallest closed inv. subspace containing M.

Hilbert space theory tells us I think that any (H, u) splits into a part absolute wrt Lebesgue meas and ~~[illegible]~~ a part singular wrt Lebesgue measure. Anyway what next? In the scattering situation how do you tell ~~what~~ when

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 bound states and so forth

$$(a - \zeta^{-1} b)(\cancel{x}) = 0 \qquad x \neq 0$$

$$a\cancel{x} = \zeta^{-1} b\cancel{x} \qquad \|x\| = |\zeta^{-1}|\|x\| \qquad |\zeta| = 1.$$

So what ~~happens~~ should happen is that we can split off $\mathbb{C}x$ from both $\cancel{X}$ and Y. ~~So how are we going to handle~~ $L = \mathbb{C}x$ $\quad aL = bL$. Assume a inclusion, b given isometry. Then $(\zeta a - b)\mathbb{C}x = 0$ $\quad b\,\mathbb{C}x \subset \mathbb{C}x$ and $b =$ mult by ζ on $\mathbb{C}x$. Then you split off

$$Y = \mathbb{C}x \oplus Y'$$
$$X = \mathbb{C}x \oplus \cancel{X}'$$

So what happens is that ~~we have~~ when we have a partial unitary $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ then ~~~~ after splitting off a max. unitary ~~~~ we have

$$0 \longrightarrow X \xrightarrow{a - \zeta^{-1} b} Y \longrightarrow E_\zeta \longrightarrow 0$$

for all $\zeta \in \mathbb{CP}^1$.

$$V = Y \ominus aX = \mathrm{Ker}(a^*)$$
$$W = Y \ominus bX = \mathrm{Ker}(b^*)$$

Claim for each $\zeta \in S^1$ $\quad v \in V$, $w \in W$ $\quad \exists!\ x \in X$

$\ni \quad (a - \zeta^{-1} b)x = v - w.$

~~The point is that one has~~ [illegible]. Is this correct dimensionally?

$$\overset{V}{\mathrm{Ker}(a^*)} + \overset{W}{\mathrm{Ker}(b^*)} + (a - \zeta^{-1} b)X = Y \quad ? \quad \text{NO}$$

$$\quad r \qquad\qquad r \qquad\qquad\quad d \qquad\quad d+r$$

$$Y = V \oplus X = X \oplus W$$

Suppose $y \perp$ to $\mathrm{Ker}(a^*)$, $\mathrm{Ker}(b^*)$, $(a - \zeta^{-1} b)X$

$$a^* y = \bar{\zeta}^{-1} b^* y \qquad y \perp \mathrm{Ker}(a^*) = \mathrm{Im}(a)^\perp$$

means $y \in \mathrm{Im}(a)$ so $\quad b^* y = y$, $\; a^* y = y$

$y \in a\cancel{X}$ $\quad y = ax_1 \quad y = bx_2$

$$a^* y = x_1 \qquad b^* y = x_2$$

$$x_1 = \bar{\zeta}^{-1} x_2$$

$\bar{\zeta}^{-1} a x_2 = a x_1 = b x_2$ leading to a bound state

The upshot is that given $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ partial unitary, not containing bound states we can always solve uniquely

$(V = X^\perp, \quad W = (bX)^\perp = \mathrm{Ker}(b^*))$

$\forall \zeta, \forall (v, w) \in V \times W, \quad \exists!\, x \quad (a - \zeta^{-1} b) x = v - w.$ NO

You need $\quad V \oplus (a - \zeta^{-1} b) X = Y \quad$ for $|\zeta| \geq 1$.

Let $y \perp V = (aX)^\perp$ i.e. $y = a x_1$

Let $y \perp (a - \zeta^{-1} b) X$ i.e. $(a^* - \bar{\zeta}^{-1} b^*) y = 0$.

Then $(a^* - \bar{\zeta}^{-1} b^*) a x_1 = 0$ or $x_1 = \bar{\zeta}^{-1} b^* a x_1$

$\bar{\zeta} x_1 = b^* a x_1 \implies |\zeta| < 1.$ (if no bound states).

Similarly need $\quad W \oplus (\zeta a - b) X = Y \quad$ for $|\zeta| \leq 1$.

Let $\quad w = (\zeta a - b)(x) \qquad W = \mathrm{Ker}(b^*)$

$$\implies 0 = (\zeta b^* a - 1) x \implies b^* a x = \frac{1}{\zeta} x \implies \left|\frac{1}{\zeta}\right| < 1.$$

and thus you get trivializations over each disk, + scattering over the circle.

Explore idea of a <u>partial port.</u>

Wait go back over partial unitaries

~~Let's under~~

654 What ideas come to mind? How about perturbation? Suppose U_0 = mult by z on $L^2(S^1)$ suppose U is a finite perturb of U_0

$$\begin{tikzcd} {} \arrow[r] & z^{-1} \arrow[r] & z^0 \arrow[rr, bend left] & z^1 \arrow[loop below] & z^2 \arrow[r] & z^3 \arrow[r] & {} \end{tikzcd}$$

what should be happening?? Perturbation region is z^0, z^1, probably. ~~So what would you want~~

You again want incoming and outgoing subspace for U. Take $H^- = \overset{(2)}{\bigoplus}_{n\le -1} z^n\mathbb{C}$ as before take zSH^+ to be $\overset{(2)}{\bigoplus}_{n\ge 2} z^n\mathbb{C}$, then Y as before

$$\cancel{X} = H^+ \ominus SH^+ = H^+ \cap S(H^-)$$

$$\begin{array}{ccc} & X & \\ H^+ & \supset & SH^+ \\ V\ \cup & & \cup\ SV \\ zH^+ & \supset & zSH^+ \\ & zX & \end{array} \qquad Y = V \oplus zX = X \oplus S(V)$$

$U = U_0 P$ P should be an op on $z^0\mathbb{C} \oplus z^1\mathbb{C}$ extended by 1. Count dimensions. Possible P are elts of U_2, dim 4. Possible partial unitaries. V seems to be fixed as $z^0\mathbb{C}$; so that $X = z\mathbb{C}$, so the only thing you can vary is a yielding dim 3. Which should be correct because there is only one parameter needed to extend the partial unitary to a unitary.

Suppose $\overset{d}{X} \underset{b}{\overset{a}{\rightrightarrows}} \overset{d+r}{Y}$ $\qquad \boxed{\dim = (d+r)^2 - r^2}$

or $2dr + (d+r)^2 - r^2$

655 possible $X^d \subset Y^{d+r}$ is $2dr$ 2 complex

possible $b: X^d \to Y^{d+r}$

$$\frac{U(d+r)}{U(r)} = d^2 + 2dr \cancel{+r^2} \cancel{-r^2} \qquad \left.\begin{array}{l} 2(d+r)-1 \\ +2(d+r)-3 \\ \vdots \\ +2(r+1)-1 \end{array}\right) = (d+r)d$$

dim of partial unitaries $+ r^2 = (d+r)^2 + 2dr$

dim of partial unitaries $= \boxed{d^2 + 4dr}$

other way of counting yields $2dr$ for choice of X and then $\left|\frac{U(d+r)}{U(r)}\right| = (d+r)^2 - r^2 = d^2 + 2dr$ for the choice of $b: X \hookrightarrow Y$. Again $\boxed{d^2+4dr}$

In the example V is given so you get d^2+2dr,

You are trying to analyze a perturbation of $U_0 =$ mult by z on $L^2(S^1)$, $U = U_0 P$ where the pert. P is unitary on $z^0\mathbb{C} \oplus z^1\mathbb{C}$ extended by 1 to the orth complement. Then $Y = z^0\mathbb{C} \oplus z^1\mathbb{C}$, $V = z^0\mathbb{C}$, $z^1\mathbb{C} = b\cancel{X}$

$$z^{-1}\mathbb{C} \oplus z^0\mathbb{C} \oplus z^1\mathbb{C} \oplus z^2\mathbb{C}$$

$$\begin{array}{ccc} H^+ & \overset{X}{\supset} & SH^+ \\ V\ \cup & & \cup\ SV \\ \cancel{U}H^+ & \supset & \cancel{U}SH^+ \\ & \cancel{U}(X) & \end{array}$$

You want to be carefully because $z^2\mathbb{C} \oplus z^3\mathbb{C} \oplus \cdots$ is probably USH^+.

$U = U_0 P$ so

$SH^+ = P^{-1}U_0^{-1}(z^2\mathbb{C} \oplus z^3\mathbb{C} \oplus$

$= P^{-1}(z\mathbb{C}) \oplus z^2\mathbb{C} \oplus \cdots$

Note that $UH^+ = U_0 P H^+$

$= U_0 H^+ = z\mathbb{C} \oplus \cdots$

So what do we have? P is this given unitary on $z^0\mathbb{C} \oplus z\mathbb{C}^1$, what does it do?

We have $UH^+ = U_0PH^+ = U_0H^+ = z\mathbb{C} \oplus z^2\mathbb{C} \oplus \cdots$

Start again $U = U_0 P$ on $L^2(S^1)$, where P a unitary operator on $z^0\mathbb{C} \oplus z^1\mathbb{C} \oplus \cdots \oplus z^k\mathbb{C}$ extended by the identity on the orth complement. The problem is to compute somehow the scattering operator, I want more namely to understand the associated vector bundle.

First question - can you find a partial unitary - it's going to be a restriction of U. You expect to get X, Y. We have $U = U_0$ on $H^- = \overset{(2)}{\bigoplus\limits_{n<0}} z^n\mathbb{C}$ and on $\overset{(2)}{\bigoplus\limits_{n\geq k+1}} z^n\mathbb{C}$. Recall

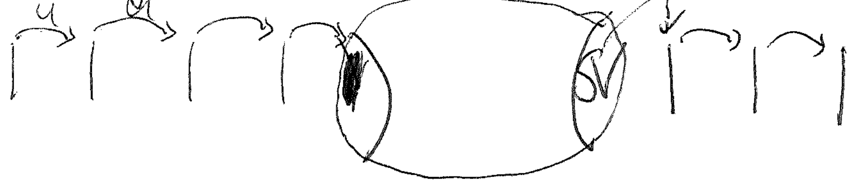
$$V = H^+ \ominus U(H^+) = H^+ \cap U(H^-) = H^+ \cap U_0(H^-) = z^0\mathbb{C}$$

Let us take tentatively the outgoing space USH^+ to be $z^{k+1}\mathbb{C} \oplus z^{k+2}\mathbb{C} \oplus \cdots$

V is what comes in from H^- under U

SV is what goes out under U into $\underset{\substack{SUH^+ \\ \| \\ SzH^+}}{\underline{USH^+}}$

Basic picture



$$V \oplus X = X \oplus SV$$

It's getting much clearer, but you a lot more to do.

657 In the perturbed situation?

~~Stable~~ You have many things to straighten out. Equivalentes pictures.

partial unitary

Start with H, U. suppose given H^-, H^+ closed subspaces $\ni$

i) $U^{-1}H^- \subset H^-$, $\bigcap U^{-n}H^- = 0$, $H^-/U^{-1}H^-$ f.d.

ii) $UH^+ \subset H^+$, $\bigcap U^n H^+ = 0$, H^+/UH^+ f.d.

iii) $H^- \perp H^+$ and $H^- \perp H^+$ f. codim in H.

Then you have $V = H^- \ominus U^{-1}H^-$, $W = H^+ \ominus UH^+$

$$H^- = \bigoplus_{n\le 0}^{(2)} U^{+n}V \qquad H^+ = \bigoplus_{n\ge 0}^{(2)} U^n W.$$

So what do we have? Let $X = (H^- \oplus H^+)^\perp$ so that we have $H = H^- \oplus X \oplus H^+$. Next claim that $\boxed{U : V \oplus X \xrightarrow{\sim} X \oplus W.}$

Proof. $H =$

$$\bigoplus_{n\le 0} U^n V \oplus X \oplus \bigoplus_{n\ge 0} U^n W$$

$$\begin{array}{ccl} H & = & U^{-1}(H^+) \oplus V \oplus X \oplus H^+ \\ u\downarrow & & \wr\downarrow \qquad\qquad\qquad\quad \wr\downarrow U \\ H & = & H^+ \oplus X \oplus W \oplus U(H^+) \end{array}$$

So U yields a "stable gain" of V and ~~with~~ W

~~Now you want to determine~~ Contraction viewpoint.

The subspace $Y = X \oplus W$ cont

658 Let's go back to showing if $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ partial unitary with no bound states i.e. ~~that~~ $0 \neq x \in X \ni ax\ bx$ dependent, then

$$0 \longrightarrow X \xrightarrow{\zeta a - b} Y \longrightarrow E_\zeta \longrightarrow 0$$

get a holom vector bundle E over $\mathbb{CP}^1$, and it's the bundle arising from the clutching fn. S.

First show $(\zeta a - b)X \oplus \underset{(aX)^\perp}{\underset{\|}{\operatorname{Ker}(a^*)}} = Y$

If $(\zeta a - b)(x) \in (aX)^\perp$

then $a^*(\zeta a - b)(x) = 0$

$\zeta x = a^* b x$ but a^*b is a contraction

so $|\zeta| \leq 1$. If $|\zeta| = 1$, then

$$\zeta a x = \underbrace{aa^*}_{\text{proj operator on } aX} bx$$

~~$|\zeta|\,\|ax\| = \|aa^*bx\|^2 + \|bx - aa^*bx\|$~~

$\|aa^*bx\| < \|bx\|$ unless $bx \in aX$

$|\zeta|\,\|x\| = \|\zeta a x\| \leq \|bx\| = \|x\| \Rightarrow aa^*bx = bx$

$\zeta a x = bx$ bd-state. $\therefore |\zeta| < 1$.

Thus $\operatorname{Ker}(a^*) \xrightarrow{\sim} E_\zeta$ for $|\zeta| \geq 1$

trivializes E over $|\zeta| \geq 1$

and $\operatorname{Ker}(b^*) \xrightarrow{\sim} E_\zeta$ for $|\zeta| \leq 1$

and this relates to $S(\zeta)$.

No bdd states $\Rightarrow$ a^*b

659 Feb 15 Picture $H, u,$ ~~[illegible]~~

$$H^- = \bigoplus_{n\geq 1}^{(2)} u^{-n} V \quad , \quad H^+ = \bigoplus_{n\geq 0} u^n W \quad , \quad H = H^- \oplus X \oplus H^+$$

orthogonal where V, W, X f.d.

Fix ζ an find eigenvector

$$\cdots + u^{-1}v_{-2} + v_{-1} + x + w_0 + u w_1 + u^2 w_2$$

$$+ \zeta^{-1}u^{-1}v_{-3} + \zeta^{-1}v_{-2} + \zeta^{-1}u(v_{-1}) + \zeta^{-1}u(x) + \zeta^{-1}u w_0 + \zeta^{-1}u^2 w_1$$

make clearer $\quad V \oplus X \xrightarrow[\sim]{u} X \oplus W$

$$(u^{-1}H^- \oplus H^+)^\perp \qquad (H^- \oplus uH^+)^\perp$$

$w_1 = \zeta^{-1}w_0$
$w_2 = \zeta^{-1}w_1$
$w_n = \zeta^{-n}w_0$

$v_{-2} = \zeta v_{-1}$
$v_{-3} = \zeta^2 v_{-1}$
$v_{-n} = \zeta^{n-1} v_{-1}$

$$\boxed{u(v_{-1} + x) = \zeta(x + w_0)}$$

$$u(x) - \zeta x = -u(v_{-1}) + \zeta w_0$$

now to analyze this carefully

There are various things to examine. Solving this eigenvalue equation. Dilating to obtain H. ~~[illegible]~~ Writing the 2×2 unitary matrix and it's inverse ~~describing~~ relating the two decompositions.

Examine dilating to obtain H. We have the C^* algebra $C(S^1)$ or group $\mathbb{Z}$. ~~Positive definite~~ You have $C(S^1)$ acting on H and a subspace Z (either X or $V \oplus X$) generating H under the action pos. def. function on $C(S^1)$ values in $\operatorname{End}(Z)$. Given by $\{ i^* z^n i \}$ $\quad i : Z \to H$. The interesting phenomena might be that $(i^* z^n i) = (i^* z\, i)^n$ for $n \geq 0$ i.e. only the contraction operator $i^* z\, i$ on Z matters. $Z = X$ ~~[illegible]~~ won't work since X can be 0.

Probably you have to take $Y = V \oplus X$.

660 Forgotten point X and $Y = X \oplus W$ are naturally modules over $\mathbb{C}[u]$

$$\begin{tikzcd}
 & (H^-)^\perp \arrow[dr, bend left, "X"] & \\
u(H^-)^\perp \arrow[ur, "V"] \arrow[dr, bend left, "uX"] & & H^+ \arrow[ul] \\
 & uH^+ \arrow[ul] \arrow[ur, "W"] &
\end{tikzcd}$$

u kills V and W and should be a contraction on X

Analyze. You have

$$Y = X \oplus W = uV \oplus uX$$

eigenvalue equation ~~is to~~ to solve is $\xi(x + w) = u(v) + u(x)$

But if $\iota^* : (H^-)^\perp \longrightarrow X$ is compatible

$$\iota^*(u\xi) = \gamma \iota^*\xi$$

$$\iota^*(u^n\xi) = \gamma \iota^*(u^{n-1}\xi) = \gamma^n \iota^*\xi$$

$$\iota^* u^n \iota x = \gamma^n x.$$

Question. how much of H, u can you recover from (X, γ)?

Set a notation.

$$H = H^+ \oplus X \oplus H^-$$

$$H^+ = \bigoplus_{n \le 0} u^{-n} V \qquad H^- = \bigoplus_{n \ge 0} u^n W$$

$$V \oplus X = (u^{-1}H^+ + H^-)^\perp$$

$$u(V \oplus X) = (H^+ + uH^-)^\perp = X \oplus W$$

eigenvectors for u

$$u^{-2}(v_2) + \xi u^{-1}(v_1) + \xi v_0 + \xi x + \xi w_0 + \xi u(w_1)$$

$$\cancel{\xi} u^{-1}(v_2) + \cancel{\xi} v_1 + \cancel{\xi} u(v_0) + u(x) + u(w_0)$$

$v_2 = \xi v_1$ $\quad w_1 = \xi^{-1} w_0$

$v_1 = \xi v_0$ $\quad w_2 = \xi^{-1} w_1$

$u(v_0 + x) = \xi(x + w_0)$

 symmetric notation above. But put

$$Y = \cancel{X} \oplus W = u(V) \oplus u(X)$$

Let $a: X \to Y$ be the inclusion, $b: X \to Y$ the restriction of u to x. Then $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ is a partial unitary operator, $\mathrm{Ker}(a^*) = W$

$$\mathrm{Ker}(b^*) = u(V) = V'$$

$$X \xrightarrow{\zeta a - b} Y \supset V'$$

$$x \longmapsto (\zeta - u)x$$

~~$(\zeta - u)(x) =$~~

$$(\zeta a - b)(x) = v'$$

$$(\zeta b^* a - 1)x = 0$$

$\Rightarrow x = 0$ if $|\zeta| < 1$.

or if $|\zeta| \leq 1$ and no bound states

Let $y \perp$ to $(\zeta a - b)X$ and $V' = \mathrm{Ker}(b^*) = (bX)^\perp$

$$y = bx \perp (\zeta a - b)X \Rightarrow (\bar{\zeta} a^* - b^*)bx = 0$$

$$\Rightarrow (\bar{\zeta} a^* b - 1)x = 0 \Rightarrow x = 0 \text{ if } |\zeta| < 1.$$

You would like a formula for S in terms of $c = b^* a$ on X.

Philosophy: I would like (X, c) to determine the essential structure. ~~What puzzles me is the relation between fact that~~

So study the structure $V \oplus X \xrightarrow[\sim]{u} X \oplus W$

So $u(V) \oplus u(X) = X \oplus W$. Set up the 2×2 matrices.

Let $Y = X \oplus W = u(V) \oplus u(X)$

$$\begin{tikzcd} V' \oplus uX & X \oplus W \arrow[l, "\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}"'] & V' \oplus uX \arrow[l, "\begin{pmatrix}\kappa & \lambda \\ \mu & \nu\end{pmatrix}"'] \end{tikzcd}$$

$$\begin{tikzcd} & V \arrow[d] & \\ X \arrow[r, "a"] & Y \arrow[r] \arrow[d, "b^*"] & W \\ & X & \end{tikzcd}$$

There is some linear algebra here – basically you have two orth. projections on the same space Y, this gives the usually Grassmannian parameters "characteristic values". But also you have an extra isomorphism between two summands from opposite sides, which allows composition, eigenvalues etc. You need to analyze this situation. Start with $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ whence operators $c = b^*a$, $c^* = a^*b$ on X. We know cc^* Look – you have a contraction on Y, namely ba^* and its adjoint ab^*.

$$\begin{tikzcd} V \oplus X \arrow[r, "u"] & \cancel{X} \oplus W \arrow[r] & u(X) + u(W) \\ & Y \arrow[u, "S"] & \overset{\frown}{H} \arrow[l, twoheadrightarrow] \end{tikzcd}$$

$$x + w \xrightarrow{u} u(x) + u(w) \xrightarrow{pr_Y \text{ onto}} u(x)$$

$$x + w \longmapsto \underbrace{u^{-1}(x) + u^{-1}(w)}_{V \hat{\oplus} X} \xrightarrow{pr_Y}$$

Look at X first, namely take x apply u and project onto X? We know that

$$\begin{tikzcd}
Y = X \oplus W & & V \oplus X \arrow[ll, "u"'] \\
X \arrow[u, "m_1"] \arrow[u, shift left=0, ""] & & X \arrow[u, "m_2"'] \arrow[ull, "b"]
\end{tikzcd}$$

with $a : X \to Y$.

so $u(x) = b(x) \longmapsto a^*b(x)$

$x \longmapsto u^{-1}(x) \longmapsto m_2^* u^{-1}(x) = b^*a(x)$.

So on X we have the contraction operator
a^*b induced by u
b^*a ———— u^{-1}.

So far it's not clear that u^n compresses ~~contracts~~ to $(a^*b)^n$

You need a notation to make this clear.

$$H^- \oplus X \oplus H^+$$

$$\cdots \oplus u^{-1}(V) \oplus V \oplus X \oplus \underbrace{W \oplus u(W) \oplus u^2(W) \oplus \cdots}_{(H^-)^{\perp}}$$

und $u(H^-)^{\perp} = (uH^-)^{\perp} \subset (H^-)^{+}$
$\qquad = X \oplus H^+$

$(H^-)^{\perp} = X \oplus H^+$

$u(H^-)^{\perp} = (uH^-)^{\perp} \subset (H^-)^{\perp}$

664. Life is hard.

Begin again.

$$H = \underbrace{\cdots \oplus u^{-1}V \oplus V}_{H^-} \oplus X \oplus \underbrace{W \oplus uW \oplus u^2 W \oplus \cdots}_{H^+}$$

$$Y = V \oplus X$$

$$\begin{array}{ccc} (u^{-1}H^- \oplus H^+)^\perp & = & V \oplus X \\ \cong \downarrow u & & \cong \downarrow u \\ (H^- \oplus uH^+)^\perp & = & X \oplus W \end{array}$$

$(H^-)^\perp = X \oplus H^+$ also H^+ closed under u

so u induces an operator c on $(H^-)^\perp / H^+ \cong X$

in fact u^n induces c^n on X.

~~$c^n(x) = i^* u^n i x$ where $i : X \to (H^-)^\perp$~~

~~$i^* u^n i \, i^* u^m i = i^* u^{n+m} i$ provided $n \geq 0$.~~

Let $i : X \longrightarrow (H^-)^\perp$ be the inclusion

i^* is the orthogonal projection onto X.

$\mathrm{Ker}(i^*)$ is $(\mathrm{Im}\, i)^\perp = H^+$.

$$u(x) = b(x) \oplus \overbrace{d(x)}^{\in W}$$

$$u^2(x) = u(c(x)) + u(d(x))$$

$$= c^2(x) + d(c(x)) +$$

$$u(x) = b(x) \oplus d(x) \in X \oplus W$$

$$u^2(x) = b^2(x) + d(b(x)) + u(d(x)) \in X \oplus W \oplus uW$$

$$u^3(x) = b^3(x) + db^2(x) + u(d(b(x))) + u^2(d(x))$$

$$u^4 = b^4 x + db^3 x + udb^2 + u^2 db + u^3 d$$

$$u\xi = \overbrace{b\xi}^{X} + \overbrace{d\xi}^{W}$$

$$u = b + d$$

$$u^2 = ub + ud = (b+d)b + ud$$

$$= b^2 + db + ud$$

$$u^3 = b^3 + db^2 + udb + u^2 d$$

$$u^4 = b^4 + db^3 + udb^2 + u^2 db + u^3 d$$

$$\therefore \quad i^* u^n i = b^n .$$

$(H^+)^\perp = H^- \oplus X$ incoming (closed ~~stable~~ under u^{-1})

u^{-n} induces an operator on $(H^+)^\perp / H^- \simeq X$

$$\overset{V \oplus X}{}$$

$$u^{-1}(x) = b^*(x) + a^*(x)$$

$$u^{-2}(x) = u^{-1} b^*(x) + b^* a^*(x) + a^{*2}(x)$$

$$u^{-1}(x) = a^* x + b^* x$$

$$u^{-2}(x) = u^{-1} a^* x + a^* b^* x + b^{*2} x$$

~~Consider now the eigenvalue equation~~

Examine the matrices

$$\begin{array}{c} V \oplus X \\ u \downarrow \uparrow u^{-1} \\ X \oplus W \end{array} \qquad u = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$\begin{array}{c} X \\ \oplus \\ W \end{array} \xleftarrow[u]{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{array}{c} V \\ \oplus \\ X \end{array} \xleftarrow[u^{-1}]{\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}} \begin{array}{c} X \\ \oplus \\ W \end{array}$$

666 Now look at eigenvector equation

$$H = \underbrace{V \oplus X}_{} \oplus W$$

$$u(v + x) = \xi(x + w)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v \\ x \end{pmatrix} = \xi\begin{pmatrix} x \\ w \end{pmatrix}$$

$$av + bx = \xi x \qquad ?$$

$$cv + dx = \xi w$$

$$(\xi - b)^{-1} a v = x$$

$$cv + d(\xi-b)^{-1} a v = \xi w$$

$$S(\xi) v = \xi^{-1} c v + \xi^{-1} d (\xi - b)^{-1} a v$$

Start again $\quad H, u \quad H = H^- \oplus X \oplus H^+$

$u^{-1}H^- \subset H^-$ ~~[illegible]~~ $\quad V = H^- \ominus u^{-1}H^-$

$u H^+ \subset H^+ \quad W = H^+ \ominus u H^+$

$$H^+ = W \oplus uW \oplus u^2 W \oplus \cdots$$ ℓ^2-direct sum.

provided $\bigcap u^n H^+ = 0$

$$H : \underbrace{\cdots\cdots \oplus u^{-1}V \oplus V}_{H^-} \oplus X \oplus \underbrace{W \oplus uW \oplus u^2 W \oplus \cdots}_{H^+}$$

$$\boxed{u : V \oplus X \xrightarrow{\sim} X \oplus W}$$

assume V, X, W f.d.

$u^{-1} v_{-1} + v_0 \oplus \{x \oplus w_0\} + u w_1 + \cdots \qquad$ ~~$u(v_{-n}) = \xi v_{-n+1}$~~

$v_{-1} + (u(v_0) + u(x)) + u w_0 \qquad \xi^{-1} u(v_{-1}) = v_0$

667 $\quad \oplus u^{-1}V \oplus V \oplus X \oplus W \oplus uW \oplus \cdots$

$$\zeta u^{-1}v_1 + \zeta v_0 + \zeta(x + w_0) + \zeta u w_1 + \zeta u^2 w_2$$

$$u^{-1}v_2 + v_1 + u(v_0 + x) + u w_0 + u^2 w_1$$

$v_1 = \zeta v_0$ $\quad$ $\boxed{u(v_0 + x) = \zeta(x + w_0)}$ $\quad$ $w_0 = \zeta w_1$

$v_2 = \zeta v_1$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $w_1 = \zeta w_2$

$$\boxed{u(v_0) - \zeta w_0 = (\zeta - u)(x)}$$

How to study this equation.

<u>Old method:</u> $\quad Y = X \oplus W = \overbrace{u(V)}^{V'} \oplus u(X)$

Then there are two $_\wedge^{\text{isometric}}$ maps $\quad X \overset{a}{\underset{b}{\rightrightarrows}} Y \quad$ $a(x) = x$, $b(x) = u(x)$.

$a^*a = b^*b = 1.$ $\quad W = (aX)^\perp = \mathrm{Ker}(a^*)$

$uV = (bX)^\perp = \mathrm{Ker}(b^*)$. $\quad$ YES.

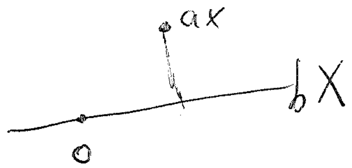
To ~~...~~ understand solns of $\quad (\zeta a - b)(x) = \underset{\mathrm{Ker}(b^*)}{v'} - w$ $\in \mathrm{Ker}(a^*)$.

Claim $\quad (\zeta a - b)X \oplus \mathrm{Ker}(b^*) = Y \quad$ for $|\zeta| < 1$

$0 = b^*(\zeta a - b)x = (\zeta\, b^*a - 1)\,x \quad$ Assume $|\zeta| = 1$

$\|\ \| \leq 1$

then and $\exists\ x \neq 0 \ni \quad \zeta\, b^*a x) = x$



$\|b^*ax\| = \|ax\|$

$\iff \quad ax \in bX \quad$ i.e. $\iff \quad ax = bx'$ $\quad$ some x'

$\Rightarrow \quad b^*a x = b^*b x' = x'$ $\quad\quad \Rightarrow \quad bb^*ax = bb^*bx' = bx'$ $= ax.$

$\Rightarrow x = \zeta b^*a x = \zeta x'$

$\therefore$ $\quad \zeta ax = bx.$

$(\zeta a - b)(x) \in \mathrm{Ker}(b^*)$ $\quad\quad$ $|\zeta| < 1$. ok

$(\zeta\, b^*a - 1)x = 0$ $\quad\quad$ $|\zeta| = 1$.

668

$\therefore\ \rho b^* a x = x$ $\qquad \|b^*ax\| < \|x\|$ unless $ax \in bX$ whence $bb^*ax = ax$

if $|\rho| \le 1$, then no go unless $bb^*ax = ax \Rightarrow \rho ax = bx$

Stay away from bound states

Go back to

$$\underbrace{u^{-1}V \oplus V}_{H^-} \oplus \overbrace{X \oplus \underbrace{W \oplus uW}_{H^+}}^{(H^-)^\perp}$$

$uH^- \supset H^-$

$u(H^-)^\perp \subset (H^-)^\perp$

$u(H^+) \subset H^+$

so u induces an endo of $(H^-)^\perp/H^+ = X \oplus H^+/H^+ = X$.

similarly $u^* = u^{-1}$ induces endos of $(H^+)^\perp/H^- = X \oplus H^-/H^- = X$.

So what is really going on? Look at

$H = H^- \oplus X \oplus H^+$ $\qquad$ restrict u, u^{-1} to X

Important point is that $uH^- \supset H^-$ so that $u(X) \subset X \oplus H^+$. Similarly $u^*(X) \subset H^- \oplus X$

Refine this result to $\begin{cases} u(X) \subset X \oplus W \\ u^*(X) \subset V \oplus X \end{cases}$ $\qquad u(x) = \beta(x) + \delta(x)$, $u^*(x) = \alpha^*(x) + \beta^*(x)$

$u: V \oplus X \longrightarrow X \oplus W$

$$X \oplus W \xleftarrow{u = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} V \oplus X \xleftarrow{u^* = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}} X \oplus W$$

Concentrate on $\beta : X \to X$ induced by u on $X \oplus H^+$

Look solving $(\rho - \beta)$

669 Go back over $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\qquad Y = aX \oplus W = V \oplus bX$

$X \xrightarrow{\lambda a - b} Y \longrightarrow E_\lambda$ $\qquad$ vector bundle over $\mathbb{C}P^1$ provided no bound states.

$(\lambda a - b)X \oplus V' = Y$ for $|\lambda| \le 1$
$(\lambda a - b)X \oplus W = Y$ for $|\lambda| \ge 1$.

Check: $(\lambda a - b)x \in V' \iff (\lambda b^* a - 1)x = 0$

If no bound states $b^* a x$ always has norm $< |x|$ so, if $x \neq 0$.

$(\lambda a - b)\cancel{X} \in W \iff (\lambda - a^* b)(x) = 0 \implies x = 0$ for $|\lambda| \ge 1$.

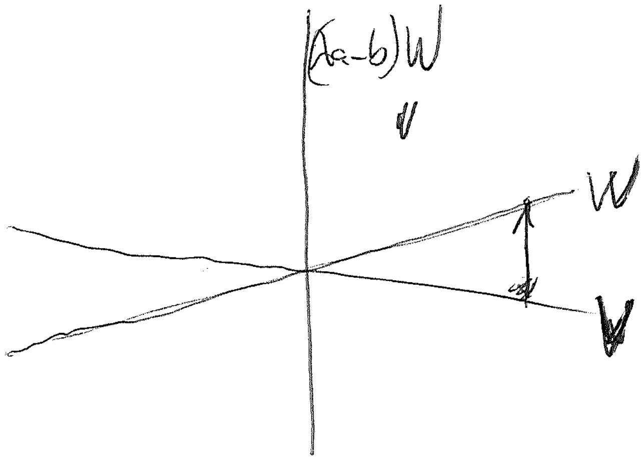
So we have trivializations $V' \xrightarrow{\sim} E_\lambda$ for $|\lambda| \le 1$
$W \xrightarrow{\sim} E_\lambda$ for $|\lambda| \ge 1$.

and for $|\lambda| = 1$ get an isom. $V' \simeq E_\lambda \simeq W$ depending on λ. If $\lambda = 0$, then can identify E_0 with $Y/bX \simeq V'$

Somehow it would be nice to invert $\lambda b^* a - 1$ and $\lambda - a^* b$.

$$\begin{tikzcd} & V' \arrow[d] \arrow[dr] & \\ X \arrow[r, "\lambda a - b"] \arrow[dr] & Y \arrow[r] \arrow[d, "b^*"] & E_\lambda \\ & X & \end{tikzcd}$$

What is going on? you have two complements for $(\lambda a - b)X$ V' and W



$$\begin{tikzcd} & W \arrow[d] \arrow[dr] & \\ X \arrow[r, "\lambda a - b"] & Y \arrow[r] \arrow[d, "a^*"] & E_\lambda \\ & X & \end{tikzcd}$$

start with v'

$v' - (\lambda a - b)(\lambda - a^* b)^{-1} a^* v' = S(\lambda) v'$

So how might I proceed?

Go back to symmetric viewpoint

$H = H^- \oplus X \oplus H^+$ orth d.s.

$uH^+ \subset H^+$ $\qquad u^{-1}H^- \subset H^-$ $\qquad$ ~~$H \supset H^-$~~

$V = H^- \ominus u^{-1}H^-$ $\qquad W = H^+ \ominus uH^+$

$$H = u^{-1}H^- \oplus V \oplus X \oplus H^+$$

$$= H^- \oplus X \oplus W \oplus uH^+$$

$$u : V \oplus X \longrightarrow X \oplus W$$

$$X \oplus W \xleftarrow{\;u = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\;} V \oplus X \xleftarrow{\;u^{-1} = \begin{pmatrix}\alpha^* & \gamma^* \\ \beta^* & \delta^*\end{pmatrix}\;} X \oplus W$$

$(H^-)^\perp = X \oplus H^+$ stable under u

u induces an operator on

$$(H^-)^\perp / H^+ = X \oplus H^+ / H^+ \cong X$$

You get u on $(H^-)^\perp$ reducing to β on X

u^n on H compresses to β^n on X

u^{*n} on H ——— $(\beta^*)^n$ on X.

Can you use this to understand the ~~largest~~ u, u^{-1} invariant subspace of H containing X.

Start with $v \in V$ apply u to get $\begin{pmatrix}\alpha(v) \\ \gamma(v)\end{pmatrix} \in X \oplus W$

Try $v + x$ $\qquad X \oplus W = uV \oplus uX$

$u(v+x) = \lambda(x+w)$

671 Feb 16, 1998

Review partial unitary $\overset{d}{X} \underset{b}{\overset{a}{\rightrightarrows}} \overset{d+r}{Y}$ $\qquad \|ax\| = \|x\| = \|bx\|$, $a^*a = 1$, $b^*b = 1$

(IDEA. A partial unitary can be completed to a unitary which has a characteristic polynomial, equivalently a ~~dia~~ positive divisor on S^1 (of degree $d+r$).
get a map $\underset{r^2}{U(r)} \longrightarrow \underset{d+r}{\text{deg } d+r \text{ div}}$)

$$aX \oplus W$$
$$\overset{\shortparallel}{V} \oplus bX$$

$$X \xrightarrow{a-\lambda b} Y \longrightarrow E_\lambda \longrightarrow 0$$

$$\overset{d}{(a-\lambda b)X} \oplus \overset{r}{\mathrm{Ker}(a^*)} = Y \qquad |\lambda| \le 1$$

$$(a-\lambda b)X \oplus \mathrm{Ker}(b^*) = Y \qquad |\lambda| \ge 1$$

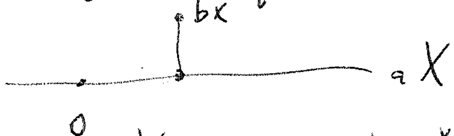
if no bound states

If $(a-\lambda b)x \in \mathrm{Ker}(a^*)$

then $0 = (1 - \lambda a^* b)x = 0$

$\exists\, x \neq 0, \lambda \quad (a-\lambda b)x = 0 \implies |\lambda| = 1.$

aa^*bx = projection of bx on aX



Case 1. $bx \notin aX \implies \|a^*bx\| < \|bx\| = \|x\|$

$\|x\| = \|\lambda a^* b x\| = |\lambda| \|a^*bx\| \implies \|\lambda\| > 1.$

Case 2. $bx \in aX \qquad aa^*bx = bx$

$x = \lambda a^* b x$
$ax = \lambda a a^* b x = \lambda b x$
bound state.

scattering of $V \oplus bW = aX \oplus W$

$$\mathrm{Ker}(b^*) \longrightarrow Y/(a-\lambda b)X \underset{\text{for } |\lambda| \le 1}{\xleftarrow{\text{isom}}} \mathrm{Ker}(a^*)$$

$$S(\lambda) : \mathrm{Ker}(b^*) \longrightarrow \mathrm{Ker}(a^*)$$

$$S(\lambda) = \left(1 - (a-\lambda b)(1-\lambda a^* b)^{-1} a^*\right)\Big|_{\mathrm{Ker}(b^*)}.$$

analytic for $|\lambda| \le 1$.

$$\begin{tikzcd} & \mathrm{Ker}(a^*) \arrow[d] & \\ X \arrow[r, "a-\lambda b"] \arrow[dr, "1-\lambda a^* b"'] & Y \arrow[r] \arrow[d, "a^*"] & E_\lambda \\ & X & \end{tikzcd}$$

672 Why unitary when $|\lambda| = 1$.

Review $\quad aX \oplus W = V \oplus bX$

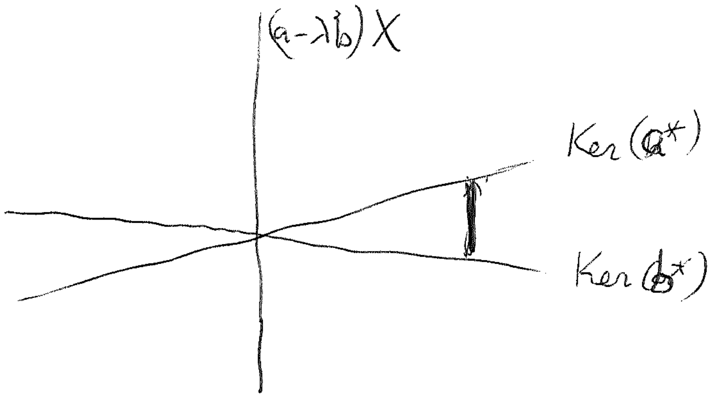
IDEA $\dim(X) = 1$ then a^*b is a scalar, so description should be easy

$$X \xrightarrow{a-\lambda b} Y \longrightarrow E_\lambda \longrightarrow 0$$

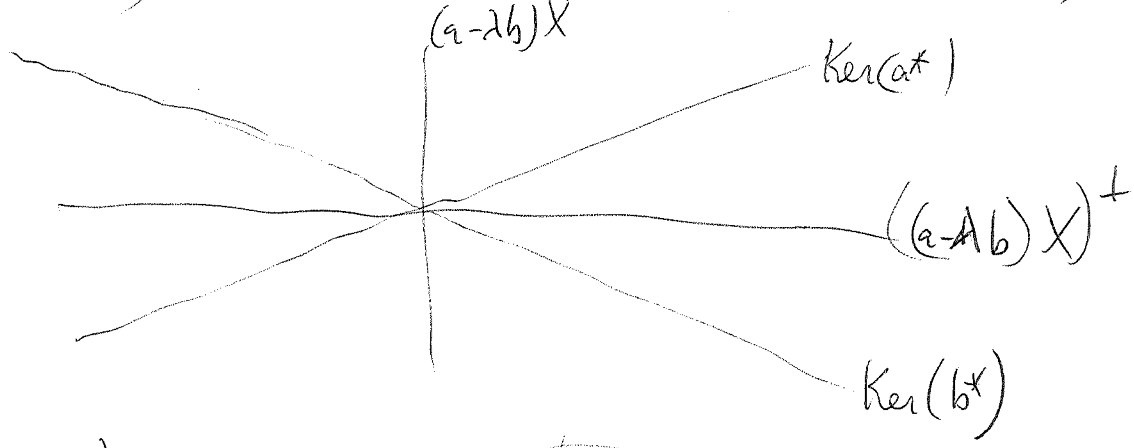
$$(a-\lambda b)X \oplus \operatorname{Ker}(a^*) = Y \qquad \text{for } |\lambda| \leq 1$$

$$(a-\lambda b)X \oplus \operatorname{Ker}(b^*) = Y \qquad \text{for } |\lambda| \geq 1$$

Assume $|\lambda| = 1$.



$$\left((a-\lambda b)X\right)^\perp = \operatorname{Ker}\left(a^* - \bar\lambda b^*\right) = \operatorname{Ker}\left(\lambda a^* - b^*\right) \quad \text{if } |\lambda| = 1$$



$$(a^* - \bar\lambda b^*)(a - \lambda b)$$

$$= a^*a - \bar\lambda b^*a - \lambda a^*b + |\lambda|^2 b^*b$$

$$= 2 - \bar\lambda b^*a - \lambda a^*b$$

eigenvector equation:

$$Y = aX \oplus W$$

$$\xrightarrow{ba^{-1}} V \oplus bX$$

$$ax + w = v + b$$

$$u^{-1}V \oplus X \oplus W$$

$$\|$$

$$V \oplus X \oplus uW$$

$$aX \oplus W = V \oplus bX$$

$$v_1 + ba^{-1}x = \xi(x + w_0)$$

$$v_1 - \xi w_0 = (\xi - ba^{-1})x$$

$$\xi u^{-1}v_1 + \xi x + \xi w_0 + \xi u(w_1)$$

$$u^{-1}(v_2) + v_2 + ba^{-1}x + u(w_0)$$

673 Go back to

$$H = H^- \oplus X \oplus H^+$$

$$\cdots\cdots \oplus u^{-1}V \oplus V \oplus \overbrace{X \oplus W}^{Y} \oplus uW \oplus u^2 W \oplus \cdots$$

Claim: u: yields an (unitary) isom $V \oplus X \xrightarrow{\sim} X \oplus W$

u - eigenvectors

$$+ u^{-2}(v_{-2}) + u^{-1}(v_{-1}) + v_0 + x + w_0 + u(w_1) + u^2(w_2) + \cdots$$

$$+ u^{-1}(v_{-2}) + v_{-1} + u(v_0) + u(x) + u(w_0) + u^2(w_1) + \cdots$$

$$v_{-1} = \xi v_0 \qquad \boxed{u(v_0 + x) = \xi(x + w_0)} \qquad w_0 = \xi w_1$$

$$v_{-2} = \xi v_{-1} \qquad\qquad w_1 = \xi w_2$$

$$\vdots$$

Claim: $\forall v \in V \quad \exists\, x, w \ni \quad u(v+x) = \xi(x+w)$

$\forall w \in W \quad \exists\, x, v \ni$

Thm: Assume [illegible] bound states $x \neq 0 \ni u(x) = \xi x$

Let $|\xi| = 1$. Then $\forall v \in V \quad \exists!\, w \in$

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before today's lecture - idea

$$\begin{tikzcd}
 & \mathrm{Ker}(a^*) \arrow[d] \arrow[dr, "\theta"] & & \\
0 \arrow[r] & X \arrow[r, "a-\lambda b"] \arrow[dr, "1-\lambda a^* b"'] & Y \arrow[r, "\pi"] \arrow[d, "a^*"] & E_\lambda \arrow[r] & 0 \\
 & & X & &
\end{tikzcd}$$

inverse of θ is

$$\pi(y) \longmapsto y - (a-\lambda b)(1-\lambda a^* b)^{-1} a^* y$$

$$1 - (a-\lambda b)(1-\lambda a^* b)^{-1} a^* = 1 - (a-\lambda b) a^* (1-\lambda b a^*)^{-1}$$

$$= 1 - (a a^* - \lambda b a^*)(1-\lambda b a^*)^{-1}$$

$$= [1 - \cancel{\lambda b a^*} - a a^* + \cancel{\lambda b a^*}](1-\lambda b a^*)^{-1}$$

674 So ~~the inverse of~~ $\mathrm{Ker}(a^*) \longrightarrow E_\lambda$ is

$$(1-aa^*)(1-\lambda b a^*)^{-1}$$

projection on $\mathrm{Ker}(a^*) = (aX)^\perp$

~~Go back to $V\oplus X \xrightarrow{\sim} X \oplus W$~~

Get other formula

$$\begin{tikzcd}
 & \mathrm{Ker}(b^*) \arrow[d] \arrow[dr, "\theta_1"] & \\
X \arrow[r, "(a-\lambda b)"] \arrow[dr, "b^*a-\lambda"'] & Y \arrow[r, "p"] \arrow[d, "b^*"] & E_\lambda \\
 & X &
\end{tikzcd}$$

inverse of θ_1 is

$py \mapsto$

$y - (a-\lambda b)(b^*a-\lambda)^{-1}b^*$

$$1-(a-\lambda b)b^*(ab^*-\lambda)^{-1} = \left(\cancel{ab^*}-\lambda - (\cancel{a}-\lambda b)b^*\right)(ab^*-\lambda)^{-1}$$

$$= -\lambda(1-bb^*)(ab^*-\lambda)^{-1} = (1-bb^*)(1-\lambda^{-1}ab^*)^{-1}$$

There has to be a simple way to get this.

$$(1-aa^*)\underbrace{(1-\lambda ba^*)^{-1}(1-bb^*)(1-\lambda^{-1}ab^*)^{-1}}$$

$$(1-\lambda ba^*)^{-1}(1-\lambda^{-1}ab^*)^{-1} - \underbrace{(1-\lambda ba^*)bb^*(1-\lambda^{-1}ab^*)^{-1}}_{b(1-\lambda a^*b)(1-\lambda^{-1}}$$

Go back to eigenvalue equation.

$$V \oplus X \oplus W \qquad u^{-1}(v_{-1}) + v_0 + x + w_0 + u(w_1)$$

$$u^{-1}(v_{-2} + v_{-1} + u(v_0 + \cancel{x}) + u(w_0) + u^2(w_1)$$

$$u(v_0+x) = g(x+w_0)$$

$$\begin{tikzcd}
\cancel{X} \oplus W & V \oplus X \arrow[l, "\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}"']
\end{tikzcd}$$

$$\alpha v + \beta x = \lambda x$$
$$\gamma v + \delta x = \lambda w$$

675 $\therefore \boxed{x = (\cancel{\lambda}-\beta)^{-1}\alpha v}$

$$\gamma v + \delta(\cancel{\lambda}-\beta)^{-1}\alpha v = \cancel{\lambda} w$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} v \\ x \end{pmatrix} = \lambda \begin{pmatrix} x \\ w \end{pmatrix}$$

$$\begin{pmatrix} v \\ x \end{pmatrix} = \lambda \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}$$

$$x = \lambda\beta^* x + \lambda\delta^* w$$

$$\boxed{x = \left((1-\lambda\beta^*)^{-1}\delta^*\right) w}$$

$\alpha\alpha^* + \beta\beta^* = 1$

Thus

$$x = (\lambda-\beta)^{-1}\alpha v$$
$$x = (1-\lambda\beta^*)^{-1}\delta^* w$$

$$\begin{matrix} X \\ \oplus \\ W \end{matrix} \xleftarrow[\sim]{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{matrix} V \\ \oplus \\ X \end{matrix}$$

Set $Y = aX \oplus \overset{\mathrm{Ker}(a^*)}{W}$

$\cancel{X} = V \oplus \cancel{X}$

676 Anyway

$$u^{-1}\mathrm{Ker}(b^*) \oplus aX \oplus \mathrm{Ker}(a^*) \xrightarrow{ba^{-1}} \mathrm{Ker}(b^*) \oplus bX \oplus u\,\mathrm{Ker}(a^*)$$

$$\begin{array}{ccccccccc} u^{-2}(v_{-1}) & & u^{-1}(v_0) & & ax & + & w_0 & & u(w_1) \\ \searrow & & \searrow & & \searrow & & \searrow & & \\ & u^{-1}(v_{-1}) & & v_0 & + & bx & \oplus & u(w) & \end{array}$$

~~eigenvalue~~ eigenvector equation

$$ax + w = \lambda(v + bx)$$

$$(a-\lambda b)x + \underset{\mathrm{Ker}\, a^*}{\underset{\cap}{w}} = \lambda \underset{\mathrm{Ker}\, b^*}{\underset{\cap}{v}}.$$

$$a^*(a-\lambda b)x = \lambda a^* v$$

$$x = (1-\lambda a^* b)^{-1}\lambda a^* v$$

$$= \lambda a^*(1-\lambda b a^*)^{-1} v$$

$$b^*(a-\lambda b)x + b^* w = 0$$

$$(b^*a - \lambda)x + b^* w = 0$$

$$x = (\lambda - b^*a)^{-1} b^* w = b^*(\lambda - ab^*)^{-1} w$$

$$w = \lambda v - (a-\lambda b)\,[\text{scribbled out}]\,x$$

$$= \lambda v - (a-\lambda b)\lambda a^*(1-\lambda b a^*)^{-1} v$$

~~$\left(\lambda a^*(1-\lambda b a^*)^{-1} v,\ (1-\lambda a^* b)^{-1} a^* w\right)$~~

$$(x, x) = \left(\lambda a^*(1-\lambda b a^*)^{-1} v,\ x\right)$$

$$= \bar{\lambda}\left(v,\ (1-\lambda a b^*)^{-1} a x\right)$$

677 Feb 17. $Y = aX \oplus W = V \oplus bX$

$$X \xrightarrow{(a-\lambda b)} Y \longrightarrow E_\lambda \longrightarrow 0$$

$$(a-\lambda b)X \oplus \mathrm{Ker}(a^*) = Y$$

$$\begin{tikzcd}
 & & \mathrm{Ker}(a^*) \arrow[d] & & \\
0 \arrow[r] & X \arrow[r, "a-\lambda b"] \arrow[dr, "1-\lambda a^*b"'] & Y \arrow[r] \arrow[d, "a^*"] & E_\lambda \arrow[r] & 0 \\
 & & X & &
\end{tikzcd}$$

$(a-\lambda b)(1-\lambda a^*b)^{-1}a^*$ projects onto $(a-\lambda b)X$ with kernel $\mathrm{Ker}(a^*)$

$$(a-\lambda b)(1-\lambda a^*b)^{-1}a^* = (a-\lambda b)a^*(1-\lambda ba^*)^{-1} = (aa^* \overset{-1\ +1}{-} \lambda ba^*)(1-\lambda ba^*)^{-1}$$

$$= 1 - (1-aa^*)(1-\lambda ba^*)^{-1}$$

So $(1-aa^*)(1-\lambda ba^*)^{-1}$ projects onto $\mathrm{Ker}(a^*)$ with kernel $(a-\lambda b)X$.

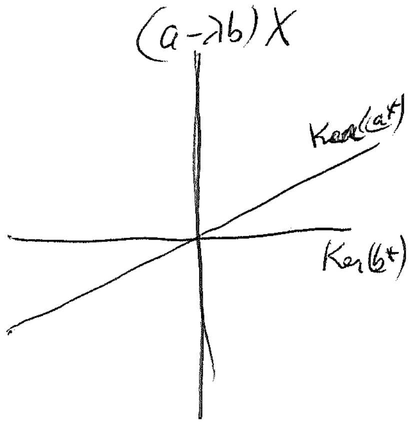
$$(1-\lambda ba^*)^{-1}(a-\lambda b) = a - \lambda b + \lambda ba^*(a-\lambda b) + (\lambda ba^*)^2(a-\lambda b)$$

$$\lambda b - \lambda^2 ba^*b + \lambda^2 ba^*b\cancel{a^*a}$$

$\therefore$ $(1-\lambda ba^*)^{-1}(a-\lambda b) = a$

Clearly $(1-\lambda ba^*)a = (a-\lambda b)$

$\therefore$ $a = (1-\lambda ba^*)^{-1}(a-\lambda b)$



$$(1-aa^*)(1-\lambda ba^*)^{-1}y = 0$$

$$(1-\lambda ba^*)^{-1}y = ax$$

$$y = (1-\lambda ba^*)ax = (a-\lambda b)x$$

$\therefore$ $S(\lambda)v = (1-aa^*)(1-\lambda ba^*)^{-1}v$?

678 Other approach from last night-

eigenvector equation $\quad Y = aX \oplus W = V \oplus bX$

$$ax + w = \lambda(v + bx)$$

$$(a - \lambda b)x + w = v'$$

$$(a - \lambda b)X \oplus \mathrm{Ker}(a^*) = Y$$

Suppose $\quad (a - \lambda b)x + w = v' \qquad$ (then $w = S(\lambda)v'$)

$$(1 - \lambda a^* b)x + 0 = a^* v'$$

$$x = (1 - \lambda a^* b)^{-1} a^* v'$$

$$= a^*(1 - \lambda b a^*)^{-1} v'$$

$$\cancel{b^*(a - \lambda b)x + b^* w}$$

$$(a - \lambda b)x = (aa^* - \lambda b a^*)(1 - \lambda b a^*)^{-1} v'$$

$$w = \left(1 - (-1 + aa^* + 1 - \lambda b a^*)(1 - \lambda b a^*)^{-1}\right) v'$$

$$\boxed{w = (1 - aa^*)(1 - \lambda b a^*)^{-1} v'}$$

analytic for $|\lambda| \leq 1$.

$$w = (1 - aa^*)(1 + \lambda b a^* + \lambda^2 b a^* b a^* + \cdots) v'$$

$$H : \cdots u^{-1}V \oplus V \oplus (aX \oplus W) \oplus uW \oplus \cdots$$

$$(uV \oplus \underbrace{bX}_{u(aX)})$$

$u(ax) = b.$

$$u^{-2}v_{-2} + u^{-1}v_{-1} + v_0 + ax + w_0 + u w_1 + u^2 w_2 = \xi$$

$$u^{-2}v_{-3} + u^{-1}v_{-2} + v_{-1} + u v_0 + bx + u w_0 + u^2 w_1 = u(\xi)$$

$$u(\xi) = \lambda^{-1}\xi$$

$$u v_0 + bx = \lambda^{-1}(ax + w_0)$$

$v_{-1} = \lambda^{-1} v_0 \qquad w_0 = \lambda^{-1} w_1$

$v_{-2} = \lambda^{-1} v_{-1} \qquad w_1 = \lambda^{-1} w_2$

$\therefore v_{-n} = \lambda^{-n} v_0, \qquad w_n = \lambda^{+n} w_0$

$$ax + w_0 = \lambda(u v_0 + bx)$$

$$(a-\lambda b)x + \underbrace{w_0}_{\in\,\mathrm{Ker}(a^*)} = \underbrace{\lambda u v_0}_{\in\,\mathrm{Ker}(b^*)}$$

---

try different thing.

$$H = \cdots \oplus u^{-1}V \oplus V \oplus X \oplus W \oplus uW \oplus \cdots$$

$$\xi = u^{-1}v_2 + v_1 + x + w_0 + u(w_1) + \cdots$$

$$u(\xi) = u^{-1}v_3 + v_2 + uv_1 + ux \qquad u(w_0)$$

$u(\xi) = \lambda^{-1}\xi$ $\qquad \lambda v_2 = v_1 \qquad \lambda w_0 = w_1$

$\qquad \lambda v_3 = v_2 \qquad \lambda w_1 = w_2$

$\qquad \vdots$

$$\lambda(uv_1 + ux) = x + w_0$$

$$(1-\lambda u)x + w_0 = \lambda u v_1 = u(\overset{v_0}{\overset{\|}{\lambda v_1}})$$

$$\cdots \oplus u^{-1}V \oplus V$$

You may want to set this up as follows

$$H: \cdots \oplus u^{-1}V \oplus V \oplus X \oplus W \oplus uW \oplus u^2W$$

$$\begin{array}{ccccc} Y & \overset{\text{df}}{=} & X \oplus W & = & uV \oplus uX \\ {\scriptstyle a}\uparrow & \nearrow{\scriptstyle c} & & & \nearrow{\scriptstyle u,\ s} \\ X & & & & X \end{array}$$

a inclusion of X in $Y = X \oplus W$
b is $u: X \to uX < Y$.

$$Y = aX \oplus \mathrm{Ker}(a^*)$$
$$= \mathrm{Ker}(b^*) \oplus bX$$

$(a - \lambda b)(x) = 0 \qquad x \neq 0 \qquad$ "bound state"

$$\|x\| = \|ax\| = \|\lambda bx\| = |\lambda|\,\|x\| \longrightarrow |\lambda| = 1.$$

680 Resolvent. If no bound state then

$(a-\lambda b) : X \longrightarrow Y$ injective

Claim $\quad (a-\lambda b)X \oplus \mathrm{Ker}(a^*) = Y \qquad |\lambda| \leq 1.$

$\qquad (a-\lambda b)X \oplus \mathrm{Ker}(b^*) = Y \qquad |\lambda| \geq 1$

$$\begin{array}{c} w \\ \shortparallel \end{array}$$

$$Y = aX \oplus \mathrm{Ker}(a^*) \qquad aa^*v + (1-aa^*)v$$

$$\shortparallel$$

$$\mathrm{Ker}(b^*) \oplus bX \qquad \xrightarrow{u = ba^{-1}} \qquad ba^*v \qquad u(1-aa^*)v$$

$$v \longmapsto$$

$$v = aa^*v + (1-aa^*)v$$

$$\downarrow u$$

$$ba^*v + u(1-aa^*)v$$

$$\shortparallel$$

$$aa^*ba^*v + (1-aa^*)ba^*v + u(1-aa^*)v$$

$$\downarrow u$$

$$(ba^*)^2 v + u(1-aa^*)ba^*v + u^2(1-aa^*)v$$

$$\downarrow u$$

$$(ba^*)^3 v + u(1-aa^*)(ba^*)^2 v + u^2(1-aa^*)(ba^*)v + u^3(1-aa^*)v$$

$$u^n(v) = \underbrace{aa^*(ba^*)^n v}_{aX} + \cancel{a}(1-aa^*)(ba^*)^n v$$

$$u(\quad)(\quad)^{n-1} v$$

$$\vdots$$

$$u^n(1-aa^*)v$$

$$\|v\|^2 = \|(ba^*)^n v\|^2 + \|(1-aa^*)(ba^*)^n v\|^2$$
$$+ \|(\quad)(ba^*)^{n-1} v\|^2$$
$$\vdots$$
$$\|(1-aa^*)v\|^2$$

How does this relate to
$(1-aa^*)(1-ba^*)^{-1}v$

OK. $aX \oplus \mathrm{Ker}(a^*)$

$\mathrm{Ker}(b^*) \oplus bX$

$$y = aa^*y + (1-aa^*)y$$
$$u(y) = \underbrace{ba^*y} + u(1-aa^*)y$$
$$= aa^*ba^*y + (1-aa^*)ba^*y + u(1-aa^*)y$$
$$u^2(y) = (ba^*)^2 y + u(\quad)ba^*y + u^2(1-aa^*)y$$

Suppose X is 1-dimensional, let x_0 be a unit vector

You get a positive def fun. on $\mathbb{Z}$; $(x_0, u^n x_0)$

The u, u^{-1} inv. ~~subspace~~ gen. by x is of rank 1. ~~What~~ ~~~~ It should ~~be~~ split off

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Feb 19. Try to get control of scattering

To under $\dim(X) = 1$, the degree, algebraically what do you have on the sheaf level.

$$0 \longrightarrow \mathcal{O}(-1)\otimes X \longrightarrow \mathcal{O}\otimes Y \longrightarrow E \longrightarrow 0$$
$$\qquad\qquad 1 \qquad\qquad\qquad 1+r$$

So you have in fact $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ ~~so~~ so you can split off the orthogonal complement of $aX+bX$.

So the real scattering can occur at most in 1 dimension. Let's check this.

$$V \oplus X \oplus W$$

$$\begin{array}{ccc} aX \oplus W = Y & & W = (aX)^\perp \\ \| & & \\ V' \oplus bX & & V' = (bX)^\perp \end{array}$$

Let $Y_0 = V' \cap W$, so that $Y_0^\perp = (aX + bX)^\perp$.

The point should be that the scattering data namely $V \oplus X \xrightarrow{\sim} X \oplus W$ determine H, u, H^+, H^- up to canonical isom. So you end up splitting off $Y_0 = V' \cap W$

Next? Assume V, W have dim $= 2$

Feb 20. Idea: X together with the contraction operator induced by u determines the nontrivial scattering. Consider

$$Y = aX \oplus W$$
$$\phantom{Y = {}} V' \oplus bX$$

Let $Y_0 = V' \cap W = \operatorname{Ker}(a^*) \cap \operatorname{Ker}(b^*)$

Then Y splits into $Y_0 \oplus Y_0^\perp$, where $Y_0^\perp = aX + bX$. Thus $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ splits into $(0 \rightrightarrows Y_0) \oplus (X \overset{a}{\underset{b}{\rightrightarrows}} Y_0^\perp)$. Thus restricts to the case of trivial scattering where $X = 0$ and $u : V \xrightarrow{\sim} U$ plus $Y = aX + bX$, whence X generates H.

683 Go back to examples.

First thing is to take $\mathbb{C}[z,z^{-1}]\otimes X$ and to equipped it with the inner product

$$(U^n x_1, U^m x_0) = (x_1, U^{m-n} x_0) = \cancel{(\quad)}$$

$$= \begin{cases} (x_1, \gamma^{m-n} x_0) & m \geq n \\ (x_1, (\gamma^*)^{n-m} x_0) & m \leq n \end{cases}$$

$$(x_1, U^n x_0) = \int z^n \Big(\sum_i \qquad \Big) \frac{d\theta}{2\pi}$$

$$\iota^* U^n \iota = \begin{cases} \gamma^n & n \geq 0 \\ (\gamma^*)^{-n} & n \leq 0 \end{cases}$$

$$= \int z^{+n} \Big(\underbrace{\sum_{k \geq 0} z^{-k} \gamma^k + \sum_{k \geq 1} z^{+k} (\gamma^*)^k} \Big) \frac{d\theta}{2\pi}$$

$$= \frac{1}{1 - z^{-1}\gamma} + \frac{z\gamma^*}{1 - z\gamma^*}$$

$$\|$$

$$\cancel{\frac{1}{1-z^{-1}\gamma}\left(1 - z^{-1}\gamma + z\gamma^*(1 - z\gamma^*)\right)\frac{1}{1 - z\gamma^*}}$$

$$\frac{1}{1-z^{-1}\gamma}\left(1 - z\gamma^* + (1 - z^{-1}\gamma) z\gamma^*\right)\frac{1}{1 - z\gamma^*}$$

$$= \frac{1}{1-z^{-1}\gamma}\left(1 - \gamma\gamma^*\right)\frac{1}{1 - z\gamma^*}$$

$$\frac{1}{1-z^{-1}\gamma} + \frac{z\gamma^*}{1-z\gamma^*}$$

$$= \frac{1}{1-z\gamma^*}\left(z\gamma^*(1-z^{-1}\gamma) + (1-z\gamma^*)\right)\frac{1}{1-z^{-1}\gamma}$$

$$= \frac{1}{1-z\gamma^*}(1-\gamma^*\gamma)\frac{1}{1-z^{-1}\gamma}$$

So we have the possibility of ~~replacing~~ getting rid of the measure. Basically we have

$$i^* f(u)\, i = \int \frac{1-\gamma^*\gamma}{(1-z^{-1}\gamma)(1-z\gamma^*)} f(z) \frac{d\theta}{2\pi} \qquad \frac{dz}{2\pi i z}$$

$$= \left\langle \frac{\sqrt{1-\gamma^*\gamma}}{1-z\gamma^*}, \frac{\sqrt{1-\gamma^*\gamma}}{1-z\gamma^*} f(z) \right\rangle_{L^2}$$

So you find the dilation of (X,γ) is $\simeq$ $L^2(S^1,X)$ via $f(u) \longleftrightarrow \frac{\sqrt{1-\gamma^*\gamma}}{1-z\gamma^*} f(z)$

Somewhere there are wave operators.

Where are V, W?

So let's try to analyze an example for class.

X one diml. $Y = aX + bX$ assume critical number is $\langle bx_0, ax_0\rangle = \langle x_0, (b^*a)x_0\rangle$.

This is too slow.

What do you want to do? Start with $H = H^- \oplus X \oplus H^+$. Then split off bound states.

Then have $L^2(S^1,V) \xleftarrow{\sim} H \xrightarrow{\sim} L^2(S^1,W)$

685 Given $H = H^- \oplus X \oplus H^+$ with

$z^{-1} H^- \subset H^-$, $\bigcap z^{-n} H^- = 0$, $V = H^- \ominus z^{-1} H^-$

get $W^- : H \longrightarrow L^2(S^1, V)$

$$\begin{array}{ccccc}
L^2(S^1, V) & \hookrightarrow & H & \hookleftarrow & L^2(S^1, W) \\
\| & & \cup & & \| \\
\bigoplus^{(2)}_{n \in \mathbb{Z}} z^n V & \sim & \bigoplus^{(2)}_{n\in\mathbb{Z}} u^n V & & \\
& & \bigoplus^{(2)}_{n\in\mathbb{Z}} u^n W & = & \bigoplus^{(2)} z^n W
\end{array}$$

More complicated thing is ? ~~Assume no bound states~~ You know what about X?

$u^{-1}(H^-) \subset H^-$ $\qquad$ $u(H^+) \subset H^+$

$H^- \subset u H^-$ $\qquad$ $u(H^+)^\perp \supset (H^+)^\perp$

$(H^-)^\perp \supset u(H^-)^\perp$ $\qquad$ $(H^+)^\perp \supset u^{-1}(H^+)^\perp$

$X + H^+ \supset u(X + H^+)$

Can write $\quad H = u^{-1}H^- \oplus V \oplus X \oplus H^+$

$$H = H^- \oplus X \oplus W \oplus uH^+$$

$$u : V \oplus X \xrightarrow{\sim} X \oplus W$$

End result.

$L^2(S^1, X)$ first consider u invariant closure of X

$\downarrow$

$L^2(S^1, V) \hookrightarrow H \hookleftarrow L^2(S^1, W)$

u inv. closure of V $\qquad$ inv. clos of W

 Keep on trying

You should be able to ~~...~~ get in general the inv. clos. of X to $L^2(S^1, \cancel{X}, \cancel{d\mu})$:

You know that ~~...~~ u^n, u^{*n} on H compress to $\gamma^n, (\gamma^*)^n$ on X for all $n \geq 0$.

$\gamma = \iota^* u \iota \qquad \iota : X \hookrightarrow H$

$$L^2\left(S^1, X, \frac{1}{1-z\gamma^*}(1-\gamma^*\gamma)\frac{1}{1-z^{-1}\gamma}\right)$$ where are the subtleties?

~~...~~ Assume nice behavior of $\frac{1}{1-z^{-1}\gamma}$ on S^1. say $\|\gamma\|^n < 1$ for some n. This is equiv. to no bound states

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So you learn that a partial unitary operator (in fin dims) should reduce to a contraction. Specifically given γ on X with $\|\gamma\| \leq 1$, and ~~...~~ $\|\gamma^n\| < 1$ for some n, ~~...~~ then it should come from a unitary ~~partial~~ unitary $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ ~~...~~ such that $aX + bX = Y$. How? I guess you define $\|ax_1 + bx_2\|^2 = \|x_1\|^2 + (x_1, \overset{\gamma}{a^*b}x_2) + (x_2, \overset{\gamma^*}{b^*a}x_1) + \|x_2\|^2$

and complete. $= \|x_1 + \gamma x_2\|^2 + (x_2, (1-\gamma^*\gamma)x_2)$

Maybe you are dilating $\begin{pmatrix} 0 & \gamma^* \\ \gamma & 0 \end{pmatrix}$ on $\begin{matrix} X \\ \oplus \\ X \end{matrix}$

$$\begin{pmatrix} \sqrt{1-\gamma^*\gamma} & \gamma^* \\ \gamma & -\sqrt{1-\gamma\gamma^*} \end{pmatrix}$$

687 You can always ~~that~~ dilate a cont. $\gamma : X \longrightarrow X'$ to a unitary, namely

$$\cancel{\begin{pmatrix}\gamma & \sqrt{1-\gamma\gamma^*}\\ \sqrt{1-\gamma^*\gamma} & \gamma^*\end{pmatrix}\begin{pmatrix}\gamma & \sqrt{1-\gamma^*\gamma}\\ -\sqrt{1-\gamma\gamma^*} & \gamma^*\end{pmatrix}}$$

$$\cancel{\begin{pmatrix}\gamma & \sqrt{1-\gamma^*\gamma}\\ -\sqrt{1-\gamma\gamma^*} & \gamma^*\end{pmatrix}\begin{pmatrix}\gamma^* & -\sqrt{1-\gamma\gamma^*}\\ \sqrt{1-\gamma^*\gamma} & \gamma\end{pmatrix} = (}$$

$$\begin{pmatrix}\sqrt{1-\gamma^*\gamma} & +\gamma^*\\ \gamma & -\sqrt{1-\gamma\gamma^*}\end{pmatrix}\begin{pmatrix}\sqrt{1-\gamma^*\gamma} & \gamma^*\\ +\gamma & -\sqrt{1-\gamma\gamma^*}\end{pmatrix}$$

It dilates to

$$X \oplus X' \xleftarrow{\begin{pmatrix}\sqrt{1-\gamma^*\gamma} & -\gamma^*\\ \gamma & \sqrt{1-\gamma\gamma^*}\end{pmatrix}} X \oplus X'$$

Assume given $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ s.t. $a^*a = 1$, $b^*b = 1$, $aX + bX = Y$.

Then
$$\|ax_1 + bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (ax_1, bx_2) + (bx_2, ax_1)$$
$$= \|x_1\|^2 + \|x_2\|^2 + (x_1, \underset{\gamma}{a^*b}\, x_2) + (x_2, \underset{\gamma^*}{b^*a}\, x_1)$$
$$= \|x_1\|^2 + \|x_2\|^2 + (x_1, \gamma x_2) + (\gamma x_2, x_1) \quad + (\gamma x_2, \gamma x_2) - \|\gamma x_2\|^2$$
$$= \|x_1 + \gamma x_2\|^2 + (x_2, (1-\gamma^*\gamma)x_2).$$
$$= \|x_1 + \gamma x_2\|^2 + \|\sqrt{1-\gamma^*\gamma}\, x_2\|^2$$
$$= \|\gamma^* x_1 + x_2\|^2 + \|\sqrt{1-\gamma\gamma^*}\, x_1\|^2$$

So you find

 You complete $X \oplus X$ with this norm to obtain Y and define a, b, to be the inclusions in_1, in_2 followed by proj. to Y. a^* should be

$$ax_1 + bx_2 \longmapsto x_1 + \gamma x_2$$

$$(ax, ax_1 + bx_2) = (x, x_1 + \gamma x_2) \quad \text{obviously.}$$

and b^* is $ax_1 + bx_2 \longmapsto \gamma^* x_1 + x_2$. We see that kernel a^* is $\simeq \sqrt{1-\gamma^*\gamma}\,X$
kernel b^* is $\simeq \sqrt{1-\gamma\gamma^*}\,X$.

Need examples now with small X.
If X has dim 1 so do V, W unless there's trivial scattering. You have in mind 2 ports. It's the question of coupling.
trans. line. has both ~~left~~ and right movers.
I think you want this on a ribbon graph

I want to put something on the $SL_2(\mathbb{Z})$ tree. there are two kinds of vertices, need 2 port and a 3 ports with cyclic symmetry.
Let's first consider a 2 port, and try to see what occurs with X of dim 1. ~~We know something~~ So V and W are of dim 2 and are split into lines, and X generically will link a line in V with one in W. Good point. $V \oplus bX = aX \oplus W$. It looks confusing. You want an analytic matrix depending on z. So how do I proceed? ~~I take~~
Let's describe the general situation
contraction of γ on X leads to a splitting according to $1 - \gamma^*\gamma$ and $1 - \gamma\gamma^*$. But I'm assuming $\dim(X) = 1$, so you should get simply ?
You have V, W

689 Try again. We begin with V, W of dim 2. and we to end up with a scattering operator

$S(\lambda): V \longrightarrow W$ $\qquad Y = V \oplus bX = aX \oplus W$

$v \in V$
$$y = a a^* y + (1 - a a^*) y \quad \text{component of } v \text{ in } W$$
$$u(y) = b a^* y + u(1 - a a^*) y$$
$$u^2(y) = (b a^*)^2 y + u(1 - a a^*) b a^* y + u^2 (1 - a a^*) y$$

You want to write this in terms of $\gamma = a^* b$

Basically $y = a a^* y + (1 - a a^*) y \in aX + \underset{W}{\mathrm{Ker}(a^*)}$

idea: Suppose you have $S(\lambda): V \longrightarrow W$, you split V, W somehow to represent left and right and then calculate the transfer matrix, which can be iterated with a length of transmission line. What do you get?

Another idea: Compare $U(p,q)$ with $U(p+q)$ in general.

First do for $p = q = 1$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(1,1)$ means $\quad g^* \varepsilon g = \varepsilon$ or
$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \varepsilon g^* \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha^* & -\beta^* \\ -\gamma^* & +\delta^* \end{pmatrix} \qquad \alpha^* \alpha - \beta^* \beta = 1 \qquad \alpha^* \beta - \beta^* \delta = 0$$
$$-\gamma^* \alpha + \delta^* \gamma = 0$$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$ means $g^* \varepsilon g = \varepsilon$

$\varepsilon g^* \varepsilon = g^{-1}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}$$

$aa^* - bb^* = 1$

$-ac^* + bd^* = 0$

$ca^* - db^* = 0$

$dd^* - cc^* = 1.$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$y_1 = c x_2 + d y_2$

$x_1 = ac^{-1} y_1 - ac^{-1} d y_2 + b y_2$

$$\boxed{\begin{aligned} x_2 &= c^{-1} y_1 - c^{-1} d y_2 \\ x_1 &= ac^{-1} y_1 + (b - ac^{-1} d) y_2 \end{aligned}}$$

Is $\begin{pmatrix} c^{-1} & c^{-1}d \\ ac^{-1} & b - ac^{-1}d \end{pmatrix} \in U(2)$?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{\Delta} = \begin{pmatrix} \bar a & -\bar c \\ -\bar b & \bar d \end{pmatrix}$$

$\bar a = \frac{d}{\Delta}$ $\quad \bar c = +\frac{b}{\Delta}$

$\bar b = +\frac{c}{\Delta}$ $\quad \bar d = \frac{a}{\Delta}$

$$\begin{pmatrix} c^{-1} & c^{-1}d \\ ac^{-1} & b - ac^{-1}d \end{pmatrix} = \Big(\Delta$$

$b - ac^{-1}d = \frac{bc - ad}{c} = \frac{-\Delta}{c}$

$$\cancel{\bar c^{-1}}\, c^{-1} d + \underbrace{\bar a}_{\frac{d}{\Delta}} \cancel{\bar c^{-1}} \underbrace{(b - ac^{-1}d)}_{\left(-\frac{\Delta}{c}\right)} \overset{?}{=} 0$$

Yes.

$$\frac{1 + |d|^2}{|c|^2} = \frac{|c|^2}{|c|^2} = 1,$$

$$|ac^{-1}|^2 + \left|\frac{\Delta}{c}\right|^2 = \frac{|a|^2 + 1}{|c|^2} = \frac{|c|^2}{|c|^2} = 1$$