Compactification maybe clear – it looks just like the closure in the Grassmannian. Thus complex subspaces W of dim n in $V^+ \oplus V^- = V_c$ ~~a~~ such that $W/W\cap\overline{W}$ is positive in $W+\overline{W}/W\cap\overline{W}$.

What do I need?? ~~[illegible]~~ At this point it ~~should~~ might be possible to handle coupling to a massive simple oscillator

Jan 7. Keep on trying to finish this ~~[illegible]~~ enquiry. ~~[illegible]~~ Define the idea of coupling two oscillators. You take direct sum of the phase spaces and Hamiltonians then introduce a perturbation of the Hamiltonian. In general the perturbation has the form $\left(\begin{array}{c|c} A & B \\ \hline B^t & C \end{array}\right)$ and we suppose $A = C = 0$, i.e. that the Hamiltonian when restricted to either ~~subs~~ factor is unchanged, i.e. you get the same motion if the other factor is tied down. So ~~if we~~ for a coupling of 2 simple operators there are 4 parameters in general, i.e. ~~[illegible]~~ ~~[illegible]~~ coefficients of $p_i q_j$ $1 \le i,j \le 2$.

Massive oscillator: $\frac{P^2}{2M} + \frac{1}{2}KQ^2$, ~~[illegible]~~ $\omega^2 = \frac{K}{M}$ where ~~[illegible]~~ M, hence K, is very large. A fixed energy surface is ~~an~~ a narrow ellipse ~~[illegible]~~ in the P direction. In the limit you get a point on the P axis moving sinusoidally. ~~[illegible]~~ For a coupling to such an ~~[illegible]~~ oscillator only 2 parameters ~~[illegible]~~ seem relevant – coeffs of $p_i P$, $q_i P$. So you have simply a given vector in the phase space of the first oscillator – strictly speaking it's a linear functional namely $\sum (c_i' q_i + c_i'' p_i)$

$$H = \underbrace{\frac{p^2}{2m} + \frac{1}{2}kq^2 + \frac{P^2}{2M} + \frac{1}{2}KQ^2 - cqQ}$$

Example from ~~$m\ddot{q} + kq$~~ yesterday ~~$M\ddot{Q} + KQ$~~ $p = m\dot{q}$, $P = M\dot{Q}$,

$$m\ddot{q} = \dot{p} = -\frac{\partial H}{\partial q} = -kq + cQ$$

$$M\ddot{Q} = \dot{P} = -\frac{\partial H}{\partial Q} = -KQ + cq$$

$$\boxed{\begin{aligned} \ddot{q} + \frac{k}{m}q &= \frac{c}{m}Q \\ \ddot{Q} + \frac{K}{M}Q &= \frac{c}{M}q \end{aligned}}$$

Let $\frac{K}{M} = \omega_0^2$ $M \to \infty$

417 How to analyze this? I think this may be inconsistent. You want M to be large, you need to restrict to an energy surface

other possibility - interaction $-cqP$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{Q} = \frac{\partial H}{\partial P} = \frac{P}{M} - cq$$

$$m\ddot{q} = \dot{p} = -\frac{\partial H}{\partial q} = -kq + cP \qquad \dot{P} = -\frac{\partial H}{\partial Q} = -KQ$$

$$\boxed{\ddot{q} + \frac{k}{m}q = \frac{c}{m}P} \qquad \ddot{P} = -K\dot{Q} = -K\left(\frac{P}{M} - cq\right)$$

$$\boxed{\cancel{\ddot{P} + \frac{K}{M}P = Kcq}}$$

$$\dddot{P} + \frac{K}{M}\dot{P} = Kc\dot{q} = Kc\frac{p}{m}$$

$$-K\ddot{Q} + \frac{K}{M}(-KQ) = Kc\frac{p}{m}$$

$$\boxed{\ddot{Q} + \frac{K}{M}Q = -c\frac{p}{m}}$$

---

check calc.

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 + \frac{P^2}{2M} + \frac{1}{2}KQ^2 - cqP$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{Q} = \frac{\partial H}{\partial P} = \frac{P}{M} - cq$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq + cP \qquad \dot{P} = -\frac{\partial H}{\partial Q} = -KQ$$

$$\ddot{q} = \frac{1}{m}(-kq + cP) \qquad \boxed{\ddot{q} + \frac{k}{m}q = \frac{c}{m}P}$$

$$\ddot{Q} = \frac{1}{M}(-KQ) - c\frac{p}{m} \qquad \boxed{\ddot{Q} + \frac{K}{M}Q = -\frac{c}{m}p}$$

so letting $K, M \to \infty$ with $\omega^2 = \frac{K}{M}$ does not affect anything.

418

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2 + \frac{P^2}{2M} + \frac{K}{2}Q^2 - cqP$$

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -kq + cP$$

$$\dot{Q} = \frac{P}{M} - cq \qquad \dot{P} = -KQ$$

eliminate the p's

$$p = m\dot{q} \qquad P = M\dot{Q} + Mcq$$

$$\dot{P} = M\ddot{Q} + Mc\dot{q} = -KQ$$

$$\boxed{\ddot{Q} + \frac{K}{M}Q = -c\dot{q}}$$

$$m\ddot{q} = \dot{p} = -kq + c(M\dot{Q} + Mcq)$$

$$\boxed{m\ddot{q} + (k - Mc^2)q = Mc\dot{Q}}$$

If we let $M \longmapsto \infty$ $Mc \longrightarrow \varepsilon$, then get

$$\boxed{\ddot{Q} + \frac{K}{M}Q = 0 \qquad m\ddot{q} + kq = \varepsilon\dot{Q}}$$

Repeat calculations

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2 + \frac{P^2}{2M} + \frac{K}{2}Q^2 - cqQ \qquad T+V$$

$$\ddot{q} + \frac{kq}{m} = \frac{c}{m}Q \qquad \ddot{Q} + \frac{KQ}{M} = \frac{+cq}{M}$$

so $$\boxed{\ddot{q} + \frac{k}{m}q = \frac{c}{m}Q \qquad \ddot{Q} + \frac{K}{M}Q = 0}$$

419 $$H = \frac{p^2}{2m} + \frac{k}{2}q^2 + \frac{P^2}{2M} + \frac{K}{2}Q^2 - cqP$$

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -kq + cP = -kq + c(M\dot{Q} + Mcq)$$

$$\dot{Q} = \frac{P}{M} - cq$$

$$\boxed{\ddot{q} + \frac{k}{m}q = \frac{cM}{m}\dot{Q} + \frac{cMc}{m}q}$$

$$P = M\dot{Q} + Mcq \qquad \frac{\dot{P}}{M} = -\frac{KQ}{M} = \cancel{M}\ddot{Q} + \cancel{M}c\dot{q}$$

$$\boxed{\ddot{Q} + \frac{K}{M}Q = -c\dot{q}}$$

$$M \to \infty, \quad \frac{K}{M} \to \omega, \quad Mc \to \varepsilon, \quad c \to 0.$$

$$\boxed{\begin{aligned}\ddot{q} + \frac{k}{m}q &= \frac{\varepsilon}{m}\dot{Q} \\ \ddot{Q} + \frac{K}{M}Q &= 0\end{aligned}}$$

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2 + \frac{P^2}{2M} + \frac{K}{2}Q^2 - cQp$$

$$\dot{q} = \frac{p}{m} - cQ \qquad \dot{p} = -kq \qquad \dot{Q} = \frac{P}{M} \qquad \dot{P} = -KQ + cp$$

$$\Downarrow$$

$$\boxed{\ddot{P} + \frac{K}{M}P = c\dot{p}}$$

$$p = m\dot{q} + cQ$$

$$\dot{p} = m\ddot{q} + c\dot{Q} = -kq$$

$$\boxed{\ddot{q} + \frac{k}{m}q = -\frac{c}{m}\dot{Q}}$$

$$\boxed{\ddot{Q} + \frac{K}{M}Q = \frac{c}{M}m\dot{q} + \frac{c^2}{M}Q}$$

$$\ddot{Q} = \frac{\dot{P}}{M} = -\frac{K}{M}Q + \frac{c}{M}(m\dot{q} + cQ)$$

thus if $M \to 0$

You need a better approach to coupling oscillators.

420 Let's concentrate on the problem of coupling & response for harmonic oscillators. Suppose you restrict to Lagrange type oscillators kinetic, ~~potential~~ energy on a configuration space $\mathbf{W}$ Allow $\mathbf{W}$ to be decomposed orthogonally wrt $\mathbf{T}$. You are trying to add one site. This should reduce to the resolvent for the forced oscillator

~~Consider~~ Lagrange picture: ~~...~~ m, k positive definite quadratic forms on E. ~~Given~~ ~~...~~ Split E orthogonally wrt m into a line L and L'. $E = L \oplus L'$

k has form $\left(\begin{array}{c|c} k/L' & \gamma \\ \hline \gamma^t & \vdots \end{array}\right)$. What do you want? I would like to understand the situation in terms of the L'-oscillator ~~...~~, the response to forcing through γ. and the frequency of the L oscillator. Can you do this electrically?

Let's first look at response to a vector.

Consider an oscillator V i.e. complex Hilbert space with pos self adjoint operator. Pick a vector γ and solve $(\partial_t + IH)\psi = F(t)\gamma$ i.e. $(s+IH)\hat{\psi} = \hat{F}(s)\gamma$.

These should take place in V_c

Jan 8 Yesterday I tried to understand coupling harmonic oscillators without much success. I think there is a geometric aspect to the problem which you don't see when you think of an abstract oscillator.

421 ~~[illegible]~~

Go back to LC circuits & review, & formulate

$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook, "d"] & C^1 \arrow[r, two heads] \arrow[d, "Z_s^{-1}"] & H^1 \\ \bar{C}_0 & C^1 \arrow[l, two heads, "d^t"'] & H_1 \arrow[l, hook'] \end{tikzcd}$$

$C^1 = H^+ \oplus H^-$

$Z_s^{-1} = (Cs) \oplus (Ls)^{-1}$

to be more precise you make C^1 a polarized Euclidean space with $C^1 \longrightarrow C_1$ the direct sum $C \oplus L^{-1} : C^{1,C} \oplus C^{1,L} \longrightarrow C_1^C \oplus C_1^L$

Identify $\bar{C}^0$ with $V = d\bar{C}^0 \subset H^+ \oplus H^-$, d = inc. d^t = projection. Upshot is that V, H splits orthog into line $\binom{1}{\omega}\mathbb{R} \subset \mathbb{R} \oplus \mathbb{R}$, $Z_s^{-1} = s \oplus s^{-1}$

$$(1+\omega^2)^{-1/2}(1 \quad \omega)\begin{pmatrix} s & \\ & s^{-1}\end{pmatrix}\begin{pmatrix}1 \\ \omega\end{pmatrix}(1+\omega^2)^{-1/2} = \frac{s + s^{-1}\omega^2}{1+\omega^2}$$

Add up to get response $\;//\; u^* \mathcal{E}^s u \qquad u : V \hookrightarrow H$

$$\sum A_\omega \frac{s+s^{-1}\omega^2}{1+\omega^2} \quad \text{on } \bar{C}^0.$$

where $A_\omega \geq 0$ are skew-symmetric $\sum A_\omega = 1$.

Q: number of modes $= \dim \bar{C}^0 + \dim H_1$
$= v - 1 + l = e = \dim(C^1)$.

~~So what do you get ...~~

Here are things to clarify. Let $H = H^+ \oplus H^-$ be a polarized Hilbert space let $0 \subset W \subset V \subset H$.

$$\begin{tikzcd} V \arrow[r, hook, "\eta"] \arrow[d, "\pi"] & H \arrow[d, "\pi'"] \\ V/W \arrow[r, hook, "\eta'"] & H/W \end{tikzcd}$$

Claim
$\pi(\eta^* \varepsilon^s \eta)^{-1}\pi^*$ and
$\eta'^*(\pi' \varepsilon^s \pi'^*)^{-1}\eta'$ are inverse

because both represent the induced by ε^s quad form on V/W

422 The Hilbert space picture is nice for seeing the eigenvalues, but confusing for the last bit, where it seems best to use quadratic forms.

Look at coupling electric circuits, better making complicated circuits of circuits. Be careful about this. Maybe a good viewpoint is to understand completely the non-Hilbert space viewpoint, namely, quadratic forms depending rationally on the parameter s.

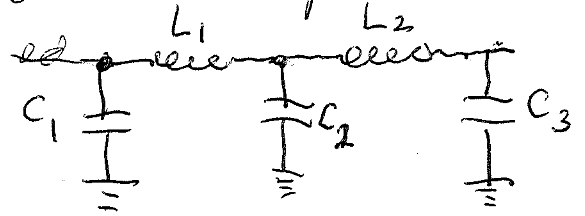
$$\sum A_\omega \frac{(1+\omega^2)s}{s^2+\omega^2} \qquad A_\omega \geq 0 \quad \sum A_\omega > 0.$$

Can you recover the polarized Hilbert space?

Start with $f(s) = \sum_1 a_\omega \frac{(1+\omega^2)s}{s^2+\omega^2}$ $\quad a_\omega \geq 0, \sum a_\omega = 1.$

There is some positive measure on the $s = i\mathbb{R}$ line invariant under $s \mapsto -s$. something involving moment problem, means integrating polynomials wrt this measure. Probably you want residues $\frac{1+\omega^2}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)$

Look at simple ladder networks.



$$Z_1 = \frac{1}{C_1 s + \frac{1}{L_1 s + Z_2}}$$

$$Z = \frac{1}{C_1 s + \frac{1}{L_1 s + \frac{1}{C_2 s + \frac{1}{L_2 s +}}}}$$

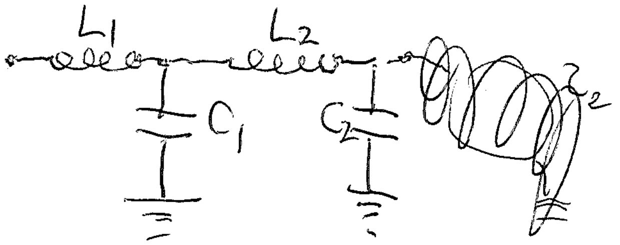
$E = L\dot{I}$

$Z = Ls$

$I = C\dot{E}$

$Z = \frac{1}{Cs}$

423



$$Z_1 = L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{Z_2}}$$

$$Z_1 = \begin{pmatrix} 1 & L_1 s \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & C_1 s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Z_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & L_1 s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C_1 s & 1 \end{pmatrix}\begin{pmatrix} Z_2 \end{pmatrix}$$

$$Z_1(s) = \begin{pmatrix} 1 + L_1 C_1 s^2 & L_1 s \\ C_1 s & 1 \end{pmatrix} Z_2(s)$$

if $Z_2 = \infty$ then

$$Z_1(s) = \frac{1 + L_1 C_1 s^2}{C_1 s} = L_1 s + \frac{1}{C_1 s} \checkmark$$

So we have

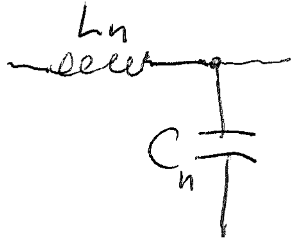
$$Z_i = \begin{pmatrix} 1 & L_i s \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ C_i s & 1 \end{pmatrix} Z_{i+1}$$

suppose $Z_i = \begin{pmatrix} A_i(s) \\ B_i(s) \end{pmatrix}$ $\qquad A_i, B_i$ polys.

$$\begin{pmatrix} 1 & L_i s \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ C_i s & 1 \end{pmatrix}\begin{pmatrix} A_{i+1} \\ B_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & L_i s \\ & 1 \end{pmatrix}\begin{pmatrix} A_{i+1} \\ C_i s A_{i+1} + B_{i+1} \end{pmatrix}$$

$$= \begin{pmatrix} A_{i+1} + L_i C_i s^2 A_{i+1} + L_i s B_{i+1} \\ C_i s A_{i+1} + B_{i+1} \end{pmatrix}$$

Change order of integers.



$$Z_{2n} = L_n s + Z_{2n-1}$$

Z

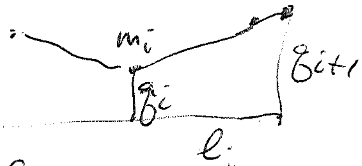
~~so what's hap~~

$$Z_n = L_n s + \frac{1}{C_n s + \frac{1}{Z_{n-1}}}$$

clear. f_n

$$\begin{pmatrix} 1 & L_n s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C_n s & 1 \end{pmatrix}$$

What about discrete string.



$$m_i \ddot{q}_i = \frac{q_{i+1}-q_i}{l_i} - \frac{q_i - q_{i-1}}{l_{i-1}}$$

$$\left(m_i s^2 + \frac{1}{l_i} + \frac{1}{l_{i-1}}\right) q_i - \frac{1}{l_i} q_{i+1} = \frac{1}{l_{i-1}} q_{i-1}$$

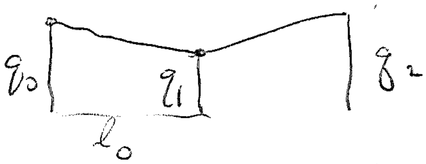
$$q_{i-1} = \left(l_i m_i s^2 + 1 + \frac{l_i}{l_{i-1}}\right) q_i - \frac{l_{i-1}}{l_i} q_{i+1}$$

Hamilton's equation might be better.

$$s q_i = \frac{p_i}{m_i} \qquad s p_i = \frac{q_{i+1}-q_i}{l_i} - \frac{q_i - q_{i-1}}{l_{i-1}}$$

Maybe ask what happens when you add a mass, i.e. get a new coord q_0 two params. m_0, l_0.

Une q



$$s^2 m_0 q_0 = \frac{q_1 - q_0}{l_0}$$

$$s^2 m_1 q_1 = \frac{q_2 - q_1}{l_1} + \frac{q_0 - q_1}{l_0}$$

solve all equations to the right to get $(u_i^s)_{i \geq 1}$ such that $u_1^s = 1$, $s^2 u_i = l_i^{-1} u_{i+1} - (l_i^{-1} + l_{i-1}^{-1}) u_i + l_{i-1}^{-1} u_{i-1}$

$i \geq 2$.

425 Then you want $(v_i^s)_{i \geq 0}$

such that $v_0^s = 1$. $\quad s^2 m_1 v_1 = \ell_1^{-1} v_2 - (\ell_1^{-1} + \ell_0^{-1}) v_1 + \ell_0^{-1} v_0$

$$s^2 m_i v_i = \ell_i^{-1} v_{i+1} - (\ell_i^{-1} + \ell_{i-1}^{-1}) v_i + \ell_{i-1}^{-1} v_{i-1}$$

$i \geq 2$.

$(v_i)_{i \geq 1} = v_1 (u_i)_{i \geq 1}$.

Recall definition $(u_i)_{i \geq 1}$ satisfies $u_1 = 1$ and $s^2 m u + k u = 0$ at all $i \geq 2$.

Better $(u_i)_{i \geq 0}$ satisfies

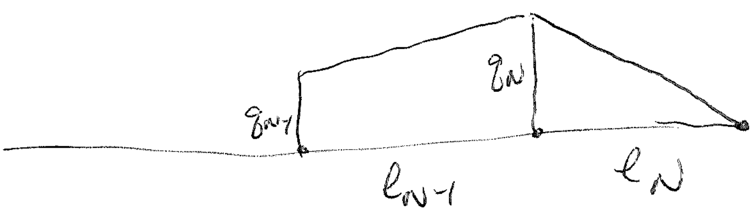
You seek a sequence of rational functions.

Let q_N be the largest site. $q_{N+1} = 0$

$$s^2 m_N q_N = \frac{q_{N-1} - q_N}{\ell_{N-1}} + \frac{-q_N}{\ell_N} \quad \Big| \quad q_{N-1} = q_N + s^2 m_N \ell_{N-1} q_N$$

$$s^2 m_i q_i = \ell_i^{-1} q_{i+1} - (\ell_i^{-1} + \ell_{i-1}^{-1}) q_i + \ell_{i-1}^{-1} q_{i-1}$$

$$q_{i-1} = (s^2 \ell_{i-1} m_i + \ell_{i-1} \ell_i^{-1} + 1) q_i - \ell_{i-1} \ell_i^{-1} q_{i+1}$$



$$s^2 m_i q_i = \ell_{i-1}^{-1} q_{i-1} - (\ell_{i-1}^{-1} + \ell_i^{-1}) q_i + \ell_i^{-1} q_{i+1}$$

$$\boxed{q_{i-1} = \ell_{i-1} (s^2 m_i + \ell_{i-1}^{-1} + \ell_i^{-1}) q_i - \ell_{i-1} \ell_i^{-1} q_{i+1}}$$

starts $\quad q_{N-1} = \ell_{N-1} (s^2 m_N + \ell_{N-1}^{-1} + \ell_N^{-1}) q_N \quad$ because $\underline{q_{N+1} = 0}$

$$\begin{pmatrix} q_{i-1} \\ q_i \end{pmatrix} = \begin{pmatrix} & -\ell_{i-1} \ell_i^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_i \\ q_{i+1} \end{pmatrix}$$

$$\underbrace{s^2 (\ell_{i-1} m_i)}_{a_i} + \underbrace{(1 + \ell_{i-1} \ell_i^{-1})}_{1 + b_i}$$

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$$\begin{pmatrix} q_{i-1} \\ q_i \end{pmatrix} = \begin{pmatrix} a_i s^2 + b_i + 1 & -b_i \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_i \\ q_{i+1} \end{pmatrix}$$

$$f_{i-1} = \frac{(a_i s^2 + b_i + 1) f_i - b_i}{f_i} = (a_i s^2 + b_i + 1) - \frac{b_i}{f_i}$$

starting with $f_N = \infty$.

Final point would be to factor ~~into~~ this quadratic matrix in s into linear matrices.

$$\begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Cs & 1 \end{pmatrix} = \begin{pmatrix} LCs^2 + 1 & Ls \\ Cs & 1 \end{pmatrix}$$

In addition ~~to~~ q_i use $\frac{q_{i+1} - q_i}{l_i} = \delta_i$

$$\boxed{q_i = \frac{\delta_i - \delta_{i-1}}{m_i s^2} \qquad \delta_i = \frac{q_{i+1} - q_i}{l_i}}$$

$s^2 m_i q_i = \delta_i - \delta_{i-1}$

$\boxed{\delta_{i-1} = -m_i s^2 q_i + \delta_i}$

$\delta_{i-1} \quad q_i \quad \delta_i \quad q_{i+1}$

$l_i \delta_i = q_{i+1} - q_i$

$\boxed{q_i = -l_i \delta_i + q_{i+1}}$

$$\begin{pmatrix} q_i \\ \delta_i \end{pmatrix} = \begin{pmatrix} -l_i & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_i \\ q_{i+1} \end{pmatrix}$$

change sign of δ.

$$\begin{pmatrix} \delta_{i-1} \\ q_i \end{pmatrix} = \begin{pmatrix} -m_i s^2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_i \\ \delta_i \end{pmatrix}$$

What formulas $\quad s^2 m_i q_i = \frac{q_{i+1} - q_i}{l_i} - \frac{q_i - q_{i-1}}{l_i}$

$\delta_i = \frac{q_i - q_{i+1}}{l_i} \qquad = \delta_{i-1} - \delta_i$

~~$\delta_{i-1} = \delta_i + s^2 m_i q_i$~~

427

$$\delta_i = \frac{q_i - q_{i+1}}{l_i}$$

$$s^2 m_i q_i = \delta_{i-1} - \delta_i$$

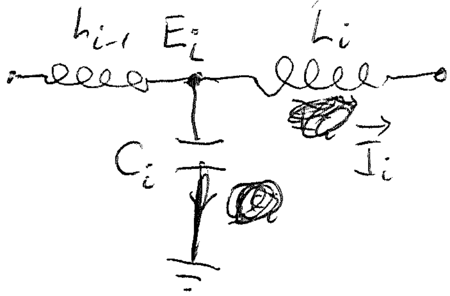
$$\boxed{\begin{aligned} q_i &= q_{i+1} + l_i \delta_i \\ \delta_{i-1} &= \delta_i + s^2 m_i q_i \end{aligned}}$$

$$q_{i-1} = q_i + l_{i-1} \delta_{i-1}$$

$$\begin{pmatrix} \delta_{i-1} \\ q_i \end{pmatrix} = \begin{pmatrix} 1 & s^2 m_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_i \\ q_i \end{pmatrix} \qquad \begin{pmatrix} \delta_i \\ q_{i+1} \end{pmatrix}$$

$$\begin{pmatrix} \delta_{i-1} \\ q_{i-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{i-1} & 1 \end{pmatrix} \begin{pmatrix} \delta_{i-1} \\ q_i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ l_{i-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & s^2 m_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s^2 m_i \\ l_{i-1} & s^2 l_{i-1} m_i + 1 \end{pmatrix}$$



$$s L_i I_i = E_i - E_{i+1}$$

$$\boxed{E_i = E_{i+1} + L_i s I_i}$$

$$\frac{1}{C_i s} E_i = Cs(I_{i-1} - I_i)$$

$$\boxed{I_{i-1} = I_i + \frac{1}{C_i s} E_i}$$

$$\begin{pmatrix} E_i \\ I_{i-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{C_i s} & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_i \end{pmatrix}$$

$$\begin{pmatrix} E_{i-1} \\ I_{i-1} \end{pmatrix} = \begin{pmatrix} 1 & L_{i-1} s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_{i-1} \end{pmatrix}$$

428 Can I now make a link with the moment problem. $d\mu$ prob. measure on $\mathbb{R}$ with finite support. $L^2(\mathbb{R}, d\mu) = H$ finite dim real H.S. get Jacobi matrix. ~~OKAY~~

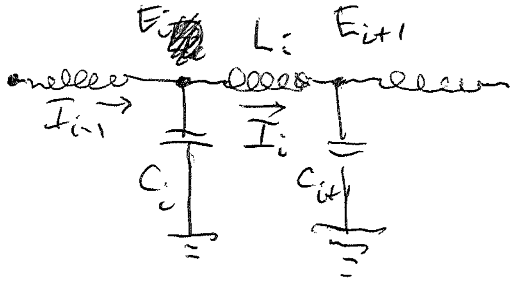
First take an abstract oscillator i.e. phase space is a complex Hilbert space E ($\Omega = \mathrm{Im}\langle , \rangle$), Hamiltonian is pos. def s.s. of H. Take real line L in E, say $L = \mathbb{R}\xi$, $\|\xi_0\| = 1$. Solve $(\partial_t + IH)\psi = f(t)\xi_0$, by L.T.

$$(s + IH)\hat{\psi} = \hat{f}\,\xi_0$$

$$\hat{\psi} = \frac{1}{s+IH}\hat{f}\,\xi_0$$

You are confused about response questions. ~~What is a response~~ This is more or less clear to me in the electrical situation, so maybe I can transport to a discrete spring.

So start with ~~a dis~~



$$E_i = L_i s I_i + E_{i+1}$$

$$\begin{pmatrix} E_i \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & L_i s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_{i+1} \\ I_i \end{pmatrix}$$

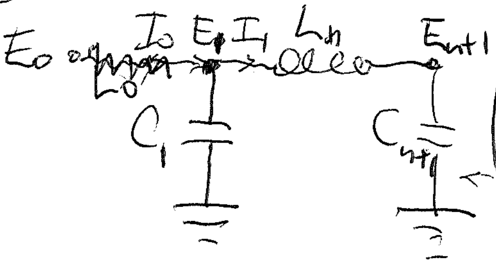
$$I_{i-1} = C_i s E_i + I_i$$

$$\begin{pmatrix} E_i \\ I_{i-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_i s & 1 \end{pmatrix} \begin{pmatrix} E_i \\ I_i \end{pmatrix}$$

Somehow the stumbling block is the link between the abstract oscillator picture and these geometric pictures. The abstract oscillator picture does not seem to express the "graph" geometry, i.e. your ~~Hilbert space~~ phase space splits according to the edges of the graph.

429 Idea is that the line somehow makes ~~removing~~ the s.a. operator ~~into~~ a <u>symmetric</u> operator with deficiency indices. Partially defined operator.
Riemann surface of genus g — attempt to make a quantum field theory out of holom. functions.
It all comes back slowly.

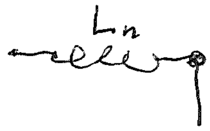
Begin with the moment problem? or with the ladder electrical network.



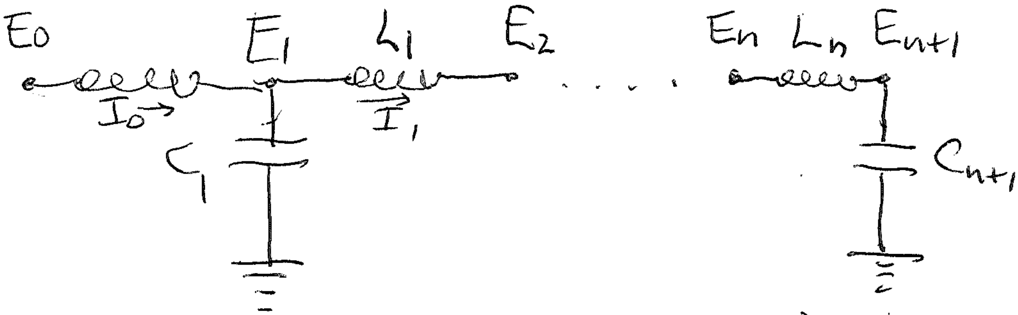
$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & L_0 s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C_1 s & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & L_n s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C_{n+1} s & 1 \end{pmatrix}\begin{pmatrix} E_{n+1} \\ 0 \end{pmatrix}$$

$$I_{n-1} - I_n =$$

$$E_{n-1} - E_n = L_n s I_n$$



$$I_n = C_n s E_n$$



$$E_i = L_i s I_i + E_{i+1} \qquad \begin{pmatrix} E_i \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & L s_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_{i+1} \\ I_i \end{pmatrix}$$

$$I_{i-1} = C_i s E_i + I_i$$

$$\begin{pmatrix} E_i \\ I_{i-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_i s & 1 \end{pmatrix}\begin{pmatrix} E_i \\ I_i \end{pmatrix}$$

$$E_n - E_{n+1} = L_n s I_n$$

$I_{n+1}=0$

$$I_n = C_{n+1} s E_{n+1}$$

$$\begin{pmatrix} E_n \\ I_n \end{pmatrix} = \begin{pmatrix} 1 & L_n s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_{n+1} \\ I_n \end{pmatrix}$$

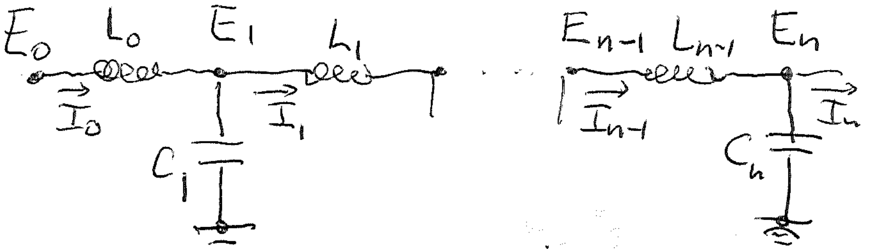
$$\begin{pmatrix} E_{n+1} \\ I_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_{n+1} s & 1 \end{pmatrix} \begin{pmatrix} E_{n+1} \\ I_{n+1} = 0 \end{pmatrix}$$

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & L_0 s \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ C_1 s & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \\ C_{n+1} s & 1 \end{pmatrix} \begin{pmatrix} E_{n+1} \\ I_{n+1} \end{pmatrix}$$

Why this calculation? What I am doing is to solve the ~~eq~~ equations of motion except for the last connection at the 0-th node. ~~[illegible]~~

I've found the response.

~~[illegible]~~ You have something like a manifold with boundary. How to formulate? Electrically



$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & L_0 s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_1 s & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & L_{n-1} s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_n s & 1 \end{pmatrix} \begin{pmatrix} E_n \\ I_n \end{pmatrix}$$

Should I be doing this concretely

431 Somehow you are missing the point with all these concrete calculations. There are other things that are strange, e.g. in the moment problem there is no ± symmetry for eigenvalues.

Maybe it would help to look at extending a partially defined symmetric operator to a self-adjoint operator. Look at a Hilbert space H and subspaces V of $H \oplus H$ which are graphs of self adjoint operators. Given In general given $T : H \to H$

$\begin{pmatrix} 1 \\ T \end{pmatrix} H$ has orth comp. $\begin{pmatrix} -T^* \\ 1 \end{pmatrix} H$.

T symm. means $T \subset T^*$. Point is to introduce

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ T^* \end{pmatrix} = \begin{pmatrix} -T^* \\ 1 \end{pmatrix}.$$

The point is that you have

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -x^* y + y^* x$$

$T \subset T^*$ means Γ_T is isotropic for this skew form.

Let's try to add partially defined maps, correspondences. Consider a self-correspondence, i.e. a type of K-module. H Hilbert space W closed subspace of $H \oplus H$. You can ask about the spectrum, i.e. z such that $az+b : W \to H$ fails to be invertible.

Let's take a chance and work with the unit circle instead of $i\mathbb{R}$. You want $az+b$ to be invertible for $z \notin S^1$. Not clear.

Try various things. You want some control over unitary + skew adjoint operators, maybe only partially defined. Let's begin with $W = \mathcal{D}_T \overset{\binom{1}{T}}{\hookrightarrow} H \oplus H$. First case to understand $\mathcal{D}_T \xrightarrow{\sim} H$ so that $W = \binom{1}{T} H$, and T ~~of~~ is skew-adjointe. Consider $\frac{1+T}{1-T}$?

The point here is to relate the Cayley transform of T and $\begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}}_{X}$. Point:

$$g\varepsilon = \frac{1+X}{1-X}\varepsilon = \frac{1+X^2+2X}{1-X^2}\varepsilon = \left(\begin{array}{c|c} \frac{1+T^2}{1-T^2} & -\frac{2T}{1-T^2} \\ \hline \frac{2T}{1-T^2} & -\frac{1+T^2}{1-T^2} \end{array}\right) \quad ?$$

Suppose T partially defined + skew symm. $\mathcal{D}_T \overset{\binom{1}{T}}{\hookrightarrow} H \oplus H$

$$\begin{pmatrix} y \\ Ty \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ Tx \end{pmatrix} = \begin{pmatrix} y^* & y^*T^* \end{pmatrix} \begin{pmatrix} Tx \\ x \end{pmatrix}$$

$$= y^*Tx + y^*T^*x = 0 \text{ if } -T \subset T^*$$

Look at situation $D_T \subset H$ and you have

$$\begin{pmatrix} y \\ Ty \end{pmatrix}^* \begin{pmatrix} Tx \\ x \end{pmatrix} = y^*Tx + (Ty)^*x = 0.$$

In other words $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_T \subset \Gamma_T^{\perp}$. Thus

$$\langle x+Tx, x+Tx \rangle = \|x\|^2 + \|Tx\|^2 \geq \|x\|^2$$

so you get a partial unitary operator $\frac{1+T}{1-T}$ from H to itself.

Look at $V \subset H \oplus H$ such that V and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V$ are $\perp$. If $\binom{x}{y} \in V$, then $\langle x+y, x+y \rangle$ $= \|x\|^2 + \cancel{\langle x,y \rangle + \langle y,x \rangle} + \|y\|^2$. Also $\|x-y\|^2 = \|x\|^2 + \|y\|^2$. ~~so we should get~~ Thus $V \underset{(1\ -1)}{\overset{(1\ 1)}{\rightrightarrows}} H$ are isometries

So if we shift from $a, b : V \to H$ to $a \pm b : V \to H$ maybe things simplify slightly.

You can simplify. ~~use~~ Transform $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V$ into $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V$. Then instead of $V \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V$ we have $V \perp \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V$ which means that if $a, b : V \rightrightarrows H$ are the two projections that $(a^*a - b^*b)(V) = 0$ i.e. $\|a(v)\|^2 = \|b(v)\|^2$ $\forall v \in V$. Thus you have isot. subspace for the indefinite hermitian for $\|\ \|_+^2 - \|\ \|_-^2$. You know that isotropic subspaces are given by partial unitaries from H to itself.

Now you maybe can reach the understanding you had 20 years ago. I especially want to understand these strings and electrical networks.

 What's involved with electrical networks?

~~Why does~~

Let's analyze the following situation. V an isotropic subspace of $H \oplus H$ for $\|\xi_+\|^2 - \|\xi_-\|^2$. Thus we give ~~[illegible]~~ two isometries $V \underset{b}{\overset{a}{\rightrightarrows}} H$. In finite dims there's no problem extending to a maximal isot. subspace, i.e. a unitary auto g of H such that $ga = b$.

You feel that somehow this situation is related to a subquotient of a polarized Hilbert space. You need examples. Start with a maximal isotropic subspace of $H \oplus H$ for the indefinite hermitian product, and cut it ~~[illegible]~~ down. Thus you have a subspace H' of codim 1 in H together with ~~an autom~~ an isom. $H' \underset{b}{\overset{in}{\rightrightarrows}} H$

You need to analyze this K-module ~~situation~~. The sheaf ~~is~~ over $\mathbb{P}^1$ should be a ~~[illegible]~~ line bundle of degree n. ~~How to be~~ sure?

$$0 \longrightarrow \mathcal{O}(-1) \otimes H' \longrightarrow \mathcal{O} \otimes H \longrightarrow F \longrightarrow 0$$

In principle the torsion subsheaf of could be $\neq 0$. Can you find a model for this situation. You have ~~[illegible]~~ H' a hyperplane in H and b a map $H' \longrightarrow H$. ~~Presumably~~ You can iterate this corresp.

$$\begin{tikzcd}
b^{-2}(H') \arrow[d] & \subset & b^{-1}(H') \arrow[d] & \subset & H' \arrow[d, "b"] & \subset & H \\
b^{-1}(H') \arrow[d] & \subset & H' \arrow[d, "b"] & \subset & H & & \\
H' & \subset & H & & & &
\end{tikzcd}$$

note that if $b : H' \to H'$ then F is a trivial line bundle + torsion sheaf assoc to b.

How might one proceed? Take complement to H ?

Maybe you can analyze when this situation is the ~~[illegible]~~ Cayley transform of a skew-symm. op.

435. Another idea: Extend b to a unitary — there should be a circle parameter.

Jan 10 Idea. via Cayley transform a partially defined skew-symmetric operator should give rise to a partial isometry (brings up Pimsner's paper reference) i.e. a K-module $H' \underset{b}{\overset{a}{\rightrightarrows}} H$ where H', H are Hilbert spaces and a, b are isometries: $a^*a = b^*b = 1_{H'}$. Say $\dim(H') = \dim(H) - 1$. Consider over the Riemann sphere the corresponding sheaf

$$0 \to \mathcal{O}(-1) \otimes H' \longrightarrow \mathcal{O} \otimes H \longrightarrow F \longrightarrow 0$$

But use decomposition into indecomposables, what you know about sheaves on $\mathbb{P}^1$. This yields a decomp. of the corresponding K-module. In the present situation F splits into $\mathcal{O}(n)$ plus torsion sheaves. Support of torsion sheaves must be on unit circle. Actually we know that in the general case F splits into $\mathcal{O}(k)$ for [redacted] different k, which suggests the possibility of a complete structure theory in finite dimensions.

How do you start? You have

$$0 \to \mathcal{O}(-1) \otimes H' \longrightarrow \mathcal{O} \otimes H \longrightarrow F \longrightarrow 0$$

so $\quad H \xrightarrow{\sim} H^0(F) \quad$ and $\quad H^0(F(-1)) \xrightarrow{\sim} \underbrace{H^1(\mathcal{O}(-2))}_{\Lambda^2 \simeq \mathbb{C}} \otimes H'$

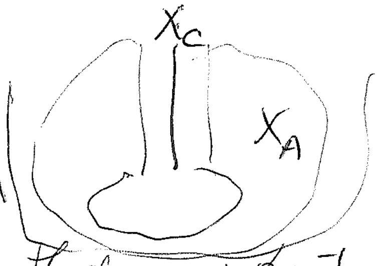
Assume $F = \mathcal{O}(n-1)$. Have

Partial isometry mentioned in Pimsner's article. Go back to free products. Codim 1 submanifold Y in X, complement disconnected. $Y \subset X$

436 Picture

$X_C \rightrightarrows X_A$

So you have $\mathbb{C} \overset{\phi}{\underset{\psi}{\rightrightarrows}} A$



and you adjoin u invertible to A so that $u\phi u^{-1} = \psi$. How is this handled by Pimsner. I guess you ~~ca~~ look for a ~~alg~~ homom. $A \longrightarrow R$. Is there any link with a partial isometry on a Hilbert space.

Is there a natural C^* ~~alg~~ algebra whose representations on a Hilbert space H are partial isometries on H.

$e^2 = e = e^*$ • b

Start with $H' \overset{a}{\underset{b}{\rightrightarrows}} H$ $\qquad a^*a = b^*b = 1_H$

The way to do this maybe is to figure out what operators you have on H. You have 4 operators

$$\begin{pmatrix} a \\ b \end{pmatrix} (a^* \ b^*) = \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix} \quad \text{on } H$$

~~So~~ what ~~do~~ do they generate. ~~so~~ 16 possible products.

$aa^*aa^* = aa^*$ $\qquad ab^*aa^*$
$aa^*ab^* = ab^*$ $\qquad ab^*ab^*$
aa^*ba^* $\qquad ab^*ba^* = aa^*$
aa^*bb^* $\qquad ab^*bb^* = ab^*$

$ba^*aa^* = ba^*$ $\qquad bb^*aa^*$
$ba^*ab^* = bb^*$ $\qquad bb^*ab^*$
ba^*ba^* $\qquad bb^*ba^* = ba^*$
ba^*bb^* $\qquad bb^*bb^* = bb^*$

You get two idempotents so there's a set of parameters here.

?

~~make~~ subsequent of polarized check, make it clean.

437 Now it's time to really understand what is happening. The situation should be basically very simple, the ingredients essential should come from the K-module situation, First understand the algebra and ignore the Hilbert space structure. Basically you have a correspondence of some sort. Wait: you need to recall what are good correspondences. The point is that ~~certain~~ there are correspondences yielding negative sheaves.

Recall: $0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1)^{\oplus 2} \longrightarrow \mathcal{O} \longrightarrow 0$

e.g. $\mathbb{C}^2 \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} \mathbb{C}$

So I need to understand the examples.

Sheaf $\mathcal{O}(n)$ $\qquad H^0(\mathcal{O}(n)) = S^n(\mathbb{C} \oplus \mathbb{C}z) = F_{n-1}$

$\qquad H^0(\mathcal{O}(n-1)) = S^{n-1}(\mathbb{C} \oplus \mathbb{C}z) = F_n$

two maps $1, z : F_{n-1} \rightrightarrows F_n$ inclusion + shift one to right

What do I take?

Go backwards. Take $H' \underset{\theta}{\overset{\hookrightarrow}{\rightrightarrows}} H$

Basically you take $a, b : W \longrightarrow V$ such that $az + b$ injective $\forall z \in S^2$, and W is of codim $\underline{1}$.

You want complex structure

Start with $a, b : W \longrightarrow V$ such $az+b$ injective $\forall z \in S^2$. Show that V has a canonical split filtration - interpret V as sections of $\mathcal{O}(n)$ W as subspace vanishing at ∞.

~~Note~~ $V = \Gamma(\mathcal{O}(n))$.

$$\mathcal{O}(n-1) \hookrightarrow \mathcal{O}(n) \xrightarrow{\text{eval at } \infty} k$$

$$\begin{tikzcd} V = \Gamma(\mathcal{O}(n)) \\ W = \Gamma(\mathcal{O}(n-1)) \arrow[u, hook] \end{tikzcd}$$

$$0 \longrightarrow \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O} \otimes V \longrightarrow L \longrightarrow 0$$

$$0 \longrightarrow W \xrightarrow{az+b} V \longrightarrow L(z) \longrightarrow 0$$

$$0 \longrightarrow W \xrightarrow{a} V \longrightarrow L(\infty) \longrightarrow 0$$

$$0 \longrightarrow W \xrightarrow{b} V \longrightarrow L(0) \longrightarrow 0.$$

Proceed by induction. ~~$V/\Gamma(\mathcal{O}(1))$~~ Suppose $\dim(V) = 2$, $\dim(W) = 1$. Have $W \overset{a}{\underset{b}{\rightrightarrows}} V$ and know these are independent, then get $W \oplus W = V$.

Take a = inclusion

$$\begin{tikzcd} b^{-1}(W) \arrow[r, hook, "b"] \arrow[d, hook] & W \arrow[d, hook] \\ W \arrow[r, hook, "b"] & V \end{tikzcd}$$

Why things are difficult - is probably because you need to use the condition that $az+b$ is <u>injective</u> for all z. To be effective you need to split the exact sequence. How to proceed?

Go back to partial isom. - complete to a unitary.

439 Jan 11. Go back to partial isometry and try to find response function. First point — a partial isometry is a ~~correspondence, a~~ K-module $W \overset{a}{\underset{b}{\rightrightarrows}} V$ such that $az+b$ is generically injective. Hence it can be split into indecomposable K-modules, in fact $\exists$ canonical split filtration describing how these types occur. Better to say that ~~[illegible]~~ the K-module is equivalent to a coh sheaf F on $\mathbb{P}^1$ generated by its section, such that $V = H^0(F)$ $W = H^0(F(-1))$
$F = \operatorname{Coker}\{\mathcal{O}(-1)\otimes W \rightrightarrows \mathcal{O}\otimes V\}$. This sheaf generalizes the $\mathbb{C}[z]$ module $\mathbb{C}[z]\otimes V/(z-b)\,\mathbb{C}[z]\otimes V$ when $a=1$. characteristic sheaf. ~~[illegible]~~: Think given F you have $H^0(F(n))$. for $n<<0$ get $H^0(F_t(n)) \simeq H^0(F_t)$ etc.

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(1) It seems that I need much better control over K-modules. perhaps learn the decomposition into indec. in Benson's book. However maybe you can do a graph type analysis related to the case where $az+b$ is invertible over the Laurent polynomials. O.K.

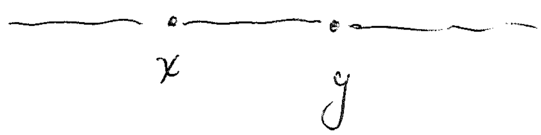
Digression: Do the graph case with L^2 chains! Assume d is an isomorphism between L^2 1-chains and L^2 0-chains. Think of the example where the graph is $\mathbb{Z} \subset \mathbb{R}$. ~~[illegible]~~ L^2 1-chains split at any vertex, hence the vertex space splits correspondingly. ~~[illegible]~~ Consider an edge. 0-chains split into left middle right - are mapped isom by d to 0-chains supported at two vertices. Get isom.

$$\begin{array}{ccc} M_x \oplus M_y & & M_x/M_x^- \oplus M_y/M_y^+ \\ \uparrow s & \Longrightarrow & \uparrow s \\ Z'_{<x} \oplus M_{\{x,y\}} \oplus Z'_{>y} & & M^+_{\{x,y\}} \subset M_{\{x,y\}} \end{array}$$

~~So again there is no scattering~~

$$Z'_{>x} \hookrightarrow M_{x,y} \times Z'_{>y}$$

First you have to be very careful that scattering do not take place in this graph situation. Then maybe look at systems where there is an H_0 nonzero.

So 

$$\begin{tikzcd} C_1 \arrow[r, "d", "\sim"'] \arrow[d, phantom, "\shortparallel"] & C_0 \arrow[d, phantom, "\shortparallel"] \\ \bigoplus\limits_{\sigma}^{(2)} M_\sigma & \bigoplus\limits_{x}^{(2)} M_x \end{tikzcd}$$

$$\begin{tikzcd} M_x \arrow[d, "d"'] & = & M_x^+ & \oplus & M_x^- \\ Z_1(\Gamma,\{x\}) \arrow[u, bend right, "s"'] & = & Z_1(\Gamma_{\geq x},\{x\}) \arrow[u, "s"] & \oplus & Z_1(\Gamma_{\leq x},\{x\}) \arrow[u, "s"'] \end{tikzcd}$$

$$\begin{tikzcd} Z_1(\Gamma,\{x,y\}) \arrow[d, "d"', shift right] & Z_1(\Gamma,\{x\}) \oplus Z_1(\Gamma,\{y\}) \arrow[l, "\sim"'] \\ M_x \oplus M_y \arrow[u, "s"', shift right] & \end{tikzcd}$$

$$Z_1(\Gamma,\{y\}) = Z_1(\Gamma_{\leq y},\{y\}) \oplus Z_1(\Gamma_{\geq y},\{y\})$$

$$Z_1(\Gamma,\{x\}) = Z_1(\Gamma_{\geq x},\{x\}) \oplus Z_1(\Gamma_{\leq x},\{x\})$$

$$\begin{tikzcd} Z_1(\Gamma_{\geq x},\{x\}) \oplus Z_1(\Gamma_{\leq y},\{y\}) \arrow[d] \\ M_x \end{tikzcd}$$

Lets think of $\quad Z_1(\Gamma,\{x,y\}) \xleftarrow{\sim} Z_1(\Gamma_{\leq x},\{x\}) \oplus M_{(x,y)} \oplus Z_1(\Gamma_{\geq y},\{y\})$

$$\begin{tikzcd} 0 \arrow[r] & Z_1(\Gamma_{\leq x},\{x\}) \oplus Z_1(\Gamma_{\geq y},\{y\}) \arrow[r] \arrow[d, "s", "d"'] & Z_1(\Gamma,\{x,y\}) \arrow[r] \arrow[d, "s"', "d"] & M_{(x,y)} \arrow[r] \arrow[l, bend right, "\mathrm{can}"'] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & M_x^- \oplus M_y^+ \arrow[r] & M_x \oplus M_y \arrow[r] & M_x^+ \oplus M_y^- \arrow[r] & 0 \end{tikzcd}$$

We get a canonical map $\quad M_x^+ \oplus M_y^- \longrightarrow M_x^- \oplus M_y^+$

441 Let's move on to K-modules – You now can view them as cosheaves over the graph $\mathbb{Z} \subset \mathbb{R}$

$az - b$

$$V \xleftarrow{b} W \xrightarrow{a} V \qquad d = az - b$$

~~Assume~~ You want to assume that $az-b$ is injective for all z (or surjective for all z?). Does the graph picture help? ~~have pointed ... Not clear what happens?? Continue~~

Go on. This should not defeat you.

Consider the case where $az - b : W \to V$ is injective for all z (including ∞ i.e. a inj). we assume codim 1 in so get $P^1 \longrightarrow P^n$ if $\dim(V) = n+1$. So what.

$$0 \longrightarrow \mathcal{O}(-1)\otimes W \longrightarrow \mathcal{O}\otimes V \longrightarrow L \longrightarrow 0$$

$$\mathcal{O}\otimes \Lambda^{n+1} V \simeq L \otimes \mathcal{O}(-n) \otimes \Lambda^n W$$

Thus get canon isom $L \simeq \mathcal{O}(n) \otimes \Lambda^{n+1} V \otimes (\Lambda^n W)^*$.

The general theory should yield ~~this~~ isom.

Look in Hilbert case. Then V is a Hilbert space of dim $\mathbb{C}^{n+1}$ can assume W is a hyperplane, $a = $ incl. and $b : W \hookrightarrow V$ is an isometry. The hypothesis that $az - b$ injective for all ~~$\mathbb{Z}$~~ means that for all $w \neq 0$, aw, bw are ind.

You might have better luck by:

The point is that V can be recovered from L as $H^0(L)$. Need some mechanism for calculating $H^0(L)$, but you have chosen the coord z on $\mathbb{P}^1$. so can Čech covering. Therefore there has to be

442 a completely standard picture for the ~~modules~~ K-module. The only interesting point will be the ~~ccc~~ interaction with the scalar product.

~~So~~ start with $a, b : \overset{n}{W} \longrightarrow \overset{n+1}{V}$ $\quad az-b$ inj $\forall$ $z \in S^2$.

Then up to a non-zero scalar $a = \cdots$ $\quad b = \cdots$.

$a = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \\ & & & 0 \end{pmatrix}$ How do I prove this?

$a, b : \overset{n}{W} \longrightarrow \overset{n+1}{V}$ $\quad az-b$ inj $\forall z$.

$\underline{n=1}$. $\quad w_0 \in \overset{1}{W} \overset{a}{\underset{b}{\rightrightarrows}} \overset{2}{V}$ $\quad a(w_0), b(w_0)$ basis for V.

$n=2$. $\quad \overset{2}{W} \quad \overset{3}{V} \quad \overset{2}{a(W)} \quad \overset{2}{b(W)}$

So you find a canonical basis for things, namely. $V = \bigoplus_{i=0}^{n} \mathbb{C} z^i$ $\quad W = \bigoplus_{i=0}^{n-1} \mathbb{C} z^i$ $\quad b = \text{inc.}$
$a = z \cdot$ shift.

$$(z - \zeta) \sum_{i \leq n-1} c_i z^i = 0$$

Now can ask about Hilbert space structures, but you want mult by z to be an isometry.

$n=1$. Here you have unit vectors $a(w_0), b(w_0)$ which span V.

ideas: C^* alg gen by b $\ni$ $bb^*b = b$

representation of this should be the same as a Hilbert space with partial isometry. Check $bb^*bb^* = bb^*$ so bb^* is a projector. Also $b^* = b^*bb^*$ so b^*b is a proj.

So bb^* projects onto bH, b^*b projects onto b^*H, have $b : b^*H \longrightarrow bH$ with inverse b^*. Basically clear.

Structure of this algebra. Observe Toeplitz is quotient by ideal gen. by $b^*b = 1$.

443 Basis b.

Can you describe this algebra. Have $\mathbb{Z}$ grading?

$$\begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix} \qquad A^3 = A \text{ have}$$

splitting of $H \oplus H$ into 3 parts $A = \pm 1$ and $A = 0$.

Go back to ~~inner~~ inner product on V. You learned that $\overset{n}{W} \underset{b}{\overset{a}{\rightrightarrows}} \overset{n+1}{V}$ $az - b$ inj. all $z \in S^2$ has standard form.

$$V = \mathbb{C} \oplus \mathbb{C}z \oplus \cdots \oplus \mathbb{C}z^n$$
$$W = \mathbb{C} \oplus \mathbb{C}z \oplus \cdots \oplus \mathbb{C}z^{n-1}$$

~~with~~ a = incl

b = shift. What is a hermitian ~~inner~~ pos. def hermitian prod. on V like? want shift to be isometry. – Hankel matrix. a_{i-j}.

Finally can you get a rational function? $\operatorname{Ker} (az - b)^*$

Real structures. Any connection with D. Joyce?

~~Any contribution~~

Situation W hyperplane in V, $b : W \to V$ linear map such that $b - z : W \to V$ injective for all z. Thus $\forall w \neq 0$ bw, w are independent.

Look at $b^t - z : V^* \to W^*$.

$$0 \to \mathcal{O}(-1) \otimes W \to \mathcal{O} \otimes V \to L \to 0$$

$V = H^0(L)$ now there's an embedding $L(-1) \to L$

$$\mathcal{O}(-2) \subset \mathcal{O}(-1) \subset \mathcal{O}$$

Pick some pt of S^2 like $z = \infty$, say, then you can ask that an element of V goes to zero at this point.

Given $W \underset{b}{\overset{a}{\rightrightarrows}} V$ such that $az - b : W \to V$ is injective $\forall z \in \mathbb{C}$ and for $z = \infty$, i.e. a is injective – equivalent to saying $a(w), b(w)$ ind. for $\forall w \neq 0$.

Form $0 \to \mathcal{O}(-1) \otimes W \xrightarrow{az - b} \mathcal{O} \otimes V \to \mathcal{L} \to 0$. We know $V \xrightarrow{\sim} H^0(L)$

so you get a hyperplane V' in V consisting of v whose sections vanish at $z=\infty$, i.e. $v \in a(W)$. Then it should follow that there is a corresp ~~subspace~~ hyperplane W' of W so that $a, b : W' \to V'$. $W' = b^{-1}a(W)$.

$$\begin{tikzcd} b^{-1}(aW) \arrow[r, "b"] \arrow[d, hook] & W \arrow[d, hook, "a"] \\ W \arrow[r, "b"] & W \end{tikzcd}$$

This is not very clear. Try again. Suppose a inclusion of a subspace $W < V$. Put $V' = W$

$$\begin{tikzcd} \overbrace{b^{-1}(V')}^{W'} \arrow[r, "b"] \arrow[d, phantom, "\cap"] & V' \arrow[d, phantom, "\cap"] \\ V' = W \arrow[r, "b"] & V \end{tikzcd}$$

Start with $\underset{\text{codim } 1}{W \subset V}$ also $b : W \to V \ni b(w), w$ ind for $w \neq 0$.

Let $V_1 = W$, $W_1 = \{w \in W \mid b(w) \in W\}$. Let

Jan 12. Consider $W \subset V$ of codim 1, $b : W \to V \ni b(w), \cancel{w}$ ind for $w \neq 0$. Let $V_1 = W$ and $W_1 = \{w \in W \mid b(w) \in W\}$. First note that $b(W) \not\subset W$, because otherwise b would have an eigenvector in W. Thus

$$\begin{tikzcd} W_1 = b^{-1}(W) \arrow[r, "b"] \arrow[d, hook] & W \arrow[d, hook] \\ V_1 = W \arrow[r, "b"'] & V \end{tikzcd}$$

so W_1 is of codim 1 in V_1 and we can proceed by induction

So the end result is a ~~sequence~~ flag $V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_n = 0$ such that $V_i + bV_i = V_{i-1}$. Then you pick $0 \neq \xi \in V_{n-1}$ and you get basis $\xi, b\xi, \dots, b^{n+1}\xi$ for V.

445 So what am I going to do? I need to get control of the almost eigenvector. We have

$V = \mathbb{C}^{n+1}$ with basis $\xi_0, \dots, \xi_n$

$W = \mathbb{C}^n$ $\qquad \xi_0, \dots, \xi_{n-1}$

b shift. In the end you have this line appearing as a quotient.

$$0 \longrightarrow \mathcal{O}(-1)\otimes W \longrightarrow \mathcal{O}\otimes V \longrightarrow L \longrightarrow 0.$$

Discuss control of this situation. V is equipped with p.d. hermitian sc. prod. such that b is an isometry; get p.d. hermitian Hankel matrix (h_{i-j}). h_0 real, $h_1, \dots, h_n$ ind cx. nos. $2n+1$ parameters needed. On the other hand you should need 1 par. to complete to a unitary matrix then ~~how man positive~~ pos. measures on S^1 with support $n+1$ points described by $2n+2$ param., so it checks. Do $n=1$.

$$\begin{pmatrix} h_0 & \alpha \\ \bar\alpha & h_0 \end{pmatrix} \qquad \left(\begin{array}{l} h_0^2 - |\alpha|^2 > 0. \\ h_0 > 0. \end{array} \right. \qquad \alpha = h_1$$

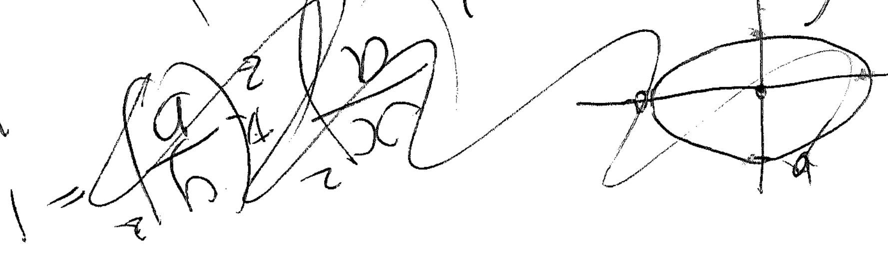
Start. You know $az - b : W \hookrightarrow V$ gives a hyperplane in V depending on z. Get same line in $V^\times$ $\operatorname{Ker}(a^t z - b^t)$. In your model this is very simple because

$$a = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \qquad za^t - b = \begin{pmatrix} z & -1 & & \\ & z & -1 & \\ & & z & -1 \\ & & & z \end{pmatrix}$$

Better

$$az - b = \begin{pmatrix} z & & \\ -1 & \ddots & \\ & \ddots & \end{pmatrix}$$

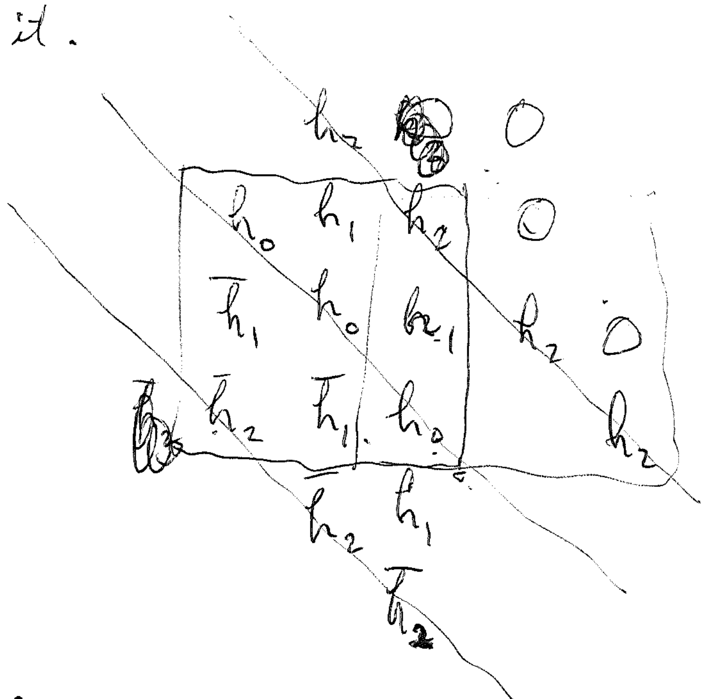
Ker of $(za-b)^t$ is $(1 \;\; z \;\; \cdots \;\; z^n)$



446 What do you want? Some sort of orthogonal complement. The idea is that you [redacted] look at the orth complements of aW, bW, [redacted] choose a unitary between the complements and then end up with a finite measure on S^1. Structurally you are stuck with an arbitrary pos. def. hermitian Hankel matrix, $2n+1$ parameters, ~~so life is easier~~

Take $n=1$. In V 2-dim'l H.S. you have two unit vectors a, b independent

$\langle b, a\rangle = h \qquad |h|<1$. There may be a scattering situation here. Namely given the inner product on $\mathbb{C}\oplus\mathbb{C}z\oplus\cdots\oplus\mathbb{C}z^n$ you should be able to extend it.



Concentrate: Take $n=1$. V 2-dim'l a, b are unit vector indep $\langle b, a\rangle = h \quad |h|<1$. Then for each $z\in S^2$ we have $\mathbb{C}(za+b)\subset V$. Orth. line.

$$\langle wa+b, za+b\rangle = \bar w z + \bar w \bar h + zh + 1.$$

$$= 1 + zh + \bar w\bar h + z\bar w|h|^2 + \bar w z(1-|h|^2)$$

$$= (1+zh)(1+\bar w\bar h)$$

447 a, b are a basis. You take

$$\langle c_1 a + c_2 b,\ c_1' a + c_2' b\rangle$$

$$= \bar{c}_1 c_1' + \bar{c}_1 c_2' \bar{h} + \bar{c}_2 c_1' h + \bar{c}_2 c_2'$$

$$= \begin{pmatrix} \bar{c}_1 & \bar{c}_2 \end{pmatrix} \begin{pmatrix} 1 & \bar{h} \\ h & 1 \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix}$$

So the orthogonal of $\mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix}$ ~~is~~ consists of those $\begin{pmatrix} c_1' \\ c_2' \end{pmatrix}$ such that

$$\begin{pmatrix} \bar{z} & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{h} \\ h & 1 \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \bar{z} c_1' + \bar{z}\bar{h} c_2' + h c_1' + c_2' = 0$$

$$= (\bar{z} + h) c_1' + (\bar{z}\bar{h} + 1) c_2' = 0$$

$$c_2' = -\frac{\bar{z} + h}{1 + \bar{z}\bar{h}} c_1'.$$

These calculations are confusing – it is probably better to discuss general issues. For example. You know that given V Hilb. sp of dim $n+1$ with a partial isometry of codim 1, ~~there~~ this is equivalent up to isom. with a $n+1 \times n+1$ Hankel matrix. Hermitian matrix of form h_{i-j} for $0 \le i, j \le n$ which is positive definite. There's a circle ~~worth~~ worth of ways to complete the partial unitary to a unitary. Real problem – how to set up things efficiently.

$$\| c_0 + c_1 z + c_2 z^2 \|^2 = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}^\dagger \begin{pmatrix} 1 & h & 0 \\ \bar{h} & 1 & h \\ 0 & \bar{h} & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$$

Is this pos. def.

$$\begin{pmatrix} 1 & h & \\ \bar{h} & 1 & h \\ & \bar{h} & 1 \end{pmatrix} = \begin{pmatrix} 1 & h & 0 \\ \bar{h} & \bar{h}h & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\bar{h}h & h \\ 0 & \bar{h} & 1 \end{pmatrix}$$

~~So if you are given~~

So a Hankel matrix $A = (a_{i-j})_{i,j \in \mathbb{Z}}$ is a multiplication by a function on S^1.

Toeplitz etc.

So suppose h_{i-j} a pos. def., herm. hankel matrix

$\sum h_n z^n$ real function on S^1

$$\int \overline{\sum c_\ell z^\ell} \sum h_n z^n \sum c_k z^k \frac{dz}{2\pi i z} = \int |f|^2 h \frac{d\theta}{2\pi}$$

$$= \sum_{\ell = n+k} \bar{c}_\ell h_n c_k = \sum \bar{c}_\ell h_{\ell-k} c_k$$

suppose $h > 0$ on S^1. $\log h$

Can write $h = |g|^2$ where g analytic invertible $|z|<1$.

Does this help me solve anything?

What next? Suppose you have an isometry on an infinite dimensional space $s^* s = 1$. Toeplitz algebra

There is no room here for anything else. Anyway something else returns - if $h(\theta)$ is a positive definite matrix function on S^1, then you have polar decomposition $h(\theta) = g(\theta)^* g(\theta)$ where $g(\theta)$ is analytic invertible in the disk. Idea must be to use h to define Hilbert space $L^2(S^1, h\frac{d\theta}{2\pi})^n$, form Hardy space closure of $z^k\xi$ $k \geq 0$ $\xi \in \mathbb{C}^n$. Get Hardy space H, and isometry mult by z, form $H \ominus zH$. Only point is why this generates. Same kind of Fredholm stuff?

Anyway look at $n = 1$. $h > 0$ pos. fn. on S^1. Look for $g \in H$ $g \perp z^k g$. $k \geq 1$.

$$\int z^k |g|^2 h \frac{d\theta}{2\pi} = 0 \qquad \text{for } k \geq 1.$$

$$\Rightarrow \quad |g|^2 h = \text{Const.} > 0 \quad \text{etc.}$$

Anyway what about $\bar{h} z^{-1} + 1 + hz$ on S^1. Can suppose $h > 0$. $1 + 2h_0 \cos\theta$

So it seems that you cannot extend the Hankel matrix $\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}$ and keep positivity. You wanted to extend it to $h(\theta) = 1 + he^{i\theta} + \bar{h}e^{-i\theta}$

$$\begin{vmatrix} 1 & h & \\ h & 1 & h \\ & h & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & h \\ h & 1 \end{vmatrix} - h\begin{vmatrix} h & h \\ 0 & 1 \end{vmatrix} = 1 - h^2 - h^2 = 1 - 2h^2$$

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450 So where next. You still don't understand the partial isometry situation. Move on to D.J. quaternionic situation. Various things come to mind ~~like~~ his dual picture to σK-modules. $A\mathbb{H}$-modules. But what's important probably is the analogues of $\mathcal{O}(n)$. My picture of $\mathcal{O}(n)$ is the K-module $\mathbb{C}^n \underset{s}{\overset{i}{\rightrightarrows}} \mathbb{C}^{n+1}$ where s is the shift

For n odd we need to take 2 of these somehow

antipodal map on S^2 is $z \longmapsto -\bar{z}^{-1}$

You want to discuss the σK-module belonging to $\mathcal{O}(1)\otimes\mathbb{H}$. Is there an analogue of $\mathcal{O}(3)\otimes\mathbb{H}$? Let's try for a σ-version of a partial isometry. Things should be basically the same. You expect to see a V and W $V = \Gamma(\mathcal{O}(2n-1)\otimes_{\mathbb{C}}\mathbb{H})$

Let's again try to understand partial isom.

Actually when is $1 + \underbrace{hz + hz^{-1}}_{2h\cos\theta} > 0$ on S^1 $\quad |2h| < 1$.

~~Observe that if $h = 1$. Then~~

$1 + 2h\cos\theta = 1 + 2h$

Do $n=1$ carefully. V 2diml a, b are unit vectors, $\langle b, a\rangle = h$. Then $V \simeq \mathbb{C}^2$ with inner product $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^\dagger \begin{pmatrix} 1 & \bar{h} \\ h & 1 \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix}$

$$\langle c_1 a + c_2 b, c_1' a + c_2' b\rangle = \bar{c}_1 c_1' + \bar{c}_1 c_2' \bar{h} + \bar{c}_2 c_1' h + \bar{c}_2 c_2'$$

451 Alternatively use orthonormal basis for V.

$$a = \begin{pmatrix}1\\0\end{pmatrix} \qquad b = \begin{pmatrix}\lambda\\ \sqrt{1-|\lambda|^2}\end{pmatrix}. \qquad W = \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$$

and $b(W) = \mathbb{C}\begin{pmatrix}\lambda\\ \sqrt{1-|\lambda|^2}\end{pmatrix}$. $\quad W^\perp = \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$

$$(bW)^\perp = \operatorname{Ker} b^\dagger = \left\{\begin{pmatrix}c_1\\c_2\end{pmatrix} \middle| \bar\lambda c_1 + \sqrt{}\, c_2 = 0\right\}$$

$$= \mathbb{C}\begin{pmatrix}\sqrt{1-|\lambda|^2}\\ -\bar\lambda\end{pmatrix}.$$

Let g be a unitary such that $g\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\lambda\\ \sqrt{1-|\lambda|^2}\end{pmatrix}$

i.e. $$g = \begin{pmatrix}\lambda & -\sqrt{1-|\lambda|^2}\,\zeta\\ \sqrt{1-|\lambda|^2} & +\bar\lambda\,\zeta\end{pmatrix}$$ where $|\zeta| = 1$.

So what are the eigenvalues of g? What are the natural questions to ask? Try for response. Charac. equation is

$$\mu^2 - (\lambda + \bar\lambda\zeta)\mu + \zeta = 0$$

this can be rearranged. But what are you aiming for? For each ζ (boundary condition) you get two roots

$$(\mu\zeta^{-1/2})^2 - \underbrace{(\lambda+\bar\lambda\zeta)\zeta^{-1/2}}_{\lambda\zeta^{-1/2}+\bar\lambda\zeta^{1/2}}(\mu\zeta^{-1/2}) + 1 = 0.$$

Your notes 1 diml determinants

You aim should be a "response function" in this situation. What should it do? its properties.

For each boundary condition ζ you get 2 roots.

In fact you get a unitary operator – get roots maybe a cyclic vector. Hankel matrices

451 Jan 13 ~~[illegible]~~ Find the response for a partial ~~[illegible]~~ isometry in the simplest case. Given V a 2-diml Hilbert space and independent unit vectors a, b. Choose basis such that $a = \begin{pmatrix}1\\0\end{pmatrix}$ $b = \begin{pmatrix}h\\ \sqrt{1-|h|^2}\end{pmatrix}$. Then

Start again: $W \overset{a}{\underset{b}{\rightrightarrows}} V$ W, V Hilbert spaces

$a^*a = 1$, $b^*b = 1$. Choose orth basis ~~[illegible]~~ $V \simeq \mathbb{C}^2$ such that $aW = \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$, $bW = \mathbb{C}\begin{pmatrix}h\\ \sqrt{1-|h|^2}\end{pmatrix}$. Then have partial unitary $\begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}h\\ \sqrt{1-|h|^2}\end{pmatrix}$ which we extend to a unitary by $\begin{pmatrix}0\\1\end{pmatrix} \longmapsto \zeta\begin{pmatrix}\sqrt{1-|h^2|}\\ -\bar{h}\end{pmatrix}$ $|\zeta| = 1$.

Maybe best is to say we have a unitary matrix

$$g = \begin{pmatrix} h & -\sqrt{1-|h|^2} \\ \sqrt{1-|h|^2} & \bar{h} \end{pmatrix}$$

and we are allowed to multiply by $\begin{pmatrix}1 & 0\\ 0 & \zeta\end{pmatrix}$

Look at the general situation from this viewpoint. Namely you have ~~[illegible]~~ a unitary g and a line ℓ, and you are interested in invariants of ~~[illegible]~~ the coset g ~~[illegible]~~ H where H = group of unitaries $(1-e) \oplus \zeta e$ e proj on ℓ. You can pick unit vector in ℓ, then get a spectral measure for g on the circle. Assume ℓ cyclic for g, and V has dim $n+1$. Possible Hankel matrices with $h_0 = 1$ are dim $2n$. possible probability measures with support $n+1$ pts give dim $2n+1$.

~~Consider pairs consisting of a cyclic unit vector~~

~~Consider pairs consisting of~~

Question: Given a partial isometry which might be the "response function" or "impedance". Discuss.

Go back to an LC network.

Scattering. H Hilbert space with a unitary operator U. Assume given "outgoing" subspace H^+ s.t. $UH^+ \subset H^+$ and an incoming subspace H^- s.t. $U^{-1}H^- \subset H^-$. Assume $H^+ \perp H^-$ and $H^+ + H^-$ is of finite codimension.

What can you say? Apply spectral theorem.

Invariant subspaces $\qquad \bigcup_{n \to -\infty} U^n H^+ \qquad \bigcup_{n \to +\infty} U^n H^-$

Does this mean no reflection

$$\begin{array}{cc} \wr\vert & \wr\vert \\ L^2(S^1, H^+/UH^+) & L^2(S^1, H^-/U^{-1}H^-) \end{array}$$

$H^- \supset U^{-1}(H^-)$

Note $(H^-)^\perp$ is also outgoing $\qquad U(H^-) \supset H^-$

$$U((H^-)^\perp) = (U(H^-))^\perp \subset (H^-)^\perp$$

Focus on the easiest case. This is $L^2(S^1)$ with ~~two lattices~~ two outgoing subspaces $H^2(S^1) \supset M$. I know in this situation that $\exists\, g(z)$ analytic in $|z| < 1$, with $|g(z)| = 1$ for $|z| = 1$. such that $M = g H^2(S^1)$. This $g(z)$ should be the response function. The zeros in the disk are involved with a type of decay.

In this situation is there a partial isometry arising from the unitary U? Typical g is $\frac{z-h}{1-\bar{h}z}$ $\quad gH^2 = (z-h)H^2$ and functions vanishing at $h \in$ disk. Consider the equation

$$\frac{z-h}{1-\bar{h}z} = e^{i\theta} \qquad \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix}(z) = e^{i\theta}$$

$$z = \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix}^{-1}(e^{i\theta}) = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} e^{i\theta} = \frac{e^{i\theta} + h}{\bar{h}e^{i\theta} + 1}$$

453 You get out of this a single disk parameter. From a codim 1 outgoing subspace you get a disk worth of response functions. There might be a V of dim 2 and W of dim 1 around. Next you might try
$$\frac{(z-h_1)(z-h_2)}{(1-\bar{h}_1 z)(1-\bar{h}_2 z)} = g(z).$$

So obviously if you ask for $\operatorname{codim}(M)=n$, then get something $2n$ diml. Related to Hankel matrices on an n-diml space. There should be a simple link!

Consider case $H^2 \supset (z-h)H^2$. It seems we two lines namely, the orthogonal complements of $(z-h)H^2$ and of zH^2. Might try $(H^2)^\perp$, $(z-h)H^2$

You are exceedingly slow. Contrast with the Bott loop group analysis. There you have lattices for $\mathbb{C}[z]$ inside $\mathbb{C}[z,z^{-1}]^n$, but no possibility of ~~inside~~ Disk. If $n=1$ you have only the lattices $\mathbb{C}[z]z^n$. ~~So you are close.~~
You need to handle a general $H^2 \supset M$ where $M = (z-h_1)\cdots(z-h_n)H^2$. This is comm. group. Not composing fractional linear transf.
Take fractional linear viewpt.
$$\begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & h \\ \bar{h}_0 z & 1 \end{pmatrix}$$

$$f(z) \longmapsto \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}(f) = \frac{f+h}{1+\bar{h}f}$$
Combine with $f \longmapsto zf$.

So the response function I want

$g(z)$
rational function of z
analytic for $|z|<1$
$|g|=1$ when $|z|=1$.

closed under composition.

What does reflection principle say? Reflect thru $|z|=1$.

$z \longmapsto \bar{z}^{-1}$ Consider $(g(z^*))^*$. This is a rational function of z $\dfrac{1}{g(\frac{1}{z})}$ So $g(z^*)^* = g(z)$.

But we want g holom for $|z|<1$, so g has zeroes insides $|z|=1$ and poles outside.

Example: SU(1,1) $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ $|a|^2-|b|^2=1$.

typically $\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \dfrac{1}{\sqrt{1-|h|^2}}$ diagonal. $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$.

So start with $g(z)$ rational as above of degree n, i.e. n zeroes inside $|z|=1$. Let $g(0)=h \in D$.

$$\begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} g(z) = \frac{g(z)-h}{1-\bar{h}g(z)} \quad \text{has } g(z)=0.$$

~~Remaining problems Yes!! What are the variants~~

You still haven't connected such a $g(z)$ to a partial isometry. try to understand

$$g(z) = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & \phi \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{z+h}{1+\bar{h}z}$$

Scattering situation. $H^2 \supset (z-h)H^2$

$$H^2 \oplus (z-h)H^2 = L^2$$

455 Is there a connection of ~~a pa~~ partial isom. to this.

$$\overset{1}{W} \underset{b}{\overset{a}{\rightrightarrows}} \overset{2}{V} \qquad a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} h \\ \sqrt{1-|h^2|} \end{pmatrix}.$$

Now extend this to a unitary operator g on V.

$$g = \begin{pmatrix} h & \sqrt{1-|h|^2} \\ \sqrt{1-|h^a|^2} & -\bar{h} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}$$

How can I proceed?

Try another viewpt. 1-diml determinants.

Consider $D = \frac{d}{dt} + A(t)$ on some interval

Try to define $\det(D)$ variationally using

$$\delta \log \det(D) = \operatorname{tr}\left(\frac{1}{D}\,\delta D\right)$$

Need situation where $\frac{1}{D}$ exists, and this requires looking ~~of~~ at ∂ conditions. Yes! ~~So how to proceed.~~

Find propagator for D. Let V be the vector space where the solutions take values. ~~[illegible]~~ Interval ~~[illegible]~~ $[0,1]$ $\qquad U(1,0) : V \longrightarrow V$ has a graph

$$\left(\frac{d}{dt} + A(t)\right) u(t,0) = 0 \qquad u(0,0) = I.$$ ~~[illegible]~~

The boundary ~~b~~ values of solutions ~~satisfy~~ is a subspace of $V \oplus V$, namely the graph $\Gamma = \left\{ \begin{pmatrix} \xi \\ U(1,0)\xi \end{pmatrix} \middle| \xi \in V \right\}$

We need to give a subspace B ~~of~~ of $V \oplus V$ which is complementary to Γ. Now you vary ~~[illegible]~~ $A(t)$ boundary conditions. In this situation you know the singularity of $D^{-1}(x,y)$ is a jump. $\frac{d}{dt} H(t) = \delta$

And you can ~~see~~ probably construct the Green's fn by linear algebra. What about a variation in the b. conds? This is tricky, you are not apparently changing D. So how can we make sense of it? ~~namely you have~~ Because D is partially defined, you may want to replace D by its graph, which is essentially the same as the graph of D^{-1}. So we should ask about determinants on graphs. Given $T: V \to V$ associate $\Gamma_T = \binom{1}{T} V$ Meaning of $tr(T^{-1}\delta T)$? tr is related to the intersection of Γ_T with Δ. determinant ~~bith~~

$$\frac{1}{\det(1-A)} = \exp\Big\{\sum_{n\geq 1} \frac{1}{n} tr(A^n)\Big\}.$$

$$-\delta \log\det(1-A) = \sum \frac{1}{n}\delta tr(A^n)$$

$$= \sum_{n\geq 1} tr(A^{n-1}\delta A) = tr\Big\{\frac{1}{1-A}\delta A\Big\}$$

You've lost the idea. The aim recall is to take a partial isom. complete it to a unitary depending on bdry conditions, then ~~take~~ charac. poly of the unitary, study this char poly as some sort of function ~~of~~ the bdry conditions. Somehow this should give the response function I want. This seems to be the unitary analogue of the LC network

Do in ~~the~~ case $n=1$. $\overset{1}{W} \underset{b}{\overset{a}{\rightrightarrows}} \overset{2}{V}$ Hilb. sp., a,b isom. a,b ind.

$V = \mathbb{C}^2$ $aW = \mathbb{C}\binom{1}{0}$ $bW = \mathbb{C}\binom{h}{\sqrt{1-|h|^2}}$ want unitaries

$$g \to g\binom{1}{0} = \binom{h}{\sqrt{1-|h|^2}}$$

$$g_\theta = \begin{pmatrix} h & -\sqrt{1-|h|^2} \\ \sqrt{1-|h|^2} & +\bar{h} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

457 $\quad \det(g_\theta) = e^{i\theta}$ uninteresting, so $\quad$ 4:30

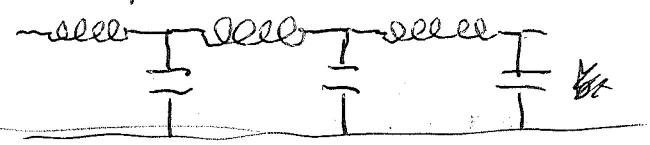
$$\lambda^2 - (h + \bar{h} e^{i\theta})\lambda + e^{i\theta} = 0.$$

Look at $\delta \operatorname{tr}(g_\theta) = \operatorname{tr}\left(\begin{pmatrix} h & -\sqrt{1-|h|^2} \\ \sqrt{1-|h|^2} & \bar{h} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \delta e^{i\theta} \end{pmatrix}\right)$

~~Alter~~ $\bar{h}\,\delta e^{i\theta}$ [redacted] ?

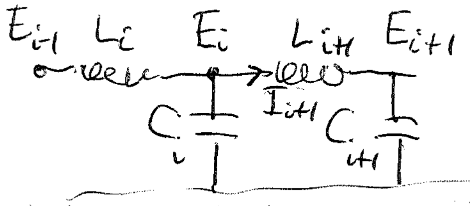
~~Different viewpoint. How am I going to proceed?~~
~~Consider a coherent sheaf.~~ $\quad W \overset{a}{\underset{b}{\rightrightarrows}} V$

Different viewpt. Go to transmission lines. $\quad$ isom. if a isom then $b^*b = 1$. So structurally $V = H^2(S^1)^m \quad b = z$



Let g be unitary on V, let $l \subset V$ be a line consider $\quad g(\zeta e_l + (1 - e_l))$

electrical response



$$E_i - E_{i+1} = L_{i+1} s I_{i+1}$$
$$I_i - I_{i+1} = C_i s E_i$$

$$\begin{pmatrix} E_i \\ I_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & L_{i+1}s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_{i+1} \\ I_{i+1} \end{pmatrix}$$

$$\begin{pmatrix} E_i \\ I_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_i s & 1 \end{pmatrix}\begin{pmatrix} E_i \\ I_{i+1} \end{pmatrix}$$

$$\begin{pmatrix} E_0 \\ I_{} \end{pmatrix} = \begin{pmatrix} 1 & L s \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ C s & 1 \end{pmatrix} \cdots$$

458 Let's examine the case of ~~an~~ an infinite Hankel matrix. This means a pos. def. translation inv. ~~...~~ hermitian inner product on $\mathbb{C}[z,z^{-1}]$. ~~...~~ measure $d\mu$ on S^1. $h_n = \int z^n d\mu$. $\langle z^m, z^n\rangle = \int z^{n-m} d\mu$

So a pos. definite Hankel is the same as a measure, S^1 Now ask ~~...~~ about $H^2(S^1, d\mu)$. This is an outgoing space $\iff$ $d\mu = f\frac{d\theta}{2\pi}$ abs. cont. ~~+ log~~ $+\int \log f < \infty$.

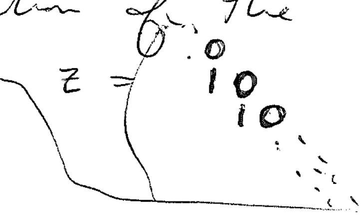
$\Longrightarrow$ easy because take g unit vector in $H^2 \ominus zH^2$. Then g analytic inside and $|g(z)| = 1$ for $|z| = 1$.

Jan 14. ~~Take~~ Coupling ~~...~~ example. ~~Take an incoming outgoing subsp~~ $L^2(S^1)$ with z. divide it into $H^- \oplus H^+ = (H^2)^\perp \oplus H^2$ and equip the two pieces ~~...~~ with the ~~...~~ induced partial isometries. Thus ~~on~~ on H^+ we have ~~an isometry~~ a unitary $H^+ \to W^+ = zH^+$ ~~$S : S^*S = 1$~~ where W^+ is a codim 1 subspace ~~defined everywhere~~, and on H^- we have ~~...~~. a unitary $W^- \xrightarrow{\sim} H^-$ where $W^- = z^{-1}H^-$ is a codim 1 subspace. Now add to get ~~...~~

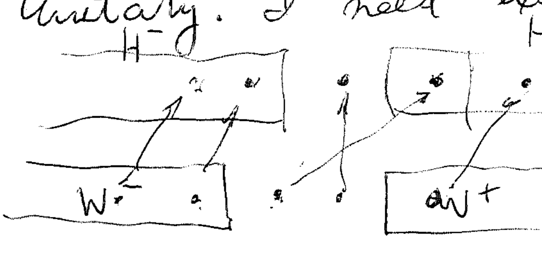
$H^- \oplus H^+ = L^2(S^1)$ $\qquad W^- \oplus W^+ = z^{-1}H^- \oplus H^+$ $\qquad$ a = inclusion $b = z$.

This ~~...~~ has deficiency indices 1,1. Look at the possible extensions to a unitary. Described by $U(1) = S^1$.

This is a special case of a perturbation of the unitary z. You have taken the matrix for z = and changed a 1 to ζ. Another possibility is to ~~cut a piece~~ separate $H^- \oplus H_0 \oplus H^+$, say



~~...~~ $(H^2)^\perp \oplus \mathbb{C} \oplus zH^2$, $W^- = z^{-1}H^-$ $W^+ = H^+$ Then we have $U(2)$ ~~...~~ acting simply transitively on the possible extensions of this partial ~~...~~ isometry to a ~~...~~ unitary. I need examples.



bound state

459 Try something else.

Suppose you have H with U such that $\exists$ ~~and~~ incoming H^- and outgoing H^+ say $\perp$ and $H^- \oplus H^+$ has finite codim. Then what structure does $H \ominus (H^- \oplus H^+)$ have?

Try coupling simple types together. Suppose ~~A~~ You have two partial isometries of dim 2. ~~Then~~ Take the direct sum and couple them end to end. The coupling should involve a single $\xi \in S^1$. ~~How~~

$$\begin{tikzcd}[column sep=small] & V^1 & & & V^2 & \\ W^1 & & W^1 \arrow[ul] & \overset{\sim}{\underset{\xi}{=}} & W^2 \arrow[u] & W^2 \arrow[ul] \end{tikzcd}$$

Consider $\overset{1}{V} \underset{b}{\overset{a}{\rightrightarrows}} \overset{2}{V}$ partial isom. ~~So these are~~

There are two distinguished lines in V. ~~But~~ ~~Let~~

Is there a means to couple the head to the tail. composition of correspondences? ~~Idea in a good case you have an exact sequence~~

Exact sequence of sheaves on S^2

$0 \longrightarrow \mathcal{O}$

We know that indecomp. K-modules of type $n, n+1$ corresp to $\mathcal{O}(n)$ $n \geq 0$.

~~Can these be~~ Can these be linked together?

~~A~~ If you have an $\mathcal{O}(n)$ and and $\mathcal{O}(m)$ can you get

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(n) \oplus \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m) \longrightarrow 0 \ ?$$

~~Klass~~

n

$$0 \longrightarrow \mathcal{O}(n) \hookrightarrow \mathcal{O}(n+m) \longrightarrow \mathcal{O}(n+m)/\mathcal{O}(n) \longrightarrow 0$$

$\exists$ if $n \geq m+1$ $\quad \mathcal{O}(m) \to \mathcal{O}(n+m)/\mathcal{O}(n)$

certain principles. There is something special about the points $(0, \infty) \in S^2$ in all of this.

You have to find a way to deal with this. Approach by simple examples, provided you don't get confused. Concentrate carefully on the dimension 2 ~~and~~ situation. First of all $\dim(V) = 1$. $W = 0$, $a, b = 0$ - one parameter ζ. Next $\dim(V) = 2$ $\dim(W) = 1$. a, b independent up to isom.

Look at the possibilities - given $W \underset{b}{\overset{a}{\rightrightarrows}} V$, ~~[illegible]~~ Better think of $W \hookrightarrow V \times V$. ~~Use ba~~ Given V Hilb. space of dim $n+1$, poss. for W lie in $\mathbb{P}^1(V^*)$, has real dim $2n$~~[illegible]~~, and then poss. for b $2n$ for bW and n^2 for the unitary isom $b: W \to bW$ So given V, the possible partial isom. have real dim. ~~[illegible]~~ $2n + 2n + n^2$. Have Unitaries on V acting center acts trivially, so if action free $n^2 + 4n - (n+1)^2 + 1$ $= n^2 + 4n - n^2 - 2n = 2n$. Another version: Given partial isometry extend to a unitary.

$$\left\{ (W, b) \;\middle|\; \begin{array}{l} W \text{ hyperplane in } V \\ \text{and } b: W \to V \\ b^* b = 1 \end{array} \right\}$$

$\xleftarrow{S^1 \text{ fibre}} \left\{ (W, g) \;\middle|\; \begin{array}{l} W \text{ hyperplane} \\ g \text{ unitary on } V \end{array} \right\}$ $\qquad$ Let $U(V)$ act on (W, g)

$$2n + (n+1)^2 = n^2 + 4n + 1$$

461 Attempt a small summary. Given V ~~and~~ Hilbert space dim $n+1$, and a partial isom. $W \overset{a}{\underset{b}{\rightrightarrows}} V$ $\dim(W) = n$. Assume W hyperplane in V a = inclusion. Choose isom $W^\perp \xrightarrow{\alpha} (bW)^\perp$ get unitary

$$g : \begin{matrix} W \\ \oplus \\ W^\perp \end{matrix} \xrightarrow{\begin{pmatrix} b & 0 \\ 0 & \alpha \end{pmatrix}} \begin{matrix} bW \\ \oplus \\ (bW)^\perp \end{matrix}$$

Changing α to $\zeta\alpha$ $|\zeta| = 1$ ~~multiples~~ changes g to $g\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$, so $\exists$ unique α ~~deg~~ s.t. $\det(g) = 1$. Now have unitary g and dist. line $W^\perp$. Assume $\underbrace{\sum g^i W^\perp = V}_{\text{generic case}}$, then you get a ~~generic~~ probability measure on S^1 support $n+1$ points adding to 0. $2n$ parameters.

Go back to **coupling**

Try first to couple 2 $\mathcal{O}(1)$'s to get an $\mathcal{O}(2)$. Here you start with $(\overset{1}{W} \rightrightarrows \overset{2}{V})$ $(\overset{1}{Y} \rightrightarrows \overset{2}{X})$

$$0 \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(n) \longrightarrow \mathcal{O}(n+1)$$

~~Given~~ Given $\overset{n}{W} \overset{a}{\underset{b}{\rightrightarrows}} \overset{n+1}{V}$ let us try to understand, get control of the line $(az+b)W^\perp$. Actually this is quotient line of V depending holom on z. Inner product on V induces one on $\mathcal{O}(1)$ over $\mathbb{P}(V^*)$. So we have to handle this cleanly.

Mixture of alg and norm

Example. $\mathbb{C}^n \overset{a}{\underset{b}{\rightrightarrows}} \mathbb{C}^{n+1}$ $a = \begin{pmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{pmatrix}$ $b = \begin{pmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{pmatrix}$

homogeneous coordinates of a ~~line in the~~ hyperplane in $\mathbb{C}^{n+1}$

$$\begin{pmatrix} c_0 & \cdots & c_n \end{pmatrix}\begin{pmatrix} z & & \\ -1 & \ddots & \\ & \ddots & z \\ & & -1 \end{pmatrix} = 0 \qquad c_0 z - c_1 = 0 \qquad c = c_0\begin{pmatrix} 1 & z & \cdots & z^n \end{pmatrix}$$

$$c_1 z - c_2 = 0$$

so for any z we get the line $\mathbb{C}(1\ z \cdots z^n) \subset (\mathbb{C}^{n+1})^*$

Now what can you do? You complete the partial isometry ba^{-1} to a unitary. Simply means

$$\begin{pmatrix} 0 & & & \xi \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & 1 & 0 \end{pmatrix}$$

Try again. Given partial isom. codim 1 $W \underset{b}{\overset{a = \text{inc}}{\rightrightarrows}} V$

choose $\alpha : W^\perp \xrightarrow{\sim} (bW)^\perp$ form.

a $\quad g = \begin{matrix} W \\ \oplus \\ W^\perp \end{matrix} \xrightarrow{\begin{pmatrix} b & 0 \\ 0 & \alpha \end{pmatrix}} \begin{matrix} bW \\ \oplus \\ (bW)^\perp \end{matrix}$ $\quad$ Altering α by $\alpha\xi$ alters g to $\begin{pmatrix} b & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$

so unique α such that $\det(g) = 1$. Then have the following structure $W^\perp$ $\quad W^\perp +$. Wait -

simplest

V Hilbert space equipped with a unitary operator g and line L. Consider filtration

$$L \subset L + gL \subset L + gL + g^2L \subset \cdots$$

I claim this depends on the coset gH_L where H_L is the subgp of $\mathcal{U}(V)$ consisting of g' such that $g' = 1$ on $L^\perp$. In effect $g'L = L$ so $gg'L = gL$..

Put $g_1 = gg'$ Then $\quad g_1 - g = gg' - g = g(g' - 1) = g(\xi e_L)$

thus $g_1 \equiv g \mod gL$ so g_1L .

463 This business still eludes me.

$$L \subset L + g^{-1}L \subset L + g^{-1}L + g^{-2}L \subset \cdots$$

$g_1 = gh$ where $(h-1)(V) \subset L$.

infact $h^{-1} = 1 - e_L + \zeta^{-1} e_L$ $\quad |\zeta| = 1$

$= 1 + (\zeta^{-1} - 1) e_L$

then $g_1^{-1}\xi = h^{-1}g^{-1}\xi = g^{-1}\xi + (\zeta^{-1}-1)e_L g^{-1}\xi$

Thus for any subspace X we have

$$g_1^{-1}X \subset g^{-1}X + L, \quad g^{-1}X \subset g_1^{-1}X + L.$$

$$\therefore \quad g_1^{-1}X + L = g^{-1}X + L.$$

Thus $L + g_1^{-1}L = L + g^{-1}L$

$$L + g_1^{-1}(L + g_1^{-1}L) = L + g_1^{-1}(L + g^{-1}L) = L + g^{-1}(L + g^{-1}L)$$

So we find $L + g_1^{-1}L + \cdots + g_1^{-k}L = L + g^{-1}L + \cdots + g^{-k}L \quad \forall k.$

Check also that $g_1 L + \cdots + g_1^k L = gL + \cdots + g^k L \quad \forall k.$

$$g_1 = gh = g(1 + (\zeta - 1)e_L)$$

so $\quad g_1\xi = g\xi + (\zeta - 1)g(e_L\xi)$

$$g_1 X + gL = gX + gL$$

so $\quad g_1 L + g_1^2 L = gL + g_1(gL) = gL + g^2 L$

$$g_1 L + g_1(g_1 L + g_1^2 L) = gL + g_1(gL + g^2L) = gL + g(gL + g^2L)$$

This makes you feel better.

Now what do you want?

You know now that a codim 1 partial isom. is effectively equivalent to a unitary of det 1 + line.

464. dimension of $\overset{\text{codim 1.}}{\text{partial}}$ isoms on V^{n+1} $\qquad$ $2n + 2n + a^2$?
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ aW bW isom.

$$(n+1)^2 - 1 + 2n = n^2 + 2n + \cancel{1 - 1} + 2n$$

~~[illegible]~~

Do you have a response function for a partial isom. of codim 1? It seems I actually get a pair consisting of $g \in SU(n+1)$ and a line L. Get invariants by conjugating by $U(n+1)$.

I still haven't found coupling.

~~[illegible]~~

Question: Given a scattering ~~situation~~ show there is a partial isometry.

Take $L^2(S^1)$ with U_0 = mult by z, and U a perturbation of finite support. Suppose you have an incoming subspace H^- and an outgoing one H^+. In fact take $H^+ = z^N H^2(S^1)$, and $H^- = z^{-k} H^2(S^1)^\perp$. Is there a partial isometry in this situation? How?

Questions. Response function for a partial isometry of codim 1.
Does $\exists$ natural partial isometry in scattering situation?
Coupling of partial isometries of codim 1.

Scattering situation. Take $L^2(S^1)$ and the ~~[illegible]~~ ~~[illegible]~~ quotient of a layer of outgoing subspaces, e.g. $H^2/\varphi(z)H^2$. If you take $\varphi(z) = z - h$ $\quad |h| < 1$ then ~~you get~~ $H^2/(z-h)H^2$ has dim 1. It's possible that the 2-diml space you want ~~the~~ is spanned by $1, \frac{z-h}{1-\bar{h}z}$ and in general you want $H^2 \ominus z\varphi H^2$