886 March 21, 98

Let's review. Treating a contraction $h : H^+ \to H^-$

You can write $h = k^* j$, where $H^+ \xrightarrow{j} Y \xleftarrow{k} H^-$, $Y =$ completion of $H^+ \oplus H^-$ under norm $\| j\xi_1 + k\xi_2\|^2$.

$$Y = H^+ \oplus \left(\sqrt{1-hh^*}\, H^-\right)^{\text{closure}} = H^- \oplus \left(\sqrt{1-h^*h}\, H^+\right)^{\text{closure}}.$$

Extreme cases: ① $\begin{pmatrix} 1 \\ h \end{pmatrix} H^+$ is a polarization of the pseudo-herm. space $H^+ \oplus H^-$, which means

$$H^+ \oplus H^- = \begin{pmatrix} 1 \\ h \end{pmatrix} H^+ + \begin{pmatrix} h^* \\ 1 \end{pmatrix} H^-$$

i.e. $\begin{pmatrix} 1 & h^* \\ h & 1 \end{pmatrix}$ invertible. Then $\begin{pmatrix} 1 & -h^* \\ -h & 1 \end{pmatrix}$ is invertible ~~hence~~ $\begin{pmatrix} 1-h^*h & 0 \\ 0 & 1-hh^* \end{pmatrix}$ is invertible, and conversely.

$\|h\|^2 = \|h^*h\| < 1$ $\quad\therefore \|h\| = \|h^*\| < 1$.

In this case $\begin{pmatrix} (1-h^*h)^{-1/2} & h^*(1-hh^*)^{-1/2} \\ h(1-h^*h)^{-1/2} & (1-hh^*)^{-1/2} \end{pmatrix}$ is a pseudo-unitary operator carrying the $\varepsilon$ polarization to the other one.

② $h$ unitary

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This is the treatment of a contraction. The next case to handle is with $\lambda$ present. Here you have a contraction $L^2(S^1, V_1) \longrightarrow L^2(S^1, V_2)$ commuting with $z$ mult.

Besides the pseudo-unitary you have the unitary relating $H^+ \oplus \sqrt{1-hh^*}\, H^- = \sqrt{1-h^*h}\, H^+ \oplus H^-$

namely $\cancel{\begin{pmatrix} h & -\sqrt{1-hh^*} \\ \sqrt{1-h^*h} & h^* \end{pmatrix}}$ $\begin{pmatrix} \sqrt{1-h^*h} & -h^* \\ h & \sqrt{1-hh^*} \end{pmatrix}$

$X = \begin{pmatrix} 0 & -h^* \\ h & 0 \end{pmatrix}$ is skew adjoint, and $0 \le -X^2 \le 1$

so $\sqrt{1+X^2} + X$ is unitary

$$\left(\sqrt{1+X^2} + X\right)\left(\sqrt{1+X^2} - X\right) = 1$$

Do we learn anything?

 ~~[illegible]~~ So consider a strict contraction $h: L^2(S^1, V') \longrightarrow L^2(S^1, V'')$ commuting with $z$.

Better to go back to situation where given $Y = aX \oplus V^+ = bX \oplus V^-$. Form

$$H = \cdots \oplus z^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{V^- \oplus bX} \oplus zV^+ \oplus \cdots$$

Eigenvector eqn. $\xi = \cdots + \lambda z^{-1} v^- + \underbrace{\lambda ax + v^+}_{\underbrace{\shortparallel}_{v^- + bx}} + \lambda^{-1} z v^+ + \cdots$

$$(\lambda a - b)x = -v^- + v^+$$

We have $L^2(S^1, V^-) \longrightarrow H \longleftarrow L^2(S^1, V^+)$

For each $v^-$ you get $S(z)v^- \in L^2(S^1, V^+)$ perp. to $z^n V^+$ for $n \geq 1$, so $S(z)v^-$ is anal for $|z| > 1$.

Know $S(\lambda)v^- = (1 - aa^*)(\lambda - ba^*)^{-1}(1 - bb^*)v^-$. ~~This is~~

~~[illegible] Assume now that~~

Assume to simplify that $\|S\| \leq 1 - \varepsilon$ and that $H$ is gen. by $V^+, V^-$. It should be possible to split off the bound states - certainly the ~~[illegible]~~ orth to $L^2(S^1, V^-) \oplus L^2(S^1, V^+)$ is contained in $X$ and invariant under $u, u^{-1}$ etc. ~~Don't [illegible]~~

~~Know you have $L^2(S^1, V^+) \to L^2(S^1, V^-)$~~

So you have $S: \underbrace{L^2(S^1, V^-)}_{H_1^{\#}} \longrightarrow \underbrace{L^2(S^1, V^+)}_{H_2^{\#}}$ strict cont.

and you know

$$H = \overbrace{H_1}^{L^2(S^1, V^-)} \oplus (1 - S^*S)^{1/2} H_2 = (1 - SS^*)^{1/2} H_1 \oplus \overbrace{H_2}^{L^2(S^1, V^+)}$$

What's happening. $H$ ~~has~~ roughly 4 pieces 2 in 2 out. What you are missing?

At this point you know everything. The analyticity of $S$ tells you that ~~the~~

$$\bigoplus_{n\leq -1} z^n V^- \perp \bigoplus_{n\geq 0} z^n V^+$$

and allows you recover $aX$ as the $\perp$ complement to these, etc.

At this point it seems I can recover from $S$ the partial unitary $V = aX \oplus V^+ = V^- \oplus bX$, assuming no bdd states.

I have the feeling that more about $X$ might be obtainable. Now look at the case where $S$ is unitary, i.e nothing is lost inside $X$. Specifically you want to consider $S$ rational function of $\lambda$ $\ni$ $S(\lambda)$ is unitary for $|\lambda| = 1$, analytic for $|\lambda| > 1$. ~~Specifically~~

First take the case $S(\lambda)$ unitary for $|\lambda| = 1$.

Then $L^2(S^1, V^-) \xrightarrow{\sim} H \xleftarrow{\sim} L^2(S^1, V^+)$

$$v^- \longmapsto S v^- \in \bigoplus_{n\leq 0} z^n V^+$$

because the image in $H$ of $V^-$ is $\perp$ to $\bigoplus_{n>0} z^n V^+$.

This gets funny. Assume $H = L^2(S^1, V^+)$. Then you have $S : L^2(S^1, V^-) \longrightarrow L^2(S^1, V^+)$

So $(Sf)(\lambda) = S(\lambda) f(\lambda)$ where $S(\lambda) : V^- \to V^+$

Assuming $S$ unit

Something you missed: essential equivalence between partial unitary and contraction. Given $X, h$ you get $Y =$ ??

889 Given a contraction $h$ on $X$ your partial unitary is:

$$Y = X \oplus \sqrt{1-hh^*}X \simeq \sqrt{1-h^*h}X \oplus X$$

So what's the assertion? Basic statement is that given a partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ such that $\overline{aX+bX} = Y$, then $Y$ is the dilation assoc. to the contraction $b^*a$ on $X$.

Begin with $X_1 \xrightarrow{a} Y \xleftarrow{b} X_2$ $\qquad a^*a=1 \quad b^*b=1$

~~$ba^*$~~:

$$\begin{tikzcd} X_1 \arrow[rr, "h"] \arrow[dr, "j"'] & & X_2 \\ & Y \arrow[ur, "k^*"'] & \end{tikzcd}$$

$\|h\| \leq 1.$

represent where $\begin{matrix} j : X_1 \to Y \\ k : X_2 \to Y \end{matrix}$ isometries

If $\overline{jX_1 + kX_2} = Y$, then $Y, j, k$ unique up to canon. isom. One has

$$\overline{jX_1 \oplus (1-kk^*)^{1/2}X_2} \xrightarrow{\sim} Y$$

$$\|$$

$$\overline{kX_2 \oplus (1-jj^*)^{1/2}X_1} \xrightarrow{\sim} Y$$

Now given an isom $X_1 \xrightarrow{\sim} X_2$, then we have a partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ with $Y = \overline{aX+bX}$

What happens? Your response function is something simple. Back to

$$\cdots + z^{-1}V^- + aX + V^+ + zV^+ + \cdots$$

$$V^- + bX^+$$

$$(\lambda a - b)x = -v^+ + v^-$$

eigen.

$$+ (\lambda z^{-1})^2 v^- + \lambda z^{-1} v^- + \lambda a x + v^+$$

$$v^- + bx + \lambda^{-1} z v^+ +$$

890 The new point is that a contraction operator on $X$ yields a ~~response~~ scattering operator ~~[illegible]~~ the ~~[illegible]~~ parts of $X$ ~~where~~ $h$ shrinks norm. To get this straight you want first $h: X_1 \to X_2$ then we have $\overline{(1-h^*h)^{1/2}X_1} \underset{h^*}{\overset{h}{\rightleftarrows}} \overline{(1-hh^*)^{1/2}X_2}$. ~~So you look at something~~

Go back to $Y = \overset{X_1}{\widehat{aX}} \oplus V^+ = \overset{X_2}{\widehat{bX}} \oplus V^-$

and assume $Y = \overline{X_1 + X_2}$.

$$\|jx_1 + kx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (\overset{h}{k^*j}\,x_1, x_2) + (x_2, \overset{h}{k^*j}\,x_1)$$
$$= \|x_2 + hx_1\|^2 + \|x_1\|^2 - \|hx_1\|^2$$

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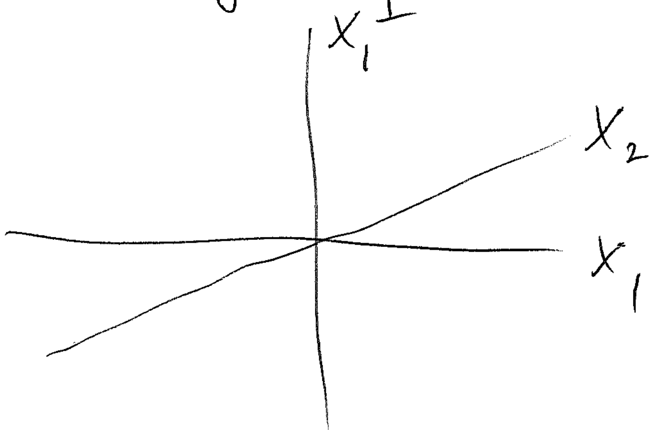
Somehow your response function amounts to a contraction operator.

$$Y = aX \oplus V^+ = bX \oplus V^-$$

First thing to analyze is

$$X_1 \xrightarrow{j} X \xleftarrow{k} X_2 \qquad j^*j = 1, \quad k^*k = 1$$

essentially the same as two involutions on $X$.




So what?

this is in Y notat.

$h = ba^*$

~~[illegible]~~

$h^*h = ab^*ba^* = aa^*$

$hh^* = ba^*ab^* = bb^*$.

891 ~~The idea is maybe that~~

Given $h: X \to X$ a contraction operator one constructs a partial unitary

$$Y = X \oplus \sqrt{1-h^*h} \quad ?$$

Start the other way. Suppose given $aX \oplus V^+ = bX \oplus V^- = Y$ such that $aX + bX = Y$. Then $Y$ is the completion of $aX \oplus bX$ for the norm.

$$\|ax_1 + bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (x_1, a^*bx_2) + (a^*bx_2, x_1)$$

$$= \|x_1 + a^*bx_2\|^2 + \underbrace{\|x_2\|^2 - \|a^*bx_2\|^2}$$

$$b^*(1-aa^*)b \quad \text{———} \quad \left\{ 1 - (a^*b)^*(a^*b) \atop {\| \atop b^*aa^*b} \right.$$

~~YES!!!~~

Start again with $H, u, Y$, $X = Y \cap u^{-1}(Y)$

$$Y = aX \oplus V^+ \qquad V^+ = Y \ominus X$$

$$= bX \oplus V^- \qquad V^- = Y \ominus u(X)$$

$$\xi = \xi^- + ax_1 + v^+ = \xi^- + bx_2 + v^-$$

$$u(\xi) = u(\xi^-) + bx_1 + u(v^+) = \lambda\xi = \lambda\xi^- + \lambda bx_2 + \lambda v^-$$

$$\underbrace{bx_1 - \lambda bx_2}_{\in u(X)} + u(\xi^-) + u(v^+) - \lambda\xi^- - \lambda v^-$$

$u(Y^\perp)$

$\cap$

$u(X)^\perp$

$\therefore \quad x_1 = \lambda x_2$

892 If $\overline{aX+bX}=Y$, then $Y$ seems to be constructed completely from $X$ and the contraction $a^*b$.

~~The scattering is.~~ $$ax_1+bx_2 = a(x_1+a^*bx_2)+(1-aa^*)bx_2$$

$$\begin{aligned} \cap\ \ Y &= aX + V^+ \\ \cup\ \ &= bX + V^- \end{aligned}$$

$$ax_1+bx_2 \doteq b(x_2+b^*ax_1)+(1-bb^*)ax_1$$

Suppose $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ partial unitary $\ni$ $\overline{aX+bX}=Y$.

$$Y = aX \oplus V^+ \qquad = bX \oplus V^-$$

$$ax_1+bx_2 = a(x_1+a^*bx_2)+(1-aa^*)bx_2 \qquad = b(b^*ax_1+x_2)+(1-bb^*)ax_1$$

~~the dual~~

$$(\lambda a-b)\,x = -v^+ + v^-$$

$$(\lambda - a^*b)\,x = +a^*v^- \qquad (\lambda ~~ba^*~~)-(\lambda\, aa^* ~~ba^*~~)$$

$$x = (\lambda-a^*b)^{-1}a^*v^-$$

$$v^+ = \underbrace{v^-}_{(1-bb^*)ax_1} - (\lambda a-b)(\lambda-a^*b)^{-1}a^*\underbrace{v^-}_{(1-bb^*)ax_1}$$

$$=\lambda(1-aa^*)(\lambda-ba^*)^{-1}\overbrace{(1-bb^*)ax_1}^{v^-}$$

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Try different notation. Suppose given $\gamma : X\to X$ $\|\gamma\|<1$. Set $Y=X^{\oplus 2}$, unitary auto of $Y$:

$$\begin{pmatrix} \gamma & -(1-\gamma\gamma^*)^{1/2} \\ (1-\gamma^*\gamma)^{1/2} & \gamma^* \end{pmatrix} \quad \begin{pmatrix} \gamma^* & (1-\gamma^*\gamma)^{1/2} \\ -(1-\gamma\gamma^*)^{1/2} & \gamma \end{pmatrix}$$

So we have $\qquad X \qquad \begin{matrix} X \\ \oplus \\ V \end{matrix}$

893 We have

$$\begin{array}{c} X \\ \oplus \\ X \end{array} \begin{array}{c} \xleftarrow{\;a=\begin{pmatrix}1\\0\end{pmatrix}\;} \\ \xleftarrow[\;b=\begin{pmatrix}\gamma\\(1-\gamma^*\gamma)^{1/2}\end{pmatrix}\;]{} \end{array} X$$

$$X^{\oplus 2} = \overbrace{\begin{pmatrix}1\\0\end{pmatrix}X}^{aX} \oplus \overbrace{\begin{pmatrix}0\\1\end{pmatrix}X}^{V^+}$$

$$= \underbrace{\begin{pmatrix}\gamma\\(1-\gamma^*\gamma)^{1/2}\end{pmatrix}X}_{bX} \oplus \underbrace{\begin{pmatrix}-(1-\gamma\gamma^*)^{1/2}\\+\gamma^*\end{pmatrix}X}_{V^-}$$

$$\lambda\begin{pmatrix}1\\0\end{pmatrix}x - \begin{pmatrix}\gamma\\(1-\gamma^*\gamma)^{1/2}\end{pmatrix}x = -\underbrace{\begin{pmatrix}0\\1\end{pmatrix}x_1}_{v^+} + \underbrace{\begin{pmatrix}-(1-\gamma\gamma^*)^{1/2}\\ \gamma^*\end{pmatrix}x_2}_{v^-}$$

Apply $a^* = (1 \;\; 0)$

$$(\lambda-\gamma)x = -(1-\gamma\gamma^*)^{1/2}x_2$$

$$x = -(\lambda-\gamma)^{-1}(1-\gamma\gamma^*)^{1/2}x_2$$

$$\begin{pmatrix}\cancel{\lambda}-\gamma\\ -(1-\gamma^*\gamma)^{1/2}\end{pmatrix}\left[-(\lambda-\gamma)^{-1}(1-\gamma\gamma^*)^{1/2}x_2\right]$$

$$= -\begin{pmatrix}0\\1\end{pmatrix}x_1 + \begin{pmatrix}-(1-\gamma\gamma^*)^{1/2}\\ \gamma^*\end{pmatrix}x_2$$

$$(1-\gamma^*\gamma)^{1/2}(\lambda-\gamma)^{-1}(1-\gamma\gamma^*)^{1/2}x_2 = -x_1 + \gamma^* x_2$$

$$x_1 = \gamma^* x_2 - (1-\gamma^*\gamma)^{1/2}\underbrace{(\lambda-\gamma)^{-1}(1-\gamma\gamma^*)^{1/2}}_{(\lambda-\gamma)^{-1}\left((1-\gamma\gamma^*)^{-1/2}\right)^{-1}}x_2$$

$$\underbrace{= \left((1-\gamma\gamma^*)^{-1/2}(\lambda-\gamma)\right)^{-1}}_{\left[(1-\gamma\gamma^*)^{-1/2}(\lambda-\gamma)(1-\gamma^*\gamma)^{-1/2}\right]}$$

 March 22, 98 Carl's B.day.

$$Y = aX \oplus V^+ = bX \oplus V^- \qquad u = ba^{-1}$$

$$(\lambda a - b)(x) = -v^+ + v^-$$

$$(\lambda - a^*b)(x) = a^*v^-$$

$$x = (\lambda - a^*b)^{-1} a^* v^- = a^*(\lambda - ba^*)^{-1} v^-$$

$$(\lambda a - b)a^* = \lambda aa^* - ba^* = \lambda - ba^* - \lambda(1 - aa^*)$$

$$(\lambda a - b)x = v^- - \lambda(1-aa^*)(\lambda - ba^*)^{-1} v^-$$

$$\therefore \quad v^+ = (1-aa^*)(1-\lambda^{-1}ba^*)\, v^-$$

$$S(\lambda) = (1-aa^*)(1-\lambda^{-1}ba^*)(1-bb^*)$$

defined + anal outside $S^1$.

what can I say about this situation? Point may be that $ba^{-1}$ when extended to $ba^*$ satisfies $(ba^*)^* ba^* = ab^*ba^* = aa^*$ and $ba^*(ba^*)^* = ba^*ab^* = bb^*$, so that $ba^*$ is a rather special kind of contraction operator on $Y$. On the other hand $a^*b$ can be a general contraction operator on $X$.

$$ba^* = \begin{pmatrix} \gamma \\ (1-\gamma^*\gamma)^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ (1-\gamma^*\gamma)^{\frac12} & 0 \end{pmatrix}$$

$$(\lambda - ba^*)^{-1} = \begin{pmatrix} \lambda - \gamma & 0 \\ -(1-\gamma^*\gamma)^{1/2} & \lambda \end{pmatrix}^{-1} = \begin{pmatrix} (\lambda-\gamma)^{-1} & 0 \\ \lambda^{-1}(1-\gamma^*\gamma)^{1/2}(\lambda-\gamma)^{-1} & \lambda^{-1} \end{pmatrix}$$

$$(1-aa^*)(\lambda - ba^*)^{-1} v^- = \begin{pmatrix} 0 & 0 \\ \lambda^{-1}(1-\gamma^*\gamma)^{1/2}(\lambda-\gamma)^{-1} & \lambda^{-1} \end{pmatrix} \begin{pmatrix} -(1-\gamma^*\gamma)^{1/2} \\ \gamma^* \end{pmatrix}$$

$$1 - bb^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma \\ (1-\gamma^*\gamma)^{1/2} \end{pmatrix} \begin{pmatrix} \gamma^* & (1-\gamma^*\gamma)^{1/2} \end{pmatrix} = \begin{pmatrix} 1-\gamma\gamma^* & \gamma(1-\gamma^*\gamma)^{1/2} \\ -(\quad)^{1/2}\gamma^* & \gamma\gamma^* \end{pmatrix}$$

895 ~~[illegible]~~ Let's get the theory straight. You have various gadgets. 1) partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ 2) contraction operator $\gamma : X^{\theta}$ 3) $S$ operator $S(\lambda): V^- \to V^+$.
How do they fit together?

Discuss $S$ rational $S(\lambda)$ unitary for $|\lambda| = 1$. analytic for $|\lambda| > 1$. $S(\lambda) : V^- \longrightarrow V^+$

$$(\lambda a - b)(x) = -S(\lambda) v^- + v^-$$

Idea: When $S(\lambda)$ unitary over $S^1$, then we have isos. $L^2(S^1, V^-) \xrightarrow{\sim} H \xleftarrow{\sim} L^2(S^1, V^+)$ assume no bdd states.

composite sends $f(\lambda) \longmapsto S(\lambda) f(\lambda)$. Thus $v^- \mapsto S(\lambda) v^-$ embeds $V^-$ into $L^2(S^1 V^+)$ into $\bigoplus_{n \le 0} z^n V^+$, in fact $S$. takes $\bigoplus_{n\le 0} z^n V^-$ ~~into~~ $\bigoplus_{n \le 0} z^n V^+$. <u>OKAY</u> so shift to $S^{-1}(\lambda) : V^+ \to V^-$. $\bigoplus_{n\le 0} z^n S V^- \subset \bigoplus_{n \le 0} z^n V^+$

$$\bigoplus_{n \le 0} z^n V^- \subset \bigoplus_{n\le 0} z^n S^{-1} V^+ \Rightarrow \bigoplus_{n>0} z^n V^- \supset \bigoplus_{n>0} z^n S^{-1} V^+$$ mult by $s$

Thus $H^+(S^1, V^-) \supset S^{-1} H^+(S^1, V^+)$. Go back to picture

$aX \oplus V^+ \oplus zV^+ \oplus \cdots$

$\oplus z^{-2} V \oplus z^{-1} V^{-1} \oplus V^- \oplus bX$

so it's $S^{-1}$: $V^+ \longrightarrow V^-$ the thing analytic inside the disk which gives the outgoing subspace

So given $S^{-1} : V^+ \to V^-$ $\qquad S^*(\lambda) = S(\lambda^*)^*$

So your basic picture before was correct namely

$$\begin{tikzcd} H^+ \arrow[r, "X"] \arrow[d, dash, "V^-"'] & SH^+ \arrow[d, dash, "V^+"] \\ zH^+ \arrow[r, "uX"] & SzH^+ \end{tikzcd} \qquad aX \oplus V^+ = V^- \oplus bX \text{ etc.}$$

896 Puzzle. Given $S(z)$ rational function $\prod \frac{z-\lambda_i}{1-\bar{\lambda}_i z}$ get

$$X = H^+ \ominus SH^+$$
$$Y = H^+ \ominus zSH^+$$

How can you interpret the cont fr. exp.

$$S = \begin{pmatrix} 1 & +h_1 \\ +\bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \ ?$$

Note that $\dim(X) = \deg(S)$. The first thing to note that $h_1 = 0 \iff S(0) = 0 \iff S = zS_1$.

~~In this case things~~ Is this the same as $V^+ \perp V^-$ ? Yes

because $V^- = \mathbb{C}1$, $V^+ = \mathbb{C}S$ and their inner produce $\int \bar{1} S \frac{d\theta}{2\pi} = S(0)$. Yes!

$$aX \oplus V^+ = V^- \oplus bX$$

If $V^+ \perp V^-$, then $V^- \subset aX$ and $V^+ \subset bX$

$$aX \oplus V^+$$

$$V^- \oplus bX$$

$$= V^- \oplus (aX \cap bX) \oplus V^+$$

A $K$-module is essentially a correspondence of a v.s. with itself. Generalizes an operator on a vector space, should $\exists$ a notion of characteristic polynomial, maybe trace, determinant. Maybe trace determinant.

Return to $aX + V^+ = bX + V^-$

assume $V^+ \perp V^-$ i.e. that $V^- \subset (V^+)^\perp = aX$

and $V^+ \subset (V^-)^\perp = bX$. ~~Then $V^- = aX + V^-$~~

Then $aX = V^- \oplus \underbrace{aX \ominus V^-}_{aX \cap (V^-)^\perp}$ $= aX \cap bX$.

so $X = \underbrace{V^- \oplus aX \cap bX}_{aX} \oplus V^+$, with $\underbrace{aX \cap bX \oplus V^+}_{bX}$

Then it seems we have

$$V^- \oplus aX \cap bX = aX$$

$$s \downarrow u$$

$$aX \cap bX \oplus V^+ = bX$$

thus we have a smaller partial unitary.

General case when $V^+, V^-$ are 1 dimensional.

If $V^+ = V^-$, then $aX = bX$ and we have a pure bound state situation.

$V^+ \neq V^-$, then $(aX + bX)^\perp = (aX)^\perp \cap (bX)^\perp$

$= V^+ \cap V^- = 0$, so $aX + bX = Y$ (stick to fin. dims). So have

$$\begin{array}{ccc} & Y & \\ aX & & bX \\ & aX \cap bX & \end{array}$$

Sit down and concentrate. $X \overset{a}{\underset{b}{\rightrightarrows}} Y$

I have to understand the structure of the K-module for $\mathcal{O}(n)$:

$$\mathcal{O}(-1) \otimes \underbrace{\Gamma(\mathcal{O}(n-1))}_{\substack{\text{basis} \\ 1, \dots, z^{n-1}}} \longrightarrow \mathcal{O} \otimes \underbrace{\Gamma(\mathcal{O}(n))}_{\substack{\text{bases} \\ 1, \dots, z^n}} \longrightarrow \mathcal{O}(n) \longrightarrow 0$$

$a = \text{inc}$
$b = \text{mult by } z$.

898 You then must explain the Hilbert space structure, the inner product.

Take $\mathcal{O}(1)$. $X \overset{a}{\underset{b}{\rightrightarrows}} Y$. In general the basic numerical invariant is $S(0) : V$

Picture

$$\begin{tikzcd} H^+ & S'H^+ \arrow[l, "aX"'] \\ zH^+ \arrow[u, "V^-"] & zS'H^+ \arrow[l, "bX"] \arrow[u, "V^+"'] \end{tikzcd}$$

$aX \oplus V^+ = V^- \oplus bX$

$\mathbb{C}S' \qquad \mathbb{C}1$

$x^2 - 2 = 0$

$$+ \; z^{-1}V^- + \overbrace{aX + V^+ + zV^+ +}^{H^+}$$

$$\underbrace{V^- + bX}_{zH^+}$$

$S' = S^{-1}$

$S : V^- \to V^+$ analytic for $|\lambda| > 1$.

$S^{-1} : V^+ \to V^-$ analytic for $|\lambda| < 1$

~~So what~~

So $Y = H^+/zS^{<}H^+ = \underbrace{H^+/zH^+}_{V^-} \oplus \underbrace{zH^+/zSH^+}_{bX}$

$= H^+/S^{<}H^+ \oplus S^{<}H^+/zS^{<}H^+$

$= aX \oplus V^+$

$\overset{\shortparallel}{S^{<}V^-}$

$\cancel{\left(\frac{m}{n}\right)^2 = 2}$

So the idea is that $S(0) = \langle 1, S' \rangle$ specifies the inner product between basic $v^- = 1$ and $S'v^- = S'$. You have $Y = aX \oplus V^+ = V^- \oplus bX$ certainly we get $Y = V^+ \oplus \underbrace{aX \cap bX}_{(V^+ \oplus V^-)^\perp} \oplus V^-$

$Y = V^- \oplus bX \supset V^+$

899 Basic question is to find a ~~port~~

<u>March 23.</u> Given

$$\begin{array}{ccc} H^+ & \xrightarrow{aX} & S^*H^+ \\ V^- \Big| & & \Big| S^*V^- = V^+ \\ zH^+ & \xrightarrow[bX]{} & zS^*H^+ \end{array}$$

$$H^+ = \bigoplus_{n\geq 0}^{(2)} z^n V^- \xrightarrow{ev_\lambda} V^-$$
$$V^+ \xrightarrow{S^*(\lambda)} V^-$$

Can you check the eigenvector equation. Pick eigenvector $\xi = v^- + \lambda^{-1}zv^- + \lambda^{-2}z^2v^- + \cdots$. Better is to take $\xi = ax_1 + v^+ = bx_2 + v^-$

Wait: Take $\xi = (1-\lambda^{-1}z)^{-1}v_0^-$ and split it into $ax +$ . Try other direction

Take $v^+ \in V^+ = S^*V^-$

Let $\xi \in \underbrace{H^+ \ominus zS^*H^+}_{Y} = aX + S^*V^- = bX + V^-$

$$\xi = ax_1 + S^*v_2^- = bx_2 + v_1^-$$

orthogonal polynomials on the circle. Basic idea

$L^2(S^1, d\mu)$ cyclic rep. of $\mathbb{Z}$ $\quad \int \bar{z}^n z^m d\mu = \phi_{m-n}$

same as pos. def function $(\phi_n)$ on $\mathbb{Z}$ with $\phi_0 = 1$. Then ~~orthonorm~~ orthonormalize $1, z, z^2, \cdots$ to get $p_0, p_1, \cdots$. $\quad p_0 = 1$.

$z^n \quad p_{n-1}$

$$\mathbb{C}p_n \overset{\perp}{\oplus} \underbrace{\{1, \ldots, z^{n-1}\}}_{F_{n-1}} = \underbrace{\{1, \ldots, z^n\}}_{F_n}.$$

$p_n \perp p_0, \ldots, p_{n-1}$

$$zp_n \perp \{z, \ldots, z^n\}$$

$$p_{n+1} \qquad zp_n$$

900 Anyway - zilch.

$p_0 = 1.$ $\quad h_1 = \langle 1, z \rangle.$ $\qquad \bar h_1 h_1$

$$\| z - h_1 \|^2 = 1 + |h_1|^2 - \langle z, h_1 \rangle - \langle h_1, z \rangle$$
$$= 1 - |h_1|^2$$

$p_1 = \dfrac{z - h_1}{\sqrt{1 - |h_1|^2}}.$ $\quad$ Set $\quad h_2 = \langle 1, z p_1 \rangle$

$z p_1 - h_2$

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$1, z, z^2, \cdots$ $\qquad \langle 1, z - h_1 \rangle = \langle 1, z \rangle - h_1 = 0$

$p_0 \; p_1$

$p_0, \; p_1 = \dfrac{z - h_1}{\sqrt{1 - |h_1|^2}}$ orthonormal

$p_0, p_1, z p_1$

$z p_1 = t_2 p_2 + t_1 p_1 + t_0 p_0$

$z^{-1} p_n \perp z p_0, \cdots, z p_{n-1}$

$\perp 1$

---

$z^{-1} p_n \perp z^{-1}, 1, \cdots, z^{n-2}$ $\qquad z p_0, \cdots, z p_{n-2} \perp p_n$

$z^{-1}, p_0, \cdots, p_{n-2}$

wait suppose you've const. $p_0, \cdots, p_n$

what about $p_{n+1}$? $\qquad p_{n+1} \in \mathbb{C} p_0 + \cdots + \mathbb{C} p_n = \mathbb{C} + \mathbb{C} z + \cdots \mathbb{C} z^{n+1}$

$+ \mathbb{C} z p_n$

$p_{n+1} =? \; z p_n + p_{n-1}$ $\qquad$ ~~Jacobi matrix~~

$c p_{n+1} = z p_n + p_n + p_{n-1}$

901 orthogonal polys on $S^1$ wrt $d\mu$

$$\underset{F_0}{\mathbb{C}1} \subset \underset{F_1}{\mathbb{C}1+\mathbb{C}z} \quad F_2 \subset \cdots \subset F_n$$

$$p_n = e_n z^n + \text{lower.} \qquad e_n > 0,\ |p_n| = 1 \quad p_n \perp F_{n-1}$$

$z p_n \in F_{n+1}$ so

$$p_{n+1} = c_0 + c_1 z + \cdots + c_n z^n + c_{n+1} z^{n+1}$$

$$z^{-1} p_{n+1} = z^{-1} c_0 + c_1 + \cdots + c_n z^{n-1} + c_{n+1} z^n$$

$$\langle z^j, z^{-1} p_{n+1} \rangle = \langle z^{j+1}, p_{n+1} \rangle = 0 \qquad \text{for } -1 \le j \le n-1$$

$$\langle p_j, z^{-1} p_{n+1} \rangle = \langle z p_j, p_{n+1} \rangle = 0 \qquad 0 \le j \le n-1$$

Take $p_{n+1}$ let $c = p_{n+1}(0)$, then

$$z^{-1} p_{n+1} = \frac{c_0}{z} + \underbrace{\text{poly of degree } n}_{a_0 + a_1 p_1 + \cdots + a_n p_n}$$

$$p_{n+1} = c_0 + a_0 z + a_1 z p_1 + \cdots + a_n z p_n$$

$$\langle p_j, z^{-1} p_{n+1} \rangle = \langle p_j, z^{-1} \rangle + a_j \qquad ?$$

~~$p_{n+1} - c p_0$~~ orth to $z, z^2, \ldots, z^n$ Look at $z p_n$

~~$z p_0, z p_1, \ldots, z p_{n-1}$~~

---

$q_0 = p_0 = 1$

$$\langle p_0, z p_0 \rangle = h_1 \qquad \langle q_0, q_0 \rangle = 1$$

$$q_1 = z - h_1 \qquad \langle q_0, q_1 \rangle = 0$$

$$\langle q_1, q_1 \rangle = \langle z, z - h_1 \rangle = 1 - |h_1|^2$$

902

$$g_2 = \cancel{\phantom{xx}}\ z g_1 + c g_0$$

$$0 = \langle g_1, g_2 \rangle = \langle g_1, z g_1 \rangle$$

$$0 = \langle g_0, g_2 \rangle = \langle g_0, z g_1 \rangle + c$$

?

$$g_2 = z g_1 - \langle g_0, z g_1 \rangle$$

---

$$z^2 = g_2 + c_1 g_1 + c_2 g_0$$

$$\langle 1, z^2 \rangle = c_2$$

$$\langle g_1, z^2 \rangle = c_1 (1 - |h_1|^2)$$

$$\langle g_2, z^2 \rangle = \|g_2\|^2$$

$$g_2 = z^2 + c_1 z + c_2$$

$$0 = \langle g_0, g_2 \rangle = \int z^2 + c_1 h_1 + c_2$$

$$\langle g_1, g_2 \rangle = \int z + c_1 + \int \bar{z} c_2$$

---

Start again with $\int z^n = \mu_n$. $n \geq 0$ $\mu_0 = 1$.

$$g_0 = 1$$

$$g_1 = z + c_1 \qquad \langle g_0, g_1 \rangle = \mu_1 + c_1 \quad \therefore c_1 = -\mu_1$$

$$g_1 = z - \mu_1$$

$$g_2 = z^2 + b_1 z + b_2 \qquad \int g_2 = \mu_2 + b_1 \mu_1 + b_2 = 0$$

$$\int \bar{z} g_2 = \mu_1 + b_1 + b_2 \bar{\mu}_1 = 0$$

903

$$b_1 = \frac{\begin{vmatrix} -\mu_2 & 1 \\ -\mu_1 & \bar\mu_1 \end{vmatrix}}{\begin{vmatrix} \mu_1 & 1 \\ 1 & \bar\mu_1 \end{vmatrix}} = \frac{-\mu_2\bar\mu_1 + \mu_1}{-|\mu_1|^2 + 1} = \frac{\mu_1 - \mu_2\bar\mu_1}{1-|\mu_1|^2}$$

$$b_2 = \cancel{\frac{\begin{vmatrix} \mu_1 & 1 \\ 1 & \bar\mu_1 \end{vmatrix}}{-1+|\mu_1|^2}} = \frac{\begin{vmatrix} \mu_1 & -\mu_2 \\ 1 & -\mu_1 \end{vmatrix}}{\begin{vmatrix} \mu_1 & 1 \\ 1 & \bar\mu_1 \end{vmatrix}} = \frac{-\mu_1^2 + \mu_2}{-|\mu_1|^2+1}$$

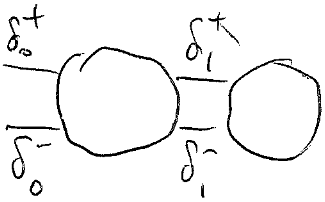
---

Let $p_0, p_1, \cdots$ be the orth sequence constructed by Gram Sch.

$\mathrm{sp}\,\{p_0, \cdots, p_n\} = \mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^n$

$$\underbrace{\mathbb{C} + \overbrace{\mathbb{C}z}^{zp_0 \quad zp_1}}_{(p_0 \quad p_1)} + \mathbb{C}z^2$$

Go back to




$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^-$$
$$u(\delta_1^-) = c\,\delta_1^+ + d\delta_0^-$$

$$\lambda\psi_1^+ = a\psi_0^+ + c\psi_1^- \qquad \psi_0^+ = \frac{\lambda}{a}\psi_1^+ - \frac{c}{a}\psi_1^-$$

$$\lambda\psi_0^- = b\psi_0^+ + d\psi_1^- \qquad \psi_0^- = \left(\frac{b}{\lambda}\left(\frac{\lambda}{a}\right)\right)\psi_1^+ + \frac{1}{\lambda}\left(d - \frac{bc}{a}\right)\psi_1^-$$

$$\begin{pmatrix} \psi_0^+ \\ \psi_0^- \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{a} & -\frac{c}{a} \\ \frac{b}{a} & \frac{1}{\lambda a} \end{pmatrix} \begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix}$$

~~$\Delta = ad - bc = 1$~~

$$\begin{pmatrix} \bar a & \bar c \\ \bar b & \bar d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ +\frac{\bar b}{a} & \frac{\lambda}{a} \end{pmatrix} \begin{pmatrix} \psi_0^+ \\ \psi_1^+ \end{pmatrix}$$

$c = -\bar b \qquad a = \sqrt{1-|b|^2}$

$$\frac{1}{a^2} + \frac{bc}{a^2} = \frac{d^2 + |b|^2}{a^2} = \frac{|a|^2}{a^2} = 1$$

$$\begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix}\begin{pmatrix}\frac{1}{a} & \frac{b}{a}\\ \frac{\bar b}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda a} & \frac{b}{a}\\ \frac{\bar b}{a} & \frac{\lambda}{a}\end{pmatrix}$$

So the real puzzle is how to relate the unitary operator from the chain of connected 2-ports to the unitary operator $z$ on $L^2(S^1, d\mu)$

Try again to understand orth poly on $S^1$. Maybe you need a symmetric treatment?

$$p_0 = 1$$
$$p_1 = \frac{z-h_1}{\sqrt{1-|h_1|^2}}$$

$$h_1 = \langle 1, z\rangle = \int z\,d\mu.$$

$$|z-h_1|^2 = 1 - \langle z, h_1\rangle - \underbrace{\langle h_1, z\rangle}_{\bar h_1 h_1}$$
$$= 1-|h_1|^2 \qquad +|h_1|^2$$

$$z^{-1}p_2 = az^{-1} + b + cz$$

$$z^{-1}p_2 = l_0 p_0 + l_1 p_1 + \frac{a}{z}$$
$$\underset{0}{(1, z^{-1}p_2)} = l_0 + 0 + (1, az^{-1})$$
$$(p_1, z^{-1}p_2) = 0 + l_1 + (p_1, az^{-1})$$

evedently, attempts to mimic a Jacobi matrix seem to fail: An alternative might be the 2x2 matrices

Let's go back in lower degrees and seriously analyze a partial unitary.

905 $\quad Y = aX \oplus V^+ = V^- \oplus bX$

assume $X$ dim $d$, $Y$ dim $d+1$. ~~I like this det eqns.~~

$$(a\lambda - b)x = -v^+ + v^-$$
$$= \quad + \ S^*(\lambda)v^+ \qquad S^*(\lambda) \text{ anal for } |\lambda| \le 1.$$

---

Consider $\quad Y = aX \oplus V^+ = V^- \oplus bX$

$$H: \quad \oplus\ z^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{\substack{\| \\ V^- \oplus bX}} \oplus\ zV^+ \oplus \cdots$$

eigenvector equation

$$\xi = +\lambda z^{-1}v^- + \underbrace{ax_1 + v^+}_{\substack{\| \\ v^- + bx_2}} + \lambda^{-1} z v^+ + \cdots$$

$$0 = u\xi - \lambda\xi = uax_1 - \lambda b x_2 \qquad x_1 = \lambda x_2, \text{ put } x = x_2$$

$$\lambda a x + v^+ = v^- + bx$$
$$(\lambda a - b)x = -v^+ + v^-$$

solve $\quad (\lambda b^* a - 1)x = -b^* v^+$

$$x = (1 - \lambda b^* a)^{-1} b^* v^+ = b^*(1 - \lambda a b^*)^{-1} v^+$$
$$v^- = v^+ + (\underset{-1}{\lambda a b^*} - \underset{+1}{bb^*})(1 - \lambda a b^*)^{-1} v^+$$
$$= (1 - bb^*)(1 - \lambda a b^*)^{-1} v^+ \qquad \text{analytic for } |\lambda| < 1.$$

---

$$\therefore \quad (a\lambda - b)x = -v^+ + v^- = -v^+ + S^*(\lambda)v^+$$
$$\boxed{(a\lambda - b)x = (S^*(\lambda) - 1)v^+}$$

What's hard to understand is the link to

$$L^2(S^1, V^-) \longrightarrow H \longleftarrow L^2(S^1, V^+)$$

~~It should be OKAY because~~ $\quad V^+ \perp z^{-1}V^- \oplus z^{-2}V^- \oplus \cdots$

so the image of $V^+$ is $L^2(S^1, V^-)$ should be in $V^- \oplus zV^- +$

906 Your picture

$$z^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{\overbrace{V^- \oplus bX}^{\parallel}} +$$

actually scattering is $\qquad\qquad \pi = (1 - aa^*)$

$$v^- = aa^* v^- + \pi v^-$$

$$u(v^-) = ba^* v^- + z\pi v^-$$

$$u^2(v^-) = (ba^*)^2 v^- + z\pi\, ba^* v^- + z^2 \pi v^-$$

too hard.

You see the scattering, ~~na~~ namely project into

$V^+ \oplus zV^+ \oplus \cdots$

and you get

$$u^n(v^-) \longmapsto z^n \pi v^- + z^{n-1}\pi\, ba^* v^- +$$

so that the limit of $z^{-n}u^n$ is

$$\pi v^- + z^{-1}\pi\, ba^* v^- + \cdots$$

$$(1-aa^*)(1-z^{-1}ba^*)^{-1} v^- \qquad \text{defined}$$

The scattering op. $S$ goes from $V^-$ to $V^+$ and it should be analytic for $|\lambda| > 1$.

$$S(\lambda)V^- \subset L^2(S^1, V^+)$$

$$S^*(\lambda)V^+ \subset L^2(S^1, V^-)$$

Now try to decipher the structure.

907 Structure ~~[illegible]~~ $H = H^2(S^1, V^-)$

$$\begin{array}{ccc} H & \overset{aX}{\supset} & S^* H \\ V^- \cup & & \cup\ S^* V^- \\ zH & \underset{bX}{\supset} & zS^* H \end{array}$$

What's wrong? Something is not working.

$L^2_{\geq 0}(S^1, V^-)$. Suppose we take $L^2_{\geq 0}(S^1)$

and ~~and~~ a closed subspace ~~[illegible]~~

Again. $H = L^2_{\geq 0}(S^1) = \bigoplus_{n \geq 0} \mathbb{C} z^n$

$K$ closed subspace $\quad zK \subset K \quad H/K$ f.d.

Then $K = f(z) H \qquad f(z) = \prod_{i=1}^{d} (z - \lambda_i) \qquad |\lambda_i| < 1$

$$= \frac{f}{z^d f^*} H \qquad S = \prod_{i=1}^{d} \frac{z - \lambda_i}{1 - \bar{\lambda}_i z}$$

So what next? Form.

$$\begin{array}{ccc} H & \overset{aX}{\supset} & K = SH \\ \mathbb{C}\quad \cup & & \cup\quad S\mathbb{C} \\ zH & \underset{bX = zaX}{\supset} & zK = zSH \end{array}$$

---

~~[illegible]~~ Go back to $Y = aX \oplus V^+ = V^- \oplus bX$.

and the eigenv. eqn. $(\lambda a - b)(x) = -v^+ + v^-$. Suppose $V^+, V^-$ 1-dimensional. ~~[illegible]~~

---

Cayley Hamilton stuff. $\frac{1}{1 - \lambda A} = e^{-\log(1 - \lambda A)}$

$$= e^{\sum_1^\infty \frac{\lambda^n}{n} A^n} \qquad \frac{1}{\det(1 - \lambda A)} = e^{\sum_1^\infty \frac{\lambda^n}{n} \operatorname{tr}(A^n)}$$

$$\lambda \frac{d}{d\lambda} \log \det(1 - \lambda A) = \sum_1^\infty \lambda^n \operatorname{tr}(A^n) = \operatorname{tr}\left(\frac{1}{1 - \lambda A}\right)$$

908 $\quad H, u, \quad Y, \quad X_1 = Y \cap u^{-1}(Y) \qquad X_1 \underset{u=b}{\overset{u}{\rightrightarrows}} Y$

$$Y_2 = Y_1 \cap u^{-1}(Y_1)$$
$$= Y \cap u^{-1}(Y) \cap u^{-1}\big(Y \cap u^{-1}(Y)\big)$$
$$= Y \cap u^{-1}(Y) \cap u^{-2}(Y).$$

$$\begin{tikzcd} Y_2 \arrow[r, hook] \arrow[d, "u"'] & Y_1 \arrow[r, hook] \arrow[d, "u"'] & Y \\ Y_1 \arrow[r, hook] \arrow[d, "u"'] & Y & \\ Y & & \end{tikzcd}$$

this should be repeatable

$$v^- = (1-bb^*)(1-\lambda ab^*)\, v^+$$
$$= v^+ - b\left(1-\lambda b^* a\right)^{-1} b^* v^+$$

Digress.

Suppose $\quad u : Y \to Y \quad$ unitary $\quad X \subset Y$

$a : X \to Y \quad$ inc., $\quad b = u : X \to Y$. $\quad$ Relate $(\lambda - ba^*)^{-1}$ to $(\lambda - u)^{-1}$. $\quad$ Point is that

$$1 = aa^* + (1-aa^*), \quad \text{so} \qquad u = \underset{ba^*}{\underset{\parallel}{uaa^*}} + \underbrace{u(1-aa^*)}_{u\pi}$$

$$\lambda - ba^* - u\pi = \lambda - u$$

$$(\lambda-u)^{-1} = (\lambda - ba^*)^{-1} + (\lambda-ba^*)^{-1}\underline{u\pi}\,(\lambda - ba^*)^{-1} + \cdots$$

$$u(1-aa^*) = (1-bb^*)u$$

You are given $u$ and $X = (\xi)^\perp$ whence

$X \underset{b}{\overset{a}{\rightrightarrows}} Y \qquad$ $a$ inclusion
$\qquad\qquad\qquad\;\;$ $b$ = rest. of $u$ to $X$.

Have eigenv. eqn. for $X$ : $\qquad (a\lambda - b)(x) = -v^+ + v^-$

$$(a\lambda - b)(x_\lambda) = -\xi + s_\lambda \xi \qquad \text{where } s_\lambda : \underset{\mathbb{C}\xi}{\underset{\parallel}{V^+}} \to \underset{\mathbb{C}u(\xi)}{\underset{\parallel}{V^-}}$$

909

So we have

$s_\lambda : V^+ \longrightarrow V^-$ analytic for $|\lambda| < 1$.

$$(b^* a\lambda - 1)x = -b^*\xi$$

$$s_\lambda \xi = \xi + (a\lambda - b)(1-\lambda b^* a)^{-1} b^*\xi$$

$$= \big((a\lambda - b)b^* + 1 - \lambda a b^*\big)(1-\lambda a b^*)^{-1}\xi$$

$$\boxed{s_\lambda \xi = (1 - bb^*)(1-\lambda a b^*)^{-1}\xi.}$$

analytic for $|\lambda| < 1$.

$s_\lambda : \underset{\mathrm{Ker}\, a^*}{V^+} \longrightarrow \underset{\mathrm{Ker}\, b^*}{V^-}$

But we also have $u : V^+ \longrightarrow V^-$

Let's write $(a\lambda - b)x = -\lambda v_0^+ + v_{-1}^-$

say $v_{-1}^- = u(v_1^+)$.

$$\lambda(ax + v_0^+) = u(ax + v_1^+)$$

Better $(a\lambda - b)x = -\overset{\in V^+}{\xi} + \overset{\in V^-}{s_\lambda \xi}$

$$\lambda(ax + \lambda^{-1}\xi) = u(ax + u^{-1}s_\lambda \xi)$$

Thus we have an eigenvector of $u$ when

$u^{-1}s_\lambda \xi = \lambda^{-1}\xi$ or $\boxed{u(\xi) = \lambda s_\lambda(\xi)}$

~~What basically happens is that specify~~

The degree checks, because $s_\lambda(\xi) = \xi + \underbrace{(a\lambda - b)(1-\lambda b^* a)^{-1} b^*\xi}_{\text{degree } \dim(X)}$

so that $\lambda u^{-1} s_\lambda - 1$ should have degree $\dim(X) + 1 = \dim(Y)$. YES.

910 Begin with $Y = aX + V^+ = V^- + bX$

$$(\lambda a - b)x = -v^+ + v^-$$

$$= -v^+ + S(\lambda)v^+$$

$\therefore V^+ \to V^-$, $S(\lambda)$ anal for $|\lambda| < 1$.

Assume $u$ unitary on $Y$ such that $ua = b$, so that $u$ induces an isom $V^+ = (aX)^\perp \to (bX)^\perp = V^-$

Let $y = ax + v^+$ satisfy $\lambda y = u(y)$.

$$\lambda ax + \lambda v^+ = bx + u(v^+)$$

$$(\lambda a - b)x = -\lambda v^+ + \underbrace{u(v^+)}_{S(\lambda)\lambda v^+}$$

Thus the eigenvector equation for $u$ reduces to $S(\lambda)\lambda v^+ = u(v^+)$. You have succeeded in compressing the characteristic equation for the operator $u$, which is $\det(\lambda - u) = 0$, ~~to~~ $S(\lambda)\lambda v^+ = u(v^+)$

$$(\lambda - u)(y) = 0 \quad \text{to} \quad (S(\lambda)\lambda - u)(v^+) = 0$$

Setting is $u$ unitary on $Y$, $X$ subspace of $Y$, $a : X \to Y$ inc., $b : X \to Y$ rest. of $u$.

Then $(\lambda - u)(y) = 0 \Rightarrow v^+ = (1 - aa^*)y$ satisfies

$$u(v^+) = \lambda S(\lambda) v^+$$

$$y = ax + v^+$$

$$(\lambda - u)(y) = (\lambda a - b)x + \lambda v^+ - u(v^+) = 0$$

$$(\lambda b^* a - 1)x + b^*(\lambda v^+) = 0$$

$$x = (1 - \lambda b^* a)^{-1} b^* \lambda v^+$$

$$u(v^+) = \lambda\Big[\underbrace{v^+ + (\lambda a - b)(1 - \lambda b^* a)^{-1} b^* v^+}_{S(\lambda)\, v^+}\Big]$$

correct but puzzling. To complete the partial unitary to a unitary?

---

Return to $H, u, Y$ cl. subspace $X = Y_1 = Y \cap u^{-1}(Y)$

Then set $Y_2 = Y_1 \cap u^{-1}(Y_1) = Y \cap u^{-1}(Y) \cap u^{-2}(Y)$ etc. and you get a sequence of partial unitaries.

$$\begin{tikzcd}
Y_2 \arrow[r, hook] \arrow[d] & Y_1 \arrow[r, hook] \arrow[d, "u"] & Y \\
Y_1 \arrow[r, hook] \arrow[d, "u"] & Y & \\
Y & &
\end{tikzcd}$$

I want to compare the $S$ operator for $Y_1 \rightrightarrows Y$ with $Y_2 \rightrightarrows Y_1$

9 12 Assume codim 1

$$\begin{tikzcd}
X \arrow[r, hook, "a"] \arrow[d, "b"] & Y \arrow[r, hook, "c"] \arrow[d, "d"] & Z \\
Y \arrow[r, hook, "c"] \arrow[d, "d"] & Z & \\
Z & &
\end{tikzcd}$$

$$Z = cY \oplus W^+ = W^- \oplus dY$$

$$Y = aX \oplus V^+ = V^- \oplus bX$$

Actually it seems that we can take $Z$ to be the pushout of $a, b$ so that provided $cY + dY = Z$, this is the case. What about inner products.

Change notation to

$$\begin{tikzcd}
aX \cap bX \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & aX \arrow[d, phantom, "\cap\, V^+"] \\
bX \arrow[r, phantom, "\underset{V^-}{\subset}"] & Y
\end{tikzcd}$$

Basic geometry should remain after removing $aX \cap bX$

Then ~~could~~ you have ~~[illegible]~~ two closed subspaces $V^-, V^+$ not ~~[illegible]~~ orth

913 March 25

~~[illegible]~~ Given $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ assume ~~$aX + bX = Y$~~

$V^{\pm}$ ~~[illegible]~~ dim 1, let $v_0^{\pm}$ be unit vect in $V^{\pm}$

Let ~~$X'$~~ $=$ ~~[illegible]~~ $b^{-1}aX = \{ax \mid u(ax) \in a\cancel{X}\}$

Then have $a' : X' \hookrightarrow aX$

$b' : X' \longrightarrow aX$

Better $T = b^{-1}aX = \{ax \mid u(ax) \in aX\}$

Then have $T \overset{\cancel{\iota}}{\underset{k}{\rightrightarrows}} aX$.

---

Set up again: Given $X \hookrightarrow Y$

Suppose $Y$ contains two codim 1 subspace $X^1, X^2$ such that $X^1 + X^2 = Y$. ~~[illegible]~~

Let $v^1, v^2$ be unit vectors $\ni (\mathbb{C}v^i)^{\perp} = X^i$. Then have an invariant $\langle v^1, v^2 \rangle = h$. Let $c : X^1 \to X^2$ be a unitary iso.

$\mathbb{C}^{n+1}$ ~~[illegible]~~ Let $u \in U_{n+1}$

$aX = \bigoplus_{i=1}^{n} \mathbb{C}e_i \oplus 0$, $v^+ = e_{n+1}$

partial unit. is $\begin{pmatrix} \\ \\ \\ \end{pmatrix}$

$bX = \bigoplus_{i=1}^{n} \mathbb{C}u(e_i)$ $v^- = u(e_{n+1})$

---

Start again. Take $u$ unit on $Y$, $X$ codim 1

$a : X \hookrightarrow Y$ inc. $b : X \to Y$ rest. of $u$ to $X$.

$b^{-1}(aX) = X \cap u^{-1}(X)$, both included in $X$ and mapped by $u$ to $X$.

914 Let $\xi_0$ be a unit v. $\perp X < Y$

$$X = (\mathbb{C}\xi_0)^\perp \qquad u^{-1}(X) = (\mathbb{C}\, u^{-1}(\xi_0))^\perp$$

$$X \cap u^{-1}(X) = (\mathbb{C}\xi_0 + \mathbb{C}u^{-1}(\xi_0))^\perp \subset (\mathbb{C}u^{-1}(\xi_0))^\perp$$

$$\cap \qquad\qquad \shortparallel$$

$$(\mathbb{C}\xi_0)^\perp = X \qquad\qquad u^{-1}(X).$$

$$\Big\downarrow S\,u$$

$$uX \cap X = (\mathbb{C}\,u(\xi_0) + \mathbb{C}\xi_0)^\perp$$

Lets try to understand

$$\xi_0 \qquad u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \xi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u(\xi_0) \qquad u^{-1}(\xi_0)$$

$$\xi_1 = u(\xi_0) = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \xi_{-1} = u^{-1}(\xi_0) = \begin{pmatrix} \bar{a} \\ b^* \end{pmatrix}$$

$$(\xi_0, \xi_1) = a \qquad (\xi_0, \xi_{-1}) = \bar{a}$$

$$(\xi_{-1}, \xi_1) = \begin{pmatrix} \bar{a} \\ b^* \end{pmatrix}^* \begin{pmatrix} a \\ c \end{pmatrix} = (a \ \ b)\begin{pmatrix} a \\ c \end{pmatrix} = a^2 + bc$$

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March 26

Let $u$ be unitary on $H$, $\xi_0$ a unit vector,

~~14~~ Consider $u^{-1}(\xi_0)$, $\xi_0$, $u(\xi_0)$.

Let $Y = \operatorname{sp}\{\xi_0, u\xi_0\}$. $\quad X = \operatorname{sp}\{\xi_0\}$

$$(a\lambda - b)(\cancel{x}) = -v^+ + S(\lambda)v^+ = -\xi_0 + S(\lambda)\ [illegible]\ \xi_0$$

$$(\lambda - u)(t\xi_0) = \lambda t\,\xi_0 - t\,u(\xi_0)$$

915 Let $u$ be unitary on $H$, $\xi_0$ unit v.

$Y = \langle \xi_0, u\xi_0 \rangle$, $X = \langle \xi_0 \rangle$, $a = m$, $b = u$.

$V^+ = (\mathbb{C}\xi_0)^\perp = \langle u\xi_0 - h\xi_0 \rangle \qquad h = (\xi_0, u\xi_0)$

$V^- = (u\xi_0)^\perp = \langle \xi_0 - \bar{h} u \xi_0 \rangle \qquad (u\xi_0, \xi_0 - \bar{h}u\xi_0) = (u\xi_0, \xi_0) - \bar{h}$

$$(\lambda - u)(\xi_0) = -s_1(u\xi_0 - h\xi_0) + s_2(\xi_0 - \bar{h}u\xi_0)$$

$$\lambda = s_1 h + s_2$$

$$+1 = +s_1 + s_2\bar{h}$$

$$\begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} h & 1 \\ 1 & \bar{h} \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \qquad \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \frac{1}{1-|h|^2}\begin{pmatrix} \bar{h} & -1 \\ -1 & h \end{pmatrix}\begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$

$$\frac{s_1}{s_2} = \frac{\lambda\bar{h} - 1}{-\lambda + h} \qquad \frac{s_2}{s_1} = \frac{\lambda - h}{1 - \bar{h}\lambda}$$

---

Now you want to handle $Y = \langle \xi_0, u\xi_0, u^2\xi_0 \rangle$

$X = \langle \xi_0, u\xi_0 \rangle \qquad V^+ = X^\perp = u^2\xi$

Maybe better to choose $\xi^+$ unit vector $\perp aX = X$

Start again with $u$ unitary on $H$, $\xi_0$ unit vector, $V^+ = \langle \xi_0 \rangle$, $V^- = \langle u\xi_0 \rangle$, $X = \langle \xi_0, u\xi_0 \rangle^\perp$

$Y = \langle \xi_0 \rangle^\perp$. Again $H \supset Y = \langle \xi_0 \rangle^\perp \supset X = \langle \xi_0, u\xi_0 \rangle^\perp$

$u(X) = \langle u\xi_0, u^2\xi_0 \rangle^\perp \subset \{u\xi_0\}^\perp$ ?

$H, u, \xi_0$ unit vector $\qquad Y = \langle \xi_0 \rangle^\perp$, $\quad u^{-1}Y = \langle u^{-1}\xi_0 \rangle^\perp$

$X = Y \cap u^{-1}Y = \langle \xi_0, u^{-1}\xi_0 \rangle^\perp$

916 $\quad H, u, \xi_0 \quad Y = \langle \xi_0 \rangle^\perp, \quad u^{-1}(Y) = \langle u^{-1}\xi_0 \rangle^\perp$

$X = Y \cap u^{-1}(Y) = \langle \xi_0, u^{-1}\xi_0 \rangle^\perp$. To compare for

$X \underset{u}{\rightleftarrows} Y$ and $Y \underset{u}{\rightleftarrows} H$ their $S$-ops.

You want to reduce the $S$ op for $Y \underset{u}{\rightleftarrows} H$ to $X \underset{u}{\rightleftarrows} Y$. Motivation should come from the original analysis of $H, u, Y$, $X = Y \cap u^{-1}Y$:

$$\begin{aligned} H &= Y^\perp \oplus X \oplus V^+ \\ & \qquad\quad \downarrow u \\ &= Y^\perp \oplus uX \oplus V^- \end{aligned}$$

$$\xi = \xi^- + x_1 + v^+ = \xi^- + u(x_2) + v^-$$

$$\lambda\xi = \lambda\xi^- + u(\lambda x_2) + \lambda v^-$$

$$u\xi = u(\xi^-) + u(x_1) + u(v^+)$$

$\pi$ projection onto $u(X)$ kernel $\pi$ is $Y^\perp + V^-$ get

$$\underbrace{\pi(\lambda-u)\xi}_{\text{assume } 0} = u(\lambda x_2 - x_1) \qquad \therefore \; x_1 = \lambda x_2$$

$$\lambda x + v^+ = ux + v^-$$

$$(\lambda-u)x = -v^+ + v^- = (S(\lambda)-1)v^-$$

Now suppose $Y^\perp = \langle \xi_0 \rangle$, As $u : V^+ \longrightarrow Y^\perp \oplus V^-$ and $u^{-1} : V^- \longrightarrow Y^\perp \oplus V^+$ we should learn something about $Y \underset{u}{\rightleftarrows} H$, we should be able to work out the scat.

917 suppose $\xi = \xi^- + \lambda x + v^+ = \xi^- + ux + v^-$

$\ni (\lambda - u)\xi = 0.$

$$\lambda\xi = \lambda\xi^- + \lambda u(x) + \lambda v^+$$
$$u\xi = u\xi^- + u(\lambda x) + uv^-$$
$$0 = (\lambda - u)\xi^- + \lambda v^+ - uv^-$$

I want the eigenvector equation for $Y \underset{u}{\rightrightarrows} H$

$$(\lambda - u)y = -w^+ + w^-$$

where $w^+ \in Y^\perp$, $w^- \in (uY)^\perp$

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Again: $H, Y, u, X = Y \cap u^{-1}(Y)$.

$$H = Y^\perp \oplus X \oplus V^+ = Y^\perp \oplus uX \oplus V^-$$

Assume $Y^\perp = \langle \xi_1 \rangle$. I want to assume know how to solve $(\lambda - u)(x) = -v^+ + v^-$, and then I would like to reduce $(\lambda - u)(y) = -w^+ + w^-$ to this. Here we have $H = Y \oplus W^+ = uY \oplus W^-$

$W^+ = \langle \xi_1 \rangle$, $W^- = \langle u\xi_1 \rangle$.

Suppose $X = 0$, so that $V^+ = V^- = Y$, $Y$ generically should be 1-dim. Assume $W^+ + W^- = Y$. Then

$H = \langle \xi_0$

$\xi_0 \quad \xi_1 \quad u\xi_1 \quad u\xi_0$

Repeat: $X = 0$ so $H = Y \oplus W^+ = W^- \oplus uY$

Suppose $H = Y \oplus W^+ = W^- \oplus uY$ so $u(W^+) = W^-$.

eigen. eqn. $(\lambda - u)y = -w^+ + w^-$. Use the basis $\xi_0, u\xi_0$ for $H$. Put $h = (\xi_0, u\xi_0)$. Then

$u\xi_0 - h\xi_0$ spans $W^+ = \langle \xi_0 \rangle^\perp$ and

$\xi_0 - \bar{h}u(\xi_0)$ —— $W^- = \langle u\xi_0 \rangle^\perp$

$$(\lambda-u)\overset{t\xi_0}{y} = -s_1(u\xi_0 - h\xi_0) + s_2(\xi_0 - \bar{h}u\xi_0)$$

$$t(\lambda\xi_0 - u\xi_0) = (s_1 h + s_2)\xi_0 + (-s_1 - s_2\bar{h})u\xi_0$$

$$t\begin{pmatrix}\lambda \\ +1\end{pmatrix} = \begin{pmatrix} s_1 h + s_2 \\ +s_1 + s_2\bar{h}\end{pmatrix} = \begin{pmatrix}1 & h \\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}s_2 \\ s_1\end{pmatrix}$$

$$\frac{s_2}{s_1} = \begin{pmatrix}1 & -h \\ -\bar{h} & 1\end{pmatrix}\lambda = \frac{\lambda - h}{1-\bar{h}\lambda}$$

~~Try to be more general.~~ The above calculations ~~$H = Y + \langle\xi\rangle$~~ done with bases $\xi_0, u\xi_0$ $\xi_0 \in Y$.

Instead use the basis $\xi_1, u\xi_1$ $\langle\xi_1\rangle = W^+$

$$H = \underset{\xi_1}{W^+} \oplus Y = uY \oplus \underset{u\xi_1}{W^-}$$

$u\xi_1 - k\xi_1$ spans $Y$ $k = \langle\xi_1, u\xi_1\rangle$

$\xi_1 - \bar{k}u\xi_1$ —— $uY$ $\langle u\xi_1, \xi_1\rangle = \bar{k}$

Put $u\xi_1 = \xi_2$. Then $\xi_2 - k\xi_1$ spans $Y$ $k = \langle\xi_1, \xi_2\rangle$

$\xi_1 - \bar{k}\xi_2$ —— $uY$

It's absurd that you should be stuck by this calculation. What's important?

$$H = Y \oplus W^+ = uY \oplus W^-$$

$$= Y \cap u^{-1}Y \oplus ? \oplus W^+ = uY \cap Y \oplus ? \oplus W^-$$

919

$$H = Y \oplus W^+ \xrightarrow{u} uY \oplus W^- \qquad Y + V^-$$

$$= \underbrace{\underbrace{(Y \cap u^{-1}Y)}_{X} \oplus V^+}_{Y} \oplus W^+ = \underbrace{\underbrace{(uY \cap Y)}_{uX} \oplus uV^+}_{uY} \oplus W^-$$

There's a scattering op. $S(\lambda) : W^+ \longrightarrow W^-$

defined by $(\lambda - u)(y) = -w^+ + S(\lambda) w^+$

See if you can organize the equations

$$\cancel{(\lambda - u)(x) = -v^+ + v}$$

$$\cancel{(\lambda - u)(x + v^+) = -w^+ + w^-}$$

$$X \oplus V^+ \oplus W^+ = uX \oplus uV^+ \oplus W^-$$

$$x_1 + v_1 + w^+ = u(x_2) + u(v_2) + w^-$$

$$u(x_1) + u(v_1) + u(w^+) = \lambda u(x_2) + \lambda u(v_2) + \lambda w^-$$

project onto $uX$ get $x_1 = \lambda x_2$

project onto $u(V^+)$ get $v_1 = \lambda v_2$

$$\overbrace{\lambda x + \lambda v^+}^{\in Y} + w^+ = \overbrace{u(x) + u(v^+)}^{\in u(Y)} + w^-$$

Important here is $(\lambda - u)(x) \in Y$ in fact

$$(\lambda - u)(x) = \underset{V^+}{-v_2^+} + \underset{V^-}{v_2^-} \quad \text{not } u(V^+)$$

920 $\quad H, u, Y, \quad X = Y \cap u^{-1}Y$

$$H = Y \oplus W^+ = uY \oplus W^-$$

$$Y = X \oplus V^+ = uX \oplus V^-$$

$$uY = uX \oplus uV^+$$

$$H = \underbrace{X \oplus V^+}_{\parallel} \oplus W^+ = uX \oplus uV^+ \oplus W^-$$

$$\overbrace{uX \oplus V^-} \oplus W^+$$

$$\begin{tikzcd} X \arrow[r, hook, "V^+"] \arrow[d, "u", "V^-"'] \arrow[dd, bend right=60, "W^-"'] & Y \arrow[r, hook, "W^+"] \arrow[d, "u", "W^-"'] \arrow[dl, phantom, "\text{cart}"] & H \\ Y \arrow[r, hook] \arrow[d, "u"] & H & \\ H & & \end{tikzcd}$$

The basic spaces are $X = Y \cap u^{-1}Y$, $Y$

$$H = Y \oplus W^+ = uY \oplus W^-$$ this defines $W^\pm \subset H$

$$Y = X \oplus V^+ = uX \oplus V^-$$ define $V^\pm \subset Y$

$$H = Y \oplus W^+ = X \oplus V^+ \oplus W^+ = \underbrace{uX \oplus V^-}_{Y} \oplus W^+$$

$$= uY \oplus W^- = \underbrace{uX \oplus uV^+}_{uY} \oplus W^-$$

~~try to analyze the eigenvector equation~~

How many subspaces: $H, X, uX, Y, uY, V^\pm$ $W^\pm$

921 Suppose you have a partial unitary $X \underset{u}{\hookrightarrow} H$ and you extend it partially to a unitary, i.e. you given $X \subset Y \subset H$ and an extension $\tilde{u}$ of $u$ to $Y$. You would like then to have? From $X \underset{u}{\hookrightarrow} H$ you get an $S_1(\lambda) : H/X \longrightarrow H/uX$. Now you've specified

$\tilde{u} : Y/X \xrightarrow{\sim} \tilde{u}Y/uX$, and what you ultimately want is $S_2(\lambda) : H/Y \longrightarrow H/\tilde{u}Y$.

Special cases: ~~suppose you go from~~ $Y = H$, i.e. You give $\tilde{u}$ on $H$ and $u$ = rest. of $u$ to $X$. You get $S_1(\lambda) : H/X \longrightarrow H/uX$ and a unitary isom $\theta : H/X \xrightarrow{\sim} H/uX$ from $\tilde{u}$. You want then $S_2(\lambda) : 0 \longrightarrow 0$. ~~Morally it should happen that $S_2(\lambda)$~~ $S_1(\lambda)$ for $|\lambda| = 1$ is a unitary from $H/X$ to $H/uX$ depending rationally on $\lambda$. Given the extension i.e. $\tilde{u} : H/X \longrightarrow H/uX$, you get ~~nothing.~~

$\lambda S_1(\lambda) = \tilde{u}$ describes the eigenvectors

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March 27. ~~Suppose given a partial unitary~~ Suppose given $H, u, X \subset H$. Relate $S$ for $Y \underset{u}{\hookrightarrow} H$ to eigenv. for $u$ on $H$.

$H = Y \oplus W^+ = uY \oplus W^-$

$\xi = y_1 + w^+ = u(y_2) + w^-$

$u(\xi) = u(y_1) + u(w^+) = \lambda u(y_2) + \lambda w^-$

project onto $u(Y)$ and you get

$y_1 = \lambda y_2$

$(\lambda - u)(y) = -w^+ + w^-$ $\quad S w^+$

$u(w^+) = \lambda w^-$

$u(w^+) = \lambda S(\lambda) w^+$

So what happens is we rewritten $(\lambda - u)(\xi) = 0$ as follows

$$\xi = y + w^+ \qquad (\lambda - u) y = -w^+ + w^-$$
$$u(w^+) = \lambda w^-$$

Solving the first equation yields $w^- = S(\lambda) w^+$ and various bound state possibilities for $Y$. The second equation then becomes $u(w^+) = \lambda S(\lambda) w^+$, so $\det(u - \lambda S(\lambda)) = 0$ should give the eigenvalues of $u$. Examine degrees. $\dim(H) = d+r$, $\dim(Y) = d$. I think the degree of $S(\lambda)$ is $d$, so $\lambda S(\lambda)$ has degree $d + r$. NO.

For example if you fix a unit vector $\xi^+ \in H$ and take $W^+ = \langle \xi^+ \rangle$ $Y = (W^+)^\perp$, then you solve $(\lambda - u) y_\lambda = (S(\lambda) - 1) w^+$

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Go over again. $Y \underset{u}{\overset{\subset}{\rightrightarrows}} H$

$$H = Y \oplus W^+ = uY \oplus W^-$$
$$\xi = y_1 + w^+ = u(y_2) + w^-$$
$$u(\xi) = u(y_1) + u(w^+)$$
$$\lambda(\xi) = \lambda u(y_2) + \lambda w^- \implies \lambda y_2 = y_1$$

so setting $y = y_2$

$$\xi = \lambda y + w^+ = u(y) + w^-$$
$$(\lambda - u)(y) = -w^+ + \underbrace{w^-}_{S(\lambda) w^+}$$
and
$$u(w^+) = \lambda w^-$$
$$\therefore \ u(w^+) = \lambda S(\lambda) w^+$$

So the spectrum of $u$ consists of bound states and $\lambda$ such that $\mathrm{Ker}(\lambda S(\lambda) - u) \neq 0$.

So what? ~~Suppose I consider triplets~~

Next consider
$$X \hookrightarrow Y \hookrightarrow H$$

an

923 Consider next $H$, $u$ and subspace $X \subset Y \subset H$. So you have a partial unitary defined on $X$ and an extension of it to $Y$. Assoc. to $u_X : X \longrightarrow H$ you get $S_X(\lambda) : H/X \xrightarrow{\sim} H/uX$ and $u_Y : Y \to H$ yields $u_{Y/X} : Y/X \xrightarrow{\sim} uY/uX$, so we have a generalized partial unitary depending on $\lambda$.

You need to generalize. You have $\underline{S_X(\lambda) : X^\perp} \longrightarrow uX^\perp$ obtained by solving

Consider $H, u$ $\quad X \subset Y \subset H$.

$$H = X \oplus V^+ \oplus W^+ = uX \oplus uV^+ \oplus uW^+$$
$$\xi = x_1 + v_1^+ \oplus w_1^+ = u(x_2) + u(v_2^+) \oplus u(w_2^+)$$

$$\lambda\xi - u\xi = \lambda u(x_2) - u(x_1) + \lambda u(v_2^+) - u(v_1^+) + \lambda u(w_2^+) - u(w_1^+)$$
$$x_1 = \lambda x_2 \qquad v_1^+ = \lambda v_2^+ \qquad w_1^+ = \lambda w_2^+$$

$$\boxed{\lambda x + \lambda v^+ + \underbrace{\lambda w^+}_{w_1^+} = u(x) + u(v^+) + \underbrace{u(w^+)}_{u(w_2^+)}}$$

$$(\lambda - u)(x + v^+) = -w_1^+ + u(w_2^+)$$

$$(\lambda - u)x = \underbrace{-\lambda v^+ - w_1^+}_{V^+ \oplus W^+} + \underbrace{u(v^+) + u(w_2^+)}_{u(V^+) \oplus u(W^+)}$$

exactly what might arise?

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & V^{-} \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & X \arrow[r, "\lambda a-b"] \arrow[dr, "\lambda b^*a-1"'] & Y \arrow[r] \arrow[d, "b^*"'] & E_\lambda \arrow[r] & 0 \\
 & & X \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

$$v_+ - (\lambda a-b)(\lambda b^*a-1)^{-1} b^* v_+$$

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try again for one hour. The basic idea is that a partial unitary?

Go back to manifolds with $\partial$. a partial unitary is like a manif. with $\partial$. You can split $M$ into a closed man. + the union of the components with nonempty bdry. You can glue together opposite pieces of the $\partial$.

Basic question: Given $X \underset{u}{\overset{\subset}{\rightrightarrows}} Y$ say $X \oplus V^+ = uX \oplus V^-$ and suppose we close part of the boundary i.e. extend $u$ to part of $V^+$. Then change $V^+$ to $V^+ \oplus W^+$. Maybe this is not the right question

Consider $X \subset Y \subset H$

$uX \subset uY \subset H$

as coupling, too hard.

925 Back to orth polys. $d\mu$ prob. meas on $S^1$. $\xi_0 = 1$. I think the basic idea is to introduce two sets of orth. polys namely. $p_0, p_1, \cdots, p_n, \cdots$ $p_i \perp p_j$ $i < j$ and $p_j = z^j$ + lower.

~~Also $q_0, \cdots, q_n$ where $q_i \perp q_j$ $i \neq j$ and~~

You want $q_n \in \mathbb{C}z^0 + \cdots + \mathbb{C}z^n$ to be orthogonal to $z, \cdots, z^n$. Better to have

$q_n = z^{-n} +$ poly in $z^{-1}$ of degree $< n$.

$\ni q_n \perp 1, z^{-1}, z^{-2}, \cdots, z^{-n}$. So you have

$$p_n \in z^n + \mathbb{C}z^{n-1} + \cdots + \mathbb{C}z^0 \qquad p_n \perp \mathbb{C}z^{n-1} + \cdots + \mathbb{C}z^0$$

$$q_n \in z^{-n} + \mathbb{C}z^{-n+1} + \cdots + \mathbb{C}z^0 \qquad q_n \perp \mathbb{C}z^{-n+1} + \cdots + \mathbb{C}z^0$$

Then ~~$z^{-1}p_{n+1} = p_n + \cdots$~~ Let $k_n = p_{n+1}(0)$

~~$z^{-1}p_{n+1} \cdots \quad p_n(0) = k_n$~~

$$p_n - z p_{n-1} \in \mathbb{C}z^{n-1} + \cdots + \mathbb{C}z^0$$

$$p_n - z p_{n-1} \perp \langle z^{n-1}, \cdots, z \rangle$$

---

Let $p_0, p_1, \cdots, p_n$ be result of orth $z^0, z^1, \cdots$

so $p_n \in z^n + F_{n-1}$ $\qquad p_n \perp F_{n-1}$ $\qquad \forall n \geq 1$.

Let $q_0, q_1, \cdots$ be $q_n \in 1 + zF_{n-1}$, $q_n \perp zF_{n-1}$

Then $p_{n+1} - zp_n \in F_n \cap (zF_{n-1})^\perp$

$\therefore$ $p_{n+1} - zp_n = h_{n+1} q_n$

926 $q_{n+1} - q_n \in zF_n \cap (zF_{n-1})^\perp$

$q_{n+1} - q_n = z h'_{n+1} p_n$

---

$p_0 = 1 = q_0$. $\quad p_1 = zp_0 - h_1 q_0$

$0 = (1, p_1) = (1, z) - h_1$

$q_1 = q_0 - h'_1 z p_0 \quad 0 = (z, q_1) = (z, 1) - h'_1$

$\therefore h'_1 = \overline{h_1}$.

$p_2 = zp_1 - h_2 q_1 \quad (z, p_2) = \underbrace{(z, zp_1)}_{=0} - h_2 \overset{0}{\overset{\shortparallel}{(z, q_1)}}$

$(1, p_2) = (1, zp_1) - h_2 (1, q_1)$

---

$p_n \in z^n + \overbrace{\langle z^{n-1}, \dots, z^0 \rangle}^{F_{n-1}} \quad p_n \perp F_n$

$q_n \in 1 + zF_{n-1} \quad q_n \perp zF_{n-1}$

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$p_{n+1} - zp_n \in F_n \quad p_{n+1} - zp_n \perp zF_{n-1}$

$\therefore p_{n+1} - zp_n = h_{n+1} q_n$

$q_{n+1} - q_n \in zF_n \quad q_{n+1} - q_n \perp zF_{n-1}$

$\therefore q_{n+1} - q_n = k_{n+1} z p_n$

$p_{n+1} = zp_n + h_{n+1} q_n$

$q_{n+1} = k_{n+1} z p_n + q_n$

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & h_{n+1} \\ k_{n+1} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

927 March 28. Anyway

$L^2(S^1, d\mu)$ $\qquad F_n = \langle z^0, \dots, z^n \rangle$

$p_n \in z^n + F_{n-1} \qquad p_n \perp F_{n-1}$

$\bar{p}_n \in z^{-n} + \bar{F}_{n-1} \qquad \bar{p}_n \perp \bar{F}_{n-1}$

$q_n = z^n \bar{p}_n \in 1 + z^n \bar{F}_{n-1} = 1 + \langle z^n, z^{n-1}, \dots, z \rangle = 1 + z F_{n-1}$

$q_n \perp z F_{n-1}. \qquad p_0 = q_0 = z^0$

$p_1 - z p_0 = h_1 \qquad \bar{p}_1 - z^{-1} \bar{p}_0 = \bar{h}_1$

$q_1 - q_0 = \bar{h}_1 z \qquad p_1 - h_1 = z p_0$

$1 = |p_1|^2 + |h_1|^2$

$p_{n+1} - z p_n \in F_n \qquad p_{n+1} - z p_n \perp z F_{n-1}$

$\therefore p_{n+1} - z p_n = h_{n+1} q_n \qquad z p_n = p_{n+1} - h_{n+1} q_n$

$h_{n+1} = p_{n+1}(0)$

OK. $\underbrace{p_n - h_n q_{n-1}}_{\perp\, zF_{n-2}} = z \underbrace{t_{n-1}}_{\in z^{n-1} + F_{n-2}} \qquad h_n = p_n(0)$

$\therefore p_n - h_n q_{n-1} = z p_{n-1} \qquad \|p_n\|^2 + |h_n|^2 \underbrace{\|q_{n-1}\|^2}_{\|p_{n-1}\|^2} = \|p_{n-1}\|^2$

orth.

$\bar{p}_n - \bar{h}_n \bar{q}_{n-1} = z^{-1} \bar{p}_{n-1} \qquad \therefore \|p_n\|^2 = (1 - |h_n|^2) \|p_{n-1}\|^2$

$q_n - \bar{h}_n z p_{n-1} = q_{n-1} \qquad \|q_n\|^2 + |h_n|^2 \|p_{n-1}\|^2 = \|q_{n-1}\|^2$

$\|p_n\|^2 = (p_n, p_n) = (z^n, p_n) = (p_n, z^n) = (1, z^n \bar{p}_n) = \int q_n \, d\mu$

928 ~~So consider next result~~ 01003

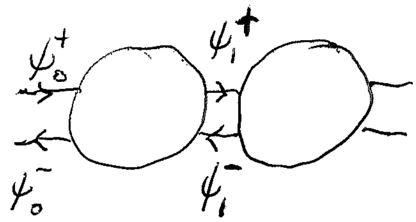
~~Question~~ Ideas: Szego det thm. First understand when $h_n = 0$ $n \gg 0$. I think this gives $\prod(1-|h_n|^2) = \lim \|g_n\|^2$. $\lim g_n$ is the predictor.

quaternionic version

There's a general puzzle concerning the doubling that takes place, either when you construct iterated port

Let's work out the details $H = L^2(S^1, d\mu)$

Approach 1 Connect ports.




$$\cancel{u\begin{pmatrix}\delta_0^+\\ \delta_1^-\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\delta_1^+\\ \delta_0^-\end{pmatrix}}$$

$$u(\psi_0^+\delta_0^+ + \psi_1^-\delta_1^-) = \lambda(\psi_1^+\delta_1^+ + \psi_0^-\delta_0^-)$$

$$= \psi_0^+(a\delta_1^+ + b\delta_0^-) + \psi_1^-(c\delta_1^+ + d\delta_0^-)$$

$$= (\psi_0^+ a + \psi_1^- c)\delta_1^+ + (\psi_0^+ b + \psi_1^- d)\delta_0^-$$

$$\lambda\psi_1^+ = a\psi_0^+ + c\psi_1^-$$

$$\lambda\psi_0^- = b\psi_0^+ + d\psi_1^-$$

$$\psi_1^- = -\frac{b}{d}\psi_0^+ + \frac{\lambda}{d}\psi_0^-$$

$$\psi_1^+ = \frac{a}{\lambda}\psi_0^+ + \frac{c}{\lambda}\left(-\frac{b}{d}\psi_0^+ + \frac{\lambda}{d}\psi_0^-\right)$$

$$= \frac{\Delta}{d\lambda}\psi_0^+ + \frac{c}{d}\psi_0^-$$

$$\begin{pmatrix}\psi_1^+\\ \psi_1^-\end{pmatrix} = \begin{pmatrix}\frac{\Delta}{d\lambda} & \frac{c}{d}\\ -\frac{b}{d} & \frac{\lambda}{d}\end{pmatrix}\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix}$$

$$\begin{pmatrix}\psi_1^+\\ \psi_1^-\end{pmatrix} = \begin{pmatrix}\frac{\Delta}{d\lambda} & \frac{c}{d}\\ -\frac{b}{d} & \frac{\lambda}{d}\end{pmatrix}\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix}$$

$\Delta = ad - bc$ say $= 1$. $\qquad \begin{pmatrix}\bar a & \bar c\\ \bar b & \bar d\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$

$\Rightarrow\ b = -\bar c \qquad \bar d = a.$

$$\left|\frac{1}{d}\begin{pmatrix}\lambda^{-1} & c\\ \bar c & \lambda\end{pmatrix}\right| = \frac{1}{d^2}(1-|c|^2) = \frac{|d|^2}{d^2} = \frac{\bar d}{d} = \frac{a}{\bar a}$$

One can simplify more by supposing $a = d = \sqrt{1-|b|^2}$

So we end up with

$$\begin{pmatrix}\psi_1^+\\ \psi_1^-\end{pmatrix} = \frac{1}{\sqrt{1-|c|^2}}\begin{pmatrix}\lambda^{-1} & c\\ \bar c & \lambda\end{pmatrix}\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix}$$

$$= \frac{1}{\sqrt{1-|c|^2}}\begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix}\begin{pmatrix}1 & c\\ \bar c & 1\end{pmatrix}\begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix}\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix}$$

$$\begin{pmatrix}\psi_n^+\\ \psi_n^-\end{pmatrix} = \begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix}\begin{pmatrix}1 & c_n\\ \bar c_n & 1\end{pmatrix}\frac{1}{\sqrt{1-|c_n|^2}}\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}\cdots$$

$$\cdots\begin{pmatrix}1 & c_1\\ \bar c_1 & 1\end{pmatrix}\frac{1}{\sqrt{1-|c_1|^2}}\begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix}\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix}$$

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Other viewpt. namely orth polys.

What's seems to be happening is that you have $h_1, h_2, \ldots$ describing two things. First - [redacted] the orthog. poly sequence

Compare ~~space~~ the Hilbert space assoc. to the coupled port with the orthogonal poly Hilbert space.

Consider

$$\begin{pmatrix}\psi_1^+\\ \psi_1^-\end{pmatrix} = \begin{pmatrix}\lambda^{-1} & c\\ \bar{c} & \lambda\end{pmatrix}\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix}$$

Looking at a Hilbert space $Y$ with basis $\delta_0^\pm, \delta_1^\pm$ 4 diml. A partial unitary

$aX = \langle \delta_0^+, \delta_1^- \rangle$ $\qquad bX = \langle \delta_0^-, \delta_1^+ \rangle$

note in this case $aX \perp bX$. Now you want to **close** the 0 end. What does this mean? Two possibilities: restrict to $\psi_0^+ = \psi_0^-$ – you get a 3 diml space.

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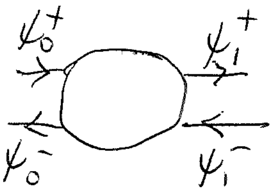
Start again. You have 


described by

$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^-$$
$$u(\delta_1^-) = c\delta_1^+ + d\delta_0^-$$

$ad - bc = 1$ $\qquad \bar{b} = -c$

$a = \sqrt{1-|b|^2}$

4 diml space $Y$ orth basis $\delta_0^\pm, \delta_1^\pm$, subspace $X = \langle \delta_0^+, \delta_1^- \rangle$ $\qquad u : X \longrightarrow uX = \langle \delta_1^+, \delta_0^- \rangle$.

$X, uX$ are $\perp$, so $V^+ = uX$, $V^- = X$ and $S(\lambda)$ should essentially be $u$.

Eigenvector equation cuts $Y$ down 2 dims:

$$\lambda\psi_1^+ = a\psi_0^+ + c\psi_1^-$$
$$\lambda\psi_0^- = -\bar{c}\psi_0^+ + a\psi_1^-$$

$$\begin{pmatrix}\psi_1^+\\ \psi_1^-\end{pmatrix} = \begin{pmatrix}\lambda^{-1} & c\\ \bar{c} & \lambda\end{pmatrix}\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix}$$

931 Now I want to close ~~[illegible]~~ the zero end to get a 1-port. ~~[illegible]~~ The problem is to make this precise. There are $\geq 2$ two possibilities. One possibity would be to extend $u$, which has been defined on $X = \langle \delta_0^+, \delta_1^- \rangle \ni uX = \langle \delta_1^+, \delta_0^- \rangle$ by adding $u(\delta_0^-) = \delta_0^+$. The other thing you might do is to impose a condition like $\psi_0^+ = \psi_0^-$ in some way.

Consider the latter. You have a 3 dim'l space with basis $\delta_0^+ = \delta_0^-$ (call this $\delta_0$) and $\delta_1^\pm$.

$u(\delta_0) = a\delta_1^+ + b\delta_0$ ? Go back to

$$\lambda\psi_1^+ = a\psi_0^+ + c\psi_1^- \qquad\qquad b = -\bar{c}$$

$$\lambda\psi_0^- = b\psi_0^+ + a\psi_1^-$$

and add the condition $\psi_0^+ = \psi_0^-$, call this $\psi_0$.

Then

$$(\lambda - b)\psi_0 = a\psi_1^-$$

$$\lambda\psi_1^+ = a\psi_0 + c\psi_1^-$$

$$= a\frac{a\psi_1^-}{\lambda - b} + c\psi_1^- = \left(\frac{a^2}{\lambda - b} + c\right)\psi_1^-$$

$$= \frac{a^2 + c\lambda - cb}{\lambda - b}\psi_1^- = \frac{1 + c\lambda}{\lambda - b}\psi_1^-$$

Compare with $\quad \dfrac{\psi_1^+}{\psi_1^-} = \dfrac{\lambda^{-1} + c}{\bar{c} + \lambda}$ from the transf. matrix. OK, YES

Do you have a 3 dim'l space with partial unitary? basis $\delta_0 = \delta_0^+ = \delta_0^-$, $\delta_1^+, \delta_1^-$ ?

932

Eigenvector equation for $u$ on $\langle \delta_0^-, \delta_0^+, \delta_1^- \rangle$

$$\lambda \psi_1^+ = a\psi_0^+ + c\psi_1^-$$
$$\lambda \psi_0^- = b\psi_0^+ + a\psi_1^-$$

$$u\left(\psi_0^-\delta_0^- + \psi_0^+\delta_0^+ + \psi_1^-\delta_1^-\right) =$$

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March 29 Compare two things.

first do the smaller thing: $Y$ has orth basis $\delta_0, \delta_1^\pm$. $X = \langle \delta_0, \delta_1^- \rangle$ $uX = \langle \delta_0, \delta_1^+ \rangle$

$$u(\delta_0) = a\delta_1^+ + b\delta_0$$
$$u(\delta_1^-) = c\delta_1^+ + d\delta_0$$

given by a unitary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ linking two orth. bases.

$$\lambda x + v^+ = u(x) + v^-$$

$$x = \psi_0\delta_0 + \psi_1^-\delta_1^-$$

$$\boxed{v^+ = \psi_1^+\delta_1^+ \qquad v^- = \phi\delta_1^-}$$

$$u(x) = \psi_0\left(a\delta_1^+ + b\delta_0\right) + \psi_1^-\left(c\delta_1^+ + d\delta_0\right)$$

$$\boxed{u(x) = \left(a\psi_0 + c\psi_1^-\right)\delta_1^+ + \left(b\psi_0 + d\psi_1^-\right)\delta_0}$$

$$\boxed{\lambda x = \left(\lambda\psi_0\delta_0 + \lambda\psi_1^-\delta_1^-\right)}$$

$$-v^+ + v^-$$

$$\lambda x - u(x) = \delta_0\left(\lambda\psi_0 - b\psi_0 - d\psi_1^-\right)$$
$$\delta_1^+\left(-a\psi_0 - c\psi_1^-\right) \qquad \delta_1^+\left(-\psi_1^+\right)$$
$$\delta_1^-\left(\lambda\psi_1^-\right) \qquad \delta_1^-\left(\phi\right)$$

933

$$(\lambda - b)\psi_0 = d\psi_1^- \qquad \lambda\psi_1^- = \phi$$

$$a\psi_0 + c\psi_1^- = \psi_1^+$$

$$\left(a\frac{d}{\lambda-b}+c\right)\frac{\phi}{\lambda} = \psi_1^+$$

$$\frac{ad+\lambda c-bc}{(\lambda-b)\lambda}\,\phi = \psi_1^+$$

$$\frac{\lambda^{-1}+c}{\lambda-b}\,\phi = \psi_1^+ \qquad S = \frac{\lambda-b}{\lambda^{-1}+c} = \frac{\lambda-b}{1-\bar b\lambda}\lambda : \psi_1^+ \mapsto \phi$$

---

next $Y$ has orth basis $\delta_0^\pm, \delta_1^\pm$, $X = \langle \delta_0^\pm, \delta_1^- \rangle$

$uX = \langle \delta_0^\pm, \delta_1^+ \rangle$

$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^-$$

$$u(\delta_1^-) = c\delta_1^+ + d\delta_0^-$$

$$u(\delta_0^-) = \delta_0^+$$

$$\lambda x = \lambda\psi_0^+\delta_0^+ + \lambda\psi_0^-\delta_0^- + \lambda\psi_1^-\delta_1^-$$

$$u(x) = \psi_0^+(a\delta_1^+ + b\delta_0^-) + \psi_1^-(c\delta_1^+ + d\delta_0^-) + \psi_0^-\delta_0^+$$

$$\lambda x - u(x) = \delta_0^+(\lambda\psi_0^+ - \psi_0^-)$$

$$\delta_0^-(\lambda\psi_0^- - b\psi_0^+ - d\psi_1^-)$$

$$\delta_1^+(-a\psi_0^+ - c\psi_1^-) \qquad -v^+$$

$$\delta_1^-(\lambda\psi_1^-) \qquad v^-$$

$$\lambda^2\psi_0^+ = b\psi_0^+ + d\psi_1^- \qquad \psi_0^+ = \frac{d}{\lambda^2-b}\psi_1^-$$

$$v^+ = \left(\frac{ad}{\lambda^2-b} + c\right)\psi_1^- = \frac{ad-bc+c\lambda^2}{\lambda^2-b}\,\frac{1}{\lambda}\,v^-$$

$$\frac{v^+}{v^-} = \frac{1+c\lambda^2}{\lambda^2-b}\,\frac{1}{\lambda}$$

$$S = \frac{v^-}{v^+} = \frac{\lambda^2-b}{1-\bar b\lambda^2}\,\lambda$$

$$\frac{\psi_1^+}{\psi_1^-} = \begin{pmatrix} \lambda^{-1} & c \\ \bar c & \lambda \end{pmatrix}\left(\frac{\psi_0^+}{\psi_0^-}\right)$$

---

Puzzle here. In the first situation you have $X \overset{2}{\underset{u}{\rightrightarrows}} \overset{4}{Y}$ and ~~pass~~ pass to a quotient - identifying $\delta_0^+, \delta_0^-$. Notice that $X, uX$ are $\perp$ originally, so $a^*b = b^*a = 0$

$X = \langle \delta_0^+, \delta_1^- \rangle \qquad uX = \langle \delta_0^-, \delta_1^+ \rangle$

Complete puzzle in organization. Yes.

Change scene. Given $X \underset{u}{\rightrightarrows} Y$ codim 1. Can you classify enlargements ~~Seems simple enough~~

$$\begin{array}{ccc} X & \underset{u}{\rightrightarrows} & Y \\ \downarrow & & \downarrow \\ X' & \underset{u'}{\rightrightarrows} & Y' \end{array}$$

~~namely~~ ~~There is essentially take~~ The orth complement of $X' \ominus X$ is a line $L$. ~~Assume~~ ~~$L \perp Y$~~. Need to extend $u$ to $L$. If $L = \langle \xi \rangle$, then $u(\xi)$ any unit vector $\perp u(X)$. Good cases $Y = X + uX$

orth poly. $\qquad H = L^2(S^1, d\mu) \qquad \int d\mu = 1.$

$F_n = \langle z^0, \dots, z^n \rangle \qquad p_0 = z^0 = 1$

$p_n \in z^n + F_{n-1} \qquad p_n \perp F_{n-1} \qquad n \geq 0$

$q_n = z^n \overline{p_n} \in z^n(z^{-n} + \overline{F_{n-1}}) = 1 + zF_{n-1} \qquad \|q_n\|^2 = \|p_n\|^2$

$q_n \perp z^n \overline{F_{n-1}} = zF_{n-1} \qquad n \geq 0$

935

$n \geq 1.$

$h_n = p_n(0)$

$$\underbrace{p_n - z p_{n-1}}_{\in F_{n-1}\ ,\ \perp\ z F_{n-2}} = h_n q_{n-1}$$

$\cancel{p_n = z p_{n-1} + h_n q_{n-1}}$

$z p_{n-1} = p_n - h_n q_{n-1}$

$\|p_{n-1}\|^2 = \|p_n\|^2 + |h_n|^2 \|q_{n-1}\|$

$\|p_n\|^2 = (1 - |h_n|^2)\|p_{n-1}\|^2$

$$\bar{p}_n - z^{-1}\bar{p}_{n-1} = \bar{h}_n q_{n-1}$$

$$q_n - q_{n-1} = \bar{h}_n z p_{n-1}$$

so

$$p_n = z p_{n-1} + h_n q_{n-1}$$

$$q_n = \bar{h}_n z p_{n-1} + q_{n-1}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Does $q_n$ converge?

$$\|q_n - q_{n-1}\|^2 = \|\bar{h}_n z p_{n-1}\|^2 = |h_n|^2 \prod_{j=1}^{n-1} (1 - |h_j|^2)$$

$$q_n - q_{n-1} = h_n z p_{n-1}$$

$$q_{n-1} - q_{n-2} = h_{n-1} z p_{n-2}$$

$\perp$

$$q_{n-r+1} - q_{n-r} = h_{n-r+1} z p_{n-r}$$

$$\|q_n - q_{n-r}\|^2 = |h_n|^2 \prod_{j=1}^{n-1}(1 - |h_j|^2) + |h_{n-1}|^2 \prod_{j=1}^{n-2} ( \quad ) + \cdots$$

$$\frac{\|q_n - q_{n-r}\|^2}{\prod^{n-r} (1 - |h_j|^2)} = |h_n|^2 \prod$$

936 Other viewpoint. Assuming $\sum |h_n|^2 < \infty$

i.e. $\prod_{n=1}^{\infty} (1-|h_n|^2) > 0$, you know that

$q_\infty$ exists $\quad q_\infty \in 1 + \overline{zF_\infty} \quad q_\infty \perp \overline{zF_\infty}$

so $\int z^j |q_\infty|^2 d\mu = (q_\infty, z^j q_\infty) = 0 \quad j \neq 0$

so $|q_\infty|^2 d\mu = \|q_\infty\|^2 \frac{d\theta}{2\pi}$

$$d\mu = \frac{\|q_\infty\|^2}{|q_\infty|^2} \frac{d\theta}{2\pi}$$

~~So you find smaller~~ $\quad d\mu = \rho \frac{d\theta}{2\pi}$

where $\rho = \dfrac{\|q_\infty\|^2}{|q_\infty(z)|^2}$

$$\log(\rho) = \log(\|q_\infty\|^2) - \log q_\infty - \overline{\log q_\infty}$$

point. $q_\infty(z) = e^{-f(z)} \qquad f(0) = 0$.

So you write $\log(\rho) = \sum_{n \in \mathbb{Z}} a_n z^n \qquad \bar{a}_n = a_{-n}$

$f(z) = \sum_{n \geq 1} a_n z^n \qquad = a_0 + f(z) + \overline{f(z)}$

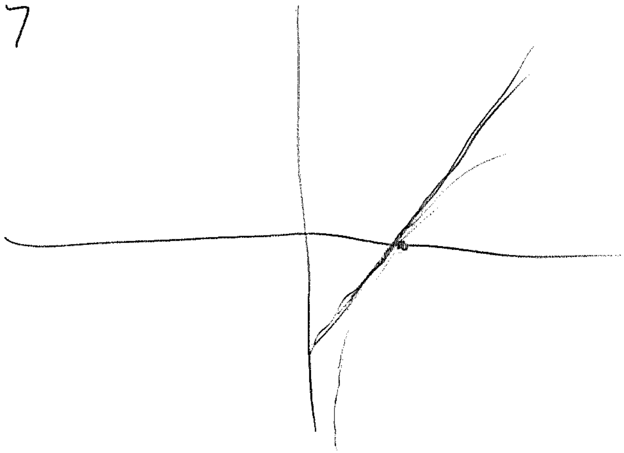
Take density $\rho(z)$ (say $> 0$).

$\log \rho(z) = a_0 + \underset{\text{anal.}}{f(z)} + \overline{f(z)} \qquad f(0) = 0$.

$\rho(z) = e^{a_0} e^{f} e^{\bar{f}} \qquad$ so $\|q_\infty\|^2 = \prod_{n=1}^{\infty} (1-|h_n|^2)$

$\|q_\infty\|^2 \frac{1}{q_\infty} \frac{1}{\bar{q}_\infty} \qquad = \exp\left\{ \int \log \rho \frac{d\theta}{2\pi} \right\}$

937



$y = \log x$

$y' = \frac{1}{x} = 1$

$y'' = -\frac{1}{x^2} < 0$

$\log x \leqslant x - 1$

$$\int \log \rho \, \frac{d\theta}{2\pi} \leqslant \int (\rho - 1) \frac{d\theta}{2\pi} = 1 - 1 = 0.$$