938 March 30, 98 Think about questions of what you might obtain from a unitary. You want a unitary having a discrete spectrum of mult. 1 e.g. $u = \frac{1+X}{1-X}$ where X is self-adjoint operator with discrete spectrum $\lambda_n \longrightarrow \infty$. The question is whether such a u arises from a sequence $h_1, h_2, \ldots$. You are very confused. Yes. Consider

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March 31, 98

Go back to orthogonal polys.

$L^2(S^1, d\mu)$ $\qquad \int d\mu = 1.$

$F_n = \langle z^0, \ldots, z^n \rangle$

$p_n \equiv z^n \mod F_{n-1}$ and $p_n \perp F_{n-1}$

$q_n \equiv 1 \mod zF_{n-1}$ and $q_n \perp zF_{n-1}$

Better: $$q_n = z^n \overline{p_n} \in 1 + z^n \overbrace{\overline{F}_{n-1}}^{\langle z^0, z^{-1}, \ldots, z^{-n+1} \rangle} = 1 + zF_{n-1}$$

Then $$p_n - zp_{n-1} \in F_{n-1} \cap (zF_{n-2})^{\perp}$$

$$p_n - zp_{n-1} = h_n q_{n-1} \qquad h_n = p_n(0)$$

$$z^n\overline{p_n} - z^{n-1}\overline{p_{n-1}} = \bar{h}_n z\, z^{n-1}\overline{q}_{n-1} \qquad \|p_{n-1}\|^2 = \|p_n\|^2 + \|h_n q_{n-1}\|^2$$

$$\shortparallel$$

$$|h_n|^2 \|q_{n-1}\|$$

$$q_n - q_{n-1} = \bar{h}_n z p_{n-1} \qquad \|p_n\|^2 = (1-|h_n|^2)\|p_{n-1}\|^2.$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

939 Try to do this with a 1-pod.

$$X \underset{u}{\overset{\subset}{\rightrightarrows}} Y \qquad Y = X \oplus V^+ = uX \oplus V^-$$

assume V^+ and V^- are 1-dimensional.

Eigenvector equation $(\lambda - u)(x) = -v^+ + \boxed{v^-}$

Zeroth case: $u(X) = X$, otherwise $\overset{S(\lambda)v^+}{\cancel{u(X)+X}} = Y \iff V^+ \cap V^- = 0$ for $|\lambda|<1$.

First Case: Suppose $V^+ \perp V^-$. Then we have

$Y = X \cap u(X) \oplus V^+ \oplus V^-$ an orth. direct sum?

better: $V^- \perp V^+ \implies V^- \subset (V^+)^\perp = X$, so

can split $X = V^- \oplus (X \cap \underset{\overset{\shortparallel}{uX}}{(V^-)^\perp}) = X \cap u(X)$.

Also we have $u^{-1}X \subset X$.

$$\begin{tikzcd} u^{-1}X \arrow[r, phantom, "\subset"] \arrow[d, "u"'] & X \arrow[d, "u"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

so $u : u^{-1}X \xrightarrow{\sim} X \cap uX$

You are now just before the point that you start yawning. Maybe a good way to proceed would be to separate u from the two decompositions. Thus you have $Y = X^+ \oplus V^+ = X^- \oplus V^-$. This is the familiar Grass. situation, dihedral gp ε, F $g = F\varepsilon$. V^+, V^- have dim 1.

Recall $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ $\quad \Gamma_T = \begin{pmatrix} 1 \\ T \end{pmatrix}$ $\quad \Gamma_T^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix}$

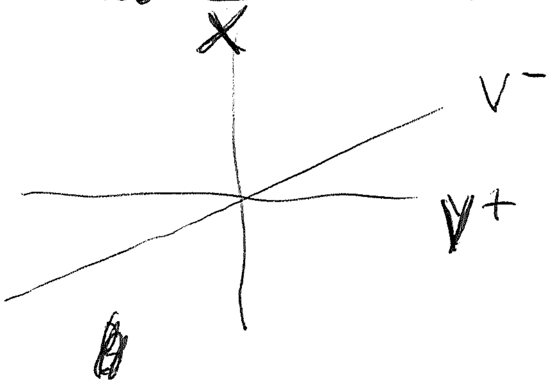
$$F \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\varepsilon}$$

$T : H^+ \to H^-$ can go ~~from~~ between subspaces of different dims

$$F(1+X) = (1+X)\varepsilon$$

$$g\varepsilon = \frac{1+X}{1-X}\varepsilon = (1+X)\varepsilon\frac{1}{1+X} = F$$

We have $Y \supset \underbrace{V^+, V^-}$ two lines.



We exclude the case $V^+ = V^-$, so that V^- projects onto a line W^- in X

V^+ ―――――――― W^+ in uX

Now you need u.

You have $X \underset{u}{\overset{\subset}{\longrightarrow}} Y$ $\qquad u^*u = 1$.

$$Y = X \oplus V^+ = uX \oplus V^-$$

Eigenvector equation $\lambda x + v^+ = u(x) + v^-$

Assume $|\lambda| < 1$, apply u^* to get $\lambda u^* x + u^* v^+ = x$

$$x - \lambda u^* x = u^* v^+$$

$$(1-\lambda u^*)\, x = u^* v^+$$

$$x = (1-\lambda u^*)^{-1} u^*(v^+)$$

$$(\lambda - u)\, x = (\lambda - u)(1-\lambda u^*)^{-1} u^*(v^+)$$

$$= (\underset{-1+1}{\lambda u^*} - uu^*)(1-\lambda u^*)^{-1} v^*$$

$$(\lambda - u)\, x = -v^+ + \underbrace{(1-uu^*)(1-\lambda u^*)^{-1}}_{S(\lambda) v^+} v^+$$

Idea: Try to write $|\lambda|^2\|x\|^2 + \|v^+\|^2 = \|x\|^2 + \|v^-\|^2$ in the form $\|v^+\|^2 - \|v^-\|^2 = (1-|\lambda|^2)\|x\|^2$. Idea here is that there should be some pseudo-herm. space

Another idea: Contraction operator. Go back to

$a,b : X \rightrightarrows Y$ $\qquad$ codim 1.

$$aX + bX = Y$$

$$\|ax_1+bx_2\|^2 = \|x_1\|^2+\|x_2\|^2+(ax_1,bx_2)+(bx_2,ax_1)$$

$$= \|x_1\|^2+\|x_2\|^2+(b^*ax_1,x_2)+(x_2,b^*ax_1)$$

$$= \|x_2+b^*ax_1\|^2+\|x_1\|^2-\|b^*ax_1\|^2$$

$$= \|x_2+b^*ax_1\|^2+\underbrace{\|x_1\|^2-(x_1,a^*bb^*ax_1)}_{(ax_1,\,(1-bb^*)ax_1)}$$

$$\|ax_1+bx_2\|^2 = \|x_1\|^2+\|x_2\|^2+(x_1,a^*bx_2)+(a^*bx_2,x_1)$$

$$= \|x_1+a^*bx_2\|^2+\|x_2\|^2-(bx_2,aa^*bx_2)$$

$$= \|x_1+a^*bx_2\|^2+(bx_2,(1-aa^*)bx_2)$$

$$1-(b^*a)^*(b^*a) = 1-a^*bb^*a = a^*(1-bb^*)a$$

I want to think carefully. The key point should be the fact that under the assumption that $\overline{aX+bX}=Y$ that $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ is determined by the contraction operator $\gamma=b^*a$ on X. Assuming codim 1, ~~that means that~~ then $1-\gamma^*\gamma$ and $1-\gamma\gamma^*$ have ranks 1, so that γ is a unitary operator operator $\gamma:\mathrm{Ker}(1-\gamma^*\gamma)\xrightarrow{\sim}\mathrm{Ker}(1-\gamma\gamma^*)$ and you ought to be able to iterate this

$$\gamma x = b^*ax$$

$$\|x\|^2 = \|ax\|^2 = (ax,bb^*ax)+(ax,(1-bb^*)ax)$$

$$= \|\gamma x\|^2 + (ax, \underbrace{(1-bb^*)}_{\text{proj onto } \mathrm{Ker}(b^*) \,=\, (\mathrm{Im}\,b)^{\perp}}ax)$$

so $\|x\|=\|\gamma x\| \iff (1-bb^*)ax=0 \iff ax\in bX$

$\iff x\in a^{-1}bX$

942 So what do you learn?

April 1 Today pursue $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ codim 1

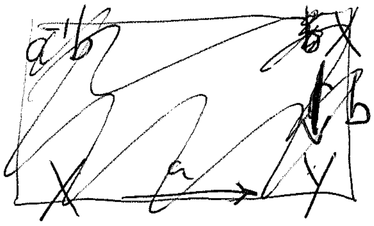
such that $aX + bX = Y$. Have

Contraction $\gamma = b^*a$ on X. $\gamma^*\gamma = a^*bb^*a$

$\|\gamma x\| = \|x\| \iff ax \in bX$

$1 - \gamma^*\gamma = a^*(1-bb^*)a$ has rank ≤ 1

The kernel is the subspace $a^{-1}bX$



$$\begin{tikzcd}
a^{-1}bX \arrow[r, "a"] \arrow[d, hook] & bX \arrow[r, "1-aa^*"] \arrow[d, hook] & V^+ \\
X \arrow[r, "a"] & Y \arrow[r, "1-aa^*"'] & V^+
\end{tikzcd}$$

$1 - \gamma\gamma^* = 1 - b^*aa^*b = b^*(1-aa^*)b$

I know $Y = aX \oplus V^+ = V^- \oplus bX$

$$\left(\qquad \right)$$

What might happen is that

$$0 \longrightarrow a^{-1}bX \xrightarrow{a} bX \longrightarrow V^+ \longrightarrow 0$$

$$0 \longrightarrow b^{-1}aX \xrightarrow{b} aX \longrightarrow V^- \longrightarrow 0$$

You combine isos. $a^{-1}bX \simeq ba^{-1}X$ ~~$a^{-1}bX$~~

$$\boxed{\begin{array}{ccc} aX & \simeq & bX \\ ax & \longmapsto & bx \end{array}}$$

$X' = X \times_{a,b} X$

$a^{-1}bX = \{x_1 \mid \exists x_2 \ni ax_1 = bx_2\}$

$$a^{-1}bX \xleftarrow[\sim]{pr_1} \{(x_1,x_2) \mid ax_1 = bx_2\} \xrightarrow[\sim]{pr_2} b^{-1}aX$$

943 $\quad X \overset{a}{\underset{b}{\rightrightarrows}} Y \qquad Y = aX \oplus V^+ = V^- \oplus b\cancel{X}$

$$V^+ = \operatorname{Ker}(a^*) = \operatorname{Im}(1-aa^*)$$

assume $\dim V^{\pm} = 1$.

define
$$\begin{tikzcd} X' \arrow[r, hook, "a'"] \arrow[d, "b'"'] & X \arrow[d, "b"] \\ X \arrow[r, hook, "a"] & Y \end{tikzcd}$$

$X' = \{(x_1, x_2) \mid ax_1 = bx_2\}$

$b'(x_1, x_2) = x_1$

$a'(x_1, x_2) = x_2$

~~$b^{-1}aX$~~ Suppose a = inclusion of $X \subset Y$ and $b = u : X \to Y$. Then $X' = X \cap u^{-1}(X)$

and
$$\begin{tikzcd} X' = u^{-1}(X) \arrow[r, phantom, "\subset"] \arrow[d, "u'"'] & X \arrow[d, "u"] \\ X \arrow[r, phantom, "\subset"] & Y \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X/X' \arrow[r] \arrow[d, "S"] & 0 \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Y/X \arrow[r] & 0 \end{tikzcd}$$

April 2. Interesting point recalled recently is that a contraction op $\gamma : X \to X$ gives rise to a partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ such that $b^*a = \gamma$, $\overline{aX + bX} = Y$. The partial unitary is unique up to canon. isom. ~~and~~ $V^+ = \sqrt{1-\gamma^*\gamma}\,X$

$$\|ax_1 + bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (\gamma x_1, x_2) + (x_2, \gamma x_1)$$

$$= \|x_2 + \gamma x_1\|^2 + \|\sqrt{1-\gamma^*\gamma}\,x_1\|^2 \qquad V^- = \sqrt{1-\gamma\gamma^*}\,X$$

$$= \|x_1 + \gamma^* x_2\|^2 + \|\sqrt{1-\gamma\gamma^*}\,x_2\|^2$$

On the other hand a contraction of $\gamma : X \longrightarrow X$ yields a partial unitary where $X' = [illegible] = \{ x \mid \|x\| = \|\gamma x\| \}$

$$= \{ x \mid (1-\gamma^*\gamma)x = 0 \}$$

Be careful. We know $0 \leq \boxed{\overset{1-}{\gamma^*\gamma}} \leq 1$, spectral theorem gives $\int_0 dE_\lambda = 1$. $X = \underbrace{\mathrm{Ker}(1-\gamma^*\gamma)}_{E_0} \oplus \underbrace{\overline{\mathrm{Im}(1-\gamma^*\gamma)}}_{1-E_0}$

Also $X = \mathrm{Ker}(1-\gamma\gamma^*) \oplus \overline{\mathrm{Im}(1-\gamma\gamma^*)}$ and

$\gamma(1-\gamma^*\gamma) = (1-\gamma\gamma^*)\gamma$ $\quad\therefore\quad$ $\gamma : \mathrm{Ker}(1-\gamma^*\gamma) \underset{\gamma^*}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Ker}(1-\gamma\gamma^*)$

~~[illegible] So from γ you get $X' \rightrightarrows X$~~

~~such that~~ Given (X, γ) you get $X' \underset{\gamma}{\overset{\hookrightarrow}{\rightarrow}} X$.

Now you have a contraction operator on X', namely γ followed by projection onto X'. This will lead to an X'' so can iterate.

Now suppose you start with $X \underset{b}{\overset{a}{\rightrightarrows}} Y$

$X' = \{ (x_1, x_2) \mid ax_1 = bx_2 \}$ $\qquad \gamma = b^*a$

$ax_1 = bx_2 \Longleftrightarrow x_2 = b^*ax_1 + ax_1 \in bX$

$\Longleftrightarrow x_1 = a^*bx_2 + bx_2 \in aX$

You want to identify X

Start again with $X \underset{b}{\overset{a}{\rightrightarrows}} Y$

Basic idea is to focus on the contraction $\gamma = b^*a$

$\|x\| = \|\gamma x\| \iff ax \in bX$

$ax = bx_1 \implies x_1 = b^*ax$

945 You want to set up something iterative. Given $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ you define $X' \rightrightarrows X$ and then $X'' \rightrightarrows X'$ etc. The idea is that on X you have γ, X' is the subspace where γ is unitary, you get γ' on X', X'' is the subspace of X' where γ' is unitary etc.

Consider (X, γ) let $X' = \mathrm{Ker}(1-\gamma^*\gamma)$

Then $\gamma X' = \mathrm{Ker}(1-\gamma\gamma^*)$. What is γ followed by projection ~~onto~~ X' ?

~~Try again~~

Say $\gamma = b^*a$ $\qquad 1-\gamma^*\gamma = 1-a^*bb^*a$

$$= a^*(1-bb^*)a$$

$X' = \mathrm{Ker}(1-\gamma^*\gamma) = \{x \mid ax = bx_1 \text{ for some } x_1\}$

$$ax = bb^*ax$$

$\gamma = b^*a$

$X' = a^{-1}bX$ $\qquad$ Restrict $\overset{b^*a}{\overset{\shortparallel}{\gamma}}$ to this

and you get b^*b

Let $x \in X'$ i.e. $ax = bb^*ax$, then

$\gamma x = b^*ax = b^*bb^*ax = \gamma x$, no help.

946. $$\begin{tikzcd} X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Y \end{tikzcd}$$ $Y = aX \oplus V^+ = bX \oplus V^-$

$\gamma = b^* a$ contraction on X.

[redacted] $$\begin{tikzcd} X' \arrow[r, "a'"] \arrow[d, "b'"'] & X \arrow[d, "b"] \\ X \arrow[r, hook, "a"] & Y \end{tikzcd}$$

$X' = \{(x_1, x_2) \mid ax_1 = bx_2\}$

$a'(x_1, x_2) = x_2$

$b'(x_1, x_2) = x_1$

Then $$\begin{tikzcd} X' \arrow[r, shift left, "a'"] \arrow[r, shift right, "b'"'] \arrow[d, "a'"'] & X \arrow[d, "a"] \\ X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Y \end{tikzcd}$$ is a partial unitary

There is something 1-sided here.

[redacted]

$ax_1 = bx_2, \; ax_3 = bx_4$

$$X'' = \{(x_1, x_2, x_3, x_4) \mid \underset{x_2}{\overset{\parallel}{a'(x_1,x_2)}} = \underset{x_3}{\overset{\parallel}{b'(x_3,x_4)}}\}$$

$$= \{(x_1, x_2, x_3) \mid ax_1 = bx_2, \; ax_2 = bx_3\}$$

$a''(x_1, x_2, x_3) = (x_2, x_3)$

$b''(x_1, x_2, x_3) = (x_1, x_2)$

Something is getting clearer, namely, the iteration [redacted] gives the successive fibre products

$$\begin{tikzcd} X & X \underset{(a,b)}{\times} X \arrow[l, shift left] \arrow[l, shift right] & X \underset{(a,b)}{\times} X \underset{(a,b)}{\times} X \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

don't get the middle face

$$\begin{tikzcd} Y & X \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

947 April 3 Sit down and study the structure of a partial unitary $X \underset{b}{\overset{a}{\rightrightarrows}} Y$. This is a self-correspondence of Y and can be iterated ~~[illegible]~~ and maybe inverted. ~~[illegible]~~

Think about correspondences. ~~[illegible]~~ In general ~~a correspondence~~ an ordered pair (p,q) of maps ~~[illegible]~~

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "p"'] & Z \\ Y & \end{tikzcd}$$

~~is~~ gives a correspondence from Y to Z and ~~[illegible]~~ another corresp. (called the transpose) from Z to Y. You probably want to associate to (p,q) the symbol $p_* q^*$ in order to express the way (p,q) carries things over Z to things over Y. ~~Correspondences basic operations [illegible]~~

Basic operations ~~on~~ correspondences are sum, transpose, and composition. So far I've been thinking of objects X, Y, etc as sets.

self correspondence $\Gamma : X \underset{q}{\overset{p}{\rightrightarrows}} Y$. This yields

$$\begin{tikzcd} X^{(2)} & X \arrow[r, "q"] \arrow[d, "p"] & Y \\ X \arrow[r, "q"] \arrow[d, "p"'] & Y & \\ Y & & \end{tikzcd}$$

$$\Gamma^n = \{(x_1, \dots, x_n) \mid qx_{i-1} = px_i\}$$

$$= X \times_{p,q} X$$

Wagner theory - discrete dynamical system, I think This is a $\mathbb{Z}$ action on a profinite set which is finitely presented, ~~[illegible]~~ maybe yielding a Markov partition, top. Markov chain Also Pimsner

Back To self-corresp. self-correspondence $X \underset{q}{\overset{p}{\rightrightarrows}} Y$ same as an oriented graph with vertices $y \in Y$ and an arrow $px \longrightarrow qx$ for $x \in X$.

9/18 Return to partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$

The correspondence ~~[illegible]~~ picture ~~[illegible]~~ shows that you can compose this corresp and its transpose to get other partial unitaries (I think). Check this

$$\begin{tikzcd}
X \times_{a,b} X^{\cancel{2}} \arrow[r, "a'"] \arrow[d, "b'"'] & X \arrow[d, "b"] \\
X \arrow[r, "a"'] & Y
\end{tikzcd}$$

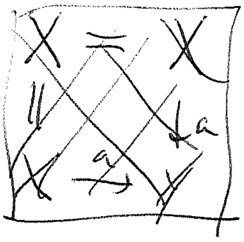
$X \underset{a,b}{\times} X \overset{\cancel{2}}{=} \{(x_1, x_2) \mid ax_1 = bx_2\}$

$a'(x_1, x_2) = x_2$

$b'(x_1, x_2) = x_1$

define $\|(x_1, x_2)\|^2 = \|x_1\|^2 = \|x_2\|^2$

Then a', b' isometric.



In general given

$$\begin{tikzcd}
X \times_Y U \arrow[r, "p_2"] \arrow[d, "p_1"'] & U \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

$f^* f = 1, \quad g^* g = 1$

define $\|(x, u)\|^2 = \|x\|^2 = \|u\|^2$
$f(x) = g(u)$

$\|x\|^2 = \|fx\|^2$, $\|u\|^2 = \|gu\|^2$, $\|fx\|^2 = \|gu\|^2$

then p_1, p_2 isometric.

So intersection seems to be O.K. What doesn't work is ~~[illegible]~~ pushout

$$\begin{tikzcd}
X \cap Y \arrow[r, "a'"] \arrow[d, "b'"'] & Z \arrow[d, "b"] \\
X \arrow[r, "a"'] & Y
\end{tikzcd}$$

$a^* a = b^* b = 1$

Then ~~[illegible]~~ is not necessarily the Hilbert space pushout, i.e. X, Z are not $\perp$ along $X \cap Y$.

Let's analyze the ~~situation~~ yet again

$Y = aX \oplus V^+ = bX \oplus V^-$ $\qquad \gamma = b^* a : X \longrightarrow X$

$1 - \gamma^* \gamma = a^* a - a^* b b^* a = a^*(1 - bb^*) a$

$$1-\gamma\gamma^* = b^*b - b^*aa^*b = b^*(1-aa^*)b$$

$$\mathrm{Ker}\,(1-\gamma\gamma^*) = \{x \mid bx \in aX\} = b^{-1}aX \qquad \gamma = b^*a$$

$$\mathrm{Ker}\,(1-\gamma^*\gamma) = \{x \mid ax \in bX\} = a^{-1}bX \qquad \gamma^* = a^*b$$

$$X' = X \times_{a,b} X = \{(x_1,x_2) \mid ax_1 = bx_2\}$$

$$\begin{tikzcd} X' = X \times_{a,b} X \arrow[r, "pr_2 = a'"] \arrow[d, "pr_1 = b'"'] & X \arrow[d, "b"] \\ X \arrow[r, "a"] & Y \end{tikzcd}$$

$$a'X' = \{x_2 \mid bx_2 \in aX\} = b^{-1}aX$$

$$b'X' = a^{-1}bX$$

$$\begin{tikzcd} \overset{b^{-1}aX}{\overset{\parallel}{a'X'}} \arrow[r, hook] \arrow[d, "b"] & X \arrow[r] \arrow[d, "b"] & X/a'X' \arrow[d, "b", "\cong"'] \\ aX \arrow[r, hook] & Y \arrow[r] & Y/aX \end{tikzcd}$$

$$\begin{tikzcd} aX \cap bX & bX \\ aX & Y \end{tikzcd}$$

We have

$$bX = (V^-)^\perp$$

$$aX = (V^+)^\perp$$

$$aX \cap bX = (V^+ + V^-)^\perp$$

~~You want to understand [illegible]~~ The issue here is [illegible] how to construct $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ from $X' \overset{a'}{\underset{b'}{\rightrightarrows}} X$. The former can be reconstructed from X equipped with the contraction $\gamma = b^*a$, namely

$$\begin{tikzcd} X' = \overset{a^{-1}bX}{\mathrm{Ker}(1-\gamma^*\gamma)} \arrow[r, shift right, "\sim", "\gamma"'] \arrow[d, "a' \text{ is the inclusion}"'] & \overset{b^{-1}aX}{\mathrm{Ker}(1-\gamma\gamma^*)} \arrow[l, shift right, "\gamma^*"'] \arrow[dl, "b' \text{ is } a'\gamma^*"] \\ X & \end{tikzcd}$$

?

~~The issue is how to reconstruct~~

We have three objects

1) $X \underset{b}{\overset{a}{\rightrightarrows}} Y$

2) X, γ

$\gamma = b^*a$

3) $X' \underset{b'}{\overset{a'}{\rightrightarrows}} X$

and constructions 1) → 2) ; 2) → 3).

Given X, γ set $X' = \mathrm{Ker}(1-\gamma^*\gamma)$, $a' : X' \hookrightarrow X$ the inclusion, b' = the composition

$$X' = \mathrm{Ker}(1-\gamma^*\gamma) \xrightarrow[\sim]{\gamma} \mathrm{Ker}(1-\gamma\gamma^*) \xhookrightarrow[\mathrm{inc}]{} X$$

I think the process 1) → 2) is reversible

Now you can add to above list

0) (Y, δ) δ contraction on Y

and discuss 0) → 1) as 2) → 3).

Thus given (Y, δ) we have

$$Y = \underbrace{\mathrm{Ker}(1-\delta^*\delta)}_{aX} \oplus \underbrace{\mathrm{Im}(1-\delta^*\delta)}_{V^+} = \underbrace{\mathrm{Ker}(1-\delta\delta^*)}_{bX} \oplus \underbrace{\mathrm{Im}(1-\delta\delta^*)}_{V^-}$$

$$b a^{-1} = \delta$$

So we have $(Y, \delta) \longmapsto (X \underset{b}{\overset{a}{\rightrightarrows}} Y)$. Can this be reversed, i.e. obtain δ from X, a, b? We know δ restricted to aX is ba^{-1}. So need to define δ on V^+. $\delta(1-\delta^*\delta) = (1-\delta\delta^*)\delta$

so $\delta\, \mathrm{Im}(1-\delta^*\delta) \subset \mathrm{Im}(1-\delta\delta^*)$ $\therefore \delta V^+ \subset V^-$

So $\delta : V^+ \longrightarrow V^-$ can be an arbitrary contraction such that $\mathrm{Ker}(1-\delta^*\delta) = \mathrm{Ker}(1-\delta\delta^*) = 0$.

Let's now review the steps, transitions. Recall given a unitary operator u on we get a partial unitary on any closed subspace. Probably that comes from a contraction of.

Given H, u, Y let $j : Y \longrightarrow H$ be the inclusion, $\delta = j^*uj$, $\delta^* = j^*u^{-1}j$. What is the corresponding $X \overset{a}{\underset{b}{\rightrightarrows}} Y$? Should be $X = Y \cap u^{-1}Y$, $a : X \to Y$ the inclusion, b = rest. of u.

$$1-\delta^*\delta = 1 - j^*u^{-1}jj^*uj = j^*u^{-1}(1-jj^*)uj$$

So $(1-\delta^*\delta)y = 0 \iff uy \in Y$

$$(y, j^*u^{-1}(1-jj^*)uy)$$
$$\|$$
$$(uy, \underbrace{(1-jj^*)uy}_{\text{proj of } uy \text{ onto } Y^\perp})$$

so $\mathrm{Ker}(1-\delta^*\delta) = Y \cap u^{-1}Y$

Also $1-\delta\delta^* = j^*u(1-jj^*)u^{-1}j$

so $(1-\delta\delta^*)y = 0 \iff u^{-1}y \in Y$ $\therefore$ $\mathrm{Ker}(1-\delta\delta^*) = Y \cap uY$.

So it's clear that ~~any~~ the partial isom assoc. to H, u, Y is the p. unit assoc. to $\delta = j^*uj$ on Y.

Next what about the eigenvector equation?

$$H = Y^\perp \oplus X \oplus V^+$$
$$H = Y^\perp \oplus uX \oplus V^-$$

$$\xi = \xi^- + x_1 + v^+ = \xi^- + u(x) + v^-$$

$$\lambda\xi^- + \lambda u(x) + \lambda v^- = u(\xi^-) + u(x_1) + u(v^+)$$

projects onto $u(X)$. $\xi^- \perp Y \supset uX$ $u(\xi^-) \perp u(Y) \supset u(X)$

$v^- \perp uX$ $u(v^+) \perp u(X)$ $\therefore$ $\lambda u(x) = u(x_1)$

$\lambda x = x_1$ $\therefore$ get $\boxed{\lambda x + v^+ = u(x) + v^-}$

952 I know that the partial unitary

~~as expressed~~ $X \underset{b=ua}{\overset{a}{\hookrightarrow}} Y$ depends only on the

contraction $\delta = j^* u j$ on Y. ~~Note~~

$$ab^* = a(ua^*) = aa^*u^{-1}$$

$ba^* = uaa^*$ = projection from Y onto aX followed by u.

Maybe what happens is that $\delta = j^* u j$

$$\begin{tikzcd}
\text{is } ba^* : aX \arrow[r] \arrow[d, phantom, "\shortparallel"] & bX \arrow[d, phantom, "\shortparallel"] \\
Y \cap u^{-1}Y \arrow[r, "u"] & Y \cap uY
\end{tikzcd}$$

direct sum with $(1-bb^*)u : V^+ \longrightarrow V^-$

What questions should you be asking?

The contraction δ has ~~a~~ spectrum unlike $X \overset{a}{\underset{b}{\rightrightarrows}} Y$

~~What ... does~~ suppose $j^*(\lambda - u)(ax) = (\lambda - \delta)(ax) = 0$

i.e. you have a solution

Review. Consider (Y, δ), $\|\delta y\| \le \|y\|$.

$$Y = \mathrm{Ker}(1-\delta^*\delta) \oplus \underbrace{\mathrm{Im}(1-\delta^*\delta)}_{V^+} = \mathrm{Ker}(1-\delta\delta^*) \oplus \underbrace{\mathrm{Im}(1-\delta\delta^*)}_{V^-}$$

$$\begin{tikzcd}
\mathrm{Ker}(1-\delta^*\delta) \arrow[d, phantom, "\shortparallel"] & \mathrm{Ker}(1-\delta\delta^*) \arrow[d, phantom, "\shortparallel"] \\
aX \arrow[r, shift left, "\delta"] & bX \arrow[l, shift left, "\delta^*"]
\end{tikzcd}$$

Let $X = \mathrm{Ker}(1-\delta^*\delta)$, let $a : X \longrightarrow Y$ inclusion

~~by~~ let $b : X \longrightarrow Y$ be $\mathrm{Ker}(1-\delta^*\delta) \xrightarrow[\sim]{\delta} \mathrm{Ker}(1-\delta\delta^*) \subset Y$

so that $ba^{-1} = \delta \mid X$. Also have

δ, δ^* between $\mathrm{Ker}(1-\delta^*\delta) = \mathrm{Ker}(1-\delta\delta^*)$

953 Review (Y, δ) $\|\delta y\| \leq \|y\|$.

$$Y = \mathrm{Ker}(1-\delta^*\delta) \oplus \underbrace{\mathrm{Im}(1-\delta^*\delta)}_{V^+} = \mathrm{Ker}(1-\delta\delta^*) \oplus \underbrace{\mathrm{Im}(1-\delta\delta^*)}_{V^-}$$

$$\mathrm{Ker}(1-\delta^*\delta) \xrightarrow[\delta]{\sim} \mathrm{Ker}(1-\delta\delta^*)$$

$$\begin{tikzcd} \underset{V^+}{\mathrm{Im}(1-\delta^*\delta)} \arrow[r, shift left, "\delta"] & \underset{V^-}{\mathrm{Im}(1-\delta\delta^*)} \arrow[l, shift left, "\delta^*"] \end{tikzcd}$$

here $\|\delta v^+\| < \|v^+\|$ for $v^+ \neq 0$.

Recall how X, γ dilates to a $Y = \overline{aX + bX}$ $\gamma = b^*a$

$$\|ax_1 + bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (\gamma x_1, x_2) + (x_2, \gamma x_1)$$

$$= \|x_2 + \gamma x_1\|^2 + (x_1, (1-\gamma^*\gamma)x_1)$$

$$Y = aX \oplus (1-\gamma^*\gamma)^{1/2}X = bX \oplus (1-\gamma\gamma^*)^{1/2}X$$

Put $\delta = ab^*$. Then $1-\delta^*\delta = 1 - ba^*ab^* = 1 - bb^*$ $=$ projection onto $(1-\gamma\gamma^*)^{1/2}X$ and $1-\delta\delta^* = 1 - ab^*ba^*$ $= 1 - aa^*$ $=$ proj on $(1-\gamma^*\gamma)^{1/2}X$. Can also say that $\mathrm{Ker}(1-\delta^*\delta) = \mathrm{Ker}(1-bb^*) = bX$ and that $\mathrm{Ker}(1-\delta\delta^*) = \mathrm{Ker}(1-aa^*) = aX$. You do indeed get the ~~situation~~ where $\delta^*\delta$ and $\delta\delta^*$ are projections

What I eventually hope to do is to calculate the S operator. So far I have defined S for a partial unitary.

954 Review. It seems that a partial unitary is equivalent to a contraction δ such $\delta^*\delta$ and $\delta\delta^*$ are projections. Namely given $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ take $\delta = ba^*$ on Y. Then $\delta^*\delta = ab^*ba^* = aa^*$ = proj on aX and $\delta\delta^* = ba^*ab^* = bb^*$ = proj on bX. In the other direction given $\delta \ni \delta^*\delta$ and $\delta\delta^*$ are projections, we have

$$Y = \cancel{Im(\delta^*\delta)} \oplus Im(1-\delta^*\delta) = \cancel{Im(\delta\delta^*)} \oplus Im(1-\delta\delta^*)$$

$$aX = Ker(1-\gamma^*\gamma) \xrightarrow{\delta/aX} Ker(1-\delta\delta^*) = bX$$

~~[illegible]~~ This construction works for ~~in~~ general ~~[illegible]~~ but one has $\delta = ba^*$ exactly where $\delta : Im(1-\delta^*\delta) \to Im(1-\delta\delta^*)$ is zero.

So a general δ on Y ~~[illegible]~~ is equivalent to a partial unitary (corresp. to [redacted] the char. valu~~es~~ 1 ~~of~~ for δ. [redacted]) Together with a strict contraction $\delta : V^+ \longrightarrow V^-$.

i.e. $\|\delta y\| < \|y\|$
$\|\delta^* y\| < \|y\|$
unless $y = 0$.

You have an eigenvector equation for the partial unitary ~~X~~ $= aX \oplus V^+ \cong bX \oplus V^-$, namely $(\lambda a - b)x = -v^- + v^+$. Is there a corresp. notion for a contraction? ~~[illegible]~~

Go back to H, u, Y $\quad \delta = j^*uj \quad j : Y \hookrightarrow H$.

Try $X = Y \cap u^{-1}(Y)$. I think I showed that $X \underset{b=u}{\overset{a=in}{\rightrightarrows}} Y$ arises from $\delta = j^*uj \quad 1 - \delta^*\delta = j^*u^{-1}(1-jj^*)uj$, so $Ker(1-\delta^*\delta) = \{y \mid uy \in Y\} = Y \cap u^{-1}(Y)$.

The thing I'm aiming for ?

955 Here's a picture that's emerging for a partial unitary. $X \underset{b}{\overset{a}{\rightrightarrows}} Y$, for a contraction Y, δ,

April 4, 98

Let review what we learned yesterday about partial unitaries + contraction.

If δ is a contraction on Y, then one gets a partial unitary by restricting δ to where it's ~~unitary~~ an isometry, i.e. $\mathrm{Ker}(1-\delta^*\delta)$. If $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ is a partial unitary, then $\delta = ba^*$ is a contraction of a special type, namely, $\delta^*\delta = ab^*ba^* = aa^*$, $\delta\delta^* = bb^*$ are projections, [illegible] which means that ~~[illegible]~~ [illegible] non isometric part of δ going from $\mathrm{Ker}(1-\delta^*\delta)$ to $\mathrm{Im}(1-\delta\delta^*)$ vanishes.

The splitting of a contraction into ~~on~~ isometric and nonisometric parts should work more generally for [illegible] $\delta : Y_1 \longrightarrow Y_2$. The basic result is then that any ~~What / Suppose That~~ contraction split uniquely into a partial unitary and a contraction without any isometric part. This is what happens generally for $\delta : Y_1 \to Y_2$. I think the only extra thing one can say is that char values > 0 can be pushed to $\underline{1}$, as well as 0, so that there are two partial isometries assoc. to δ.

Next suppose $Y_1 = Y_2 = Y$. What should you ask? What's important? ~~[illegible]~~ Let δ be a contraction on Y. Wait. In the past you went from partial unitary to $S(\lambda)$. ?

If you have a δ on Y $\ni$ $\delta^*\delta, \delta\delta^*$ proj., then

956 you know there is an equivalent partial unit.

$$Y = aX \oplus V^+ = bX \oplus V^- \qquad \delta = ba^*$$

so you have an $S(\lambda)$. There should be a formula for S involving only δ, in fact $S(\lambda)$ should be the ~~[illegible]~~ component of the resolvent $(\lambda - \delta)^{-1}$. ~~Component of resolvent~~

$$(\lambda a - b)x = -v^- + v^+$$

$$(\lambda - a^*b)x = -a^*v^-$$

$$x = (\lambda - a^*b)^{-1}(-a^*)v^-$$

$$v^+ = v^- - (\lambda a - b)a^*(\lambda - ba^*)^{-1}v^-$$

$$= [\lambda - \cancel{ba^*} - \lambda aa^* + \cancel{ba^*}](\lambda - ba^*)^{-1}v^-$$

$$\boxed{\begin{aligned} v^+ &= (1-aa^*)(1-\lambda^{-1}ba^*)^{-1}v^- & |\lambda| > 1. \\ v^- &= (1-bb^*)(1-\lambda a^*b)^{-1}v^+ & |\lambda| < 1 \end{aligned}}$$

How does one remember this? ~~You have~~

~~[illegible]~~ Suppose things are defined ~~[illegible]~~ on $|\lambda| = 1$ near the point of interest. Then the correspondence between v^+, v^- is unitary so one should be the adjoint of the other

$$(1-bb^*)(1-\lambda a^*b)^{-1}(1-aa^*)$$

957 ~~Not a good point.~~ $1-\delta^*\delta = 1-(ba^*)^*(ba^*)$
$= 1 - aa^*$

$$\delta(\lambda) = (1-\delta^*\delta)(1-\lambda^{-1}\delta)^{-1}(1-\delta\delta^*) \quad : V^- \to V^+ \quad |\lambda|>1$$

$$\delta^*(\lambda) = (1-\delta\delta^*)(1-\lambda\delta^*)^{-1}(1-\delta^*\delta) \quad : V^+ \to V^- \quad |\lambda|<1.$$

So given the contraction δ on Y of the special type, $\delta(\lambda)$, $\delta^*(\lambda)$ are the outside matrix coefficients of the resolvents for δ, δ^*.

Question, Do you want to get involved with adding to the partial unitary ba^{-1} a general contraction h from V^+ to V^-. So δ becomes $ba^* + h(1-aa^*)$

Go back to $X \overset{a}{\underset{b}{\rightrightarrows}} Y$. You have contraction of $\delta = ba^*$ which is $ba^{-1} : aX \xrightarrow{b} bX$ extended by 0 to V^+. Then corresponding partial unitary $Y = \text{Ker}(1-\delta^*\delta) \oplus \text{Im}(1-\delta^*\delta)$ $= \text{Ker}(1-\delta\delta^*) \oplus \text{Im}(1-\delta\delta^*)$ is the original one, since $\delta^*\delta = ab^*ba^* = aa^* =$ proj on aX etc. ~~Question~~

You also have the contraction $\gamma =$ a^*b ~~a^*b~~ ? which arises when you reconstruct $\overline{aX+bX} \subset Y$.

Motivation: A contraction γ on X yields a partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ where $Y =$ completion of $X \oplus X$ with norm

$$\|ax_1 + bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + ($$

How to set up. Suppose $X = Y \cap u^{-1}Y \underset{u|X}{\overset{a = \text{inc.}}{\rightrightarrows}} Y$

$bx = u(ax)$ $\qquad \gamma = a^*b = a^*ua$

958 Take $\gamma = a^*b$. $\gamma^*\gamma = b^*aa^*b$

$\|\gamma x\| = \|x\| \iff bx \in aX \iff x \in b^{-1}aX$

$$\begin{tikzcd}
X' = X \times_{a,b} X \arrow[r, "pr_2 = a'"] \arrow[d, "b' = pr_1"'] & X \arrow[d, "b"] \\
X \arrow[r, "a"] & Y
\end{tikzcd}$$

$X' = \{(x_1, x_2) \mid ax_1 = bx_2\}$

$a'(x_1, x_2) = x_2$

$b'(x_1, x_2) = x_1$

Ultimately you end up with $a'X' = b^{-1}aX$ and $b'X' = a^{-1}bX$. If $x_2 \in b^{-1}aX$, i.e. $bx_2 = ax_1$, then $x_1 = a^*ax_1 = a^*bx_2$, so

Recall following idea.

$$Y \leftleftarrows X \leftleftarrows X \times_{a,b} X \leftleftarrows X \times_{a,b} X \times_{a,b} X \leftleftarrows$$

$$X \times_{a,b} X \times_{a,b} X = \{(x_1, x_2, x_3) \mid ax_1 = bx_2, ax_2 = bx_3\}$$

I need an organizing principle. Maybe try starting with H, u, Y, $X = Y \cap u^{-1}(Y)$, $a : X \xhookrightarrow[\text{incl}]{} Y$, $b : X \longrightarrow Y$ rest. of u. ~~so~~ that $bx = uax$, $\forall x \in X$.

[redacted]

Start again with H, u, Y, $X = Y \cap u^{-1}(\cancel{X})$, $X \overset{a}{\underset{b}{\rightrightarrows}} Y$

$ax = x$ $bx = ux$.

$$\begin{tikzcd}
X' = X \times_{a,b} X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d, "b"] \\
X \arrow[r, hook, "a"] & Y
\end{tikzcd}$$

$X' = b^{-1}aX = X \cap u^{-1}X$

$= Y \cap u^{-1}Y \cap u^{-1}Y \cap u^{-2}Y$

$= Y \cap u^{-1}Y \cap u^{-2}Y$

What are you doing? You have $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ and have formed

$$\begin{tikzcd} X' = X \times_{a,b} X \arrow[r, "a' = pr_2"] \arrow[d, "b' = pr_1"'] & X \arrow[d, "b"] \\ X \arrow[r, hook, "a"] & Y \end{tikzcd}$$

$a'(x_1, x_2) = x_2$

$X' = b^{-1}aX = X \cap a^{-1}X$

What δ on X is such that $\mathrm{Ker}(1-\delta^*\delta) = aX$? Ans.

$\delta = ba^*$ $\qquad \delta^*\delta = ab^*ba^* = aa^* = 1$ on aX.

What γ on X is such that $\mathrm{Ker}(1-\gamma^*\gamma) = a'X' = b^{-1}aX$

$\boxed{\gamma = a^*b}$ for $\gamma^*\gamma = b^*aa^*b = 1$ on $b^{-1}aX$.

Review: You are given $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ and form $X' = X \times_{a,b} X$. Identify X with aX and $x' = (x_1, x_2)$ with $a'x' = x_2 \in b^{-1}aX$. Then $\gamma = a^*b$ is a contraction on X such that $\mathrm{Ker}(1-\gamma^*\gamma) = b^{-1}X$. $\qquad \gamma^*\gamma = b^*aa^*b$ is 1 on $b^{-1}aX$.

What is the general situation? I have $X \overset{}{\underset{b}{\rightrightarrows}} Y$

a = inclusion
a^* = projection on X
b = restriction of

$X' = b^{-1}X$ ~~is where~~ is the subspace on which $\gamma = a^*b$ is an isom.

$Y = aX \oplus V^- \neq bX \oplus V^+$

$bx \in X \iff bx \perp V^+$
$\iff x \perp b^*V^+$

960 This stuff is still not clear as I would like. I have $X \underset{b}{\overset{\hookrightarrow}{\rightarrow}} Y$ with $b^*b = 1_X$, and I get a sequence of subspaces of X namely $X, b^{-1}X, b^{-2}X, b^{-3}X, \ldots$

Note: $b^{-2}X = \{x \in X \mid bx \in X \text{ and } b^2x \in X\}$. Note $X \supset b^{-1}X \supset b^{-2}X \supset \ldots$. Next let $V^+ = (X)^\perp$.

Then $bx \in X \Longleftrightarrow bx \perp V^+ \Longleftrightarrow x \perp b^*V^+$

$$0 = (bx, V^+) = (x, b^*V^+)$$

Given y we have $y \in X \Leftrightarrow y \perp V^+$

and $y \in X,\ by \in X \Longleftrightarrow y \perp V^+ + b^*V^+$

$y \in X,\ by \in X,\ b^2y \in X \Longleftrightarrow y \perp V^+ + b^*V^+ + (b^*)^2V^+$

Start with $X \underset{b}{\overset{\hookrightarrow}{\rightarrow}} Y$ $\quad b^*b = 1$.

$$Y = X \oplus V^+ = bX \oplus V^-$$

~~Wait. Go back to $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\quad a^*a = 1_X = b^*b$~~

~~get $Y = aX \oplus V^+ = bX \oplus V^-$.~~

~~Let $\delta = ba^*$ $\quad b^{-1}X = \{x \mid bX \in$~~

Consider

$$b^{-1}X = \{x \in X \mid bx \in X\} = \{x \in X \mid \overset{x \perp b^*V^+}{\overline{bx \perp V^+}}\} = (b^*V^+)^\perp \cap X$$

$$= (V^+ + b^*V^+)^\perp.$$

$$\begin{tikzcd} b^{-1}X \arrow[r, phantom, "\subset"] \arrow[d, "b'"'] & X \arrow[d, "b"] \\ X \arrow[r, phantom, "\subset"] & Y \end{tikzcd} \qquad \begin{tikzcd} b(b^{-1}X) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & bX \arrow[d, phantom, "\cap"] \\ X \arrow[r, phantom, "\subset"] & Y \end{tikzcd}$$

$V^- = (bX)^\perp = \mathrm{Ker}(b^*)$

$0 = (y, bX) = (b^*y, X) \Rightarrow b^*y = 0$

V^+

$$b(b^{-1}X) = (V^+ + V^-)^\perp$$

961 $b(b^{-1}X) = \{bx \mid x\in b^{-1}X\}$

$= \{bx \mid x, bx\in X\}. = \{x_2 \mid \exists x_1 \;\; bx_1 = x_2\}$

$X\cap bX = \{x_2\in X \mid \exists x_1 \;\; x_2 = bx_1\}$

So what seems to happen is that

$b^{-1}X$ is the subspace of X $\perp$ b^*V^+

$b(b^{-1}X)$ ———————— $\perp$ a^*V^-

Try different approach – You might do differently Various approaches.

April 5 ~~Still the [illegible]~~

Consider H, u, Y sat.

$Y \neq uY$ $\quad Y^1 = Y$ $\quad Y^2 = Y\cap u^{-1}Y$ $\quad Y^3 = Y\cap u^{-1}Y\cap u^{-2}Y$

There's probably a "good" alg. situation here with $\mathbb{Z}$ symmetry. Notice that decreasing $Y^1, Y^2, Y^3, --$ goes via orthogonal complements into an increasing system for $Y^\perp$

Go back over orth polynomials and try to generalize.

$L^2(S^1, d\mu)$ $\quad \int d\mu = 1.$ $\quad p_0 = q_0 = 1.$

$F_n = \langle z^0, \cdots, z^n\rangle$ $\quad p_n \in z^n + F_{n-1}$ $\quad p_n \perp F_{n-1}$

$q_n \in 1 + zF_{n-1}$ $\quad q_n \perp zF_{n-1}$

Then

$$\left.\begin{array}{l} p_n - zp_{n-1} \in F_{n-1} \\ p_n - zp_{n-1} \perp zF_{n-2} \end{array}\right\} \Rightarrow \begin{array}{c} p_n - zp_{n-1} = h_n q_{n-1} \\ h_n = p_n(0) \end{array}$$

$$\left.\begin{array}{l} q_n - q_{n-1} \in zF_{n-1} \\ q_n - q_{n-1} \perp zF_{n-2} \end{array}\right\} \Rightarrow q_n - q_{n-1} = h_n' z p_{n-1}$$

Orth. relation.

$$p_n - h_n q_{n-1} = z p_{n-1} \qquad \|p_n\|^2 + |h_n|^2 \|q_{n-1}\|^2 = \|p_{n-1}\|^2$$

$$q_n - h'_n z p_{n-1} = q_{n-1} \qquad \|q_n\|^2 + |h'_n|^2 \|p_{n-1}\|^2 = \|q_{n-1}\|^2$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ h'_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Try to generalize to $\overbrace{Y \cap u^{-1} Y}^{X}$ $\quad Y \quad$ $\overbrace{Y + uY}^{Z}$

$$F_{n-2} \qquad F_{n-1} \qquad F_n$$

Let p_0, q_0 be unit vectors in Y such that $p_0 \perp X$, $q_0 \perp uX$. Let $p_1 \in up_0 + Y \subset Z$, $p_1 \perp Y$. Let $q_1 \in q_0 + uY$, $q_1 \perp uY$. Then

$$\left.\begin{array}{l} p_1 - up_0 \in Y \\ p_1 - up_0 \perp uX \end{array}\right\} \Rightarrow p_1 - up_0 = hq_0$$

$$\left.\begin{array}{l} q_1 - q_0 \in uY \\ q_1 - q_0 \perp uX \end{array}\right\} \Rightarrow q_1 - q_0 = h' u p_0$$

$$p_1 - hq_0 = up_0 \qquad \|p_1\|^2 + |h|^2 = 1$$

$$q_1 - h' u p_0 = q_0 \qquad \|q_1\|^2 + |h'|^2 = 1.$$

So I am starting with $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\quad Y = aX \oplus V^+ = bX \oplus V^-$ where $V^+ = \langle p_0 \rangle$, $V^- = \langle q_0 \rangle$, $\|p_0\| = \|q_0\| = 1$. Z should be determined by Y Wait. The critical thing is Y and uY need not ~~be~~ intersect perpendicularly.

$$\begin{tikzcd} X \arrow[r, hook, "a"] \arrow[d, "b"'] & Y \arrow[d, "u"] \\ Y \arrow[r, hook] & Z \end{tikzcd}$$

963 2 dim'l space $Z \ominus X$ with orthog. bases [illegible] up_0, q_1 and q_0, p_1

~~Diagonalize what~~

Point missed Z is determined by Y and the contraction operator δ given by u, which should be $\gamma = ba^*$ on aX and a contraction of $V^+ \to V^-$. ~~So~~ Now $V^+ = \langle p_0 \rangle$ and the contraction $V^+ \to V^-$ is induced by u, i.e. $\delta p_0 = q_0 (q_0, up_0)$

Go over all this again. $H, u, Y,$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[r, shift right=1.5, "u"'] & Y \arrow[r, hook] \arrow[r, shift right=1.5, "u"'] & Z \\ Y \cap u^{-1} Y \arrow[u, equal] & Y \arrow[u, equal] & Y + uY \arrow[u, equal] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook, "(p_0)"] \arrow[d, "(q_0)"'] & Y \arrow[d, "u", "(q_1)"'] \\ Y \arrow[r, hook, "(p_1)"'] & Z \end{tikzcd}$$

two orthogonal bases for $Z \ominus X$ namely p_0, q_1 and q_0, p_1. $u(q_0)$ ~~is~~ not used.

Can you describe δ? δ is

$$\begin{tikzcd} Y \arrow[r, "u"] & Z \arrow[r, "pr"] & Y \\ X \arrow[u, phantom, "\cup"] \arrow[r, "u"] & Y \arrow[u, phantom, "\cup"] \arrow[ur, "1"'] & \end{tikzcd}$$

$\delta = u$ on X need then δ on p_0. But

$u(p_0) = p_1 - h q_0$ so $\delta p_0 = -h q_0$

$\boxed{(q_0, up_0) = -h}$

$q_1 = h' up_0 + q_0$

$(q_0, q_1) = h'(q_0, up_0) + \|q_0\|^2$

$= h'(-h) + 1$

orth

$q_1 - h' up_0 = q_0$

$\cancel{(q_1, up_0)} - \overline{h'} = (q_0, up_0)$

Check this: You have $q_1 - h' up_0 = q_0$ apply $(up_0, \ )$

$0 - h'(up_0, up_0) = (up_0, q_0)$ $\quad -h' = (up_0, q_0) = \overline{(q_0, up_0)} = -\overline{h}$

964 Review. Consider H, u, Y, $X = Y \cap u^{-1}Y$, $Z = Y + uY$

so
$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "u|X"'] & Y \arrow[d, "u|Y"] \\ Y \arrow[r, hook] & Z \end{tikzcd}$$
is bicartesian. Assume both indices ~~are~~ 1. Let

$Y = X \oplus \langle p_0 \rangle$ $\quad \|p_0\| = 1$

$Y = uX \oplus \langle q_0 \rangle$ $\quad \|q_0\| = 1$

Then $Z = Y + \langle up_0 \rangle = uY + \langle q_0 \rangle$.

Let $p_1 \in up_0 + Y$ $\quad p_1 \perp Y$

$q_1 \in q_0 + uY$ $\quad q_1 \perp uY$

Then $p_1 - up_0 \in Y$ and is $\perp$ to uX $\quad (p_1 \perp Y \supset uX$; $up_0 \perp uX)$

so $p_1 - up_0 = hq_0$ $\quad h = (q_0, hq_0) = \cancel{(q_0, p_1)} - (q_0, up_0)$

$\|p_1\|^2 + |h|^2 = \|p_0\|^2 = 1$ $\quad \therefore h = -(q_0, up_0)$

Also $q_1 - q_0 \in uY$ and is $\perp$ to uX $\quad \left(\begin{matrix} q_1 \perp uY \supset uX \\ q_0 \perp uX \end{matrix} \right)$

so $q_1 - q_0 = h' up_0$

$\overline{h'} = (h'up_0, up_0) = (q_1 - q_0, up_0) = -(q_0, up_0) = h$

Thus
$$p_1 = up_0 + hq_0$$
$$q_1 = \bar{h} up_0 + q_0$$
$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\|p_1\|^2 = \|q_1\|^2 = 1 - |h|^2.$$

I don't think $X = Y \cap u^{-1}Y$ was used. Enough is X, uX codim 1 in Y and. $Z = Y + uY$.

So you could have $Y + uY = Y \Rightarrow uY \subset Y$

Then $uY = Y$ or $(uY = uX)$ NO so u unitary and $h = 1$.

965 All this has to be written up, but the key point in my mind is the doubling that seems to arise when you think of coupling 2-ports. This might be related to real structure - maybe Dominic Joyce's stuff.

Suppose you're given $(h_n)_{n\in\mathbb{Z}}$ ∋ $|h_n| < 1$ all n. You look for a Hilbert space with orth basis $\tilde{p}_n$ and also $\tilde{q}_n$.

Basic equations are

$$\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \frac{1}{\sqrt{1+|h_n|^2}} \begin{pmatrix} u\tilde{p}_{n-1} \\ \tilde{q}_{n-1} \end{pmatrix}$$

What happens actually?

$$\begin{tikzcd} X \arrow[r, hook, "p_0"] \arrow[r, shift right=2, "u"'] & Y \\ & q_0 \end{tikzcd}$$

$p_0 \in Y,\ p_0 \perp X$. $p_1 \in u(p_0) + Y$ $\quad p_1 \perp Y$

$q_0 \in Y,\ q_0 \perp uX$ $\quad q_1 \in q_0 + uY$ $\quad q_1 \perp uY$

$$p_1 - up_0 = hq_0 \qquad 0 - (q_0, up_0) = h$$

$$q_1 - q_0 = h'up_0 \qquad 0 - (q_0, up_0) = (h'up_0, up_0) = \overline{h'}$$

Introduce $\|p_1\|^2 + |h|^2 = 1$.

$$\|p_1\|\tilde{p}_1 - up_0 = hq_0$$

$$\|q_1\|\tilde{q}_1 - q_0 = \bar{h}\,up_0$$

$$\begin{pmatrix} \tilde{p}_1 \\ \tilde{q}_1 \end{pmatrix} = \frac{1}{\sqrt{1-|h|^2}} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} up_0 \\ q_0 \end{pmatrix}.$$

$$\therefore \begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \frac{1}{\sqrt{1-|h_n|^2}} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} u\tilde{p}_{n-1} \\ \tilde{q}_{n-1} \end{pmatrix}$$

966 ~~[illegible]~~ Is it possible to eliminate the g's.

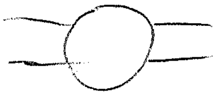
Doubling procedure $\left(\text{Related to } \left(\frac{1+X}{\sqrt{1-X^2}}\right)^2 = \frac{1+X}{1-X} \ ? \right)$.

Introduce $u^{1/2}$ somehow. Question; Given u on H is $\begin{pmatrix} & 1 \\ u & \end{pmatrix}$ on $H^{\oplus 2}$ unitary: $\begin{pmatrix} & 1 \\ u & \end{pmatrix}^* = \begin{pmatrix} 0 & u^{-1} \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}\begin{pmatrix} 0 & u^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$ OKAY.

So the idea will to ~~[illegible]~~ double the Hilbert space H, there should be a $\mathbb{Z}_2$ action.



$d = a \quad c = -\bar{b}$

$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^-$$
$$u(\delta_1^-) = c\delta_1^+ + d\delta_0^-$$

$$Y = \langle \delta_0^+, \delta_1^-, \delta_0^-, \delta_1^+ \rangle$$

$$X = \langle \delta_0^+, \delta_1^- \rangle \qquad uX = \langle \delta_1^+, \delta_0^- \rangle$$

observe $Y = X \overset{\perp}{\oplus} uX$ so that $V^+ = uX$ and $V^- = X$. $S(\lambda)$ should be the constant op. u or u^{-1} possibly multiplied by $\lambda^\alpha, \lambda^\gamma$

$$(\lambda - u)(x) = -v^+ + v^-$$

$$y = x_1 + v^+ = u(x) + v^-$$

$$\lambda y = \lambda u(x) + \lambda v^-$$

$$u(y) = u(x_1) + u(v^+)$$

$\therefore x_1 = \lambda x$

$\lambda x + v^+ = u(x) + v^-$

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$$x = \psi_0^+\delta_0^+ + \psi_1^-\delta_1^- ,$$

$$ux = \psi_0^+\left(a\delta_1^+ + b\delta_0^-\right) + \psi_1^-\left(c\delta_1^+ + d\delta_0^-\right)$$

$$(\lambda - u)x = \overline{-v^+ + v^-} = \left(a\psi_0^+ + c\psi_1^-\right)\delta_1^+ + \left(b\psi_0^+ + d\psi_1^-\right)\delta_0^-$$

so $+v^- = \lambda x$ so $S(\lambda)v^+ = \lambda u^{-1} v^+$

$-v^+ = -ux$

$$\begin{tikzcd} S(\lambda) = \lambda u^{-1} : V^+ \arrow[r] & V^- \arrow[d, phantom, "\shortparallel"] \\ uX & X \end{tikzcd}$$

Another example. Take Y 3 diml X 1 dim.

Rough idea: $S(\lambda)$ for $|\lambda| = 1$ is unitary valued but converts to $U(1,1)$ when $V^\pm$ are polarized

$$0 \to \mathcal{O}(-1)\otimes X \longrightarrow \mathcal{O}\otimes Y \longrightarrow \overset{2}{E} \longrightarrow 0$$

E rank 2 degree 1 $E = \mathcal{O}\oplus\mathcal{O}(1)$

$(\lambda - u)(x) \subset X + u(X)$, so you have an orth. line.

i.e. $X + u(X) < Y$.

First ~~suppose~~ consider $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $a = \begin{pmatrix}1\\0\end{pmatrix}$ $b = \begin{pmatrix}h\\1\end{pmatrix}$

$b^* = (\bar{h}\ 1)$

$$(\lambda a - b)(x) = \begin{pmatrix}\lambda - h\\1\end{pmatrix}x = -\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\-h\end{pmatrix}\overset{c}{}$$

$$\left(\bar{h}\lambda - |h|^2 + 1\right)x = -1$$

$$x = \frac{-1}{\bar{h}(\lambda - h) + 1}$$

$$\begin{pmatrix}\lambda - h\\1\end{pmatrix}\frac{-1}{\bar{h}(\lambda - h) + 1} = -\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\-\bar{h}\end{pmatrix} c$$

$$-\frac{\lambda-h}{\lambda\bar h+1-|h|^2}=c$$

$$-\frac{1}{\lambda\bar h+1-|h|^2}=-1-\bar h c$$

$$1+\bar h\frac{(-1)(\lambda-h)}{\lambda\bar h+1-|h|^2}=\frac{\cancel{\lambda\bar h}+1-\cancel{|h|^2}-\cancel{\bar h\lambda}+\cancel{|h|^2}}{\lambda\bar h+1-|h|^2}$$

$$\begin{pmatrix}1\\-\bar h\end{pmatrix}\frac{1}{\sqrt{1-|h|^2}}\,c\sqrt{1-|h|^2}$$

$$S(\lambda)=-\frac{\lambda-h}{\lambda\bar h+1-|h|^2}\sqrt{1-|h|^2}\qquad b^*=\frac{1}{\sqrt{\ }}\begin{pmatrix}\bar h & 1\end{pmatrix}$$

$$\begin{tikzcd} X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Y \end{tikzcd}\qquad a=\begin{pmatrix}1\\0\end{pmatrix}\qquad b=\begin{pmatrix}h\\1\end{pmatrix}\frac{1}{\sqrt{1+|h|^2}}$$

$$(\lambda a-b)x=\begin{pmatrix}\lambda-h/\sqrt{\ }\\-1/\sqrt{\ }\end{pmatrix}x=-\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1/\sqrt{\ }\\-\bar h/\sqrt{\ }\end{pmatrix}c$$

$$\frac{\bar h(\lambda-h/\sqrt{\ })-1/\sqrt{\ }}{\sqrt{\ }}x=-\frac{1}{\sqrt{\ }}$$

$$x=\frac{-1}{\bar h\lambda-\sqrt{1+|h|^2}}$$

$$\left(\lambda-h/\sqrt{1+|h|^2}\right)\frac{-1}{\bar h\lambda-\sqrt{1+|h|^2}}=\frac{1}{\sqrt{1+|h|^2}}c$$

$\in SU(1,1)$

$$c=\frac{-\lambda\sqrt{1+|h|^2}+h}{\bar h\lambda-\sqrt{1+|h|^2}}=\begin{pmatrix}\sqrt{1+|h|^2} & -h\\-\bar h & +\sqrt{1+|h|^2}\end{pmatrix}(\lambda)\ ?$$

h bad symbol

$$c=\frac{-\lambda\sqrt{1+|h|^2}+h}{\bar h\lambda-\sqrt{1+h^2}}=\frac{\sqrt{1+|h|^2}\,\lambda-h}{\sqrt{1+|h|^2}-\bar h\lambda}$$

969 April 6.

Review. $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ yields $\underbrace{\delta_0 = ba^*}_{}$ special contraction on Y to which we can add a contraction $V^+ \xrightarrow{h} V^-$

What you need to do ultimately is to work out the relation between partial unitaries and contractions. You want to be able to proceed up and down, and to understand exactly what happens.

For example, suppose given (Y, δ) you get

$$\begin{tikzcd}[column sep=small]
Y = & \mathrm{Ker}(1-\delta^*\delta) & \oplus & \mathrm{Im}(1-\delta^*\delta) & = & \mathrm{Ker}(1-\delta\delta^*) & \oplus & \mathrm{Im}(1-\delta\delta^*) \\
& aX \arrow[rrrr, bend right, "\delta_0 = ba^*"'] & \oplus & V^+ \arrow[rrrr, bend right, "\delta_h"'] & & bX & \oplus & V^-
\end{tikzcd}$$

What contraction do you consider on $X = aX$? Probably a^*b. Leave for the moment

Above just went from Y, δ down. Next discuss going up to $Y \rightrightarrows Z$, Picture is

$$\begin{tikzcd}
H, u, Y, \; X = Y \cap u^{-1}Y \arrow[r, hook, shift left, "a"] \arrow[r, shift right, "u_X"'] & Y \arrow[r, hook, shift left, "j"] \arrow[r, shift right, "u_Y"'] & \underbrace{Y + uY}_{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, hook, "a"] \arrow[d, "u_X"'] & Y \arrow[d, "u_Y"] \\
Y \arrow[r, hook, "j"'] & Z
\end{tikzcd}$$

$\delta = j^* u_Y$

Check: $\|\delta y\| = \|y\|$ if $u(y) \in Y$ i.e. $y \in Y \cap u^{-1}Y$

$u_Y a = j u_X$

$j^* u_Y a = j^* j u_X = u_X$

$\delta a = u_X$

$Z = Y \oplus W^+ = u_Y Y \oplus W^-$

970 $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ Assume $Y = \overline{aX + bX}$ $\gamma = a^*b$

$$\|ax_1 + bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (ax_1, bx_2) + (bx_2, ax_1)$$

$$(x_1, \gamma x_2) + (\gamma x_2, x_1)$$

$$= \|x_1 + \gamma x_2\|^2 + (x_2, (1-\gamma^*\gamma)x_2)$$

$$= \|x_2 + \gamma^* x_1\|^2 + (x_1, (1-\gamma\gamma^*)x_1)$$

$$Y = aX \oplus V^+ = bX \oplus V^-$$

$$\|y\|^2 = \|a^*y\|^2 + ?$$

$y = ax_1 + bx_2$ $a^*y = x_1 + \gamma x_2$

$$(1-aa^*)y = (1-aa^*)bx_2$$

$$\|(1-aa^*)y\|^2 = ([illegible] \; bx_2, (1-aa^*)bx_2)$$

$=$

$y = ax_1 + bx_2$ $a^*y = x_1 + \gamma x_2$

$b^*y = \gamma^* x_1 + x_2$

$$y = \underbrace{aa^*y}_{a(x_1 + \gamma x_2)} + \underbrace{(1-aa^*)y}_{(1-aa^*)bx_2}$$

You want to map x_2 to $-a\gamma x_2 + bx_2$

$\phi(x) = -a\gamma x + bx$ $a^*\phi(x) = 0$

$$\|\phi x\|^2 = (\cancel{-a\gamma x} + bx, -a\gamma x + bx)$$

$$= (x, -b^*a\gamma x + x)$$

$$= (x, (1-\gamma^*\gamma)x)$$

In principle you could work this out.

~~So basically you want~~ to

971 Review: $$\begin{tikzcd} X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Y \end{tikzcd}$$ $Y = \overline{aX + bX}$, $\gamma = a^*b$

$$\|ax_1 + bx_2\|^2 = \|x_1 + \gamma x_2\|^2 + (x_2, (1-\gamma^*\gamma)x_2)$$

$$= \|\gamma^* x_1 + x_2\|^2 + (x_1, (1-\gamma\gamma^*)x_1)$$

The first expression says that $Y \simeq X \oplus (1-\gamma^*\gamma)^{1/2}X$

You want to make this isom. explicit.

$$y = ax_1 + bx_2 \xmapsto{a^*} x_1 + \gamma x_2$$

$$ax \longleftarrow\!\shortmid \; x \in X$$

$$y \xmapsto{a^*} x_1 + \gamma x_2 \longmapsto a(x_1 + \gamma x_2) = aa^*y$$

$$(1 - aa^*)(ax_1 + bx_2) = (1 - aa^*)bx_2 = a(-\gamma x_2) + bx_2$$

Define $\phi : X \longrightarrow Y$, $\phi(x) = (-a\gamma + b)x$

Then $a^*\phi(x) = (-\gamma + a^*b)x = 0$

and $\|\phi(x)\|^2 = (-a\cancel{\gamma x} + bx, -a\gamma x + bx) = (x, (-\gamma^*\gamma + 1)x)$

$$= \|(1-\gamma^*\gamma)^{1/2}x\|^2$$

Thus what you learn is

$$Y \simeq X \oplus (1-\gamma^*\gamma)^{1/2}X$$

$$y \longmapsto (a^*y \; ,$$

$$X \oplus \overline{(1-\gamma^*\gamma)^{1/2}X} \longrightarrow Y$$

$$x_1 + (1-\gamma^*\gamma)^{1/2}x_2 \longmapsto ax_1 + (b - a\gamma)x_2$$

$(x_1' + \gamma x_2')$ $\qquad$ $ax_1' + bx_2'$

$+($

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$$X \oplus \overrightarrow{(1-\gamma^*\gamma)^{1/2} X} \xrightarrow{\text{isom.}} Y$$

$$x_1 + (1-\gamma^*\gamma)^{1/2} x_2 \longmapsto a x_1 + (b - a\gamma) x_2$$

[illegible]

To show onto let $y = a x_1' + b x_2'$, then

$$y = a(x_1' + \gamma x_2') + (b - a\gamma) x_2'$$

comes from $(x_1' + \gamma x_2') + (1-\gamma^*\gamma)^{1/2} x_2'$ on the left.

Similarly we have

$$[redacted] \; (1-\gamma\gamma^*)^{1/2} X \oplus X \xrightarrow{\text{isom}} Y$$

$$(1-\gamma\gamma^*)^{1/2} x_1 + x_2 \longmapsto (a - b\gamma^*) x_1 + b x_2$$

~~So~~

$$\begin{aligned} & a x_1' + b x_2' \\ & \qquad \| \\ & (a - b\gamma^*) x_1' + b(\gamma^* x_1' + x_2') \end{aligned}$$

Put together

$$X \oplus \overline{(1-\gamma^*\gamma)^{1/2} X} \xrightarrow{\sim} \overline{(1-\gamma\gamma^*)^{1/2} X} \oplus X$$

$$\begin{tikzcd} x_1 + (1-\gamma^*\gamma)^{1/2} x_2 \arrow[r, mapsto] \arrow[d, mapsto] & (x_1 - \gamma x_2) + (\gamma^* x_1 + (1-\gamma^*\gamma) x_2) \\ a x_1 + (b - a\gamma) x_2 & \end{tikzcd}$$

$$\begin{aligned} & \qquad \| \\ & a(x_1 - \gamma x_2) + b x_2 \\ & (a - b\gamma^*)(x_1 - \gamma x_2) + b(\gamma^*(x_1 - \gamma x_2) + x_2) \end{aligned}$$

973 April 7, $a^*a = b^*b = 1$

Take $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ such that $\overline{aX + bX} = Y$, show how Y, a, b can be constructed from X, γ $\gamma = a^*b$. Y is the completion of $X + X$ with respect to a suitable herm. inner product:

$$\|ax_1 + bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (ax_1, bx_2) + (bx_2, ax_1)$$

$$= \|x_1 + \gamma x_2\|^2 + (x_2, (1-\gamma^*\gamma)x_2)$$

$$= \|\gamma^* x_1 + x_2\|^2 + (x_1, (1-\gamma\gamma^*)x_1)$$

If $y = ax_1 + bx_2$, then $a^*y = x_1 + \gamma x_2$

$$y = aa^*y + (1-aa^*)y$$

so $(1-aa^*)y = ax_1 + bx_2 - a(x_1 + \gamma x_2)$

$$= (b - a\gamma)x_2$$

$$\|y\|^2 = \underset{\|}{\|a^*y\|^2} + \|(1-aa^*)y\|^2$$

$$= \|x_1 + \gamma x_2\|^2 + (x_2, (1-\gamma^*\gamma)x_2)$$

$$\therefore \|(b-a\gamma)x_2\|^2 = \|(1-aa^*)y\|^2$$

Actually $((b-a\gamma)x_2, (b-a\gamma)x_2)$ $\qquad a^*(b-a\gamma) = \underset{=0}{\gamma - \gamma}$

$$= (bx_2, (b-a\gamma)x_2)$$

$$= (x_2, (1-\gamma^*\gamma)x_2).$$

974 So you have

$$X \oplus \overline{(1-\gamma^*\gamma)^{1/2}X} \xrightarrow{\sim} Y$$

$$x_1' + (1-\gamma^*\gamma)^{1/2}x_2' \longmapsto ax_1' + (b-a\gamma)x_2'$$

$$(x_1+\gamma x_2) + (1-\gamma^*\gamma)^{1/2}x_2 \longleftarrow\!\shortmid (ax_1 + bx_2)$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$$

$$a(x_1+\gamma x_2) + (b-a\gamma)x_2$$

Similarly you have

$$\overline{(1-\gamma\gamma^*)^{1/2}X \oplus X} \xrightarrow{\sim} Y$$

$$(1-\gamma\gamma^*)^{1/2}x_1' \oplus x_2' \longmapsto (a-b\gamma^*)x_1' + bx_2'$$

$$(1-\gamma\gamma^*)^{1/2}x_1 \oplus \gamma^* x_1 + x_2 \longleftarrow\!\shortmid ax_1+bx_2$$

$$\parallel$$

$$(a-b\gamma^*)x_1 + b(\gamma^* x_1 + x_2)$$

So what is the contraction δ on Y

Review. Back to $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\quad \overline{aX+bX} = Y \quad \gamma = a^*b$

$$\|ax_1+bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (ax_1, \overset{\gamma}{\downarrow} bx_2) + (bx_2, ax_1)$$

$$= \|x_1+\gamma x_2\|^2 + (x_2, (1-\gamma^*\gamma)x_2)$$

$$= \|\gamma^* x_1 + x_2\|^2 + (x_1, (1-\gamma\gamma^*)x_1)$$

$$ax_1 + bx_2 \xmapsto{aa^*} a(x_1+\gamma x_2)$$

$$ax_1 + bx_2 = a(x_1+\gamma x_2) + \overbrace{(b-a\gamma)x_2}^{\text{has norm}^2 =}$$

are orthogonal $\quad a^*(b-a\gamma) = a^*b - \gamma = 0.$

Thus

$$\boxed{\begin{aligned} X \oplus \overline{(1-\gamma^*\gamma)^{1/2}X} &\xrightarrow{\sim} Y \\ x_1' \oplus (1-\gamma^*\gamma)^{1/2}x_2' &\longmapsto ax_1' + (b-a\gamma)x_2' \end{aligned}}$$

875 And similarly

$$\boxed{\begin{array}{l} (1-\gamma\gamma^*)^{1/2}X \oplus X \xrightarrow{\sim} Y \\ (1-\gamma\gamma^*)^{1/2}x_1' \oplus x_2' \longmapsto (a-b\gamma^*)x_1' + bx_2' \end{array}}$$

What is the contraction operator $\gamma = ba^*$ on Y.

The unitary part is $ba^{-1} : aX \longrightarrow bX$. [illegible]

[illegible] What sort of relation

Try to recover the general picture. Go back to H, u, Y and

$$\begin{tikzcd} X \arrow[r, hook, shift left, "a"] \arrow[r, shift right, "b=u_X"'] & Y \arrow[r, hook, shift left, "a'"] \arrow[r, shift right, "b'=u_Y"'] & Z \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook, "a"] \arrow[d, "u_X=b"'] & Y \arrow[d, "u_Y=b'"] \\ Y \arrow[r, hook, "a'"] & Z \end{tikzcd}$$

$X = Y\cap u^{-1}Y$, $Z = Y + uY$

Eigenvector equation $\quad Y = X\oplus V^+ = V^-\oplus uX$

$$H = Y^\perp \oplus X \oplus V^+ = Y^\perp \oplus uX \oplus V^-$$

$$\xi = \xi^- + x_1 + v^+ = \xi^- + u(x) + v^-$$

$$\lambda\xi = \lambda\xi^- + \lambda u(x) + \lambda v^-$$

$$u(\xi) = u(\xi^-) + u(x_1) + u(v^+)$$

project onto $u(X)$. kills $Y^\perp$ as $uX \subset Y$

V^- as $V^- \perp uX$

$u(Y^\perp)$ as $uX \perp u(Y^\perp)$

$u(V^+)$ as $uV^+ \perp uX$.

get $x_1 = \lambda x$ yielding

$$\lambda x + v^+ = u(x) + v^-$$

$$\boxed{(\lambda - u)(x) = -v^+ + v^-}$$

976 ~~Go back to the beginning~~

Problem: From (X, γ) you construct $X \overset{a}{\underset{b}{\rightrightarrows}} Y$, with $V^+ = \mathrm{Ker}(a^*) = (1-\gamma\gamma^*)^{1/2}X$ and $V^- = \mathrm{Ker}(b^*) = (1-\gamma\gamma^*)^{1/2}X$. Then you have

$S(\lambda): V^+ \longrightarrow V^-$ defd & analytic for $|\lambda|<1$, and

$S^{-1}(\lambda): V^- \longrightarrow V^+$ ———— $|\lambda|>1$, ..

$$(\lambda a - b)x = -v^+ + v^-$$

$$(\lambda b^* a - 1)x = -b^* v^+$$

$$x = (1-\lambda b^* a)^{-1} b^* v^+$$

$$v^- = v^+ + (\lambda a - b)(1-\lambda \underbrace{b^* a}_{\gamma^*})^{-1} b^* v^+$$

Can you understand $(1-\lambda ab^*)^{-1}$ on Y ?

April 8 Continue Consider

$X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\qquad Y = aX \oplus V^+ = V^- \oplus bX$. Form (H, u)

$$z^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{} \oplus zV^+ +$$
$$\oplus \overbrace{V^- \oplus bX}^{}$$

$$H = \bigoplus_{n\le -1}^{(2)} z^{+n}V^- \oplus \overbrace{aX \oplus V^+}^{Y} \oplus \bigoplus_{n\ge 1}^{(2)} z^n V^+$$

Assume $\overline{aX + bX} = Y$. Then (H, u) is generated by aX or by $uaX = bX$. H completion of $\bigoplus_{n\in\mathbb{Z}} z^n X$ $= \mathbb{C}[z, z^{-1}] \otimes X$ with

$$(x_1, z^n x_2) = (x_1, \gamma^n x_2) \qquad n \ge 0$$
$$= (x_1, \gamma^{*n} x_2) \qquad n \le 0.$$

$$H = H^- \oplus X \oplus H^+$$

$$uH^+ \subset H^+ \Rightarrow u(H^- + X) \supset H^- + X$$
$$\Rightarrow u^{-1}(H^- + X) \subset H^- + X$$

$$uH^+ \subset H^+$$
$$uX \subset X + V^+ \subset X + H^+$$
$$u^{-1}H^- \subset H^-$$
$$u^{-1}X \quad ?$$

Idea. Given (X, γ) you get H by completing $\mathbb{C}[z, z^{-1}] \otimes X$ wrt $\|\sum z^n x_n\|^2 =$ whatever arises $(x, u^n x')_H = \begin{cases} (x, \gamma^n x')_X & n \geq 0 \\ (x, \gamma^{*|n|} x')_X & n \geq 0 \end{cases}$

Can decompose orthogonally

$$H = \underbrace{\overset{(2)}{\bigoplus_{n \leq -1}} z^n (1 - \gamma\gamma^*)^{1/2} X}_{H^-} \oplus X \oplus \underbrace{\overset{(2)}{\bigoplus_{n \geq 0}} z^n (1 - \gamma^*\gamma)^{1/2} X}_{H^+}.$$

why because this generalizes

$$Y = V^- \oplus bX = aX \oplus V^+$$

details.

$$\|\sum u^n x_n\|^2 = \sum_{p,q} (x_p, u^{q-p} x_q)$$

$$= \sum_{p,n} (x_p, u^n x_{p+n})$$

First thing to do would be to discuss $\mathbb{C}[z] \otimes X$.

$$\|\sum_{n \geq 0} z^n x_n\|^2 = \|x_0 + z \overbrace{\sum_{n \geq 1} z^{n-1} x_n}^{w}\|^2$$

$$= \|x_0\|^2 + \|w\|^2 + (x_0, zw) + (zw, x_0)$$

There's some identity here to be found.

978 But what are you trying to do? What picture?

$$L^- = L^2(S^1, \overbrace{(1-\gamma\gamma^*)^{1/2}X}^{V^-}) \longleftrightarrow H \longleftrightarrow L^2(S^1, \overbrace{(1-\gamma^*\gamma)^{1/2}X}^{V^+}) = L^+$$

What you will get here is $H = H^{\text{bound}} \oplus \overline{L^+ + L^-}$.
and $\overline{L^+ + L^-}$ will be described by a contraction between L^+ and L^-. The contraction must be given by $S(\lambda)$ L^∞ function on S^1 values in contractions $V^+ \to V^-$.

You want to work out the details. So where to start. Let's examine the situation of $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ where at least the scattering seems simpler. $Y = aX \oplus V^+ = V^- \oplus bX$ o.d.s.

$$H = \quad \cdots \oplus z^{-1}V^- \oplus \overbrace{\boxed{\begin{array}{c} aX \oplus V^+ \\ \| \\ V^- \oplus bX \end{array}}}^{Y} \oplus zV^+ \oplus \cdots$$

eigenvector equation $\xi = \xi' + ax_1 + v^+ = \xi' + bx + v^-$

$$\lambda\xi = \lambda\xi' + \lambda bx + \lambda v^- \qquad \lambda x = x_1.$$

$$u(\xi) = u(\xi') + bx_1 + u(v^+)$$

$$(\lambda a - b)x = -v^+ + v^-$$

Take $\| x_0 + zx_1 + z^2x_2 + \cdots \|^2$

$$= \| x_0 + \gamma x_1 + \gamma^2 x_2 + \cdots \|^2 \quad + \quad ?$$

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$$\overbrace{\left(x_0 + z x_1 + z^2 x_2 + \cdots\right)}^{\xi} =$$

$$x_0 + z\overbrace{\left(x_1 + z x_2 + \cdots\right)}^{\xi'}$$

$$\|\xi\|^2 = \|x_0\|^2 + \|\xi'\|^2 + \left(x_0, z\xi'\right) + \left(z\xi', x_0\right)$$

$$\left(x_0, z\xi'\right) = ($$

$$\left(x_0, z^{n+1} x_n\right) = \left(x_0, \gamma^{n+1} x_n\right.$$

$$\|\xi\|^2 = \|x_0\|^2 + \|\xi'\|^2 + \left(x_0, \gamma x_1 + \gamma^2 x_2 + \cdots\right) + \left(\gamma x_1 + \gamma^2 x_2 + \cdots, x_0\right)$$

$$\|\xi\|^2 = \|x_0 + \gamma x_1 + \gamma^2 x_2 + \cdots\|^2$$

$$+$$

$$\xi^0 = x_0 + z\,\xi^1$$

Assume $\|\gamma\| < 1$.

~~$\|x_0 + x_1 z + x_2 z^2\|^2 = \|x_0\|^2 + \|x_1\|^2 + \|x_2\|^2$~~

$$\left(u^n x, u^m x'\right)_H = \left(x, u^{m-n} x'\right)_H$$

$$= \int z^{m-n}\left(x, \left(\sum_{k \geq 1} (\gamma^*)^k z^{+k} + \sum_{k\geq 0} \gamma^k z^{-k}\right) x'\right)_X$$

$$= \int z^{m-n}\left(x, \underbrace{\left(\frac{\gamma^* z}{1-\gamma^* z} + \frac{1}{1-\gamma z^{-1}}\right)}_{} x'\right)$$

$$\frac{1}{1-\gamma^* z}\Big(\underbrace{\gamma^* z(1-\gamma z^{-1}) + 1-\cancel{\gamma^* z}}_{1-\gamma^*\gamma}\Big)\frac{1}{1-\gamma z^{-1}}$$

980 Thus given $f,g \in \mathbb{C}[z,z^{-1}]\otimes X$ you have

$$\big(f(u),g(u)\big)_H = \int \left(f(z), \frac{1}{1-\gamma^* z}(1-\gamma^*\gamma)\frac{1}{1-\gamma z^{-1}}\, g(z)\right)_X \frac{d\theta}{2\pi}$$

$$= \int\left((1-\gamma^*\gamma)^{1/2}\frac{1}{1-\gamma z^{-1}}f(z),\ (1-\gamma^*\gamma)^{1/2}\frac{1}{1-\gamma z^{-1}}g(z)\right)\frac{d\theta}{2\pi}$$

Other possibility is

$$\frac{\gamma^* z}{1-\gamma^* z}+\frac{1}{1-\gamma z^{-1}} = \frac{1}{1-\gamma z^{-1}}\Big(\underbrace{(1-\gamma z^{-1})\gamma^* z + (1-\gamma^* z)}_{1-\gamma\gamma^*}\Big)\frac{1}{1-\gamma^* z}$$

$$\big(f(u),g(u)\big)_H = \int\left(\left((1-\gamma\gamma^*)^{1/2}\frac{1}{1-\gamma z^{-1}}f(z),\ (\quad)^{1/2}\frac{1}{1-\gamma z^{-1}}g(z)\right)\right)_X\frac{d\theta}{2\pi}$$

So what does this tell me? You get isometric maps.

$$H \longrightarrow L^2(S^1,X)$$

$$f \longmapsto (1-\gamma^*\gamma)^{1/2}\frac{1}{1-\gamma z^{-1}}f(z)$$

$$x \longmapsto (1-\gamma^*\gamma)^{1/2}\frac{1}{1-\gamma z^{-1}}\,x \quad \in H^2_-$$

Notice that ~~as long as~~ provided $\frac{1}{1-\gamma z^{-1}}$ extends to $|z|=1$, we get

$$H \longrightarrow L^2(S^1, (1-\gamma^*\gamma)^{1/2}X) \qquad \text{isometric}$$

$$x \longmapsto (1-\gamma^*\gamma)^{1/2}\frac{1}{1-\gamma z^{-1}}\,x \quad \in H^2_-(S^1,V^-)$$

What else

$$H \longrightarrow L^2(S^1, (1-\gamma\gamma^*)^{1/2}X)$$

$$x \longmapsto (1-\gamma\gamma^*)^{1/2}\frac{1}{1-\gamma^* z}\,x \quad \in H^2_+(S^1,V^+)$$

What next? All this should amount to a simple transform.

981 $\frac{1}{1-\gamma^* z}x =$

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$$\frac{1}{1-\gamma z^{-1}}(1-\gamma\gamma^*)\frac{1}{1-\gamma^* z}$$

April 9. General picture: Given X, γ take corresp. dilation $H, u, j : X \to H$ such that $j^* u^n j = \gamma^n$ $n \geq 0$. $H =$ completion of $\mathbb{C}[z,z^{-1}]\otimes X = L^2(S^1, d\mu)$ $d\mu$ is $\mathcal{L}(X)$-valued measure on S^1. Have decomp

$$H = H^2_-(S^1,V^-)\oplus X \oplus H^2_+(S^1,V^+)$$

$V^+ = \overline{(1-\gamma^*\gamma)^{1/2}X}$

$V^- = \overline{(1-\gamma\gamma^*)^{1/2}X}$

$$H = H^{bound}\oplus \overline{L^2(S^1,V^-)+L^2(S^1,V^+)}$$

So we are approaching a picture of a Hilbert space (module?) constructed from a contraction $S : L^2(S^1,V^-) \longrightarrow L^2(S^1,V^+)$. $S = S(\lambda)$ analytic for $|\lambda| > 1$. Point $S \in L^\infty(S^1, Hom(V^-,V^+))$ extends analytically outside S^1. Formula to prove is

$$S(\lambda)v = (1-aa^*)(1-\lambda^{-1}ba^*)^{-1}v^- .$$ W

$aX + V^+$

$V^- + bX$

$$v^- = aa^*v^- + (1-aa^*)v^-$$

$$u(v^-) = ba^*v^- + z(1-aa^*)v^-$$

$$u(v^-) = aa^*ba^*v^- + (1-aa^*)(ba^*v^- + zv^-)$$

$$u^2(v^-) = aa^*(ba^*)^2v^- + (1-aa^*)((ba^*)^2v^- + z(ba^*)v^- + z^2v^-)$$

$$\lim_{n\to\infty} z^{-n}u^n(v^-) = (1-aa^*)(1 + z^{-1}ba^* + z^{-2}(ba^*)^2 + \dots)v^-$$