538 ~~me~~ Here's another idea: pencil like in pencil of hyperplane sections in Lefschetz theory. You should understand pencils in the Grassmannian or their analogs. Recall in Lefschetz theory you have a smooth projective variety X, $\mathcal{L} = j^*\mathcal{O}(1)$ for an embedding $X \hookrightarrow \mathbb{P}^N$ = hyperplanes in $H^0(X,\mathcal{L})$

Wait.

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}(V) \\ \cup & & \cup \\ X \cap H & & H = \mathbb{P}(W) \end{array}$$

W hyperplane in V.

Given $X \xrightarrow{f} \mathbb{P}_1(V)$, get $\mathcal{L} = f^*(\mathcal{O}(1))$ and

$$\Gamma(X,\mathcal{L}) \longleftarrow \Gamma(\mathbb{P}_1(V), \mathcal{O}(1)) = V^*$$

So if $\mathcal{L}$ very ~~ample~~ on X, then have canonical map

$$X \longrightarrow \mathbb{P}_1(\Gamma(X,\mathcal{L})^*)$$

So a ~~principal of~~ hyperplane section ~~corresponds to a~~ of X corresponds to a hyperplane in $\Gamma(X,\mathcal{L})^*$, i.e. a line in $\Gamma(X,\mathcal{L})$, and a pencil of hyperplane sections in X should correspond to a 2 dim'l subspace in $\Gamma(X,\mathcal{L})$.

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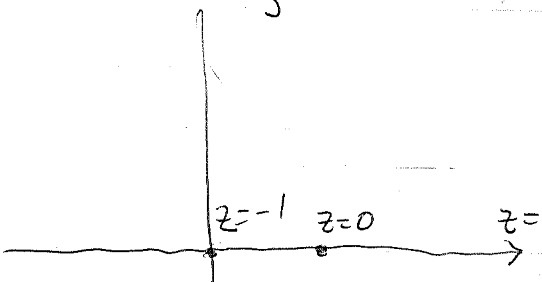
539 So what ~~happens~~ might happen is that, my rational functions of s ~~might be~~ is related to pushing forward a pencil to the proj. line.

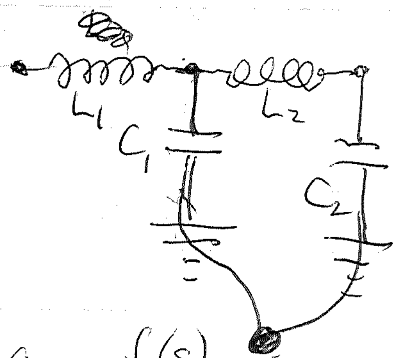
So where can I start?

Can we relate s-situation, $SL_2(\mathbb{R})$, to the z-situation $SU(1,1)$? This may be impossible because of the ~~difference between~~ time signals take.

$f(s)$ You need $\boxed{z = \frac{s-1}{s+1}}$



$$\frac{f(s)-1}{f(s)+1}$$

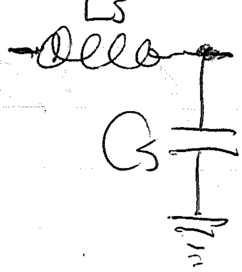


Here's an idea: Take an $f(s)$ arising from LC ladder

Take a single LC $\qquad \frac{E_0}{I_1} = \frac{s^2+\omega^2}{L^{-1}s} \qquad \omega = \frac{1}{\sqrt{LC}}$

$$Ls + \frac{1}{Cs}$$
$$\parallel$$
$$\frac{LCs^2+1}{Cs} = \frac{s^2+(LC)^{-1}}{L^{-1}s}.$$

What would you like to do?
Consider the 2 port



$$Z_0 = Ls + \frac{1}{Cs + \frac{1}{Z_1}}$$

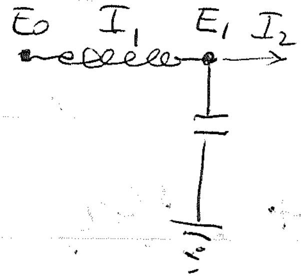
YES

$$(Z_0) = \begin{pmatrix}1 & Ls\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ Cs & 1\end{pmatrix}(Z_1)$$

So you can easily describe this 2 port by a

I want to shift from s to $z = \frac{s-1}{s+1}$

The problem. to compare this LC 2-port with a partial unitary. The LC 2-port has 3 vertices 2 edges.



$$E_0, I_1, E_1, I_2$$

$$E_0 - E_1 = LsI_1$$
$$I_1 - I_2 = CsE_1$$

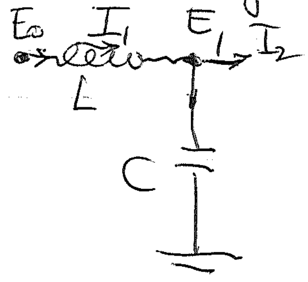
$$\begin{pmatrix} E_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_2 \end{pmatrix}$$

So we have a 4 dimensional ~~space~~ space with 2 conditions

$$0 \longrightarrow E_s \longrightarrow \mathbb{R}^4 \xrightarrow{\ as-b\ } \mathbb{R}^2$$

Problem E has type $\mathcal{O}(-1)^{\oplus 2}$. You would ~~like to~~

Start again - you are considering a simple 2-port described by



$$E_0 - E_1 = LsI_1$$
$$I_1 - I_2 = CsE_1$$

$$\begin{pmatrix} E_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_2 \end{pmatrix}$$

 So you have 4 diml space of (E_0, I_1, E_1, I_2)
2 conditions Ker is 2 diml, so get a rank 2 vector bundle over S^2.

$$0 \longrightarrow \mathcal{E}_s \longrightarrow \mathbb{R}^4 \xrightarrow{as-b} \mathbb{R}^2 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(-1)^4 \longrightarrow \mathcal{O}^2 \longrightarrow 0$$

$\mathcal{E}$: rank 2, degree -4

$$\mathcal{E} = \mathcal{O}(-2) \oplus \mathcal{O}(-2)$$
$$\mathcal{E} = \mathcal{O}(-3) \oplus \mathcal{O}(-1)$$

What should be the question? You are looking at a 2 port because maybe it is related to a partial unitary of rank 1. But actually it looks like a 1-port should be related to a partial unitary? You need to make sense, organize the phenomena. Maximal isotropic subspace of a symplectic vector space, resp. an indefinite, Hermitian space should be similar. Can you fit one inside the other? Maybe not.

Try. Take symplectic space V of real dim 2. Then any line is max. isot. $\mathbb{P}_1(V) = \mathbb{P}_1(\mathbb{R})$ which is the unit circle in $\mathbb{C}$ essentially. Thus in dim. 2

Can identify $PSL_2(\mathbb{R})$ $PU(1,1)$

Jan 26, 98 Review the puzzle. To get beyond the S^2 type response
maybe look at moment problem.
may be look at reflection of a 1-port coupled to a transmission line.

E_{i-1} I_i E_i

$$E_{i-1} = E_i + L_i s I_i$$

$$I_i = I_{i+1} + C_i s E_i.$$

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$$-d_x E = \lambda s I \qquad d_x^2 E = \lambda s(-d_x) I$$
$$-d_x I = \gamma s E \qquad = \gamma\lambda s^2 E$$

$$\left(\partial_x^2 - \gamma\lambda s^2\right) E = 0 \qquad s = -i\omega$$

$$\left(\partial_x^2 + \gamma\lambda\omega^2\right) E = 0 \qquad \gamma\lambda\omega^2 = k^2$$

$$kx = \omega t \qquad x = \frac{\omega}{k} t \qquad c = \left(\frac{\omega}{k}\right) = \frac{1}{\sqrt{\lambda\gamma}}$$

So what do we get — take $\lambda = 1, \gamma = 1$.

$$\partial_x E + sI = 0 \qquad ik - i\omega$$
$$\partial_x I + sE = 0$$

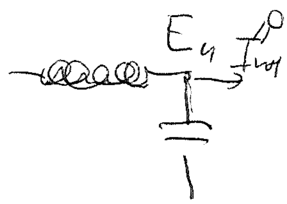
~~So we have zilch~~ $\left(\partial_x^2 + \omega^2\right) E = 0$

Better to use s

$$\partial_x E + sI = 0 \qquad \partial_x^2 E = s^2 E$$
$$\partial_x I + sE = 0 \qquad E = ae^{sx} + be^{-sx}$$
$$-\partial_x E = sae^{sx} + sbe^{-sx}$$
$$I = -ae^{sx} + be^{-sx}$$

$$\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} a & b \\ -a & b \end{pmatrix}$$

$$= a\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{sx} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-sx}$$



We want to get a reflection coefficient from a 1-port LC circuit

$E_x, I_x \quad E_0, I_1$

$$\partial_x E = -sI \qquad x < 0$$
$$E_x\big|_{x=0} = E_0 \qquad \partial_x I = -sE \qquad x < 0$$
$$I_x\big|_{x=0} = I_1 \qquad \frac{E_0}{I_1} = \begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ C_1 s & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \\ C_n s & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix} E_x \\ I_x \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{sx} + b \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-sx} \underset{\text{at } x=0}{\approx} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

~~So we choose~~ Let $Z(s) = L_1 s + \frac{1}{C_1 s + \cdots}$

So we get our solution is $\frac{a+b}{-a+b} = Z(s)$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = Z \qquad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} Z = \frac{Z-1}{Z+1}$$

maybe we want $\frac{b}{a} = \frac{1+Z}{1-Z}$. You want a reflection coeff. Go back to

$$\left(\frac{b}{a}\right) = \frac{Z+1}{Z-1}$$

$$\partial_x E = -sI$$

What does $\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{sx}$ mean? $e^{s(t+x)}$

means $\begin{pmatrix} E \\ I \end{pmatrix}(x,t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega(x+t)}$ outgoing

other is $\begin{pmatrix} E \\ I \end{pmatrix}(x,t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega(x-t)}$ $e^{s(t-x)}$ incoming

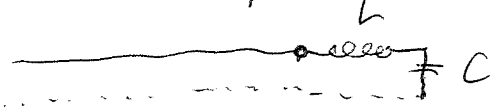
solution is $\begin{pmatrix} E \\ I \end{pmatrix}(x,t) = ($

Basically the solution is

$$\begin{pmatrix} E_x \\ I_x \end{pmatrix} = \underbrace{\left(\frac{a}{b}\right)}_{\frac{Z+1}{Z-1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{s(t+x)} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{s(t-x)}$$

Example

$\partial_x E_x = -s I_x$
$\partial_x I_x = -s E_x$



$$\begin{pmatrix} E_x \\ I_x \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{s(t+x)} + b \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{s(t-x)}$$

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ 0 \end{pmatrix} e^{st}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(\frac{a}{b} \right) = Ls + \frac{1}{Cs}$$

$$\left(\frac{a}{b} \right) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \left(Ls + C^{-1} s^{-1} \right)$$

$$= \frac{Ls + C^{-1}s^{-1} - 1}{Ls + C^{-1}s^{-1} + 1} = \frac{Ls^2 - s + C^{-1}}{Ls^2 + s + C^{-1}}$$

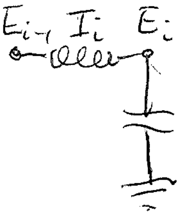
$$Ls^2 - s + C^{-1} = 0 \qquad s = \frac{1 \pm \sqrt{1 - 4LC^{-1}}}{2L}$$

$$\frac{s^2 - L^{-1}s + (LC)^{-1}}{s^2 + L^{-1}s + (LC)^{-1}} \quad \omega_0^2$$

$$\text{disc} = (L^{-1})^2 - 4(LC)^{-1}$$

Point is that if $4LC^{-1} > 1$ then have 2 zeroes of ~~[illegible]~~ numerator in $\mathrm{Re}(s) > 0$, while if $4LC^{-1} < 1$ then have 2 positive real roots

545 Simpler example — go over formula again.



$$E_{i-1} = E_i + L_i s I_i \qquad\rightsquigarrow\qquad -\partial_x E = \lambda s I$$
$$I_i = I_{i-1} + C_i s E_i \qquad\qquad -\partial_x I = \gamma s E$$

$\lambda = \gamma = 1.$
$$-\partial_x E = s I \qquad \begin{pmatrix} E \\ I \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{sx} + b \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-sx}$$
$$-\partial_x I = s E$$

$$\begin{pmatrix} E \\ I \end{pmatrix}(x,t) = \underbrace{a \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{s(t+x)}}_{\text{outgoing}} + \underbrace{b \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{s(t-x)}}_{\text{incoming}}$$

suppose we take

$$E_0 = \overset{Z}{Ls}\, I_0$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\left(\frac{a}{b}\right) = Z \qquad \frac{a}{b} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}(Z) = \frac{Z-1}{Z+1}$$

In this example $Z = Ls$
$$\frac{Ls-1}{Ls+1} = \frac{s-L^{-1}}{s+L^{-1}}$$

$$Z = \frac{1}{Cs} \quad\Bigg|\quad g = \frac{1-Cs}{1+Cs} = \frac{-s+C^{-1}}{s+C^{-1}} \quad \text{analytic for } \operatorname{Re}(s) > 0$$

What are you in the process of doing? You have achieved an understanding of LC circuits, but you haven't managed yet to ~~[illegible]~~ find the partial (unitary, skew-adjoint?) operator that should be around.

~~So let's see what happens [illegible]~~

Suppose we connect an n port to n transmission lines, or maybe we should think of a bundle of n transmission lines (coaxial cables) coming into an n-port. A single cable yields

$$-\partial_x E = lsI$$
$$-\partial_x I = csE$$

l = inductance density
c = cap — density

$$\partial_x^2 E = ls(-\partial_x I) = lscsE = lcs^2E.$$

Recall that if E has values in V, then I has values in E^*, so that $l : V^* \longrightarrow V$, $c : V \longrightarrow V^*$ can be generalized to pos definite quadratic forms. Then $lc : V \longrightarrow V$ is an operator on V, and $cl : V^* \longrightarrow V^*$ should be the transpose operator on V^*. This is like a harmonic oscillator where you have $m : V \longrightarrow V^*$ $k : V \longrightarrow V^*$, but here you have $c : V \longrightarrow V^*$ and $l^{-1} : V \longrightarrow V^*$. So you get eigenspace decomposition in general orthogonal wrt both quadratic forms. Signals travel at the speed λ where λ^2 is the eigenvalue of lc.

Let's try to understand the scattering operator for this situation. We have response function $G_s : V \longrightarrow V^*$, $E \longmapsto I$ for our n port

$$G_s = \sum_{\omega \geq 0} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega, \qquad \sum a_\omega > 0.$$

Suppose we assume $c = l^{-1}$, whence $V = V^*$ effectively

$$E = e^{sx} \qquad -se^{sx} = lsI \qquad I = -l^{-1}e^{sx}$$
$$e^{-sx} \qquad se^{-sx} = lsI \qquad I = l^{-1}e^{-sx}$$

$$\begin{pmatrix} E \\ I \end{pmatrix}(x,t) = \underbrace{\begin{pmatrix} 1 \\ -l^{-1} \end{pmatrix} a e^{s(t+x)}}_{\text{outgoing}} + \underbrace{\begin{pmatrix} 1 \\ l^{-1} \end{pmatrix} b e^{s(t-x)}}_{\text{incoming}}$$

$$\begin{pmatrix} E \\ GE \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad ?$$

what you mean is that

$$\begin{pmatrix} 1 \\ G \end{pmatrix} V = \begin{pmatrix} a+b \\ c(-a+b) \end{pmatrix} V$$

i.e. $\quad G = c(-a+b)(a+b)^{-1}$

or $\quad c^{-1}G = (-ab^{-1}+1)b\big((ab^{-1}+1)b\big)^{-1}$

$$= \frac{-ab^{-1}+1}{ab^{-1}+1} \qquad c^{-1}G = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}(ab^{-1})$$

$$ab^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}(c^{-1}G) = \frac{-c^{-1}G+1}{c^{-1}G+1}$$

$$ab^{-1} = \frac{-\ell G_s + 1}{\ell G_s + 1} \qquad \begin{aligned} G_s &: V \longrightarrow V^* \\ \ell &: V^* \longrightarrow V \end{aligned}$$

take $\ell = 1$, $\quad G_s = \frac{1}{Ls} \qquad ab^{-1} = \dfrac{-\frac{1}{Ls}+1}{\frac{1}{Ls}+1} = \dfrac{Ls-1}{Ls+1} = \dfrac{s-L^{-1}}{s+L^{-1}}$

Essentially no further understanding. Next.

Jan 27, 1998 focus on scattering maybe?

situation: Take a 1-port with response fn. $\frac{I}{E} = G_s$ & connect to a transmission line to obtain a reflection coefficient. ~~What~~ You want a clean picture of the partial unitary, maybe a link with Dominic Joyce's modules. & line eqn. $\cancel{-\partial_x E = s}$

$$-\partial_x E = \ell s I$$
$$-\partial_x I = cs E$$

$$\cancel{\ell s E} \quad \underset{(-s)}{(-\partial_x)} \hat{E} e^{sx} = \ell s \hat{I} e^{sx} \qquad \begin{pmatrix} 1 \\ -\ell \end{pmatrix} e^{s(t+x)}$$

$$-\partial_x E = l\,\partial_t I \qquad \begin{pmatrix}1\\-c\end{pmatrix}e^{s(t+x)} \qquad \begin{pmatrix}1\\c\end{pmatrix}e^{s(t-x)}$$
$$-\partial_x I = c\,\partial_t E$$

$$\begin{pmatrix}E\\I\end{pmatrix}(x,t) = e^{s(t+x)}\begin{pmatrix}1\\-c\end{pmatrix} a + e^{s(t-x)}\begin{pmatrix}1\\c\end{pmatrix} b$$

outgoing incoming

$$\begin{pmatrix}1\\G_s\end{pmatrix}E = \begin{pmatrix}1\\-c\end{pmatrix}a + \begin{pmatrix}1\\c\end{pmatrix}b = \begin{pmatrix}1&1\\-c&c\end{pmatrix}\begin{pmatrix}\cancel{a}\\ba^{-1}\end{pmatrix}$$

$$G_s = \cancel{c(-a+b)(a+b)^{-1}}\ \cancel{\ \ }\ c(-1+ba^{-1})(1+ba^{-1})$$

$$lG_s = \frac{-1+ba^{-1}}{1+ba^{-1}} = \frac{ba^{-1}-1}{ba^{-1}+1} = \begin{pmatrix}1&-1\\1&1\end{pmatrix}(ba^{-1})$$

$$ba^{-1} = \begin{pmatrix}1&1\\-1&1\end{pmatrix}(lG_s) = \cancel{\ \ \ } \ \frac{lG_s+1}{-lG_s+1}$$

So none of this is clear.

First notice that your ~~...~~ transmission line requires continuous time. Thus you have a basic incompatibility.

You are facing a puzzle. Response fn. for a ladder network

$$\begin{pmatrix}E_{i-1}\\I_i\end{pmatrix} = \begin{pmatrix}1+L_iC_is^2 & L_is\\ C_is & 1\end{pmatrix}\begin{pmatrix}E_i\\I_{i+1}\end{pmatrix}$$

$$\begin{pmatrix}E_0\\I_1\end{pmatrix} = \begin{pmatrix}1 & L_1s\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ C_1s & 1\end{pmatrix}\cdots\cdots\begin{pmatrix}1 & \\ C_ns & 1\end{pmatrix}\begin{pmatrix}E_n\\0\end{pmatrix}$$

$$E_{i-1} = E_i + L_i s I_i$$
$$I_i = I_{i+1} + C_i s E_i$$

deg $(I_{n+1}) = -1$
degree $(E_n) = 0$
degree $I_n = 1$
degree $E_{n-1} = 2$

so degree $E_{n-j} = 2j$ even in s
$I_{n-j} = 2j+1$

549 So E_0 has degree $2n$ and is even
I_1 has degree $2n-1$ —— odd.

$n=1$

$$\begin{pmatrix} E_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1+LCs^2 & Ls \\ Cs & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ 0 \end{pmatrix}$$

So $$\frac{E_0}{I_1} = \frac{\text{deg } 2n \text{ even}}{\text{deg } 2n-1 \text{ odd}} = \frac{a_0}{s} + \sum_{j=1}^{n-1} \frac{s(1+\omega_j^2)}{s^2+\omega_j^2} a_j + a_\infty s$$ degree $2n$

$$\frac{E_1}{I_1} = \frac{\text{deg } 2n-2 \text{ even}}{\text{deg } 2n-1 \text{ odd}} = \frac{c_0}{s} + \sum_1^{n-1} \text{——}$$ degree $2n-1$

$$\frac{I_1}{E_1} = \frac{\text{deg } 2n-1}{\text{deg } 2n-2} = \sum_{j=1}^{n-1} \frac{}{s^2+\omega_j^2} a_j + a_\infty s$$ $2n-1$

$$\frac{I_2}{E_1} = \frac{\text{deg } 2n-3}{\text{deg } 2n-2} = \underbrace{\qquad\qquad}$$ $2n-2$

$$\frac{E_1}{I_2} = \frac{\text{deg } 2n-2}{\text{deg } 2n-3}$$ $2n-2$

$$\frac{E_2}{I_2}$$ $2n-3$

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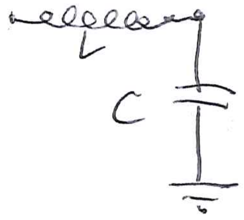
next, can you find a partial s.a. op.

~~[illegible]~~ I need a partial s.a. op. Consider the case

E_0, I_1, E_1



$$\frac{E_0}{I_1} = Ls + \frac{1}{Cs} = \frac{LCs^2+1}{Cs}$$

$$= \frac{s^2+\omega^2}{L^{-1}s} \qquad \omega = \frac{1}{\sqrt{LC}}$$

Here ~~[illegible]~~ we have $E_0, I_1, E_1, \cdots, I_n, E_n$ $2n+1$ variables

$$\begin{cases} E_{i-1} = E_i + L_i s I_i \\ I_i = I_{i+1} + C_i s E_i \end{cases}$$

$2n$ equations.

How can you interpret this? Kronecker module

$$\ell_s \longrightarrow R^{2n+1} \xrightarrow{as-b} \mathbb{R}^{2n}$$

How can you interpret this? What is a partial s.a. operator? Examine the situation — you have managed to get something roughly $2n$ dimensional where s appears linearly. ~~[illegible]~~

try another viewpoint, say a symplectic viewpoint. Symplectic viewpoint. Think

Ultimately you will need to handle a maximal isotropic subspace of a symplectic v.s. depending on a frequency parameter. Remember the pencil idea.

Start with $\dim(W) = 2$. W symplectic

Start with V real (1-dim'l?) and $q_s : V \to V^*$ quadratic form depending rationally on s. q_s essentially a rational function, and it can have any degree. What method needed to understand? Complexify, have S^2 for the s variable, and $S^2 = \mathbb{P}_1(\mathbb{C}^2)$ for the variable max isot. subspace

551 Not much to see here. ~~So consider instead~~
You have a rational function i.e. $S^2 \xrightarrow{f} S^2$
whence $f^*\mathcal{O}(1) = \mathcal{O}(d)$ where d is the degree.

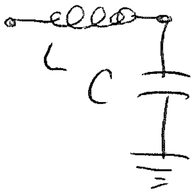
$$f(s) = \frac{p(s)}{q(s)} \qquad p, q \text{ polynomials.}$$

Next idea — find a partial s.a. operator.

$$\frac{E_0}{I_1} = \frac{\deg 2n}{\deg 2n-1} \quad \text{remove } L_1 s \text{ get} \quad \frac{E_1}{I_1} = \frac{\deg 2n-2}{\deg 2n-1}$$

You want a phase space picture

Question: Consider



$$\frac{E_0}{I_1} = Ls + \frac{1}{Cs} = \frac{s^2+\omega^2}{L^{-1}s}$$

rational fn. of degree 2 - need $0 \to \mathcal{O}(-2) \to \mathcal{O} \otimes R^3 \to \mathcal{O}(1) \otimes R^2 \to 0$

You would like to find a correspondence indicating a partial operator and then you want to find something self-adjoint.

Jan 28. Problem: Given an LC 1-port, to find an associated partial operator. From a 1-port you get a rational function

Start again and concentrate.

Consider an LC-port and the assoc. impedance form

$$\frac{E}{I} = \sum_{0 \le \omega \le \infty} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega \qquad a_\omega \ge 0 \quad \sum a_\omega > 0.$$

This impedance form amounts to what? a finite subset F of $[0,\infty]$ together with a nonnegative quadratic form $a_\omega \neq 0$ for each $\omega \in F$ such $\sum a_\omega > 0$. Such a thing is a kind of measure | quadratic form valued measure.

It appears that you are dealing with a 2nd order Diffl operator, and that you maybe want to extract a square root.

Digress to understand time dependent response of a 1-port. $Z = \frac{\hat{E}}{\hat{I}} = \frac{s^2+\omega^2}{L^{-1}s}$ $\omega = \frac{1}{\sqrt{LC}}$

Thus if $E = \hat{E}e^{st}$ $I = \hat{I}e^{st}$, then

$\boxed{L^{-1}s\hat{E} = (s^2+\omega^2)\hat{I}}$ $\underbrace{L^{-1}\partial_t E = (\partial_t^2+\omega^2)I}$

[redacted]

Thus if you give $I(t)$, you solve a 1st order DE for E, and if you give $E(t)$, ——— 2nd ——— ——— I. Somewhere in this situation I would to see a partial operator, try skew-symmetric real operator. The idea is that any boundary condition $\frac{E}{I} = ir$ leads to a characteristic eqn.

$\frac{s^2+\omega^2}{L^{-1}s} = ir$ $s^2 - irL^{-1}s + \omega^2 = 0$

$$s = \frac{irL^{-1} \pm \sqrt{-r^2L^{-2} - 4\omega^2}}{2}$$

Set $L = 1$ to simplify.

$$s = \frac{ir \pm i\sqrt{r^2+4\omega^2}}{2}$$

purely imag roots, good. $r = 0, \infty$

$r = 0 \Rightarrow s = \pm i\omega$

$r = \infty \Rightarrow s = 0$ or ∞

Try again: ~~[illegible]~~.

You have this impedance function $\frac{E}{I} = Z = \frac{s^2+\omega^2}{L^{-1}s}$, function or better quadratic form depending on the frequency parameter. This is a rational function of degree 2 e.g. simple poles at $s = 0, \infty$. So it determines a line bundle over $S^2 = \mathbb{P}^1$ namely $\mathcal{O}(2) = \mathcal{O}Z \overset{\text{NO}}{\subset}$ field K $= \mathbb{C}(s)$ of meromorphic fns of s. So I get a correspondence $\Gamma(\mathcal{O}(2)(-1)) \rightrightarrows \Gamma(\mathcal{O}(2))$. In general a ladder $L_1, C_1, \ldots, L_n, C_n$ all $\neq 0$ yields response function $\frac{E_0}{I_1} = Z$ which is of degree $2n$ (denom degree $2n-1$ numerator of degree $2n$), so $\mathcal{O}(2n+1) = \mathcal{O}Z \overset{\text{NO}}{\subset} K$, whence a correspondence $\Gamma(\mathcal{O}(2n)) \rightrightarrows \Gamma(\mathcal{O}(2n+1))$, and this is all defined over the reals.

You need to study this correspondence, why, for an n-port you will have an impedance matrix function which you want to be able to reduce to a correspondence. What type - n is the rank of the v.b. so you will have something like $a, b : \mathbb{R}^{r+n} \to \mathbb{R}^n$. ~~[illegible]~~ All this confusion with duality.

$$W^n \overset{a}{\underset{b}{\rightrightarrows}} V^{r+n} \qquad 0 \to \mathcal{O}(-1) \otimes W^n \xrightarrow{a\lambda - b} \mathcal{O} \otimes V^{r+n} \to \mathcal{E}^r \to 0$$

Assume $\mathcal{E}^r$ is a v.b. i.e. $a\lambda - b$ is injective for all $\lambda \in S^2$. Ask what this means. In particular a, b injective so we have two codim r subspaces of V namely aW, bW which are isomorphic, so

$$\begin{array}{ccccccccc}
0 & \longrightarrow & aW & \longrightarrow & V & \longrightarrow & V/aW & \longrightarrow & 0 \\
 & & \wr\| & & & & & & \\
0 & \longrightarrow & bW & \longrightarrow & V & \longrightarrow & V/bW & \longrightarrow & 0
\end{array}$$

Alternate picture namely

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "\wr s"] & V \arrow[d, "\wr s"] \\ W' \arrow[r, hook] & V' \end{tikzcd}$$

Maybe the important thing is a stable isomorphism between V/W, V'/W'. Can you get these organized? ~~Can you get started?~~

Start again: Suppose you have $W \overset{a}{\underset{b}{\rightrightarrows}} V$ such that $a\lambda - b$ inj $\forall \lambda$.

This gives you a vector bundle. Suppose you choose complements

$$V = aW \oplus V_1$$
$$V = bW \oplus V_2$$

$$\begin{tikzcd} & V_1 \arrow[d] & \\ W \arrow[r, "a\lambda - b"] & V \arrow[r] & E_\lambda \\ & V_2 \arrow[u] & \end{tikzcd}$$

You get some sort of rational map $V_1 \xrightarrow{\sim} E_\lambda \xleftarrow{\sim} V_2$

too complicated

$$V_1 \xrightarrow{\sim} E_\lambda \xleftarrow{\sim} V_2$$

Try another approach: Start with $Z(s) = \dfrac{s^2+\omega^2}{L^{-1}s}$

Look at this as a map $s \longmapsto Z(s)$ from $\mathbb{C} \cup \infty \longrightarrow \mathbb{C} \cup \infty$ $\quad \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$ preserves [redacted] $i\mathbb{R} \cup \infty$, $Re(s) > 0$. So if [redacted] you change to the disk picture you have a rational function of z preserving $|z| < 1$ and $|z| = 1$. So all zeroes inside $|z| = 1$ and poles outside, f is a product of $\dfrac{z-\alpha}{1-\bar{\alpha}z}$ $\quad |\alpha| < 1$.

Somehow you have to concentrate on this problem. You have a rational function

555 Start again. You have a response function such as $Z(s) = \frac{s^2+\omega^2}{L^{-1}s} = \frac{1}{L^{-1}}\left(s + \frac{\omega^2}{s}\right)$: ~~maps~~ preserves $\mathrm{Re}(s) < 0, = 0, > 0$ resp. You know this rational function determines a line bundle $\mathcal{O}Z = \mathcal{O}(2)$ <u>NO</u>, it's $\mathcal{O} + \mathcal{O}Z = \mathcal{O}(2)$, so you have $0 \to \mathcal{O}(-2) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(2) \to 0$

What's going on? You are probably being very stupid, certainly you have made one dumm mistake. Your mistake involves identifying a rational function $Z(s)$ with a line bundle. To straighten things out let's consider the general n-port case. This presents you with $Z(s) = \sum_{0 \le \omega \le \infty} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$ where a_ω is a quadratic form on a real vector space U such that $a_\omega \geqslant 0$ and $\sum a_\omega > 0$. Question: Can you obtain from this a vector bundle over $\mathbb{P}^1(\mathbb{C})$ of rank $r = \dim(U)$? Idea: For each s, $Z(s)$ is a "symm" map $U_c \to U_c^*$ except where $Z(s)$ has a poles. The graph of $Z(s)$ is a maximal isotropic subspace of $U \oplus U^*$. It should happen that this ~~extends nicely over~~ vector bundle we have for real $s \neq 0, \infty$ in fact extends over the Riemann sphere in general. This is something to be proved. There should be no problem because of the simple poles and the niceness of a_ω. So what do you find?

What happens when $r = 1$. Then the line $\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} Z \\ 1 \end{pmatrix}$ in $U_c \oplus U_c^* = \mathbb{C}^2$ as s varies generates a line subbundle of $\mathcal{O} \oplus \mathcal{O}$. It should turn out to be $\mathcal{O}(-d)$ where d is the degree of the rational fn Z. Why? $0 \to \mathcal{L}^{-1} \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{L} \to 0$ ask where $\begin{pmatrix} Z \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ become dependent, i.e. where $Z = \infty$, get poles of Z giving you the degree of Z.

556 Thus you have a line bundle of the right degree. So now assume that this works in general.

Jan 29. Some progress was made toward the end of yesterday, namely in the rank 1 case you get a line bundle over the Riemann sphere: $\begin{pmatrix} E \\ I \end{pmatrix}\mathbb{C} = \begin{pmatrix} Z \\ 1 \end{pmatrix}\mathbb{C}$. This gives a sub line bundle of $\mathcal{O}^{\oplus 2}$ and the degree of the quotient ~~line~~ bundle is the number of points where $\begin{pmatrix} Z \\ 1 \end{pmatrix}\mathbb{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{C}$, i.e. the number of poles of Z. Thus you get a canonical correspondence. Suppose d is the degree so that you have

$$0 \longrightarrow \mathcal{O}(-d) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathcal{O}(d) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}(-d) \longrightarrow \mathcal{O} \otimes \mathbb{C}^{d+1} \longrightarrow \mathcal{O}(1) \otimes \mathbb{C}^{d} \longrightarrow 0$$

dualize

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-d) \arrow[r] \arrow[d] & \mathcal{O}^{\oplus 2} \arrow[r] \arrow[d] & \mathcal{O}(d) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(-1)^{\oplus d} \arrow[r] & \mathcal{O}^{d+1} \arrow[r] & \mathcal{O}(d) \arrow[r] & 0
\end{tikzcd}$$

Better notation.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{O} \oplus \mathcal{O} \arrow[r] & \mathcal{M} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{O} \otimes H^0(\check{\mathcal{L}})^\vee \arrow[r] \arrow[u] & \mathcal{O}(1) \otimes H^0(\check{\mathcal{L}}(-1))^\vee \arrow[r] & 0
\end{tikzcd}$$

So if I dualize I get $\check{\mathcal{L}}$ equipped with 2 sections

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}^\vee \arrow[r] \arrow[d] & \mathcal{O}^{\oplus 2} \arrow[r] \arrow[d] & \check{\mathcal{L}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(-1) \otimes H^0(\check{\mathcal{L}}(-1)) \arrow[r] & \mathcal{O} \otimes H^0(\check{\mathcal{L}}) \arrow[r] & \check{\mathcal{L}} \arrow[r] & 0
\end{tikzcd}$$

It seems you end up with a correspondence $W \overset{a}{\underset{b}{\rightrightarrows}} V$ defining the line bundle $\check{\mathcal{L}}$ and a 2 dimension subspace of V which spans $\check{\mathcal{L}}$.

Now you have a way to linearize ~~spectra~~ Z. So next consider the general case meaning rank r. You have a general response function $Z(s) = \sum \frac{s(\ )}{s^2+\omega^2} a_\omega$ $\quad a_\omega \geq 0 \quad \sum a_\omega > 0$.

Here $a_\omega : V^* \to V$ is a symmetric quadratic form so that $s \mapsto \binom{Z}{1} V^* \subset \begin{matrix} V \\ \oplus \\ V^* \end{matrix}$ is a rational map from $\mathbb{P}^1$ to the Lagrangian symplectic Grassmannian of maximal isotropic subspaces of $V \oplus V^*$. By completeness this extends to a regular map $\mathbb{P}^1 \longrightarrow Sgr(V \oplus V^*)$. Thus we have a vector bundle E of rank ~~n~~ over $\mathbb{P}^1$ $\quad E_s = \binom{Z_s}{1} V^* \subset \begin{matrix} V \\ \oplus \\ V^* \end{matrix}$. The v.b. will give rise to a correspondence, say surjective type $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ ~~surjective~~

$$0 \longrightarrow E_s \longrightarrow X \xrightarrow{as-b} Y \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & \mathcal{O} \otimes X \arrow[r] \arrow[d] & \mathcal{O}(1) \otimes Y \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathcal{O} \otimes \begin{pmatrix} V \\ + \\ V^* \end{pmatrix} & &
\end{tikzcd}$$

maybe not surj.

~~[illegible] what happens?? [illegible]~~

The geometric situation consists of a symplectic vector space $V \oplus V^*$ and a subbundle of the trivial bundle $\mathcal{O} \otimes (V \oplus V^*)$ which is Lagrangian at each $s \in \mathbb{P}^1$. So we ~~have~~ now have a negative vector bundle E over $\mathbb{P}^1$ of rank = dim V, so we get a K-module:

$$0 \longrightarrow E \longrightarrow \mathcal{O} \otimes X \longrightarrow \mathcal{O}(1) \otimes Y \longrightarrow 0$$

~~and we can read~~

$$0 \longrightarrow E(-2) \longrightarrow \mathcal{O}(-2) \otimes X \longrightarrow \mathcal{O}(-1) \otimes Y \longrightarrow 0$$

$$\therefore \quad X = H^1(E(-2)) \qquad Y = H^1(E(-1))$$

 Is there something about metrics we can say. This s variable I am playing with is like the time variable for an oscillator. What is the model you would like? The poles of $Z(s)$ are like the eigenvalues of a skew-symmetric real ~~operator~~ which via Cayley transform becomes an orthogonal operator. This connects up with the dihedral group gen by ε, F.

So what comes next?

Idea: To give a subquotient V/W of a polarized Euclidean space $H = H^+ \oplus H^-$ is very close to giving an orthogonal representation of $\mathbb{Z}/2 * \mathbb{Z}/3$.
Complex case: A rep of $\mathbb{Z}/3$ is a decomp. into 3 eigenspaces with eigenvalues $1, e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}$.

All I've accomplished so far is to get a vector bundle over $\mathbb{P}^1$ from a response function. The question whether there is additional structure from inner products? Recall what happens starting from

$$Z_s = \sum_{0<\omega<\infty} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega \qquad a_\omega : P^* \to P \quad \text{symm.} \geq 0 \quad \sum a_\omega > 0.$$

I consider the graph $E_s = \begin{pmatrix} Z_s \\ 1 \end{pmatrix} P^* \subset \begin{matrix} P_c \\ \oplus \\ P_c^* \end{matrix}$ and get a vector bundle ≤ 0 over $\mathbb{CP}^1$, whence

$$0 \to E \to \mathcal{O} \otimes X \to \mathcal{O}(1) \otimes Y \to 0$$

$$X = H^1(E(-2)) \qquad Y = H^1(E(-1))$$

In the case where $E = \mathcal{O}(-n)$ $\dim(Y) = \mathbb{C}^n$

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$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \mathcal{O}^{n+1} \longrightarrow \mathcal{O}(1)^n \longrightarrow 0$$

$$0 \longrightarrow E \longrightarrow \mathcal{O}\otimes X \longrightarrow \mathcal{O}(1)\otimes Y \longrightarrow 0$$

$$0 \longrightarrow E(-1) \longrightarrow \mathcal{O}(-1)\otimes X \longrightarrow \mathcal{O}\otimes Y \longrightarrow 0$$

$$Y \xrightarrow{\sim} H^1(E(-1))$$

$$0 \longrightarrow E(-2) \longrightarrow \mathcal{O}(-2)\otimes X \longrightarrow \mathcal{O}(-1)\otimes Y \longrightarrow 0$$

$$H^1(E(-2)) \xrightarrow{\sim} H^1(\mathcal{O}(-2))\otimes X = X.$$

If $E = \mathcal{O}(-n)$, then $Y = H^1(\mathcal{O}(-(n+1)))$ dual to $H^0(\mathcal{O}(n-1)) \cong \mathbb{C}^n$

What can you do? I could try to bring in scattering op. This means replacing Z, a symmetric matrix, by its Cayley transform which should be symplectic. ~~Recall the~~

Review examples. $cl = 1$.

$$-\partial_x E = ls I$$
$$-\partial_x I = cs E$$

$$\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} a \\ -ca \end{pmatrix} e^{sx} + \begin{pmatrix} b \\ cb \end{pmatrix} e^{-sx}$$

set $x = 0$

$$\frac{Z-l}{Z+l}$$

$$\begin{pmatrix} ZI \\ I \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix}\left(\frac{a}{b}\right) \qquad \left(\frac{a}{b}\right) = \begin{pmatrix} c & -1 \\ c & 1 \end{pmatrix}(Z) = \frac{c(Z)-1}{c(Z)+1}$$

All that $Z \mapsto \frac{Z-1}{Z+1}$ does is to convert a purely imaginary impedance $Z(s)$ (i.e. when $s \in i\mathbb{R}$) to a

$Z(s)$ is in general a symmetric complex matrix which is purely imag. (hence skew adjoint) for $s \in i\mathbb{R}$. Thus $\frac{Z-1}{Z+1}$ is unitary. I haven't done anything but a frac. lin transf

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moment problem. $d\mu$ measure on $\mathbb{R}$ with finite moments, same as matrix $\int x^i x^j d\mu = m_{i+j}$ which is ≥ 0. Let $\phi_n(x)$ be the seq. of orth polys.

$\phi_1(x) = 1,\ \phi_2(x),\ ,$ $\quad H = L^2(\mathbb{R}, d\mu),\quad A = x\cdot \text{ s.a.}$

$F_1 < F_2 \subset \cdots$ $\quad F_p = \mathbb{C}[x].$ Clearly

$AF_p \subset F_{p+1}$ $\quad \int (x\phi_p, F_j)\, d\mu = (\phi_p, xF_j) \subset (\phi_p, F_{j+1}) = 0$ if $j < p-2$

$$x\phi_p = a_p \phi_{p+1} + b_p \phi_p + c_{p}\phi_{p-1}$$

$$a_p = (x\phi_p, \phi_{p+1}) \qquad c_p = (x\phi_p, \phi_{p-1}) = (\phi_p, x\phi_{p-1}) = a_{p-1}$$

$$J_n = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ & & & a_{n-1} & b_n \end{pmatrix}$$

so you get a J-matrix which is symmetric and ≥ 0.

count dims. $2n-2$ J_n's describes all $(n-1)$pt measures

561 What am I going to do?
Basically I think you have

Count again. dim of J_n = n b's $n-1$ a's
$= 2n-1.$

$m_0, \ldots, m_{2n-2}$ | $2n-1$ things.

puzzle. Start with $\int - d\mu$ restrict to degree $< n$
get $\mu_0, \ldots, \mu_{2n-2}$ moments. total $2n-1$.

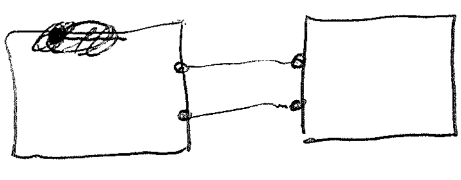
$$\sum_{i=1}^{n} a_i \delta(x-\lambda_i)$$

---

ideas here — you want to look at symmetric measures (inv. under $x \mapsto -x$; these have all $b_i = 0$) to obtain analogs of LC circuits

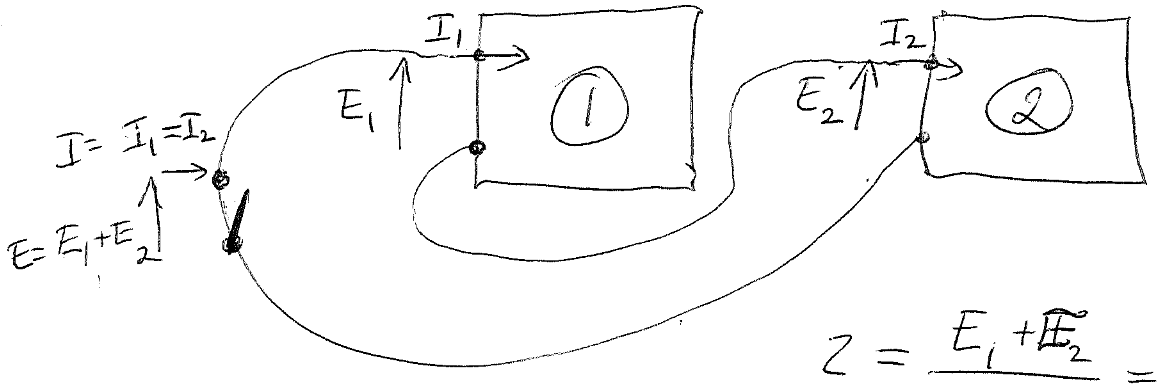
coupling: Any n port is a symplectic v.s. $V^* \oplus V$ of dim $2n$ together with the max isot. subspace $\binom{Z}{1} V^*$ depending on s. (Analogy Riemann surface with boundary circle yields symplectic v.s. $C^\infty(S^1)/\mathbb{C}$, $\int_{S^1} f dg$, and get maximal isot subspace, namely bdry values of analytic functions on the interior (assumes $g=0$). to get max isot subspace). What is coupling about?

Example: Take two 1-ports with imped functions $Z_1 = \frac{E_1}{I_1}$ $Z_2 = \frac{E_2}{I_2}$



562 When the coupling is done we should be left with some sort of characteristic poly, torsion sheaf on S^2 giving the normal modes.

Basic example I have is to combine Z_1, Z_2 either $Z_1 + Z_2$ or $\frac{1}{Z_1^{-1}+Z_2^{-1}}$ and then set the result $= 0$ or ∞. (These are the 1-port boundary conditions defd over $\mathbb{R}$.) So how to understand this. $Z_1 = \frac{E_1}{I_1}$ $Z_2 = \frac{E_2}{I_2}$

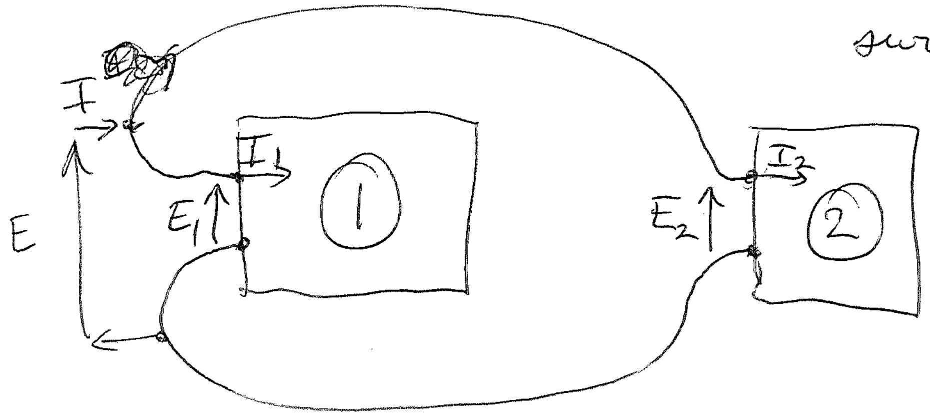


$$Z = \frac{E_1 + E_2}{I} = Z_1 + Z_2 = 0$$

switch ~~on~~ closed

$Z_1 + Z_2 = \infty$

switch open



$$E = E_1 = E_2 \qquad I = \cancel{I_2} + I_1$$

$$Z = \frac{E}{I} = \frac{E}{I_1 + I_2} = \frac{1}{\frac{I_1}{E_1} + \frac{I_2}{E_2}} = \frac{1}{Z_1^{-1} + Z_2^{-1}}$$

563 ~~So~~ it seems that coupling amounts to ~~[illegible]~~ passing to a subquotient of V

Try carefully to

Let's try once more to formulate the basic problem. Given a r-port, it has a response function $Z(s) : V^* \longrightarrow V$, $r = \dim V$, which is rational in s and satisfies certain pos. conditions. Do these reality + positivity condition have a good interpretation?

First thing we can do is to self couple, ~~[illegible]~~ which should lead to a characteristic polynomial, or characteristic torsion sheaf depending on the self coupling. So you want to analyze self-coupling and this should involve an arbitrary subspace of V, i.e. ~~[illegible]~~ a O subquotient. ~~[illegible]~~

~~[illegible]~~ $\dim V = 1$, there are two subspaces

~~[illegible]~~

Maybe ask different questions. Symplectic approach?

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determinants. You know that a port gives rise to a response which is a vector bundle over $\mathbb{P}^1$. You want to understand self-coupling. Self-coupling yields a ~~[illegible] is that To~~ characteristic poly., or charact torsion sheaf on $\mathbb{P}^1$ depending on the boundary. You got analogy of a manifold with boundary. ~~the~~ Problem - can you recover response from this det. funct.

There's also the picture of a Dirac op. on a man.

 on a manifold with ∂ needing boundary values to make a s.a. op. Leads to ~~such~~ partial s.a. operator with deficiency indices.

Look at rank 1 case. ~~[illegible]~~ In this case you have $Z = \frac{E}{I}$ a rational function of s. ~~[illegible]~~

Yesterday you learned, ~~[illegible]~~ but did not carefully check, that there is a partial s.a. operator in this situation, a moment problem which ~~[illegible]~~ lies behind this. The measure on $\mathbb{R}$ is even - ~~[illegible]~~ invariant under $x \to -x$.

~~[illegible] learned yesterday~~

What can you learn about Z by self coupling. Take rank 1. Then there are two possible self couplings namely $Z = 0, \infty$ i.e. leave the circuit open ~~and~~ or short about it. ~~So as far~~ The only information here are the poles and zeroes of Z, these ~~[illegible]~~ determine Z up to a nonzero constant. Consider next rank 2. ~~This~~ ~~[illegible] Here we have a rank 2 vector bundle~~ ~~[illegible]~~ Recall self-coupling arises from any ~~[illegible]~~ subspace of V. What is Z? rational map $s \mapsto Z_s$ from $\mathbb{P}^1\mathbb{R}$ to max. isot. subspaces of $V^* \oplus V$, so you get $0 \longrightarrow \mathcal{E} \hookrightarrow \mathcal{O} \otimes (V^* \oplus V)$

Lagrangian subbundle of the trivial symplectic ~~[illegible]~~ v.b. fibre $V^* \oplus V$. What is self-coupling? ~~So if you [illegible]~~

Study self-coupling. You have V embedded as subquotient of $H^+ \oplus H^-$ and then you take a zero subquotient of V, i.e. a subspace. This gives you a subspace of $H^+ \oplus H^-$ meaning what?

565 Suppose you have $Z_s : V^* \to V$ ~~describing~~ the impedance function of an r port. What does it mean to self-couple this r-port, for example to ~~attach~~ ~~[illegible]~~ wires between some of the external vertices. ~~by~~

~~[illegible]~~ A wire ~~[illegible]~~ joining 2 vertices forces the vertex voltage function to be equal at these ~~edges~~ vertices and the vertex current function to sum to zero ~~over~~ at these vertices. Seems to be passing to a subspace of V^* and corresponding quotient space of V.

Simpler example suppose you have ~~a~~ $r+1$ external vertices and you ~~[illegible]~~ collapse two vertices together. Case 1. The new vertex is considered internal so that there are now $r-1$ external vertices. Case 2: The new vertex is considered external. Begin with 2nd case: ~~[illegible] subspace of V~~ $I = I_x + I_y$, $E_{\circ} = E_x = E_y$ so we have a subspace of V and a corresp. quotient space of V^*. ~~[illegible]~~ $V' \subset V$ hyperplane where $E_x = E_y$, then $V'^* \twoheadleftarrow V^* \xleftarrow{Z_s^{-1}} V \hookleftarrow V'$. ~~To~~

To handle Case 1 you ~~might~~ can look at what happens when you move an external vertex to an internal one. Then ~~[illegible]~~ again you ~~[illegible]~~ pass from the vertex current space V^* to ~~[illegible]~~ the hyperplane of vertex currents zero at the dist. vertex, so V is replaced by a quotient by a line.

Start again. ~~The [illegible] problem~~ An interesting problem is how much information about an r-port can be obtained from the various ways of connecting it to itself. Describe the process. Suppose the r-port leads to Lagrangian subbundle $\mathcal{E}$ of $\mathcal{O}\otimes(V^*\oplus V)$ (where $\dim(V)=r$ of course). A self-connection should lead to characteristic torsion sheaf, better might be positive divisor on $\mathbb{P}^1$, probably should be a section of $\mathcal{O}(\Lambda^r\mathcal{E})^\vee$. Can you see how to obtain this?

Describe your picture of self-connection. First passing to a subspace of V – this corresponds to soldering external vertices together, then you change the external vertices to internal ones. Thus have $W\subset V$. So how can you get out a divisor for a Lagrangian subbundle $\mathcal{E}\subset\mathcal{O}\otimes(V^*\oplus V)$. We have typical symplectic quotient

$$\begin{array}{ccc} V^*\oplus W & \subset & V^*\oplus V \\ \downarrow & & \downarrow \\ W^*\oplus W & \subset & W^*\oplus V \end{array}$$

but you have a maximal isotropic subspace $E\subset V^*\oplus V$, generically E is the graph of a non-degenerate quadratic form on V which then induced one on W. Maybe there's a ~~divisor~~ discriminant you can take of the induced q. form on W. If so, you maybe get a number for generic E, and this should be to a divisor. We know $\dim V = \operatorname{rank}\mathcal{E}$.

Try to understand things naively. Start with

$F\subset V^*\oplus V$ max isot. $F=\begin{pmatrix} Z \\ 1\end{pmatrix}V^*$ where

$Z: V^*\longrightarrow V$ is symm. non-degenerate.

Along comes $W \subset V$

$$\begin{array}{ccccc} W & \hookrightarrow & V & \twoheadrightarrow & V/W \\ & & \downarrow{\scriptstyle Z^{-1}} & & \\ W^* & \twoheadleftarrow & V^* & \hookleftarrow & (V/W)^* \end{array}$$

basic fact is the direct sum result

$$\begin{array}{ccccc} V & = & W & \oplus & W^0 \\ {\scriptstyle Z^{-1}}\downarrow & & {\scriptstyle Z_W^{-1}}\downarrow & & \downarrow{\scriptstyle Z_{W^0}^{-1}} \\ V^* & = & W^* & \oplus & (W^0)^* \end{array}$$

so what to do ?? First question: discriminant of a quadratic form, there's probably a ~~[illegible]~~ Maslov line bundle around also (but this is maybe quantum mechanical). ~~So the~~. $\underbrace{\mathcal{E} \subset \mathcal{O} \otimes (V^* \oplus V)}$

Problem is? ~~a~~ closing a port by connecting it to itself. You have maybe decided that this means?

Start again. Given V of dim r, $Z_s : V^* \longrightarrow V$ symmetric generically non degenerate. ~~Also subspace~~ Get Lagrangian vector bundle ~~[illegible]~~ $\mathcal{G} \subset \mathcal{O} \otimes (V^* \oplus V)$ $\quad \mathcal{G}_s = \begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^*$

But you have also a subspace W of V.
Given $V^* \oplus V$ symplectic, $W \subset V$ subspace whence a symplectic quotient situation

$W \oplus W^* \qquad V^* \oplus V \qquad (V/W)^* \oplus (V/W)$

There's no canonical way to map between these
However ~~an isotropic~~ a Lagrangian subspace might do

 something. What next? ~~Question: I~~

So the issue is maybe that we have a symplectic quotient together with a Lagrangian subspace.

(Another germ of IDEA: the relation between an operator and the kernel representing the operator. This should be involved when 2-ports are connected in a chain — you might go over Fourier integral operators formalism with the aim of getting insight into phase space picture for response functions. What happens roughly is that the kernel $k(x,y)dy$ is singular normal to the Δ so you need to blowup the Δ essentially to control things, maybe blowup is replaced by F.T. : $e^{i\xi(x-y)}k(x,\xi)$ but probably you want to handle Gaussians only.)

Anyway there are things called Fourier integral operators. Basic examples arise with wave equations. Solutions of wave equation have singularities propagating via classical mechanics. This means a flow on phase space.

Anyway — you have symplectic space $V^* \oplus V$ and a Lagrangian subspace $E = \text{graph}\begin{pmatrix} Z \\ 1 \end{pmatrix} V^*$. First step is restriction to W

$$\begin{tikzcd} W \arrow[r, hook] & V \arrow[d, "Z^{-1}"] \\ W^* & V^* \arrow[l, two heads] \end{tikzcd}$$

picture this as soldering ext'l nodes forcing $E_x = E_y$

 So what's happening? ~~It seems that~~ You have symp. v.s. $V^* \oplus V$ and generic Lagrangian subspace G, ~~get the~~ then G is graph of a symm. map $Z: V^* \longrightarrow V$ which is an isom. ~~For~~ get $\Lambda^{max} Z : (\Lambda^{max} V)^* \longrightarrow (\Lambda^{max} V)$ so get quadratic form on a 1-dimensional space. In our case Z depends rationally on s, get rational function of s.

On $H^+ \oplus H^-$ have $A_s = s\pi_+ + s^{-1}\pi_-$ ~~=~~

$$\begin{tikzcd}
V = \bar{C}^0 \arrow[r, hook] & H = C^1 \arrow[d, "S"] \\
V^* = \bar{C}_0 & C_1 \arrow[l, two heads]
\end{tikzcd}$$

Try again. $Sp_{2n}(\mathbb{R}) \sim U_n$ so $\pi_1 = \mathbb{Z}$, the double covering ~~is~~ the metaplectic group and it is the thing that arises when you want to lift some action of $Sp_{2n}(\mathbb{R})$, ~~Thus~~ I think the natural action on Lagrangian subspaces, to some line bundle relevant for quantum mechanics.

What is the stabilizer of the Lag. subspace V^* in $V \oplus V^*$. ~~[illegible] on the sp~~ If G_{V^*} is the stabilizer of V^*, then get split surjection $G_{V^*} \longrightarrow GL(V^*)$, and ~~[illegible]~~ ~~G~~ the kernel fixes V^* and $V \oplus V^*/V^*$ so ~~is the~~ acts simply-transitively on the Lagrangian subsp comp to V^*, same as quad. form $g: V \to V^*$. Thus

~~$\dim \left(\begin{matrix}\text{Lagrangian}\\ \text{subspaces}\end{matrix}\right) = \dim Sp_{2n} + \frac{n(n+1)}{2} \quad n^2 + n(n+1)$~~

$$\dim Sp_{2n} = \dim U_n + \dim \begin{pmatrix}\text{complex symm}\\ n\times n \text{ matrices}\end{pmatrix}$$
$$= n^2 + n(n+1) = 2n^2+n$$
$$\dim G_{V^*} = n^2 + \frac{n(n+1)}{2}$$

~~HUA~~ +

$$\therefore \dim(\text{Symp Grass}) = \frac{n(n+1)}{2}$$

$$\text{Symp Grass} = Sp_{2n}(\mathbb{R}) \Big/ GL_n(\mathbb{R}) \times \text{symm matrices}^{\text{real } n\times n}$$
$$\sim U_n / O_n$$

[redacted] OKAY.

Look at differently. The metaplectic group acts on $L^2(\mathbb{R}^n)$, not the symplectic group. How to see this? Weil made systematic study which among other things yields the Θ fn. proof of quadratic reciprocity. $SL_2(\mathbb{R}) = Sp_2(\mathbb{R})$. You have the Fourier transform on $L^2(\mathbb{R})$ which corresp to $\begin{pmatrix} 0 & -1 \\ +1 & 0\end{pmatrix}$ in $SL_2(\mathbb{R})$

$$f \mapsto \hat f(\xi) = \int e^{ix\xi} f(x) \frac{dx}{\sqrt{2\pi}}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\hat{\hat f}(y) = \int e^{iy\xi} \frac{d\xi}{\sqrt{2\pi}} \int e^{ix\xi} f(x) \frac{dx}{\sqrt{2\pi}}$$
$$= \int f(x)dx \, \frac{1}{2\pi} \underbrace{\int e^{i(y+x)\xi}\, d\xi}_{2\pi\,\delta(y+x)}$$
$$= f(-y).$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(i) = i$$
$$ai+b = -c+di$$
$$a=d \quad b=-c.$$
$$ad-bc = a^2+b^2 = 1.$$

$Sp_2(\mathbb{R}) \sim SO_2$. Something's wrong because I thought ~~the~~ you needed a double cover of $Sp_2(\mathbb{R})$ to act on $L^2(\mathbb{R})$

$$\int_{-R}^{R} e^{i\xi x}\, d\xi = \frac{e^{i\xi x}}{i\cancel{x}}\Bigg]_{-R}^{R} = \frac{e^{iRx}}{i\cancel{x}} - \frac{e^{-iRx}}{+i\cancel{x}}$$

$$\frac{e^{iRx} - e^{-iRx}}{ix} = \frac{2i\sin(Rx)}{ix} = 2\frac{\sin Rx}{x}$$

What has probably happened is that your F.T. choice for the ~~lifting~~ action of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is not the correct one i.e. the one you get by ~~making~~ lifting locally.

Symp Grass find L_1 $\left.\begin{matrix} 2n-1 \\ +\,2n-3 \\ + \;\cdots \\ \vdots \\ 1 \end{matrix}\right) = n^2$ max. isot. flags $n-1+n-2+\cdots+1 = \frac{n(n-1)}{2}$

$n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

~~Return to fundamental soln~~

Situation: You have n-port $Z_s : V^* \longrightarrow V$, better description is a Lagrangian bundle $G \subset \mathcal{O}\otimes(V^*\oplus V)$ over $\mathbb{P}^1$. You want to self-connect, or close, this n-port, I think this means giving a subspace W of V representing external nodes connected together, and then looking at the "poles", modes with vertex currents are 0 to get a divisor.

So given $Z_s : V^* \longrightarrow V$ & non-degenerate quadratic form, I restrict Z_s^{-1} to W, get a quadratic form $W \hookrightarrow V \xrightarrow{Z_s^{-1}} V^* \longrightarrow W^*$ on W, then look at poles - zero vertex currents. I am confused.

Try something more geometric. Suppose given (V, Z_s) internalizing all external ~~nodes~~ nodes means ~~form~~ restricting to 0 vertex currents i.e. where

$G_s = \begin{pmatrix} Z_s \\ 1 \end{pmatrix} V^* \cap \begin{pmatrix} 1 \\ 0 \end{pmatrix} V^*$. This means you look at

$$G_s \subset \mathcal{O}\otimes(V^*\otimes V) \longrightarrow \mathcal{O}\otimes V$$

where this map vanishes. Most confusing. However ~~notice that~~ it is natural to ask about the zeroes and poles of Z_s

 Try again. You have $G \subset O\otimes(V^*\oplus V)$, in fact an exact sequence

$$0 \longrightarrow G \longrightarrow O\otimes(V^*\oplus V) \longrightarrow G' \longrightarrow 0$$

of vector bundles, with G, G' have rank $r = \dim(V)$. ~~and dra~~ Obvious thing to do is to consider the projections $G \longrightarrow O\otimes V^*$, $G \longrightarrow O\otimes V$

$$\begin{pmatrix} E \\ I \end{pmatrix} \longmapsto I \qquad \begin{pmatrix} E \\ I \end{pmatrix} \longmapsto E$$

Kernel of former is a ~~voltage~~ state with 0 current | pole of Z.
Kernel of latter is a state with 0 voltage | zero of Z

It seems like the thing to do is to form Λ^{max}.

$$\Lambda_O^{r}(G) \subset \Lambda_O^{r}\left(O\otimes(V^*\oplus V)\right) = \bigoplus_{p+q=r} O\otimes \Lambda^p V^* \otimes \Lambda^q V$$

$$\Lambda^p V^* = \Lambda^h V^* \otimes \Lambda^q V$$

This looks interesting - why?

What happens with the exterior alg of a symplectic vector space. Take S, ω symplectic, then in the exterior alg you get L mult. by ω and the adjoint, whence a repn. of sl_2.

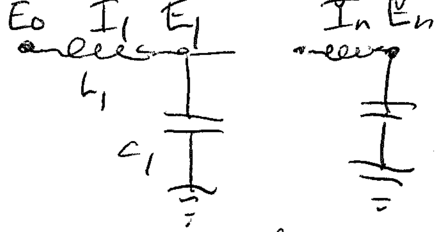
Obvious question: Given $Z_s = \sum'_{0\le\omega\le\infty} \frac{s(1+\omega^2)}{s^2+\omega^2}\, a_\omega$ $\quad a_\omega \geq 0,\ \sum a_\omega > 0$

what is the degree of G? Obvious guess is $\operatorname{rk}(a_0) + \operatorname{rk}(a_\infty) + 2\sum'_{0<\omega<\infty} \operatorname{rk}(a_\omega)$

$$\begin{array}{ccc} & & H = H^+ \oplus H^- = \bigoplus H_\omega \\ & & \big\downarrow \\ W\cap W^\perp & \subset\!\longrightarrow & W^+ = \bigoplus_\omega (W^+)_\omega \end{array}$$

dim of minimum H?

573 Suppose $r=1$. The formula for the degree d is correct = no. of poles, but dim H might be off by 1.



$$Z = L_1 s + \frac{1}{C_1 s + \frac{1}{\ }}$$

This graph has $2n$ edges, $n+2$ vertices.
So dim $H = 2n$.

$$\underbrace{U = \bar{C}^0}_{n+1} \xrightarrow{d} \underbrace{C^1 = H}_{2n}$$

$$\{(E^0, \dots, E^n)\}$$

$$W \supseteq \{(E^1, \dots, E^n)\}$$

So in this example $W^\perp, W$ have dim n. ~~and~~

What is the degree of Z? $2n$

$$\begin{pmatrix} E_{i-1} \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & L_i s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & L_i s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_i s & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_{i+1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & L_n s \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ C_n s & 1 \end{pmatrix} \begin{pmatrix} E_n \\ 0 \end{pmatrix}}_{\begin{pmatrix} E_n \\ C_n s E_n \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 0 \\ C_{n-1} s & 1 \end{pmatrix} \begin{pmatrix} (1 + L_n C_n s^2) E_n \\ C_n s E_n \end{pmatrix} =$$

$$I_i = I_{i+1} + C_i s E_i$$

$$E_{i-1} = E_i + L_i s I_i$$

deg $I_{n+1} = -1$

deg $E_n = 0$

deg $I_n = 1$

deg $E_{n-1} = 2$

$I_{n-1} = 3$

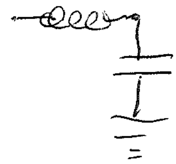
degree $I_1 = 2n - 1$

degree $E_0 = 2n$

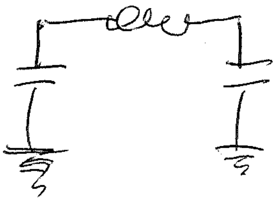
$Z = \frac{E_0}{I_1}$ has degree $2n$

deg $E_i = 2(n-i)$

deg $I_i = 2(n-i) + 1$

574 As a check take  $Z = \frac{E_0}{I_1} = Ls + \frac{1}{Cs}$

degree 2.



$$\frac{1}{C_1 s + \frac{1}{L_2 s + \frac{1}{C_2 s}}}$$

$$= \frac{1}{C_1 s + \frac{C_2 s}{L_2 C_2 s^2 + 1}} = \frac{L_2 C_2 s^2 + 1}{C_1 L_2 C_2 s^3 + C_1 s + C_2 s}$$

So what can you say in general?

$$Z = \sum \cdots a_\omega + a_\infty s$$

You seem to get a minimum H and quotient space. Ask what is the meaning of Part of $W^\perp$, H is the $\omega = \infty$ part.

$$H \xrightarrow{p} H/W = W^\perp$$

$$s\, p\pi_+ p^* + s^{-1} p\pi_- p^*$$

apply spectral thm to $p\pi_+ p^*$ get

$$\bigoplus_\omega \frac{s + s^{-1}\omega^2}{1+\omega^2}\pi_\omega \quad \text{on } \bigoplus (W^\perp)_\omega$$

$\omega = \infty$ part represents where $p\pi_- p^* = +1$

i.e. $W^\perp \cap H^-$. Can remove whence $W^\perp \cap H^- = 0$.

$\omega = 0$ part rep. where $p\pi_+ p^* = 1$ i.e. $W^\perp \cap H^+$.

It seems that given (V, Z) there is a canonical dilation to $W \subset U \subset H$, $V = U/W$ and $H = H^+ \oplus H^-$. You want a simple construction. You have a 2 step process. First from $Z = \sum_{\omega \in F} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$ $a_\omega \geq 0$, $\sum a_\omega > 0$

you get $W^\perp = \bigoplus_{\omega \in F} (W^\perp)_\omega$ and $V \hookrightarrow W^\perp$

Let's hope to find a K-module as in the rank 1 case. Recall that if

$$Z = \begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ C_1 s & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \\ C_n s & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \simeq \frac{E_0}{I_1}$$

where E_0 has degree $2n$, I_1 ______ $2n-1$, then Z has degree $2n$ and $2n$ is $\dim(H)$, so the K-module should have dims ~~2n~~ $2n+1$, $2n$:

$$0 \longrightarrow \underset{\displaystyle \mathcal{O}(-2n)}{\overset{\shortparallel}{\mathcal{E}}} \longrightarrow \mathcal{O}^{2n+1} \longrightarrow \underset{\displaystyle \mathcal{O}(1)\otimes H}{\overset{\shortparallel\,?}{\mathcal{O}(1)^{2n}}} \longrightarrow 0$$

Is there an obvious $(2n+1)$-diml space?
Maybe take $\frac{1}{Z} = \frac{I_1}{E_0}$ but this has the same degree $2n$.

In the rank 1 example what happens?
You start with $Z = \sum_F \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$ $\quad F \subset [0,\infty]$, $a_\omega \geq 0$, $\sum a_\omega = 1$.
What is the degree of Z? Answer: the number of poles

$$rk(a_0) + 2\sum_{0<\omega<\infty} rk(a_\omega) + rk(a_\infty)$$

In this rank 1 case ~~the $\{a_\omega \neq 0\}$~~ each nonzero ω contributes $\mathbb{R}$ to $W^\perp$. Each $a_\omega \neq 0$ with $0<\omega<\infty$ contributes $\mathbb{R}$ to H^+ and H^-.
Look at $\omega=0$. $\frac{1}{s}a_0$ contributes $\mathbb{R}$ to H^+
$s\,a_\infty$ ______ $\mathbb{R}$ to H^-.
So the degree $(Z) = \dim(H)$. The same count might work in the rank r case.

So what remains to be done? ~~Life is~~ You are trying to find a Kronecker correspondence explaining the vector bundle. At the moment you ~~would~~ ~~do~~ need a good construction of the vector bundle. You need to prove it $\exists$. You probably can appeal to algebraic geometry, but the poles are simple, so the issue should be fairly clear. Also the result about the degree should be clear. Keep a going.

You want to construct an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}\otimes\mathbb{R}^{n+d} \longrightarrow \mathcal{O}(1)\otimes\underbrace{H}_{\mathbb{R}^d} \longrightarrow 0$$

You are looking for a natural space extending H in an whole pencil of ways. <u>IDEA</u>: Your s parameter - is it related to the spectral flow on the Grassmannian? i.e. $X \mapsto tX$, apparently $t = s^2$.

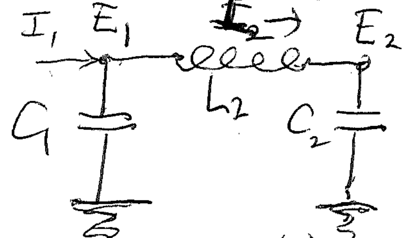
$$H = H^0(\check{\mathcal{E}}(-1)) \qquad \mathbb{R}^{n+d} = H^0(\check{\mathcal{E}})$$

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, two heads] & H \arrow[d, two heads] \\ V = U/W = U\cap W^{\perp} \arrow[r, hook] & W^{\perp} = H/W \end{tikzcd}$$

Is there a reasonable way to combine H with V which is a ~~subquotient~~ of H.

What would you like to find? partial operator, where V, V^* are related to deficiency and H yield the partial operator

577 Example in the rank 1 case



$\{(I_1, E_1, \ldots, I_n, E_n)\}$ 2n dimil v.s.

$$\begin{cases} I_i = I_{i+1} + C_i s E_i \\ E_{i-1} = E_i + L_i s I_i \end{cases}$$ 2n-1 equations.

$$\begin{array}{c}\mathcal{O}(-3)\\ \parallel \\ \mathcal{E}\end{array} \longrightarrow \mathcal{O}\otimes\mathbb{R}^{2n} \longrightarrow \mathcal{O}(1)\otimes\mathbb{R}^{2n-1} \longrightarrow 0$$

$$Z = \frac{E_1}{I_1} = \frac{1}{C_1 s + \dfrac{1}{L_2 s + \dfrac{1}{C_2 s}}} = \frac{C_2 L_2 s + 1}{\left(1 + C_1 s\left(L_2 s + \frac{1}{C_2 s}\right)\right) C_2 s} = \frac{\deg 2}{\deg 3}.$$

Is there a partial s.a. operator around behind this?

Let's study a partial s.a. operator. I do not know where to ~~start~~. Let's try a more complicated version of a LC ladder, why not a Sturm-Liouville DE.

$$-\partial_x^2 u + Vu = \lambda u.$$

simple eigenvalue problem. Assume over some interval, say $[0,\infty)$, maybe $V\uparrow$ so that for each λ get ! $u_\lambda(x)$ up to constant factor vanishing at ∞. Boundary conditions at $x=0$. e.g. $u(0)=0$ or $u'(0)=0$.

Probably

$$\partial_x \begin{vmatrix} u_1 & u_1' \\ u_2 & u_2' \end{vmatrix} = \begin{vmatrix} u_1 & u_1'' \\ u_2 & u_2'' \end{vmatrix} = \begin{vmatrix} u_1 & (V-\lambda)u_1 \\ u_2 & (V-\lambda)u_2 \end{vmatrix} = 0.$$

What are some of the ideas? Introduce $L^2(0,\infty)$ close the operator $(-\partial_x^2 + V)$ to obtain a symm. op. T. Then $(u_1, Tu_2) = (Tu_1, u_2)$ for $u_1, u_2 \in \mathcal{D}_T$

$\mathcal{D}_T \ni u$ means probably that u abs. cont with L^2 u'. and $u(0)=0$. $T^* \supset T$.

 ~~What next?!~~ Try this. Take (V, Z) and impose a boundary condition; i.e. $Z = 0$, or $Z = \infty$, or maybe even $Z = i\omega_0$ $\omega_0 \in \mathbb{R}$. Then there should be a Hilbert space with s.a. operator around. ~~YES~~!

~~[illegible] $SL_2(\mathbb{Z})$ [illegible]~~

Keep up the struggle. Maybe the idea is that the minimal H's you get from Z and Z^{-1} although the same dimension (degree d) do not coincide

Actually what about $\mathcal{U} \oplus W^\perp$. We have

$$\mathcal{U} + W^\perp \supset W + W^\perp = H, \text{ and } \mathcal{U} \cap W^\perp = V$$

so that $\dim(\mathcal{U} \oplus W^\perp) = \underline{\dim H} + \dim V = d + r$.

and there is at least one map from $\mathcal{U} \oplus W^\perp$ to H.

$\mathcal{U}$ $\qquad$ H

W $\qquad$ $W^\perp$

can you think of examples. e.g.

$$\begin{array}{c} W \\ \downarrow \\ \mathcal{U} = \bar{C}^0 \longrightarrow C^1 \longrightarrow \end{array}$$

Question: Within the symplectic framework, how does a symplectic quotient mix with a Lagrangian subspace.