 Feb. 1. Consider $V = U/W$ a subquotient of $H = H^+ \oplus H^-$. We then get a Lagrangian subspace $F_s = \begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* \subset \begin{matrix} V \\ \oplus \\ V^* \end{matrix}$ depending rationally on s.

Lagrangian subbundle $F \subset \mathcal{O} \otimes (V \oplus V^*)$ over P^1. You want ~~[illegible]~~ to construct the canonical resolution of F:

$$0 \longrightarrow F \longrightarrow \mathcal{O} \otimes \underbrace{H^0(F^*)^*}_{\dim = d+r} \longrightarrow \mathcal{O}(1) \otimes \underbrace{H^0(F^*(-1))^*}_{\dim d} \longrightarrow 0$$

where $d = \dim(H)$ and $r = \dim(V)$. My guess is that

$$H^0(F^*(-1))^* = H, \quad H^0(F^*)^* = U \oplus W^\perp$$

Let's discuss simpler cases: $U = H$ or $W = 0$. $W = 0$ is the case where $V = \bar{C}^0 \subset C^1 = H$, i.e. all nodes of the circuit are external. You are used to working with $U = H$ and the proj $p: H \to W^\perp$, so consider this case. On $H \overset{\prime\prime}{=} H^+ \oplus H^-$ you have $A_s = s\pi_+ \oplus s^{-1}\pi_-$

$$A_s = s\pi_+ + s^{-1}\pi_- \quad \text{on} \quad H = H^+ \oplus H^-$$

$$A_s^{-1} = s^{-1}\pi_+ + s\pi_-$$

$$pA_s^{-1}p^* = s^{-1}(p\pi_+p^*) + s(p\pi_-p^*) \quad \text{on} \quad W^\perp$$

$$= \sum \frac{s^{-1}\omega^2 + s}{1+\omega^2} \pi_\omega \quad \text{on} \quad W^\perp = \bigoplus (W^\perp)_\omega$$

Where are you? You have

Better: go back to case $V = U = \bar{C}^0 \xrightarrow{i} H = C^1$

$W = 0$
$W^\perp = H$.

$$A_s = s\pi_+ + s^{-1}\pi_- \quad \text{on} \quad H^+ \oplus H^-$$

$$i^* A_s i = s(i^*\pi_+ i) + s^{-1}(i^*\pi_- i) \quad \text{on} \quad V$$

$i^* A_s i$ is a quadratic form on V, gives us a map $V \to V^* = V$, so we have $F_s = \begin{pmatrix} 1 \\ i^* A_s i \end{pmatrix} V \subset \begin{matrix} V \\ \oplus \\ V^* \end{matrix}$

To understand this you use sp. thm. for $i^*\pi_+ i$

$$i^* A_s i = \bigoplus_\omega \frac{s + s^{-1}\omega^2}{1+\omega^2} \pi_\omega \quad \text{on} \quad V = \bigoplus V_\omega$$

What happens is that everything here is a direct sum. All you have is 1 subspace U in $H = H^+ \oplus H^-$, [redacted] i.e. an orth rep. of dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$. ~~So you have~~ Cases to look at

$$U = V = \begin{pmatrix} 1 \\ \omega \end{pmatrix} H^+ \subset \begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix} \Leftarrow \mathbb{R}^2$$

$$V = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \frac{1}{\sqrt{1+\omega^2}} \mathbb{R} \subset \mathbb{R}^2$$

Wait. $V = \mathbb{R}$ $\quad \iota : V \to H \quad \iota = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \frac{1}{\sqrt{1+\omega^2}}$

God what a mess.

Take $H = \mathbb{R}^2$. Take $H = \mathbb{R}^2$. $V = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \mathbb{R}$

quad. form is $v \longmapsto s v^2 + s^{-1}\omega^2 v^2 = (s + s^{-1}\omega^2) v^2 \quad v \in \mathbb{R}$

normalize at $s=1$ to be canon. form to get $v \mapsto \frac{s+s^{-1}\omega^2}{1+\omega^2} v^2$

Then $F_s = \begin{pmatrix} 1 \\ \frac{s+s^{-1}\omega^2}{1+\omega^2} \end{pmatrix} \mathbb{R} \subset \mathbb{R}^2 = V \oplus V^*$ $\qquad \frac{LCs^2+1}{Cs}$

have poles at $s = 0, \infty$. $\quad Z_s^{-1} = \frac{s+s^{-1}\omega^2}{1+\omega^2}$ has deg 2.

have $F = \mathcal{O}(-2)$. You want to embed F_s into $\mathbb{R}^3 = V \oplus H$. You have chosen two maps $F_s \to \mathbb{R}$ from a 3 diml space.

There should be a better viewpoint. Let's try something Lagrangian, start with H and $sQ_+ + s^{-1}Q_-$ on $H^+ \oplus H^-$. Then what do we get? This quad. gives ~~a~~ a Lagr. subbundle $F = F_+ \oplus F_- \subset (H^+ \oplus H^{+*}) \oplus (H^- \oplus H^{-*})$

So $F_+ = \text{graph} \begin{pmatrix} 1 \\ sQ_+ \end{pmatrix} \subset \mathcal{O} \otimes (H^+ \oplus H^{+*})$. Obviously

$F_+ = \mathcal{O}(-1) \otimes H^+$

581 Abstract question: Given quadratic form on H with certain non-deg. properties, then get induced q.f. on any ~~subquotient~~. Can you formulate this symplectically? First question is: given Lagrangian subspace of a symplectic vector space and an isotropic subspace, when does the symplectic vector space splits naturally.

Symplectic quotient by an isotropic subspace.

Take

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & X/Y & \to & 0 \\ & & \downarrow^{0} & & \downarrow^{s} & & & & \\ 0 & \leftarrow & Y^* & \leftarrow & X^* & \leftarrow & Y^0 & \leftarrow & 0 \end{array}$$

So an interesting ~~point~~ is that given a Lagrangian subspace Y, and the ~~quotient~~ X/Y are dual, so you get another symplectic space $Y \oplus X/Y$. So F and ~~the~~ quotient bundle $\mathcal{O} \otimes (V^* \oplus V)$ are self-dual.

I just learned that a Lagrangian subspace Y and the ~~quotient~~ X/Y are dual, so that the assoc. graded $Y \oplus X/Y$ is naturally symplectic.

Next a Lagrangian subspace by itself has no intrinsic notion of positivity. But what is positive? Once $V \oplus V^* = X$ is chosen i.e. Lagrangian subspace + Lag. complement, then ~~another La~~ any Lagrangian Y complementary to both both V, V^* is described by a non-deg. quadratic form $g : V \to V^*$, so it has a signature. ~~What also~~

Can you really get to the bottom of the situation. ~~Yes.~~

You have to analyze things further. Initially you have $0 \subset W \subset U \subset H = H^+ \oplus H^-$
you get $Q_s = sQ_+ \oplus s^{-1}Q_-$ pos. quad form on H,
you find the induced q.f. on $U/W = V$. This gives a Lagrangian subbundle F of $\mathcal{O}\otimes(V^* \oplus V)$. The problem is to find, compute, describe the canonical resolution

~10:45 Jean drives Cindy to work

$$0 \to F \to \mathcal{O}\otimes H^0(F^*)^* \longrightarrow \mathcal{O}(1)\otimes H^0(F^*(-1))^* \to 0$$

then to Bicester Village, I'm alone with Anne-Marie

11:00. So how to proceed? I propose to use a $\cong H$ symplectic approach. One has ~~and~~ on $F_1 \subset \mathcal{O}\otimes(H\oplus H^*)$ to begin with, and ends with $F \subset \mathcal{O}\otimes(V\oplus V^*)$. What's the geometry? From quad form viewpt. you have

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & H \arrow[d] \\ U/W \arrow[r, hook] & H/W \end{tikzcd}$$

Question: Should you be looking at orthogonal subspaces for the quadratic form. Thus you have Q_s on $H^+ \oplus H^- = H$ singular at $s = 0, \infty$ but
11:15 these singularities separate in the direct sum. So you can form W_s^0. You can take $0 \subset W \subset U \subset H$ and split this filtration using Q_s. So what.

If $A_s = s\pi_+ \oplus s^{-1}\pi_-$ on $H^+ \oplus H^-$
then $W_s^0 = A_s^{-1}(W^\perp)$. Some understanding might be achieved in this fashion.

But you would like a symp. approach passing from F_1 Lagrangian in $\mathcal{O}\otimes(H\oplus H^*)$ to F Lagrangian in $\mathcal{O}\otimes(V\oplus V^*)$, where V subquot of H.

583 You have to well understand this stuff with subquotients. Consider

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, hook, "\iota"] & H \arrow[r, two heads, "p"] \arrow[d, shift right, "Q_s"'] & H/W \arrow[r] & 0 \\
0 & W^* \arrow[l] & H^* \arrow[l, two heads, "{}^t\iota"'] \arrow[u, shift right, "Q_s^{-1}"'] & (H/W)^* \arrow[l, hook', "{}^tp"'] & 0 \arrow[l]
\end{tikzcd}$$

What is the best way, a good way to describe inducing a pos quad form Q_s to the subquotient $V = U/W$. Simplest formula

$$Q_1(v) = \inf_{w \in W} Q(v+w)$$

(critical) stationary value

11:34

Is there a way to translate stationary value, critical value into Lagrangian subspace terms? Pull-back, intersect?

11:45 I should be able to settle this. Basically you have $0 \subset W \subset U \subset H$, $V = U/W$, $A : H \to H^*$ and you end up with $A_1 : V \to V^*$. Process

$$\begin{tikzcd}
U/W & U \arrow[l, two heads, "p_1"'] \arrow[r, hook, "i"] \arrow[d, shift left, "\iota^t A \iota"] & H \arrow[d, "A"] \\
(U/W)^* \arrow[u, "p_1(\iota^t A \iota)^{-1} p_1^t"] \arrow[r, hook, "p_1^t"'] & U^* \arrow[u, shift left, "(\iota^t A \iota)^{-1}"] & H^* \arrow[l, "\iota^t"']
\end{tikzcd}$$

alt.

$$\begin{tikzcd}
H \arrow[r, "p"] & H/W \arrow[d, shift left, "(pA^{-1}p^t)^{-1}"] & U/W \arrow[l, hook', "\iota_1"'] \arrow[d, "\iota_1^t (pA^{-1}p^t)^{-1} \iota_1"] \\
H^* \arrow[u, "A^{-1}"] & (H/W)^* \arrow[l, hook', "p^t"'] \arrow[u, shift left, "pA^{-1}p^t"] \arrow[r, two heads, "\iota_1^t"'] & (U/W)^*
\end{tikzcd}$$

So you've described the process of ~~[illegible]~~ inducing $A: H \to H^*$ to a $A_1 : V \to V^*$ where V is a subquotient, namely

$$A_1 = \left(p_1 (\iota^t A \iota)^{-1} p_1^t\right)^{-1} = \iota_1^t \left(p A^{-1} p^t\right)^{-1} \iota_1$$

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, "p_1"] & H \arrow[d, "p"] \\ U/W \arrow[r, hook, "\iota_1"] & H/W \end{tikzcd}$$

Now might $\exists$ a symplectic version of this formula? You have a Lagrangian subspace Y of the symplectic space X. What's the symplectic
12:02 analog of a sub quotient U/W of V? Try a symplectic subquotient: Y/Z where $Z \subset Y \subset X$ are symplectic subspaces, i.e. the restriction of the symp. form on X to Y and to Z is non degenerate. 12:13 Alicia returns.

Suppose $X = H \oplus H^*$, does a ~~[illegible]~~ ~~subquotient~~ of H determine a symplectic sub ~~quotient~~ of X. Probably No. You have $0 \subset W \subset U \subset H$

$0 \subset U^\circ \subset W^\circ \subset H^*$.

But there's a notion of symplectic quotient, starting from an isotropic subspace. ~~What is an isotr~~ Above you have wrong definition of symp. quotient.

 Let's start with $W \subset H$

$$\begin{tikzcd} H \oplus H^* & H \oplus W^\circ \arrow[l, hook'] \arrow[r, two heads] & H/W \oplus W^\circ \end{tikzcd}$$

Work abstractly. Given X symplectic consider symplectic flag manifold. $G = KAN$ Iwasawa decomp for $Sp_{2n}(\mathbb{R})$. I think that $K \cap B = T$ and $B = TAN$. $SL_2(\mathbb{R}) = SO(2) \times \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

[redacted] I think that the symp. flag man. consists of flags of isotropic subspaces. Count dimensions

$$\dim Sp_{2n} = 2n^2 + n = \underbrace{n^2}_{K = U_n} + \underbrace{(n^2+n)}_{\substack{\text{complex symm.} \\ n\times n \text{ mat.}}}$$

symp. flag man. has dim $= \dim(K/T) = n^2$ if $T = O(1)^n$

$$2n-1 + 2n-3 + \cdots + 3 + 1 = n^2$$

$$F_1 \subset F_2 \subset \cdots \subset F_n$$

Assume this is true, i.e. that the only interesting symplectic quotients arise from isot. subspaces.

First case

$$X \hookleftarrow M \twoheadrightarrow Y$$

M: subspace $\ni$ $\omega(M, x) = 0 \implies x \in M$

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d] & M \arrow[r, hook] \arrow[d] & X \\ N \arrow[r, hook] \arrow[d] & Y & \\ Z & & \end{tikzcd}$$

$$\begin{tikzcd} W \oplus H^* \arrow[r, hook] \arrow[d] & U \oplus H^* \arrow[r, hook] \arrow[d] & H \oplus H^* \arrow[d] \\ W \oplus U^* \arrow[r, hook] \arrow[d] & U \oplus U^* \arrow[r, hook] \arrow[d] & H \oplus U^* \arrow[d] \\ W \oplus W^* \arrow[r, hook] & U \oplus W^* \arrow[r, hook] & H \oplus W^* \end{tikzcd}$$

586 Supposedly there is notion of symplectic ~~quotient~~: pass to super Lag subspace, divide by null subspace. Alternative is to divide by an isotropic subspace and pass to subspace where skew form is well defined.

~~Alternatives~~ Question: Given isot subspaces $W \subset U \subset X$ is there someway to relate $U/W \oplus (U/W)^*$ to X? Suppose U Lagrangian. Then

$$\begin{array}{ccc} U^0 & \hookrightarrow & X \\ \downarrow & & \\ 0 = U^0/U & & \end{array}$$

It seems that there is an angle here that is not a consequence of symplectic philosophy. ~~This~~ Namely, if $W \subset U$ are isotropic subspaces of X symplectic, then there doesn't seem to be a natural symplectic space with max isotropic subspace U/W. Simpler: If U is isotropic in X, there doesn't seem to be a natural symplectic space with maximal isotropic subspace U.

IDEA: maximal isotropic subspaces are related to the boundary of the symmetric space - you vaguely recall picking a polarization, describing another polarization via a complex symmetric contraction which you can diagonalize via the action of U_n leading to eigenvalues $0 \le c_1 \le \dots \le c_n < 1$. You can let intervals [redacted] of the c_i tend to 1 at different rates

 Go back to p581 question: Given symplectic X, a Lagrangian subspace, and an isotropic subspace, can you say anything? (Motivation: Quadratic form on V which is non deg on W defines splitting: $V = W \oplus W^{\circ}$.)

This is insufficient information, e.g. the isotropic subspace can be extended to a Lagrangian subspace, and generically two Lag. subspaces describe X in hyperbolic form.

Example: If $X = H \oplus H^*$, $W \subset H$, then $W \oplus W^{\circ}$ is Lagrangian in X. Suppose Lag. subspace F is graph of $Q: H \to H^*$

$$\begin{tikzcd} W \arrow[r, hook] & H \arrow[r] \arrow[d, "S"] & H/W \\ W^* & H^* \arrow[l] & W^{\circ} \arrow[l] \end{tikzcd}$$

So $F \cap (W \oplus W^{\circ}) = \left\{ \begin{pmatrix} h \\ Qh \end{pmatrix} \in \begin{matrix} W \\ \oplus \\ W^{\circ} \end{matrix} \right\}$ i.e. $\langle h, Qh \rangle = 0$.

Thus in this example Q non degenerate on $W \iff \text{graph}_Q$ and $W \oplus W^{\circ}$ are $\pitchfork$.

How can I handle $W \subset U \subset H$?

$W^{\circ} > U^{\circ}$ are max isotropic.

$$\begin{tikzcd} & H \oplus H^* \arrow[d, dash] & \\ & U \oplus W^{\circ} \arrow[dl, dash, "d"'] \arrow[dr, dash] & \\ W \oplus W^{\circ} \arrow[dr, dash] & & U \oplus U^{\circ} \arrow[dl, dash, "d"] \\ & W \oplus U^{\circ} \arrow[d, dash] & \\ & 0 & \end{tikzcd}$$

$\leftarrow$ two Lag. complements for F

588 Main problem: Given $W \subset \mathcal{U} \subset H = H^+ \oplus H^-$

get $F \subset \mathcal{O}\otimes(V\oplus V^*)$ Lagrangian subbundle. $r = rk(F) = \dim(V)$, $d = \deg(F) = \dim H$. in minimal situation.

All this has to be checked carefully and written out. The problem is to construct the "dual" canonical resolution of F:

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] & \mathcal{O}\otimes \underbrace{H^0(F^*)^*}_{\dim = d+r} \arrow[r] & \mathcal{O}(+1)\otimes \underbrace{H^0(F^*(-1))^*}_{\dim\ d} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
W \arrow[d] & W \arrow[d, "d"] \\
\mathcal{U} \arrow[r, hook, "i"] \arrow[d] & H \arrow[d, "p"] \\
V^n = \mathcal{U}\cap W^\perp \arrow[r, hook, "\iota_1"] & W^\perp
\end{tikzcd}$$

It looks like $H^0(F^*)^* \cong \mathcal{U}\oplus W^\perp$

The square

$$\begin{tikzcd}
\mathcal{U} \arrow[r, hook, "i"] \arrow[d, "p_1"] & H \arrow[d, "p"] \\
\mathcal{U}\cap W^\perp \arrow[r, hook, "\iota_1"] & W^\perp
\end{tikzcd}$$

is fixed. You are after the graph of an operator on V n and you have a formula

$$\left(p_1(\iota^* A \iota)^{-1} p_1^*\right)^{-1} = \iota_1^*(pA^{-1}p^*)^{-1}\iota_1$$

So what?

Feb 2. Very little progress toward finding the correspondence. So where to begin.

Let's make more precise the degree structure.

Start with $Z_s = \sum_1 \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$. Look at the graph of $Z_s : V^* \to V$ $\quad F_s = \begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* \subset \begin{matrix} V \\ \oplus \\ V^* \end{matrix}$

589 Then want intersection with $\begin{pmatrix} 1 \\ 0 \end{pmatrix} V^*$

~~so~~ so what actually happens is we have

$$\begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* = F_s \subset \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \qquad \text{and you have } \begin{pmatrix} 1 \\ 0 \end{pmatrix} V$$

want to know what its like to handle a pole say at $s = -i\omega$. We have

$$\frac{s(1+\omega^2)}{s^2+\omega^2} = \frac{1+\omega^2}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)$$

can you see what happens?

near $s = i\omega$ we have $Z_s = \sum_{\text{poles} \neq i\omega} g(s) + \frac{1+\omega^2}{2} a_\omega \frac{1}{s-i\omega}$

analytic near ω.

$$Z_s = g_s + a'_\omega \frac{1}{s-i\omega} : V^* \longrightarrow V$$

Want to understand how $\begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* = \begin{pmatrix} s-i\omega \\ a'_\omega + (s-i\omega) g_s \end{pmatrix} V^*$

at $s = i\omega$ get $\begin{pmatrix} 0 \\ a'_\omega \end{pmatrix} V^*$ in other words the intersection of F_s with $\begin{pmatrix} 1 \\ 0 \end{pmatrix} V^*$. Take $\omega = 0$

Then you have $F_s = \begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* = \begin{pmatrix} s \\ a' + s g_s \end{pmatrix} V^*$ and you want its intersection with $\begin{pmatrix} 1 \\ 0 \end{pmatrix} V^*$, so look at $\begin{pmatrix} 1 \\ 0 \end{pmatrix} V^* \hookrightarrow \begin{matrix} V^* \\ \oplus \\ V \end{matrix} \xrightarrow{(0\ 1)} V$ and we have

the map $V^* \xrightarrow{a' + s g_s} V$. Somehow the part of V the goes to ∞ as $s \longrightarrow i\omega$ is described by a_ω which is why you get the degree you do.

590 Let's go over carefully how to calculate the intersection of $\{F_s\} = F$ vector bundle over $S^2 = \mathbb{C}P^1$ contained in $\mathcal{O} \otimes (V^* \oplus V)$ with the subbundle $\mathcal{O} \otimes V^*$. This means the intersection of the map $F \hookrightarrow \mathcal{O} \otimes (V^* \oplus V) \longrightarrow \mathcal{O} \otimes V$ ~~with the zero or better the zero set of the ... map~~ with 0 computed properly. Locally around $s = i\omega$ (take $\omega = 0$) this map is $V^* \xrightarrow{a' + s g_s} V$, which is non sing except at $s = 0$ where it is a', and you

$$\cancel{\begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* = \begin{pmatrix} g_s + a'_\omega \frac{1}{s - i\omega} \\ 1 \end{pmatrix} V^* = \begin{pmatrix} a'_\omega + (s - i\omega) g_s \\ s - i\omega \end{pmatrix} V^*}$$

~~compose with $(0 \ 1) : V \oplus V^* \longrightarrow V^*$~~

we have to find the fibre of this at $s = i\omega$

Split V^* into $\mathrm{Ker}(a'_\omega)$ and a complement $\mathrm{Im}(a'_\omega)$.

Then in the limit you should get

$$\begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* = \begin{pmatrix} g_s + \frac{a'_\omega}{s - i\omega} \\ 1 \end{pmatrix} V^* \supset \begin{pmatrix} g_{i\omega} \\ 1 \end{pmatrix} \mathrm{Ker}(a'_\omega) \quad \begin{pmatrix} a'_\omega V \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} g_s (s - i\omega) + a'_\omega \\ s - i\omega \end{pmatrix} V^* \supset \begin{pmatrix} a'_\omega V^* \\ 0 \end{pmatrix}$$

Presumably $\begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* \longrightarrow \begin{pmatrix} g_{i\omega} \\ 1 \end{pmatrix} \mathrm{Ker}(a'_\omega) \oplus \begin{pmatrix} a'_\omega V^* \\ 0 \end{pmatrix}$

So ~~...~~ the intersection with $\begin{pmatrix} 1 \\ 0 \end{pmatrix} V$ is $\begin{pmatrix} a'_\omega V^* \\ 0 \end{pmatrix}$ as expected.

Can we compare $s=0$, $s=\infty$ in some good way?

$$\begin{tikzcd} W \arrow[r, hook, "i"] \arrow[d, "p_1"'] & H \arrow[d, "p"] \\ V = U\cap W^\perp \arrow[r, hook, "\iota_1"] & W^\perp \end{tikzcd}$$

$A_s = s\pi_+ + s^{-1}\pi_-$ on $H^+ \oplus H^-$

$$\boxed{\iota_1^*(pA_s^{-1}p^*)^{-1}\iota_1 = \left(p_1(i^*A_s i)^{-1}p_1^*\right)^{-1}}$$

$$pA_s^{-1}p^* = s^{-1}(p\pi_+p^*) + s(p\pi_-p^*) = \sum \frac{s^{-1}+s\omega^2}{1+\omega^2}\pi_\omega \quad \text{on } W^\perp = \oplus (W^\perp)_\omega$$

$$\iota_1^*(pA_s^{-1}p^*)^{-1}\iota_1 = \sum \frac{s(1+\omega^2)}{s^2+\omega^2}\underbrace{\iota_1^*\pi_\omega\iota_1}_{a_\omega}$$

When is H minimal? You have $W^\perp$ minimal when $(W^\perp)_\omega = \operatorname{Im}(\iota_\omega)$ $\quad \iota_1 : V \longrightarrow \oplus (W^\perp)_\omega$

$\iota_{1,\omega} = \pi_\omega \iota_1$

You have H_ω. Take spectral thm for ~~$pA^{-1}p^*$~~ $p\pi_+p^*$ s.a. $0 \le \cdot\cdot \le 1$.

The thing to understand is when H is minimal
This should be easy in either picture

Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525

UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

Remark that the "pole" frequencies, i.e. the poles of $\iota_1^*(p_*(A_s))$ are determined by the eigenvalues of W, whereas the "zero" frequencies, i.e. the poles of $p_{1*}(\iota^*(A_s))$ are determined by U. Let's try to understand minimality. Now when is H minimal as far as $W^\perp$ is concerned, i.e. when is H the minimal dilation of the operator $p\pi_+p^*$ to a projection. For $0<\omega<1$ H_ω is twice the size of $W^\perp_\omega$

$$p\pi_+p^* = \frac{1}{1+\omega^2}$$

$\omega=0$ means $p\pi_+p^* = 1$

or $p\pi_-p^* = 0$ or

$$\pi_-(W^\perp_0) = 0$$

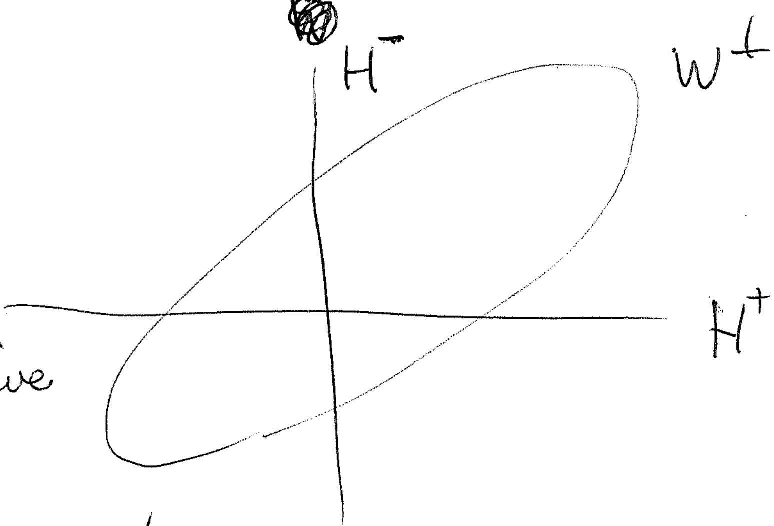
$$\boxed{W^\perp_0 = H^+ \cap W^\perp}$$

$\omega=\infty$ means $p\pi_+p^* = 0$

i.e. $\pi_+p^* = 0$ $\pi_+W^\perp_\infty = 0$

or $W^\perp_\infty \subset H^-$

$$\boxed{W^\perp_\infty = H^- \cap W^\perp}$$



from the poles of $\iota_1^*(p_*A_s)$ you get the minimal $(W^\perp)_\omega$ which will give you

$$\underset{0<\omega<\infty}{2\,\text{rank}\, a_\omega} + \underset{H^+\cap(W^\perp)}{rk(a_0)} + \underbrace{rk\, a_\infty}_{H^-\cap(W^\perp)}$$

for the dim of the minimal H. What about the other side?

593 Now look at $U = \bigoplus U_\omega$

eigenspaces $\quad \iota^* A_s \iota = s(\iota^*\pi_+\iota) + s^{-1}(\iota^*\pi_-\iota)$

Feb 3. Go back to your response function Z and see if you can construct a Hilbert space from the polar data and maybe another from the zero data. Except you must bring in $s = 0, \infty$ somehow into the picture.

Consider then a rational $Z_s =$ say Z_s corresponds to a measure of finite support.

Look at the moment problem. Given the moments $\mu_n = \int x^n d\mu(x)$, recover $d\mu(x)$. Stieltjes found

$$\int \frac{d\mu(x)}{z - x} = \int d\mu \frac{1}{z(1-\frac{x}{z})} = \sum_{n \geq 0} \frac{\mu_n}{z^{n+1}}$$

This is convergent for $|z| > R$ = amplitude of Supp $d\mu$

Take finite measure and construct cont. frac. How does this proceed?

$$f(z) = \sum \frac{a_j}{\lambda_j - z} + a_\infty z$$

$$f(z) = a_\infty z + \frac{1}{\quad}$$

$z = a + bi$

$$\frac{1}{2i}\left(\frac{1}{x - z} - \frac{1}{x - \bar{z}}\right) = \frac{\cancel{x} - \bar{z} - \cancel{x} + z}{2i(x-z)(x-\bar{z})} = \frac{\operatorname{Im}(z)}{(x-a)^2 + b^2}$$

 I want to consider

$$f(z) = \overbrace{\int \frac{d\mu(x)}{x-z}}^{f_1(z)^{-1}} + b_1 + a_1 z \qquad c_1 \text{ real} \quad c_2 \geq 0$$

$$= b_1 + a_1 z + \cfrac{1}{b_2 + a_2 z + \cfrac{1}{\;}}$$

$$\operatorname{Im}(z) > 0 \implies \operatorname{Im}\left(-\frac{1}{z}\right) > 0$$

$$f_1(z) = a_1 z - b_1 + \underbrace{\int \frac{d\mu(x)}{x-z}}_{-\frac{1}{f_2(z)}} \qquad a_1 > 0 \quad b_1 \in \mathbb{R}$$

$$f_1(z) = a_1 z - b_1 - \cfrac{1}{a_2 z + b_2 - \cfrac{1}{a_3 z + b_3 -}}$$

$$f_1(z) = \begin{pmatrix} a_1 z - b_1 & -1 \\ 1 & 0 \end{pmatrix} f_2(z) = \frac{(a_1 z - b_1) f_2 - 1}{f_2}$$

Try next

$$\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \begin{pmatrix} a_1 z - b_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \qquad \begin{aligned} \xi_0 &= (a_1 z - b_1)\xi_1 - \eta_1 \\ \eta_0 &= \xi_1 \end{aligned}$$

$$\xi_0 - (a_1 z - b_1)\xi_1 + \xi_2 = 0$$

$$\xi_0 + b_1 \xi_1 + \xi_2 = a_1 z \xi_1$$

So you have a positive diagonal matrix $(a_1, a_2, \dots)$ and a symmetric J matrix.

$$\begin{matrix} +b_1 & 1 & \\ 1 & b_2 & 1 \end{matrix}$$

leading to ~~real eigenvalues~~ roots.

595 Review what's happening. You start with an $f(z) \in \mathbb{C}(z)$ such that $f(\mathbb{R} \cup \infty) \subset \mathbb{R} \cup \infty$, $f(\{\operatorname{Im} z > 0\}) \subset \{\operatorname{Im} z > 0\}$, and constructed its cont. fraction rep.

$$f(z) = a_1 z - b_1 - \frac{1}{|a_2 z - b_2} - \frac{1}{|a_3 z - b_3} - $$

where $a_1 \geq 0$, $a_2 > 0, \ldots, a_n > 0$, $b_1, \ldots, b_n \in \mathbb{R}$

from this I get Jacobi system.

Febr. 4 Go over Jacobi matrix theory.

Start first with $f(z) \in \mathbb{C}(z)$ $f(\mathbb{R} \cup \infty) \subset \mathbb{R} \cup \infty$ $(\therefore f \in \mathbb{R}(z))$, $f(\{\operatorname{Im}(z) > 0\}) \subset \{\operatorname{Im}(z) > 0\}$.

~~equiv.~~ $\cancel{f(z) = a_1 z - b_1 + }$

Let d = degree f $\qquad$ $az \quad z \to \infty$

$$f_1(z) = f(z) = a_1 z - b_1 + \overbrace{O\left(\frac{1}{z}\right)}^{g_1(z)}$$

$a_1 \neq 0$ then $\deg g_1 = d - 1$.

$a_1 = 0$ —— $\deg g_1 = d$

But $g_1(\infty) = 0$, $f_2 = \frac{-1}{g_1}$ has a pole at $z = \infty$.

$$f_1 = a_1 z - b_1 - \frac{1}{a_2 z - b_2}.$$

The goal should be the underlying "Hilbert space structure" for these formulas. Apparently de Branges has completely worked this out at least for rank 1. Program should be to find the appropriate general setup for the algebraic case

analyze carefully, begin

 What are some of the variables? There an eigenvalue parameter, call it λ, ~~or~~ s, or z. frequency parameter, which places us over $\mathbb{CP}^1$ for the algebraic stuff. There is also a circle ~~[illegible]~~ $\mathbb{P}^1(\mathbb{R})$ and the disk on either side. ~~[illegible]~~
In the moment problem situation, you treat $\lambda = \infty$, $s = \infty$ specially, moreover this is a point on the ~~circle~~ distinguished ~~[illegible]~~ $\mathbb{RP}^1$.

Key idea ~~is~~ to couple ~~[illegible]~~ simple 2 ports, a simple 2-port should be of degree 1 in the eigenvalue parameter, examples? for LC circuits the simple ports are described by $\begin{pmatrix} 1 & as \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ as & 1 \end{pmatrix}$ with $a > 0$. ~~[illegible]~~ what about moment problem?

$$\cancel{f(z) = a_1 z - b_1 - \frac{1}{\;}}$$

$$f(z) = a_1 z - b_1 - \cfrac{1}{a_2 z - b_2 - \cfrac{1}{a_3 z - b_3 - \cfrac{1}{\;}}}$$

$$\frac{\xi_0}{\eta_0} = a_1 z - b_1 - \frac{\eta_1}{\xi_1} = a_1 z - b_1 - \frac{1}{\frac{\xi_1}{\eta_1}} = \frac{(a_1 z - b_1) f_1 - 1}{f_1}$$

$$= \begin{pmatrix} a_1 z - b_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \qquad \eta_0 = \xi_1$$

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} a_1 z - b_1 & -1 \\ +1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\xi_0 = (a_1 z - b_1)\xi_1 - \xi_2$$

$$\xi_0 + b_1 \xi_1 + \xi_2 = a_1 z \xi_1$$

 So the simple 2 ports have the form $\begin{pmatrix} az-b & -1 \\ 1 & 0 \end{pmatrix}$

i.e. $w \longmapsto \begin{pmatrix} az-b & -1 \\ 1 & 0 \end{pmatrix}(w) = \dfrac{(az-b)w-1}{w} = az-b-\dfrac{1}{w}$

Now you want to look at a product of these

$$\begin{pmatrix} a_2z-b_2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1z_1-b_1 & -1 \\ 1 & 0 \end{pmatrix} = \cancel{\begin{pmatrix} & \\ & \end{pmatrix}}$$

$$\begin{pmatrix} (a_2z-b_2)(a_1z_1-b_1)-1 & -a_2z+b_2 \\ a_1z_1-b_1 & -1 \end{pmatrix}$$

there are a lot of minuses making this ugly.

but it's clear you are iterating

$$\xi_{i+1} = (a_iz-b_i)\xi_i - \xi_{i-1}$$

so ~~deg~~

$$\begin{array}{ll} & \text{deg} \\ \xi_0 = 0 & -1 \\ \xi_1 = 1 & 0 \\ \xi_2 = a_1z-b_1 & 1 \\ \xi_3 = (a_2z-b_2)(a_1z-b_1)-1 & 2 \end{array}$$

<u>OKAY</u> ~~How [illegible] to think~~

Thus $\begin{pmatrix} az-b & -1 \\ 1 & 0 \end{pmatrix}$ raises degree (in z) by 1

This in $SL_2(\mathbb{R})$ for z real.

598 I don't know what to make out of this!

~~[illegible]~~ Can $\begin{pmatrix} az-b & -1 \\ 1 & 0 \end{pmatrix}$ be factored.

$$w \mapsto -\frac{1}{w} \longmapsto -b-\frac{1}{w} \longmapsto az-b-\frac{1}{w}$$

$$\begin{pmatrix} 1 & az-b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & a'z-b' \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a'z-b' & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a'z+b & 1 \end{pmatrix}$$

This means

Feb 5. 1 hour on maths.

I want to ~~[illegible]~~ clean up the relationship between the continued fraction & the moment problem. The idea: a measure with finite moments leads to a Hilbert space of polynomials - this might be the de Branges spaces. ~~of polynomials~~

Describe what happens. Move to s plane suppose have inner product on $\mathbb{R}[s] \ni$ mult. by s symmetric. Then get orthogonal polynomials.

$$\phi_0, \phi_1, \phi_2, \ldots$$

$$s\phi_0 = b_0\phi_0 + a_0\phi_1$$

$$s\phi_1 = a_0\phi_0 + b_1\phi_1 + a_1\phi_2$$

$$s(\phi_0\ \phi_1 \cdot) = (\phi_0\ \phi_1 \cdots)\begin{pmatrix} b_0 & a_0 & \\ a_0 & b_1 & a_1 \\ & a_1 & b_2 \end{pmatrix}$$

Point evaluator. To keep thing simple suppose $s \to -s$ symmetry i.e. all $b_j = 0$.

599 The key idea is probably the point evaluator. In any case there is a finite amount of data to get straight.

Point: Given $d\mu$ you get inner product on polys: $\mathbb{R}[s]$, so if you look at polys of degree $< n$, you have an $n \times n$ Jacobi matrix, a measure with ~~n point~~ supported on n points, you have a standard way to "close" the partial herm. operator $s: V_{n-1} \longrightarrow V_n$ to a hermitian operator. Probably what you ~~need~~ need to finish the picture is the point evaluator formula

---

one project - correlate J-matrix and cont. fr. exp.

start with $f_1(s) = a_1 s + \cfrac{1}{a_2 s + \cfrac{1}{a_3 s +}}$

$$f_1(s) = a_1 s + \frac{1}{f_2(s)} = \begin{pmatrix} 1 & a_1 s \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(f_2)$$

$$= \begin{pmatrix} a_1 s & 1 \\ 1 & 0 \end{pmatrix}(f_2)$$

thus if

$$\begin{pmatrix} \eta_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} a_1 s & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \eta_2 \\ \xi_2 \end{pmatrix} \Rightarrow \xi_1 = \eta_2$$

sim $\xi_0 = \eta_1$

and so we end up with

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} a_1 s \xi_1 + \xi_2 \\ \xi_1 \end{pmatrix} \qquad \xi_0 = a_1 s \xi_1 + \xi_2$$

In general then we end up this way with the standard skew adjoint operator $\xi \mapsto \sigma\xi - \sigma^{-1}\xi$ where σ is the shift, and there is the pos. s.a diagonal operator a. Then to ~~[illegible]~~ understand the operator [illegible] $\sigma - \sigma^{-1} - as$.

Alternate notation. $f_1(\omega) = a_1 \omega - \frac{1}{f_2} = \begin{pmatrix} a_1\omega & -1 \\ 1 & 0 \end{pmatrix}$

$$\xi_0 = a_1 \omega \xi_1 - \xi_2 \qquad \text{or} \qquad \xi_0 + \xi_2 = a_1 \omega \xi_1$$

600 How to organize? Equations

$$\xi_{n-1} + \xi_{n+1} = \omega a_n \xi_n \qquad \forall n.$$

operator $\sigma + \sigma^{-1} - \omega a$ in

Suppose you ~~restrict~~ to ξ having support $\{1, \dots, n\}$

then $\sigma + \sigma^{-1}$ is compressed to this subspace

One way to handle a is to ~~factor~~

$$a^{-1/2}(\sigma + \sigma^{-1} - \omega a)\, a^{-1/2} = a^{-1/2}\sigma + (a^{-1/2}\sigma)^* - \omega$$

So that the ~~eigenvalues~~ critical freq are the eigenvalues of the s.a. operator $a^{-1/2}\sigma + (a^{-1/2}\sigma)^*$

Focus upon the increasing family of Hilbert spaces that you get from the orthogonal polys.

Review what you learned. You have a p.f.

$$f_1(\omega) = a_1\omega - \cfrac{1}{a_2\omega - \cfrac{1}{\ \ }} = a_1\omega - \frac{1}{f_2}$$

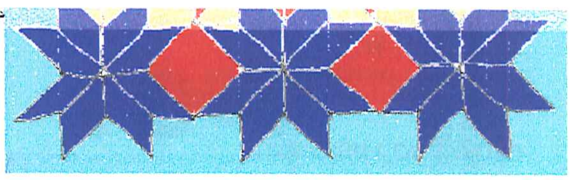
$$f_1 = \begin{pmatrix} 1 & a_1\omega \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(f_2)$$

$$= \begin{pmatrix} a_1\omega & -1 \\ 1 & 0 \end{pmatrix}(f_2) \qquad \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} a_1\omega & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

leads to

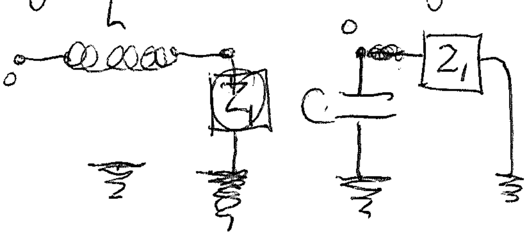
$$\xi_0 + \xi_2 = a_1\omega\,\xi_1$$
$$\xi_{n-1} + \xi_{n+1} = a_n\omega\,\xi_n$$

You want something smooth



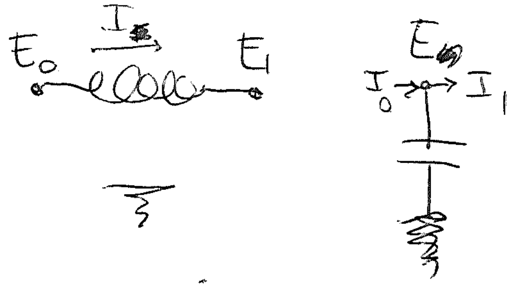
601

Here is something occurring to me. Recall that the s version leads to a standard skew-symmetric op, roughly $\sigma - \sigma^{-1}$ and a positive definite one $a = (a_n)$. What's rather nice is the picture of coupled 2 ports that emerges. You have the standard type symplectic structure coupled with the diagonal terms. Diagonal terms are



$$Z_0 = Ls + Z_1 \qquad \frac{1}{Z_0} = Cs + \frac{1}{Z_1} \qquad Z_0 = \frac{1}{Cs + \frac{1}{Z_1}}$$

$$Z_0 = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix} \qquad Z_0 = \begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix} Z_1$$



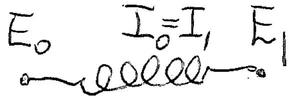
P = net power in
$= E_0 I - E_1 I$

$$P = EI_0 - EI_1$$

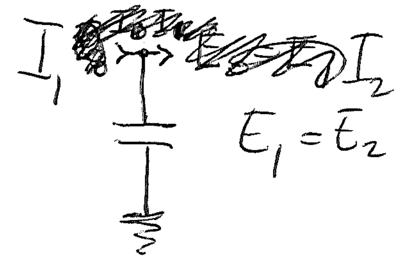
UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525

602 This seems like an interesting point, but perhaps not really important. It seems advisable ~~necessary~~ now to forego the electrical picture, because the inductance capacitance distinction seems not to be basic. **YES.** just odd versus even in numbering.



$$E_0 I_0 - E_1 I_1 = (E_0 - E_1) I_1$$



$$E_1 I_1 - E_2 I_2 = E_1 (I_1 - I_2)$$

write in terms of $\xi_0 = E_0$, $\xi_1 = I$, $\xi_2 = E_1$

It seems the skew form is not really completely skew symmetric because of edge effects. **YES**

$$\sum p_i \Delta q_i = \sum p_i (q_i - q_{i+1}) \qquad - \sum \Delta p\, q$$

Feb 6. Yesterday you ended with confusion over the symplectic business. Go over some of the ideas. First you wrote the continued fraction in the s variable

$$f_1 = a_1 s + \frac{1}{f_2} \qquad f_1 = \begin{pmatrix} a_1 s & 1 \\ 1 & 0 \end{pmatrix} f_2$$

leading to

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} a_1 s & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

or

$$\xi_0 = a_1 s \xi_1 + \xi_2$$
$$\xi_0 - \xi_2 = s a_1 \xi_1$$

Thus ignore boundary effects

$$\xi_{n-1} - \xi_{n+1} = s a_n \xi_n$$
$$(\sigma - \sigma^{-1}) \xi = s (a \xi)$$

where $(\sigma\xi)_n = \xi_{n-1}$ is the forward shift. What I liked about this is the combination of the skew symmetric operator $\sigma - \sigma^{-1}$ and the ~~[illegible]~~ positive symmetric operator a, which means we have a harmonic oscillator structure, phase space picture. (Also write σ^* instead of σ to handle the Toeplitz (half space) version).

$\sigma - \sigma^{-1}$ is a standard type skew-symm. op. You want to ~~[illegible]~~ link it to the coupling of 2 ports. ~~[illegible]~~ This brings in power somehow, you were identifying power ~~[illegible]~~ EI somehow with the symplectic form. But power is a quadratic function on

---

Feb 8. Return to the symplectic stuff

$$f_1(s) = a_1 s + \frac{1}{f_2} = \begin{pmatrix} 1 & a_1 s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(f_2) = \begin{pmatrix} a_1 s & 1 \\ 1 & 0 \end{pmatrix}(f_2)$$

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} a_1 s & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \xi_0 = s a_1 \xi_1 + \xi_2$$

$$\xi_{n-1} - \xi_{n+1} = s a_n \xi_n$$

which means studying the operator $(\sigma - \sigma^*)(\xi) = s a \xi$

$\sigma - \sigma^*$ skew symm, a pos symmetric, which means we have a harmonic oscillator, at least if $\sigma - \sigma^*$ is non-degenerate which should be true in even degrees.

$$\begin{pmatrix} & -1 & \\ 1 & & -1 \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

certainly true for $n \geq 1$,

$$\begin{pmatrix} & -1 & & \\ 1 & & -1 & \\ & 1 & & -1 \\ & & 1 & \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Yes

~~[illegible]~~ I want to use the ~~[illegible]~~ idea that $\sigma - \sigma^{*-1}$ is a standard coupling.

604 I want to make something out of the symplectic ~~coupl~~ coupling idea. Actually you mean Hamilton's principle. Hamilton's principle say the classical motion in time is stationary for the action $\int p\,dq - H\,dt$, thus

$$\delta \int_{t_0}^{t_1} (p\dot{q} - H)\,dt = \int \left(\delta p\, \dot{q} + p\,\delta\dot{q} - \delta p \frac{\partial H}{\partial p} - \delta q \frac{\partial H}{\partial q} \right) dt$$

$$= \int \left(\delta p \left(\dot{q} - \frac{\partial H}{\partial p} \right) + \left(-\dot{p} - \frac{\partial H}{\partial q} \right) \delta q \right) dt$$

$$- \left[p\,\delta q \right]_{t_0}^{t_1} = 0$$

11:45

So $\dot{q} = \frac{\partial H}{\partial p}$ $\quad \dot{p} = -\frac{\partial H}{\partial q}$ $\quad$ and $\boxed{(p\,\delta q)(t_1) = (p\,\delta q)(t_0)}$

Is there any significance to the last fact? ~~The problem is that~~ What is the meaning of $p\,\delta q$? What does it mean? $p\,\delta q$ is a bilinear form. $p\,\delta q$ is a quadratic function of $(p,q) \in V^* \oplus V$. Fundamentally it defines the duality between V^* and V, i.e. between position and momentum space. The condition says this duality is preserved under time evolution. This implies that the symplectic structure is preserved on $V^* \oplus V$. How can I phrase things? This is strange. Suppose I consider two symplectic spaces in split form $V \oplus V^*$, $W \oplus W^*$ and suppose I give an isomorphism $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} W \\ \oplus \\ W^* \end{matrix} \xleftarrow{\sim} \begin{matrix} V \\ \oplus \\ V^* \end{matrix}$

Then the meaning of $p^t\delta q = p'^t \delta q'$ is? here $q \in V$, $p \in V^*$, $q' \in W$, $p' \in W^*$

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \qquad p'^t \delta q' = (q^t c^t + p^t d^t)(a q + b p)$$

$$= q^t(c^t b + a^t d)p + q^t \underbrace{(c^t a)}_{} q + p^t \underbrace{(d^t b)}_{} p$$

certainly $\neq 0$ for dim $= 1$

605 suppose $H = 0$ so that p, q constant in time

$$\int_{t_0}^{t_1} (p\dot{q})\, dt = 0. \quad \text{No help.}$$

Is my view of Hamilton's principle wrong?

I have $$\int_{t_0}^{t_1} (p\dot{q} - H(p,q))\,dt = A$$

The action functional on the space $\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}$ of paths $[t_0, t_1] \longrightarrow V \oplus V^*$. This action function is a quadratic function on the path space. Stationary means

$$\delta A = \frac{\delta A}{\delta q}\delta q + \frac{\delta A}{\delta p}\delta p = 0$$

Keep $q(t)$ fixed, what is $\frac{\delta A}{\delta p}$?

$$\delta A = \int_{t_0}^{t_1} \delta p\left(\dot{q} - \frac{\partial H}{\partial p}\right) dt$$

Keep $p(t)$ fixed

$$\delta A = \int_{t_0}^{t_1} \left(p\, \delta\dot{q} - \delta q \frac{\partial H}{\partial q}\right) dt$$

$$= \cancel{-}\,[p\,\delta q]_{t_0}^{t_1} - \int \left(\dot{p} + \frac{\partial H}{\partial q}\right) \delta q \, dt$$

I guess what might happen is that p ~~might~~ jumps at the endpoints.

The problem is clear, namely, the action functional is a quadratic function of the path in phase space, so it depends only on the symmetrization of A.

Symmetrize $\int p\dot{q}\, dt - \int H\, dt$

to get $$\frac{1}{2}\int (p_1\dot{q}_2 + p_2\dot{q}_1)\, dt - \frac{1}{2}\int \big(H(p_1,q_1) + H(p_2,q_2)\big)\, dt$$

$$= -\frac{1}{2}(p_1 q_2 + q_2 p_1)\Big]_{t_1}^{t_2} - \frac{1}{2}\int (\dot{p}_1 q_2 + \dot{p}_2 q_1) - H(p_1,q_1) - H(p_2,q_2)$$

606 What you've decided to do is to try to sort out this business of the symplectic coupling on the discrete level. How does this work? How am I do this? How can you proceed? What are the basic ideas? Let's start with the coupling idea - the change from a quadratic form to a symplectic transformation. This is fairly basic. We have $V \oplus W$ configuration space for the port and a quadratic form on $V \oplus W$, i.e. an isom $V \oplus W \xrightarrow{\sim} V^* \oplus W^*$ whose graph is maximal isotropic for the symp. form $\omega_{can} \oplus (-\omega_{can})$ on $(V \oplus V^*) \oplus (W \oplus W^*)$. In good case this ~~graph is a~~ maximal isotropic subspace is also the graph of an ~~isom~~ symplectomorphism $V \oplus V^* \simeq W \oplus W^*$

Formulas. $\begin{pmatrix} \alpha & \beta^t \\ \beta & \gamma \end{pmatrix}$ quadratic form. $\begin{matrix} V^* \\ \oplus \\ W^* \end{matrix} \longleftarrow \begin{matrix} V \\ \oplus \\ W \end{matrix}$

$$\begin{pmatrix} 1 & 0 \\ \alpha & \beta^t \\ 0 & 1 \\ \beta & \gamma \end{pmatrix} : \begin{matrix} V \\ \oplus \\ V^* \\ \oplus \\ W \\ \oplus \\ W^* \end{matrix} \longleftarrow \begin{matrix} V \\ \oplus \\ W \end{matrix}$$

$$= \begin{pmatrix} 0 & 1 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \beta^t \end{pmatrix}^{-1} : \begin{matrix} W \\ \oplus \\ W^* \end{matrix} \qquad \begin{matrix} V \\ \oplus \\ V^* \end{matrix}$$

$$\begin{pmatrix} 0 & 1 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(\beta^t)^{-1}\alpha & (\beta^t)^{-1} \end{pmatrix} = \begin{pmatrix} -(\beta^t)^{-1}\alpha & (\beta^t)^{-1} \\ \beta - \gamma(\beta^t)^{-1}\alpha & \gamma(\beta^t)^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \alpha & \beta^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \beta & \gamma \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \alpha & \beta^t \end{pmatrix} \begin{pmatrix} -\beta^{-1}\gamma & \beta^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\beta^{-1}\gamma & \beta^{-1} \\ \beta^t - \alpha\beta^{-1}\gamma & \alpha\beta^{-1} \end{pmatrix}$$

 So you have

$$\begin{pmatrix} \alpha & \beta^t \\ \beta & \gamma \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\beta^{-1}\gamma & \beta^{-1} \\ \beta^t - \alpha\beta^{-1}\gamma & \alpha\beta^{-1} \end{pmatrix}$$

$$\begin{pmatrix} -d^t & b^t \\ c^t & -a^t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} -(\beta^t)^{-1}\alpha & (\beta^t)^{-1} \\ \beta - \gamma(\beta^t)^{-1}\alpha & \gamma(\beta^t)^{-1} \end{pmatrix}$$

Some things to consider: This map goes from symmetric matrices to symplectic matrices, so it's a kind of Cayley transform.

Look at the 1 dim'l case.

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\frac{\gamma}{\beta} & \frac{1}{\beta} \\ \beta - \frac{\alpha\gamma}{\beta} & \frac{\alpha}{\beta} \end{pmatrix}$$

$$\frac{-\gamma\alpha}{\beta^2} - \left(\frac{\beta^2 - \alpha\gamma}{\beta^2}\right)$$

sign is wrong.

symplectic: $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\qquad g^t J g = J$

$$g^{-1} = J^{-1} g^t J \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$$

Look at C.T.

$$g = \frac{1+X}{1-X} \qquad g^t = \frac{1+X^t}{1-X^t}$$

$$\frac{1-X}{1+X} = g^{-1} = J^{-1} g^t J = \frac{1 + J^{-1}X^tJ}{1 - J^{-1}X^tJ}$$

$$J^{-1}X^tJ = -X \qquad X^tJ = -JX = (X^tJ)^t$$

Thus $X = SJ$ where S symm.

$$X^t = -JS = -J^{-1}XJ$$

608 Feb 8 $\quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad g \in \text{Symp}: \ g^t J g = J$

$$\iff g^{-1} = J^{-1} g^t J$$

$$g = \frac{1+x}{1-x} \Rightarrow g^t = \frac{1+x^t}{1-x^t}$$

so $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Symp $\iff g^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c^t & -a^t \\ d^t & -b^t \end{pmatrix}$

$$= \begin{pmatrix} +d^t & -b^t \\ -c^t & a^t \end{pmatrix}$$

symp $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$

now suppose $g = \frac{1+x}{1-x} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}(x) \qquad x = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}(g) = \frac{g-1}{g+1}$

$g = -1 + \frac{2}{1-x} \qquad J^{-1} g^t J = J^{-1}\frac{1+x^t}{1-x^t}J = \frac{1 + J^{-1}x^t J}{1 - J^{-1}x^t J} \qquad g^{-1} = \frac{1-x}{1+x}$

so want $\quad J^{-1}x^t J = -x \qquad$ i.e. $x \in$ Lie Sp

$x^t J + Jx = 0 \qquad (Jx)^t = x^t(-J) = Jx$

Thus Lie Sp $= sp$ is $\cong$ the space of symm. matrices.

via $x \longmapsto Jx = h \quad \therefore x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sp \iff \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$

is symm. i.e. $d = -a^t$, $b^t = b$, $c^t = c$

$$h = \begin{pmatrix} \alpha & \beta^t \\ \beta & \gamma \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta^t \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta^t \end{pmatrix} = x$$

$h \longmapsto x = J^{-1}h \qquad h = Jx$

$$\frac{1+x}{1-x} = \frac{1+J^{-1}h}{1-J^{-1}h} = (1-J^{-1}h)^{-1}(1+J^{-1}h)$$

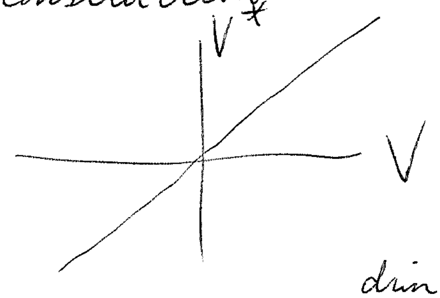
$$= (J-h)^{-1}(J+h)$$

So it seems that the correspondence you want is not the Cayley transform. Work it out for $SL_2(\mathbb{R}) = Sp_2(\mathbb{R})$. Is it involved with ~~transition~~ transmission, scattering?

609 Anyway what happens?

Basic object must be Lagrangian subspaces

quadratic form q on V is ~~...~~ same as Lagrangian subspace of $V \oplus V^*$, transversal to V^* by the graph construction



$$\begin{pmatrix}1\\ q\end{pmatrix}^t \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ q\end{pmatrix} = \begin{pmatrix}1 & q^t\end{pmatrix}\begin{pmatrix}-q\\ 1\end{pmatrix} = -q+q^t$$

$$\dim \text{Lag Grass} = \frac{n(n+1)}{2}$$

Let X, Y be Symp v.s. $X \oplus Y$ with form $\omega_X \oplus (-\omega_Y)$. ~~Symplectic for~~ $\Gamma \subset X \oplus Y$ trans. to X, Y. Γ graph of ~~...~~ invertible $f : X \to Y$. Then f is a symplectic iso $\Leftrightarrow$ Γ Lagrangian.

$$\omega_{\cancel{Y}}(f(x_1), f(x_2)) \overset{?}{=} \omega_X(x_1, x_2)$$

$$\omega_X(x_1, x_2) - \omega_Y(fx_1, fx_2) = 0$$

i.e. $\begin{pmatrix}x\\ fx\end{pmatrix}$ is Lagrangian.

$\cancel{X} = \mathbb{R}^2 = V \oplus V^*$ $\qquad Y = W \oplus W^* = \mathbb{R}^2$

$$X = \begin{matrix}V\\ \oplus\\ V^*\end{matrix} \qquad Y = \begin{matrix}W\\ \oplus\\ W^*\end{matrix} \qquad X \oplus Y = \begin{matrix}V\\ \oplus\\ V^*\\ \oplus\\ W\\ \oplus\\ W^*\end{matrix} \supset \begin{pmatrix}1 & 0\\ 0 & 1\\ a & b\\ c & d\end{pmatrix}$$

$$X \oplus Y = \begin{matrix}V\\ \oplus\\ W^*\\ \oplus\\ V^*\\ \oplus\\ W\end{matrix} \qquad J = \begin{pmatrix} & & 1 & \\ & & & -1\\ -1 & & & \\ & +1 & & \end{pmatrix} \to \begin{pmatrix}0 & 0 & & 1\\ 0 & -1 & & \\ -1 & 0 & & \\ 0 & 0 & 1 & \end{pmatrix} \to \begin{pmatrix}0 & 0 & 0 & 1\\ & & & 1\\ & & -1 & \\ -1 & & & \end{pmatrix}$$

60

$$J = \left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right) \longmapsto \begin{pmatrix} & & -1 & \\ 1 & & & \\ & 1 & & \\ & & & -1 \end{pmatrix} \longmapsto \begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

$$X \oplus Y = \begin{matrix} V \\ \oplus \\ W^* \\ \oplus \\ V^* \\ \oplus \\ W \end{matrix} \supset \begin{pmatrix} 1 & 0 \\ c & d \\ 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} V \\ \oplus \\ V^* \end{pmatrix}$$

It looks like we want $\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} -d^{-1}c & d^{-1} \\ a - bd^{-1}c & bd^{-1} \end{pmatrix}$

$$\begin{matrix} V^* \\ \oplus \\ W \end{matrix} \xleftarrow{\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}} \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \xleftarrow{\begin{pmatrix} 1 & 0 \\ -d^{-1}c & d^{-1} \end{pmatrix}} \begin{matrix} V \\ \oplus \\ W^* \end{matrix}$$

Is it clear that $\begin{pmatrix} -d^{-1}c & d^{-1} \\ a - bd^{-1}c & bd^{-1} \end{pmatrix}$ is symmetric

when $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$?

$ad^t - bc^t = 1$ $\quad ab^t = ba^t$

$d^t a - c^t b = 1$ $\quad cd^t = dc^t$

$\cancel{d^t a}$ $\quad d^{-1}c = c^t (d^t)^{-1} = (d^{-1}c)^t$

$d^t b = b^t d \implies \cancel{bd^{-1}}$ $\quad bd^{-1} = (d^t)^{-1} b^t = (bd^{-1})^t.$

$c^t a = a^t c$

$a - b\underbrace{c^t (d^t)^{-1}}_{(d^{-1}c)^t = d^{-1}c.} = (d^{-1})^t$

Go back to C.T. but on $V \oplus V^*$. A quadratic function on $V \oplus V^*$ is $\begin{pmatrix} \alpha & \beta \\ \beta^t & \gamma \end{pmatrix} : \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \longleftarrow \begin{matrix} V \\ \oplus \\ V^* \end{matrix}$

function on $V \oplus W$ is $\begin{matrix} V^* \\ \oplus \\ W^* \end{matrix} \xleftarrow{\begin{pmatrix} \alpha & \beta \\ \beta^t & \gamma \end{pmatrix}} \begin{matrix} V \\ \oplus \\ W \end{matrix}$

6 11 Get thoroughly confused. The $U(n,n)$ theory might be simpler? How can you possibly do this?

I guess I would like to understand the <u>successive</u> coupling arising from continued fractions.

Here's what we did above $X = \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{matrix} W \\ \oplus \\ W^* \end{matrix} = Y$

symplectic iso $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} = \begin{pmatrix} ad^t - bc^t & -ab^t + ba^t \\ cd^t - dc^t & da^t - cb^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^t a - b^t c & d^t b - b^t d \\ -c^t a + a^t c & a^t d - c^t b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form $X \oplus Y$ with $\omega_X \oplus (-\omega_Y)$

$$X \oplus Y = \begin{matrix} V \\ \oplus \\ V^* \\ \oplus \\ W \\ \oplus \\ W^* \end{matrix} \simeq \begin{matrix} V \\ \oplus \\ W^* \\ \oplus \\ V^* \\ \oplus \\ W \end{matrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{pmatrix} \begin{pmatrix} V \\ \oplus \\ V^* \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ c & d \\ 0 & 1 \\ a & b \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_1^* \\ w_1 \\ w_1^* \end{pmatrix} \wedge \begin{pmatrix} v_2 \\ v_2^* \\ w_2 \\ w_2^* \end{pmatrix} = \begin{matrix} +v_1 v_2^* - v_1^* v_2 \\ -w_1 w_2^* + w_1^* w_2 \end{matrix}$$

$$\begin{pmatrix} v_1 \\ w_1^* \\ v_1^* \\ w_1 \end{pmatrix} \wedge \begin{pmatrix} v_2 \\ w_2^* \\ v_2^* \\ w_2 \end{pmatrix} = \begin{matrix} v_1 v_2^* + w_1^* w_2 \\ - v_1^* v_2 - w_1 w_2^* \end{matrix}$$

6/2 Go backwards

$$V \oplus W^* \oplus V^* \oplus W \supset \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \\ \beta^t & \gamma \end{pmatrix} \begin{pmatrix} V \\ \oplus \\ W^* \end{pmatrix}$$

$$V \oplus V^* \oplus W \oplus W^* \supset \begin{pmatrix} 1 & 0 \\ \alpha & \beta \\ \beta^t & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V \\ \oplus \\ W^* \end{pmatrix}$$

$$V \oplus V^* \xrightarrow{\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}^{-1}} V \oplus W^* \xrightarrow{\begin{pmatrix} \beta^t & \gamma \\ 0 & 1 \end{pmatrix}} W \oplus W^*$$

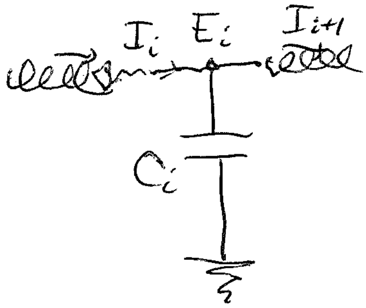
$$\cancel{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & \beta^t \end{pmatrix} \begin{pmatrix} \beta^{-1} & -\beta^{-1}\gamma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta^{-1} & -\beta^{-1}\gamma \\ \alpha\beta^{-1} & \beta^t - \alpha\beta^{-1}\gamma \end{pmatrix}}$$

something went wrong because you want $\beta = d^{-1}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \beta^t & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta^{-1}\alpha & (\beta)^{-1} \end{pmatrix} = \begin{pmatrix} \beta^t - \gamma\beta^{-1}\alpha & \gamma\beta^{-1} \\ -\beta^{-1}\alpha & \beta^{-1} \end{pmatrix}$$

---

try transmission line approach $\qquad \omega_0 = \frac{1}{\sqrt{LC}}$



$$E_{i-1} - E_i = L_i \partial_t I_i$$

$$I_i - I_{i+1} = C_i \partial_t E_i$$

$$E_{x-\Delta x} - E_x = (l \Delta x) \partial_t I_x$$

$$I_x - I_{x+\Delta x} = (c\Delta x) \partial_t E_x$$

$$\boxed{\begin{aligned} -\partial_x E(x,t) &= l \partial_t I(x,t) \\ -\partial_x I(x,t) &= c \partial_t E(x,t) \end{aligned}}$$

$$\partial_x^2 E = -l \partial_t \partial_x I$$

$$\boxed{\partial_x^2 E = lc \partial_t^2 E}$$

$\frac{1}{\sqrt{lc}}$

613 assume $lc = 1$ $\cancel{e^{i\omega(x-t)}}$

$$(\partial_x^2 - \partial_t^2) E = 0$$

$$-\partial_x E =$$

$$\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ \ \end{pmatrix} e^{i\omega(x-t)}$$

$$-\partial_x (E e^{st}) = ls (I e^{st})$$
$$-\partial_x (I e^{st}) = cs (E e^{st})$$

$$\partial_x E + lsI = 0$$
$$\partial_x I + csE = 0$$

$$\partial_x^2 E - s^2 E = 0$$

$$E = a e^{sx} + b e^{-sx}$$
$$-\partial_x E = -sae^{sx} + sbe^{-sx} = lsI$$
$$I = -cae^{sx} + cbe^{-sx}$$

$$\cancel{\begin{pmatrix} E \\ I \end{pmatrix} = e^{sx+st} \begin{pmatrix} 1 \\ c \end{pmatrix}} a +$$

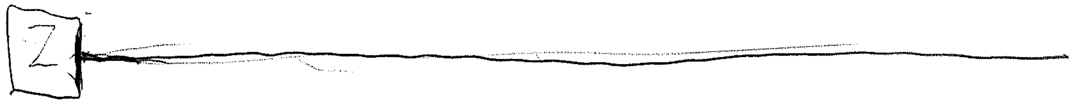
$$\boxed{\begin{pmatrix} E \\ I \end{pmatrix}(x,s) = e^{sx} \begin{pmatrix} 1 \\ -c \end{pmatrix} A + e^{-sx} \begin{pmatrix} 1 \\ c \end{pmatrix} B}$$

A, B const.

$$\begin{pmatrix} E \\ I \end{pmatrix}(x,t) = \overset{\text{incoming}}{e^{s(x+t)}} \begin{pmatrix} 1 \\ -c \end{pmatrix} A + \overset{\text{outgoing}}{e^{s(t-x)}} \begin{pmatrix} 1 \\ c \end{pmatrix} B$$

$$\frac{B}{A}$$

$$\begin{pmatrix} E \\ I \end{pmatrix}(0,t) = e^{st} \begin{pmatrix} 1 \\ -c \end{pmatrix} A + e^{st} \begin{pmatrix} 1 \\ c \end{pmatrix} B$$
$$= e^{st} \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$



$$-Z(s) = \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} c & -1 \\ c & 1 \end{pmatrix} \begin{pmatrix} -Z \\ \ \end{pmatrix} = \frac{-cZ - 1}{-cZ + 1}$$

$$\frac{B}{A} = \frac{cZ - 1}{cZ + 1}$$

$$\boxed{\frac{B}{A} = \frac{Z - l}{Z + l}}$$

614 Now I want to study a 2-port from the viewpoint of scattering.

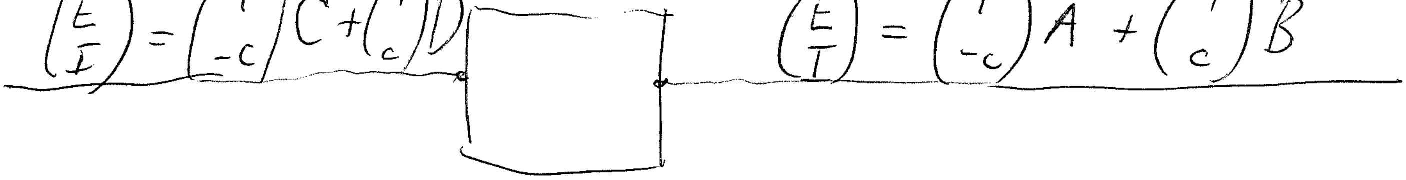
equations $\partial_x E + l\,\partial_t I = 0$ $\qquad lc = 1$

$\partial_x I + c\,\partial_t E = 0$

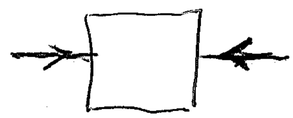
solutions

$$\begin{pmatrix}E\\I\end{pmatrix}(x,t) = e^{s(t+x)}\begin{pmatrix}1\\-c\end{pmatrix}A + e^{s(t-x)}\begin{pmatrix}1\\c\end{pmatrix}B$$

$\begin{pmatrix}E\\I\end{pmatrix} = \begin{pmatrix}1\\-c\end{pmatrix}C + \begin{pmatrix}1\\c\end{pmatrix}D$ $\qquad$ $\begin{pmatrix}E\\I\end{pmatrix} = \begin{pmatrix}1\\-c\end{pmatrix}A + \begin{pmatrix}1\\c\end{pmatrix}B$



Study 2 port.

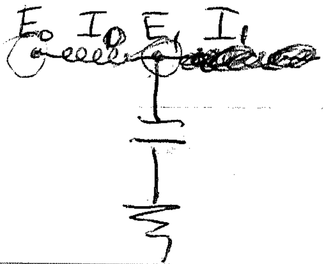


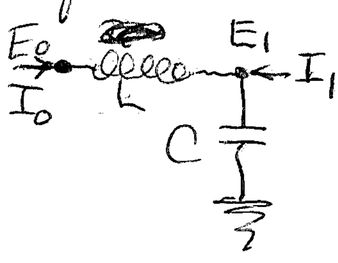
quadratic form $\Gamma_s = \begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}$?

Response $I = \Gamma_s E$ $\qquad \begin{pmatrix}I_0\\I_1\end{pmatrix} = \Gamma_s\begin{pmatrix}E_0\\E_1\end{pmatrix}$

There is some confusion because ? I haven't really explained the duality between current space and voltage space. Duality is given by the quadratic function?

Consider ~~exam~~ example of a ladder circuit.





$E_0 - E_1 = LsI_0$

$I_0 + I_1 = CsE_1$

615 so have 4 dim'l space with coordinates E_0, I_0, E_1, I_1 2 equations $\left\{\begin{array}{l} E_0 - E_1 = Ls I_0 \\ I_0 + I_1 = Cs E_1 \end{array}\right.$ F_s

get 2 dim'l subspace for any s. ~~What is the response~~ Check F_s is Lagrangian.
What is the skew form. — power into circuit?

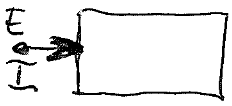
$$\begin{pmatrix} E_0 \\ I_0 \\ E_1 \\ I_1 \end{pmatrix}^t \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right) \begin{pmatrix} E_0' \\ I_0' \\ E_1' \\ I_1' \end{pmatrix} = \begin{array}{l} E_0 I_0' - E_0' I_0 \\ E_1 I_1' - E_1' I_1 \end{array}$$

power into circuit is a quadratic function

$$\begin{pmatrix} E_0 \\ I_0 \\ E_1 \\ I_1 \end{pmatrix} \longmapsto E_0 I_0 + E_1 I_1$$ which sets

up a duality between $\left|\begin{array}{l} E_0 \\ E_1 \end{array}\right.$ space and $\left|\begin{array}{l} I_0 \\ I_1 \end{array}\right.$ space.

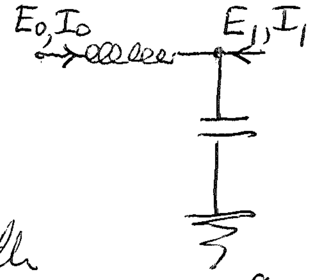
and then you take the corresponding symplectic form
Already I should do this for a 1-port.



2 dim'l space of $\begin{pmatrix} E \\ I \end{pmatrix} \in R^2$
for each s get $F_s = \begin{pmatrix} z_s \\ 1 \end{pmatrix} \mathbb{R} \subset \mathbb{R}^2$

so I get a line bundle over the s plane $\cup \infty$

start again: Go back to



$E_0 - E_1 = Ls I_0$
$I_0 + I_1 = Cs E_1$

You have for each s a
2 dim'l subspace of 4 space with
coords E_0, I_0, E_1, I_1. So you have a correspondence
between voltage space + current space, and a corresp
between E_0, I_0 and E_1, I_1. Generically these
correspondences should be isomorphisms. Find formulas.

616

$$\boxed{\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_0 \end{pmatrix} \qquad \begin{pmatrix} E_1 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Cs & +1 \end{pmatrix}\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}}$$

$$\therefore \begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ Cs & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix} = \begin{pmatrix} 1+LCs^2 & Ls \\ Cs & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} E_1 \\ -I_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -Cs & 1 \end{pmatrix}\begin{pmatrix} 1 & -Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & -Ls \\ -Cs & 1+LCs^2 \end{pmatrix}\begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

$$E_1 = (Cs)^{-1}(I_0 + I_1)$$

$$E_0 = E_1 + Ls I_0 = ((Cs)^{-1} + Ls) I_0 + (Cs)^{-1} I_1.$$

$$\boxed{\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} (Cs)^{-1} + Ls & (Cs)^{-1} \\ (Cs)^{-1} & (Cs)^{-1} \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \end{pmatrix}}$$

$\det = LC^{-1}$

$$I_0 = (Ls)^{-1} E_0 - (Ls)^{-1} E_1$$

$$I_1 = Cs E_1 - (Ls)^{-1} E_0 + (Ls)^{-1} E_1$$

$$= (Cs + (Ls)^{-1}) E_1 - (Ls)^{-1} E_0$$

<u>Yes.</u>

$$\boxed{\begin{pmatrix} I_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} (Ls)^{-1} & -(Ls)^{-1} \\ -(Ls)^{-1} & Cs + (Ls)^{-1} \end{pmatrix}\begin{pmatrix} E_0 \\ E_1 \end{pmatrix}}$$

$\det = CL^{-1}$

So what should the formulas be?

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$$

$$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \end{pmatrix}$$

$$I_0 = cE_1 - dI_1 \qquad I_0 + dI_1 = cE_1$$

$$E_1 = c^{-1}I_0 + c^{-1}dI_1$$

$$E_0 = a(c^{-1}I_0 + c^{-1}dI_1) - bI_1$$

$$= ac^{-1}I_0 + (ac^{-1}d - b)I_1$$

$$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} ac^{-1} & ac^{-1}d - b \\ \underset{c^{-1}}{\cancel{ac^{-1}d-b}} & c^{-1}d \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}$$

Reminded of Legendre transform = poor man's F.T. or L.T.

$V \qquad V^*$

$F \longmapsto \hat{F}$

$\hat{F}(\lambda) =$ stationary value of

$\langle \lambda, v \rangle - F(v) \qquad v \in V$

critical values: $\lambda = F'(v)$ solve for $v = v(\lambda)$

and then $\hat{F}(\lambda) = \langle \lambda, v(\lambda) \rangle - F(v(\lambda))$. So what.

~~[illegible]~~ What might be important is when

$F = \log \det.$?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & b^t \\ -c^t & a^t \end{pmatrix}$$

$$cd^t = dc^t$$

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$$

$$E_1 - \cancel{dI_1} = c^{-1}I_0 + c^{-1}dI_1$$

$$E_0 = ac^{-1}I_0 + \begin{pmatrix} ac^{-1}d \\ -b \end{pmatrix} I_1$$

$$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \underbrace{\begin{pmatrix} ac^{-1} & \overset{(c^{-1})^t}{ac^{-1}d - b} \\ c^{-1} & c^{-1}d \end{pmatrix}}_{\frac{ad}{c^2} - \frac{1}{c^2} = \frac{bc}{c^2} = \frac{b}{c}} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}$$

$$E_0 = aE_1 - bI_1$$

$$bI_1 = -E_0 + aE_1 \qquad I_1 = -b^{-1}E_0 + b^{-1}aE_1$$

$$I_0 = cE_1 - d(-b^{-1}E_0 + b^{-1}aE_1)$$

$$= db^{-1}E_0 + (c - db^{-1}a)E_1$$

$$\begin{pmatrix} I_0 \\ \overline{I_1} \end{pmatrix} = \begin{pmatrix} db^{-1} & \overset{(-b^{-1})^t}{c - db^{-1}a} \\ -b^{-1} & b^{-1}a \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \end{pmatrix}$$

$$(c^{-1})^t = ac^{-1}d - b \text{ ?}$$

$$1 = ac^{-1}dc^t - bc^t$$

$$= ad^t - bc^t$$

6 18 Question: Suppose you know that

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_1 \\ -I_1 \end{pmatrix} \implies \begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} ac^{-1} & (c^{-1})^t \\ c^{-1} & c^{-1}d \end{pmatrix} \begin{pmatrix} I_0 \\ +I_1 \end{pmatrix}$$

$$\begin{pmatrix} I_0 \\ +I_1 \end{pmatrix} = \begin{pmatrix} db^{-1} & -(b^{-1})^t \\ -b^{-1} & b^{-1}a \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \end{pmatrix}$$

There are problems with

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$$

note $c=0$ here

$$\begin{pmatrix} I_0 \\ -I_1 \end{pmatrix} = \underbrace{\begin{pmatrix} (Ls)^{-1} & -(Ls)^{-1} \\ -(Ls)^{-1} & (Ls)^{-1} \end{pmatrix}}_{\text{singular.}} \begin{pmatrix} E_0 \\ E_1 \end{pmatrix}$$

Feb 9

$$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta^t \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}$$

Assume β invertible, and solve for $\begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$ in terms of $\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$

$$\beta^{-1}E_1 = I_0 + \beta^{-1}\gamma I_1$$
$$I_0 = \beta^{-1}E_1 - \beta^{-1}\gamma I_1$$
$$E_0 = \alpha(\beta^{-1}E_1 - \beta^{-1}\gamma I_1) + \beta^t I_1$$
$$= \alpha\beta^{-1}E_1 + (\beta^t - \alpha\beta^{-1}\gamma) I_1$$

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha\beta^{-1} & \alpha\beta^{-1}\gamma - \beta^t \\ \beta^{-1} & \beta^{-1}\gamma \end{pmatrix}}_{= \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta^t \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} ac^{-1} & c^{-1} \\ c^{-1} & c^{-1}d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha\beta^{-1} & \alpha\beta^{-1}\gamma - \beta^t \\ \beta^{-1} & \beta^{-1}\gamma \end{pmatrix}$$

$$\alpha = ac^{-1} = \alpha\beta^{-1}(\beta^{-1})^{-1} = \alpha$$
$$\beta = c^{-1} = (\beta^{-1})^{-1} = \beta$$
$$\gamma = c^{-1}d = (\beta^{-1})^{-1}\beta^{-1}\gamma = \gamma$$

619 $\begin{pmatrix} \alpha & \beta^t \\ \beta & \gamma \end{pmatrix} \rightsquigarrow \begin{pmatrix} \alpha\beta^{-1} & \alpha\beta^{-1}\gamma - \beta^t \\ \beta^{-1} & \beta^{-1}\gamma \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\alpha = \alpha^t,\ \gamma = \gamma^t$

claim $\uparrow$ symplectic i.e.

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\alpha\beta^{-1}(\beta^{-1}\gamma)^t - (\alpha\beta^{-1}\gamma - \beta^t)(\beta^{-1})^t = 1$$

$$(\beta^{-1}\gamma)^t\alpha\beta^{-1} - (\alpha\beta^{-1}\gamma - \beta^t)^t\beta^{-1} = 1$$

$$-\alpha\beta^{-1}(\alpha\beta^{-1}\gamma - \beta^t)^t + (\alpha\beta^{-1}\gamma - \beta^t)(\alpha\beta^{-1})^t$$

$$-\alpha\beta^{-1}\gamma(\beta^{-1})^t\alpha + \alpha + (\alpha\beta^{-1}\gamma - \beta^t)(\beta^{-1})^t\alpha = 0$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Updownarrow$$

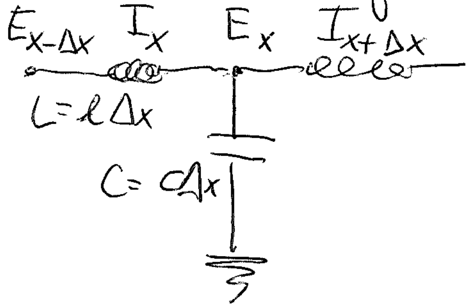
$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

nothing new from this.

You have to straighten out the symp. stuff later

620 today's lecture

continuous limit of ladder network



$$E_{x-\Delta x} - E_x = l\Delta x\, \partial_t I_x$$

$$I_x - I_{x+\Delta x} = c\Delta x\, \partial_t E_x$$

$$\boxed{\begin{aligned} \partial_x E + l\, \partial_t I &= 0 \\ \partial_x I + c\, \partial_t E &= 0 \end{aligned}}$$

$$\partial_x^2 E = -l\, \partial_x \partial_t I = lc\, \partial_t^2 E$$

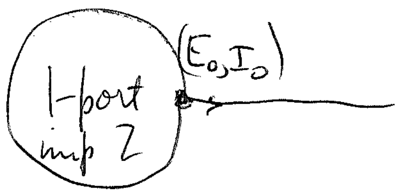
wave eqn speed $\frac{1}{\sqrt{lc}}$

take $lc=1$ $\quad E = f(x+t) + g(x-t)$

time dep. e^{st} $\quad | \quad \partial_x^2 E = s^2 E$

$$E = \left(Ae^{sx} + Be^{-sx}\right)e^{st}$$

$$\begin{pmatrix} E \\ I \end{pmatrix} = \underbrace{e^{s(x+t)}\begin{pmatrix} 1 \\ -c \end{pmatrix} A}_{\text{incoming from right}} + \underbrace{e^{s(-x+t)}\begin{pmatrix} 1 \\ c \end{pmatrix} B}_{\text{outgoing to right}}$$

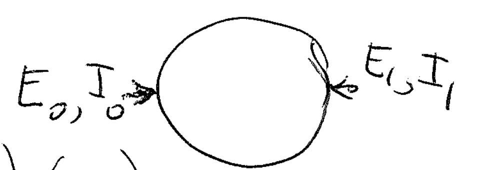


$$-Z = \frac{E_0}{I_0} = \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix}\left(\frac{A}{B}\right)$$

$$\frac{A}{B} = \begin{pmatrix} c & -1 \\ c & 1 \end{pmatrix}(-Z) = \frac{+cZ+1}{+cZ-1} = \frac{Z+l}{Z-l}$$

reflection coeff $\quad \frac{Z-l}{Z+l}$

Examine 2-port



$$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \end{pmatrix} \qquad \begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$$

621

$$\beta I_0 + \gamma I_1 = E_1$$

$$I_0 = \beta^{-1}E_1 - \beta^{-1}\gamma I_1$$

$$E_0 = \alpha(\beta^{-1}E_1 - \beta^{-1}\gamma I_1) + \beta I_1$$

$$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \end{pmatrix}$$

$$E_1 = \gamma I_0 + \delta I_1$$

$$\gamma I_0 = E_1 - \delta I_1$$

$$I_0 = \gamma^{-1}E_1 - \gamma^{-1}\delta I_1$$

$$E_0 = \alpha\gamma^{-1}E_1 - \alpha\gamma^{-1}\delta I_1 + \beta I_1$$

$$\boxed{\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} \alpha\beta^{-1} & \alpha\beta^{-1}\gamma - \beta \\ \beta^{-1} & \beta^{-1}\gamma \end{pmatrix}\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}}$$

$$cE_1 = dI_1 + I_0$$

$$E_1 = c^{-1}I_0 + c^{-1}dI_1$$

$$E_0 = ac^{-1}I_0 + ac^{-1}dI_1 - bI_1$$

$$\frac{ad-bc}{c} = c^{-1}$$

$$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \underbrace{\begin{pmatrix} ac^{-1} & ac^{-1}d - b \\ c^{-1} & c^{-1}d \end{pmatrix}}_{\text{symmetric}}\begin{pmatrix} I_0 \\ I_1 \end{pmatrix}$$

determ. $= \frac{b}{c}$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha\beta^{-1} & \alpha\beta^{-1}\gamma - \beta \\ \beta^{-1} & \beta^{-1}\gamma \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} ac^{-1} & \overset{c^{-1}}{\overset{\parallel}{ac^{-1}d - b}} \\ c^{-1} & c^{-1}d \end{pmatrix}}$$

given 1-1 corresp between $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$ $\ni$ $c^{-1}\ \exists$ and $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ symm. $\ni$ $\beta^{-1}\ \exists$.

622 $$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot = \begin{pmatrix} ac^{-1} & ac^{-1}d - b \\ c^{-1} & c^{-1}d \end{pmatrix} \cdot \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}$$

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha\gamma^{-1} & \alpha\gamma^{-1}\delta - \beta \\ \gamma^{-1} & \gamma^{-1}\delta \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$$

Q. Is there a ~~possible~~ possibility that somewhere in this algebra lurks convolution of kernels to describe composition of operators?

Go on now to scattering which involves a similar transformation.

$$\begin{pmatrix} E \\ I \end{pmatrix} = Ae^{sx}\begin{pmatrix} 1 \\ -c \end{pmatrix} + Be^{-sx}\begin{pmatrix} 1 \\ c \end{pmatrix} \qquad x > 0$$

$$\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$ value at $x = 0$ of the ~~diff~~ solution of the transmission line equations for $x \leqq 0$

$\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$ ———————— $x \geqq 0$.

and say $$\begin{pmatrix} E_0 \\ E_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}$$

But we want a different basis. What am I trying to do?

623 Consider instead $\boxed{(-\partial_x^2+V)\,u = k^2 u}$

V of compact support.

on the left you have two basic solutions e^{ikx}, e^{-ikx} and on the right also e^{ikx} e^{-ikx}

Get SL_2 matrix.

incoming on left

incoming on the right

$$\psi = e^{ikx} \longleftrightarrow \overset{\varphi}{A e^{ikx}} + B\overset{\varphi^*}{e^{-ikx}}$$

$$\psi^* = e^{-ikx} \longleftrightarrow C e^{ikx} + D e^{-ikx}$$

$$\begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix}$$

---

incoming

$$e^{ikx} \longleftrightarrow A e^{ikx} + B \overset{\varphi_{inc}}{e^{-ikx}} \qquad e^{-ikt}$$

$$e^{-ikx} \longrightarrow C e^{ikx} + D e^{-ikx}.$$

$$-\frac{B}{D} e^{-ikx} \longleftrightarrow -\frac{BC}{D} e^{ikx} - B e^{-ikx}$$

$$\boxed{\overset{inc}{e^{ikx}} - \frac{B}{D} e^{-ikx} \longleftrightarrow \frac{1}{D} e^{ikx}}$$

$$\boxed{\frac{1}{D} e^{-ikx} \longleftrightarrow \frac{C}{D} e^{ikx} + \overset{inc}{e^{-ikx}}}$$

$\cancel{e^{ikx} \; \frac{BC}{D} e^{-ikx} \longleftrightarrow \frac{C}{D} e^{ikx}}$ scattering matrix

is something like

$$\begin{pmatrix} -\frac{B}{D} & \frac{1}{D} \\ \frac{1}{D} & \frac{C}{D} \end{pmatrix}$$

be intelligent. have solution on the left

$$e^{s(x+t)}\begin{pmatrix}\ell\\-1\end{pmatrix}h_1 + e^{s(-x+t)}\begin{pmatrix}\ell\\1\end{pmatrix}h_2$$

boundary values

$$\begin{pmatrix}E_0\\I_0\end{pmatrix}=\begin{pmatrix}\ell&\ell\\-1&1\end{pmatrix}\begin{pmatrix}h_1\\h_2\end{pmatrix}$$

and we have soluton on the right

$$e^{sx}\begin{pmatrix}\ell\\-1\end{pmatrix}k_1 + e^{-sx}\begin{pmatrix}\ell\\1\end{pmatrix}k_2$$

with values at $x = D$.

$$\begin{pmatrix}E_1\\-I_1\end{pmatrix}=\begin{pmatrix}\ell&\ell\\-1&1\end{pmatrix}\begin{pmatrix}k_1\\k_2\end{pmatrix}$$

Then $\begin{pmatrix}E_0\\I_0\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}E_1\\-I_1\end{pmatrix}$

so $\begin{pmatrix}\ell&\ell\\-1&1\end{pmatrix}\begin{pmatrix}h_1\\h_2\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\ell&\ell\\-1&1\end{pmatrix}\begin{pmatrix}k_1\\k_2\end{pmatrix}$?

Find another approach. Look on the ~~left~~ right at "the" incoming solution

$$\begin{pmatrix}E_1\\-I_1\end{pmatrix}=\begin{pmatrix}\ell\\-1\end{pmatrix}$$

$$\begin{pmatrix}E_0\\I_0\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\ell\\-1\end{pmatrix}=\begin{pmatrix}\ell&\ell\\-1&1\end{pmatrix}($$

625 Basically you should only have to consider the 1-sided case of scattering.

$$\begin{pmatrix} E \\ I \end{pmatrix}(x,t) = e^{s(x+t)}\begin{pmatrix} 1 \\ -1 \end{pmatrix} A + e^{s(-x+t)}\begin{pmatrix} 1 \\ 1 \end{pmatrix} B$$

here E, I, A, B are vectors, ~~so the~~ at $x=0$ ($t=0$ also)

$$\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

If the n-fold transmission line is connected to an n-port with imp. Z, then we have $\begin{pmatrix} E \\ I \end{pmatrix} \subset \begin{pmatrix} Z \\ -1 \end{pmatrix} V^*$ so

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \subset \begin{pmatrix} Z \\ -1 \end{pmatrix} V^*$$

$$\begin{pmatrix} A \\ B \end{pmatrix} \in \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} Z \\ -1 \end{pmatrix} V^* = \begin{pmatrix} Z+1 \\ Z-1 \end{pmatrix} V^*$$

$$\therefore \begin{pmatrix} B \\ A \end{pmatrix} \subset \begin{pmatrix} Z-1 \\ Z+1 \end{pmatrix} V$$

So the scattering operator is $\frac{Z-1}{Z+1}$, essentially Cayley transform. Recall that

$$Z = \sum \underbrace{\frac{s(1+\omega^2)}{s^2+\omega^2}}_{\frac{1+\omega^2}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)} a_\omega \qquad a_\omega \text{ quad form} \geq 0 \text{ on } V.$$

Looks like we should worry about Siegel UHP.

Note

626 The point is perhaps that for the n-fold transmission line you have ~~de~~ probably chosen an inner product on V the voltage space.

$$\begin{cases} \partial_x E + l\,\partial_t I = 0 \\ \partial_x I + c\,\partial_t E = 0 \end{cases} \qquad \begin{cases} E \in V & \text{dim } n \\ I \in V^* & \text{dim } n. \end{cases}$$

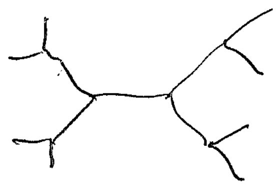
$$\partial_x^2 E - (lc)\,\partial_t^2 E = 0$$

$$l : V^* \longrightarrow V \qquad c : V \longrightarrow V^* \qquad \text{pos. def. q-f.}$$

even for $n = 1$. $(E, I) \in V, V^*$ dual but not canon. isom.

Conclude that from an invariant viewpoint you might as well suppose $l = c = 1$. So what do you learn? The reflection coeff is $\frac{Z-1}{Z+1}$ operator on $V = V^*$. Strictly $\frac{Z-l}{Z+l}$

$$V^* \xrightarrow{Z-l} V \xleftrightarrow{(Z+l)^{-1}} V^*$$



But now see if you can get

$$g = \frac{Z-1}{Z+1} \qquad Z_s = \sum \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega \qquad \sum a_\omega > 0.$$

$$\cancel{\|(Z+1)v\|^2 = (v, Z_s^* Z_s v)}$$

$$(v, (Z+1)v) = \|v\|^2 + \sum \underbrace{\frac{s(1+\omega^2)}{s^2+\omega^2}}_{\text{has Re}} \underbrace{(v, a_\omega v)}_{\geq 0}$$

$$\frac{1+\omega^2}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)$$

$$\mathrm{Re}\left(\frac{1}{s-i\omega}\right) = \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{\bar{s}+i\omega}\right) = \frac{1}{2}\frac{s+\bar{s}}{(s-i\omega)(\bar{s}+i\omega)}$$

$$|s|^2 + |\omega|^2 + \cancel{\omega(s-\bar{s})} \; 2\omega\,\mathrm{Im}\, s$$

$$= \frac{\mathrm{Re}(s)}{\mathrm{Re}(s)^2 + (\omega + \mathrm{Im}(s))^2}$$

627

$$\mathrm{Re}\left(\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right)$$

?

$$= \mathrm{Re}\left(\frac{2s}{s^2+\omega^2}\right)$$

$$\mathrm{Re}\left(\frac{1}{s-i\omega}\right) = \frac{1}{2}\left(\frac{1}{s-i\omega}+\frac{1}{\bar{s}+i\omega}\right)$$

$$= \frac{1}{2}\,\frac{s+\bar{s}}{|s|^2+\omega^2+\underbrace{i\omega s - i\omega\bar{s}}_{i\,2i\,\mathrm{Im}(s)\,\omega}}$$

$$= \cancel{4}\,\frac{\mathrm{Re}(s)}{\mathrm{Re}(s)^2+\mathrm{Im}(s)^2-2\,\mathrm{Im}(s)\,\omega+\omega^2}$$

$$= \frac{\mathrm{Re}(s)}{\mathrm{Re}(s)^2+\left(\mathrm{Im}(s)-\omega\right)^2} \qquad > 0$$

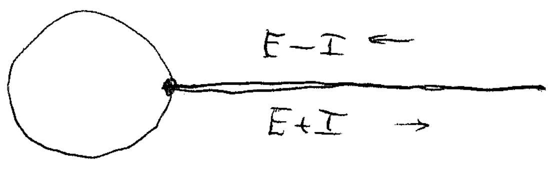
Feb 10 what can you do about lecture?

n-fold transmission line $(E, I) \in V$

$$\partial_x E + \partial_t I = 0$$
$$\partial_x I + \partial_t E = 0$$

$$(\partial_x + \partial_t)(E+I) = 0$$
$$(\partial_x - \partial_t)(E-I) = 0$$

$$B e^{s(-x+t)}$$
$$\parallel$$
$$E+I = f(x-t)$$
$$E-I = g(x+t)$$
$$\parallel$$
$$A e^{s(x+t)} \quad \text{inc.}$$



$$\begin{pmatrix}1 & 1\\ +1 & -1\end{pmatrix}\begin{pmatrix}E\\ I\end{pmatrix} = \begin{pmatrix}B\\ A\end{pmatrix}$$

$$-Z = \frac{E_0}{I_0} = \cancel{\frac{B+A}{B-A}}$$

impred. of port

values at $x=0$

$$\begin{pmatrix}E\\ I\end{pmatrix} = \frac{1}{2}\begin{pmatrix}+1 & +1\\ +1 & -1\end{pmatrix}\begin{pmatrix}B\\ A\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}B\\ A\end{pmatrix} = -Z$$

$$S = \frac{B}{A} = \begin{pmatrix}+1 & +1\\ +1 & -1\end{pmatrix}(-Z) = \frac{-Z+1}{-Z-1} = \frac{Z-1}{Z+1}$$

628

$$S(s) = \frac{Z_s - 1}{Z_s + 1} \qquad Z_s = \sum_{0 \le \omega < \infty} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$$

$v \in \mathbb{C}^n$ $\quad (v, (Z_s+1)v) = \|v\|^2 + \sum \underbrace{\frac{s(1+\omega^2)}{s^2+\omega^2}} \underbrace{(v, a_\omega v)}_{\geq 0 \text{ at least one} > 0.}$

$\therefore (Z_s+1)^{-1} \ \exists$ for $Re(s) > 0$

$$\underbrace{\frac{1+\omega^2}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)}$$

takes $Re(s) > 0$ into $Re(\cdot) > 0$.

$\boxed{\text{so } S(s) \text{ is analytic for } Re(s) > 0.}$

Another point ~~$s \in i\mathbb{R} \infty$~~

$$Z_s = \underbrace{\sum_{\omega < 0} \cdots}_{h(s)} + s a_\infty$$

$h(s)$ analytic at ∞. $\qquad h(\infty) = 0$

$$\frac{Z_s - 1}{Z_s + 1} = \frac{h + s a_\infty - 1}{h + s a_\infty + 1} = 1 - \frac{2}{s a_\infty + 1 + h}$$

$\rho > 0$

$$a_\infty = \left(\begin{array}{c|c} \rho & 0 \\ \hline 0 & 0 \end{array}\right) \qquad s a_\infty + 1 + h = \left(\begin{array}{c|c} s\rho + * & * \\ \hline * & * \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & -ca^{-1}b + d \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d - ca^{-1}b \end{pmatrix}$$

629. Spend time on scattering.

An idea: You know that connecting an r-port to an r-fold transmission line transform the impedance Z to its Cayley transform $\frac{Z-1}{Z+1} = S$:

$$\partial_x E + \partial_t I = 0 \qquad \text{here take } V = V^*,\ l = c = 1.$$
$$\partial_x I + \partial_t E = 0$$

$$(\partial_x + \partial_t)(E+I) = 0 \qquad \underset{\text{outgoing}}{E+I} = B e^{s(-x+t)} \qquad \frac{E_0+I_0}{E_0-I_0} = \frac{B}{A}$$

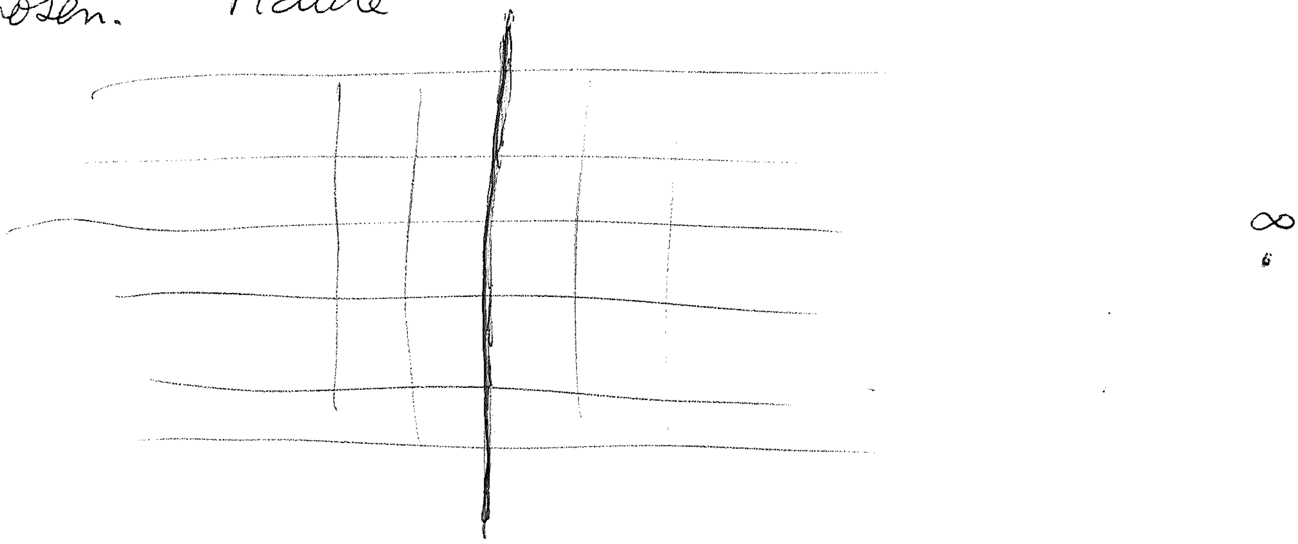
$$(\partial_x - \partial_t)(E-I) = 0 \qquad \underset{\text{incoming}}{E-I = A e^{s(x+t)}}$$

$$S = \frac{B}{A} = \frac{E_0+I_0}{E_0-I_0} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\frac{E_0}{I_0} = -Z\right) = \frac{-Z+1}{-Z-1} = \frac{Z-1}{Z+1}$$

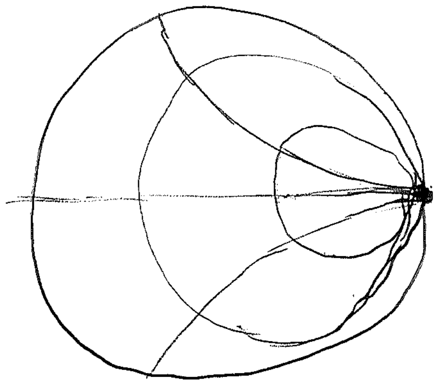
This S is a merom. function of s analytic on $\{Re(s) \geq 0\} \cup \infty$, unitary-valued on the boundary. It might be nice to find a good proof of this. There's also this reality condition $Z(s)^* = Z(\bar{s})$ together with $Z(-s) = -Z(s)$, which implies $Z(s)^* = -Z(-\bar{s})$ so that $Z(s)$ is skew hermitian on $i\mathbb{R}\cup\infty$, hence its C.T. is unitary for $s \in i\mathbb{R} \cup \infty$. In fact $Z(it)$ is i times a real symm. matrix, but I don't think this means very much.

You seem to have some problem relating the s variable to the z variable describing the disk. These are the variables describing the Riemann sphere. The difference

630 between them somehow reflects that in the s-picture a point of the circle is chosen. Picture



This must be understood later. It's intriguing to speculate the meaning of about de Branges spaces in this picture



But the program for the moment should be to analyze scattering operators. Insight should come from loop group theory. The point maybe is that you are interested in loops with unitary values which extend analytically over the interior. So what should the assertion be? Basically - any rational scattering operator (unitary values analytic in the interior) corresponds to a partial unitary, (stable isomorphism?) This ~~can't be~~ might not be quite correct, since the s parameter ~~behaves differently~~ looks different.

631 Feb 11. Let's review. I consider a port, LC network inside, use frequency parameter s, so that time dependence is e^{st}, "real" frequencies are ~~[illegible]~~ $s = i\omega \in i\mathbb{R}$. There's a response function, which properly speaking is a vector subbundle F of $\mathcal{O}\otimes(V\oplus V^*)$ over the Riemann sphere, thus $F_s \subset V\oplus V^*$. This is one theorem, which you must get into a good form. How? Ideally you should derive it from varying the proof deriving the structure of the impedance fn Z_s. What form might this take? You ~~[illegible]~~ start with $H = H^+ \oplus H^-$ polarized ~~[illegible]~~ Euclidean space, with operator $s\pi_+ \oplus s^{-1}\pi_-$, and then suitably induce this operator to a subquotient $V = U/W$, $W \subset U \subset H$. Maybe good to see if this induction can be easily described in vector bundle terms. ~~[illegible]~~ There's an alternate viewpoint ~~[illegible]~~ consisting of coupling the port to a trans. line and looking at the scattering operator $S = \frac{Z-1}{Z+1}$. Supposedly F is the ~~[illegible]~~ vector bundling over the Riemann sphere assoc. to ~~[illegible]~~ the clutching function S.

(IDEA. ~~[illegible]~~ Is there a de Branges theory arising from ~~[illegible]~~ curves, the idea being to generalize from ~~[illegible]~~ a Hilbert space of polynomials?)

~~[illegible]~~ Scattering should be simpler. Instead of $\begin{pmatrix} E \\ I \end{pmatrix}$ you look at $\begin{pmatrix} E-I \\ E+I \end{pmatrix}$. ~~So~~ Let's start ~~with yield~~ where? Instead of a symmetric map ~~quadratic fn~~ $Z^{-1} : V \longrightarrow V^*$ we ?

 Still unclear. You have voltage space V current space V^* and $Z_s^{-1} : V \longrightarrow V^*$ symmetric. Initially you think of V and s as real, so $s \mapsto Z_s^{-1}$ can be viewed as a rational map to complex symmetric from the Riemann "s" sphere. If $\mathrm{Re}(s) > 0$, then $\mathrm{Re}(Z_s^{-1}) > 0$. So you have a rational map into the Siegel UHP from $\mathrm{Re}(s) > 0$, and the boundary $\mathrm{Re}(s) = 0$ goes to Lagrangian subspaces.

You have the above symplectic approach. ~~Now the point is to change the~~ Next change to scattering picture. $Z = \frac{E}{I} \longmapsto \frac{E-I}{E+I} = \frac{Z-1}{Z+1} = S$

$$F_s = \begin{pmatrix} Z_s \\ 1 \end{pmatrix} V \subset V \oplus V$$

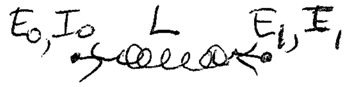
$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} F_s = \begin{pmatrix} Z_s - 1 \\ Z_s + 1 \end{pmatrix} V = \begin{pmatrix} S \\ 1 \end{pmatrix} V \subset \begin{matrix} V \\ \oplus \\ V \end{matrix}$$

Start again. Consider a port with voltage space V, current space V^*. Look at response. This should be for each s a Lagrangian subspace $F_s \subset V \oplus V^*$. This is the symplectic picture. But there is also the scattering picture

You start with $W \subset U \subset H^+ \oplus H^-$, H polarized Euclidean space, then from $s\|h_+\|^2 + s^{-1}\|h_-\|^2$ you get an induced quad form on $V = U/W$ whose form we have analyzed. Simple poles i.e. res. ≥ 0.

You understand a lot, but the details are incomplete. You hope the scattering picture is

633 better. ~~[illegible]~~ The scattering picture requires an isom $c: V \simeq V^*$ and it's natural to take c to ~~[illegible]~~ be Z_1^{-1}. ~~What the shit~~ You have then E, I in the same space V. Let's understand scattering operators for simple circuits.



$$E_0 - E_1 = LsI_0 \qquad I_0 = -I_1$$

So inside the 4 diml space of (E_0, I_0, E_1, I_1) you have the 2 plane satisfying $I_0 = I_1$ $\quad E_0 - E_1 = LsI_0$

But we want to use the coordinates $E_0 \pm I_0$ and $E_1 \pm I_1$.

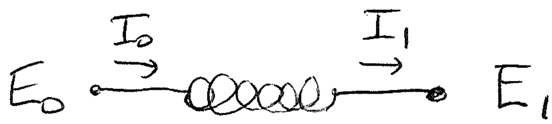
$$\cancel{\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_1 \end{pmatrix}} \qquad \begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$$

$$\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix} = \begin{pmatrix} 1 & -Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

$E + I = e^{s(-x+t)} A$

$E - I = e^{s(x+t)}$

Let's use



$$I_0 = I_1$$
$$E_0 - E_1 = LsI_1$$

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$$

634

$$\begin{pmatrix} E_0+I_0 \\ E_0-I_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\left(\frac{-1}{2}\right)\begin{pmatrix} E_1+I_1 \\ E_1-I_1 \end{pmatrix}$$

$$= \left(\frac{-1}{2}\right)\begin{pmatrix} 1 & 1+Ls \\ 1 & Ls-1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \left(-\frac{1}{2}\right)\begin{pmatrix} -2-Ls & Ls \\ -Ls & Ls-2 \end{pmatrix}$$

$E+I = e^{s(-x+t)}$ const
$E-I = e^{s(x+t)}$ const.

$$\begin{array}{c} \text{inc.} \\ \begin{pmatrix} E_0+I_0 \\ E_0-I_0 \end{pmatrix} \\ \text{out} \end{array} = \underbrace{\begin{pmatrix} 1+\frac{1}{2}Ls & -\frac{1}{2}Ls \\ \frac{1}{2}Ls & 1-\frac{1}{2}Ls \end{pmatrix}}_{\text{this lies in } SU(1,1) \text{ for } s\in i\mathbb{R}.} \begin{array}{c} \text{out} \\ \begin{pmatrix} E_1+I_1 \\ E_1-I_1 \end{pmatrix} \\ \text{inc.} \end{array}$$

$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \ni |a|^2-|b|^2=1.$

What's the relation between $SU(1,1)$ and $U(2)$

$$(E_0+I_0) = a(E_1+I_1) + b(E_1-I_1)$$
$$(E_0-I_0) = \bar{b}(E_1+I_1) + \bar{a}(E_1-I_1)$$

$$(E_1+I_1) = \frac{1}{a}(E_0+I_0) - \frac{b}{a}(E_1-I_1)$$
$$(E_0-I_0) = \bar{b}(\qquad\qquad) + \bar{a}(E_1-I_1)$$
$$= \frac{\bar{b}}{a}(E_0+I_0) + \underbrace{\left(-\frac{\bar{b}b}{a}+\bar{a}\right)}_{\frac{1}{a}}(E_1-I_1)$$

$$\begin{pmatrix} E_1+I_1 \\ E_0-I_0 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{\bar{b}}{a} & \frac{1}{a} \end{pmatrix}}_{\text{unitary}} \begin{pmatrix} E_0+I_0 \\ E_1-I_1 \end{pmatrix}$$

det $\frac{1}{a^2} + \frac{|b|^2}{a^2} = \frac{a\bar{a}}{a^2} = \frac{\bar{a}}{a}$

in the example.

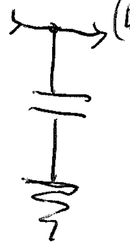
$$\begin{pmatrix} \frac{1}{1+\frac{1}{2}Ls} & \frac{+\frac{1}{2}Ls}{1+\frac{1}{2}Ls} \\ \frac{\frac{1}{2}Ls}{1+\frac{1}{2}Ls} & \frac{1}{1+\frac{1}{2}Ls} \end{pmatrix}$$

identity if $s=0$.

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ if $s=\infty$.

This doesn't look very helpful

Next try $(E_0, I_0) \to (E_1, I_1)$



$E_0 = E_1$

$I_0 - I_1 = Cs\,E_1$

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$$

$$\begin{pmatrix} E_0+I_0 \\ E_0-I_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix} \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \left(+\frac{1}{2}\right) \begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+Cs & 1 \\ 1-Cs & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} 1/2$$

$$= \begin{pmatrix} \frac{2+Cs}{2} & \frac{Cs}{2} \\ -\frac{Cs}{2} & \frac{2-Cs}{2} \end{pmatrix}$$

$$\begin{array}{cc} a & b \end{array}$$
$$\begin{pmatrix} 1+\frac{1}{2}Cs & \frac{1}{2}Cs \\ -\frac{1}{2}Cs & 1-\frac{1}{2}Cs \end{pmatrix}$$
$$\begin{array}{cc} \bar{b} & \bar{a} \end{array}$$

$$\begin{pmatrix} E_1 + I_1 \\ E_0 - I_0 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1+\frac{1}{2}Cs} & \dfrac{-\frac{1}{2}Cs}{1+\frac{1}{2}Cs} \\ \dfrac{-\frac{1}{2}Cs}{1+\frac{1}{2}Cs} & \dfrac{1}{1+\frac{1}{2}Cs} \end{pmatrix} \begin{pmatrix} E_0 + I_0 \\ E_1 - I_1 \end{pmatrix}$$

Would it help to do a 1-port.

$$Z = Ls + \frac{1}{Cs} = \frac{LCs^2+1}{Cs}$$

$$\frac{Z-1}{Z+1} = \frac{\frac{LCs^2+1}{Cs} - 1}{\frac{LCs^2+1}{Cs} + 1} \simeq \frac{LCs^2 - Cs + 1}{LCs^2 + Cs + 1}$$

As a check note that roots of num. are

$$s = \frac{C \pm \sqrt{C^2 - 4LC}}{2LC}$$

If $C^2 - 4LC \leq 0$, then $Re(s) = \frac{\varepsilon}{2LC} > 0$

> 0, then $\frac{C + \sqrt{C^2-4LC}}{2LC} > 0$

and the prod of the roots is $\frac{1}{LC}$ so other root also > 0

In the example, $\det = \frac{\bar{a}}{a} = \frac{1 - \frac{1}{2}Cs}{1 + \frac{1}{2}s}$ or $\frac{1-\frac{1}{2}Ls}{1+\frac{1}{2}Ls}$

again root is $s = \frac{2}{C}$ or $\frac{2}{L}$ in RHP.

So where to start?

Your aim is to control the vector bundle.

Roughly