1 Need to review reflection positivity.
<u>Can you first</u> understand Gaussian case?

So what do I do?

Review program. Given $aX \oplus V^+ = V^- \oplus bX = Y$ of type $\mathcal{O}(n)$, get line bundle $L_z = Y/(az-b)X$, get holom section from v_0^- (unit v_0 in ~~basis for~~ V^+), allowing sections y to ~~yield~~ functions:

$$(az-b)x = -y + \tilde{y}(z)\,v_0^-$$

solve
$$(1-zb^*a)x = b^*y \qquad b^*(1-zab^*)^{-1}$$

$$\tilde{y}(z) = \Big(v_0^-,\ \underbrace{y + (az-b)(1-zb^*a)^{-1}b^*y}_{\left(1-zab^* + (az-b)b^*\right)(1-zab^*)^{-1}y}\Big)$$

$$= (v_0^-, (1-zab^*)^{-1}y).$$

$$\int |\tilde{y}(z)|^2 \frac{d\theta}{2\pi} = \int \left(y, (1-z^{-1}ba^*)^{-1}(1-bb^*)(1-zab^*)^{-1}y\right)\frac{dz}{2\pi i z}$$

$$\frac{1}{z-ba^*}(1-bb^*)\frac{1}{1-zab^*} = ?$$

$$\frac{1}{z-ba^*} + ab^*\frac{1}{1-zab^*} = \frac{1}{z-ba^*}\left(1-zab^* + (z-ba^*)ab^*\right)\frac{1}{1-zab^*}$$

Do residue calculation. $\frac{1}{1-zab^*}$ analytic for $|z| \le 1$.

so you evaluate. put $z = ba^*$ $\quad 1-ba^*ab^* = 1-bb^*$.

$$\int \left(y, \frac{1}{\cancel{1-}zba^*}y\right)\frac{dz}{2\pi i z} \quad \text{analytic outside } |z|=1 \text{ except at } \infty$$

This yields the isom. embedding $Y \hookrightarrow L^2(S^1)$

There's too much calculation here.

Forgot $g(z) = \det(1-zab^*)$

2 other versions of the calculation. The ~~previous~~ above ~~uses~~ can be understood better by ~~attaching~~ ~~extending~~ the canonical extension

$$H = \cdots \oplus u^{-1}V^- \oplus aX \oplus V^+ \oplus uV^+ \oplus \cdots$$
$$\oplus u^{-1}V^- \oplus V^+ \oplus bX \oplus uV^+ \oplus \cdots$$

by attaching incoming + outgoing subspaces.

Other ~~sketches~~ pictures: X, γ $\quad \gamma$ contraction

~~Then~~ Form $H =$ completion of $\bigoplus_{n\in\mathbb{Z}} u^n X$ $\qquad \gamma^* u^n \gamma = \gamma^n$, $n \geq 0$.

~~Note~~
$$V^+ = (1-\gamma^*\gamma)^{1/2} X$$
$$V^- = (1-\gamma\gamma^*)^{1/2} X.$$

$$\underbrace{\|x_0 + u x_1\|^2}_{a x_0 + b x_1} = \|x_0\|^2 + (x_0, \gamma x_1) + (\gamma x_1, x_0) + \|x_1\|^2$$
$$= \|\underbrace{x_0 + \gamma x_1}_{a^*(x_0+ux_1)}\|^2 + \|(1-\gamma^*\gamma)^{1/2} x_1\|$$

$$\gamma = a^* b$$
$$1 - \gamma^*\gamma = 1 - b^* a a^* b$$
$$= b^* \pi_+ b$$

$$\int \left\| (1-\gamma\gamma^*)^{1/2} \frac{1}{1-z\gamma^*} x \right\|^2 \frac{dz}{2\pi i z} = \int \left(x, \frac{1}{1-\bar{z}^{-1}\gamma} (1-\gamma\gamma^*) \frac{1}{1-z\gamma^*} x \right) \frac{dz}{2\pi i z}$$
$$= \|x\|^2 \quad \text{residues inside } S^1.$$

Opinion: It looks as if ~~your your problem~~ a proper explanation of these calculations will involve residue calculus. There should be a link between quasi-determinants and residues.

One parameter version. ~~If~~ You want ~~[illegible]~~ to replace the nearly unitary $\overset{\text{contraction}}{a^*b}$ with a nearly hermitian operator with ~~[illegible]~~ imaginary part, call it, β.

$$\int_{-\infty}^{\infty} \left\| (?)^{1/2} \frac{1}{\omega - \beta} x \right\|^2 \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(x, \frac{1}{\omega - \beta^*} (?) \frac{1}{\omega - \beta} x \right)$$

close counter in UHP $\qquad$ want spectrum of β in LHP

so sing. at $\omega = \beta^*$ in UHP. $\qquad$ ~~[illegible]~~ $\quad ? = i\underbrace{(\beta - \beta^*)}_{2i\,\mathrm{Im}(\beta)} \geq 0$

3 $X \underset{b}{\overset{a}{\rightrightarrows}} X$ $\quad -i\lambda = \frac{1-z}{1+z} \quad z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda+i}$ maybe λ should be ∞

$$a = i\varepsilon + A \qquad b = i\varepsilon - A$$

$$\begin{aligned} az-b &= (i\varepsilon+A)z - (i\varepsilon - A) \\ &= (iz-i)\varepsilon + (z+1)A \\ &= \left(i\frac{z-1}{z+1}\varepsilon + A\right)(z+1) \\ &= (\lambda\varepsilon - A)(-z-1). \end{aligned}$$

~~$a^*a \ldots (i\varepsilon^* + A^*)(\varepsilon + A)$~~

Assume $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ given and Y has scalar product.

$$\|ax\|^2 = \|i\varepsilon x + Ax\|^2 = \|\varepsilon x\|^2 + (Ax, i\varepsilon x) + (i\varepsilon x, Ax) + \|Ax\|^2$$
$$\|bx\|^2 \qquad - \qquad\qquad - \qquad\qquad -$$

so $\|ax\| = \|bx\| \iff (Ax, \varepsilon x) = (\varepsilon x, Ax) \quad \forall x.$

$ax = 0 \implies ax = bx = 0 \implies$ $a+b = 2i\varepsilon$, $a-b = 2A$ kills x,

Assume that $\varepsilon x = Ax = 0 \implies x = 0$. Then you have a partial unitary.

Want no bound states. Since a, b and ε, A are expressable in terms of each other its clear that $az-b$ injective $\forall z \in \mathbb{C}\cup\infty \iff \lambda\varepsilon - A$ injective $\forall \lambda \in \mathbb{C}\cup\infty$.

So what can you do?

So far no scalar prod on X, except that the partial unitary requires $\|x\|^2 = \|ax\|^2 = \|\varepsilon x\|^2 + \|Ax\|^2$ so this is the mistake!! How do you correct?

$$a = (i\varepsilon + A)h^{-1/2} \qquad h = \overset{1}{\varepsilon^*\varepsilon} + A^*A$$
$$b = (i\varepsilon - A)h^{-1/2}$$

4 Suppose we start with ~~X, Y Hilb. spaces~~

$X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ $\quad$ X, Y Hilb. spaces

$\varepsilon^*\varepsilon = 1$ $\quad$ (ε is an emb.)

$\varepsilon^* A = A^*\varepsilon$

$\lambda\varepsilon - A$ injective $\forall \lambda \in \mathbb{C}$.

$$\|(\iota\varepsilon \pm A)x\|^2 = \|x\|^2 \pm \|Ax\|^2 = \|(1+A^*A)^{1/2}x\|^2$$

$$\|(\iota\varepsilon \pm A)(1+A^*A)^{-1/2}x\|^2 = \|x\|^2$$

$$a = (\iota\varepsilon + A)(1+A^*A)^{-1/2}$$

$$b = (i\varepsilon - A)(1+A^*A)^{-1/2}$$

$$ab^* = (\iota\varepsilon + A)((1+A^*A)^{-1/2}(1+A^*A)^{-1/2})(-\iota\varepsilon^* - A^*)$$

$$b^*a = (1+A^*A)^{-1/2}\underbrace{(-i\varepsilon^* - A^*)(\iota\varepsilon + A)}_{\varepsilon^*\varepsilon - iA^*\varepsilon - i\varepsilon^*A - A^*A}(1+A^*A)^{-1/2}$$

$$= 1 - A^*A - 2i(\varepsilon^*A)$$

does $A^*\varepsilon = \varepsilon^*A$ commute with A^*A $\quad$ NO

$A^*\varepsilon A^*A =$

$$b^*a = h^{-1/2}(-\iota\varepsilon^* - A^*)(i\varepsilon + A)\,h^{-1/2}$$

$$1 = h^{-1/2}(-\iota\varepsilon^* + A^*)(i\varepsilon + A)\,h^{-1/2}$$

$$1 + b^*a = h^{-1/2}(-2i\,\varepsilon^*)(\iota\varepsilon + A)\,h^{-1/2}$$

$$= h^{-1/2}(+2 - 2i\,\varepsilon^*A)\,h^{-1/2}$$

$$1 - b^*a = h^{-1/2}(2A^*)(i\varepsilon + A)\,h^{-1/2}$$

5 $(1-b^*a)(1+b^*a)^{-1} = h^{-1/2}(iA^*\varepsilon + A^*A)(1-i\varepsilon^*A)^{-1}h^{+1/2}$?

Let's look at ~~eigenvalue~~ eigenvectors

There are two lines in Y naturally presented, namely $\operatorname{Ker}\varepsilon^*$, $\operatorname{Ker}A^*$. These are the orth complements in Y of the subspace $(\lambda\varepsilon - A)X$ for $\lambda = \infty, 0$. Look at

$$\begin{tikzcd}
 & & e_0 \in \operatorname{Ker}(\varepsilon^*) \arrow[d, hook] \arrow[dr] & & \\
0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, "\lambda - \varepsilon^* A"'] & Y \arrow[r] \arrow[d, "\varepsilon^*"] & L_\lambda \arrow[r] & 0 \\
 & & X & &
\end{tikzcd}$$

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e_0$$

$$(\lambda - \varepsilon^*A)x = -\varepsilon^* y$$

$$y - (\lambda\varepsilon - A)(\lambda - \varepsilon^*A)^{-1}(+\varepsilon^*)y = \tilde{y}(\lambda)e_0$$

$$= y - (\lambda\varepsilon - A)\varepsilon^*(\lambda - A\varepsilon^*)^{-1}y$$

$$= \{\lambda - \cancel{A}\varepsilon^* - (\lambda\varepsilon - \cancel{A})\varepsilon^*\}(\lambda - A\varepsilon^*)^{-1}y$$

$$= (1 - \varepsilon\varepsilon^*)(1 - \lambda^{-1}A\varepsilon^*)^{-1}y$$

$$\tilde{y}(\lambda) = (e_0, (1 - \lambda^{-1}A\varepsilon^*)^{-1}y)$$

this has poles on real axis.

You propose to modify ~~Ker ε^*~~ ε^* a bit

$$b^* = h^{-1/2}(-i\varepsilon^* - A^*)$$

$\varepsilon^* - iA^*$.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr] & Y \arrow[r] \arrow[d, "\varepsilon^* - iA^*"] & L_\lambda \arrow[r] & 0 \\ & & X & & \end{tikzcd}$$

$$(\varepsilon^* - iA^*)(\lambda\varepsilon - A) = \lambda - \varepsilon^* A - i\lambda A^*\varepsilon + iA^*A$$
$$= \lambda + iA^*A - (1+i\lambda)A^*\varepsilon$$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\lambda\varepsilon A"] \arrow[dr, "\cancel{\lambda A^*}\varepsilon - A^*A"'] & Y \arrow[d, "A^*"] \\ & & X \end{tikzcd}$$

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e_0'$$
$$(\lambda A^*\varepsilon - A^*A)x = -A^*y$$
$$x = -(\lambda A^*\varepsilon - A^*A)^{-1}A^*y$$

$$y + (\lambda\varepsilon - A)(\lambda A^*\varepsilon - A^*A)^{-1}A^*y = \tilde{y}(\lambda)e_0'$$

$$y + (\lambda\varepsilon - A)A^*(\lambda\varepsilon \cancel{A}^* - AA^*)^{-1}y \quad \text{NO good}$$

$$\{\lambda\varepsilon A^* - AA^* + (\lambda\varepsilon - A)A^*\}(\lambda\varepsilon A^* - AA^*)^{-1}y$$

$$(\lambda A^*\varepsilon - A^*A)^{-1}A^* \overset{?}{=} A^*(\lambda\varepsilon A^* - A\cancel{A}^*)^{-1}$$

$$\Uparrow \quad \text{if both invertible}$$

$$A^*(\lambda\varepsilon A^* - AA^*) = (\lambda A^*\varepsilon - A^*A)A^* \qquad \text{OK}$$

$(\lambda\varepsilon - A)(\varepsilon^*(\lambda\varepsilon - A))^{-1}\varepsilon^*$ is OK

$(\lambda\varepsilon - A)(A^*(\lambda\varepsilon - A))^{-1}A^*$ is not OK.

7 Curious $\quad 1-xy$ invertible $\Longleftrightarrow 1-yx$ invertible

$$\left(1+y(1-xy)^{-1}x\right)(1-yx) = 1 - yx + y(\cancel{1-xy})^{-1}\underbrace{x(1-yx)}_{\cancel{(1-xy)}x}$$

$\varepsilon - xy$ invertible $\Longleftrightarrow \varepsilon - yx$ is inv.

need $[\varepsilon, X] = 0$

$$\left[\alpha + y(\varepsilon-xy)^{-1}x\right](\varepsilon - yx) = yx + \overset{1}{\alpha}(\varepsilon - yx) = \varepsilon$$

idea was
$$(\varepsilon - xy)^{-1} = \varepsilon^{-1}(1-xy\varepsilon^{-1})^{-1}$$
$$= \varepsilon^{-1} + \varepsilon^{-1}xy\varepsilon^{-1} + \varepsilon^{-1}(xy\varepsilon^{-1})^2 + \cdots$$

$$\left[\varepsilon^{-1} + \varepsilon^{-1}y(\varepsilon - xy)^{-1}x\right](\varepsilon - yx) = 1 - \varepsilon^{-1}yx + \varepsilon^{-1}yx = 1.$$

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[dr, "AA^*"'] & Y \arrow[d, "A^*"] \\ & X \end{tikzcd}$$

$$(\lambda\varepsilon - A)x = -y + \overset{c}{\cancel{\varepsilon}}\, e_0'$$

$$A^*Ax = A^*y$$
$$x = (A^*A)^{-1}A^*y$$

you can't do
$$(A^*A)^{-1}A^*y = A^*(AA^*)^{-1}y$$

$$y - A(A^*A)^{-1}A^*y$$

$$\left(e_0', y + (\lambda\varepsilon - A)\left(A^*(\lambda\varepsilon - A)\right)^{-1}A^*y\right)$$
$$= \left(e_0', y + \lambda\varepsilon(\lambda A^*\varepsilon - A^*A)^{-1}A^*y\right)$$

8 Think. What might you be missing?
~~[illegible]~~:

$$0 \longrightarrow X \xrightarrow{\lambda\varepsilon - A} Y \longrightarrow L_\lambda \longrightarrow 0$$

Any $y \in Y$ gives a holom. section of $\{L_\lambda\}$. If $y \neq 0$ section vanishes n points, get equiv between $\mathbb{P}(Y)$ and divisors ≥ 0 of degree n. ~~[illegible]~~ Your goal is to represent Y as functions. One way to do this is to select a $\neq 0$ section e_0, then solve

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda) e_0$$

Why can you solve this? Look at homogeneous equation

$$(\lambda\varepsilon - A)x = c e_0$$

Linear alg. You have a K-module of type $O(n)$ meaning $\dim Y = n+1$, $\dim X = n$, $a\lambda - b$ injective $\forall$ all λ includ ∞. Why does any line in Y arise from some λ and line in X.

$$\mathbb{P}X \times \mathbb{P}^1 \longrightarrow \mathbb{P}Y$$

So the simple idea seems to be to add this e_0 to get

$$\begin{matrix} X \\ \oplus \\ 1 \end{matrix} \xrightarrow{(\lambda\varepsilon - A \;\; e_0)} Y$$

Inverting this is equiv. to solving the equation

$$y = (A - \lambda\varepsilon)x + c e_0$$

 Take a J-matrix version.

$$\begin{pmatrix} -\lambda+b_1 & a_1 & & & e_1 \\ a_1 & -\lambda+b_2 & a_2 & & e_2 \\ & a_2 & \ddots & a_{n-1} & \vdots \\ & & a_{n-1} & -\lambda+b_n & \vdots \\ & & & a_n & e_{n+1} \end{pmatrix}$$

Observe the determinant is a poly of degree n assuming $e_{n+1} \neq 0$.

Let's go over what has been learned. You have this K-module $\lambda\varepsilon - A : X \to Y$ of type $\mathcal{O}(n)$ say.

$$\begin{tikzcd} & & \mathbb{C}e \arrow[d, hook] \arrow[dr] & & \\ 0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[r] & L_\lambda \arrow[r] & 0 \end{tikzcd}$$

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e$$

$$y = (A - \lambda\varepsilon)x + \tilde{y}(\lambda)e.$$

~~[illegible]~~ There is something funny here because the ~~determinant has~~ characteristic poly has degree $\leq n$. It seems that I want $e_{n+1} \neq 0$. i.e. $e \neq 0$ at ∞.

Anyway, so wlr

$$(i\varepsilon - A)^*(\lambda\varepsilon - A) = (-i\varepsilon^* - A^*)(\lambda\varepsilon - A)$$
$$= -i\lambda + i(\varepsilon^* A) - \lambda(A^*\varepsilon) + A^*A$$
$$= (-i\lambda + A^*A) + (i-\lambda)\varepsilon^* A$$

my problem is to find a non-hermitian extension of A

$$A^*(i\varepsilon + A)(1 - i\varepsilon^* A)^{-1}$$

given $X \overset{\varepsilon}{\underset{A}{\rightrightarrows}} Y$ want $z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda + i}{\lambda + i}$

$a = i\varepsilon + A$ $\quad \|ax\|^2 = \|\varepsilon x\|^2 + (i\varepsilon x, Ax) + (Ax, i\varepsilon x) + \|Ax\|^2$

$b = i\varepsilon - A$ $\quad \|bx\|^2 \quad - \quad -$

$\cancel{\varepsilon^* A = A^* \varepsilon}$

so $\cancel{\sqrt{\|ax\|^2} = \|\varepsilon x\|^2 + \|Ax\|^2 = \|(\varepsilon^*\varepsilon + A^*A)^{1/2} x\|^2}$

$\therefore \quad (\varepsilon x, Ax)_Y = (Ax, \varepsilon x)_Y \qquad \forall x.$

also $\|x\|^2 = \|\varepsilon x\|^2 + \|Ax\|^2$.

what is ε^* ?

$\cancel{(\varepsilon^* x, y)_Y \neq (x, \varepsilon y)}$

$$(\varepsilon^* y, x)_X = (y, \varepsilon x)_Y$$

$$\|$$

$$(\varepsilon(\varepsilon^* y), \varepsilon x) + (A\varepsilon^* y, Ax)$$

What is ε^* ?

$$(\varepsilon^* y, x)_X = (y, \varepsilon x)_Y$$

$$\boxed{(a^* y, x) = (y, (i\varepsilon + A)x)_Y}$$

$$a^* a = (-i\varepsilon^* + A^*)(i\varepsilon + A) = \varepsilon^*\varepsilon + A^*A$$

11 You must understand perfectly what a partial hermitian operator is. Usual picture is a densely defined operator $D \longrightarrow H \oplus H$ satisfying an equal to its annihilator condition.

How? $D = \left\{ \begin{pmatrix} \xi \\ A\xi \end{pmatrix} \;\middle|\; \xi \in \mathcal{D}_A \right\}$

D is a closed subspace of $H \oplus H$ such that $pr_1 : D \longrightarrow H$ is injective and has ~~closed~~ dense image. From C^* module theory ~~you~~ ~~want~~ it is not.. assume existence of adjoint.

$\mathcal{D}_T \Rightarrow \Gamma_T \subset H \oplus H$ $\qquad \mathcal{D}$

$$(\xi_1, T\xi_2) - (T^*\xi_1, \xi_2) = 0$$

$$\begin{pmatrix} \xi_1 \\ +T^*\xi_1 \end{pmatrix} \perp \begin{pmatrix} +T\xi_2 \\ -\xi_2 \end{pmatrix} \qquad \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \xi_2 \\ T\xi_2 \end{pmatrix}$$

general case. Take $H_1 \oplus H_2$ $\qquad \Gamma_{T^*} \perp \underbrace{\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}}_{J} \Gamma_T$

The good case is when $\Gamma_{T^*} \oplus \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \Gamma_T = \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix}$

i.e. $\begin{pmatrix} 1 & -T \\ T^* & 1 \end{pmatrix} \qquad \begin{matrix} D_{T^*} \\ \oplus \\ D_T \end{matrix} \longrightarrow \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix}$

shift to $\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = 1 + \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$

12 partial unitary = subspace of $H \oplus H$, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ isotropic for $\|\xi_1\|^2 - \|\xi_2\|^2$ hermitian form.

partial hermitian = subspace Γ of $H \oplus H$ isotropic for hermitian form $\xi \mapsto \operatorname{Im}(\xi_1, \xi_2)$ and such that $p_1 : \Gamma \to H$ is injective.

Check 2nd description. Polarize $\xi \xrightarrow{\operatorname{Im}} (\xi_1, \xi_2)$

$$\frac{1}{4} \sum_{k=0}^{4} i^{-k} Q(\xi + i^k \eta)$$

$$= \frac{1}{4} \sum_{k=0}^{4} i^{-k} \frac{(\xi_1 + i^k \eta_1, \xi_2 + i^k \eta_2) - (\xi_2 + i^k \eta_2, \xi_1 + i^k \eta_1)}{2i}$$

$$= \frac{(\xi_1, \eta_2) - (\xi_2, \eta_1)}{2i} = \frac{1}{2i}\left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right)$$

Suppose $\Gamma = \begin{pmatrix} 1 \\ T \end{pmatrix} W$ $\qquad W = p_1(\Gamma)$

isotropic means

$$0 = \left(\begin{pmatrix} w \\ Tw \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w' \\ Tw' \end{pmatrix} \right)$$

$$= (w, Tw') - (Tw, w')$$

$az - b$

$= a\left(\frac{1+i\lambda}{1-i\lambda}\right) - b$

$= a(1+i\lambda) - b(1-i\lambda) \ldots$

$= a-b + i(a+b)\lambda$

$i\frac{a-b}{a+b} - \lambda$

Now do what?

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ i & +i \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ i & +i \end{pmatrix} \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

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Remaining step go between unitary + hermitian pictures via C.T.

unitary
$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] & H \\ H & \oplus \end{tikzcd}$$

$$Q(\xi) = \|\xi_1\|^2 - \|\xi_2\|^2$$
$$= \left(\xi, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xi\right)$$

$$az - b = a\left(\frac{1+i\lambda}{1-i\lambda}\right) - b = \frac{a(1+i\lambda) - b(1-i\lambda)}{1-i\lambda} = \frac{(a-b) + i\lambda(a+b)}{\cdot\cdot}$$
$$= \left(i\lambda + \frac{a-b}{a+b}\right)\left(\frac{1}{1-i\lambda}\right)(a+b)$$
$$= \left(\lambda - i\,\frac{a-b}{a+b}\right)\left(\frac{i}{1-i\lambda}\right)(a+b)$$

$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{tikzcd} X \arrow[r, "\binom{a}{b}"] \arrow[d] & H \oplus H \arrow[r, "\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}"] & H \oplus H \\ {} \end{tikzcd}$$

Check.
$$\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} x = \begin{pmatrix} i(a-b)x \\ (a+b)x \end{pmatrix}$$

$$\cancel{\|\xi_1\|^2 - \|\xi_2\|^2 = \|(a-b)x\|^2 - \|(a+b)x\|^2}$$
$$\cancel{= \|ax\|^2 - (ax, bx) - (bx, ax) + \|bx\|^2}$$
$$\cancel{- \|ax\|^2 + (\quad) + (\quad) - \|ax\|^2}$$
$$\cancel{= -2}$$

$$\mathrm{Im}(\xi_1, \xi_2) = \mathrm{Im}\big(i(a-b)x, (a+b)x\big)$$
$$= \pm\mathrm{Re}\big((a-b)x, (a+b)x\big)$$
$$= \|ax\|^2 - \|bx\|^2$$

$\varepsilon = a+b$
$A = i(a-b)$

The good setting seems to be an isotropic subspace X of $Y \oplus Y$ for the hermitian form $\|\xi_1\|^2 - \|\xi_2\|^2$, and then you want to intersect this with graph of mult by z. In other words you have the basic sequence

$$0 \to \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

over $\mathbb{P}^1$ tensored with Y. You combine ~~the~~ a hermitian scalar product on Y with the symplectic? structure on $H^0(\mathcal{O}(1))$. Not symplectic but rather $U(1,1)$ structure to get the pseudo scalar product on $Y \otimes H^0(\mathcal{O}(1))$. Then what? X isotropic, find induced v.b. on X°/X. LC circuits are similar, maybe identically except it's a real version.

Partial unitary on H = subspace of $H \oplus H$ isotropic for hermitian form $\|\xi_1\|^2 - \|\xi_2\|^2$

Partial hermitian op on H = subspace of $H \oplus H$ projecting nonsingularly on first factor, i.e. pr_1 injective which isotropic for skew hermitian form $\left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right)$

$= (\xi_1, \eta_2) - (\xi_2, \eta_1)$. This should be determined by purely imag. function $(\xi_1, \xi_2) - (\xi_2, \xi_1) = 2i \, \mathrm{Im}(\xi_1, \xi_2)$

$\therefore$ $\Gamma \subset H \oplus H$ isotropic subspace such that $\forall (\xi_1, \xi_2) \in \Gamma$ $(\xi_1, \xi_2) \in \mathbb{R}$.

Thus $\Gamma = \Gamma_A = \left\{ \begin{pmatrix} \xi \\ A\xi \end{pmatrix} \middle| \xi \in D_A \right\}$.

$(\xi, A\xi)$ real

15 Review. You have achieved some understanding of partial hermitian operators. A hermitian operator A on H can be identified with a subspace Γ of $H \oplus H$ which is maximal isotropic for the hermitian form $\xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \longmapsto \operatorname{Im}(\xi_1, \xi_2)$. Corresp herm. bilinear form is found by polarization

proj $p_1 : \Gamma \xrightarrow{\sim} H$

$$\frac{1}{4}\sum_{k=0}^{3} i^{-k} Q(\xi + i^k \eta) = \frac{1}{4}\sum_{k=0}^{3} i^{-k} \underbrace{\frac{(\xi_1 + i^k\eta_1, \xi_2 + i^k\eta_2) - (\xi_2 + i^k\eta_2, \xi_1 + i^k\eta_1)}{2i}}$$

$$= \frac{(\xi_1, \eta_2) - (\xi_2, \eta_1)}{2i} = \frac{1}{2i}\left(\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right)$$

$\frac{1}{i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ is hermitian.

~~[illegible]~~ Isotropic means $\left(\begin{pmatrix}\xi\\ A\xi\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\eta\\ A\eta\end{pmatrix}\right)$

$= (\xi, A\eta) - (A\xi, \eta)$ vanishes,

A partial hermitian is subspace $X \subset H \oplus H$ isotropic for same herm. form and such that $p_1 : X \hookrightarrow H$

So the study of partial hermitian operators should reduce to partial unitaries

Now time to sort out previous problems, where you could handle partial unitaries but not partial hermitians. C.T. $z = \frac{1-(-i\lambda)}{1+(-i\lambda)} = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda + i}{\lambda - i}$

$i(1-i\lambda)(az - b) \equiv i\big(a(1+i\lambda) - b(1-i\lambda)\big) = i\big(a - b + i\lambda(a+b)\big)$

$= \left(\lambda - i\frac{a-b}{a+b}\right)(a+b)$.

$\varepsilon = a+b$

$A = i(a-b)$

$$X \xrightarrow{\begin{pmatrix}a\\ b\end{pmatrix}} \begin{matrix}H\\ \oplus\\ H\end{matrix} \xrightarrow{\frac{1}{\sqrt{2}}\begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix}} \begin{matrix}H\\ \oplus\\ H\end{matrix}$$

$$\left(\begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\right)\frac{1}{2}$$

$$\frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

Notice that $\begin{pmatrix} a \\ b \end{pmatrix} : X \to \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ also $\begin{pmatrix} \varepsilon \\ A \end{pmatrix}$ do not use scalar prod on X. So you should equip X with the ~~inner~~ scalar product $\|x\|^2 = \|(a+b)x\|^2$ to arrange that $\varepsilon^* \varepsilon = 1$.

Try to discuss the general theory of the eigenvector equation etc.

$$X \subset X^\circ \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \qquad X^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, \left(\begin{pmatrix} ax \\ bx \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = 0 \right\}$$

$$X^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, a^* y_1 = b^* y_2 \right\}. \qquad (ax, y_1) = (bx, y_2)$$

$$a^* y_1 = b^* y_2.$$

e.g. $\begin{pmatrix} V^+ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ V^- \end{pmatrix}$, eigenvector equation

$$W^\circ = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix} X}_{W} \oplus \begin{pmatrix} V^+ \\ V^- \end{pmatrix}$$

$$X \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \xrightarrow{(z \;\; -1)} Y$$

$(z \;\; -1)$ has kernel $\left\{ \begin{pmatrix} \xi \\ z\xi \end{pmatrix} \right\} = L_z$

dim $Y \oplus Y = 2n+2$
dim $W = n$
dim $W^\circ = n+2$

So you assume $\forall z$ $az - b$ inj, this means no bound states. $W^\circ \cap L_z$ should be a line in $W^\circ / W = V^+ \oplus V^-$. The line is clear namely the correspondence between V^+, V^- given by the eigenvalue equation

$$(az - b)x = -v^+ + v^-$$

So simple.

17 What happens in the electrical setting. Somehow $U(n,n)$ becomes $Sp(2n,\mathbb{R})$.

Go back to partial hermitian setting/ and find what you did wrongly.

$$X \xrightarrow{\lambda\varepsilon - A} Y$$

Wait, better idea. Is it possible to derive the formula $\tilde{y}(z) = (e_0^-, (1-za^*b)^{-1}y)$ without the tricks used before? This is the solution of $(az-b)x = -y + \tilde{y}(z)e_0^-$. You want maybe to ~~look~~ work in the ~~category~~ double $Y\oplus Y$

In $Y\oplus Y$ you have $\Gamma_z = \begin{pmatrix}1\\z\end{pmatrix}Y$, $W = \begin{pmatrix}a\\b\end{pmatrix}X$

$W + \Gamma_z$ codim 1 in $Y \oplus Y$

$$\underset{n}{\begin{pmatrix}ax\\bx\end{pmatrix}} + \underset{n+1}{\begin{pmatrix}y\\zy\end{pmatrix}} + \underset{1}{\begin{pmatrix}0\\ce_0^-\end{pmatrix}} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(z \quad -1)\cdot \qquad (az-b)x - ce_0^- = y$$

ax

bx No.

Consider partial herm. case

$$\underbrace{\begin{pmatrix}\varepsilon\\A\end{pmatrix}X}_{W}, \quad \underbrace{\begin{pmatrix}1\\\lambda\end{pmatrix}Y}_{\Gamma_\lambda}, \qquad W^0$$

you want elements e of Y whose sections are nonvanishing in the UHP

$$e \in (\varepsilon\lambda - A)x \implies \lambda \text{ lower half plane?}$$

 partial hermitian ops. $W = \begin{pmatrix}\varepsilon\\A\end{pmatrix}X \subset \begin{matrix}Y\\\oplus\\Y\end{matrix}$

isotropic ~~for~~: $\left(\begin{pmatrix}\varepsilon x\\Ax\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\varepsilon x'\\Ax'\end{pmatrix}\right) = (\varepsilon x, Ax') - (Ax, \varepsilon x') = 0$

Note this doesn't depend on any inner product on X.

What is: W° ? $= \left\{\begin{pmatrix}y_1\\y_2\end{pmatrix} \,\middle|\, (y_1, Ax') - (y_2, \varepsilon x') = 0 \quad \forall x'\right.$.

For example. $\begin{pmatrix}0\\y_2\end{pmatrix}$ with $(y_2, \varepsilon X) = 0$. $\begin{pmatrix}(AX)^\perp\\(\varepsilon X)^\perp\end{pmatrix}$

Is it true that $\begin{pmatrix}\varepsilon\\A\end{pmatrix}X \oplus \begin{matrix}(AX)^\perp\\\oplus\\(\varepsilon X)^\perp\end{matrix} = W^\circ$? NO.

Let $\begin{pmatrix}y_1\\y_2\end{pmatrix} \in W^\circ$ i.e. $(y_1, Ax) = (y_2, \varepsilon x) \quad \forall x$.

We have $Y = \varepsilon X + (\varepsilon X)^\perp$, so $y_1 = \varepsilon x + y_1'$ $\quad y_1' \in \varepsilon X^\perp$

$$\begin{pmatrix}\varepsilon\\A\end{pmatrix}x = \begin{pmatrix}\varepsilon x\\Ax\end{pmatrix} \in \begin{matrix}(AX)^\perp\\\oplus\\(\varepsilon X)^\perp\end{matrix} \qquad \begin{matrix}A^*\varepsilon x = 0\\ \varepsilon^* A x = 0.\\ \text{possible}\end{matrix}$$

so it is possible for $Ax \in (\varepsilon X)^\perp$. Anyway continue.

You want something say involving $\lambda = \pm i$

$\varepsilon = a + b \qquad\qquad i\varepsilon - A = 2ib$

$A = i(a-b) \qquad\quad i\varepsilon + A = 2ia$

$$(i\varepsilon - A)^*(\lambda\varepsilon - A) = (-i\varepsilon^* - A^*)(\lambda\varepsilon - A)$$

$$= -i\lambda\,\varepsilon^*\varepsilon + \underbrace{i\varepsilon^* A - A^*\lambda\varepsilon}_{(i-\lambda)\,\varepsilon^* A} + \underbrace{A^*A}_{1-\varepsilon^*\varepsilon}$$

$$= 1 + (-1-i\lambda)\,\varepsilon^*\varepsilon + (i-\lambda)\,\varepsilon^* A$$

$$= 1 + (\lambda - i)(i\varepsilon^*\varepsilon - \varepsilon^* A)$$

19 Your problem again. You know that $(az-b)(x)=0 \implies x=0$ for $|z|\neq 1$.

because $zax = bx \implies \|zax\| = \|bx\|$

$$\underset{|z|\|x\|}{\|zax\|} = \underset{\|x\|}{\|bx\|}$$

Try $(\lambda\varepsilon - A)x = 0 \qquad \lambda\varepsilon x = Ax$

$$(Ax, \varepsilon x) = (\varepsilon x, Ax)$$

$$(\lambda\varepsilon x, \varepsilon x) \qquad (\varepsilon x, \lambda\varepsilon x) \implies (\lambda-\bar\lambda)\|\varepsilon x\|^2 = 0.$$

$$(i\varepsilon-A)^*(\lambda\varepsilon-A) = 1 + (\lambda-i)(-i\varepsilon^*\varepsilon - \varepsilon^* A)$$

provided $\varepsilon^*\varepsilon + A^*A = 1$

$$(\lambda\varepsilon - A)x = -y + \tilde{g}(\lambda)e_0^- \qquad (i\varepsilon-A)^* e_0^- = 0$$

$$y + (\lambda\varepsilon-A)\Big[1+(\lambda-i)(-i\varepsilon^*\varepsilon - \overbrace{\varepsilon^* A}^{A^*\varepsilon})\Big]^{-1}(i\varepsilon-A)^*(-y)$$

$$y + (\lambda\varepsilon-A)\Big[1+(\lambda-i)\underbrace{(-i\varepsilon^*-A^*)}_{(i\varepsilon-A)^*}\varepsilon\Big]^{-1}(i\varepsilon-A)^*(-y)$$

$$y - (\lambda\varepsilon-A)(i\varepsilon-A)^*\Big[1+(\lambda-i)\,\varepsilon(i\varepsilon-A)^*\Big]^{-1}y$$

$$\underbrace{\Big[1+(\lambda-i)\varepsilon(i\varepsilon-A)^* - (\lambda\varepsilon-A)(i\varepsilon-A)^*\Big]}\Big[1+(\lambda-i)\varepsilon(i\varepsilon-A)^*\Big]^{-1}y$$

$$\underbrace{(\cancel{\lambda\varepsilon} - i\varepsilon - \cancel{\lambda\varepsilon} + A)(i\varepsilon-A)^*}$$

$$(-i\varepsilon+A)(-i\varepsilon^* - A^*)$$

proj onto V^-

$$\boxed{\tilde{y}(\lambda) = \Big(e_0^-, \Big[1+(\lambda-i)\,\varepsilon(i\varepsilon-A)^*\Big]^{-1}y\Big)}$$

OKAY

20 Can you find an interpretation of trick using the double? Basic trick.

$$\begin{tikzcd}
X \arrow[r, "az-b"] \arrow[dr, "1-zb^*a"'] & Y \arrow[d, "-b^*"] \\
& X
\end{tikzcd}$$

$$(az-b)x = -y + \tilde{y}(z)e_0^-$$

$$(1-zb^*a)x = b^*y$$

$$x = (1-zb^*a)^{-1}b^*y = b^*(1-zab^*)^{-1}y$$

$$\tilde{y}(z)e_0^- = y + (az-b)b^*(1-zab^*)^{-1}y$$

$$= (1 - \cancel{zab^*} + \cancel{azb^*} - bb^*)(1-zab^*)^{-1}y$$

$$\boxed{\tilde{y}(z) = \left(e_0^-, (1-zab^*)^{-1}y\right)}$$

here you have used $(1-zb^*a)^{-1}\,\exists \Leftrightarrow (1-zab^*)^{-1}\,\exists$.

Goes back to $X \underset{q}{\overset{p}{\rightleftarrows}} Y$

$$\begin{tikzcd}
& Y \arrow[d, "\binom{q}{1}"] \arrow[dr, "1-pq"] & \\
X \arrow[r, "\binom{1}{p}"] \arrow[dr, "1-qp"'] & X\oplus Y \arrow[r, "(-p\ 1)"] \arrow[d, "(1\ -q)"] & Y \\
& X &
\end{tikzcd}$$

2×2

$$(1-pq)^{-1} = 1 + p(1-qp)^{-1}q$$

 ~~[scribbled out]~~ Review. Consider $Y \oplus Y$ with hermitian form $\xi \mapsto \|\xi_1\|^2 - \|\xi_2\|^2$. Then an isotropic subspace W has the form $\begin{pmatrix} a \\ b \end{pmatrix} X$ where $\|ax\| = \|bx\|$ for all x. The annihilator is
make this $\|x\|$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \, (ax, y_1) = (bx, y_2) \quad \forall x \right\}$$

$$(x, a^* y_1 - b^* y_2) = 0$$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \, a^* y_1 = b^* y_2 \right\}.$$

$$\begin{pmatrix} a a^* y_1 \\ b b^* y_2 \end{pmatrix}$$

Suppose $a^* y_1 = b^* y_2$. ~~Then $a^* a a^* y_1$~~

write

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (1 - aa^*) y_1 + a a^* y_1 \\ (1 - bb^*) y_2 + bb^* y_2 \end{pmatrix} = \begin{pmatrix} (1-aa^*) y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (1-bb^*) y_2 \end{pmatrix}$$

So $W^0 = W \oplus \begin{pmatrix} V^+ \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ V^- \end{pmatrix}$. $\Gamma_z = \begin{pmatrix} 1 \\ z \end{pmatrix} Y$

Γ_z isotropic for $|z| = 1$. What can we do?

$Y \oplus Y$ contains $W, \Gamma_z, \begin{pmatrix} V^+ \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ V^- \end{pmatrix}$

Generically

$$Y \oplus Y = W + \begin{pmatrix} V^+ \\ 0 \end{pmatrix} + \Gamma_z$$
$$= W + \begin{pmatrix} 0 \\ V^- \end{pmatrix} + \Gamma_z$$

Fundamental problem. Consider p. herm. ~~situation~~

[redacted]

$$0 \longrightarrow X \xrightarrow{\lambda \varepsilon - A} Y \longrightarrow E_\lambda \longrightarrow 0$$

Find an element $e_0 \in Y$ which trivializes E over the UHP including the real axis. ~~[scribbled out]~~

zeroes of section e_0 are in LHP should be eigenvalues of some **extension** of A. This is tricky because there are $n = \dim X$ zeroes of e_0. Yes!

 Problem: Consider p. herm. situation

$$0 \longrightarrow X \xrightarrow{\lambda\varepsilon - A} Y \longrightarrow E_\lambda \longrightarrow 0$$

To find $e_0 \in Y$ such that the corresponding section of E_0 has its zeroes in the LHP. These zeroes should be the eigenvalues of some ~~point of~~ compression of A to X. Example. Adjust the scalar product on X so that $\varepsilon^*\varepsilon = 1$. ~~Then~~

Better: Let $\varepsilon^* : Y \to X$ ~~be~~ $\ni$ $\varepsilon^*\varepsilon = 1$. ~~Then the section~~ and e_0 generates $\operatorname{Ker}(\varepsilon^*)$.

Then section $\neq 0$ when $\lambda - \varepsilon^* A$ nonsing. usual calculation will work. Check this $\eta\varepsilon = 1$.

$$(\lambda\varepsilon - A)x = -y + \tilde{y}(\lambda)e_0 \qquad \eta(e_0) = 0$$

$$(\lambda - \eta A)x = -\eta y$$

$$\lambda \neq 0$$

$$x = -(\lambda - \eta A)^{-1}\eta y = -\eta(\lambda - A\eta)^{-1}y$$

$$y - (\lambda\varepsilon - A)\eta(\lambda - A\eta)^{-1}y = [\lambda - \cancel{A\eta} - (\lambda\varepsilon - \cancel{A})\eta](\lambda - A\eta)^{-1}y$$

$$= \lambda(1 - \varepsilon\eta)(\lambda - A\eta)^{-1}y$$

$$\tilde{y}(\lambda) = (e_0, (1 - \lambda^{-1}A\eta)^{-1}y)$$

Observe that ηA and $A\eta$ have the same spectrum $\neq 0$. Now explain ~~~~ your trick calculation in quasi-det terms.

Use ε' instead of η. Idea you start with

$$X \xrightarrow{\lambda\varepsilon - A} \overset{e}{\underset{\cup}{}} Y$$

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, "e"] & \\
X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, "\lambda p\varepsilon - pA"'] & Y \arrow[r] \arrow[d, "p"] & E_\lambda \\
 & Y/\mathbb{C}e &
\end{tikzcd}$$

You want $p\varepsilon : X \to Y/\mathbb{C}e$ to be an isom., then define $\varepsilon' = (p\varepsilon)^{-1}p : Y \to X$

So it seems that we have one splitting $Y = \varepsilon X \oplus \mathbb{C}e$

and have arranged ~~another splitting~~

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, "e"] & \\
X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, "\lambda - \varepsilon' A"'] & Y \arrow[r] \arrow[d, "\varepsilon'"] & E_\lambda \\
 & X &
\end{tikzcd}$$

How do I relate $\lambda - \varepsilon' A$ to $\lambda - A\varepsilon'$?

You form $X \oplus Y$

It seems maybe worthwhile using $Y \oplus Y$

where one factor is $\mathbb{C}e \oplus \varepsilon X$.

$$\begin{tikzcd}
 & Y \arrow[d] & \\
X \arrow[r, "\begin{pmatrix} 1 \\ \lambda\varepsilon - A \end{pmatrix}"] & X \oplus Y \arrow[r] \arrow[d] & Y \\
 & X &
\end{tikzcd}$$

$$(1 - A\varepsilon')^{-1} = 1 + A(1 - \varepsilon' A)^{-1}\varepsilon'$$

$$\begin{tikzcd}
 & Y \arrow[d, "\begin{pmatrix}\varepsilon' \\ 1\end{pmatrix}"] & \\
X \arrow[r, "\begin{pmatrix}1 \\ A\end{pmatrix}"] & X \oplus Y \arrow[r, "(-A \;\; 1)"] \arrow[d, "(1 \;\; -\varepsilon')"] & Y \\
 & X &
\end{tikzcd}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc}$$

$$\begin{tikzcd}
X \arrow[r, "\begin{pmatrix}1 \\ A\end{pmatrix}"] & X \oplus Y \arrow[r, "(-A \;\; 1)"] \arrow[d, equal] & Y \\
X & X \oplus Y \arrow[l, "(1 \;\; -\varepsilon')"'] & Y \arrow[l, "\begin{pmatrix}\varepsilon' \\ 1\end{pmatrix}"']
\end{tikzcd}$$

Main statement is that any matrix coefficient of the inverse matrix is the inverse of the quasi-determinant

$$\boxed{\langle 1 | \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} | 1 \rangle = \frac{d}{ad-bc} = \left(a - bd^{-1}c\right)^{-1}}$$

Is there a way to fit

$$(1-pq)^{-1} = \underset{a}{1} + \underset{b}{p}(\underset{d}{1-qp})^{-1}\underset{-c}{q} \qquad \begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix}$$

into this ~~[illegible]~~ picture

$$\begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -q \end{pmatrix} (1 \;\; p)$$

Is $\begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix}$ invertible? det $= 1$ in comm. case

$$\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & p \\ -q & 1-qp \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -p \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} 1-pq & -p \\ q & 1 \end{pmatrix}$$

$$\boxed{(A^{-1})_{ij} = (ij \text{ quasi-det})^{-1}}$$

Your problem.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "e"] & \\ X \arrow[r, Rightarrow, "\varepsilon - \lambda^{-1}A"] \arrow[dr] & Y \arrow[r] \arrow[d, "\varepsilon'"] & E_\lambda \\ & X & \end{tikzcd}$$

You need to get $(e, \underbrace{(1-\lambda^{-1}A\varepsilon')^{-1}}_{\text{this requires}} y)$

$$\begin{tikzcd} X \arrow[r, shift left, "\lambda^{-1}A"] & Y \arrow[l, shift left, "\varepsilon'"] \end{tikzcd}$$

Ingredients

$$\begin{tikzcd} & V \arrow[d, shift left, "i"] \\ X \arrow[r, "b-a"] & Y \arrow[u, shift left, "i^*"] \arrow[d, shift left, "b^*"] \\ & X \arrow[u, shift left, "b"] \end{tikzcd}$$

Abstract calculation

To solve

$$(b-a)x = -y + iv$$

$$(1-b^*a)x = -b^*y$$

$$x = -(1-b^*a)^{-1}b^*y = -b^*(1-ab^*)^{-1}y$$

$$y + (b-a)x = y - (b-a)b^*(1-ab^*)^{-1}y$$

$$= (1-ab^* - (b-a)b^*)(1-ab^*)^{-1}y$$

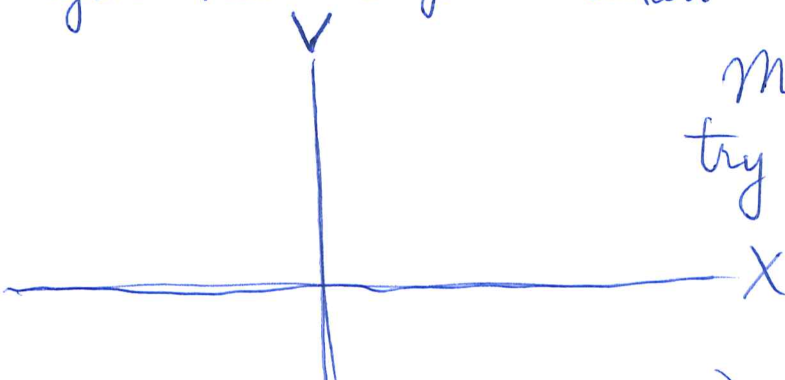
$$iv = \underbrace{(1-bb^*)}_{ii^*v}(1-ab^*)^{-1}y$$

$$\boxed{v = i^*(1-ab^*)^{-1}y}$$

This is a perturbation calculation which should be fairly general. You begin with the splitting

$$Y = bX \oplus iV \qquad \begin{tikzcd} X \oplus V \arrow[r, shift left, "(b\ \ i)"] & Y \arrow[l, shift left, "\begin{pmatrix} b^* \\ i^* \end{pmatrix}"] \end{tikzcd}$$

and you have a perturbation $b-a$ of b.



Maybe you should try

$$\begin{tikzcd} \begin{pmatrix} X \\ V \end{pmatrix} \arrow[r, shift left, "(b-a\ \ i)"] & Y \arrow[l, shift left, "\begin{pmatrix} b^* \\ i^* \end{pmatrix}"] \end{tikzcd}$$

$$(b-a\ \ i)\begin{pmatrix} b^* \\ i^* \end{pmatrix} = 1 - ab^*$$

27 You start with the isomorphism and inverse

$$Y \xrightarrow{\begin{pmatrix} b^* \\ i^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V \end{matrix} \xrightarrow{(b \;\; i)} Y$$

and you have the pert. of b:

$$Y \xrightarrow{\begin{pmatrix} b^* \\ i^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V \end{matrix} \xrightarrow{(b-a \;\; i)} Y \qquad (b-a \;\; i)\begin{pmatrix} b^* \\ i^* \end{pmatrix} = 1 - ab^*$$

so you get $(b-a \;\; i)^{-1} = \begin{pmatrix} b^* \\ i^* \end{pmatrix}(1-ab^*)^{-1}$

---

back to p. herm. $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ split $Y \underset{(\varepsilon \;\; j)}{\overset{\begin{pmatrix} \varepsilon^* \\ j^* \end{pmatrix}}{\rightleftarrows}} \begin{matrix} X \\ \oplus \\ V \end{matrix}$

$$Y \xrightarrow{\begin{pmatrix} \varepsilon^* \\ j^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V \end{matrix} \xrightarrow{(\lambda\varepsilon - A \;\; j)} Y \qquad (\lambda\varepsilon - A \;\; j)\begin{pmatrix} \varepsilon^* \\ j^* \end{pmatrix} = \lambda - A\varepsilon^*$$

we need to choose $\varepsilon^* : Y \longrightarrow X$ so that $\varepsilon^*\varepsilon = 1$

such a choice can be altered by an element of X

~~Such~~ These ε^* for an affine space of dim n. So what is de Branges choice?

$$\left(\begin{matrix} b_1 & a_1 & \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 \\ & & a_3 \end{matrix}\right)$$

$$\underbrace{\begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \end{pmatrix}}_{\text{choice for } \varepsilon^*}\begin{pmatrix} b_1 & a_1 & \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 \\ 0 & 0 & a_3 \end{pmatrix}$$

$$\begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & b_3 & a_3 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 \\ & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$A\varepsilon^* = \begin{pmatrix} b_1 & a_1 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 \\ 0 & a_2 & b_3 & a_3 \end{pmatrix} \qquad \begin{pmatrix} b_1c_1 + a_1c_2 \\ a_1c_1 + b_2c_2 + a_2c_3 \\ a_2c_2 + b_3c_3 \\ a_3c_3 \end{pmatrix}$$

28

$$\begin{pmatrix} b_1 & a_1 & b_1c_1+a_1c_2 \\ a_1 & b_2 & a_1c_1+b_2c_2 \\ & a_2 & a_2c_2 \end{pmatrix} = A\varepsilon^*$$

You should look at $\varepsilon^* A$

$$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{pmatrix}\begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \\ & a_2 \end{pmatrix} = \begin{pmatrix} b_1 & a_1+c_1a_2 \\ a_1 & b_2+c_2a_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & c_1 \\ & 1 & & c_2 \\ & & 1 & c_3 \end{pmatrix}\begin{pmatrix} b_1 & a_1 & \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 \\ & & a_3 \end{pmatrix} = \begin{pmatrix} b_1 & a_1 & c_1a_3 \\ a_1 & b_2 & a_2+c_2a_3 \\ 0 & a_2 & b_3+c_3a_3 \end{pmatrix}$$

It looks as if you want $c_1 = c_2 = 0 \qquad c_3 = i$

Then $\varepsilon^* A$.

Now work out the details: Start with $\varepsilon, A : X \to Y$

you want to find $\varepsilon' : Y \to X \qquad \varepsilon'\varepsilon = 1$.

I want a natural choice of ε'.

If we choose ~~...~~ scalar prod on X so that $\varepsilon^*\varepsilon = 1$, i.e. ε is an isometry. Then we can

$1 - \varepsilon\varepsilon^* = \iota\iota^* \qquad \varepsilon' = \varepsilon^* +$

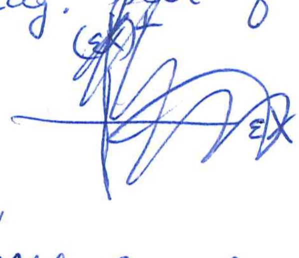
29 It looks like I'm not getting ~~a calculation~~ something straightforward. What do you want? I need an ε' such that $\varepsilon'\varepsilon = 1$, equivalently a ~~line~~ complementary to εX, εX corresp. to $\lambda = \infty$.

What do you want? You need a $\neq 0$ elt e ~~line~~ in Y which provides ~~the~~ a section of the line bundle. You want $e \notin \varepsilon X$ so that $\mathbb{C}e \oplus \varepsilon X = Y$ whence $1 = ee' + \varepsilon\varepsilon'$ $\quad \varepsilon'\varepsilon = 1_X$. So you want a line whose assoc. section ~~is non~~ vanishes only in the LHP. Then get $\varepsilon'\varepsilon = 1$.

so
$$Y \xrightarrow{\begin{pmatrix}\varepsilon' \\ e'\end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C}\end{matrix} \xrightarrow{(\varepsilon - \lambda^{-1}A \;\; e)} Y \qquad \begin{aligned}&(\varepsilon - \lambda^{-1}A)\varepsilon' + ee' \\ &= 1 - \lambda^{-1}A\varepsilon'\end{aligned}$$

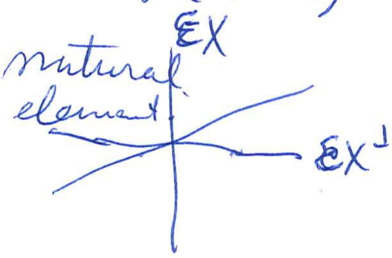
The choice is the element e, because εX is fixed. So you ask for a natural ~~element~~ line outside εX. The orthogonal complement, but this yields $\varepsilon^* A$ on X which is hermitian, we want something with a negative imag. part of rank 1.

$$\varepsilon' = \varepsilon^* + f \qquad f\varepsilon = 0 \qquad f : Y/\varepsilon X \to X$$



$$\varepsilon' A = \varepsilon^* A + fA$$

So $f(Y/\varepsilon X)$ is a line in X. Is there a



So you have to find a line in X.

So things look very interesting indeed. Let's review carefully. You have a partial hermitian operator $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ of $O(n)$ type. Y is a Hilbert space of dim $n+1$, X has dim n, $\lambda\varepsilon - A$ is injective $\forall\lambda$ including ∞. Partial herm. means $\begin{pmatrix}\varepsilon\\A\end{pmatrix}X \subset Y\oplus Y$ is isotropic wrt $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

i.e. $\left(\begin{pmatrix}\varepsilon\\A\end{pmatrix}(x'),\ \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\varepsilon\\A\end{pmatrix}(x)\right) = 0 \quad \forall x, x'.$

$$(\varepsilon x', Ax) = (Ax', \varepsilon x)$$

You have line bundle $E_\lambda = Y/(\lambda\varepsilon - A)X$ over $\mathbb{P}^1$, you wish to find a natural section ~~vanishing~~ only in the LHP. Such a section not vanishing at ∞ amounts to an element of $e \in Y$ $e \notin \varepsilon X$.

Get splitting $Y \xrightarrow{\begin{pmatrix}\varepsilon'\\e'\end{pmatrix}} \begin{matrix}X\\ \oplus\\ \mathbb{C}\end{matrix} \xrightarrow{(\varepsilon\ \ e)} Y$ $\quad 1 = \varepsilon\varepsilon' + ee'$ etc. $\begin{pmatrix}\varepsilon'\\e'\end{pmatrix}(\varepsilon, e) = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

Then to solve $(\lambda\varepsilon - A)x + ce = y$

Apply ε' $\quad (\lambda - \varepsilon'A)x = \varepsilon' y$ $\quad$ can always be solved when $\det(\lambda - \varepsilon'A) \neq 0$.

So the issue becomes to find a natural choice of e or ε'. Have the orthogonal complement of εX. $\mathrm{Ker}(\varepsilon^*)$ once scalar product on X chosen so that ε is an isometry - If you take $\varepsilon' = \varepsilon^*$ get $\det(\lambda - \varepsilon^* A) = 0$ which has real roots as $\varepsilon^* A$ is hermitian.

31 ~~What to do?~~

What to do? Possibilites $\left\{\begin{array}{l}\text{Look at } \left(\binom{\varepsilon}{A}X\right)^{\circ} \text{ in } \begin{array}{c}Y\\ \oplus \\ Y\end{array} \\ \text{Use vanishing at } \infty \text{ filtration of } Y\end{array}\right.$

Try 2nd. The idea is to seek ε' in the form $\varepsilon^* + f$ where $f: Y/\varepsilon X \longrightarrow X$. This means you need to find a line in X if $f \neq 0$. So far I have singled out $\lambda = \infty$. From K-module theory you do get natural complementary ~~filtrations~~ flags by looking order of vanishing at 0 and ∞. $\lambda = i$ might play a special role.

I know from K-module theory that we can identify Y with $\mathbb{C} + \mathbb{C}\lambda + \cdots + \mathbb{C}\lambda^n$, X degree $< n$, $\varepsilon = \text{inc}$, $A = \lambda$ in an essentially unique way (non-zero scalar). The $\mathbb{C}$ is sections vanishing to order n at $\lambda = \infty$. This gives degree filtration on polys. Notice that translation doesn't change this. YES

Have J matrix picture of the $\boxed{\text{partial}}$ $\boxed{\text{herm.}}$ op.

Let's go on to W°/W. W is an isotropic subspace of $Y \oplus Y$, ~~so you need to~~ To extend to an isot subspace should be the same as giving a ~~self adj~~ hermitian extension of the partial herm. op. But there is a complex projective line of n dim subspaces containing W and contained in W°. Is there a natural point? One

~~That~~ You discussed before extending the J matrix

There's should be some condition. You discussed possible $\varepsilon' A$, mainly possible ε' (n dims)

The Cayley transform should take W°/W into $\begin{array}{c}V^+\\ \oplus\\ V^-\end{array}$

Hopefully there are extensions of the partial ~~hermitian~~ herm. to a nearly hermitian op $A\varepsilon'$ on Y.

32 Oct 2, 98 Review. p. herm. $X \overset{\varepsilon}{\underset{A}{\rightrightarrows}} Y$

$$W = \begin{pmatrix}\varepsilon\\ A\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix} \qquad W^\circ = \left\{ \begin{pmatrix}y_1\\ y_2\end{pmatrix} \,\middle|\, \left(\begin{pmatrix}y_1\\ y_2\end{pmatrix}, \begin{pmatrix}0&1\\ -1&0\end{pmatrix}\begin{pmatrix}\varepsilon x\\ Ax\end{pmatrix}\right) = 0 \iff (y_1, Ax) = (y_2, \varepsilon x) \ \forall x \right\}$$

~~Clearly this is projective line~~ W°/W is 2-diml with hermitian form type $1, -1$. Let $W \subset V \subset W^\circ$. If $p_1 : V \xrightarrow{\sim} Y$ then V is the graph of an extension of [redacted] (ε, A) to Y. $p_1 W = \varepsilon X$ so use gen.^{e} for $(\varepsilon X)^\perp$. Take $y_1 = e \in (\varepsilon X)^\perp$

$(e, Ax) = (y_2, \varepsilon x)$. There's a unique $y_2 \in \varepsilon X$ with the appropriate property and any multiple of e can be added to y_2. So you get an affine line of possible V. ~~So the description~~ In the J matrix description

$$\begin{bmatrix} b_1 & a_1 & \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 \end{bmatrix}$$

any element of $\mathbb{C}$.
Anyway what else?

Review. Given $X \overset{\varepsilon}{\underset{A}{\rightrightarrows}} Y$ $\quad W = \begin{pmatrix}\varepsilon\\ A\end{pmatrix} X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$

W° consists of $\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ $\quad (y_1, Ax) = (y_2, \varepsilon X) \quad \forall x$

Add to W $\begin{pmatrix}e\\ \varepsilon x_2\end{pmatrix}$ where e spans $(\varepsilon X)^\perp$
and εx_2 satisfies $(e, Ax) = (\varepsilon x_2, \varepsilon x) \ \forall x$

y_2 determined up to ~~element~~ a multiple of e.

$y_2 = \varepsilon x_2 + ce$. This defines extn. $\tilde{A}$ of $A\varepsilon^{-1}$

$(e, \tilde{A} e) = 0$

33 Review. $W = \begin{pmatrix}\varepsilon \\ A\end{pmatrix} X$ $\quad W^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (y_1, Ax) = (y_2, \varepsilon x) \ \forall x \in X \right\}$

$p_1 : W \xrightarrow{\sim} \varepsilon X$, e unit vector $\perp \varepsilon X$, take $y_1 = e$

$\exists!\, x_0 \ni (e, Ax) = \underbrace{(\varepsilon x_0, \varepsilon x)}_{\substack{\| \\ (x_0,\, x)}} \quad \forall x$ $\quad$ if $\varepsilon^* \varepsilon = 1$.

Then get ~~a hermitian~~ $\begin{pmatrix} e \\ \varepsilon x_0 \end{pmatrix} \in W^\circ$. Can describe all extensions of A to a hermitian op $\tilde{A}$ on Y, by $\tilde{A}\varepsilon = A$, $\tilde{A} e = \varepsilon x_0 + ce$ $\quad c \in \mathbb{R}$. You need $\begin{pmatrix} e \\ \varepsilon x_0 + ce \end{pmatrix}$ to be isotropic: $\mathrm{Im}(e, \varepsilon x_0 + ce) = \mathrm{Im}(c) = 0$.

I want nearly hermitian operator with neg. Imag part. Probably want $c = -i$ so that $(e, \tilde{A} e) = -i$ up to a positive scalar. ~~But you have~~

Go back to C.T. to ~~figure out~~ find what corresponds to $z = \infty$, $\lambda = i$ $\qquad V^+ = \mathrm{Ker}\, a^*$

$$W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \qquad W^\circ = W \oplus \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$$

$$az - b = a\frac{1+i\lambda}{1-i\lambda} - b \sim a(1+i\lambda) - b(1-i\lambda)$$
$$-i\underbrace{(a-b)}_{A} + \lambda\underbrace{(a+b)}_{\varepsilon}$$

b arises from $z = \infty$, $\lambda = +i$

$$V^- = \big((i\varepsilon - A)X\big)^\perp$$

~~If you want~~ Go back to $W = \begin{pmatrix}\varepsilon \\ A\end{pmatrix} X$ and

$$W^\circ = \begin{pmatrix}\varepsilon \\ A\end{pmatrix} X + \mathbb{C}\begin{pmatrix} e \\ \varepsilon x_0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ e \end{pmatrix}$$

~~...~~ Remaining problem: to connect $\big((i\varepsilon - A)X\big)^\perp$ with the J-matrix

$$\det \lambda - \underbrace{\begin{pmatrix} b_1 & a_1 & & & \\ a_1 & \ddots & \ddots & & \\ & \ddots & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n & a_n \\ & & & a_n & i \end{pmatrix}}_{A} = (\lambda - i) d_n - a_n^2 d_{n-1}$$

$$\det\begin{pmatrix} \lambda-b_1 & -a_1 & & \\ -a_1 & \ddots & & \\ & & \ddots & -a_{n-1} \\ & & -a_{n-1} & \lambda-b_n \end{pmatrix}$$?? This doesn't look so promising.

You have this way to produce $\tilde{A}$ extending $A\varepsilon^{-1}$ such that $\tilde{A}-\tilde{A}^*$ rank 1 purely imag.

You now have clear picture of nearly hermitian extensions of $A\varepsilon^{-1}$.

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puzzle. What happens. Review.

have p. herm. $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ of type $\mathcal{O}(n)$

$W = \begin{pmatrix}\varepsilon\\A\end{pmatrix}X \subset \begin{matrix}Y\\ \oplus\\ Y\end{matrix}$ isotropic for

$$\left(\xi, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\xi'\right) = (\xi_1, \xi_2') - (\xi_2, \xi_1')$$

$$W^0 = \left\{\begin{pmatrix}y_1\\y_2\end{pmatrix} \middle| (y_1, Ax) = (y_2, \varepsilon x) \quad \forall x\in X\right\} \supset W$$

e unit vector in $(\varepsilon X)^\perp$, define x_0 by

$$\underbrace{(e, Ax)}_{\text{linear func. on } X} = (\varepsilon x_0, \varepsilon x)$$

Then $\begin{pmatrix}e\\ \varepsilon x_0\end{pmatrix} \in W^0$

$$W^0 = \begin{pmatrix}\varepsilon\\A\end{pmatrix}X \oplus \mathbb{C}\begin{pmatrix}e\\ \varepsilon x_0\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}0\\e\end{pmatrix}$$

~~Extending the partial herm. $A\varepsilon^{-1}$ to $\tilde{A}$ amounts to $\begin{pmatrix}\varepsilon\\A\end{pmatrix}X$ $\tilde{A}$~~

Consider subspaces: $\overset{n}{W} \subset \overset{n+1}{V} \subset \overset{n+2}{W^0}$ $\ni$ $V = \Gamma_{\tilde{A}}$

$\tilde{A} : Y \to Y$ extends $A\varepsilon$. $\tilde{A}\Big| \begin{matrix}\varepsilon x = Ax \\ e = \varepsilon x_0 + ce\end{matrix}$

$$(\varepsilon x + e, \tilde{A}(\varepsilon x + e)) = (\varepsilon x + e, Ax + \varepsilon x_0 + ce)$$

$$= \underbrace{(\varepsilon x, Ax)}_{\text{real}} + (\varepsilon x, \varepsilon x_0) + \underbrace{(e, Ax)}_{(\varepsilon x_0,\, \varepsilon x)} + (e,e)c$$

Im part = Im c.

So $\tilde{A}$ is nearly real hermitian.

J-matrix picture of $\tilde{A}$.

$$\begin{pmatrix} b_1 & a_1 & & & \\ a_1 & \ddots & \ddots & & \\ & \ddots & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n & a_n \\ & & & a_n & c \end{pmatrix}$$

Now ~~one~~ you might use $\tilde{A}$ to construct an isometric embedding

$$\tilde{y}(\lambda) = (e, (\lambda - \tilde{A})^{-1} y)$$

of Y into $L^2(\mathbb{R})$.

Look at poles: $\det(\lambda - \tilde{A}) = 0$

$$\det(\underbrace{\lambda - \tilde{A}}_{M_{n+1}}) = (\lambda - \underbrace{c}_{b_{n+1}}) \det(\lambda - \underbrace{\varepsilon^* A}_{M_n}) - a_n^2 \det(\lambda - M_{n-1})$$

$$d_{n+1} = (\lambda - b_{n+1}) d_n - a_n^2 d_{n-1}$$

$$\frac{d_{n+1}}{d_n} = \lambda - b_{n+1} - \frac{a_n^2}{\left(\frac{d_n}{d_{n-1}}\right)}$$

~~Something is wrong.~~

~~[illegible]~~ The preceding is reasonable but it does not yield the relation you want, you expect from de Branges + scattering. ~~The point~~

The important point involves choosing a ~~[illegible]~~ ~~line in Y whose~~ section of the line bundle.

~~Idea:~~ Let's try to use the line orthogonal to $(i\varepsilon - A)X$ in Y.

Go back to J-matrix & try to combine the $\iota\varepsilon$ with the imag part of c.

$$\begin{matrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \\ & & & a_n \end{matrix}$$

What do you need. ~~[illegible]~~

$$Y \xrightarrow{\begin{pmatrix}\varepsilon' \\ e'\end{pmatrix}} X \oplus \mathbb{C} \xrightarrow{(\lambda\varepsilon - A \;\; e)} Y$$

Present understanding. $(\varepsilon \;\; e)\begin{pmatrix}\varepsilon' \\ e'\end{pmatrix} = \varepsilon\varepsilon' + ee' = 1.$

$$(\lambda\varepsilon - A \;\; e)\begin{pmatrix}\varepsilon' \\ e'\end{pmatrix} = \lambda - A\varepsilon' \qquad \begin{pmatrix}\varepsilon' \\ e'\end{pmatrix}(\varepsilon \;\; e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

~~[illegible]~~ Can you calculate ε', e, e' for the line $(\iota\varepsilon - A)X^{\perp}$?

$$A = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \\ 0 & a_2 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A^* = \begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \end{pmatrix} \qquad \varepsilon^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\iota\varepsilon^* + A^* = \begin{pmatrix} b_1 + i & a_1 & 0 \\ a_1 & b_2 + i & a_2 \end{pmatrix} \qquad (\iota\varepsilon^* + A^*)\varepsilon = \begin{pmatrix} b_1 + i & a_1 \\ a_1 & b_2 + i \end{pmatrix}$$

$$(\iota\varepsilon^* + A^*)\varepsilon = i + A^*\varepsilon \qquad \text{invertible}$$

$$\varepsilon' = (i + A^*\varepsilon)^{-1}(\iota\varepsilon^* + A^*) = \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \;\; \underbrace{\begin{pmatrix} b_1 + i & a_1 \\ a_1 & b_2 + i \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ a_2 \end{pmatrix}}_{\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix}}\right)$$

$$\operatorname{Ker}(\varepsilon') = \begin{pmatrix} -\xi_1 \\ -\xi_2 \\ 1 \end{pmatrix}$$

$$(i\varepsilon^* + A^*)\varepsilon = i + A^*\varepsilon$$
$$= i + \varepsilon^* A$$
$$= \varepsilon^*(i\varepsilon + A)$$

$$\underbrace{\begin{pmatrix} 1 & 0 & \xi_1 \\ 0 & 1 & \xi_2 \\ 0 & 0 & 1 \end{pmatrix}}_{\begin{pmatrix}\varepsilon' \\ e'\end{pmatrix}} \underbrace{\begin{pmatrix} 1 & 0 & -\xi_1 \\ 0 & 1 & -\xi_2 \\ 0 & 0 & 1 \end{pmatrix}}_{(\varepsilon \;\; e)} = \mathrm{Id}$$

$A\varepsilon'$ has kernel $\mathbb{C}e$

What is $\varepsilon' A$.

$$\begin{pmatrix} 1 & 0 & \xi_1 \\ 0 & 1 & \xi_2 \end{pmatrix} \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \\ & a_2 \end{pmatrix} = \begin{pmatrix} b_1 & a_1 + \xi_1 a_2 \\ a_1 & b_2 + \xi_2 a_2 \end{pmatrix}$$

Try again.

$$A = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & & b_n \\ & & & a_n \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ & 0 & \\ & & 1 \\ & & 0 \end{pmatrix}$$

$$i\varepsilon^* + A^* = \begin{pmatrix} b_1+i & a_1 & & & \\ a_1 & b_2+i & a_2 & & 0 \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ 0 & & & a_{n-1} & b_n+i & a_n \end{pmatrix}$$

So $\varepsilon' = (\iota + \varepsilon^* A)^{-1}(\iota\varepsilon^* + A^*)$

$$= \begin{pmatrix} I & \xi \end{pmatrix}$$

$$(\iota + \varepsilon^* A)\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_n \end{pmatrix}$$

$$\varepsilon' = \varepsilon^* + (\iota + \varepsilon^* A)^{-1} x_0 \langle e_{n+1}|$$

$$(e_{n+1}, Ax) = (x_0, x)$$

$$\varepsilon' A = \varepsilon^* A + (\iota + \varepsilon^* A)^{-1} x_0 \langle e_{n+1}| A$$

The point somehow

$$A\varepsilon' = A\varepsilon^* + \underbrace{A(\iota + \varepsilon^* A)^{-1}}_{(\iota + A\varepsilon^*)^{-1} A} x_0 (e_{n+1}|$$

$$(\iota + A\varepsilon^*)^{-1} A x_0 (e_{n+1}|$$

Review the logic. You have a holom line bundle E over $\mathbb{P}^1$ with fibres $Y/(\lambda\varepsilon - A)X = E_\lambda$. Any $y \in Y$ gives a holom. section which has n zeroes if ~~thus if~~ $y \neq 0$. Get divisors of degree n on $\mathbb{P}^1$ equivalent to lines in $\mathbb{P}^1$.

If $f \notin \varepsilon X$, then $Y = \mathbb{C}f \oplus \varepsilon X$

we get

$$Y \xrightarrow[\sim]{\begin{pmatrix} \varepsilon' \\ f' \end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow[\sim]{(\varepsilon \;\; f)} Y$$

$$Y \xrightarrow{\begin{pmatrix} \varepsilon' \\ f' \end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{(\lambda\varepsilon - A \;\; f)} Y \qquad (\lambda\varepsilon - A)\varepsilon' + ff' = \lambda - A\varepsilon'$$

divisor is roots of $\frac{1}{\lambda}\det(\lambda - A\varepsilon') = \det(\lambda - \varepsilon' A)$

In simpler terms

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "ff"] & & \\ X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr, bend right, "\lambda - \varepsilon' A"'] & Y \arrow[r] \arrow[d, "\varepsilon'"] & E_\lambda \arrow[r] & 0 \\ & X & & \end{tikzcd}$$

39 ~~What is the point?~~

Different approach. Calculate $\ker(\lambda \varepsilon^* - A^*)$

$$\begin{pmatrix} b_1-\lambda & a_1 & & & \\ a_1 & b_2-\lambda & \ddots & & \\ & \ddots & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n-\lambda & a_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ \\ u_{n+1} \end{pmatrix} = 0$$

$$(b_1-\lambda)u_1 + a_1 u_2 = 0$$

$$a_1 u_1 + (b_2-\lambda)u_2 + a_2 u_3 = 0$$

solve starting from $u_1 = 1$.

$$u_2 = \frac{\lambda - b_1}{a_1} \qquad u_3 = \frac{(\lambda - b_2)u_2 - a_1 u_1}{a_2}$$

$$d_j = \det(\lambda - M_j) \qquad M_j = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{j-1} \\ & & a_{j-1} & b_j \end{pmatrix}$$

$$d_{j+1} = (\lambda - b_{j+1}) d_j - a_j^2 d_{j-1}$$

$$d_{j+1} = (\lambda - b_{j+1}) d_j - a_j^2 d_{j-1} \qquad j \geq 0.$$

$$u_{j+1} = \frac{d_j}{a_1 \cdots a_j} \qquad d_j = (\lambda - b_j) d_{j-1} - a_{j-1}^2 d_{j-2}$$

$$\underline{a_j u_{j+1}} = (\lambda - b_j) u_j - a_{j-1} u_{j-1}$$

$$\frac{d_j}{a_1 \cdots a_{j-1}} = (\lambda - b_j)\frac{d_{j-1}}{a_1 \cdots a_{j-1}} - a_{j-1}\frac{d_{j-2}}{a_1 \cdots a_{j-2}}$$

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$$\begin{pmatrix} b_1-\lambda & & \\ & & \\ & & b_n-\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} 0 \\ \\ 0 \\ a_n \end{pmatrix} u_{n+1} = 0$$

$$\begin{pmatrix} 0 \\ \vdots \\ a_n u_{n+1} \end{pmatrix} = (\lambda - M_n)\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

J-matrix

$$A = \begin{bmatrix} b_1 & a_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \\ & & & a_n \end{bmatrix} \qquad \operatorname{Ker}(\lambda \varepsilon^* - A^*).$$

$$\begin{bmatrix} b_1-\lambda & a_1 & & & \\ a_1 & b_2-\lambda & & & \\ & & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n-\lambda & a_n \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ \\ u_{n+1} \end{bmatrix} = 0$$

$$(b_1-\lambda)u_1 + a_1u_2 = 0 \qquad a_1u_2 = (\lambda-b_1)u_1$$

$$a_1u_1 + (b_2-\lambda)u_2 + a_2u_3 = 0 \qquad a_2u_3 = (\lambda-b_2)u_2 - a_1u_1$$

$$\frac{a_1u_2}{u_1} = \lambda - b_1 \qquad \frac{a_2u_3}{u_2} = \lambda - b_2 - a_1^2 \Big/ \frac{a_1u_2}{u_1}$$

$$a_{j-1}u_{j-1} + (b_j-\lambda)u_j + a_ju_{j+1} = 0$$

$$a_ju_{j+1} = (\lambda-b_j)u_j - a_{j-1}u_{j-1}$$

$$\begin{pmatrix} a_ju_{j+1} \\ u_j \end{pmatrix} = \begin{pmatrix} \frac{\lambda-b_j}{a_{j-1}} & -a_{j-1} \\ a_{j-1}^{-1} & 0 \end{pmatrix}\begin{pmatrix} a_{j-1}u_j \\ u_{j-1} \end{pmatrix}$$

So the recursion formula is

$$\begin{pmatrix} a_j u_{j+1} \\ u_j \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\lambda - b_j}{a_{j-1}} & -a_{j-1} \\ \frac{1}{a_{j-1}} & 0 \end{pmatrix}}_{\det = 1} \begin{pmatrix} a_{j-1} u_j \\ u_{j-1} \end{pmatrix} \qquad \begin{array}{l} u_1 = 1 \\ u_0 = 0 \end{array}$$

$$\begin{pmatrix} a_1 u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} \lambda - b_1 & \\ & \end{pmatrix}$$

$$\begin{pmatrix} u_{j+1} \\ u_j \end{pmatrix} = \begin{pmatrix} \frac{\lambda - b_j}{a_j} & -\frac{a_{j-1}}{a_j} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_j \\ u_{j-1} \end{pmatrix}$$

So inductively u_{j+1} is a poly $\dfrac{\lambda^j}{a_j \cdots a_1} = \dfrac{\det(\lambda - M_j)}{a_j \cdots a_1}$

~~So never get down to~~

$$(M_n - \lambda)\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -a_n u_{n+1} \end{pmatrix}$$

$$\boxed{(\lambda - M_n)\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n-1 \\ a_n u_{n+1} \end{pmatrix}}$$

$\operatorname{Ker}(\lambda \varepsilon^* - A^*)$ gen. by $\begin{pmatrix} u_1 \\ \vdots \\ u_{n+1} \end{pmatrix}$

 Here's the question:

Go back to the idea, the important idea, which is the zeroes of the section of $\mathcal{O}(n)$ given by a generator of $\mathrm{Ker}(i\varepsilon^* - A^*)$.

$$\begin{tikzcd} X \arrow[r, "\lambda\varepsilon - A"] \arrow[dr] & Y \arrow[d, "\iota\varepsilon^* - A^*"] \\ & X \end{tikzcd}$$

$$\begin{aligned}(\iota\varepsilon^* - A^*)(\lambda\varepsilon - A) &= i\lambda - \lambda A^*\varepsilon - i\varepsilon^* A + A^*A \\ &= i\lambda + A^*A - (\lambda + i)\varepsilon^* A \\ &= 1 + A^*A + \underbrace{i\lambda - 1}_{i(\lambda+i)} - (\lambda+i)\varepsilon^* A \\ &= 1 + A^*A + (\lambda+i)(i - \varepsilon^* A)\end{aligned}$$

This is no help. Perhaps A^*A is not good. Better might be to have $(\varepsilon^*A)^2 = A^*\varepsilon\varepsilon^*A = A^*A - A^*\underbrace{\pi}_{e^*e}A$

Somehow this is too hard

$$(\iota\varepsilon^* - A^*)(-i\varepsilon - A) = 1 + A^*A$$

Adopt de Branges approach. Namely consider the Hilbert space with the orthonormal basis given by the sequence of polynomials $u_1, u_2, \ldots, u_n$

$$(\iota\varepsilon^* - A^*)(\lambda\varepsilon - A) = \lambda(\iota - A^*\varepsilon) - \underbrace{(\iota\varepsilon^* - A^*)A}$$

$$iA^*\varepsilon - A^*A = A^*(\iota\varepsilon - A)$$

$$\lambda - \underbrace{(i - A^*\varepsilon)^{-1}A^*}(\iota\varepsilon - A)$$

$$A^*(i - \varepsilon A^*)^{-1}(\iota\varepsilon - A)$$

$$(\iota\varepsilon^* - A^*)(\lambda\varepsilon - A) = \lambda(\iota\varepsilon^* - A^*)\varepsilon - (\iota\varepsilon^* - A^*)A$$

So we get

$$((\iota\varepsilon^* - A^*)\varepsilon)^{-1}(\iota\varepsilon^* - A^*)A$$

whose spectrum we want. If you change A by $c\varepsilon$, this operator changes by c.

Go back to the de Branges approach where you have $\overset{u_1}{p_1}, p_2, \ldots, \overset{u_{n+1}}{p_{n+1}}$ polynomials in λ recursion relations. Form

$$\sum_{j=1}^{n} p_j(\lambda)\, p_j(\mu)$$

I think ~~I~~ begin to understand now. You know about extending $A\varepsilon^{-1}$ to a nearly hermitean operator which will then give an L^2 representation. This must be what de Branges does. ~~You know about~~ Work this picture out and correlate with point evaluators.

Recursion relations

$$\lambda p_j = a_j p_{j+1} + b_j p_j + a_{j-1} p_{j-1}$$

$$\sum_{j=1}^{n} \lambda p_j(\lambda) p_j(\mu) = \sum_{j=1}^{n} a_j p_{j+1}(\lambda) p_j(\mu) + \cancel{b_j p_j(\lambda) p_j(\mu)} + \sum_{j=0}^{n-1} a_j p_j(\lambda) p_{j+1}(\mu)$$

$$-\sum_{j=1}^{n} p_j(\lambda)\, \mu p_j(\mu) - \sum_{j=1}^{n} p_j(\lambda) a_j p_{j+1}(\mu) + \cancel{p_j(\lambda) b_j p_j(\mu)} + \sum_{j=0}^{n-1} p_{j+1}(\lambda) a_j p_j(\mu)$$

$$= a_n\left(p_{n+1}(\lambda) p_n(\mu) - p_n(\lambda) p_{n+1}(\mu)\right)$$

$$\sum_{j=1}^{n} p_j(\lambda) p_j(\mu) = a_n\left(\frac{p_{n+1}(\lambda) p_n(\mu) - p_n(\lambda) p_{n+1}(\mu)}{\lambda - \mu}\right)$$

44 Try carefully.

Go back to $W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$

$$W^0 = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \, (y_1, Ax) = (y_2, \varepsilon x), \forall x \in X \right\}$$

~~define $x_0 \in X$ so that~~ y

Let e_{n+1} be a unit vector $\in (\varepsilon X)^\perp$ $\qquad x_n = a_n e_n$

Let $x_n \in X \ni (e_{n+1}, Ax) = (\varepsilon x_n, x)$

Then $W^0 = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \oplus \mathbb{C}\begin{pmatrix} e_{n+1} \\ x_n \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix}$

In terms of the J-matrix.

$$\left[\begin{array}{ccccc|c} b_1 & a_1 & & 0 & & 0 \\ a_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & a_{n-1} & & 0 \\ 0 & & a_{n-1} & b_n & & a_n \\ & & & a_n & & c \end{array}\right]$$

A

Now I think can use this $\tilde{A}$ to ~~give a~~ get an isometric embedding

Review:

$(\lambda - \alpha) x = -y + \cdot$

Idea:

$$(e, (\lambda - \alpha)^{-1} y) = \tilde{y}(\lambda)$$

$$\int_{-\infty}^{\infty} |\tilde{y}(\lambda)|^2 \frac{d\lambda}{2\pi} = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left((\lambda-\alpha)^{-1} y, e\right)\left(e, (\lambda-\alpha)^{-1} y\right)$$

$$= \int_{\boxed{\ }} \frac{d\lambda}{2\pi} \left(y, \underbrace{(\lambda-\alpha^*)^{-1} \cdot |e\rangle\langle e| \, (\lambda-\alpha)^{-1}}\, y\right) \;\; \text{''} \;\; \frac{2\pi i}{2\pi}\left(y, |e\rangle\langle e| (\alpha^*-\alpha)^{-1} y\right)$$

residue

$\cancel{\frac{2\pi i}{2\pi}(\alpha^*-\alpha)^{-1} |e\rangle\langle e|}$

$i\,|e\rangle\langle e| = b(\alpha^*-\alpha)$

Now the requirement is $-i(\alpha^* - \alpha) = |e\rangle\langle e|$

α^* spec. in UHP $\qquad -2\,\dfrac{\alpha-\alpha^*}{2i} = -2\,\mathrm{Im}(\alpha)$

Idea now is to let the imginary part go to ∞.

$$\lambda-\alpha = \begin{bmatrix} \lambda-b_1 & & & \\ & \ddots & & \\ & & \lambda-b_n & -a_n \\ & & -a_n & \lambda-c \end{bmatrix}$$

$2\,\mathrm{Im}\,\alpha = \mathrm{Im}\,c.$

$-i(\alpha^*-\alpha) = -i(\bar{c}-c) = -2\,\mathrm{Im}\,c$

So what you try to do then is to

~~Review perturb~~

$$\lambda-\alpha = \left(\begin{array}{cccc|c} \lambda-b_1 & -a_1 & & & 0 \\ -a_1 & \lambda-b_2 & & & \\ & -a_2 & \ddots & & \\ & & & \lambda-b_n & -a_n \\ \hline 0 & & & -a_n & \lambda-c \end{array}\right) \qquad \alpha = \left(\begin{array}{c|c} M_n & \\ & a_n \\ \hline a_n & c \end{array}\right)$$

$$(\lambda-\alpha)^{-1} = \left(\begin{array}{c|c} \lambda-M_n & g \\ \hline g^* & \lambda-c \end{array}\right)^{-1} = \left(\begin{array}{c|c} \cancel{(\lambda-M_n)^{-1}} \; g(\lambda-c)^{-1}g^* & \\ \hline & \end{array}\right)$$

$$(\lambda-M_n)^{-1} - g\frac{1}{\lambda-c}g^* \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Big/ ad-bc$$

$$\left(\lambda-M_n - g\frac{1}{\lambda-c}g^*\right)^{-1}$$

arrange $c \to \infty$ $\quad \dfrac{ad-bc}{-c}$ $= b - ac^{-1}d$

But actually you want $\langle e | (\lambda-\alpha)^{-1}$

So what?

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1} = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$$ assume d invertible.

$$\begin{pmatrix}\lambda - M & g \\ g^* & \lambda - c\end{pmatrix}^{-1}$$

$$\begin{pmatrix}1 & -bd^{-1}\\0&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\-d^{-1}c&1\end{pmatrix}$$

$$= \begin{pmatrix}a-bd^{-1}c & 0\\ c & d\end{pmatrix}\begin{pmatrix}1&0\\-d^{-1}c & 1\end{pmatrix} = \begin{pmatrix}a-bd^{-1}c&0\\0&d\end{pmatrix}$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1} = \begin{pmatrix}1&0\\-d^{-1}c&1\end{pmatrix}\begin{pmatrix}(a-bd^{-1}c)^{-1}&0\\0&d^{-1}\end{pmatrix}\begin{pmatrix}1&-bd^{-1}\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}(a-bd^{-1}c)^{-1} & 0\\ -d^{-1}c(a-bd^{-1}c)^{-1} & d^{-1}\end{pmatrix}\begin{pmatrix}1&-bd^{-1}\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}(a-bd^{-1}c)^{-1} \\ \underline{-d^{-1}c(a-bd^{-1}c)^{-1}}\end{pmatrix}$$

$$-d^{-1}c(a-bd^{-1}c)^{-1} = -\frac{1}{\lambda-c}\, g^*\left(\lambda - M - g\frac{1}{\lambda-c}g^*\right)^{-1}$$

Let's try for a new direction.

Consider

$$\begin{tikzcd} & & V \arrow[d] \\ 0 \arrow[r] & X \arrow[r, "az-b"] & Y \arrow[r] \arrow[d, "b^*"] & {} \\ & & X \end{tikzcd}$$

$$\begin{pmatrix}b^*\\e_0^*\end{pmatrix} \quad (b-az \quad e_0)$$

$$Y \longrightarrow X \oplus \mathbb{C} \longrightarrow Y$$

$$(b-az \quad e_0)\begin{pmatrix}b^*\\e_0^*\end{pmatrix} = 1-zab^*$$

$$(b-az)x + \tilde{y}(z)e_0 = y$$ has solution

$$\begin{pmatrix}x\\c\end{pmatrix} = \begin{pmatrix}b^*\\e_0^*\end{pmatrix}(1-zab^*)^{-1}y$$

 and you get an isom. embedding.

$$\tilde{y}(z) = e_0^*(1-zab^*)^{-1}y$$

$$\int \frac{d\theta}{2\pi} |\tilde{y}(z)|^2 = \int \frac{d\theta}{2\pi i} \left(y, \frac{1}{\cancel{1-}ba^*} e_0 e_0^* \frac{1}{1-zab^*} y\right)$$

$$= \left(y, e_0 e_0^* \frac{1}{1-ba^*ab^*} y\right)$$ more arguments needed.

Principle: The element of Y you use to trivialize the line bundle over the UHP ~~determines~~ determines the ~~zeros~~ poles. ~~If~~ So if you want the results

Take an LC circuit, form the corresp J-matrix. Calculate the ~~out~~.

~~Take a partial unit matrix~~

Put into words the problem. Take a J-matrix determine its response.

Take a partial unitary $\quad aX \oplus V^+ = bX \oplus V^-$

The response function is a map ~~[illegible]~~ $S(z): V^- \to V^+$, it really depends upon the line V^-. ~~[illegible]~~

~~[illegible] partial hermitian~~

~~Want~~ Given $W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ Also have

$(az-b)x \qquad (z \;\; -1) : \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \longrightarrow Y$

So consider $\quad \overset{n}{W = \begin{pmatrix} a \\ b \end{pmatrix} X} \qquad \overset{n+1}{\begin{pmatrix} 1 \\ z \end{pmatrix} Y} \qquad \overset{n+2}{W^0} = \overset{n}{W} \oplus \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$

$W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \hookrightarrow W^0/W$

dim 1

$$W^\circ \cap \binom{1}{z} Y = \operatorname{Ker}\left\{ W^\circ \xrightarrow{(z\ \ -1)} Y \right\}$$

$$= \left\{ \begin{pmatrix} ax + v^+ \\ bx + v^- \end{pmatrix} \;\middle|\; z(ax+v^+) = bx + v^- \right\}$$

$$(az-b)\,x = -zv^+ + v^-$$

$$v^- = (1-bb^*)(1-zab^*)^{-1} z v^+$$

Move on to herm. setting.

$$W = \binom{\varepsilon}{A} X \qquad \binom{1}{\lambda} Y$$

max. isot. for λ real ~~isotropic~~

$$W^\circ \cap \binom{1}{\lambda} Y \hookrightarrow W^\circ / W$$

line.

Example: Suppose

$$A = \begin{pmatrix} 0 & a_1 & & \\ a_1 & 0 & a_2 & \\ & a_2 & 0 & \\ & & & a_3 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & & 0 & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ & & & 0 \end{pmatrix}$$

$$W^\circ = \binom{\varepsilon}{A} X \oplus \mathbb{C}\begin{pmatrix} e_4 \\ a_3 e_3 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ e_4 \end{pmatrix}$$

$$\Big\downarrow (\lambda\ \ -1)$$

$$(\lambda\varepsilon - A)X + \mathbb{C}(\lambda e_4 - a_3 e_3) + \mathbb{C} e_4$$

$$(\lambda - \tilde{A})Y$$

$$W^\circ \cap \binom{1}{\lambda} Y = W^\circ \cap \operatorname{Ker}(\lambda\ \ -1).$$

$$\binom{\varepsilon}{A} x + \binom{e_4}{a_3 e_3} x_4 + \binom{0}{e_4} \in \operatorname{Ker}(\lambda\ \ -1)$$

$$(\lambda\varepsilon - A)x + (\lambda e_4 - a_3 e_3)x_4 = e_4$$

$$\left[\lambda - \begin{pmatrix} \cdot & a_1 & & \\ a_1 & & a_2 & \\ & a_2 & & a_3 \\ & & a_3 & \end{pmatrix}\right] \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

49 We agreed that solution of

$$\lambda u_1 - a_1 u_2 = 0 \qquad a_1 u_2 = \lambda u_1$$

$$-a_1 u_1 + \lambda u_2 - a_2 u_3 = 0 \qquad a_2 u_3 = \lambda u_2 - a_1 u_1$$

$$-a_2 u_2 + \lambda u_3 - a_3 u_4 = 0 \qquad a_3 u_4 = \lambda u_3 - a_2 u_2$$

$$-a_3 u_3 + \lambda u_4 = 1 \qquad \underset{a_4 u_5}{1} = \lambda u_4 - a_3 u_3$$

$$\left[\lambda - \begin{pmatrix} & & \\ & \tilde{A} & \\ & & \end{pmatrix}\right]\begin{pmatrix} u_1 \\ \\ \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_4 u_5 \end{pmatrix}$$

we take $x_1 = 1 = u_1$ Then

find $x_2 = u_2$

$x_3 = u_3$

$x_4 = u_4$

satisfies $(\lambda E - A)\begin{pmatrix} u_1 \\ x \\ u_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -a_3 u_4 \\ \lambda u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_4 u_5 \end{pmatrix}$

so we have ~~the element~~ the element

$$\begin{pmatrix} x \\ \cancel{\tilde{A} x} \\ A x \end{pmatrix} + \left(\cancel{\begin{matrix} 0 \\ 0 \\ 0 \\ a_4 u_5 \end{matrix}}\ \begin{matrix} 0 \\ \\ e_4 (a_4 u_5) \end{matrix}\right) \in W^0 \cap \operatorname{Ker}(\lambda - 1)$$

$$\begin{pmatrix} u_4 e_4 \\ a_3 u_4 e_3 \end{pmatrix} + \begin{pmatrix} 0 \\ (a_4 u_5) e_4 \end{pmatrix}$$

seems to consist of the ~~the~~ components u_4, $a_4 u_5$

50 Oct 5 Missing point. CT does not commute with taking response function?

Consider $W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$, form $W^\circ \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y$ and its image $Z_z \subset W^\circ / W$. On the other hand can apply $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ to get

$$W' = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix} \quad \text{and find } W'^{\circ'} \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y$$

and its image $Z'_\lambda \subset W'^{\circ'}/W'$. These are all related by the unitary transformation ~~These are all related by the unitary~~ so there is an isom

$$\begin{matrix} W^\circ/W & \xrightarrow{\ \sim\ } & W'^{\circ'}/W' \\ \text{IS} & & \text{IS} \\ V^+ \oplus V^- & & \mathbb{C}\begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix} \end{matrix}$$

$$\begin{pmatrix} \varepsilon \\ A \end{pmatrix} x + \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \qquad \begin{matrix} y_2 = Ae_1 & & Ae_n \\ y_1 = e_1, \dots, e_n \end{matrix}$$

$$W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} x \qquad W^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \ (y_1, Ax) = (y_2, \varepsilon x) \quad \forall x \right\}$$

$$y_1 = e_{n+1} \qquad y_2 = a_n e_n + c\, e_{n+1}$$

$$W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \ni \begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} + \overset{x_{n+1}}{c_1} \begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix}$$

$$\lambda(\varepsilon x + e_{n+1} x_{n+1}) = Ax + e_n a_n x_{n+1} + e_{n+1} c$$

$$\left[\lambda - \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & \ddots & \ddots & & \\ & \ddots & \ddots & a_{n-1} & \\ & & a_{n-1} & b_n & a_n \\ & & & a_n & 0 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ c \end{pmatrix}$$

You know the vector $\begin{pmatrix} u_1 \\ | \\ u_{n+1} \end{pmatrix}$ of orth polys ~~g~~
satisfies $(\lambda - M_{n+1})\, u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_{n+1} u_{n+2} \end{pmatrix}$ Take all $b_k^* = \cancel{0}$

u_λ

$\therefore$ get $\begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} u_{n+1} + \begin{pmatrix} 0 \\ a_{n+1} e_{n+1} \end{pmatrix} u_{n+2}$

$W^0 \cap \binom{1}{\lambda} Y$ ~~spanned~~ by $\begin{pmatrix} \varepsilon \\ A \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} u_{n+1} + \begin{pmatrix} 0 \\ a_{n+1} e_{n+1} \end{pmatrix} u_{n+2}$

apply $\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$ or $\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$ to get

$$\begin{pmatrix} i\varepsilon + A \\ i\varepsilon - A \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} i e_{n+1} + a_n e_n \\ i e_{n+1} - a_n e_n \end{pmatrix} u_{n+1} + \begin{pmatrix} e_{n+1} \\ -e_{n+1} \end{pmatrix} a_{n+1} u_{n+2}$$

spanning the corresponding line in the ~~two~~ Z picture
We need the image of this ~~in middle & W~~
in $V^+ \oplus V^-$. Looks messy.

~~Apparently what happens is that~~

Try the other direction. Take V^+ which should be ~~the~~ $\operatorname{Ker}(-i\varepsilon^* - A^*)$ spanned by u^{-i}

~~$\operatorname{Ker}(\lambda\varepsilon^* - A)$~~ $$(\lambda - \tilde{A})\, u^\lambda = e_{n+1} a_{n+1} u^\lambda_{n+2}$$

$$\therefore \quad (\lambda \varepsilon^* - \varepsilon^* \tilde{A})\, u^\lambda = 0.$$

Find

$$\begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} 1 \\ z \end{pmatrix} Y \qquad \left(\begin{pmatrix} a \\ b \end{pmatrix} X + \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix} \right) \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y$$

$$\begin{pmatrix} \xi \\ z\xi \end{pmatrix} = \begin{pmatrix} ax + v^+ \\ bx + v^- \end{pmatrix} \qquad (az-b)x = -zv^+ + v^-$$

Suppose $\mathrm{Im}\left\{ W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \to W^0/W = \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix} \right\}$ is $\begin{pmatrix} 1 \\ \phi(z) \end{pmatrix} V^+$

~~i.e.~~ $\cancel{(az-b)x = z}$ ~~$v^- = \phi(z) v^+$. Then~~

$$\cancel{(az-b)x = -zv^+ + \phi(z)v^+}$$
$$\cancel{= (\phi(z) - z) v^+}$$

$$\cancel{(z - a^*b)x}$$

$$z^{-1} S(z)^{-1} v^-$$
$$= z^{-1}(1-a^*a)(1-z^{-1}ba^*)^{-1} v^-$$

$$(az-b)x = -zv^+ + v^-$$
$$(1 - zb^*a)x = zb^*v^+$$
$$x = zb^*(1-zab^*)^{-1}v^+$$
$$v^- = z\left((1-zab^*) + (az-b)b^* \right)(1-zab^*)^{-1} v^+$$

$$v^- = \underbrace{z(1-bb^*)(1-zab^*)^{-1} v^+}_{S(z)v^+} \qquad \text{where } S(z) : V^+ \to V^-$$

$\therefore$ L_z line in $\begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix}$ is $\begin{pmatrix} v^+ \\ zS(z)v^+ \end{pmatrix}$

$$L_z = \mathrm{Im}\left(W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \to W^0/W = \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix} \right). \quad \text{Note}$$

$$L_0 = \begin{matrix} V^+ \\ \oplus \\ 0 \end{matrix} \qquad L_\infty = \begin{matrix} 0 \\ \oplus \\ V^- \end{matrix}$$

The logic here is that $z \mapsto L_z$ is ~~a~~ regular map from z Riemann spheres to ~~...~~ $\mathbb{P}_1(W^\circ/W)$, i.e. rational functions of z after one chooses some sort of coords on W°/W. ~~These~~ $\exists$ a herm. form on W°/W, i.e. a ~~real~~ unit circle in $\mathbb{P}_1(W^\circ/W)$ which is ~~...~~ the image of $|z|=1$. ~~...~~

$$W = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \qquad W^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, (y_1, Ax) = (y_2, \varepsilon x) \right\}$$

$$= W \oplus \mathbb{C}\begin{pmatrix} e_{n+1} \\ a_n e_n \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix}$$

$$W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \ni \begin{pmatrix} \varepsilon x + \{e_{n+1}\} x_{n+1} + 0 \\ Ax + \{e_n a_n\} x_{n+1} + e_{n+1} c \end{pmatrix}$$

Then $x_1, \dots, x_{n+1}$ satisfy.

$$\lambda x = \tilde{A} x + e_{n+1} c$$

and we have the solution $x_i = u_i^\lambda \qquad i = 1, \dots, n+1$

$$\left[\lambda - \begin{pmatrix} 0 & a_1 & & & \\ a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_n \\ & & & a_n & 0 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ \vdots \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n+1} u_{n+2}^\lambda \end{pmatrix}$$

$$L_\lambda = \underline{\text{Image of } W^\circ \cap \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \text{ in } W^\circ/W} \simeq \begin{pmatrix} e_{n+1} \\ e_n a_n \end{pmatrix} \mathbb{C} \oplus \begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix} \mathbb{C}$$

line $\boxed{\begin{pmatrix} e_{n+1} \\ e_n a_n \end{pmatrix} u_{n+1}^\lambda + \begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix} a_{n+1} u_{n+2}^\lambda}$

 Try other approaches. What is the link between $\operatorname{Ker}(\lambda\varepsilon^* - A^*)$ and $W^\circ \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y$?

$$W = \begin{pmatrix}\varepsilon\\ A\end{pmatrix} X \qquad \begin{pmatrix}y\\ \lambda y\end{pmatrix} \in W^\circ \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y$$

i.e. $\quad (y, Ax) = (\lambda y, \varepsilon x) \qquad \forall x$

or $\quad ((A^* - \lambda\varepsilon^*)y, x) = 0 \qquad \forall x.$

Thus $\quad W^\circ \cap \begin{pmatrix}1\\ \lambda\end{pmatrix} Y = \left\{ \begin{pmatrix}y\\ \lambda y\end{pmatrix} \,\middle|\, (\lambda\varepsilon^* - A^*)y = 0 \right\}.$

$$u^\lambda = \begin{pmatrix}u_1^\lambda\\ \vdots\\ u_{n+1}^\lambda\end{pmatrix} \qquad (\lambda - \tilde{A})u^\lambda = \begin{pmatrix}0\\ \vdots\\ 0\\ a_{n+1}u_{n+2}^\lambda\end{pmatrix}$$

So $\forall \lambda$ we have $\begin{pmatrix}u^\lambda\\ \lambda u^\lambda\end{pmatrix} \in W^\circ$, ~~can~~ can ask about hermitian pairing $\left(\begin{pmatrix}u^\lambda\\ \lambda u^\lambda\end{pmatrix}, \begin{pmatrix} & 1\\ -1 & \end{pmatrix} \begin{pmatrix}u^\mu\\ \mu u^\mu\end{pmatrix} \right)$

$$= \mu(u^\lambda, u^\mu) - \bar{\lambda}(u^\lambda, u^\mu) = (\mu - \bar{\lambda})(u^\lambda, u^\mu).$$

$$- (u^\lambda, \tilde{A}u^\mu) + (\tilde{A}u^\lambda, u^\mu)$$

$$= (u^\lambda, (\mu - \tilde{A})u^\mu) - ((\lambda - \tilde{A})u^\lambda, u^\mu)$$

$$= u_{n+1}^{\bar{\lambda}} a_{n+1} u_{n+2}^\mu - a_{n+1} u_{n+2}^{\bar{\lambda}} u_{n+1}^\mu$$

$$= \boxed{a_{n+1} \begin{vmatrix} u_{n+1}^{\bar{\lambda}} & u_{n+1}^\mu \\ u_{n+2}^{\bar{\lambda}} & u_{n+2}^\mu \end{vmatrix} = (\mu - \bar{\lambda})(u^\lambda, u^\mu)}$$

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$$\begin{tikzcd}
 & & \mathbb{C}v_0^+ \arrow[d] & & \\
0 \arrow[r] & X \arrow[r, "\lambda\varepsilon - A"] \arrow[d, "i\varepsilon + A"'] & Y \arrow[r] \arrow[d, "i\varepsilon^* + A^*"] & L_\lambda \arrow[r] & 0 \\
 & Y \arrow[r, "\lambda\varepsilon^* - A^*"] & X & &
\end{tikzcd}$$

$$z = \frac{-\lambda + i}{\lambda + i}$$

$$(az - b)X = \big(a(-\lambda + i) - b(\lambda + i)\big)X$$
$$= \big(\lambda(-a-b) + i(a-b)\big)X$$
$$= \big(\lambda(a+b) - i(a-b)\big)X$$

$$(i\varepsilon^* + A^*)(\lambda\varepsilon - A) = \lambda(A^*\varepsilon) - i(\varepsilon^* A) + i\lambda\varepsilon^*\varepsilon - A^*A$$

$$\underbrace{\underbrace{(i\varepsilon^*\varepsilon + A^*\varepsilon)}_{(i\varepsilon^* + A^*)\varepsilon}}_{\varepsilon^*(i\varepsilon + A)}\lambda - \underbrace{(i\varepsilon^* + A^*)A}_{A^*(i\varepsilon + A)}$$

seems that $(\mu\varepsilon^* + A^*)(\lambda\varepsilon + A) = (\lambda\varepsilon^* + A^*)(\mu\varepsilon + A)$

Start again. You want to know when $(i\varepsilon^* + A^*)(\lambda\varepsilon - A)$ is singular, i.e.

$$\| $$
$$(\lambda\varepsilon^* + A^*)(i\varepsilon + A)$$

$$\cancel{ab^* x = \bar{z} X}$$
$$\cancel{\Longleftrightarrow \; b^* x = \bar{z} a^* x \quad \text{and} \quad (1 - aa^*)x = 0}$$
$$\cancel{\Longleftrightarrow \; (\bar{z} a^* - b^*)x = 0 \quad \text{and} \quad x \in V^+}$$
$$\cancel{\Longleftrightarrow \; x \perp (az - b)X \quad \text{and} \quad x \perp aX}$$

$$z b^* a x = x \qquad (az - b)x = y + cv^-$$

$$z(bx', ax) = (x', x) \quad \forall x' \qquad \text{no meaning.}$$

OK. OK so what happens?

Let's try again. Consider a partial unitary

$Y = aX \oplus V^+ = V^- \oplus bX \qquad a^*a = b^*b = 1.$

equiv. a subspace $\begin{pmatrix} a \\ b \end{pmatrix} X \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$ isotropic wrt $\|y_1\|^2 - \|y_2\|^2$.

Assume of type $O(n)$, $az - b$ always inj.

$$Y \xrightarrow{\begin{pmatrix} b^* \\ e^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{\begin{pmatrix} b & e \end{pmatrix}} Y$$

inverse isos.
e unit v. sp V^-

perturbation

$$Y \xrightarrow{\begin{pmatrix} b^* \\ e^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ \mathbb{C} \end{matrix} \xrightarrow{\begin{pmatrix} b-az & e \end{pmatrix}} Y$$

$$\begin{pmatrix} b-az & e \end{pmatrix}\begin{pmatrix} b^* \\ e^* \end{pmatrix} = \underbrace{bb^* + ee^*}_{1} - zab^*$$

$\forall y \ \exists!$ $\quad (b-az)x + \tilde{y}(z)e = y$

answer: ~~$\tilde{y}(z) = e^*$~~

$$\begin{pmatrix} x \\ \tilde{y}(z) \end{pmatrix} = \begin{pmatrix} b^* \\ e^* \end{pmatrix} (1 - zab^*)^{-1} y = \begin{pmatrix} (1 - zb^*a)^{-1} b^* y \\ e^*(1 - zab^*)^{-1} y \end{pmatrix}$$

Next point is that $y \longmapsto e^*(1 - zab^*)^{-1} y$ is isom. embed of Y into $L^2(S^1)$.

$$\int |\tilde{y}(z)|^2 \frac{d\theta}{2\pi} = \int \left((1 - zab^*)^{-1} y, e\right)\left(e, (1 - zab^*)^{-1} y\right)$$

$$= \int \left(y, \frac{1}{\cancel{z}-\cancel{z}\,ba^*} e e^* \frac{1}{\cancel{1}-zab^*} y\right) \frac{dz}{2\pi i \cancel{z}}$$

$$= \|y\|^2.$$

57 You want the same thing ~~...~~ in the hermitian picture. So you consider

$$\begin{array}{ccc} & & \\ & & \downarrow e \\ X & \xrightarrow{\lambda\varepsilon - A} & Y \\ & & \downarrow \varepsilon^* \end{array}$$

$$\begin{aligned}\lambda\varepsilon - A &= \lambda(a+b) - i(a-b) \\ &= (\lambda - i)a + (\lambda+i)b \\ &= -(-\lambda+i)a + (\lambda+i)b \\ &\sim b - \left(\frac{-\lambda+i}{\lambda+i}\right)a\end{aligned}$$

$$\begin{array}{ccc} X & \xrightarrow{\lambda\varepsilon - A} & Y \\ & & \downarrow + i\varepsilon^* + A^* \\ & & X \end{array}$$

$\varepsilon = a+b$

$A = i(a-b)$

$i\varepsilon + A = 2ib.$

$$\underbrace{(i\varepsilon^* + A^*)(\lambda\varepsilon - A)} = i\lambda\,\varepsilon^*\varepsilon + (\lambda - i)\varepsilon^* A - A^*A$$

$$= i\lambda\,\varepsilon^*\varepsilon + (\lambda - i)\varepsilon^* A - (1 - \varepsilon^*\varepsilon)$$

$$\Big(\underbrace{(i\varepsilon^* + A^*)}\varepsilon\Big)\lambda - (i\varepsilon^* + A^*)A$$

invertible because you can suppose $\varepsilon^*\varepsilon = 1$.

then you have $i + \underbrace{A^*\varepsilon}_{\text{herm.}}$ So it should be

true that $\alpha + i\beta$ is invertible where $\alpha = \alpha^* > 0$ and $\beta = \beta^*$, namely ~~...~~ $(\alpha + i\beta)x = 0$

$\Rightarrow (x, \alpha x) + i(x, \beta x) = 0 \quad \therefore (x, \alpha x) = 0 \Rightarrow x = 0.$

$$(i\varepsilon^* + A^*)\varepsilon = \varepsilon^*(i\varepsilon + A)$$

$$(i\varepsilon^* + A^*)A = A^*(i\varepsilon + A)$$

so we have $\underbrace{\big((i\varepsilon^* + A^*)\varepsilon\big)^{-1}\big((i\varepsilon^* + A^*)A\big)}$

$$i\varepsilon + A = i(a+b) + i(a-b) = 2ia$$
$$i\varepsilon^* + A^* = i(a^*+b^*) - i(a^*-b^*) = 2ib^*$$

$$(i\varepsilon^* + A^*)\varepsilon = 2ib^*(a+b) = 2i(1+b^*a)$$
$$(i\varepsilon^* + A^*)A = 2ib^*(+i)(a-b) = -2(b^*a-1)$$
$$= 2(1-b^*a)$$

$$\left[(i\varepsilon^*+A^*)\varepsilon\right]^{-1}\left[(i\varepsilon^*+A^*)A\right] = \left(2i(1+b^*a)\right)^{-1}(2)(1-b^*a)$$

$$= \frac{1}{i}\,\frac{1-b^*a}{1+b^*a} \qquad \frac{1}{i}\,\frac{1-z^{-1}}{1+z^{-1}} = \frac{1}{i}\,\frac{z-1}{z+1} = i\,\frac{1-z}{1+z} = \lambda.$$

Apparently $\boxed{(i\varepsilon^*+A^*)\varepsilon \quad \text{and} \quad (i\varepsilon^*+A^*)A}$ commute.

In this setting we should have an extra condition relating $\varepsilon^*\varepsilon$ and A^*A.

$$\varepsilon^*\varepsilon A^*\varepsilon$$

$$(i\varepsilon^*+A^*)\varepsilon = 2i(1+b^*a)$$
$$(-\varepsilon^*+iA^*)A = 2i(1-b^*a)$$

$$i\,\varepsilon^*\varepsilon + iA^*A = 4i$$

$$\varepsilon^*\varepsilon + A^*A = 4$$

$$(a^*+b^*)(a+b) + (a^*-b^*)(a-b)$$

$$= \begin{matrix} a^*a + b^*a + a^*b + b^*b \\ a^*a - b^*a - a^*b + b^*b \end{matrix} = 4.$$

$$A^*\varepsilon A^*\varepsilon$$
$$= A^*\varepsilon\varepsilon^*A$$
$$=$$

If you assume $(i\varepsilon^*+A^*)\varepsilon = \varepsilon^*(i\varepsilon+A)$

$$\Vert$$

$$A^*\varepsilon + i(4-A^*A) = \text{const} + \underbrace{A^*(\varepsilon - iA)}$$

$$\varepsilon^*A - iA^*A = (\varepsilon^* - iA^*)A$$