499 Stable isomorphisms of f.d. Hilb spaces.

By a stable isom between K and L we mean an isom $K \oplus W \xrightarrow{\sim} W \oplus L$ for some W. I think we can then define a scattering operator $g(z) : K \longrightarrow L$ ~~which~~ rational function z analytic for $|z| \leq 1$ and unitary valued for $|z| = 1$. But this does not determine W ~~unless~~ when there are bound states. We need to understand the algebra. There are lots of things to prove such as the existence of the scattering. Recall how done.

$$\oplus\, u^{-2}K \oplus u^{-1}K \oplus W \oplus L \oplus uL \oplus u^{2}L$$

$$\downarrow u$$

$$\oplus\, u^{-2}K \oplus u^{-1}K \oplus K \oplus W \oplus uL \oplus u^{2}L$$

So we ~~define~~ obtain Hilbert space rep of $C(S^1)$. Now can look for eigenvectors in the Gelfand sense. Ultimately you get out of this a coherent analytic sheaf over S^2. This you must be precise about, because you need a precise K-module. Concentrate on the sheaf F. The rank will be $\dim(K) = \dim(L)$; call this r. V should be $K \oplus W = W \oplus L$. Recall that if $0 \to \mathcal{O}(-1) \otimes W \to \mathcal{O} \otimes V \to F \to 0$, then $V = H^0(F)$ has ~~dim~~ $= \deg(F) + \operatorname{rank}(F)$ $\therefore$ $\dim(W) = \deg(F)$.

You see the coherent sheaf as follows. You see its Čech complex, namely, the above Hilb. space gives the sections over S^1. Then inside this are sections extending to either disk. These will be incoming + outgoing subspace, and they intersect to give $V = H^0(F)$.

 What can you hope for? You might be able to put metrics on Joyce's modules.

For the moment you need to be able to calculate. First show that bound states split off.

What happens if $W = 0$. Then we give an isomorphism of $K \xrightarrow{\sim} L$. Sheaf is $\mathcal{O}^r$ $r = \dim(K) = \dim(L)$ want $\mathbb{Z}/3$ to act. Need a stable isom. $\mathbb{C}^3 \oplus W \xrightarrow{\sim} W \oplus \mathbb{C}^3$

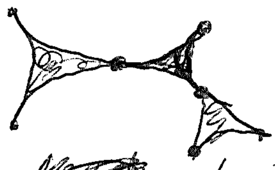
Will have $S(z) \in M_3(\mathbb{C})$ $\ni$ $S(z) \in U(3)$ for $|z| \in S^1$.

Jan 21. ~~[illegible]~~

try to understand whether you can get anything interesting on the $\Gamma = Sl_2(\mathbb{Z})$ tree, or whether any ~~[illegible]~~ Γ invariant circuit over the tree necessarily has ~~[illegible]~~ continuous spectrum. so let's ~~[illegible]~~ look at ~~the question~~ how ~~to~~ describe ~~[illegible]~~ something equivariant over the tree. ~~So consider [illegible] begin by looking at~~

begin where? finite LC circuit 2-port over the edges finite LC circuit 3-port for the vertices. Former must have $\mathbb{Z}/2$ symmetry, latter $\mathbb{Z}/3$ symmetry. First look at the real case (LC). So where do I start? The ~~[illegible]~~ method will be just as before to use the impedance function on the edges, except now you have to ~~understand how to con~~ find how to handle the vertices. So let us see what to do.

to get started let us connect identical 3-ports together to see what happens



first look at 2 ~~[illegible]~~ ports.

First you need to handle the group $\mathbb{Z}$ and maybe next $\mathbb{Z}/2 * \mathbb{Z}/2$ before proceeding to the general case. ~~Next go~~

~~[illegible]~~ You have description of a general 2-port, namely a positive def quadratic form on $\mathbb{R}^2$ depending ~~on~~ s in a specific rational way. ~~With luck it will be clear~~

501 So you have this tree acted on by $PSL_2(\mathbb{Z})$. $= \mathbb{Z}/2 * \mathbb{Z}/3$, and you want ~~to construct~~ some idea of what we can construct by surgery. Look at the simpler case of $\mathbb{Z}/2 * \mathbb{Z}/2$, or even $\mathbb{Z}$ which might be simpler.

Digress to make notes on Top. paper refereed by D. Salamon. Novikov Morse theory: manifold X with a "Morse function" $X \longrightarrow S^1$. In this situation Novikov, in analogy to Witten, constructed a chain complex out of critical point data, but the complex consists of f.g. free $\mathbb{Z}[t,t^{-1}]$-modules. Presumably the complex is supposed to calculate the homology of $\tilde{X} = X \times_{S^1} \mathbb{R}$. Choose on X a generic metric, let Σ be the inverse image of a generic point of S^1 say 0, and look at the gradient flow arising from the Morse function. Watch the trajectory of a point of Σ. If ~~one doesn't hit a~~ the trajectory doesn't get swallowed by a critical point, then it comes back to Σ. So you get a partially defined $f : \Sigma \longrightarrow \Sigma$ such that the fixpts of f^k for any k yield closed trajectories of the gradient flow. Apparently there's a zeta function arising from these fixpts; D. Fried (84) studied this for a nonsingular flow and transversal Σ. In the present paper a connection is made with R-torsion. One assumes the chain complex of finite free $\mathbb{Z}[t,t^{-1}]$-mods is acyclic, so then the torsion is defined and should be something like an element of $\mathbb{Z}[t,t^{-1}]$. The ~~...~~ main theorem I think says that the two natural ~~...~~ expressions for ζ differ multiplicatively by the torsion. Ideas in the proof include the Lefschetz Fixpt theorem, extending the partial operator f on homology somehow.

Return to tree acted on by $PSL_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$. Actually, take simpler tree, namely $\mathbb{R}$ acted on by $\mathbb{Z}$. ~~...~~ Each edge is a 2-port and there is ~~...~~ coupling at each vertex. Now I need to describe LC 2-ports, a general LC 2-port. Real polarized vector space V, $V = V^+ \oplus V^-$ equipped with a ~~...~~ quadratic form which is a rational function of s simple poles at $s = \pm i\omega$ non-negative residues etc. V^+, V^- same dimensions. Coupling should be an isomorphism ~~...~~ $V^+ \simeq V^-$. Idea: can you replace the quadratic form on $V = V^+ \oplus V^-$ by some sort of map $V^+ \longrightarrow V^-$ which can be combined with the coupling isom $V^+ \xrightarrow{\sim} V^-$ to be iterated ~~by the~~ ~~...~~ i.e. like a transfer matrix. ~~... can you replace~~

Simple puzzle: Quadratic form q_s on $V^+ \oplus V^-$ together with isomorphism $V^+ \simeq V^-$ yields operator $V^+ \xrightarrow{A} V^+$ depending on s, which can be iterated k times then trace taken. These traces can be assembled into $-\log\det(1-\lambda A) = \sum_{1}^{\infty} \frac{\lambda^k}{k} \operatorname{tr}(A^k)$

503 So what next?

Let's try to understand the coupling.

$q_s : V \longrightarrow V^*$ symmetric positive

$\Gamma_{q_s} \subset V \oplus V^*$ Lagrangian subspace

significance of splitting $V = V^+ \oplus V^-$

$V^* = V^{+*} \oplus V^{-*}$

have $\begin{cases} \theta : V^+ \longrightarrow V^- \\ \theta^t : V^{-*} \longrightarrow V^{+*} \end{cases}$. With appropriate sign $\Gamma_\theta + \Gamma_{\theta^t}$ might be Lagrang.

This is interesting. How do I explain coupling? Take $\oplus$ and pass to a quotient. You have the quadratic form on $V = V^+ \oplus V^-$ and coupling amounts to restriction to the graph of $\theta : V^+ \to V^-$. I'm thinking of V^+ as the potentials, so ~~...~~ you are setting voltages equal (up to θ). You still don't have a map $V^+ \to V^+$.

Would it be clearer if you ~~...~~ represented V as a ~~subquotient~~ of a polarized Hilbert space.

Consider then a quadratic form on $\mathbb{R}^2 = V^+ \oplus V^-$

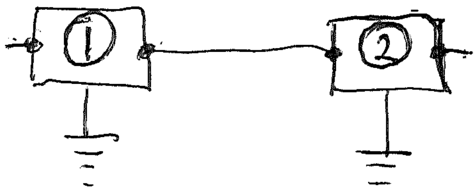
$$q_s = \sum_{\omega \in S} a_\omega \frac{s(1+\omega^2)}{s^2+\omega^2} \qquad a_\omega \geq 0 \quad \sum a_\omega > 0$$

A quadratic form has 3 coeff $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\begin{cases} a > 0 \\ ac - b^2 > 0. \end{cases}$

~~...~~ Suppose given $\theta : V^+ \longrightarrow V^-$. I propose to restrict q_s to the graph of θ. ~~...observe that~~ Maybe all that's important is that we have a line in $\mathbb{R}^2$ to restrict to. No – you want to ~~be able to~~ get an operator on V^+.

Problem: Given q_s on $\mathbb{R}^2 = V^+ \oplus V^-$

You haven't gotten anywhere. Try something simpler. Given V_1, V_2, V_3 and pos. def. q forms on $V_1 \oplus V_2$ and $V_2 \oplus V_3$ explain how to get a quadratic form on $V_1 \oplus V_3$. To avoid disconnected graphs, assume always connected with basept. Then we have 2 circuits.



I am thinking of series connections

(1) gives $(E_1, E_2) \in V_1 \oplus V_2 \xmapsto{q_s} (I_1, I_2) \in V_1^* \oplus V_2^*$

$$(E, q_s(E)) = \text{power} \quad E_1 I_1 + E_2 I_2$$

So what does coupling mean. We consider

(E_1, E_2, I_1, I_2) graph of $q_s^{(1)}$ $\qquad$ (E_3, E_4, I_3, I_4) graph of $q_s^{(2)}$ $\qquad$ 4 ind. variables

impose conditions $E_2 = E_3$, $I_2 = -I_3$

imposing $E_2 = E_3$ is restricting to a subspace of the voltage space, but then you ignore $E_2 = E_3$ which means passing to a ~~quotient~~ space.

So you have a subquotient of the direct sum.

Can you get this any better?? First of all ~~concentrate~~ on the geometry. You have two ~~circuits~~ networks and a common subquotient.

Puzzle What is the significance of a real v.s. V equipped with quad form $g_s(\sigma)$ rational in s ~~etc~~.
You would like to say that there is some Hilbert space around with a partial ~~[illegible]~~ self adjoint? op.
~~You need something again.~~ You are missing this picture. Try coupling positive quadratic forms on $\mathbb{R}^2$.

First point. Given a pos. def quad. form on $V_1 \oplus V_2$ and a pos. def quad form on $V_2 \oplus V_3$ there is an induced quadratic form on $V_1 \oplus V_3$, namely

$$\begin{tikzcd} \overset{x}{V_1} \oplus \overset{y}{V_2} \oplus \overset{z}{V_3} \arrow[r, hook, "1\times\Delta"] \arrow[d, two heads] & \underbrace{(V_1 \oplus V_2)}_{p} \oplus \underbrace{(V_2 \oplus V_3)}_{g} \\ V_1 \oplus V_3 & \end{tikzcd}$$

$$r(x,z) = \inf_{y \in V_2} \big(p(x,y) + g(y,z) \big)$$

$$p(x+y) = \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad g(y+z) = \begin{pmatrix} y \\ z \end{pmatrix}^t \begin{pmatrix} d & e \\ e & f \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$ax^2 + \underbrace{2bxy + cy^2 + dy^2 + 2eyz}_{(c+d)y^2 + (2bx + 2ez)y} + fz^2$$

$$(c+d)\left[y^2 + 2\left(\frac{bx+ez}{c+d}\right) y + \left(\frac{bx+ez}{c+d}\right)^2 \right] - \frac{(bx+ez)^2}{c+d}$$

$$ax^2 - \frac{b^2x^2 + 2be\,xz + e^2z^2}{c+d} + fz^2$$

$$\left(a - \frac{b^2}{c+d}\right)x^2 - 2\,\frac{be}{c+d}\,xz + \left(f - \frac{e^2}{c+d}\right)z^2$$

Is there some way to compose things, like replacing quadratic forms by scattering matrices.

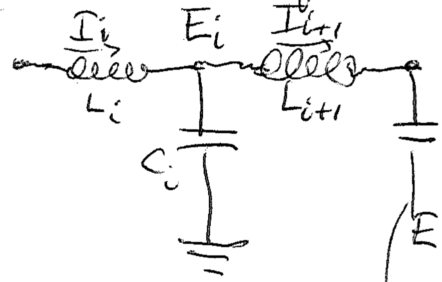
$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} d & e \\ e & f \end{pmatrix} \longmapsto \begin{pmatrix} a & b & 0 \\ b & c+d & e \\ 0 & e & f \end{pmatrix}$$

restrict to $y = -\frac{bx+ez}{c+d}$

$$\begin{pmatrix} 1 & -\frac{b}{c+d} & 0 \\ 0 & -\frac{e}{c+d} & 1 \end{pmatrix} \begin{pmatrix} a & b & \\ b & c+d & e \\ & e & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-b}{c+d} & \frac{-e}{c+d} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{b}{c+d} & 0 \\ 0 & -\frac{e}{c+d} & 1 \end{pmatrix} \begin{pmatrix} a - \frac{b^2}{c+d} & \frac{-be}{c+d} \\ \frac{-be}{c+d} & \frac{-e^2}{c+d} + f \end{pmatrix} = \begin{pmatrix} a - \frac{b^2}{c+d} & \frac{-be}{c+d} \\ \frac{-be}{c+d} & f - \frac{e^2}{c+d} \end{pmatrix}$$

There's a clear analogy with Jacobi matrices. You want to find something that composes nicely.

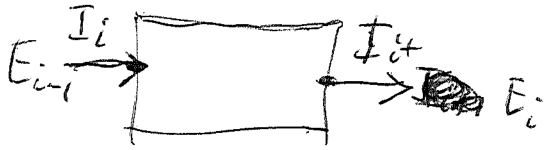


$C_i s E_i = I_i - I_{i+1}$ $\quad I_i = I_{i+1} + C_i s E_i$

$L_i s I_i = E_{i-1} - E_i$ $\quad E_{i-1} = E_i + L_i s_i I_i$

$$\begin{pmatrix} E_{i-1} \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & L_i s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_i \end{pmatrix} \qquad \begin{pmatrix} E_i \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_i s & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_{i+1} \end{pmatrix}$$

$$\begin{pmatrix} E_{i-1} \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & L_i s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_i s & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_{i+1} \end{pmatrix}$$



So maybe the point is not to look at $E_1, E_2 \longmapsto I_1, I_2$ but rather the IVP $E_1/I_1 \rightsquigarrow E_2/I_2$

508. ~~Symplectic space~~ Take a symplectic isom.

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \xrightarrow{\sim} \begin{matrix} W \\ \oplus \\ W^* \end{matrix}$ Look at its graph $\subset \begin{matrix} V \oplus W \\ \oplus \\ (V\oplus W)^* \end{matrix} = \begin{matrix} X \\ \oplus \\ X^* \end{matrix}$

~~Also~~ What does symplectic mean.

$$g^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$$

Look at Γ_g

$$\begin{matrix} W \\ \oplus \\ W^* \end{matrix} \xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \qquad \begin{pmatrix} 1 & 0 \\ a & b \\ 0 & 1 \\ c & d \end{pmatrix} \begin{pmatrix} V \\ \oplus \\ V^* \end{pmatrix} \subset \begin{pmatrix} V \\ \oplus \\ W \\ \oplus \\ V^* \\ \oplus \\ W^* \end{pmatrix}$$

$$\begin{pmatrix} 1 & a & 0 & c \\ 0 & b & 1 & d \end{pmatrix} \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & +1 & & \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & b \\ 0 & 1 \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & c \\ 0 & b & 1 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -c & -d \\ -1 & 0 \\ +a & b \end{pmatrix}$$

$$= \begin{pmatrix} -ac+ca & 1-ad+bc \\ -bc-1+da & -bd+db \end{pmatrix} = 0$$

Look at

$$\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b^{-1}a & b^{-1} \end{pmatrix} = \begin{pmatrix} -b^{-1}a & b^{-1} \\ c-db^{-1}a & db^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & a & 0 & c \\ 0 & b & -1 & d \end{pmatrix} \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & b \\ 0 & -1 \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & c \\ 0 & b & -1 & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ c & d \\ -1 & 0 \\ -a & -b \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & -1 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b^{-1}a & b^{-1} \end{pmatrix} = \begin{pmatrix} b^{-1}a & -b^{-1} \\ c-db^{-1}a & db^{-1} \end{pmatrix}$$

$-b^{-1}$

507 So I ask about the transfer matrix instead.

So given $\begin{pmatrix} I_1 \\ -I_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$ you solve for E_1, I_1 in terms of E_2, I_2

$$I_1 = aE_1 + bE_2 \qquad I_1 = ab^{-1}(-I_2 - cE_2) + bE_2$$

$$-I_2 = bE_1 + cE_2 \qquad E_1 = b^{-1}(-I_2 - cE_2)$$

$$I_1 = -\frac{a}{b}I_2 + \left(b - \frac{ac}{b}\right)E_2 \qquad \begin{pmatrix} -\frac{a}{b} & b - \frac{ac}{b} \\ -\frac{1}{b} & -\frac{c}{b} \end{pmatrix}$$

$$E_1 = -\frac{1}{b}I_2 - \frac{c}{b}E_2$$

$$\det = +\frac{ac}{b^2} + \left(\cancel{b^2} - \frac{ac}{b^2}\right) = 1,$$

as long as ~~[illegible]~~ the trans. coeff

Basically, what happening is that you have

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idea here: A quadratic form on $V_1 \oplus V_2$ is ~~[illegible]~~ a kind of Lagrangian subspace of $(V_1 \oplus V_2) \oplus (V_1 \oplus V_2)^*$, ~~and~~ and generically should be the graph of a ~~sf~~ symplectic transf. $V_1 \oplus V_1^* \longrightarrow V_2 \oplus V_2^*$

Jan 22

Complex version: ~~[illegible]~~ identify a ~~partial~~ unitary with a max isotropic subspace of $H^+ \oplus H^-$ wrt $\|h_+^+\|^2 - \|h_-^-\|^2$. Now suppose you have $H^\pm = H_1^\pm \oplus H_2^\pm$

So the ~~[illegible]~~ indef H. space H is the direct sum of H_1 and H_2. ~~[illegible]~~ The graph of a isometric isom. $T : H_1 \to H_2$ should be isotropic, except you should reverse signs on H_2

$$(H, \varepsilon) = (H_1, \varepsilon_1) \oplus (H_2, -\varepsilon_2) \qquad \text{consider } \begin{pmatrix} 1 \\ T \end{pmatrix} : H_1 \xrightarrow{\sim} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$$

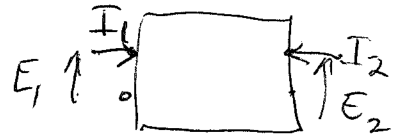
~~[illegible]~~ $$\begin{pmatrix} h_1 \\ Th_1 \end{pmatrix}^* \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{pmatrix} \begin{pmatrix} h_1 \\ Th_1 \end{pmatrix} = h_1^* \varepsilon_1 h_1 - (Th_1)^* \varepsilon_2 (Th_1)$$

509 To analyze 2-ports. First statement should concern the relation between $SL_2(\mathbb{R}) = Sp_2(\mathbb{R})$ and the space of pos. def quad forms on $\mathbb{R}^2$.- The same as harmonic oscillators with phase space $\mathbb{R}^2$. Both are 3 diml over $\mathbb{R}$ and connected. $SL_2(\mathbb{R}) \sim SO(1)$ whereas pos def q f are contractible. Try yesterday's notation:

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} \qquad \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$$

$$I_1 = cE_2 + d(-cE_1 + aI_1)$$

$$(\underbrace{1 - da}_{-bc})I_1 = -dcE_1 + cE_2 \qquad I_1 = +\frac{d}{b}E_1 \; \ldots \; \frac{1}{b}E_2$$



$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_2 \\ -I_2 \end{pmatrix} \qquad ad - bc = 1.$$

$$E_1 = aE_2 - bI_2 \qquad -bI_2 = E_1 - aE_2$$

$$I_1 = cE_2 - dI_2 \qquad I_2 = \left(-\frac{1}{b}\right)E_1 + \frac{a}{b}E_2$$

$$= cE_2 - d\left(-\frac{1}{b}E_1 + \frac{a}{b}E_2\right)$$

$$= \frac{d}{b}E_1 + \underbrace{\left(c - \frac{da}{b}\right)}_{\frac{cb-da}{b} = -\frac{1}{b}}E_2$$

$$\boxed{\begin{aligned} I_1 &= \frac{d}{b}E_1 + \left(-\frac{1}{b}\right)E_2 \\ I_2 &= \left(-\frac{1}{b}\right)E_1 + \frac{a}{b}E_2 \end{aligned}}$$

What does positive mean? $\frac{d}{b} > 0 \qquad \frac{ad-1}{b^2} = \frac{bc}{b^2} = \frac{c}{b} > 0$

$a = \frac{1}{d}(1 + bc) \qquad$ so if $\frac{c}{b} > 0$ then a, d same sign

510 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \longmapsto \begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \frac{d}{b} & -\frac{1}{b} \\ -\frac{1}{b} & \frac{a}{b} \end{pmatrix}$

Conversely assume

$$I_1 = AE_1 + BE_2$$
$$I_2 = BE_1 + CE_2$$

$BE_2 = -AE_1 + I_1$ $\qquad E_2 = -\frac{A}{B}E_1 + \frac{1}{B}I_1$

$I_2 = BE_1 + C\left(-\frac{A}{B}E_1 + \frac{1}{B}I_1\right)$ $\qquad I_2 = \frac{B^2-AC}{B}E_1 + \frac{C}{B}I_1$

$BE_1 = I_2 - CE_2$

$I_1 = AE_1 + B($

$= A\left(-\frac{C}{B}E_2 + \frac{1}{B}I_2\right) + BE_2$

$$\boxed{\begin{aligned} E_1 &= \left(-\frac{C}{B}\right)E_2 + \frac{1}{B}I_2 \\ I_1 &= \left(\frac{B^2-AC}{B}\right)E_2 + \left(\frac{A}{B}\right)I_2 \end{aligned}}$$

$\frac{AC}{B^2} + \frac{B^2-AC}{B^2} = 1$

$$\boxed{\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} -\frac{C}{B} & -\frac{1}{B} \\ \frac{B^2-AC}{B} & -\frac{A}{B} \end{pmatrix} \begin{pmatrix} E_2 \\ -I_2 \end{pmatrix}}$$

$$\boxed{\begin{aligned} \begin{pmatrix} A & B \\ B & C \end{pmatrix} &= \begin{pmatrix} \frac{d}{b} & -\frac{1}{b} \\ -\frac{1}{b} & \frac{a}{b} \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} -\frac{C}{B} & -\frac{1}{B} \\ \frac{B^2-AC}{B} & -\frac{A}{B} \end{pmatrix} \end{aligned}}$$

what role does pos. play.

$$\begin{pmatrix} E_1 \\ -I_1 \end{pmatrix} = \begin{pmatrix} -\frac{C}{B} & \frac{1}{B} \\ \frac{AC-B^2}{B} & -\frac{A}{B} \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}$$

So what is the program. In general ~~terms~~ you want to relate the Sp and the indef. U-theories. You want to recover the Hilbert space.

~~[illegible]~~ Principle: The response of a port does not see the bound states.

You should try to classify 2-ports. Suppose you start with $V = V_1 \oplus V_2$ equipped with Q_s. $Q_s = \left(\begin{array}{c|c} A & B \\ \hline B & C \end{array}\right)$. Assuming $B \neq 0$, one can replace Q_s by a symplectic transformation depending on s. What you want to do is to factor this symplectic transf. into simpler ones. Is this possible?

You think that a 1-port gives rise to a coherent sheaf of rank 1 over the Riemann sphere, so a 2 port should give rise to a coherent sheaf of rank 2, i.e. a rank 2 vector bundle which has the form $\mathcal{O}(k) \oplus \mathcal{O}(l)$ together with a torsion sheaf — this should be the bound state part. Anyway what happens?

Let's analyze ranke 1. Let's go over the general case V/W subquotient of $H = H^+ \oplus H^-$. On H consider q.f. $s\|h^+\|^2 + s^{-1}\|h^-\|^2$ and descend to the quotient space H/W.

What do you have to get straight first. Given a nondeg q.f. on a vector space how it descends to a subspace and ~~or~~ quotient space.

Let $W \subset V$ let $V \xrightarrow[\sim]{q} V^*$ be a quadratic form whose ~~restriction~~ q^W to W is n.d.

$$\begin{tikzcd} W \arrow[r, hook, "\iota"] \arrow[d, "\wr"'] & V \arrow[r, two heads, "p"] \arrow[d, "\wr", "q"'] & V/W \\ W & V^* \arrow[l, "\iota^t"'] & (V/W)^* \arrow[l, hook', "p^t"] \end{tikzcd}$$

$q^W = \iota^t q \iota$

Claim: 1) q induces a n.d. quad form on V/W: $q^{V/W} : V/W \xrightarrow{\sim} (V/W)^*$. Specifically $w \mapsto q^V(v+w)$ has a unique stationary point w_0 and $q^{V/W}(v+W) = q^V(v+w_0)$. $q^{V/W}(v+W)$ is the stationary value of q^V on the coset $v+W$.

2) $(q^{V/W})^{-1} = p q^{-1} p^t$

Do I need to amplify 1) further? What else might I say? Splitting the filtration. Examine $W^\perp$ wrt q. $W^\perp = \{v \mid \iota^t q(v) = 0\} = q^{-1}\,\mathrm{Im}(p^t)$

~~Start with W~~ bilinear form $\langle v_1, q v_2\rangle$

$$\begin{tikzcd} W \arrow[r, hook, "i"] \arrow[d, "\wr"'] & V \arrow[d, "q"] & W^\perp \arrow[l, hook'] \arrow[d] \\ W^* & V^* \arrow[l, "i^*"'] & (V/W)^* \arrow[l, hook'] \end{tikzcd}$$

$i^* q i$

Define $W^\perp = \{v \mid \langle W, qv\rangle = 0\}$

Claim $W^\perp$ comp to W in V.

$W \subset V$, $q : V \to V^*$ bilinear form $\langle v_1, q v_2\rangle$ assume q restricted to W is n.d. – this means $\iota^t q \iota : W \hookrightarrow V \xrightarrow{q} V^* \twoheadrightarrow W^*$ is an isom.

then $W \oplus W^\perp = V$ $\qquad W^\perp = \{v \in V \mid \langle W, qv\rangle = 0\}$

Let $w \in W \cap W^\perp$ $\quad \langle w_1, qw\rangle = 0 \quad \forall w_1$

$\Rightarrow w \in \mathrm{Ker}\, \iota^t q \iota \Rightarrow w = 0$. Given $v \in V$ $\exists!$ w such that $\langle w_1, qw\rangle = \langle w_1, qv\rangle \quad \forall w_1$

$\Rightarrow v - w \in W^\perp \quad \therefore \quad v \in W + W^\perp$.

next we have $qW^\perp \subset (V/W)^* = \operatorname{Ker} \iota^t$

$$W^\perp = q^{-1}(W^\circ). \quad \therefore \; q(W^\perp) \subset W^\circ.$$

$$\cancel{W} = W \oplus W^\perp$$

First $W^\perp$ is a complement to W

2nd qW is ———— to $W^\circ = (V/W)^*$

$$\begin{tikzcd} W \arrow[r, hook, "i"] \arrow[d, "s"', "\iota^t q i"'] & V \arrow[d, "q"] & W^\perp \arrow[l, hook'] \arrow[d, "q"] \\ W^* \arrow[r, hook, "i'"] & V^* & W^\circ \arrow[l, hook'] \end{tikzcd}$$

OKAY, what do we find. $\langle v_1, qv_2 \rangle$ bilinear form on V non-degenerate when rest, to W. Then $V = W \oplus W^\perp$ in fact $(V, q) = (W, q^W) \oplus (W^\perp, q^{W^\perp})$ so q is non-deg $\iff q^{W^\perp}$ is non-deg. So what am I learning?

Take our subquotient $0 \subset W \subset V \subset H$

What happens is that you can do everything relative to $s = 1$. In any case you have a splitting of this filt. $H = W \oplus (V \ominus W) \oplus (H \ominus V)$ this splitting depends on s, but it can be referred to $s = 1$. Really clean

You have on $H = H^+ \oplus H^-$ the quadratic form $s\|h^+\|^2 + s^{-1}\|h^-\|^2$, so what do you do? Gauss ortho-normalization? So how can I analyze all this??

~~You start~~ You go over the idea of the ~~oth~~

~~[illegible]~~. What sort of things do you want ~~[illegible]~~ to accomplish? Clean up the result describing the quadratic form on V/W. Recover H from V/W and q_s.

Recap. $Q(v_1, v_2)$ ~~[illegible]~~ bilinear form on V

$W^\perp = \{ v \mid Q(w, v) = 0 \ \forall w \in W \}$. Assume $Q(w_1, w_2)$ non-degenerate i.e. $\forall \lambda \in W^* \ \exists!\, w$ such that

$$\langle w_1, \lambda \rangle = Q(w_1, w) \quad \forall w_1 \in W.$$

$$\langle w_1, Qw \rangle \qquad\qquad Q : W \xrightarrow{\sim} W^*.$$

Then $W \oplus W^\perp \xrightarrow{\sim} V$, why?

~~$W^\perp \to V \quad W$~~

~~$0 \to (V/W)^* \to V^* \to W^* \to 0$~~

If $w \in W \cap W^\perp$, then $Q(w_1, w) = 0 \quad \forall w_1$

$$\implies w = 0.$$

Given $v \in V$ have $w_1 \mapsto Q(w_1, v) \in W^*$ know $\exists!$ $w \ni Q(w_1, v - w) = 0 \quad \forall w_1 \implies \forall v \ \exists!\, w \ni v - w \in W^\perp$. Then $Q(w_1 + z_1, w_2 + z_2)$

$$= Q(w_1, w_2) + Q(z_1, z_2) + \cancel{Q(w_1, z_2)} + Q(z_1, w_2)$$

as $z_2 \in W^\perp$

5/5 $W \subset V$ $Q(v_1, v_2)$ bilinear on V

s.t. $Q(w_1, w_2)$ nondeg on W i.e.

$w \mapsto Q(-, w)$ is an isom $W \longrightarrow W^*$.

Set $W^\perp = \{ v \in V \mid Q(W, v) = 0 \}$

Claim $W \oplus W^\perp = V$: ~~$W \cap W^\perp = 0$~~

$w \in W \cap W^\perp \implies Q(W, w) = 0$

$\implies w = 0$ since $Q(-, w)$ on W determines w

next given v have $Q(-, v)$ on W, know $\exists w$

$Q(-, v) = Q(-, w) \implies Q(-, v - w) = 0 \implies v - w \in W^\perp$

$\implies v \in W + W^\perp$.

Critical question: Is $Q(W^\perp, W) = 0$?

$$V = W \oplus W'$$

$$\downarrow \text{ triangular}$$

$$V^* = W^* \oplus W'^*$$

$$Q\left(\begin{pmatrix} x \\ x' \end{pmatrix}, \begin{pmatrix} y \\ y' \end{pmatrix}\right) = \begin{pmatrix} x \\ x' \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} = xy + x'y + x'y'$$

Let $W = \left\{ \begin{pmatrix} * \\ 0 \end{pmatrix} \right\}$

$$Q\left(W, \begin{pmatrix} y \\ y' \end{pmatrix}\right) = Q\left(\begin{pmatrix} * \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ y' \end{pmatrix}\right) = * y = 0 \quad \forall x \iff y = 0$$

$$\therefore \quad W^\perp = \left\{ \begin{pmatrix} 0 \\ * \end{pmatrix} \right\}.$$

$$\underset{W^\perp}{\begin{pmatrix} 0 \\ x' \end{pmatrix}}, \ \underset{W}{\begin{pmatrix} y \\ 0 \end{pmatrix}}: \quad Q\left(\begin{pmatrix} 0 \\ x' \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}\right) = x'y \neq 0.$$

5/6 Statement: $W \subset V$, Q (skew-) symmetric bilinear ~~quadratic~~ form on V whose restriction to W is n.d. Then $W^\perp = \{v \mid Q(w,v)=0,\ \forall w\}$ is a complement to W, i.e. $V = W \oplus W^\perp$ and $Q = Q_W \oplus Q_{W^\perp}$.

Moreover Q is non-deg $\Leftrightarrow$ $Q_{W^\perp}$ is n.d.

What else do you need? Let's take $Q_s(h) = s\|h^+\|^2 + s^{-1}\|h^-\|^2$ on $H = H^+ \oplus H^-$

Start with $W \subset H$, you aim to compute Q_s induced on H/W.

First the restriction to W.

Jence of a canonical splitting $H = \bigoplus_{\omega \in S} H_\omega$ compatible with

$H = H^+ \oplus H^-$

$H = W \oplus W^\perp$

such that if $V_\omega = H_\omega \cap V$ $0 < \omega < \infty$, then

V_ω = graph of ωu, $u : H_\omega^+ \xrightarrow{\sim} H_\omega^-$ orth isom

$g=1$ $F = \varepsilon = +1$ or -1.

$V_0 = H_0^+$ | $(V^\perp)_0 = H_0^-$

$V_\infty = H_\infty^-$ | $(V^\perp)_\infty = H_\infty^+$

I guess I should also consider $V^\perp$ since I need both. Go back to F

Maybe you can play the game with F, ε

517 Consider alternate approaches.

Go back to passing a q. form to a ~~quotient~~ space. Look at this on a Euclidean space

You have V Euclidean with $(\ ,\ )$ scalar product and $Q(v_1, v_2) = (v_1, Av_2)$ A symmetric > 0. Next have subspace $W \subset V$, and you want induced form on $W, W^\perp$

$$V = W \oplus W^\perp \qquad \text{orth for } (\ ,\ ).$$

$$i : W \hookrightarrow V \qquad \text{get } i^*Ai \text{ rep. } Q_W.$$

Next take $z \in W^\perp$ and choose w to minimize $(z+w, A(z+w))$.

$$\delta(z+w, A(z+w)) =$$

$$(\delta w, A(z+w)) + (z+w, A\delta w) = 2(\delta w, A(z+w))$$

i.e. $A(z+w) \in W^\perp$ or $\boxed{z+w \in A^{-1}W^\perp}$

Then you want the values of $(-, A-)$ on this element $z+w$.

$$(z+w, A(z+w)) = (z, A(z+w))$$

$$\begin{tikzcd}
W \arrow[r, hook, "i"] \arrow[d, "i^*Ai"'] & V \arrow[r, two heads, "j^*"] \arrow[d, "A"] & W^\perp \\
W^* & V^* \arrow[l, two heads, "i^*"'] & W^\perp \arrow[l, hook', "j"'] \arrow[u, "j^*A^{-1}j"']
\end{tikzcd}$$

can I prove that $(j^*A^{-1}j)^{-1}$ is the quadratic form induced on V/W.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & W^{\perp} \arrow[d, "j"] \arrow[dr, "j^*A^{-1}j"] & & \\
0 \arrow[r] & W \arrow[r, "Ai"] \arrow[dr, "i^*Ai"'] & V \arrow[r, "j^*A^{-1}"] \arrow[d, "i^*"] & W^{\perp} \arrow[r] & 0 \\
 & & W \arrow[d] & ?? & \\
 & & 0 & \text{etc.} &
\end{tikzcd}
$$

You have q.f (v, Av) on V to be pushed down to $V/W \simeq W^{\perp}$. $\min\limits_{v \in z+W} (v, Av)$. Let v_0 be the stationary point. Then $(W, Av_0) = 0$ $\quad Av_0 \in W^{\perp}$
so $v_0 \in A^{-1}W^{\perp}$.

Go back to $v + w_0 \mapsto v + W$

$$
\begin{tikzcd}
W \arrow[r, hook, "\iota"] \arrow[d, "S"', "\iota^t A \iota"'] & V \arrow[r] \arrow[d, "A"] & V/W \arrow[d] \\
W^* & V^* \arrow[l, "\iota^t"'] & (V/W)^* \arrow[l, hook'] \\
 & & W^0 \arrow[u, equal]
\end{tikzcd}
$$

sends $v + W$ to $A(v+w_0)$ where $v + w_0$ unique in $A^{-1}(W^0)$

you take $v + W \in V/W$ choose $v + w_0$ to minimize the q.f on this cosets so $\langle \iota W, A(v+w_0) \rangle = 0$
$\iota^t A(v+w_0) = 0$, so $\quad A(v+w_0) \in (V/W)^*$

So it seems you are using $\quad V = W \oplus A^{-1}(W^0)$

$$v = w_0 + A^{-1}(A(v+w_0))$$

519 So the result says that given $Q(v) = (v, Av)$ on V
then ~~restriction~~ of $Q(v)$ to W is (w, i^*Aiw)
and push forward of $Q(v)$ to $W^\perp$ is $(j^*(A^{-1})j)^{-1}$.

$$(z + w_0, A(z+w_0))$$

min means exactly that $\underbrace{A(z+w_0)}_{z'} \in W^\perp$

$(z, z') = ?$ where $z' = A(z+w_0)$ & $z' \in W^\perp$

then $z + w_0 = A^{-1}j(z')$

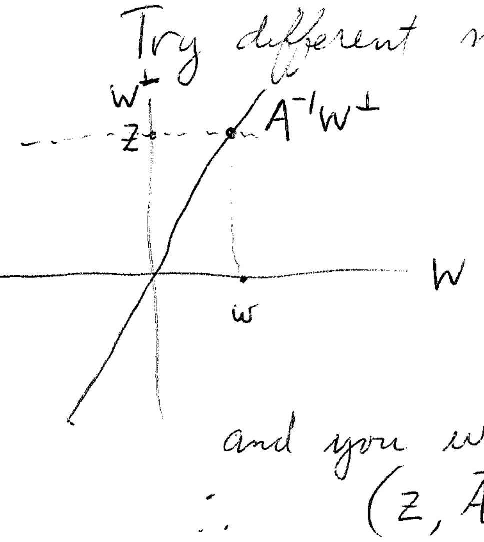
and $z = (j^*A^{-1}j)(z')$.

i.e. $\boxed{z' = (j^*A^{-1}j)^{-1} z}$.

Jan 23 Situation: V Euclidean space with pos. def. quadratic form (v, Av), W subspace. Have $V/W \simeq W^\perp$. What is the orthogonal complement of W for bilinear for $B(v_1, v_2) = (v_1, Av_2)$? Clearly it is $A^{-1}(W^\perp)$ So we have $V = W \oplus A^{-1}(W^\perp)$.
To compute the ~~induced~~ push forward quadratic form For $p: V \to V/W$ which we can identify with proj $j^*: V \to W^\perp$, $j: W^\perp \to V$ inclusion. Answer: given $z \in W^\perp$ you write $z = -w + A^{-1}(z') \in W \oplus A^{-1}(W^\perp)$ and take the values $(A^{-1}z', A(A^{-1}(z'))) = (A^{-1}z', z')$

Thus we have

Try different notation



$z + W$
$\ni$
$z + w = A^{-1}(z')$

want ~~$z \mapsto (z', Az')$~~

$z \mapsto (A^{-1}z', AA^{-1}z') = (z', A^{-1}z')$

and you write this $(z, \bar{A}z)$ get in trouble with

$\therefore (z, \bar{A}z) = (z', A^{-1}z')$?

$$(z', A^{-1}z') = (z', z+w) = (z', z)$$

$\therefore$ $\bar{A}z = z'$ and $j^*A_j^{-1}(\bar{A}z)$

$= j^*A^{-1}(z') = j^*(z+w) = z$

So it works, but it isn't clean yet.

Summarize: Problem of quad. forms versus s.a. operators

Look abstractly. General problem — Given V with two positive quadratic forms $(-,-)$ and $(-,A-)$. Then you are given a filtration $0 \subset F_1 \subset \cdots \subset F_k = V$. Each quadratic form gives a splitting. ~~There's a general~~ The A orthogonal splitting can be expressed in terms of the Euclidean one by a ~~general~~ triangular matrix with positive diagonal entries. ~~What might be useful~~

~~What is it~~

What might be useful. A itself gives rise to a filtration by increasing λ E_λ projections from the spectral theorem. $A = \int \lambda \, dE_\lambda$. And you have the joint (Schreier) filtration combining ~~of~~ the two filtrations (F_p) and $(E_\lambda V)$.

Concentrate on cleaning this up as much as possible. The goal should be a dilation statement (something like GNS) for subquotients of a polarized Hilbert space. Key idea: Q quadratic form on V nondeg on W yields a quadratic form on V/W as follows V is the direct sum of W and W^0

IDEA non symmetric bilinear forms occur naturally with Ext = RHom on K^0_* Bondal, tilting autos of derived category

A quadratic form Q on V which is non degenerate on a subspace W yields a splitting $V = W \oplus W^\circ$ orthogonal wrt Q, hence a quadratic form Q'' on $V/W \simeq W^\circ$.

$Q''(v + W)$ = stationary value of $Q(v+w)$ for $w \in W$.

(So you have two descriptions of the induced form on the quotient space.

Do the above for symm. or skew-symm. bil. forms.

$B(v_1, v_2)$. $\qquad {}^{\perp}W = \{v \in V \mid B(w, v) = 0\}$.

$$\begin{array}{ccccc} W & \xrightarrow{i} & V & \xrightarrow{k} & W^* \\ w & \longmapsto & w & \longmapsto & B(\end{array}$$

ki is inv.

$\therefore V = W \oplus {}^{\perp}W$

$$B(w_1 + x_1, w_2 + x_2) = B(w_1, w_2) + B(x_1, x_2)$$

How can you describe B'' on V/W.

Given v_1, v_2 write $v_i = w_i + x_i$ and

examine $B(v_1, v_2) = B(w_1, w_2) + B(x_1, x_2)$

this is a (quad) function of (w_1, w_2)

$$\delta\, B(w_1, w_2) = B(\delta w_1, w_2) + \underbrace{B(w_1, \delta w_2)}$$

vanishes for all $\delta w_1 \Leftrightarrow w_2 = 0$. $\qquad$ vanishes for all $\delta w_2 \Leftrightarrow w_1 = 0$

~~Now suppose~~

$$\begin{array}{ccccc} W & \xrightarrow{i} & V & & W^{\perp} \\ & & \big\downarrow A & & \\ W^* & \twoheadleftarrow & V^* & & \end{array}$$

$$\begin{tikzcd} W \arrow[r, hook, "i"] & V \arrow[r, two heads, "p"] \arrow[d, shift right, "B"'] & V/W \\ W^* & V^* \arrow[l, two heads, "i^t"'] \arrow[u, shift right, "B^{-1}"'] & (V/W)^* \arrow[l, hook', "p^t"'] \end{tikzcd}$$

$$(V/W)^* = W^\circ$$

$$B^{-1}(W^\circ) = W^\perp$$

$$\begin{array}{ccc} V & = & W \oplus B^{-1}W^\circ \\ \downarrow & & \downarrow B \\ V^* & = & BW \oplus W^\circ \end{array}$$

Let's leave and move to polarizations.
But first I need a clean statement.

V with $(\ ,\ )$ $\qquad B(v_1, v_2) = (v_1, Av_2)$ $\quad A$ symm.

$W \overset{i}{\subset} V \quad \Rightarrow \quad i^*Ai : W \to W$ invertible

Claim that the induced quadratic form on V/W is $(j^*A^{-1}j)^{-1}$ $\qquad W^\perp \xrightarrow{j} V \xrightarrow{A^{-1}} V \xrightarrow{j^*} W^\perp$

~~…~~ The real assertion I need is that given V with $(\ ,\ )$, A s.a. >0, W then the s.a. of induced on V/W is $(j^*A^{-1}j)^{-1}$ where $j : W^\perp \hookrightarrow V$. Maybe better is i^*Ai for restriction and $(p A^{-1}p^*)^{-1}$ for a projection.

$$W \xrightarrow{i} V \xrightarrow{p} W^\perp$$

$$i^*Ai \qquad\qquad (pA^{-1}p^*)^{-1}$$

$$\boxed{i^*(A) \qquad\qquad p_*(A).}$$

Can we define

f

523 ~~So we have a coherent sheaf.~~

Consider $H = H^+ \oplus H^-$ and a subquotient V/W of H.

$$\begin{tikzcd} V \arrow[r, hook, "i"] \arrow[d, "p"] & H \arrow[d, "p'"] \\ V \cap W^\perp \arrow[r, hook, "i'"] & W^\perp \end{tikzcd}$$

$A_s = s\pi_+ + s^{-1}\pi_-$

$i^* A_s i = s(i^*\pi_+ i) + s^{-1}(i^*\pi_- i)$

add to $\mathbb{1}$

hence they commute can be simultaneously diagonalize.

Claim $\boxed{p_*(i^*(A_s)) = i'^*(p'_*(A_s))}$

$\therefore \; V = \bigoplus_{0\le\omega\le\infty} V_\omega$ orth direct sum

where
$$i^*\pi_+ i = \bigoplus_\omega \frac{1}{1+\omega^2}\pi_\omega$$

$$i^*\pi_+ i = \bigoplus_\omega \frac{\omega^2}{1+\omega^2}\pi_\omega$$

$$i^*(A_s) = i^* A_s i = \bigoplus_\omega \frac{s+s^{-1}\omega^2}{1+\omega^2}\pi_\omega$$

$$p_*(i^*(A_s)) = \left(p\, i^*(A_s)^{-1} p^*\right)^{-1}$$

$$\boxed{p_*(i^*(A_s)) = \left(\sum_\omega \frac{s(1+\omega^2)}{s^2+\omega^2} p\pi_\omega p^*\right)^{-1}}$$

$$p'_*(A_s) = \left(p'(s\pi_+ \oplus s^{-1}\pi_-)^{-1} p'^*\right)^{-1}$$
$$= \left(p'(s^{-1}\pi_+ + s\pi_-)p'^*\right)^{-1}$$
$$= \left(s^{-1}(p'\pi_+ p'^*) + s(p'\pi_- p'^*)\right)^{-1}$$

sum add to $\mathbb{1}$.

524 Recovery. Given $\sum_\omega \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$

with $a_\omega \geq 0$. and $\sum a_\omega = 1$. Each a_ω is a s.a. operator $0 \leq a_\omega \leq 1$ so can be dilated ~~from~~ to a projection in an essentially unique way.

a_ω $\qquad$ $-1 \leq \overbrace{2a-1}^{\alpha} \leq 1$

$1-a_\omega$

$$\begin{pmatrix} \frac{1+\alpha}{2} & \sqrt{1-\alpha^2} \\ \sqrt{1-\alpha^2} & \frac{1-\alpha}{2} \end{pmatrix} = \begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix}$$

$\sqrt{1-(2a-1)^2}$

$= \sqrt{-4a^2+4a} =$

What are you doing. You have ~~$a^*a \geq 0$~~ $a = a^*$

$0 \leq a \leq 1$ i.e. $a^2 \leq 1$ and you propose to ~~write~~

~~find $\sum a_\omega = 1$~~ Wait. You hav

Start again with $a_\omega \geq 0$ $\sum a_\omega = 1$. on Z

You want then to find maps $\iota_\omega : Z \to V_\omega$

such that $\iota_\omega^* \iota_\omega = a_\omega$. Can assume $Z \twoheadrightarrow V_\omega$

So you let V_ω = completion of Z ~~with~~ wrt $(z, a_\omega z)$

and ι_ω the canon map. Then

$$(\iota_\omega z_1, \iota_\omega z_2) = (z_1, a_\omega z_2)$$
$$\parallel$$
$$(z_1, \iota_\omega^* \iota_\omega z_2)$$

$\therefore$ $a_\omega = \iota_\omega^* \iota_\omega$. Put together $Z \xrightarrow{\iota = (\iota_\omega)} \bigoplus_\omega V_\omega$

$$\begin{pmatrix} \iota_\omega^* & \cdots \end{pmatrix} \begin{pmatrix} \iota_\omega \\ \vdots \\ \vdots \end{pmatrix} = \sum \underbrace{\iota_\omega^* \iota_\omega}_{\iota^* \pi_\omega \iota} = \sum a_\omega = 1.$$

$$\iota = \sum \pi_\omega \iota = \sum \iota_\omega$$

$$\iota_\omega^* \iota_{\omega'} = \iota^* \pi_\omega \pi_{\omega'} \iota = \delta_{\omega\omega'} \iota^* \pi_\omega \iota$$

5.25 This seems to work well.

I would like a connection with ~~[illegible]~~ S^2 if possible. ~~You have a rational m~~

You would like to construct H directly from $\sum_{0\le\omega\le\infty} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$ $\quad a_\omega \geq 0, \sum a_\omega = 1$

how can the sum of the residues be 1

$$\frac{s(1+\omega^2)}{s^2+\omega^2} = \frac{1+\omega^2}{2}\left(\frac{1}{s+i\omega} + \frac{1}{s-i\omega}\right)$$

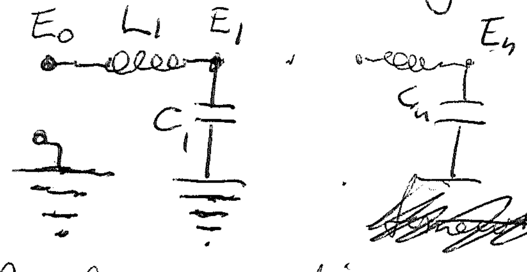
$$(1+\omega^2)\, a_\omega = \text{Res}\, \frac{s(1+\omega^2)}{s^2+\omega^2} \quad \text{in finite plane}$$

~~It would~~ Question: Whether you can directly recover H from ~~these~~ V/W and the quadratic form. Looks miraculous, but the GNS philosophy says it ought to work. So what do we hope for? Is there any link to K-modules.

Take a line $L = \mathbb{R}$ with $f(s) = \sum \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$
$a_\omega > 0, \ \sum a_\omega = 1$

~~Then~~ Suppose there are n-ω's $\omega_1 < \cdots < \omega_n$, say $0, \infty$ not included to simplify. So f has $2n$ poles, its denominator should be $\prod_{j=1}^{n}(s^2+\omega_j^2)$ and ~~numerator~~ numerator has ~~should have~~ degree $2n-1$. Thus $s=\infty$ is a zero and there ~~are~~ should be $2n-1$ roots of the numerator. Now each $a_\omega > 0$ gives rise to a line L_ω, so $\bigoplus L_\omega$ has dim n and then H will have dim $2n$ since each L_ω sits inside $H^+_\omega \oplus H^-_\omega$ as a gr[illegible]. The idea now is maybe the fact that?

Wait. There should be a ~~self~~ partial adjoint operator with deficiency indices (1,1) in this situation somewhere. Now the ω's we have $\pm\omega_1, \dots, \pm\omega_n$ ~~are associated to an operator A~~ belong to the modes where ~~one~~ the boundary condition $f(s) = \infty$ is choose | ~~≠~~0 voltage but 0 current. Try model



$$Z_0(s) = L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{Z_1(s)}}$$

~~Somewhere here.~~

~~I have a picture here of an n-port~~

This graph has $n+2$ vertices and $2n$ edges

The voltage space has dim $n+1$

$$n+2-2n = 1-(n-1) \checkmark$$

$e = 2n \qquad v = n+2 \qquad \ell = n-1$

~~So we have nothing to do~~ The basic picture here should involve circuits connected together. So how to proceed?? Actually we know that for each s, there is a line in (E_0, I_1) space given by.

$$\begin{pmatrix} E_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ C_1 s & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & L_n s \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ C_n s & 1 \end{pmatrix} \begin{pmatrix} E_n \\ I_{n+1} \end{pmatrix} \qquad I_{n+1} = 0$$

You have to find a mechanism to explain things. Lets start again.

So ~~what~~ where do I begin?

527 Given a rational function $f(s) = \sum \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$ with $a_\omega \geq 0$, $\sum a_\omega = 1$, you want to find a partial self-adjoint operator. What is the dim of the space of $f(s)$. Suppose $0 < \omega_1 < \cdots < \omega_n < \infty$ then $a_1, \dots, a_n > 0$ and $\sum a_i = 1$. Looks like $2n-1$ constants. $\left(\begin{matrix} E_0(s) \text{ has degree } 2n \\ I_1(s) \text{ ——— } 2n-1. \end{matrix}\right.$

What are you trying to do? I remember vaguely that the real theory may use singularities at both $0, \infty$. What is the basic picture?

You might try ~~...~~ simplex case

The chain of $SL_2(\mathbb{R})$ matrices should be equivalent to ~~...~~ a coupling of q. forms - graphs has vertices $E_0, \dots, E_n$ in addition to ground and currents $I_1, \dots, I_n$ and $2n$ edges. Wait $v = n+2$, $e = 2n$

$1 - l = v - e = n+2 - 2n = 2-n$ $\qquad l = 1+n-2 = n-1.$

$$\begin{array}{ccc} n+1 & & 2n \\ \bar{C}^0 & \subset & C^1 \\ \big\downarrow {\scriptstyle E_0} & & \\ \mathbb{R} & & \end{array}$$

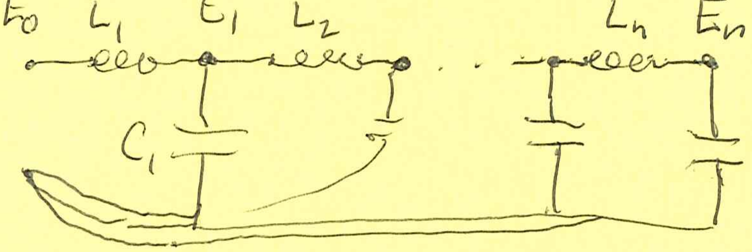
~~Look~~ the fundamental problem is to reconstruct the polarized Hilbert space C^1 in some form. This requires

$$\begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix} = \begin{pmatrix} s & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ & 1 \end{pmatrix} \begin{pmatrix} s & \\ & 1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 0 \\ C_1 s & 1 \end{pmatrix} = \begin{pmatrix} s & \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \\ C_1 & 1 \end{pmatrix} \begin{pmatrix} s & \\ & 1 \end{pmatrix}$$

528 ~~[illegible]~~ Jan 24

Problem: ~~Which~~ Partial s.a. operator to explain response from an LC circuit. Take circuit.



$n+2$ vertices include ground

$2n$ edges

$$\begin{pmatrix} I_1 \\ E_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ L_1 s & 1 \end{pmatrix} \begin{pmatrix} 1 & C_1 s \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & C_n s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n+1} = 0 \\ E_n = 1 \end{pmatrix}$$

e.g. $n=1$

$$\begin{pmatrix} I_1 \\ E_0 \end{pmatrix} = \begin{pmatrix} 1 & \\ L_1 s & 1 \end{pmatrix} \begin{pmatrix} C_1 s \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 s \\ 1 + L_1 C_1 s^2 \end{pmatrix}$$

response function has form $f(s) = \sum \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$ $\quad a_\omega \geq 0$, $\sum a_\omega > 0$

with n ω's (say $0, \infty$ excluded).

$$\frac{C_1 s}{1 + L_1 C_1 s^2} = \frac{s L_1^{-1}}{s^2 + (L_1 C_1)^{-1}} \qquad \omega^2 = \frac{1}{L_1 C_1} \qquad a_\omega = \frac{L_1^{-1}}{1 + (L_1 C_1)^{-1}}$$

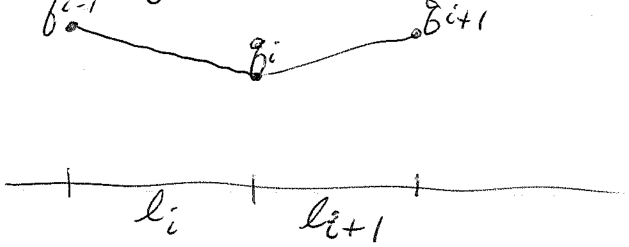
~~I think there's a pos Jacobi matrix~~ around here. ~~On an $n+2$ dim'l space.~~

My idea is that the rational function $f(s)$ has degree $2n$ so corresponds to a section of $\mathcal{O}(2n)$. Hence we should be looking for a K-module of dims $2n$, $2n+1$. There is a ~~[illegible]~~ Jacobi matrix around on $\mathbb{R}^n$ roughly; — this gives: $2n+1$ parameters.

529 It seems that $O(2n)$ is the wrong to consider. Maybe exploit the fact that $f(-s) = -f(s)$. Take quotient of s plane by $s \mapsto -s$. Fixpts: $-s = s$, $2s = 0$, $s = 0, \infty$. quotient is s^2-plane, then natural function f is equivalent to a section of $O(n)$, means looking for K-modules of dims $n, n+1$. Significance of the fact that the ~~K module~~ roots $s^2 = -\omega^2$ are on $\mathbb{R}_{\leq 0}$?

$$\mu_i = \frac{q_{i-1} - q_i}{l_i}$$

string formulas



$$m_i s^2 q_i = -\frac{q_i - q_{i-1}}{l_i} + \frac{q_{i+1} - q_i}{l_{i+1}}$$

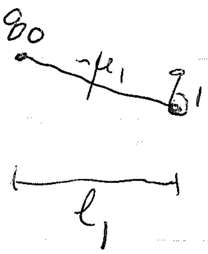
$$m_i s^2 q_i = \mu_i - \mu_{i+1}$$

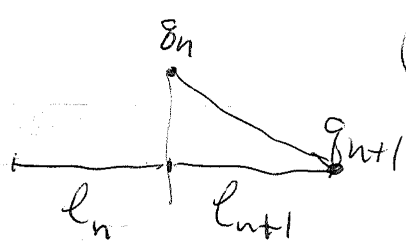
recursion relation
$$q_{i-1} = q_i + l_i \mu_i$$
$$\mu_i = \mu_{i+1} + m_i s^2 q_i$$

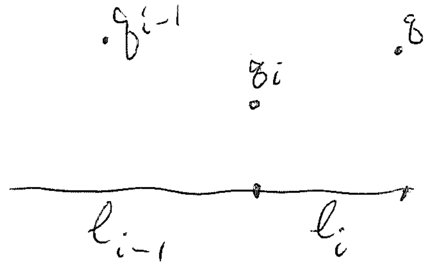
$$\begin{pmatrix} q_{i-1} \\ \mu_i \end{pmatrix} = \begin{pmatrix} 1 & l_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_i \\ \mu_i \end{pmatrix} = \begin{pmatrix} 1 & l_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m_i s^2 & 1 \end{pmatrix} \begin{pmatrix} q_i \\ \mu_{i+1} \end{pmatrix}$$

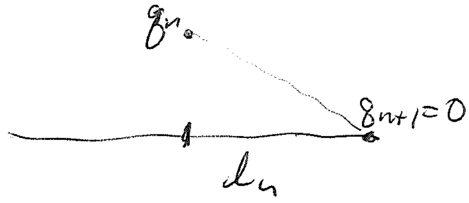
$$\begin{pmatrix} q_0 \\ \mu_1 \end{pmatrix} = \begin{pmatrix} 1 & l_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ m_1 s^2 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & l_n \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ m_n s^2 & 1 \end{pmatrix} \cancel{\begin{pmatrix} q_n = 0 \\ \mu_{n+1} \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} 0 \\ \mu_{n+1} \end{pmatrix}} \begin{pmatrix} 1 & l_{n+1} \\ & 1 \end{pmatrix} \begin{pmatrix} q_{n+1} \\ \mu_{n+1} \end{pmatrix}$$









$$m_i \ddot{q}_i = -\frac{q_i - q_{i+1}}{l_i} + \frac{q_{i-1} - q_i}{l_{i-1}}$$

$$= -\mu_i + \mu_{i-1}$$

$$\boxed{\begin{aligned} \mu_{i-1} &= \mu_i + m_i s^2 q_i \\ q_i &= q_{i+1} + l_i \mu_i \end{aligned}}$$

$$\cancel{q_n = l_n \mu_n}$$

$$\cancel{\mu_{n-1} = \mu_n + m_n s^2 q_n}$$

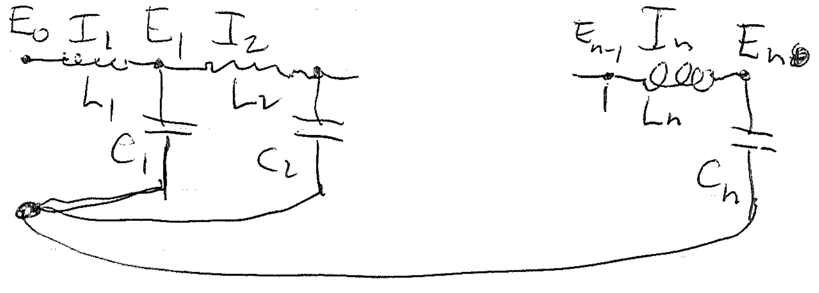
$$\cancel{q_{n-1} = q_n + l_{n-1} \mu_{n-1}}$$

$$\begin{pmatrix} q_0 \\ \mu_0 \end{pmatrix} = \begin{pmatrix} 1 & l_0 \\ & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ \mu_0 \end{pmatrix} = \begin{pmatrix} 1 & l_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m_1 s^2 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ \mu_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & l_0 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m_1 s^2 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & l_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m_n s^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & l_{n+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overset{0}{\overset{\|}{q_{n+1}}} \\ \mu_n \end{pmatrix}$$

This leads basically nowhere.
What is your aim?? Trying to connect LC response to K-modules.

There is an obvious space of dim $2n+1$ namely $E_0, I_1, E_1, I_2, E_2, \cdots, E_{n-1}, I_n, E_n$. We have the equations relating them



$i = 1, \ldots, n$ w $I_{n+1} = 0$

$$I_i = I_{i+1} + C_i s E_i$$

$$E_{i-1} = E_i + L_i s I_i$$

$i = 1, \ldots, n$

unique solution depending on E_n, thus get a line in this $2n+1$ diml subspace dep on E

531 So the only question now is ~~what to do about~~ whether the map has ~~the requ~~ K form, but this should be obvious

$$0 \longrightarrow \ell_s \longrightarrow \mathbb{R}^{2n+1} \xrightarrow[as-b]{} \mathbb{R}^{2n}$$

$$as : (E_0, I_1, \cdots I_n, E_n) \longmapsto (L_1 s I_1, C_1 s E_1, \cdots, C_n s E_n)$$

$$b : (\qquad\qquad) \longmapsto (E_0 - E_1, I_1 - I_2, \cdots, E_{n-1} - E_n, + I_n)$$

~~Maybe you need something~~ ~~Hatch~~ Maybe the general LC situation can be put in an $as-b$ form

---

Repeat what you've learned. Consider

$E_0 \quad I_1 \quad E_1 \quad I_2 \cdots \quad E_{n-1}, I_n, E_n \quad$ $2n+1$ variables

$$\begin{cases} E_{i-1} = E_i + L_i s I_i & i = 1, \dots, n \\ I_i = I_{i+1} + C_i s E_i & i = 1, \dots, n \quad \text{with } I_{n+1} = 0 \end{cases}$$

$2n$ equations, get line ℓ_s

Does this example generalize? Your previous analysis amounts to

$$\begin{array}{ccl} V^{n+1} & \hookrightarrow & H^{2n} = H^n_+ \oplus H^n_- \\ {\scriptstyle E_0}\downarrow & & \\ \mathbb{R} & & \end{array}$$

Somehow you have replaced

~~Let~~ Look at H with the qf $s\|h_+\|^2 + s^{-1}\|h_-\|^2$

get $s q_+ + s^{-1} q_-$ where q_+, q_- are orth forms.

response function?

532 Discuss the problem. You want to find a Hilbert space interpretation of an r-port that leads to a partial unitary of codim r, hence to an r-diml vector bundle over S^2. This may be too optimistic, but is worth pursuing. Let's examine carefully the case we can compute.

Suppose we start with an r-port i.e. real vector space of dim r equipped with a quadratic form depending rationally on s of the form $f(s) = \sum_{0 \le \omega \le \infty} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega \qquad a_\omega \geqslant 0 \quad \sum a_\omega > 0$.

Take $r = 1$. Suppose $\{\omega \mid a_\omega \neq 0\}$ has card n label them $\omega_1 \leqslant \omega_2 < \cdots < \omega_n$ and to simplify suppose $0, \infty$ not included. Identify $V = \mathbb{R}$ so that $\sum a_\omega = 1$. Actually what is f?

$f_s : V \longrightarrow V^*$. f is rational in s of degree $2n$

To write $f = \frac{f}{g}$ quotient of two polys you convert to denom. $\prod_{i=1}^{n} (s^2 + \omega_i^2)$ of deg $2n$

and the numerator has degree $2n-1$. (If $\omega_1 > 0$, $\omega_n = \infty$ you ~~make~~ convert denom. to $\prod_{i=1}^{n-1} (s^2+\omega_i^2)$ of degree $2n-2$

and numerator has degree $2n-1$ & is divisible by s. so degree is $2n-1$.) ~~So now take denom.~~

If $\omega_1 = 0$, $\omega_n < \infty$ you convert to denom. $s \prod_{i=2}^{n-1} (s^2+\omega_i^2)$ of degree $2n-1$, numerator has degree $2n-2$, so degree is $2n-1$.

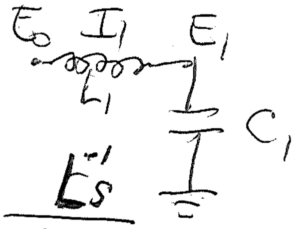
$$\frac{1}{s} + \frac{sb}{s^2+\omega^2} = \frac{s^2(b) + s^2+\omega^2}{s(s^2+\omega^2)}$$

OKAY

533 Where's the action? You have a ~~rational~~ functions of degree $2n$. A better way to put it is to say you have

$$q : \mathcal{O}\otimes V \longrightarrow \mathcal{O}(2n)\otimes V^*$$

~~So study this~~ Basic example.



$$\frac{E_0}{I_1} = Ls + \frac{1}{Cs} \qquad \frac{I_1}{E_0} = \frac{1}{Ls+\frac{1}{Cs}} = \frac{L^{-1}s}{s^2+\omega^2} \qquad \omega = \frac{1}{\sqrt{LC}}$$

This rational function has degree 2.
graph has 3 vertices 2 edges

$$\begin{array}{ccc} C^0 & \xrightarrow{\sim} & C^1 \\ {\scriptstyle E_0}\downarrow & & \\ \mathbb{R} & & \end{array}$$

$\mathcal{O}(-n)\otimes V \longrightarrow \mathcal{O}(n)\otimes V^*$
so you have a bilinear form on ?

Problem: How can you go from the response function to the Hilbert space.

Approach: Look for a partial unitary

Possible approaches - maybe you should work on ?

Question: Link between moment problem and LC chain

Think of $q_s : V \longrightarrow V^*$ as being a map

$\mathcal{O}\otimes V \xrightarrow{q} \mathcal{O}(+2n)\otimes V^*$ or $\mathcal{O}(-n)\otimes V \longrightarrow \mathcal{O}(n)\otimes V^* = (\mathcal{O}(-n)\otimes V)^*$

$\mathcal{O}(2n)\otimes V \xleftarrow{q^{-1}} \mathcal{O}\otimes V^*$

so get

$$\mathcal{O}(-n)\otimes V \xrightarrow{q} \mathcal{O}(n)\otimes V^* \qquad ?$$
$$\mathcal{O}(n)\otimes V \xleftarrow{q^{-1}} \mathcal{O}(-n)\otimes V^*$$

~~What is the moment~~ Moment problem H Euclidean A s.a. ξ cyclic vector, get moments $(\xi, A^n\xi)$, appropriate positive matrix on polynomials of degree $<$ dim H.

534 can orthonormalize e $\xi, A\xi, \ldots, A^{n-1}\xi$

find A given by symm Jac. matrix | ~~orth. basis~~

orth polys wrt μ

Jan 25, 98

What's obvious – what's taking so long

What's missing.

Consider a 1-port LC network

Here's the problem. You have a description for the Green's functions arising from n-port LC networks, but you want the underlying Hilbert space.

The LC network is a configuration space picture of something you are seeking. You have the Green's function of a 2nd order ~~operator~~ DE, and you seek something like a phase space with a first order DE.

What can you do?

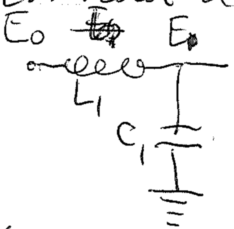
Develop your hunch that partial unitaries are the fundamental object.

Return to harmonic oscillators

Inverse spectral problem.

Symplectic versus orthogonal

Consider a 1-port LC network



$$\frac{E_0}{I_1} = L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{L_2 s + \cfrac{1}{C_2 s + \cdots \cfrac{1}{C_n s}}}}$$

What do I know?

What do you see?

You see

$${}^{n+1}\bar{C}^0 \xrightarrow{d} {}^{2n}C^1 = {}^{n}H^+ \oplus {}^{n}H^-$$

$$E_0 \downarrow$$

$$\mathbb{R}$$

But what you want is the vector space of dim $2n+1$ cons. of $(E_0, I_1, E_1, \ldots, I_n, E_n)$. Call this V^{2n+1}

What else? $2n$ equations one for each edge

$$E_{i-1} = E_i + L_i s I_i$$

$$I_i = I_{i+1} + C_i s E_i$$

this gives $\ell_s \hookrightarrow V^{2n+1} \xrightarrow{as-b} W^{2n}$ (onto)

~~Whence~~ so you get a line bundle over $\mathbb{P}^1$ isomorphic to $\mathcal{O}(-2n)$

In terms of the data $\bar{C}^0 \overset{d}{\hookrightarrow} C^1 = H^+ \oplus H^- \ldots$ $E_0 f$

it is not easy to see a vector space of dim $2n+1$. ~~mapping onto~~ ~~However~~

Maybe you should ~~[illegible]~~ form symplectic spaces

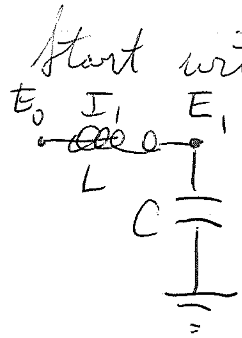
$$\begin{tikzcd}
 & \overset{n+1}{\bar{C}^0} \arrow[r, hook] \arrow[ddl] & \overset{2n}{C^1} \arrow[r, two heads] & \overset{n-1}{H^1} \\
 & \oplus & \oplus & \\
\mathbb{R} & \bar{C}_0 & C_1 \arrow[l, two heads] \arrow[r] & H_1 \\
\oplus & & & \\
\mathbb{R} \arrow[uur] & & &
\end{tikzcd}$$

Aim now to get away from configuration space. Try to organize things — maybe kinematics + dynamics

Problem to be solved. Given a suitable response function find the appropriate ~~operator~~ partial operator. ~~That~~ Suppose you assume ~~you are seek~~ the answer is a partial unitary. Then ~~that~~ you can ask if the response function can be tranformed to a partial unitary. This seems clear.

536 Additional point. Is it possible to break things into a chain of simple things

Start with



$$\frac{E_0}{I_1} = Ls + \frac{1}{Cs} = \frac{s^2+(LC)^{-2}}{Cs}$$

3 diml configuration space

vertex space + loop space) NO

$n+1=2$ $n-0$

In general you

$$\overset{v-1}{\bar{C}^0} \hookrightarrow \overset{e}{C^1} \twoheadrightarrow \overset{l}{H^1} \qquad v-e=1-l$$

$$\bar{C}_0 \twoheadleftarrow C_1 \leftrightarrow H_1$$

The configuration space of the circuit is the space $\bar{C}^0 \oplus H_1$ which has dimension $v-1+l=e$

If I want I can think of phase space as being either $C^1 \oplus C_1$ or $(\bar{C}^0 \oplus H_1) \oplus (\bar{C}_0 \oplus H^1)$

The latter is a bit non canonical since the sequence $\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$ doesn't split canonically. So you have a situation where the "dilated" phase space is $C^1 \oplus C_1$ carries response of very simple type s, s^{-1}.

<u>Then we have</u> a

Situation: Take a real symplectic ~~vect~~ ~~-~~ The objects to consider are going to be a real symplectic vector together with a rational map from the projective line over $\mathbb{R}$ into the symplectic Grassmannian satisfying some positivity condition. Make this precise.

537 Let M be a real symplectic vector space (f.d)
Then we have the symplectic Grassmannian consisting of maximal isotropic subspaces of M.
If $\dim(M) = 2d$, then $Sgr(M) \subset Gr_d(M)$. If

$$M = \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \quad \text{with} \quad \begin{pmatrix} v_1 \\ \lambda_1^t \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ \lambda_2^t \end{pmatrix} = +\lambda_1 v_2 - v_1^t \lambda_2^t \quad ?$$

then any point of $Gr_d(M)$ complementary to V^* is the graph $\binom{1}{T}V$ of a map $T: V \longrightarrow$
know $\binom{1}{T}V$ is max. isot. $\Longleftrightarrow$ T

$$\left(\binom{1}{T} v_1\right)^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \binom{1}{T} v_2 = \cancel{t}\, v_1^t T v_2 + (Tv_1)^t v_2$$

~~Better~~ $\cancel{M = \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \ni \begin{pmatrix} v \\ w^t \end{pmatrix} \quad \begin{pmatrix} v_1 \\ w_1^t \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ w_2^t \end{pmatrix}}$

$\cancel{= (v_1^t \; w_1) \begin{pmatrix} w_2^t \\ -v_2 \end{pmatrix}}$

$$M = \begin{matrix} V \\ \oplus \\ V^* \end{matrix} \ni \begin{pmatrix} v \\ \lambda^t \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ \lambda_1^t \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ \lambda_2^t \end{pmatrix}$$

$$= (v_1^t \;\; \lambda_1) \begin{pmatrix} \lambda_2^t \\ -v_2 \end{pmatrix} = v_1^t \lambda_2^t - \lambda_1 v_2 = \lambda_2 v_1 - \lambda_1 v_2$$

$\cancel{(Tv_2)^t v_1 = (Tv_1)^t v_2}$

$\dim Sgr(M) = \frac{d(d+1)}{2}$

$\cancel{v_2 \; T \; v_1}$

$\dim Sp(M) = \frac{2d(2d+1)}{2} = 2d^2 + d$

$\dim$ stabilizer of $\underset{GL_d}{A}X + \underset{\text{symm}}{B} = d^2 + \frac{d(d+1)}{2}$