465 Jan 15, 97. ~~[illegible]~~

~~[illegible]~~ Link scattering and partial isom.

Scattering situation. $H^2(S^1) \supset \varphi H^2(S^1)$. Is there a partial isometry assoc to this layer. ~~[illegible] simplest example~~ Do some examples of scattering.

Try perturbation viewpt. Start with U_0 = mult by z on $L^2(S^1)$, and consider a perturbation U_1 of U_0. This means that ~~[illegible]~~ $U_1 = U_0(U_0^{-1}U_1)$ where $U_0^{-1}U_1$ is a unitary ~~with~~ such that $U_0^{-1}U_1 - 1$ has finite support as a matrix. ~~This means~~ Simplest case is to let

$$U_0^{-1}U_1\,(\xi^n) = \begin{cases} \xi^n & n \neq 0 \\ \xi & n = 0. \end{cases}$$

More complicated is $U_0^{-1}U_1(\xi^n) = \xi^n \quad n \neq 0, 1$

~~[illegible]~~ given by ~~a [illegible]~~ some 2×2 matrix.

e.g. I've put the perturbation on the right of U_0. Suppose ~~[illegible]~~

$$U_0^{-1}U_1 \text{ sends } \begin{matrix} \xi_0 & \text{to} & \xi_1 \\ \xi_1 & \text{to} & \xi_0 \end{matrix}$$

Then U_1 sends $\begin{matrix} \xi_0 & \text{to} & \xi_2 \\ \xi_1 & & \xi_1 \end{matrix}$ bound ~~state~~

How do you analyze? Eigenvectors $u_\xi^\pm$ incoming and outgoing. Is it obvious these exist, i.e. that you can solve ~~the~~ equations: $U_1 u_\xi = \xi u_\xi$

$u_\xi = \sum c_n \xi_n$. To solve $U_1 u_\xi = \xi u_\xi$

$$U_1 U_0^{-1} U_0 u_\xi = \xi u_\xi$$

$$U_0 u_\xi = \xi U_0 \quad\quad U_0 - \xi)$$

?

To solve $U_1 v^\xi = \xi v^\xi$

where $U_1 = U_0(U_0^{-1}U_1) = U_0 + U_0 K$

Then $(U_0 + U_0 K) v^\xi = \xi v^\xi$

$$(\xi - U_0) v^\xi = U_0 K v^\xi$$

$$(\xi - U_0 - U_0 K) v^\xi = 0$$

should be a Volterra eq.

So how do we handle this?

Perturbation might be harder than necessary.

You want to ~~solve the equation~~ find $\psi(n)$ such that $\psi(n+1) = U_1 \psi(n)$ $\forall n$

$\psi(n) = U_1^n \psi(0)$ transform

$$\hat{\psi}_\xi = \sum_{n \geq 0} \xi^{-n} \psi(n)$$

$\cancel{U_1 \hat{\psi}_\xi = \xi \sum \xi^{-n-1} \psi(n+1) = \xi \hat{\psi}_\xi}$

$$U_1 \hat{\psi}_\xi = \sum_{n \geq 0} \xi^{-n} \underbrace{U_1 \psi(n)}_{\psi(n+1)} = \xi \sum_{n \geq 0} \xi^{-n+1} \psi(n-1)$$

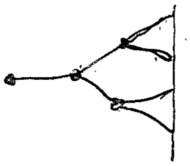
$$= \xi \sum_{n \geq 1} \xi^{-n} \psi(n) = \xi(\hat{\psi}_\xi - \psi_0)$$

$\xi^{-1} U_1 \hat{\psi}_\xi = \hat{\psi}_\xi - \psi_0$ $\qquad \psi_0 = (1 - \xi^{-1} U_1) \hat{\psi}_\xi$

$$\boxed{\hat{\psi}_\xi = \frac{1}{1 - \xi^{-1} U_1} \psi_0}$$

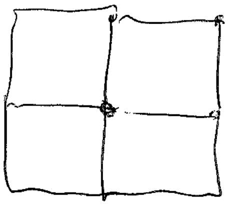
Now put in that $U_1 U_0^{-1} = 1 + K$

467 Digress. What is the resistance of a tree?



$$\frac{1}{2},\quad \left(1+\frac{1}{2}\right)\frac{1}{2},\quad \left(1+\frac{3}{4}\right)\frac{1}{2}$$

$$\frac{3}{4} \qquad \frac{7}{8}$$



Use $L^2(\mathbb{Z}\times\mathbb{Z}) \simeq L^2(S^1\times S^1)$

green's function $\dfrac{1}{\sin^2\left(\frac{\theta_1}{2}\right) + \sin^2\left(\frac{\theta_2}{2}\right)}$

logarithmic singularity.

Go back to discrete "wave eqn". $\psi(n+1) = U_1\psi(n)$

Transform $\quad \hat{\psi}_\xi = \sum_{n\geq 0} \xi^{-n}\psi(n)$

$$\xi^{-1}U_1\hat{\psi}_\xi = \sum_{n\geq 0} \xi^{-n-1}\psi(n+1) = \hat{\psi}_\xi - \psi(0)$$

$$\boxed{\hat{\psi}_\xi = \frac{1}{1-\xi^{-1}U_1}\psi(0)}$$

Now use that $U_1U_0^{-1} = 1+K$.

$$U_1 = U_0 + KU_0$$

$$1-\xi^{-1}U_1 = 1-\xi^{-1}U_0 - \xi^{-1}KU_0$$

Look at inhomog. eqn.

$$(1-\xi^{-1}U_1)\hat{\psi}_\xi = 0$$

$$(1-\xi^{-1}U_0)\hat{\psi}_\xi = \xi^{-1}KU_0\hat{\psi}_\xi$$

$$(1-\xi^{-1}U_0-\xi^{-1}U_0K)\hat{\psi}_\xi = 0$$

$$\left(1-\frac{1}{1-\xi^{-1}U_0}\xi^{-1}U_0K\right)\hat{\psi}_\xi = 0$$

$$\hat{\psi}_\xi = (G\xi^{-1}U_0K)\psi_\xi$$

468 $$\hat{\psi}_\zeta = \frac{1}{1-\zeta^{-1}U_1}\psi(0)$$

$$= \frac{1}{1-\zeta^{-1}u_0 - \zeta^{-1}u_0 K}\psi(0)$$

$$U_1^n - U_0^n = \sum_{i=1}^{n} u_1^{i-1}(u_1-u_0)\,u_0^{n-i}$$

$U_0^{-n}U_1^n$ does this converge as $n \to +\infty$ and $n \to -\infty$

Look at $U_1^n U_0^{-n}$ as $n \to +\infty$.

If you apply this to a vector of finite support, then you know that for n large $K\,U_0^{-n}\xi = 0$, hence

$$u_1 U_0^{-1} U_0^{-n}\xi = U_0^{-n}\xi \quad \text{or} \quad U_1^{n+1}U_0^{-n-1}\xi = U_1^n U_0^{-n}\xi$$

level of Laurent polys. the wave operators exist.

Make some examples: First how about the eigenvectors.

$$(1-\zeta^{-1}U_0)\,\xi^\zeta = 0$$

$$\xi^\zeta = \sum c_n \xi_n$$

$$(1-\zeta^{-1}u_0)\,\xi^\zeta = \sum c_n\xi_n - c_n\zeta^{-1}\xi_{n+1}$$

$$= \sum (c_n - \zeta^{-1}c_{n-1})\,\xi_n \qquad \therefore \boxed{c_n = \zeta c_{n-1}}$$

$$c_n = \zeta^n c_0.$$

Is it clear that $\forall \xi\ \exists\ \xi^\zeta$ for U_1?

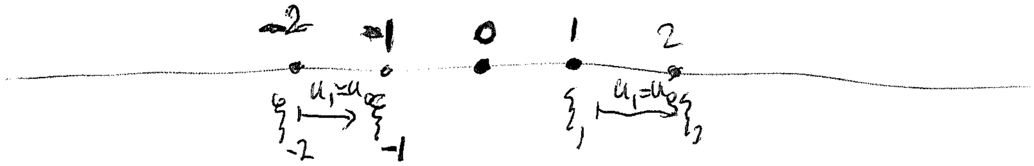
469 discrete scattering. put the perturbation into a box and see if you can find in the box a partial isometry. Yes!

Simple example. Suppose the perturbation ~~limited to the region~~ has support of length 1.

See if you can set up the wave operator.

Idea given ξ consider $U_1^n U_0^{-n}\xi$ as n increases. $U_1^n \overbrace{(U_1 U_0^{-1})}^{1+K} U_0^{-n}\xi$

Assume K supported at $n=0,1$.



$U_1=(1+K)U_0$

$K=U_1U_0^{-1}-1.$

for what ξ_n do we have $W^-\xi_n=\xi_n$.

Thus $\boxed{U_1U_0^{-1}}\xi_n=\xi_n$ for $n\le -1$

$U_1-U_0=KU_0$ means that $U_1\xi_n\neq U_0\xi_n$
$\Longrightarrow n=-1$ or 0.

Conventions so far: $U_1-U_0=KU_0$ $\quad |K|=\{0,1\}$, $|KU_0|=\{-1,0\}$.

Try $V=\mathbb{C}\xi_{-1}\oplus\mathbb{C}\xi_0$. Then ξ_1 is a dist. line in V. $U_1^{-1}\xi_1$ another dist. line.

Situation. Suppose you have H^- incoming, H^+ outgoing, H^-, H^+ perpendicular, let $V=(H^-)^\perp\cap(H^+)^\perp$, so that $H=H^-\oplus V\oplus H^+$. Orth basis ξ_n $n\le -1$ for H^- $\ni U\xi_n=\xi_{n+1}$, $n\le -2$. Orth basis ξ_n, for $n\ge 1$ H^+ such that $U\xi_n=\xi_{n+1}$. Look at $U(\xi_{-1})$. Is this in V

470 Go back to a partial ~~unitary~~ isom $\overset{n}{W} \underset{b}{\overset{a}{\rightrightarrows}} \overset{n+1}{V}$ and enlarge V as follows. We have two lines in W, $\mathrm{Ker}(a^*)$, $\mathrm{Ker}(b^*)$. Call one L^+ and the other L^-. Combine $z^{-1}H^2_-(S^1, L^-) \oplus V \oplus z H^2_+(S^1, L^+)$

$\oplus z^{-2}L^- \oplus z^{-1}L^- \oplus V \oplus zL^+ \oplus z^2L^+ \oplus \cdots\cdots$ Easy enough. and it seems to work.

Dominic Joyce version. Need a real version. I think you want to look at $L^2(S^1, H)$ $H = \mathbb{C}^2$. You have singled out $\{0, \infty\}$ on S^1.

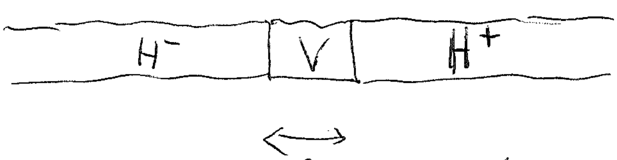
Try to find the complexification of what you are seeking. You will have $L^2(S^1)\otimes\mathbb{C}^2$ with a funny σ. Is there a unitary operator like $\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$? Then run same perturbation analysis.

Try for $H\otimes\mathcal{O}(1)$ V 4 diml

Inside V you have two dim 2 subspaces and an isom between them. You need σ acting. V^σ is 4 diml W^σ 1-diml. Get dims straight.

For $L=\mathcal{O}(1)$ you have $W = \Gamma(L(-1)) = \mathbb{C}$ $V = \Gamma(L) = \mathbb{C}^2$. and $W\oplus W \xrightarrow[(a,b)]{\sim} V$. In the σ-case the σK-module is simply $\mathbb{R} \longrightarrow \mathbb{H}$

Next go back to $L^2(S^1)$ + perturbation leading to $\overset{1}{W} \rightrightarrows \overset{2}{V}$. What was the point. Something entering from left exiting from right



$\updownarrow$ 2 dims vertically

$\leftrightarrow$ 2 dim here probably reversed by σ

471 Anyway what happens to z? The basic unitary. What do we have geometrically. You ~~take the~~ ultimately are dealing with functions on S^1 with values in $\mathbb{C}^2$ which are σ invariant, where σ is $z \mapsto -z$ on S^1 and something like ^{t} on $\mathbb{C}^2$. Do we have anything corresponding to U? Guess that you want $\Gamma(S^1, H \otimes \mathcal{O}(1))$. It's hard to tell, but suppose you think of some space made out of pairs of antipodal points on S^2. You are going to get a

Discuss carefully. You have over S^2 a line bundle $\mathcal{O}(1)$ everything with σ action except that $\sigma^2 = -1$ on $\mathcal{O}(1)$. You have σ-invariant covering consisting of the disks $|z| \leq 1$ and $|z| \geq 1$ intersecting along the circle. It's meaningful to discuss the smooth sections of $\mathcal{O}(1)$ over ~~the~~

Let $W \underset{b}{\overset{a}{\rightrightarrows}} V$ be a partial isom. of codim 1

Then can form $H = H^- \oplus V \oplus H^+$ with unitary U as follows

$$\bigoplus_{n<0} U^n \operatorname{Ker}(a^*) \oplus \operatorname{Im} b \oplus \underbrace{\operatorname{Ker}(b^*) \oplus \bigoplus U^n \operatorname{Ker}(b^*)}_{n \geq 0}$$

$$\downarrow U \qquad \qquad \downarrow ab^{-1} \qquad \qquad \downarrow U$$

$$\overbrace{\bigoplus_{n<0} U^n \operatorname{Ker}(a^*) \oplus \operatorname{Ker}(a^*)} \oplus \operatorname{Im}(a) \oplus \overbrace{\bigoplus_{n>0} U^n \operatorname{Ker}(b^*)}$$

We have extended the partial unitary to a unitary

Actually another possibility ~~would~~ might be to ~~form a~~ extend ab^{-1} to a contraction and dilate somehow.

However what I really want to do is to try to understand if there's a σ version. Take simple case where $\dim(V) = 2$. We are given

$$V = \mathrm{Im}(b) + \mathrm{Ker}(b^*)$$

$$V = \mathrm{Ker}(a^*) + \mathrm{Im}(a)$$

Now how can σ act? How can σ act? say $\sigma^2 = -1$ on V.

Jan 16. Try to find a σ-version of a partial unitary with $\dim(V) = 2$.

$$V = \mathrm{Im}(b) \oplus \mathrm{Ker}(b^*)$$

$$\mathrm{Im}(b) \xrightarrow{ab^{-1}} \mathrm{Im}(a)$$

$$V = \mathrm{Ker}(a^*) \oplus \mathrm{Im}(a)$$

We want σ to act on V say $\sigma^2 = -1$

σ must ~~act on W~~ respect the ~~correspondence~~ K-module $W \overset{a}{\underset{b}{\rightrightarrows}} V$ in some sense, a, b are related to $0, \infty$ pair of anti podal points. It seems logical that σ should interchange a, b. Take $V = \mathbb{H}$ with $\sigma = j$. Pick a line $\mathrm{Im}(b) = \mathbb{C}(\alpha + j)$ $\alpha \in \mathbb{C}$. Take

$$\mathrm{Im}(a) = \sigma\, \mathrm{Im}(b) = \mathbb{C}(-1 + \bar{\alpha} j) = \mathbb{C}\left((-\bar{\alpha}^{-1}) + j\right)$$

~~It~~ seems that ~~this is~~ $\mathrm{Im}(a)$ is $\mathrm{Im}(b)^\perp$.
$\therefore$ Get nothing at all.

Anyway let's go over coupling. Maybe a first step might be to ~~look at the~~ write up various aspects of the partial unitary situation. Review basic construction start with f.d Hilbert spaces W, V and two isometries $W \overset{a}{\underset{b}{\rightrightarrows}} V$, then form

$$H = \bigoplus_{n>0} U^n \mathrm{Ker}(b^*) \oplus \underbrace{\mathrm{Im}(a) \oplus \mathrm{Ker}(a^*)}_{V} \oplus \bigoplus_{n>0} U^n \mathrm{Ker}(a^*)$$

$$\mathrm{Im}(a) \xrightarrow{ba^*} \mathrm{Im}(b), \qquad \bigoplus_{n>0} U^n \mathrm{Ker}(b^*) \xrightarrow{U} \bigoplus_{n>0} U^n \mathrm{Ker}(b^*) \oplus \mathrm{Ker}(b^*), \qquad \mathrm{Ker}(a^*) \oplus \bigoplus_{n>0} U^n \mathrm{Ker}(a^*) \xrightarrow{U} \bigoplus_{n>0} U^n \mathrm{Ker}(a^*)$$

$$\bigoplus_{n>0} U^n \mathrm{Ker}(b^*) \oplus \underbrace{\mathrm{Ker}(b^*) \oplus \mathrm{Im}(b)}_{\substack{\parallel \\ V}} \oplus \bigoplus_{n>0} U^n \mathrm{Ker}(a^*)$$

473 What actually appears is that you have $H = H^- \oplus W \oplus H^+$ orthogonal decomp.

H^- incoming $\quad U^{-1}H^- \subset H^-$ $\quad \bigcup_{n\to+\infty} U^n H^- = H$, $\bigcap_{n\to-\infty} U^n H^- = 0$

H^+ outgoing $\quad U H^+ \subset H^+$ $\quad \bigcup_{n\to-\infty} U^n H^+ = H$, $\bigcap_{n\to+\infty} U^n H^+ = 0$.

Put $V = W \oplus (H^+ \ominus UH^+)$

$=$ orthogonal complement of $H^- \oplus UH^+$.

What do you want to say. That the basic object is a ~~[illegible]~~ H, U, H^+, H^- ~~[illegible]~~.

$$\boxed{\begin{array}{l} U^{-1}H^- \subset H^- \\ U H^+ \subset H^+ \\ H^+ \perp H^- \\ \bigcap_n U^{-n} H^- = 0 \\ \bigcap_n U^n H^+ = 0 \end{array}}$$

Then $W = \perp$ comp. of $H^- \oplus H^+$

$V = W \oplus (K^+)$ $\quad K^+ = H^+ - UH^+$

$H^- \oplus W \oplus K^+ \oplus U(H^+)$

$H^- \oplus U(K^-) \oplus U(W) \oplus U(H^+)$

In this situation you have two complements for $H^- \oplus U(H^+)$ namely $W \oplus$ ~~[illegible]~~ K^+ and $U(K^-) \oplus U(W)$

So get a partial unitary.

How partial unitaries arise. Given $H \overset{U}{\circlearrowleft}$ and a subspace W of H let $V =$ ~~$W \oplus$~~ $W + U(W)$

Is it possible to classify? Maybe what's relevant here is the scattering operator.

Problem: Classify partial unitaries. ~~What happen~~ of codim 1 to begin ~~[illegible]~~ with. You have ~~[illegible]~~ decomposition into indecomposable K-modules. It is possible that the answer is ~~[illegible]~~ a certain type of scattering operator.

474 Review what you've done. To a partial unitary $W \overset{a}{\underset{b}{\rightrightarrows}} V$ you've associated a scattering situation

Need discussion - Problem is to completely under partial unitaries, then go back to partial hermitian operators.

Consider the basic construction starting from

$$W \overset{a}{\underset{b}{\rightrightarrows}} V \quad \bigoplus_{n<0} U^n(\mathrm{Ker}\, a^*) \oplus \mathrm{Ker}(b) \oplus \mathrm{Ker}(b^*) \oplus \cdots$$

$$\cdots \quad \mathrm{Ker}(a^*) \oplus \mathrm{Im}(a) \oplus \bigoplus_{n>0} U^n \mathrm{Ker}(b^*)$$

ab^{-1}

What is the scattering operator? If there are no bound states ~~what~~ you have

$$\bigoplus_{n\in\mathbb{Z}} U^n(\mathrm{Ker}\, a^*) \xrightarrow{\sim} H \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} U^n\, \mathrm{Ker}(b^*)$$

and you know this is given by a bdd measurable map $S^1 \xrightarrow{\varphi} \mathrm{Unit}(\mathrm{Ker}\, a^*, \mathrm{Ker}\, b^*)$

So how do I calculate this map? This is the basic invariant of a partial unitary.

There's lots to do before it's all clean. Look for eigenvector. $W^* \overset{a^t}{\underset{b^t}{\leftleftarrows}} V^*$ $L_\lambda = \mathrm{Ker}(\lambda a^t - b^t)$ generically this is a line in V^* depending on λ. You want a map $\mathrm{Ker}(a^*) \longrightarrow \mathrm{Ker}(b^*)$ depending on λ. Basically you have two ~~quotient~~ lines a ? What do you have? $V = \mathrm{Im}\, b + \mathrm{Ker}(b^*) = \mathrm{Ker}(a^*) + \mathrm{Im}(a)$. You should probably consider $\mathrm{Ker}(b^*)$ and $\mathrm{Ker}(a^*)$ as the quotient lines of $V/\mathrm{Im}\, b$, $V/\mathrm{Im}\, a$ of V. But then $V/(a\lambda+b)W$ is a variable quotient line joining them.

475 Continue: ~~You have aW, bW~~ You start with $W \underset{b}{\overset{a}{\rightrightarrows}} V$ and you want some sort of map (ultimately unitary) ~~from~~ linking $V/bW = \mathrm{Ker}(b^*)$ and $V/aW = \mathrm{Ker}(a^*)$. Thus you want a line inside $V/bW \times V/aW$ depend on λ. We have a ~~quotient~~ line $V/(a\lambda - b)W$. Maybe what you need is to give the complements $\mathrm{Ker}(b^*)$ and $\mathrm{Ker}(a^*)$. Then you do have $\mathrm{Ker}(b^*) \times \mathrm{Ker}(a^*) \subset V$, can follow with $V \to V/(a\lambda-b)W$

(Earlier today you had a faint idea - find the bound states by intersecting $W \xrightarrow{(a,b)} V \times V$ with diagonal, or a variant of the diagonal - graph of s.)

This looks promising. Check as follows. What you need are complements to $\mathrm{Im}(b)$, $\mathrm{Im}(a)$. Then you get a correspondence between these complements depending on λ i.e. $\mathrm{Im}(b)^\perp \longrightarrow V/(a\lambda-b)W \longleftarrow \mathrm{Im}(a)^\perp$

Notice a choice of complements allows us to extend ab^{-1}: $\mathrm{Im}(b) \longrightarrow \mathrm{Im}(a)$ to a map $V \longrightarrow V$, which has eigenvalues. Compare viewpoints. **OKAY**

$$\begin{tikzcd}
 & & \mathcal{O}\otimes \mathrm{Im}(b)^\perp \arrow[d, hook] & & \\
0 \arrow[r] & \mathcal{O}(-1)\otimes W \arrow[r] & \mathcal{O}\otimes V \arrow[r] & F \arrow[r] & 0 \\
 & & \mathcal{O}\otimes \mathrm{Im}(a)^\perp \arrow[u, hook] & &
\end{tikzcd}$$

~~Take the~~

$$\begin{tikzcd}
 & & \mathrm{Ker}(b^*) \arrow[d, hook] \arrow[dr, "\sim\ \text{isom at } \lambda=0"] & & \\
0 \arrow[r] & W \arrow[r, "a\lambda - b"] & V \arrow[r] & F(\lambda) \arrow[r] & 0
\end{tikzcd}$$

at $\lambda = 0$ $\quad (a\lambda - b)W = bW$ is comp to $\mathrm{Ker}(b^*) = (bW)^\perp$

Probably you can estimate norm $\quad \|b^* a\| \le 1$

$$b^*(a\lambda - b) = (b^*a)\lambda - 1$$

OK for $|\lambda| < 1$.

476 Basically OKAY, namely for $|\lambda|<1$ you get

$$\begin{tikzcd}
\mathrm{Ker}(b^*) \arrow[d, hook] \arrow[dr, "\simeq"] & \\
V \arrow[r] & F(\lambda) \\
\mathrm{Ker}(a^*) \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

So we get a function $\mathrm{Ker}(a^*) \longrightarrow \mathrm{Ker}(b^*)$ depending on λ. Needs more work tomorrow. What about now?

Compute. Suppose $\qquad \mathrm{Im}(b) \oplus \mathrm{Ker}(b^*) \cong V$

$\qquad\qquad\qquad\qquad\quad \mathrm{Ker}(a^*) \oplus \mathrm{Im}(a) = V$

$$W \xrightarrow{\Delta} \begin{matrix} W \\ \oplus \\ W \end{matrix} \xrightarrow{\begin{pmatrix} b & a \end{pmatrix}} \begin{matrix} V \\ \oplus \\ V \end{matrix} \longleftarrow \begin{matrix} \mathrm{Ker}(b^*) \\ \oplus \\ \mathrm{Ker}(a^*) \end{matrix} \xleftarrow{?}$$

So V 2-dim with 2 splitting

$$V = \begin{matrix} \mathrm{Im}(b) \\ \oplus \\ \mathrm{Ker}(b^*) \end{matrix} \longrightarrow \begin{matrix} \mathrm{Ker}(a^*) \\ \oplus \\ \mathrm{Im}(a) \end{matrix}$$

$$V = \begin{matrix} \mathrm{Im}(b) \\ \oplus \\ \mathrm{Im}(a) \end{matrix} \;\cancel{\oplus}\; \begin{matrix} \mathrm{Ker}(b^*) \\ \oplus \\ \mathrm{Ker}(a^*) \end{matrix}$$

get 2×2 inv. matrix to describe splittings.

coupling: $\quad W \overset{a}{\underset{b}{\rightrightarrows}} V \qquad X \overset{a'}{\underset{b'}{\rightrightarrows}} Y$

$$\boxed{\begin{tikzcd}
\mathrm{Im}(b) \oplus \mathrm{Ker}(b^*) \arrow[d, "ab^{-1}\,\cong"'] \arrow[dr, "\text{given}\,\cong"] & \mathrm{Im}(b') \oplus \mathrm{Ker}(b'^*) \arrow[d, "a'b'^{-1}\,\cong"] \\
\mathrm{Ker}(a^*) \oplus \mathrm{Im}(a) & \mathrm{Ker}(a'^*) \oplus \mathrm{Im}(a')
\end{tikzcd}}$$

coupling

477 These coupling pictures have to be translated into matrices.

ideas for tomorrow. Correlate your LC circuit response function with ~~unitary~~ partial unitary response function. How can these be connected? Restrict to 1 ports. I think that LC circuit response function is equivalent to a partial hermitian operator – at least you can go from the LC response fn. to a continued fraction (There may be some pitfalls.

Question: Take a lagrange type harmonic oscillator

$$L = \frac{1}{2}\dot{q}^t m \dot{q} - \frac{1}{2} q^t k q \quad \Big| \quad p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \Big| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$m\ddot{q} + kq = 0$ Now add a external dependent on q.

$V = -q^t F$ $\quad \boxed{m\ddot{q} + kq = F(t)}$ $\quad s = e^{-i\omega t}$

$$\hat{q}(s) = \int_0^\infty e^{-st} q(t)\,dt \qquad F(t) = \mathrm{Re}\left(\hat{F}(s)e^{st}\right)$$

$$\boxed{\hat{q}(s) = (ms^2+k)^{-1}\hat{F}(s).}$$

general response to a periodic applied force dep. on q.

Next a ~~quotient~~ space of position space $V \longrightarrow V/W$

Say $q \longmapsto q_1 = (1\,0\,0\,0)q$. Assume $\begin{pmatrix} q_1 \\ \vdots \\ q_L \end{pmatrix} \longrightarrow \begin{pmatrix} q_1 \\ \vdots \\ q_k \end{pmatrix}$

This isn't as straightforward as I thought

$F(t) \in V^*$ $\quad V$ position space in gen

$F(t) \in (V/W)^*$

478 Jan 17 Lag type harm osc. $q(t) \in V \ \forall t$

K.E. $= \frac{1}{2}\dot{q}^t m \dot{q}$ P.E. $= \frac{1}{2} q^t k q$ $m, k : V \to V^*$ symm. pos. def.

D equations of motion $m\ddot{q} + kq = 0$.

applied force $F(t)$ depending on t, $F(t) \in V^*$, new

~~eqn~~ DE $m\ddot{q} + kq = F$ altered pot. energy $\frac{1}{2}q^t k q - q^t F$

Solve via L.T. ~~$(ms^2+k)\hat{q}$~~ $\boxed{\hat{q}(s) = \frac{1}{ms^2+k}\hat{F}(s)}$ so

~~[illegible]~~ you have a map $V^* \to V$ namely $(ms^2+k)^{-1}$

so the response is a ~~[illegible]~~ quadratic form on V^* dep. rationally on s. Let V/W be a quotient space (~~[illegible]~~ particles you are looking at), $(V/W)^* = W^\circ \subset V^*$

Then get

$$\begin{array}{ccc} (V/W)^* & \hookrightarrow & V^* \\ & & \downarrow (ms^2+k)^{-1} \\ V/W & \leftarrow & V \end{array}$$

Conclude the response seen from outside is the restriction of $(ms^2+k)^{-1}$ to $(V/W)^*$. Now spectral theory for symm. ops, say V splits canonically $V = \bigoplus V_\omega$ $V_\omega = \operatorname{Ker}(-m\omega^2+k)$

$$j : V \to V/W, \quad V_\omega \hookrightarrow V, \quad j_\omega : V_\omega \to V/W$$

~~$j(ms^2+k)^{-1}j^t = \sum_\omega \frac{1}{-m\omega^2+k} j_\omega j_\omega^t$~~

~~$\frac{1}{ms^2+k} = \bigoplus_\omega \frac{1}{ms^2}$~~

$$V = \bigoplus V_\omega \quad \text{where} \quad V_\omega = \{ \xi \mid \omega^2 m \sigma = k\sigma \}$$

where $(m^{-1}k)^{1/2} = \omega$.

$$(ms^2+k)^{-1} = (s^2 + m^{-1}k)^{-1} m^{-1}$$

$V = \bigoplus V_\omega$ gives $\iota_\omega : V_\omega \to V$ and $\iota_\omega^t : V^* \to V_\omega^*$

$\iota_{\omega_1}^t \iota_{\omega_2} = \delta_{\omega_1 \omega_2}$

$\sum_\omega \iota_\omega \iota_\omega^t = 1_V$

$$\frac{1}{ms^2+k} = \sum_\omega \iota_\omega \frac{1}{s^2+\omega^2} m^{-1} \iota_\omega^t$$

≥ 0 on V

$$= \sum_\omega \frac{1}{s^2+\omega^2} (\iota_\omega m^{-1} \iota_\omega^t)$$

~~pos. def. on V_ω~~

$$\frac{1}{ms^2+k} = \sum_\omega \frac{1}{s^2+\omega^2} \underbrace{\iota_\omega m^{-1} \iota_\omega^t}_{\text{add up to } m^{-1}: V^* \to V}$$

$$j \frac{1}{ms^2+k} j^t = \sum_\omega \frac{1}{s^2+\omega^2} j_\omega m^{-1} j_\omega^t \qquad V_\omega \xhookrightarrow{\iota_\omega} V \xrightarrow{j} V/W$$

$$\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) = \frac{2i\omega}{s^2+\omega^2}$$

Start again $q(t)\in V$, K.E. $= \frac{1}{2}\dot{q}^t m \dot{q}$, P.E. $= \frac{1}{2} q^t k q - q^t F(t)$ $\quad m: V \to V^*$, $k:$

$$m\ddot{q} + kq = F(t) = Re(\hat{F}(s) e^{st}) \qquad s = -i\omega$$

$$\hat{q}(s) = (ms^2+k)^{-1}\hat{F}(s). \qquad \omega = \sqrt{\frac{k}{m}}$$

m, k simultaneously diagonalize, ~~orth~~ canonical splitting $V = \bigoplus_{\omega>0} V_\omega \qquad V_\omega = \{\xi\in V \mid \underbrace{(m\omega^2-k)(\xi)}_{=0}\}$

$$(ms^2+k)^{-1} : V^* \longrightarrow V$$

$$V = \oplus V_\omega$$

$$\begin{tikzcd} V_\omega \arrow[r, "\iota_\omega"] & V \\ V_\omega^* & V^* \arrow[l, "\iota_\omega^t"'] \arrow[u, "(ms^2+k)^{-1}"'] \end{tikzcd}$$

Point is that $(ms^2+k)^{-1} = \sum_\omega \frac{1}{}$

$$(ms^2+k)^{-1} = (m(s^2+m^{-1}k))^{-1} = (s^2+m^{-1}k)^{-1}m^{-1} :$$

$$\begin{tikzcd} V^* \arrow[r] \arrow[d, "\iota_\omega^t"'] & V \\ \oplus V_\omega^* \arrow[r] & \oplus V_\omega \arrow[u, "\iota_\omega"'] \end{tikzcd}$$

Again: the aim is to describe response functions assoc. to a Lagrange type harmonic oscillator

$m, k: V \longrightarrow V^*$ symm pos. def.

$$\hat{q} = (ms^2+k)^{-1}\hat{F}$$

$$\begin{tikzcd} (V/W)^* \arrow[r, "m_\omega^{-1}"] \arrow[d, "j^t"'] & V/W \\ V^* \arrow[r, "(ms^2+k)^{-1}"'] & V \arrow[u, "j"'] \end{tikzcd}$$

$$j(ms^2+k)^{-1}j^t : (V/W)^* \longrightarrow (V/W)$$

But $V = \bigoplus_\omega V_\omega \qquad m = \oplus m_\omega \qquad k = \oplus k_\omega$

$$V^* = \bigoplus_\omega V_\omega^* \qquad \boxed{\omega^2 m_\omega = k_\omega}$$

$$ms^2+k = \bigoplus_\omega m_\omega(s^2+\omega^2) \quad \text{on } \bigoplus V_\omega$$

$$(ms^2+k)^{-1} = \underbrace{\bigoplus_\omega \frac{1}{s^2+\omega^2}\, m_\omega^{-1}} \;:\; \bigoplus V_\omega^* \to \bigoplus V_\omega$$

$$\sum_\omega \frac{1}{s^2+\omega^2}\, \iota_\omega m_\omega^{-1} \iota_\omega^t$$

$$V_\omega \xrightarrow{\iota_\omega} V$$
$$V_\omega^* \xleftarrow{\iota_\omega^t} V^*$$

$$j(ms^2+k)^{-1}j^t = \sum \frac{1}{s^2+\omega^2}\, j\iota_\omega m_\omega^{-1} \underbrace{\iota_\omega^t j^t}_{{}^t(j\iota_\omega)}$$

So the response function is ~~apparently~~ of the form

$$\sum_\omega \frac{1}{s^2+\omega^2} A_\omega$$

where A_ω is a non-negative q.f. on V^*. $A_\omega = j\iota_\omega m_\omega^{-1}\,{}^t(j\iota_\omega)$

$$\sum A_\omega = j m^{-1}\,{}^t j \qquad \text{values at } s=\infty$$

$$\sum \frac{1}{\omega^2} A_\omega = \sum j\iota_\omega \frac{1}{\underbrace{\omega^2 m_\omega}_{k_\omega}}\,{}^t(j\iota_\omega) = j k^{-1} j^t$$

∴ value at $s=0$.

Curious path in the space of pos. definite quad forms from m^{-1} to k^{-1}, not really: if $j=1$, then it's the inverse of the linear path ms^2+k.

Contrast with the response function for an LC-circuit.

$$\left(\frac{s^{-1}+\omega^2 s}{s^2+\omega^2}\right)^{-1} = \frac{1+\omega^2}{s^{-1}+\omega^2 s} = \frac{s(1+\omega^2)}{s^2+\omega^2} = \underbrace{\frac{2s}{s^2+\omega^2}}_{\frac{1}{s-i\omega}+\frac{1}{s+i\omega}}\, \frac{1+\omega^2}{2}$$

$$\boxed{\sum_\omega A_\omega \frac{s(1+\omega^2)}{s^2+\omega^2}} \quad \text{versus} \quad \boxed{\sum \frac{1}{s^2+\omega^2} A_\omega}$$

481 Go over response for an electric circuit.

$$E = L\dot{I} \qquad I = C\dot{E}$$

$$\begin{array}{ccc} \bar{C}^0 & \xrightarrow{d} & C^1 = C^{1,C} \oplus C^{1,L} \\ & & \Big\downarrow Z_s^{-1} = \begin{pmatrix} Cs & 0 \\ 0 & (Ls)^{-1} \end{pmatrix} \\ \bar{C}_0 & \xleftarrow{\partial = {}^t d} & C_1 = C_1^C \oplus C_1^L \end{array}$$

Get ${}^t d\, Z_s^{-1} d : \bar{C}^0 \longrightarrow \bar{C}_0$. Write $d = \begin{pmatrix} d^C \\ d^L \end{pmatrix}$

$${}^t d\, Z_s^{-1} d = \begin{pmatrix} {}^t d^C & {}^t d^L \end{pmatrix} \begin{pmatrix} Cs & \\ & (Ls)^{-1} \end{pmatrix} \begin{pmatrix} d^C \\ d^L \end{pmatrix} = s\left({}^t d^C C d^L\right) + s^{-1}\left({}^t d^L L d^L\right)$$

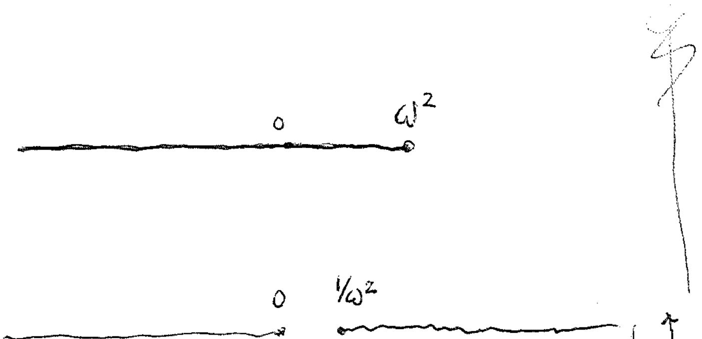
This is confusing, but becomes better when you adopt Hilbert picture: $V \subset H^+ \oplus H^-$ $\quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix}$

$F = +1$ on $V \mid -1$ on $V^\perp$. Know ~~...~~

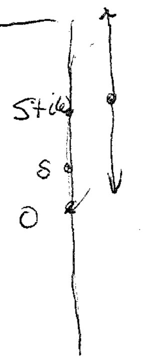
Compare variables. $\dfrac{s(1+\omega^2)}{s^2+\omega^2}$, $\dfrac{1}{s^2+\omega^2}$ rational function of s. If $s \in i\mathbb{R}$, former in $i\mathbb{R}$ latter in $\mathbb{R}$. What about $\mathrm{Re}(s) > 0$. Apparently $s \mapsto \dfrac{1}{s^2+\omega^2}$ maps $\mathrm{Re}(s) > 0$?

$$s \longmapsto s^2 \longmapsto s^2+\omega^2$$

$$\{\mathrm{Re}(s) > 0\} \mapsto \{s^2 \notin \mathbb{R}_{\le 0}\} \longmapsto \{s^2+\omega^2 \notin \omega^2 + \mathbb{R}_{\le 0}\}$$

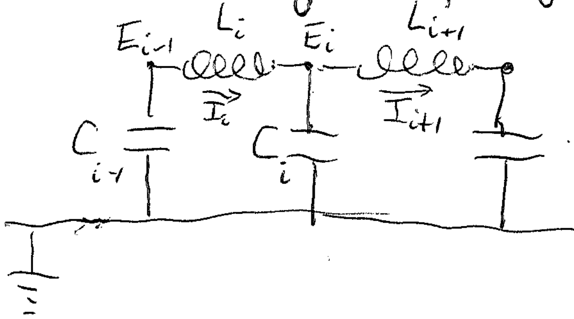


$$\frac{2s}{s^2+\omega^2} = \frac{1}{s-i\omega} + \frac{1}{s+i\omega}$$



 The surprise is that the response of the harmonic oscillator leaves much to be desired.

Examples of coupling



$$C_i s E_i = I_i - I_{i+1}$$

$$I_i = \underline{I_{i+1} + C_i s E_i}$$

$$L_i s I_i = E_{i-1} - E_i$$

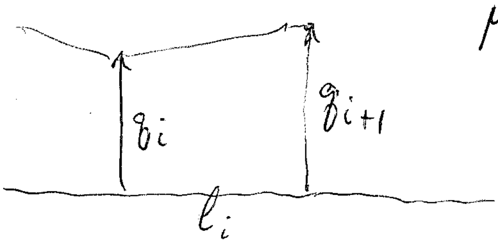
$$E_{i-1} = E_i + L_i s I_i$$

$$\begin{pmatrix} E_{i-1} \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & L_i s \\ & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_i \end{pmatrix} \qquad \begin{pmatrix} E_i \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_i s & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_{i+1} \end{pmatrix}$$

$$\begin{pmatrix} E_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_1 s & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ C_n s & 1 \end{pmatrix} \begin{pmatrix} E_n \\ I_{n+1} \end{pmatrix}, \quad I_{n+1} = 0$$

$$\frac{E_0}{I_1} = L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{L_2 s + \cfrac{1}{C_2 s + \cdots}}}$$

string picture



$$\mu_i = \frac{q_i - q_{i+1}}{l_i} \qquad q_i$$

$$m_i s^2 q_i = \mu_{i+1} - \mu_i$$

$$m_i s^2 q_i = \underbrace{\frac{q_{i+1} - q_i}{l_i}}_{-\mu_i} - \underbrace{\frac{q_i - q_{i-1}}{l_{i-1}}}_{-\mu_{i-1}}$$

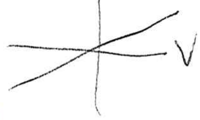
$$\boxed{\mu_{i-1} = \mu_i + m_i s^2 q_i}$$

$$q_{i+1} - q_i = l_i(-\mu_i)$$

$$\boxed{q_i = q_{i+1} + l_i \mu_i}$$

483 Can you relate ~~[illegible]~~ LC circuits to partial unitaries?

Idea – start from quadratic form rational in s with some positivity property. First significant point is that a quadratic form on V is equivalent to a maximal isotropic subspace of $V \oplus V^*$, V^* (symplectic) ~~[illegible]~~ which is transversal to V^*.



So the point may be that you have some ~~[illegible]~~ sort of holom map ~~[illegible]~~ into ~~[illegible]~~ ?

Look carefully at dim 1 case. from an LC circuit we get a rational function of s of the form $f(s) =$

$$\sum_{0 \le \omega \le \infty} a_\omega \frac{s(1+\omega^2)}{s^2+\omega^2} \quad \text{with} \quad a_\omega \ge 0 \qquad \sum a_\omega = 1$$

~~[illegible]~~ Properties: $f(s)$ analytic off $i\mathbb{R}$

$\mathrm{Re}(s) \gtreqless 0 \implies \mathrm{Re}(f(s)) \gtreqless 0.$

$\overline{f(s)} = f(\bar{s})$.

$f(-s) = -f(s)$

~~[illegible]~~ What can you say about the degree of f, i.e. the number of poles $=$ $2 \times$ number of $a_\omega > 0$ for $0 < \omega < \infty$ $+1$ ~~[illegible]~~ if $a_0 > 0$ $+1$ if $a_\infty > 0$.

number of parameters is the degree.

Next look at partial ~~[illegible]~~ unitaries. Here the ~~[illegible]~~ we get $g(z)$ rational function of z such that

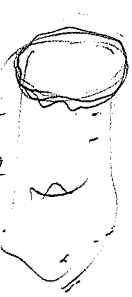
~~[illegible]~~ $|z| \gtreqless 1 \implies |g(z)| \gtreqless 1.$ ~~[illegible]~~ I think $g(z)$ has the form $c \prod_{i=1}^{n} \frac{z-\alpha_i}{1-\bar{\alpha}_i z}$, $|c| = 1$, $|\alpha_i| < 1$. Degree of g = ~~number of times~~ degree of map $S^1 \to S^1$ $= n$.

~~[illegible]~~ number of parameters to ~~[illegible]~~ describe g is $2n+1$ real parameters, ~~maybe 2n+1 if you count~~ c is the constant in S^1!

 There is this Schwartz lemma to construct by coupling.

Another idea is $U(1,1)$ picture. Partial unitary should give by scattering an analytic map ~~into the dis~~ of the unit disk into a symmetric space for $U(p,p)$ maximal isotropic on the bdry?

Go back to partial unitaries, describe how to calculate scattering. Next point Go back over your approach to CFT. Oriented circle is space. Consider smooth functions modulo constants with skew form $\int f\,dg$ obvious real structure. When the circle is boundary of the disk you get a polarization. When the circle is the boundary of surface with g holes you find not enough bdry values of holom. functions. ~~Then try Θ functions.~~ Then the point was to try $\log f$ where f is invertible holom. function.



Anyway continue! ~~Let us decide what to do.~~ Either make notes for electric circuits, or understand partial unitaries better.

Basic construction given $W \underset{b}{\overset{a}{\rightrightarrows}} V$. ~~What's important here is~~

$$\begin{array}{c|c|c} \oplus\, u^{-2}\mathrm{Ker}(a^*) \oplus u^{-1}\mathrm{Ker}(a^*) \oplus & \overbrace{\mathrm{Im}(b) \oplus \mathrm{Ker}(b^*)}^{V} & \oplus\, u\,\mathrm{Ker}(b^*) \oplus \cdots \\ \oplus \qquad\qquad \oplus & \mathrm{Ker}(a^*) \oplus \mathrm{Im}(a) & \oplus\, u\,\mathrm{Ker}(b^*) \end{array}$$

$$\mathrm{Im}(b) \xrightarrow{ab^{-1}} \mathrm{Im}(a)$$

The idea here is that we have isos (no bd states)

$$\bigoplus_{n\in\mathbb{Z}} u^n\,\mathrm{Ker}(a^*) \xrightarrow{\sim} H \xleftarrow{\sim} \bigoplus u^n\,\mathrm{Ker}(b^*)$$

such an isom should be given by $g(z) : S^1 \longrightarrow \mathrm{Unit}\begin{pmatrix}\mathrm{Ker}(a^*),\\ \mathrm{Ker}(b^*)\end{pmatrix}$

You need control of this

You want to compute scattering.

First case $\dim(V)=2$ $\dim(W)=1$. You want a ~~map~~ ~~unitary~~ map $\mathrm{Ker}(a^*) \longrightarrow \mathrm{Ker}(b^*)$ depending on z.

Idea:

$$\begin{tikzcd} & \mathrm{Ker}(a^*) \arrow[d, phantom, "\cap"] \arrow[dr, "\sim"] & & \\ W \arrow[r, "az+b"] & V \arrow[r] & L_z \arrow[r] & 0 \\ & \mathrm{Ker}(b^*) \arrow[u, phantom, "\cup"] \arrow[ur] & & \end{tikzcd}$$

Alternative approach — look at inclusion of outgoing subspaces

$$\begin{tikzcd}[column sep=small] \overset{n}{\mathrm{Im}(b)} \arrow[dr] & \oplus & \overset{1}{\mathrm{Ker}(b^*)} \arrow[dr] & \oplus & U\mathrm{Ker}(b^*) \arrow[dr] & \oplus & \cdots \\ \mathrm{Ker}(a^*) & \oplus & \mathrm{Im}(a) & \oplus & U\mathrm{Ker}(b^*) & \oplus & \cdots \end{tikzcd}$$

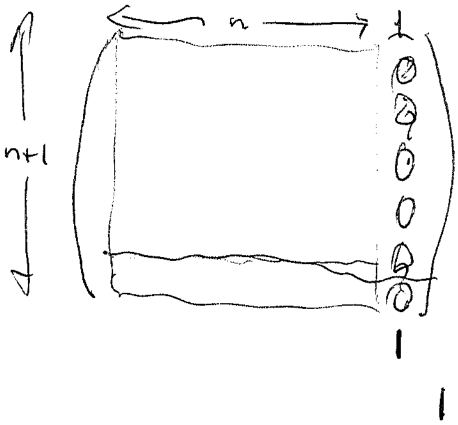
So the idea is to take $\mathrm{Im}(b) \oplus \left(\mathrm{Ker}(b^*) \oplus U\,\mathrm{Ker}(b^*) \oplus \cdots \right)$

~~[illegible]~~ Make a shrewd guess. Start with $bW \subset V$

Choose basis so that $bW = \mathbb{C}^n \subset \mathbb{C}^{n+1} = V$.

$|a_{11}|^2 + |a_{12}|^2 = 1$

$n=1$ $\quad \mathbb{C} \overset{b}{\subset} \mathbb{C}^2$



$$a\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\a_{12}\end{pmatrix}$$

$$S = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

this is an isometry $\quad S^*S = 1$.

OKAY — this clarifies things, namely in general given $W \underset{b}{\overset{a}{\rightrightarrows}} V$ partial unitary you can form a partial isom

$$bW \oplus \mathrm{Ker}(b^*) \oplus U\,\mathrm{Ker}(b^*)$$

$$aW \xrightarrow{ab^{-1}} \mathrm{Ker}(a^*)$$

and so forth.

486 You feel somehow that it should be possible to write down eigenvectors for this operator. ~~Is it possible for me to regard~~

$$S = \begin{pmatrix} a_{11} & & & \\ a_{12} & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \\ & & & \ddots \end{pmatrix}$$

Use the structure theory

What is the spectrum of S

all of $|z| < 1$.

$\mathrm{Ker}(S^* - \lambda) \neq 0$

$$S^* \sum_{n \geq 0} c_n S^n \xi = \sum_{n \geq 1} c_n S^{n-1} \xi + c_0 S^* \xi$$

$$\lambda \sum_{n \geq 0} c_n S^n \xi = \sum_{n \geq 1} \lambda c_{n-1} S^{n-1} \xi$$

need $c_0 S^* \xi = 0$ and $c_n = \lambda c_{n-1} \Rightarrow c_n = \lambda^n c_0$

$$\sum_{n \geq 0} \lambda^n S^n \xi \qquad \text{where } S^* \xi = 0.$$

$$\xi = \begin{pmatrix} \bar{a}_{12} \\ -\bar{a}_{11} \\ 0 \\ 0 \end{pmatrix}$$

$$S\xi = \begin{matrix} a_{11} \bar{a}_{12} \\ a_{12} \bar{a}_{12} \\ -\bar{a}_{11} \end{matrix}$$

$$S^* - \lambda = \begin{pmatrix} \bar{a}_{11} - \lambda & \bar{a}_{12} & 0 & & \\ & -\lambda & 1 & 0 & \\ & & -\lambda & 1 & 0 \\ & & & -\lambda & 1 \end{pmatrix}$$

$$(\bar{a}_{11} - \lambda) c_1 + \bar{a}_{12} c_2 = 0$$
$$-\lambda c_2 + c_3 = 0$$
$$-\lambda c_3 + c_4 = 0$$

$|a_{11}|^2 + |a_{12}|^2 = 1$

If $|a_{12}|^2 = 0$, then $aW = bW$.

$$\bar{a}_{12} c_2 = (\lambda - \bar{a}_{11}) c_1 \qquad c_3 = \lambda c_2$$

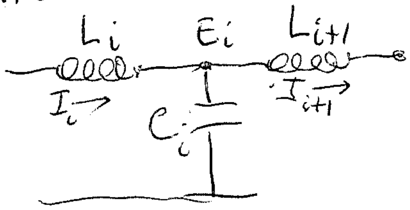
$$c_2 = \left(\frac{\lambda - \bar{a}_{11}}{\bar{a}_{12}} \right) c_1 \qquad c_4 = \lambda c_3$$

487 So it seems we have a eigenvector

~~these~~ $z^0, z^1, z^2, \ldots$ as basis for $\mathrm{Ker}(b^*) \cup \mathrm{Ker}(d^*)$

Then got $$1+\frac{\lambda-\bar{a}_{11}}{\bar{a}_{12}}\left(z+\lambda z^2+\lambda^2 z^3+\cdots\right)$$

What do we need to connect transmission lines. First derive transmission line equations.



$$C_i s E_i = I_i - I_{i+1}$$
$$L_i s I_i = E_{i-1} - E_i$$

$$\boxed{\begin{aligned}\gamma s E &= -\partial_x I\\ \lambda s I &= -\partial_x E\end{aligned}}$$

γ capac / length

λ ind / length

$$\gamma\,\partial_t E = -\partial_x I$$
$$\lambda\,\partial_t I = -\partial_x E$$

$$\lambda\gamma\,\partial^2_{tt}E = -\lambda\partial^2_{tx} I = \partial^2_{xx}E$$

$$\left(\partial^2_{xx} - \lambda\gamma\,\partial^2_{tt}\right)E = 0$$

$\frac{1}{\sqrt{LC}}$

$$e^{i(kx-\omega t)}\hat{E} \qquad -k^2+\lambda\gamma\omega^2 = 0$$

$$\left(\frac{\omega}{k}\right) = \frac{1}{\sqrt{\lambda\gamma}} = c$$

freq. indep.

On a ~~[illegible]~~ transmission line you have waves travelling in two directions. Assume $\gamma=\lambda=1$

$$\partial_t E + \partial_x I = 0$$
$$\partial_t I + \partial_x E = 0$$

$$E = \hat{E}e^{ik(x-t)} \qquad \partial_t E = -ik\hat{E}e^{ik(x-t)}$$
$$I = \hat{I}e^{ik(x-t)} \qquad \partial_x I = ik\hat{I}e^{ik(x-t)}$$

$$\hat{E} = \hat{I}.$$

$$\begin{pmatrix}E\\ I\end{pmatrix} = \begin{pmatrix}\hat{E}\\ \hat{I}\end{pmatrix}e^{-ik(x+t)} \qquad \hat{E} = -\hat{I}.$$

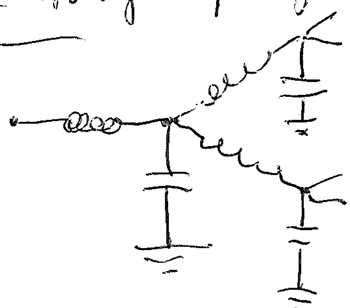
So if I use transmission line need both directions.

488 Jan 18, 98 Return to computing scattering ~~matrix~~ operator for a partial ~~an incomplete~~ unitary. Your missing the ~~...~~ resolvent type calculation.

IDEA: Cramer's rule maybe gives ~~giving~~ a formula for the eigenvector. In the case of ~~...~~ a Jacobi matrix the ~~successive~~ orthogonal polys might be given by characteristic polys.

Eigenvectors - really the incoming and outgoing ones.
Maybe there's a Toe matrix appropriate to the partial unitary based on ~~...~~ expressing it as an iterated coupling

Examples: Study on the $SL_2(\mathbb{Z})$ tree the LC circuit with capacitances at the vertices and inductances for the edges. Mechanically considering a discrete string vibrating transversally to the plane with masses at the vertices and weightless string along the edges. Other possibility ~~...~~ would be a ~~uniform mass density~~ string with mass (uniform to begin with) so that one has waves. You maybe can consider ~~...~~ vibrations in the plane of the string to get a different 3 way coupling at the vertex. Limitations - All these things seem to ~~...~~ bring in the C^* alg generated by $SL_2(\mathbb{Z})$ which should contain free subgroups of finite ~~...~~ index - in fact, there should be a map $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6 \to \mathbb{Z}/12$, whence a free subgroup of index 12.

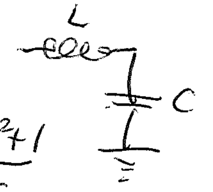


$$Z = Ls + \frac{1}{Cs + \frac{2}{Z}} = Ls + \frac{Z}{CsZ+2}$$

$$Z(CsZ+2) - Ls(CsZ+2) = Z$$

$$(Cs)Z^2 + (\underbrace{2 - LCs^2 - 1}_{1-LCs^2})Z - 2Ls = 0$$

$$\begin{aligned} b^2 - 4ac &= (1-LCs^2)^2 - 4(Cs)(-2Ls) \\ &= 1 - 2LCs^2 + L^2C^2s^4 + 8LCs^2 \\ &= 1 + 6LCs^2 + (LCs^2)^2 \end{aligned}$$

489 natural osc. for  is $\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$

$$Z = Ls + \frac{1}{Cs} = \frac{LCs^2+1}{Cs}$$

$$LC(-\omega_0^2) + 1 = 0 \qquad \text{assume } \omega_0 = 1$$

$$Z = \frac{-(1-s^2) \pm \sqrt{1+6s^2+s^4}}{2Cs} \qquad s^2 = -3 \pm \sqrt{8} = -3 \pm \underbrace{2\sqrt{2}}_{2.808}$$

$$\simeq \frac{-(1-s^2) \pm \sqrt{(s^2+5.8)(s^2+.2)}}{2Cs}$$

$$2Z(s) = 2Ls + \frac{2}{Cs + \frac{2}{Z}} = (2L)s + \frac{1}{\left(\frac{C}{2}\right)s + \frac{1}{Z}}$$

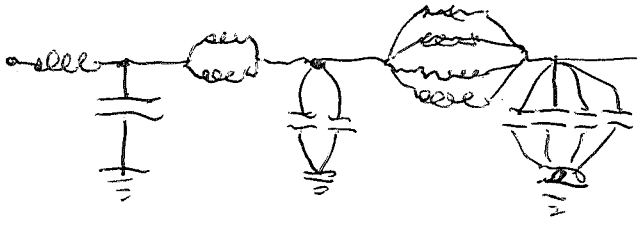
$$Z = Ls + \frac{1}{Cs + \frac{1}{\frac{1}{2}Z}}$$

$$= Ls + \frac{1}{Cs + \frac{1}{\frac{1}{2}Ls + \frac{1}{2Cs + \frac{4}{Z}}}}$$

$$\cancel{\begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2}Ls \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2Cs & 1 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} 1+L^2C^2s^2 & Ls \\ Cs & 1 \end{pmatrix}\begin{pmatrix} 1+L^2C^2s^2 & \frac{1}{2}Ls \\ 2Cs & 1 \end{pmatrix}}$$

By symmetry



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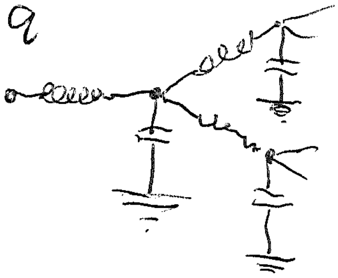
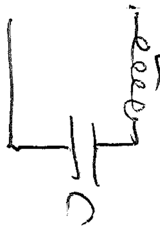


$$Z(s) = Ls + \frac{1}{Cs + \frac{2}{Z(s)}}$$



$$Z = Ls + \frac{Z}{ZCs+2}$$

$$Z(ZCs+2) = Ls(ZCs+2) + Z$$

LCs^2+1

$\omega^2 = \frac{1}{LC}$

$$Z^2(Cs) + (2-1-LCs^2)Z - 2Ls = 0.$$

$$Cs\,Z^2 + (1-LCs^2)Z - 2Ls = 0$$

$$Z = \frac{-(1-LCs^2) \pm \sqrt{(1-LCs^2)^2 + 8LCs^2}}{2Cs.}$$

$$b^2 - 4ac = (1-LCs^2)^2 - 4(Cs)(-2Ls)$$

$$= 1 - 2LCs^2 + L^2C^2s^4 + 8LCs^2$$

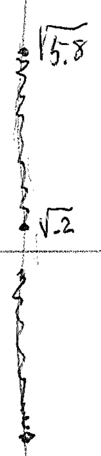
$$= 1 + 6LCs^2 + L^2C^2s^4$$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$$= 1 + 6\left(\frac{s}{\omega_0}\right)^2 + \left(\frac{s}{\omega_0}\right)^4$$

$$\left(\frac{s}{\omega_0}\right)^2 = \frac{-6 \pm \sqrt{36-4}}{2} = \frac{-6 \pm 4\sqrt{2}}{2} = -3 \pm 2\sqrt{2}$$

$$b^2 - 4ac = \left(\left(\frac{s}{\omega_0}\right)^2 + 5.8\right)\left(\left(\frac{s}{\omega_0}\right)^2 + .2\right)$$



Observation concerning a ribbon graph. Put a uniform massive ~~string~~ at each edge – associate with each side of the edge waves travelling ~~counter~~ in the pos. direction.

Couple at a vertex an incoming waves to the appropriate outgoing wave in the cyclic order.

Philosophy. A partial unitary of codim 1 can be coupled to a transmission line to yield a scattering function (What is the meaning of a real partial unitary?)

What is a 2-port? A partial unitary of codim 2 where $\mathrm{Ker}(a^*)$ and $\mathrm{Ker}(b^*)$ are each split into orthogonal lines. Similarly for a 3 port.

Simplest kind of 1-port has $\dim(V) = 1$ $W=0$.

$$u^{-1}\,\mathrm{Ker}(a^*) \oplus \mathrm{Im}(b) \oplus \mathrm{Ker}(b^*)$$

$$\oplus\ \mathrm{Ker}(a^*) \oplus \mathrm{Ker}\, a$$

have $\mathrm{Ker}(a^*) = \mathrm{Ker}(b^*) = V$ and the scattering matrix is the identity. Next comes $\dim(V) = 2$. $\dim(W) = 1$.

You might get something cleaner provided you specified $\mathrm{Ker}(a^*)$ and $\mathrm{Ker}(b^*)$ first.

Data an isomorphism $\mathrm{Ker}(a^*) \oplus W \simeq W \oplus \mathrm{Ker}(b^*)$

Can you compose? Given $\mathrm{Ker}(b^*) \oplus X \simeq X \oplus \mathrm{Ker}(c^*)$, then get $\mathrm{Ker}(a^*) \oplus W \oplus X \xrightarrow{\sim} W \oplus \mathrm{Ker}(b^*) \oplus X \xrightarrow{\sim} W \oplus X \oplus \mathrm{Ker}(c^*)$.

491 So what? Try then to identify an unitary isomorphism $K \oplus W \simeq W \oplus L$ with a scattering operator $g(z) : K \xrightarrow{\sim} L$ analytic $|z| < 1$ unitary boundary values. You form.

$$\cdots \oplus u^{-2}K \oplus u^{-1}K \oplus W \oplus L \oplus uL \oplus \cdots$$

$$\searrow^{u} \qquad \searrow^{1} \qquad \searrow^{u}$$

$$\cdots \oplus u^{-1}K \oplus K \oplus W \oplus uL \oplus$$

This is somehow the Eilenberg trick.

Notice that an isom. $K \oplus W \simeq W \oplus L$ is ~~the~~ ~~same as~~ a stable isomorphism between K and L. Does a stable isomorphism between K and L lead to an isomorphism $K[t, t^{-1}] \xrightarrow{\sim} L[t, t^{-1}]$? Does a stable isom have a characteristic poly? Certainly you get a K-module

$$\begin{tikzcd} & K \oplus W \arrow[d, equal] \\ W \arrow[ur, hook] \arrow[dr, hook] & V \arrow[d, equal] \\ & W \oplus L \end{tikzcd}$$

How much can you do algebraically? Or is this all linked to Hilb. space. Actually what do you learn $K \oplus W \simeq W \oplus L$ yield

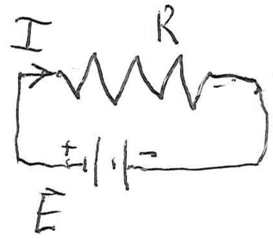
$$K \oplus K \oplus W \simeq K \oplus W \oplus L \simeq W \oplus L \oplus L.$$

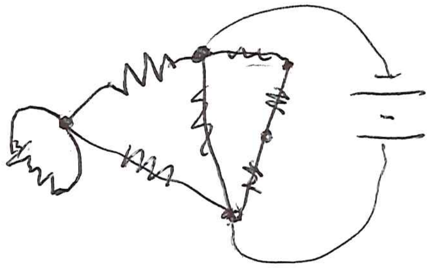
$$K[u^{-1}] \oplus W \simeq W \oplus L$$

491 for tomorrow's lecture.

resistance networks.

Ohm's Law $E = RI$





conn graph

$e = \dim C_1$

$v-1$ conditions

Bott's approach. Introduce ~~metrics~~ inner products on C^0, C^1. V ?

Look, you have to straighten out the difficulty arising from Hilbert space viewpoint and quadratic form viewpoint. ~~Quadratic form we~~

~~Hilbert viewpoint~~ LC case

Key argument

$$V = \bar{C}^0 \xhookrightarrow{d} C^1 \qquad V \to V/W$$

$$\begin{array}{ccc} V & \xhookrightarrow{\iota} & H \\ \gamma\downarrow & & \downarrow \\ V/W & \hookrightarrow & H/W \end{array}$$

On H we have a quadratic form $s\|h_+\|^2 + s^{-1}\|h_-\|^2$

restrict to V get $\langle v, (s\iota_+^*\iota_+ + s^{-1}\iota_-^*\iota_-)v\rangle$.

Given pos. def. form Q on H, how do you get the induced form on V/W? Probably minimize Q on the coset $v+W$.

$$\bar{Q}(v+W) = \inf_w Q(v+w) \qquad \text{Coset}$$

say $Q(v) = \langle v, Qv\rangle = v^t Q v$

 Assume $Q(v+w)$ min. at w_0. Then

$$\delta(v^t Q v) = (\delta v)^t Q v + v^t Q(\delta v)$$
$$= (\delta w)^t Q v + w^t Q\, \delta w$$
$$= 2(\delta w)^t Q v$$

?

$\therefore$ $Qv \perp W$.

Start again. Given $v \in V$. Let w_0 be $\ni$ $(v+w_0)^t Q (v+w_0)$ is min $\Longrightarrow$ $Q(v+w_0) \perp W$.

$$\begin{tikzcd} W \arrow[r, hook] & \overset{v}{V} \arrow[r] \arrow[d, "Q"] & V/W \\ W^* & \underset{Qv}{V^*} \arrow[l] & \end{tikzcd}$$

to ~~solve~~ find w_0 such that

$$w^t Q(v+w_0) = 0 \ \forall w$$

$$w^t Q v + w^t Q w_0 = 0$$

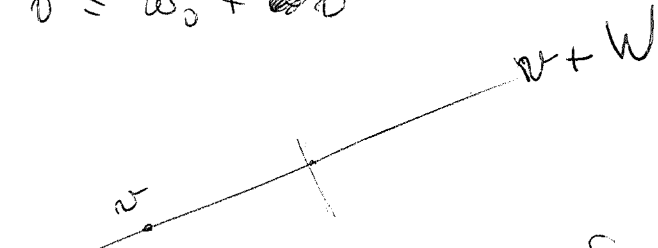
want $Qv + Qw_0 \in W^\perp$

Question: Given $Q(v,v')$ symm. pos. def bilinea form. Choose $w_0 \in W$ $\ni$ $Q(v+w_0, v+w_0)$ min.

Then ~~$Q(v+w_0, W)=0$~~. $Q(W, v+w_0) = 0$

i.e. $v + w_0 \in W^\perp$ wrt Q. Thus you write

$v = w_0 + v'$



Point is $Q(v,-)$ on W is represented $Q(w_0,-)$

$\therefore$ $Q(v - w_0, W) = 0$.

So from concrete viewpoint. you take v to $Q(v,-)$, then invert $Q|W$

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & H \arrow[d] \\ V/W \arrow[r, hook] & H/W \end{tikzcd}$$

quadratic form pos. def on H restrict to V then push down to V/W or first push down to H/W then restrict to V/W

in both cases you get the ~~restrict~~ of the q form to the orthogonal complement of $W \subset V$. How do you calculate it? Take $v + W \in V/W$, ~~then~~ if $v \overset{Q}{\perp} W$ nothing further needed, otherwise look at $Q(v,-)$ ~~in~~ W^*, use Q rest. to W gives iso $W \to W^*$ to write $Q(v,-) = Q(w,-)$, and then $v - w \overset{Q}{\perp} W$. Notice to do explicitly you need to invert $W \xrightarrow{\sim} W^*$

So start with $Q = s\|h_+\|^2 + s^{-1}\|h_-\|^2$ on H. Look at ~~at~~ the arb. subspace W and calculate the restriction ?

Given $K \oplus W \xrightarrow{\sim} W \oplus L$ unitary iso f.d. Hilb. spaces.

Use to define a unitary U on $\bigoplus_{n\le 0} U^n K \oplus W \oplus \bigoplus_{n\geq 1} U^n L = H$

Look at this as a ~~mod~~ module over $\mathbb{C}[u,u^{-1}]$ containing K and L, so we get maps.

$$\bigoplus_{n\in\mathbb{Z}} U^n K \longrightarrow H \longleftarrow \bigoplus_{n\in\mathbb{Z}} U^n L$$

commuting with U. You also have projections in the other direction. Algebraically it might be interesting to ask when the bound states split off leaving the scattering isom.

Actually something subtle seems to be ~~~~ happening here. You seem to be getting a lot of structure out of a correspondence. Suppose you were to consider $K, L \subset V$ and an isom $V/K \simeq V/L$. in ~~contrast~~ contrast to what you looked at already $\mathrm{Im}(b)$, $\mathrm{Im}(a) \subset V$ and ~~~~ the isomorphism $ab^{-1}: \mathrm{Im}(b) \xrightarrow{\sim} \mathrm{Im}(a)$. You seem now to have some kind of spectral theory

494 Jan 19. today's lecture in 4 hours.

Let's spend the next few hours getting formulas straight. Subspace $V \subset H^+ \oplus H^-$. ~~Suppose~~ $\cancel{V = \begin{pmatrix}1\\1\end{pmatrix} H^+}$

$\varepsilon = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ $\quad \varepsilon = \varepsilon^* = \varepsilon^{-1}$ $\quad F = \begin{matrix}+1 & \text{on } V\\ -1 & \text{on } V^\perp\end{matrix}$ $\quad F = F^* = F^{-1}$

group generated by 2 inv. is a dihedral group. $g = F\varepsilon$

$\varepsilon g \varepsilon^{-1} = \varepsilon F \varepsilon \varepsilon = \varepsilon F = g^{-1}$. Get splitting into indecomposable orthogonal reprs. of dihedral group.

$g = +1$ $\quad \varepsilon = F = 1$
$\qquad\quad\;\; \varepsilon = F = -1$

$g = -1$ $\quad \varepsilon = 1 = -F$
$\qquad\quad\;\; \varepsilon = -1 = F$

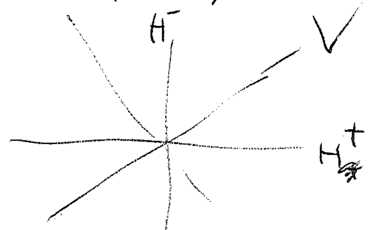
~~one dim~~ V 1-diml.

$0 < \theta < \pi/2 < \theta < \pi$

$$g = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \quad \varepsilon = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{on } \mathbb{R}^2$$

$$F = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \qquad V = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\mathbb{R} \qquad V^\perp = \begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}$$

$$= \begin{pmatrix}1\\ \tan\theta\end{pmatrix}\mathbb{R}$$



$V = \begin{pmatrix}1\\ T\end{pmatrix} H^+$ put $X = \begin{pmatrix}0 & -T^*\\ T & 0\end{pmatrix}$ skew adj.

$$g = \frac{1+X}{1-X}$$

$V = \begin{pmatrix}1\\ T\end{pmatrix} H^+ \qquad V^\perp = \begin{pmatrix}-T^*\\ 1\end{pmatrix} H^-$

$$F\begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix} = \begin{pmatrix}1 & +T^*\\ T & -1\end{pmatrix} = \begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$F(1+X) = (1+X)\varepsilon$$

$$1+X = F(1+X)\varepsilon = F\varepsilon(1-X) = g(1-X).$$

$$F = \frac{(1+X)^2}{1-X^2}\,\varepsilon = \begin{pmatrix} 1-T^*T & -2T^* \\ 2T & 1-TT^* \end{pmatrix} \begin{pmatrix} 1+T^*T & \\ & 1+TT^* \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1-T^*T}{1+T^*T} & \cancel{+2T^* / 1+TT^*} \\ \cancel{\frac{2T}{1+T^*T}} & -\frac{1-TT^*}{1+TT^*} \end{pmatrix}$$

not what's important

Back to stuff on quadratic forms.

$$\begin{array}{ccc} V & \hookrightarrow & H^+ \oplus H^- = H \\ \downarrow & & \downarrow \\ V/W & \hookrightarrow & H/W \end{array}$$

Have $Q_s(h) = s\|h_+\|^2 + s^{-1}\|h_-\|^2$ on H. Get ~~the~~ induced quadratic form on V/W.

$v \mapsto (v' \mapsto \langle Tv, v'\rangle)$

$V \xrightarrow{T} V^*$ same as $\langle Tv_1, v_2\rangle$ bilinear form.

$V = V^{**} \xrightarrow{T^t} V^*$ $\quad v \mapsto (\lambda \mapsto \langle \lambda, v\rangle) \mapsto \langle \lambda, Tv\rangle$

$(v' \mapsto Tv' \mapsto \langle Tv', v\rangle)$

$T^t(v)$ is $v' \longmapsto \langle Tv', v\rangle$ $\qquad T = T^t \iff \langle Tv', v\rangle = \langle Tv, v'\rangle$.

T nondeg. when T an isom. $\quad T^{-1}: V^* \longrightarrow V = (V^*)^*$

$T^{-1}(\lambda_1, \lambda_2) = \langle \lambda_1, T^{-1}\lambda_2\rangle = \langle T(T^{-1}\lambda_1), T^{-1}\lambda_2\rangle$

Focus upon your aim. $\quad s, s^{-1}$

$V \qquad H \qquad q_s : H \longrightarrow H^*$

$V/W \qquad H/W$

$$\begin{array}{ccc} V & \xrightarrow{i} & H \\ \downarrow & & \downarrow q_s \\ V^* & \xleftarrow{i^*} & H^* \end{array}$$

$i^* q_s i$

s, s^{-1}

$g: H \to H^*$

$$\begin{tikzcd}
W \arrow[r, hook] & H \arrow[r, "j"] \arrow[d, shift right, "g"'] & H/W \arrow[d, shift right] \\
W^* & H^* \arrow[l, two heads] \arrow[u, shift right, "g^{-1}"'] & (H/W)^* \arrow[l, "j^t"'] \arrow[u, shift right, "j g^{-1} j^t"']
\end{tikzcd}$$

How to do this?

$$\begin{tikzcd}
& (H/W)^* \arrow[d, "j"] \arrow[dr] & \\
W \arrow[r] \arrow[dr, "\varsigma"'] & H \arrow[r] \arrow[d, "j"] & H/W \\
& W^* &
\end{tikzcd}$$

$\lambda = \cos\theta$

$V \subset H^+ \oplus H^-$ $\qquad \frac{1}{2}(F\varepsilon + \varepsilon F) = \lambda \in (-1,1)$

Remove where $g = F\varepsilon$ is -1. $\qquad \omega = \frac{\sqrt{1-\lambda^2}}{\lambda}$

$F\varepsilon = -1$ splits into
$\varepsilon = +1 \quad F = -1 \quad V^\perp \cap H^+$
$\varepsilon = -1 \quad F = +1 \quad V \cap H^-$

$V^\perp = H^+,\ H^- = 0$
$V = H^-,\ H^+ = 0$

$V = \begin{pmatrix} 1 \\ T \end{pmatrix} H^+ \qquad V^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} H^-$

$X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ skew adj. $\qquad F\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{\varepsilon}$

$F(1+X) = (1+\cancel{X})\varepsilon$

$F = (1+X)\varepsilon(1+X)^{-1} = \overbrace{(1+X)(1-X)^{-1}}^{g}\varepsilon \qquad TT^* = \omega^2$

497 Lecture Jan 20, 98

Idea: $W \subset V$ Euclidean space, Q quadratic form (pos. def) on V, ~~$Q(v)$~~ A the corresp. pos. s.a. op:
$Q(v) = (v, Av)$.

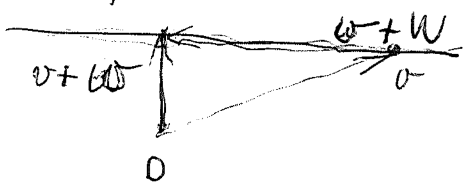
(1) Restriction of Q to W has the corresp op = the composition $W \overset{i}{\hookrightarrow} V \xrightarrow{A} V \xrightarrow{i^*} W$

(2) Induced by Q form on $V/W = W^\perp$ has the corresp op ~~composition~~ inverse of
$$W^\perp \underset{j}{\hookrightarrow} V \xrightarrow{A^{-1}} V \xrightarrow[j^*]{} W^\perp$$

Proof of (1) $\quad Q(w) = (w, Aw) = (iw, Aiw) = (w, \underline{i^*Ai}w)$

(2) What is the induced form on V/W

take coset $v + W$



so you select $w \in W$ such that $Q(v+w)$ is minimum and then take this minimum value.
$$q^{V/W}(v) = \inf_{w \in W} Q(v+w) \qquad \text{~~at a min point.~~}$$

$$Q(v+w) = (v+w, A(v+w))$$

At the minimum point $(v+w, AW) = 0$ or $A(v+w) \perp W$ or $A(v+w) \in W^\perp$. or $v + w \in A^{-1} W^\perp$ Thus

$$\begin{array}{ccccccc}
A(v+w) & \longmapsto & A(v+w) & \longmapsto & v+w & \longmapsto & v \\
\cap & & \cap & & \cap & & \cap \\
W^\perp & \underset{j}{\hookrightarrow} & V & \xrightarrow{A^{-1}} & V & \xrightarrow{j^*} & W^\perp
\end{array}$$

 Make [redacted] several attempts. ~~Start with~~ Try again -

$$\begin{tikzcd}
W \arrow[r, "i"] \arrow[d, "{}^t i Q i"'] & V \arrow[r, two heads, "p"] \arrow[d, shift right, "Q"'] & V/W \\
W^* & V^* \arrow[l, "{}^t i"] \arrow[u, shift right, "Q^{-1}"'] & (V/W)^* \arrow[l, "p^t"] \arrow[u, "pQ^{-1}p^t"']
\end{tikzcd}$$

$$(w_1, i^t Q i w_2) = (iw_1, Qiw_2)$$

~~Try~~ First method: Assume ${}^t iQi$ is an isom. Then

Consider

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, "i"] & V \arrow[r, "p"] \arrow[d, "Q"] & V/W \arrow[r] & 0 \\
0 & W^* \arrow[l] & V^* \arrow[l, "i^t"] & (V/W)^* \arrow[l, "p^t"] & 0 \arrow[l]
\end{tikzcd}$$

Assume i^tQi is an isom. Then $r = (i^tQi)^{-1} i^t Q$ is a retraction for i: $ri = 1$, so the top exact sequence. Also $Qi(i^tQi)^{-1}$ is a section of i^t, so the bottom exact sequence splits. ~~Note that so In this case we do get an induced map~~ What happens?

I know that QiW is a complement for $W^\circ = (V/W)^*$

~~QiW ... W ... V~~

$$K = \mathrm{Ker}({}^t iQ) = Q^{-1}(W^\circ).$$

$$V = iW \oplus Q^{-1}(W^\circ)$$

$$V^* = QiW \oplus W^\circ$$

Still unclear.