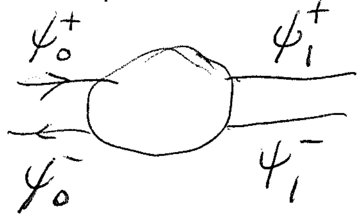
831 March 13, 98

I'm still puzzled by the situation when $|S(\lambda)| \leq 1-\varepsilon$ for $|\lambda|<1$. I think this automatically means continuous spectrum. You need examples. $S(\lambda)=0$ trans. line.




$$\begin{pmatrix} \psi_0^- \\ \psi_0^+ \end{pmatrix} = \begin{pmatrix} \frac{\Delta}{\lambda a} & \frac{b}{a} \\ \frac{c}{a} & \frac{\lambda}{a} \end{pmatrix} \begin{pmatrix} \psi_1^- \\ \psi_1^+ \end{pmatrix}$$

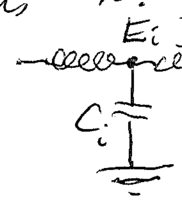
~~Now take~~ Complete with a transmission line on the right: $\begin{pmatrix} \psi_1^- \\ \psi_1^+ \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}$ Then get $\frac{\psi_0^-}{\psi_0^+} = \frac{b}{\lambda}$

Basically the idea should be that you couple two half lines together and calculate the resolvent.

Let's try to understand a linear graph of 2 ports like 20 years ago. You have transfer matrices $T_n$ s.t. $\psi_n = T_n \psi_{n+1}$ $\forall n$ and now I know there's a simple unitary operator around whose spectrum is the main object of interest. I could compare ~~what I want doing~~ this discrete situation with a Dirac operator where the potential is locally constant except at integer points $x$.

Consider a trans line, ~~better~~

x ldx x+dx T




$$I_x - I_{x+dx} = c\,dx\, \partial_t E_x \qquad \partial_x I + c\partial_t E = 0 \qquad lc=1$$

$$E_x - E_{x+dx} = l\,dx\, \partial_t I_x \qquad \partial_x E + l\partial_t I = 0$$

$$(\partial_x + \partial_t)(E+I) = 0 \qquad (\partial_x - \partial_t)(I-E) = 0$$

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$$(\partial_x + s)(E+I) = 0$$
$$(\partial_x - s)(E-I) = 0$$

If time dep $e^{st}$

$$\begin{pmatrix} E+I \\ E-I \end{pmatrix} = \begin{pmatrix} Ae^{-sx} \\ Be^{sx} \end{pmatrix}$$

transfer matrix from $x=0$ to $x=1$ is

$$\begin{pmatrix} A \\ B \end{pmatrix} \longmapsto \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

can you find a unitary operator here?

$$\begin{pmatrix} \psi_0^+ \\ \psi_0^- \end{pmatrix} = \begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix}$$

$$\begin{pmatrix} \psi_0^- \\ \psi_0^+ \end{pmatrix} = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} \psi_1^- \\ \psi_1^+ \end{pmatrix}$$

$\lambda = e^{-s}$

$$S_0 = e^{-2s} S_1$$

~~My feeling is that~~ Do things inside $SU(1,1)$. Given $S(z)$ analytic for $|z|<1$ and satisfying $|S(z)|<1$, look at $S(0)$

$$\tilde{S}(z) = \frac{S(z) - S(0)}{1 - \overline{S(0)}\, S(z)} = \begin{pmatrix} 1 & -S(0) \\ -\overline{S(0)} & 1 \end{pmatrix} S(z)$$

$$\tilde{S}(0) = 0 \implies \tilde{S}(z) = z S_1(z).$$

$$S_1(z) = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -S(0) \\ -\overline{S(0)} & 1 \end{pmatrix} S(z)$$

$$S_0(z) = \begin{pmatrix} 1 & S_0(0) \\ \overline{S_0(0)} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} S_1(z).$$

and you can proceed in general.

$$S_0(z) = \begin{pmatrix} 1 & h_0 \\ \overline{h_0} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & h_1 \\ \overline{h_1} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots$$

<u>This</u> sort of thing will work (?) for any $|S(z)| < 1$
$|z| < 1.$

Compare
$$\partial_x E + l\,\partial_t I = 0 \qquad\qquad lc = 1$$
$$l \cdot (\partial_x I + c\,\partial_t E = 0)$$

$$(\partial_t + \partial_x)(E + lI) = 0$$
$$(\partial_x - \partial_t)(E - lI) = 0.$$

$$\begin{pmatrix} E + lI \\ E - lI \end{pmatrix} = \begin{pmatrix} e^{-s(x-t)} A \\ e^{s(x+t)} B \end{pmatrix} \qquad \text{set } t = 0.$$

transfer matrix

$$\begin{pmatrix} E + lI \\ E - lI \end{pmatrix}_{x=1} = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix} \begin{pmatrix} E + lI \\ E - lI \end{pmatrix}_{x=0}$$

$$\begin{pmatrix} 1 & l \\ 1 & -l \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix}_{x=0} = \begin{pmatrix} e^{+s} & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} 1 & l \\ 1 & -l \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix}_{x=1}$$

$$\begin{pmatrix} 1 & l \\ 1 & -l \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} E + I \\ E - I \end{pmatrix}_{x=0} = \underline{\qquad\qquad\qquad} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} E+I \\ E-I \end{pmatrix}_{x=1}$$

834 $\begin{pmatrix} E+I \\ E-I \end{pmatrix}_{x=0} = \underbrace{\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -\ell & -\ell \\ -1 & 1 \end{pmatrix}}\begin{pmatrix} e^{s} & \\ & e^{-s} \end{pmatrix}\begin{pmatrix} 1 & \ell \\ 1 & -\ell \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E+I \\ E-I \end{pmatrix}_{\cancel{x}=1}$

$$\frac{1}{\sqrt{4\ell}}\begin{pmatrix} \ell+1 & \ell-1 \\ \ell-1 & \ell+1 \end{pmatrix}\begin{pmatrix} e^{s} & \\ & e^{-s} \end{pmatrix}\begin{pmatrix} \ell+1 & -\ell+1 \\ -\ell+1 & \ell+1 \end{pmatrix}\frac{1}{\sqrt{4\ell}}$$

$$\underbrace{\begin{pmatrix} \frac{\ell+1}{2\sqrt{\ell}} & \frac{\ell-1}{2\sqrt{\ell}} \\ \frac{\ell-1}{2\sqrt{\ell}} & \frac{\ell+1}{2\sqrt{\ell}} \end{pmatrix}}\begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix}\begin{pmatrix} \frac{\ell+1}{2\sqrt{\ell}} & \frac{-\ell+1}{2\sqrt{\ell}} \\ \frac{-\ell+1}{2\sqrt{\ell}} & \frac{\ell+1}{2\sqrt{\ell}} \end{pmatrix}$$

what can I say about this thing? Recall $\ell > 0$. It is a typical element in $SU(1,1)$ of the form

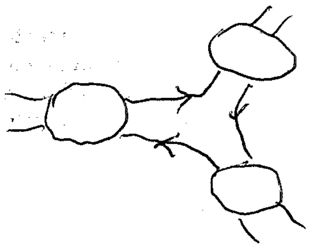
$$\begin{pmatrix} \frac{1}{\sqrt{1-t^2}} & \frac{t}{\sqrt{1-t^2}} \\ \frac{t}{\sqrt{1-t^2}} & \frac{1}{\sqrt{1-t^2}} \end{pmatrix} \qquad -1 < t < 1$$

$$\frac{\ell^{1/2} - \ell^{-1/2}}{2}$$ goes from $-\infty$ to $\infty$ as $\ell$ goes from $0$ to $\infty$

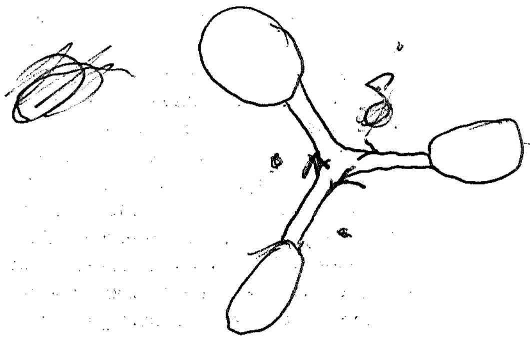
$$\frac{t}{\sqrt{1-t^2}}$$ goes from $-\infty$ to $+\infty$ as $t$ goes from $-1$ to $+1$.

So far I have checked that if I couple transmission line segments of (length and) ~~time~~ 1, then the transfer matrix should be a product of the type I want.

 Idea: Green's function. Go back to



where you define $u$ on $\ell^2(\Gamma)$. You know then ~~[illegible]~~ that $\lambda - u$ is invertible on $\ell^2$ for $|\lambda| \neq 1$. So you get a solution of the ~~[illegible]~~ $(\lambda - u)(\psi) = \delta$, where $\delta$ in any basis element. What does this amount to?



$$(\lambda - u)\psi = \delta$$

You need to write

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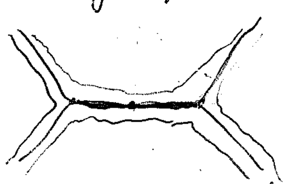
March 14. $\Gamma = PSL_2(\mathbb{Z})$. You look at $\ell^2(\Gamma)$ and you should have a unitary operator on this corresp to the thing you've been ~~studying~~ studying. ~~Left~~

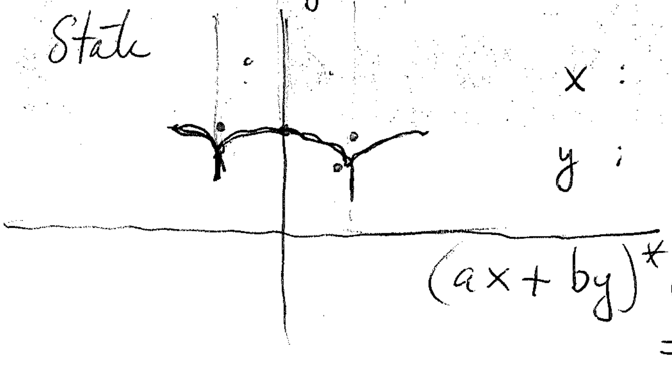
Have ribbon graph. ~~So~~ $\Gamma$ acts on this tree struct



fdl domain $\mathbb{Z}/3 \; \mathbb{Z}/2$

$\Gamma$ acts simply transitively on oriented edges. Stabilizer of ~~the~~ any big edge is $\mathbb{Z}/2$. ~~So~~

~~The~~ State



$x : \tau \mapsto \tau + 1$

$y : \tau \mapsto -\frac{1}{\tau}$

$$(ax + by)^*(ax + by) = (x^{-1}\bar{a} + y^{-1}\bar{b})(ax + by)$$
$$= |a|^2 + |b|^2 + \bar{a}b\, x^{-1}y^{-1} + \bar{b}a\, yx$$

836 $\quad$ If $x^{-1}y^{-1} = yx$ i.e. [illegible] $yxyx = 1$

All I need to do is to arrange $\bar{a}b = \bar{b}a$

So the unitary operator exists nicely. So now everything works...

What is the fundamental domain for $SL_2(\mathbb{Z})$ on UHP? Consider unimodular vectors in $\mathbb{Z}\oplus\mathbb{Z}$. i.e. direct summands. i.e. pairs $\binom{x}{y}$ rel. prime. A vertex of the tree $T$ is a line $\mathbb{Z}\binom{x}{y}$ which is a summand of $\mathbb{Z}\oplus\mathbb{Z}$. Thus $x, y$ rel. prime. A 1-simplex is a pair of ind. lines. Define size of $\mathbb{Z}\binom{x}{y}$ to be $\max\{|x|, |y|\}$. Euclidean alg $\Rightarrow$ given $\begin{matrix}x\\y\end{matrix}$ $\exists$ smaller

$(x_0, x_2) = 1$ $\quad$ with [illegible]

$$\cancel{x_1 = q_1 x_0 + x_2}$$
$$\cancel{x_2 = q_2 x_1 + x_3}$$
$$\cancel{x_3 = q_3 x_3 + x_4}$$
$$\cancel{\frac{x_1}{x_0} = q_1}$$

then

$$x_0 = q_0 x_1 + x_2$$
$$x_1 = q_1 x_2 + x_3$$

$$\frac{x_0}{x_1} = q + \frac{x_2}{x_1}$$

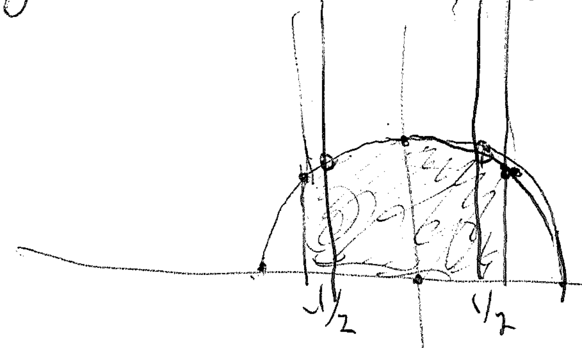
$$\frac{x_0}{x_1} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{\ }}}$$

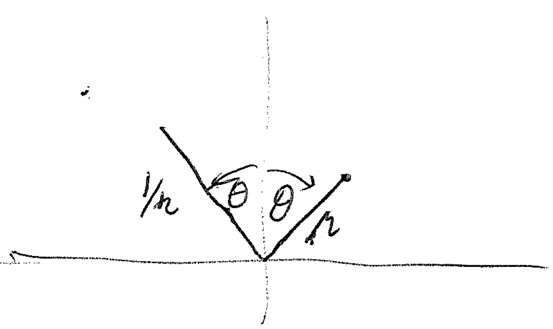
837 What is ~~the~~ a fundamental domain for $\Gamma = PSL_2(\mathbb{Z})$ acting on UHP?

Let $(a,b)=1$. can find $x, y \in \mathbb{Z} \ni$ $ax + by = 1$ by Euclidean algorithm. $(x,y)$ unique up to $\mathbb{Z}(-b,a)$. ~~Assume $|a| \le |b|$~~ can arrange $0 \le x < |b|$, and solution is unique.

Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, row operations ~~amount to E~~ amount to mult. by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. You should Euclidean algorithm for $a, c$ is achieved by left mult by these matrices. so modulo the action of the subgroup generated by these you get to case $a = \pm 1$, $c = 0$, so they generated clearly.

So basic idea is to take $\tau$ move it by trans. to $-1 < |\tau| \le 1$, if $|\tau| > 1$ done, if not. Basically you ~~can~~ consider $|\tau|$. If $|\tau| \geq 1$, then move by translation to fund. domain. If $|\tau| \leq 1$ apply $\tau \mapsto -\frac{1}{\tau}$ to get a $\left|-\frac{1}{\tau}\right| > 1$.



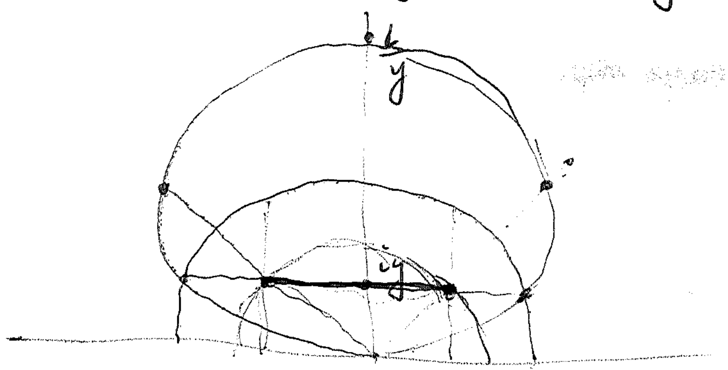




can suppose $-\frac{1}{2} \leq \mathrm{Re}(\tau) \le \frac{1}{2}$ suppose $|\tau| < 1$ also

$\tau, -\tau^{-1}$ Point is that for $|\tau| \leq 1$ $\mathrm{Im}(\tau) > 0$ then $\mathrm{Im}(-\tau^{-1}) > \mathrm{Im}(\tau)$ $\frac{-1}{x+iy} = \frac{-(x-iy)}{x^2+y^2} = \frac{-x+iy}{x^2+y^2}$

838 Look at $\tau = x+iy$ $|x| \leq \frac{1}{2}$

$$-\frac{1}{\tau} = -\frac{1}{x+iy} = \frac{-x+iy}{x^2+y^2}$$




$$\frac{1}{2}+iy \rightsquigarrow \frac{-\frac{1}{2}+iy}{\frac{1}{4}+y^2}$$

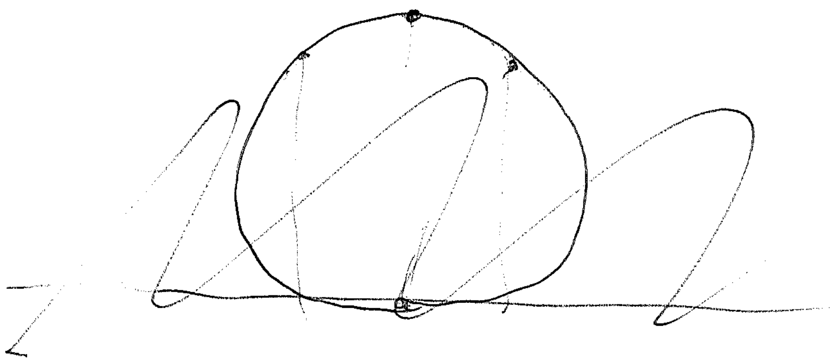
$$y \rightsquigarrow \frac{y}{\frac{1}{4}+y^2} \geq 4y$$

Now $y < \frac{\sqrt{3}}{2} \Rightarrow \frac{1}{4}+y^2 < 1.$

$\tau = re^{i\theta}$

$$x+iy \longmapsto \frac{-x+iy}{x^2+y^2}$$

$$\mathrm{Im}\left(\frac{-x+iy}{x^2+y^2}\right) = \frac{y}{x^2+y^2} \geq \frac{y}{\frac{1}{4}+y^2}$$



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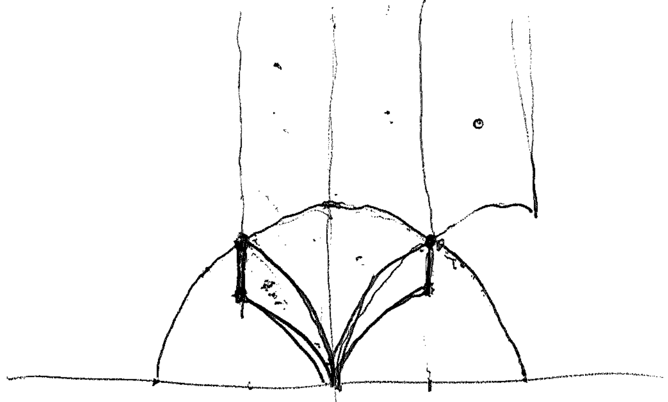
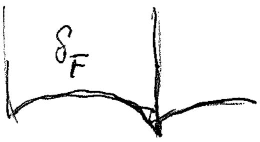


Now you need the generators ~~for~~ in $\Gamma$. I have to make clear the states. Think of there being a basis vector for each ~~triangle~~ image of the fundamental domain $F$ i.e. oriented edge.



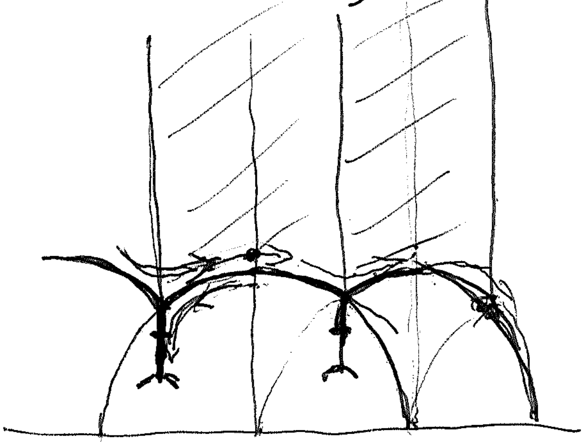

$u = a(\text{trans}) + b(\text{ref})$.

$s$ = 180° rotation around $i$ $\quad \tau \mapsto -\frac{1}{\tau}$

$s_1$ = 120° rotation about $e^{i\pi/3} = \omega$

$\text{trans} = s_1^{-1} s$

$a(s_1^{-1}s) + b(s)$



0

$$(a\lambda + b\mu)^*(a\lambda + b\mu) = 1.$$

$$|a|^2 + |b|^2 + \bar{a}b\,\lambda^*\mu + a\bar{b}\,\mu^{-1}\lambda$$

so we need $\lambda^{-1}\mu = \mu^{-1}\lambda$ $\quad$ involution because s.a. and unitary

$\bar{a}b + a\bar{b} = 0$ $\quad$ eg. $a$ real, $b$ imag.

$\lambda = s_1^{-1}s \qquad \mu = s$

$\lambda^{-1}\mu = s^{-1}s_1 s \qquad \mu^{-1}\lambda = s^{-1}s_1^{-1}s$

840 Do it this way. When is a linear comb. $ax + by$ of two unitaries $x, y$ a unitary

$$(ax+by)^*(ax+by) = |a|^2+|b|^2 + \bar{a}b\,x^{-1}y + \bar{b}a\,y^{-1}x$$

want $x^{-1}y = y^{-1}x$ and this must be of order 2.

so we have

$x(as+b)y$ where $sy = x$ ; $ax+by = (a+byx^{-1})x$
$= x(a+bx^{-1}y)$

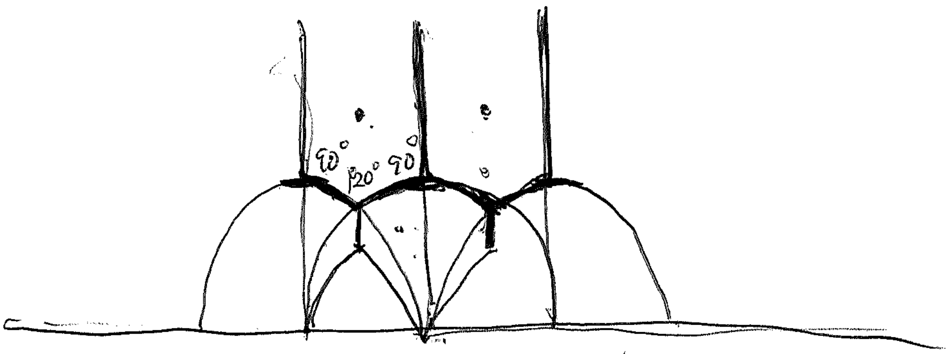
Yes. So the basic idea is that you have an involution $s$ which you combine with something

$\cos\theta + is\sin\theta$ for some angle $\theta$.

No doubt it works.

So the linear comb. involves two group elements differing by an involution. In $\Gamma$ this means probably translation and order 3.

$$(x^{-1}y)^2 = x^{-1}yx^{-1}y = 1 \iff (yx^{-1})^2 = 1.$$




next to understand spectrum.

Green's function $(\lambda-u)^{-1}\delta$ $\quad (\lambda-u)^{-1} \in n(\Gamma)$

$\sum \lambda^{-n-1}u^n$ where $u = a(\text{transl}) + b(\text{-120° rot})$

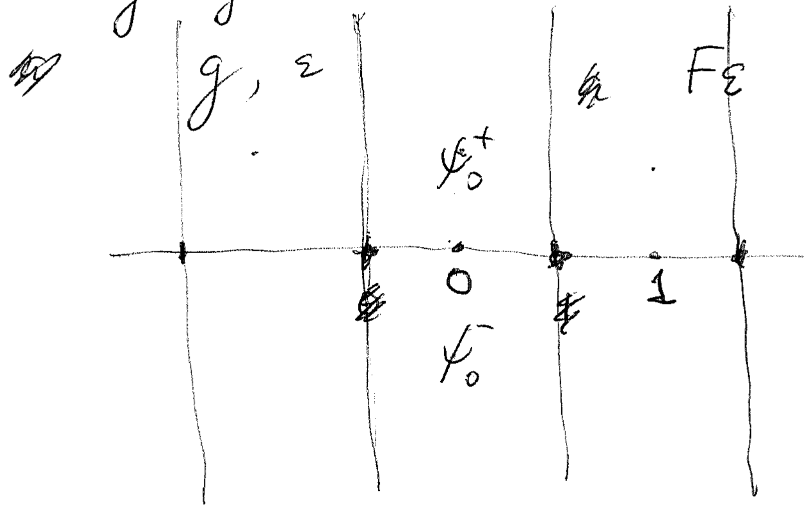
Look at dihedral group generated by $\varepsilon, F$

$g = \dfrac{1+X}{1-X}$ $\qquad 1+X = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}$

$$F(1+X) = (1+X)\varepsilon = \varepsilon(1-X)$$

$$Fg = \varepsilon \qquad g = F\varepsilon$$

841 So consider $ax+by$ unitary so that $x^{-1}y = y^{-1}x$ order 2. Possibilities for $x,y$




$aF\varepsilon + b\varepsilon$

$u$

$\varepsilon$ rotates around 0
$F$ —— $\frac{1}{2}$

$$g^{-n}\varepsilon g^{n} = g^{-2n}\varepsilon$$ two conj classes of elts order 2.

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March 15, 1998

Question: Once you have $u$ you can consider its spectral theory: the "eigenspace" for the eigenvalue $\lambda$. This is a representation of $\Gamma$

$$(aF\varepsilon + b\varepsilon)^2 = (aF+b)\varepsilon(aF+b)\varepsilon \quad \times$$
$$= (aF+b)(-aF+b)$$
$$= b^2 - a^2 + (ab-ba)F$$

You've got the usual business of left & right mult. on the group. So what are you going to use?

$$u(\delta_0^+) = a\delta_1^+ + b\delta_0^-$$

$F\varepsilon \quad F$
$g \quad g\varepsilon$

$$aF\varepsilon + b\varepsilon = ag + b\varepsilon$$
$$(aF\varepsilon + b\varepsilon)^2 = a^2g^2 + b^2 + ab(g\varepsilon + \varepsilon g) \quad \text{~~commuting with~~}$$
$$(g+g^{-1})\varepsilon$$

$$(sg + tg\varepsilon)^2 = g(s+t\varepsilon)g(s+t\varepsilon)$$
$$= g($$

842 Basically you have a unitary operator defined on a Hilbert with orth basis $\delta_n^{\pm}$ by

$$u(\delta_n^+) = a\delta_{n+1}^+ + b\delta_n^- \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_2$$
$$u(\delta_n^-) = c\delta_n^+ + d\delta_{n-1}^-$$

obvious commutes with translation $\delta_n^{\pm} \longmapsto \delta_{-1}^{\pm}$

You want it to commute with ~~refle~~ $\delta_n^{\pm} \longrightarrow \delta_{-n}^{\mp}$

i.e. $u(\delta_{-n}^-) = a\delta_{-n-1}^- + b\delta_{-n}^+ \implies a = d \quad b = c.$

$u(\delta_{-n}^+) = c\delta_{-n}^+ + d\delta_{-n+1}^+ \implies$

Now you have the dihedral group operating on this Hilbert space and it acts simply transitively on the orth basis $\{\delta_n^{\pm}\}$. $\varepsilon : \delta_n^{\pm} \longrightarrow \delta_{-n}^{\mp}$

$g : \delta_n^{\pm} \longrightarrow \delta_{n+1}^{\pm}$. Then $F = g\varepsilon : \delta_n^{\pm} \longrightarrow \delta_{-n}^{\mp} \longrightarrow \delta_{-n-1}^{\mp}$

is rotation around $+\frac{1}{2}$

$$\langle e \rangle = \delta_0^+ \implies \langle g^n \rangle = \delta_n^+$$
$$\langle g^n \varepsilon \rangle = \delta_n^-$$

Then right mult by $g$ is $\delta_n^+ g = \delta_{n+1}^+$

$$\delta_n^- g = \langle g^n \varepsilon g \rangle = \langle g^{n-1}\varepsilon \rangle = \delta_{n-1}^-$$

and right mult by $\varepsilon$ is $\delta_n^+ \varepsilon = \delta_n^-$, $\delta_n^- \varepsilon = \delta_n^+$

Then

$$u(\delta_n^+) = \delta_n^+(ag + b\varepsilon)$$
$$u(\delta_n^-) = \delta_n^-(b\varepsilon + ag) \qquad \text{so it works.}$$

Then $u$ is right mult by $ag + b\varepsilon$ $\qquad |a|^2 + |b|^2 = 1$, $\bar{a}b + \bar{b}a = 0$.

So now how do I handle the spectrum.

843 $u = ag + b\varepsilon$ should satisfy a quadratic equation over $\mathbb{C}(g)$ maybe $\mathbb{C}[g, g^{-1}]$.

$$u + u^{-1} = ag + b\varepsilon + \bar{a}g^{-1} + \bar{b}\varepsilon \qquad \text{say } \begin{pmatrix} a = \bar{a} \\ \bar{b} = -b \end{pmatrix}$$
$$= a(g + g^{-1}).$$

You want to understand the spectrum of $u$. It sits over ~~the~~ two bands of spectrum of $g$ ~~..., $g = e^{i\theta}$~~ ~~where~~ of $g = e^{i\theta}$, then solve $\cos(\varphi) = a\cos\theta$

$$\lambda + \lambda^{-1} = a(z + z^{-1})$$

Algebraically you have for each $\lambda$ a 2 dim'l eigenspace for $u$, which must be the representation of the dihedral group belonging to $\frac{\lambda+\lambda^{-1}}{2a}$.

What should you ask? What should matter? You should link the spectrum of $u$ to the Green's function $(\lambda - u)^{-1}$, really, the jump in this function as you cross the unit circle. Note that as $u$ is right mult by $ag + b\varepsilon$, then $(\lambda - u)^{-1}$ is right mult. by an element in some ~~...~~ sort of completion of $\mathbb{C}\Gamma$, i.e. you have

$$(\lambda - u)^{-1} = \text{right mult by } \sum_{n \geq 0} \lambda^{-n-1}(ag + b\varepsilon)^n$$

for $|\lambda| > 1$, and this will be something like

$\sum_{\gamma \in \Gamma} \psi_\gamma(\lambda)\langle\gamma\rangle$. Solution of $\psi(\lambda - u) = \langle 1 \rangle$

You want to focus on the jump as $\lambda$ crosses $|\lambda| = 1$. What method do you have? What is the ~~...~~ nice situation? What do you want more than anything? You want to divide, i.e.

844 a "hypersurface" codim 1 surface, then look at the line of boundary values.

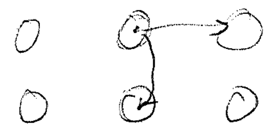
You want the response on one side, i.e. $S(\lambda)$ to be analytic for $|\lambda| < 1$, to have unitary boundary values, the boundary values should be analytic on $|\lambda| = 1$ except at $\lambda = -1$. Then when you take $\frac{S(\lambda)-1}{S(\lambda)+1}$ you get a nice fn of $\lambda$

Your goal is to find a unitary operator with discrete spectrum related to zeroes of zeta in the critical strip.

Let's try to analyze the jump in $(\lambda - u)^{-1}$.

Try to decribe the Green's function. Use codim 1 splitting. Consider $(\lambda-u)^{-1}$. Sets up an isomorphism between? ~~So what we do is to~~

Let's use a splitting. $(\lambda-u)^{-1}$ is an isomorphism between $\mathbb{C}\delta_0^+ \oplus \mathbb{C}\delta_0^-$ and $\ell^2$ things



Try transfer matrix approach $(\lambda - u)\psi = 0$ is replaced by $\psi_n = T^{-n}\psi_0$.

$$\begin{pmatrix} \psi_n^- \\ \psi_n^+ \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar{b}}{a} & \frac{\lambda}{a} \end{pmatrix} \begin{pmatrix} \psi_{n+1}^- \\ \psi_{n+1}^+ \end{pmatrix}$$

Look carefully. The basic idea is that

$\lambda + \lambda^{-1} = a(z + z^{-1})$ where $\lambda$ is the eigenvalue of $u = (ag + b\varepsilon)$ and $z$ is the eigenvalue of $g$. We concerned with $|\lambda| > 1$ to begin, and there is one root $z$ such that $|z| > 1$. Now continue to $|\lambda| = 1$. Part of the $\lambda$-unit circle, namely $\left|\frac{\lambda+\lambda^{-1}}{2}\right| \leq a$, leads to $|z| = 1$. What's the link between $z$ and $S(\lambda)$.

845 OKAY idiot

$$z\begin{pmatrix}\psi_0^- \\ \psi_0^+\end{pmatrix} = T\begin{pmatrix}\psi_0^- \\ \psi_0^+\end{pmatrix}$$

$$\delta = \frac{\lambda^{-1}\delta + b}{\bar b \delta + \lambda} \qquad a = \bar a \quad \bar b = -b$$

$$z\begin{pmatrix}\delta \\ 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar b}{a} & \frac{\lambda}{a}\end{pmatrix}\begin{pmatrix}\delta \\ 1\end{pmatrix}$$

$$\bar b\delta + \lambda = \lambda^{-1} + b\delta^{-1}$$

$$\boxed{\begin{aligned}\lambda - \lambda^{-1} &= b(\delta + \delta^{-1}) \\ \lambda + \lambda^{-1} &= a(z + z^{-1})\end{aligned}}$$

$$z = \frac{1}{a}(\bar b \delta + \lambda) \qquad az\delta = \lambda^{-1}\delta + b$$

$$az - \lambda = \bar b \delta$$

$$\boxed{\delta = \frac{az - \lambda}{\bar b} = \frac{b}{az - \lambda^{-1}}}$$

$$a^2z^2 - az(\lambda + \lambda^{-1})^{+1} = +|b|^2$$

$$a^2(z^2+1)$$

I think now you are in a position to understand $\delta(\lambda)$. You find $\delta(\lambda)$ for $|\lambda| > 1$ by first finding $z$ from $z + z^{-1} = \frac{\lambda + \lambda^{-1}}{a}$. You want the branch such that $z = a\lambda + \ldots$ This is the root such that $T\begin{pmatrix}\delta \\ 1\end{pmatrix} = z\begin{pmatrix}\delta \\ 1\end{pmatrix}$ with $|z| > 1$.

~~So you understand~~ From $\lambda - \lambda^{-1} = b(\delta + \delta^{-1})$ you ought to be able to understand $\delta(\lambda)$ for $|\lambda| > 1$.

$$\delta - \frac{\lambda - \lambda^{-1}}{b} + \delta^{-1} = 0$$

$$\delta = \left(\frac{\lambda - \lambda^{-1}}{+2b}\right) \pm \sqrt{\left(\frac{\lambda - \lambda^{-1}}{2b}\right)^2 - 1}$$

$$= \lambda\left[\frac{1 - \lambda^{-2}}{2b} \pm \sqrt{\left(\frac{1 - \lambda^{-2}}{2b}\right)^2 - 1}\right.$$

$$= \frac{\lambda}{2b}\left[(1 - \lambda^{-2}) \pm \sqrt{(1 - \lambda^{-2})^2 - 4b^2}\right.$$

$$1 - 4b^2 - 2\lambda^{-2} + \lambda^{-4}$$

8/16

$$s^2 - 2fs + 1 = 0$$

$$s = f \pm \sqrt{f^2 - 1}$$

$$= f\left(1 - \sqrt{1 - f^{-2}}\right) \qquad |f| >$$

$$= f\left(1 - \left(1 + \left(\tfrac{1}{2}\right)(-f^{-2}) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(-f^{-2})^2 + \cdots\right)\right)$$

$$= \tfrac{1}{2}f^{-1} + \tfrac{1}{8}f^{-3} + \cdots$$

Go back to relation between $s, \lambda$ ~~and~~

$$\lambda - \lambda^{-1} = b(s + s^{-1})$$

$$s = \lambda - \lambda$$

$\lambda + \lambda^{-1}$ $\qquad \lambda - a(z + z^{-1}) + \lambda^{-1} = 0$

$$s = \frac{\lambda^{-1}s + b}{\bar{b}s + \lambda} = (\lambda\bar{b})^{-1} + \frac{b - \bar{b}^{-1}}{\bar{b}s + \lambda}$$

$$\begin{array}{r|l} & \lambda^{-1}\bar{b}^{-1} \\ \bar{b}s + \lambda & \lambda^{-1}s + b \\ & \lambda^{-1}s + \bar{b}^{-1} \\ \hline & b - \bar{b}^{-1} \end{array}$$

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_1 + }}$$

$$x = a_1 + \frac{1}{a_2 + \frac{1}{x}} = a_1 + \frac{x}{a_2x + 1}$$

$$a_2x^2 + \cancel{x} = a_1a_2x + a_1 \cancel{+x}$$

$$a_2x^2 - a_1a_2x + a_1 = 0$$

$$x = a_1 + \frac{1}{x} \qquad x^2 = a_1x + 1$$

847 March 16, 1996. There's a puzzle here namely solving $\frac{\lambda+\lambda^{-1}}{a} = z+z^{-1}$. It's possible this is related to elliptic functions in a simple way. These are the eigenvalues of $T = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar b}{a} & \frac{\lambda}{a} \end{pmatrix}$   $a=\sqrt{1-|b|^2}$

$b = it \qquad a = \sqrt{1-|b|^2} = \sqrt{1-t^2}$

Look at $S = \frac{\lambda^{-1}S + b}{\bar b S + \lambda}$   $T\begin{pmatrix} S \\ 1 \end{pmatrix} = z\begin{pmatrix} S \\ 1 \end{pmatrix}$

$$\frac{1}{\lambda a}S + \frac{b}{a} = zS \qquad S = \frac{b}{az-\lambda^{-1}}$$

$$\frac{\bar b}{a}S + \frac{\lambda}{a} = z \qquad S = \frac{az-\lambda}{\bar b}$$

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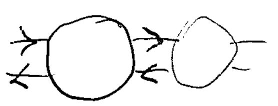
formulas. You basically want to understand, make precise, pin down the response function for the half line. This involves the Hilb. space $Y$



with basis $\delta_n^\pm$ $n \geq 0$ and the partial unitary $u$ defined on $X = (\delta_0^-)^\perp$ with image $uX = (\delta_0^+)^\perp$ given by

$$u(\delta_n^+) = a\delta_{n+1}^+ + b\delta_n^- \qquad n \geq 0$$
$$u(\delta_{n+1}^-) = c\delta_{n+1}^+ + d\delta_n^- \qquad n \geq 0$$

There's an eigenvector equation for $\psi = \sum_{n \geq 0} \psi_n^\pm \delta_n^\pm$

$$\lambda\psi = \lambda\overbrace{\Big(\psi_0^+\delta_0^+ + \sum_{n\geq 1}\psi_n^\pm\delta_n^\pm\Big)}^{x} + \lambda\tilde\psi_0^-\delta_0^-$$

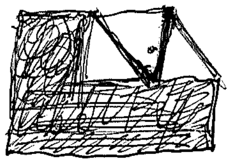
$$= \tilde\psi_0^+\delta_0^+ + \overbrace{\sum_{n\geq 0}\psi_n^+\left(a\delta_{n+1}^+ + b\delta_n^-\right) + \sum_{n\geq 0}\psi_{n+1}^-\left(c\delta_{n+1}^+ + d\delta_n^-\right)}^{u(x)}$$

848 You want a complete analysis of the periodically coupled 2 post. ~~You want~~ Various methods. Laplace transform for studying the half-line. Green's function?

basis $\delta_n^{\pm}$ $\quad n \in \mathbb{Z}$

$$u(\delta_n^+) = a\delta_{n+1}^+ + b\delta_n^-$$
$$u(\delta_n^-) = c\delta_n^+ + d\delta_{n-1}^-$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$$

Suppose you work on the half-line. Then you must decide what to solve. Remark first that $u$ yields partial unitary on the half line, special in the sense that $V^+ \perp V^-$.

What should I do? 

Why not use $(\lambda - u)^{-1}$ $\exists$ for $|\lambda| \neq 1$.

Focus upon $\psi_0 = \begin{pmatrix} \psi_0^+ \\ \psi_0^- \end{pmatrix}$. Maybe look at possible $(\lambda-u)^{-1}\psi_0$. ~~We can look at~~

Let's try to sort out the relations holding between the half line problem - Dirichlet problem, ? ~~Start with yields~~ Look at $(\lambda - u)\psi = f$. The fact is that $u$ has a special form.

Consider Green's function. You have to be resourceful in analyzing the edge effects. Basically what happens is that there is a unique decaying solution ~~to~~ the left and one to the right.

 Now begin analysis. ~~[illegible]~~ Since $\lambda - u$ is invertible it sets up an isom. between $\{\psi_0\}$ and those $\psi$ satisfying $(\lambda-u)\psi = 0$ except at ~~[illegible]~~ $n=0$. Now analyze the edge or cut using the form of $u$. Now $[(\lambda-u)\psi]^+_n$ depends on $\psi^+_n, \psi^+_{n-1}, \psi^-_n$.

$$[(\lambda-u)\psi]^+_n = \lambda\psi^+_n - a\psi^+_{n-1} - c\psi^-_n \qquad 2$$
$$[(\lambda-u)\psi]^-_n = \lambda\psi^-_n - b\psi^+_n - d\psi^-_{n+1}$$
$$\cancel{(\lambda-u)(\quad \psi^+_n\delta^+_n}$$

$$u\left(\sum_n \psi^\pm_n \delta^\pm_n\right) = \sum_n \psi^+_n\left(a\delta^+_{n+1} + b\delta^-_n\right) + \sum_n \psi^-_n\left(c\delta^+_n + d\delta^-_{n-1}\right)$$

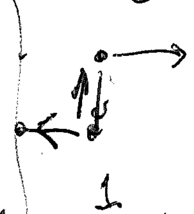
~~[illegible] What ? [illegible] you need $\lambda\psi_n$~~

Suppose $[(\lambda-u)\psi]_n = 0 \quad n \neq 0$. What you want is to be able to split the solutions $\psi$ of $[(\lambda-u)\psi]_n = 0 \quad \forall n \neq 0$ into two subspaces, roughly those with support $> 0$ and those with suppose $< 0$. From the transfer matrix picture this should be possible, i.e. the solution decaying $\to +\infty$ is a multiple of $\binom{1}{s}$ + corresponds to $|z| > 1$ for $T_1$.

So life goes on. Given that $[(\lambda-u)\psi]^\pm_n = 0 \quad n \neq 0$ can you divide things up properly.

$$[(\lambda-u)\psi]^-_1 = \lambda\psi^-_1 - b\psi^+_1 - d\psi^-_2 = 0 \qquad \text{gives } \psi^+_2 \text{ from } \psi^\pm_1$$
$$[(\lambda-u)\psi]^+_1 = \lambda\psi^+_1 - a\psi^+_0 - c\psi^-_1 = 0 \qquad \text{gives}$$

8570 Suppose you consider all $\psi$ satisfying $[(\lambda-u)\psi]_n = 0$ for $\forall n \geq 1$. Included in this subspace are all $\psi$ with ~~support~~ $\psi_n = 0$ $\forall n \geq 0$ i.e. support $\subset \{n \leq -1\}$. Quotient space is all $\psi_n$ vanishing for $n < 0$ i.e. support $n \geq 0$. ~~Basically you would like to~~ Ask about $\psi$ support $\subset \{n \geq 0\}$ $\ni (\lambda - u)\psi = 0$ in $(n > 0)$. This stuff is subtle. YES.



Ultimately you have a 4 diml space with 2 relations.

$$\lambda\psi_1^+ - a\psi_0^+ - c\psi_1^- = 0$$
$$\lambda\psi_0^- - b\psi_0^+ - d\psi_1^- = 0$$

March 17, ~~Problems / Discrete~~

The problem to understand is what happens when you cut the line in two. ~~pieces~~ You have been trying to cut at $n=0$, but ~~this~~ this seems to involve ~~cutting~~ three pieces: $n<0$, $n=0$, $n>0$. Cutting should mean splitting the Hilbert space in two. ~~Morse theory~~ This seems to be hard to correlate with Green's function

What you don't understand very well is the link between $(\lambda-u)^{-1}$ and $(\lambda - a^* b)^{-1}$ for the partial unitary given by $u$ on the half space. These should be close because they both satisfy the eigenvector equation away from the boundary. Take following viewpoint - use

807 the existence of the partial unitary resolvent to construct the Green's function.

$\exists$ something special about the partial unitary

$$H^+ = \mathbb{C}\delta_0^- \oplus aX \qquad aX \text{ spanned by } \delta_n^+, \delta_{n+1}^-, \ n \geq 0$$

$$= \mathbb{C}\delta_0^+ \oplus uX \qquad uX \text{ spanned by } \delta_{n+1}^+, \delta_n^-, \ n \geq 0$$

is that $\mathbb{C}\delta_0^-$ and $\mathbb{C}\delta_0^+$ are $\perp$. This is related to the fact that $S(\lambda)$ vanishes at $\lambda = 0$ (or $\infty$?)

~~Suppose you want to construct~~

Idea: Consider $\psi = (\lambda - u)^{-1} f$ where $f_n = 0$ $n \geq 0$.
Then $\psi$ restricted to $H^+$ should satisfy the eigenvector equation for the ~~$n \geq 0$,~~ partial unitary so $\psi$ should be ~~possible to identify~~ determined by the boundary values.

So consider $H^+ = \left\{ \sum_{n \geq 0} \psi_n^\pm \delta_n^\pm \right\} = \mathbb{C}\delta_0^- \oplus \alpha X$ domain of partial unitary

$$= \mathbb{C}\delta_0^+ \oplus \beta X \quad \text{range}$$

Start with $\psi \in H$ satisfying

$[(\lambda - u)\psi]_n = 0$ for $n \geq 0$.

In general given $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\qquad Y = aX \oplus V^+$

$$= V^- \oplus bX$$

the eigenv. eqn. is $\lambda(ax + v_0^+) = v_{-1}^- + bx$

or $(\lambda a - b)x = v_{-1}^- - \lambda v_0^+$

unique solution for any $v_{-1}^-$ and $\lambda \ni |\lambda| > 1$.

Suppose that $(\lambda - u)\psi = 0$ say for $n \geq -1$.
but we are only going to look at $(\psi_n, n \geq 0)$.
Then

852 ~~Situation~~ $Y \subset H$ with $u$

Let $aX = Y \cap u^{-1}(Y)$ $\qquad bX = u(Y) \cap Y$

$ba^{-1} = u : aX \xrightarrow{\sim} bX$. $\qquad$ Then $Y = aX \oplus \mathrm{Ker}(a^*)$

$\qquad\qquad = \mathrm{Ker}(b^*) \oplus bX$

$$\begin{tikzcd}[column sep=small]
u^{-1}(Y) & & Y & & u(Y) \\
& X \arrow[ul, dash] \arrow[ur, dash] & & u(X) \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Let $(\lambda - u)\xi = 0$ in $H$. ~~Look at~~

$$Y = \mathrm{Ker}(a^*) \oplus aX$$
$$= \mathrm{Ker}(b^*) \oplus bX$$

Write
$$\xi = \xi^- + ax + v_0^+ \qquad \in Y^\perp \oplus aX \oplus \mathrm{Ker}(a^*)$$
$$= \xi^- + v_{-1}^- + bx'$$

Then ~~$u(\xi) = \lambda u(\xi) + \lambda v$~~

$$\lambda\xi = \lambda\xi^- + \lambda v_{-1}^- + bx'$$

$\therefore \lambda x' = x$

$$u(\xi) = \underbrace{u(\xi^- + v_0^+)}_{\substack{\cap \\ Y^\perp + \mathrm{Ker}(b^*)}} + bx$$

$$H = H^- \oplus V = H^- \oplus aX \oplus \mathrm{Ker}(a^*)$$
$$= H^- \oplus \mathrm{Ker}(b^*) \oplus bX.$$

$$\xi = \xi^- + \overbrace{ax' + v_0^+}^{\xi^+} \qquad u(\xi) = u(\xi^- + v_0^+) + bx'$$
$$= \xi^- + \overbrace{v_{-1}^- + bx} \qquad \lambda(\xi) = \lambda(\xi^-) + \lambda v_{-1}^- + \lambda bx$$

$\therefore$ $x' = \lambda x$

853 Again. Given $u$ on $H$ and a closed subspace $Y$, let $X = Y \cap u^{-1}(Y)$ (= domain of partial unitary on $Y$ induced by by $u$). Then have

$$H = Y^\perp \oplus X \oplus (Y \ominus X)$$
$$= Y^\perp \oplus u(X) \oplus (Y \ominus u(X))$$

orthogonal direct sums. Let $\xi \in H$ satisfy $(\lambda - u)(\xi) \in Y^\perp$. ~~Thus~~ Write $\xi = \xi^- + x' + v_0^+$
$= \xi^- + u(x) + v_{-1}^-$. Then

$$\lambda \xi = \lambda \xi^- + \lambda u(x) + \lambda v_{-1}^-$$
$$u(\xi) = u(\xi^-) + u(x') + u(v_0^+)$$

Check that $(\lambda - u)(\xi)$, $\xi^-$, $u(\xi^-)$, $v_{-1}^-$, $u(v_0^+)$ are all perpendicular to $u(X)$. Yes. So you conclude that $\lambda x = x'$, ~~and we have~~ and so $\lambda x + v_0^+ = u(x) + v_{-1}^-$

---

Try again
$$H = Y^\perp \oplus aX \oplus \mathrm{Ker}(a^*)$$
$$= Y^\perp \oplus bX \oplus \mathrm{Ker}(b^*)$$

$$\xi = \xi^- + ax + v_0^+ = \xi^- + bx'' + v_{-1}^-$$
$$\lambda\xi = \lambda\xi^- + \underset{\text{orth}}{\lambda b x''} + \lambda v_{-1}^-$$
$$u(\xi) = u(\xi^-) + \underset{\text{orth}}{bx} + u(v_0^+)$$

~~these are~~

$$\lambda a x'' + v_0 = bx'' + v_{-1}^-$$

$\therefore$ you find $\lambda x'' = x$.

854 Try again $H$, $u$, $Y$ closed subspace

$$H = Y^\perp \oplus X \oplus (Y \ominus X) \qquad X = Y \cap u^{-1}(Y)$$
$$= Y^\perp \oplus u(X) \oplus (Y \ominus u(X))$$

$$\xi = \xi^- + x_1 + y_1 = \xi^- + u(x_2) + y_2$$

Assume $(\lambda - u)\xi \in Y^\perp$. Then

$$u(\xi) = u(\xi^-) + u(x_1) + u(y_1)$$
$$\lambda(\xi) = \lambda\xi^- + \lambda u(x_2) + \lambda y_2$$

$$\lambda u(x_2) - u(x_1) = \overset{\in u(X^\perp)}{u(\xi^-)} + u(y_1) - \overbrace{\lambda\xi^-}^{Y^\perp} - \overbrace{\lambda y_2}^{\in u(X)^\perp}$$
$$+ \underbrace{(\lambda - u)(\xi)}_{Y^\perp}$$

$\xi^- \in Y^\perp \subset X^\perp$

$\Rightarrow u(\xi^-) \in u(X^\perp) = u(X)^\perp$ $\qquad Y \supset X, u(X)$

$Y^\perp \subset (uX)^\perp = u(X^\perp)$ $\qquad Y^\perp \subseteq X^\perp$ and $u(X)^\perp$

$\therefore \lambda x_2 = x_1$

$\therefore$ find $\boxed{\lambda x_2 + y_1 = u(x_2) + y_2}$

Go over it again. You have $H = Y^\perp \oplus Y$ and

$X = Y \cap u^{-1}(Y)$ $\qquad u(X) = u(Y) \cap Y$

$$Y = X \oplus Z' = u(X) \oplus Z''$$

855 Assume $(\lambda - u)\xi \in Y^{+}$. Write

$$\xi = \xi^- + x_1 + z' \in Y^\perp \oplus X \oplus Z'$$

$$\xi = \xi^- + u(x_2) + z'' \in Y^\perp \oplus u(X) \oplus Z''$$

$$\underbrace{(\lambda - u)\xi}_{\in Y^\perp} = \underbrace{\lambda \xi^-}_{\in Y^\perp} - \underbrace{u(\xi^-)}_{\in (uY)^\perp} + \underbrace{\lambda u(x_2) - u(x_1)}_{\in u(X)} + \underset{u(X)^\perp}{\underset{\cap}{\lambda z''}} - \underset{u(X^\perp) = u(X)^\perp}{\underset{\wedge}{u(z')}}$$

$$Y \supset u(X) \implies Y^\perp \subset u(X^\perp) = u(X)^\perp$$

$$Y \supset X \implies Y^\perp \subset X^\perp \implies u(Y^\perp) \subset u(X)^\perp$$

$$\therefore \quad u(\lambda x_2 - x_1) = 0 \qquad x_1 = \lambda x_2$$

$$\lambda x_2 + z' = u(x_2) + z''$$

---

In general given $H, u$ and $Y$. Then compare

$Y \qquad u(Y) \qquad u^{-1}(Y)$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] & Y \arrow[d, "f"] \\ u^{-1}(Y) \arrow[r, hook] & Y + u^{-1}(Y) \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook, "u"] \arrow[d, "f"] & Y \arrow[d, phantom, "\cap"] \\ Y \arrow[r, "u"] & u(Y) + Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & u^{-1}(Y) \arrow[r, "\overset{u}{\sim}"] \arrow[d] & Y \arrow[d, "f"] \\ Y \arrow[r] \arrow[rr, bend right, "u"'] & Y + u^{-1}(Y) \arrow[r] & u(Y) + Y \end{tikzcd}$$

$$+ \, u^{-1}(V^-) + aX + V^+ + u(V^+) + \cdots$$

$$\boxed{V^- + bX}$$

$$Y$$

$$\lambda \xi = +\lambda u^{-1}(v_{-1}^-) + \lambda a x + \lambda v_0^+ + \lambda u(v_1^+)$$

$$u(\xi) \quad u^{-1}(v_{-2}^-) + v_{-1}^- + bx + u(v_0^+)$$

856

$$H = Y^{\perp} \oplus aX \oplus V^{+}$$
$$= Y^{\perp} \oplus V^{-} \oplus bX$$

$$\xi = \xi^{-} + ax_1 + v^{+}$$
$$\xi = \xi^{-} + v^{-} + bx_2$$

$$\lambda\xi = \lambda\xi^{-} + \lambda b x_2 + \lambda v^{-}$$
$$u(\xi) = u(\xi^{-}) + bx_1 + u(v^{+}) \qquad x_1 = \lambda x_2$$

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$$Y = aX + V^{+} \qquad \xi = ax_1 + v^{+} \qquad u(\xi) = u(v^{+}) + bx_1$$
$$= V^{-} + bX \qquad \xi = v^{-} + bx_2 \qquad \lambda(\xi) = \lambda v^{-} + \lambda b x_2$$

$$\lambda\xi = \lambda u^{-1}(v_{-1}^{-}) + \lambda a x_1 + \lambda v^{+} + \lambda u(v_1^{+})$$

$$u(\xi) = u^{-1}(v_{-2}^{-}) + v_{-1}^{-} + bx_1 + u(v^{+})$$

leads to $\quad \lambda a x + \lambda v^{+} = v^{-} + bx$

$$(\lambda a - b)x = v^{-} - \lambda v^{+}$$

---

Subtly different from

$$H = Y^{\perp} \oplus aX \oplus V^{+} \qquad \xi = \xi^{-} + ax_1 + v^{+}$$
$$= Y^{\perp} \oplus V^{-} \oplus bX \qquad \xi = \xi^{-} + v^{-} + bx_2$$

eigenvector equation is $\quad u(\xi) = \lambda\xi$

$$u(\xi) = bx_1 + u(\xi^{-} + v^{+}) \quad \text{— } \perp \text{ to } \boxed{bX}$$
diff in $Y^{\perp}$ →
$$\lambda(\xi) = \lambda b x_2 + \lambda(\xi^{-} + v^{-}) \quad \text{— } \perp \text{ to } bX$$

$\therefore \lambda x_2 = x_1 \quad$ so you get

$$a\lambda x_2 + v^{+} = v^{-} + bx_2$$

857 Another time. $H, u, Y$,

$X = Y \cap u^{-1}(Y)$. Then $Y = X \oplus V^+$
$= u(X) \oplus V^-$

Let $(\lambda - u)\xi \in Y^\perp$. Write $\xi = \xi^- \oplus x_1 \oplus v^+$
$= \xi^- \oplus u(x_2) \oplus v^-$

Then
$$u(\xi) = u(\xi^-) + u(x_1) + u(v^+)$$
$$\lambda(\xi) = \lambda\xi^- + \lambda u(x_2) + \lambda v^-$$
$$\underset{\substack{\Uparrow \\ u(X)}}{u(x_1 - \lambda x_2)} = \underbrace{u(\xi) - \lambda\xi}_{Y^\perp} \underbrace{- u(\xi^-)}_{u(Y^\perp)} \underbrace{- \lambda\xi^-}_{Y^\perp} \underbrace{- u(v^+)}_{uV^+} \underbrace{- \lambda v^-}_{V^-}$$

Observe $u(X) \subset Y, u(Y)$ so $u(X) \perp Y^\perp, u(Y^\perp), V^-$
as $X \perp V^+ \Rightarrow u(X) \perp u(V^+)$. $\therefore$ $u(x_1 - \lambda x_2) = 0$
$\Rightarrow x_1 = \lambda x_2$. Thus $\xi$ projected onto $Y$ satisfies
$$\xi - \xi^- = \boxed{\lambda x + v^+ = u(x) + v^-}$$
$$(\lambda - u)x = -v^+ + v^- \quad \text{in } Y$$
where $v^+ \perp X$ and $v^- \perp u(X)$. Let
$j : X \longrightarrow Y$ be the inclusion. ~~Then~~ Maybe $p : \cancel{X \to X}$
the projection.
$$p(\lambda - u)x = pv^-$$
$$\begin{aligned} &\parallel \\ &(\lambda - pu)x \qquad \therefore \quad x = (\lambda - pu)^{-1} p v^- \\ &\qquad\qquad\qquad\qquad\quad = p(\lambda - up)^{-1} v^- \end{aligned}$$

$$v^+ = v^- - (\lambda - u)p(\lambda - up)^{-1} v^-$$
$$= (\lambda - \cancel{u}p - (\lambda - \cancel{u})p)(\lambda - up)^{-1} v^- = \lambda(1 - p)(\lambda - up)^{-1} v^-$$
for $|\lambda| > 1$.

858 The assertion: If $(\lambda-u)\xi \in Y^{\perp}$, then the proj. of $\xi$ on $Y$ has the form

$$pr_Y(\xi) = \lambda x + v^+ = u(x) + v^-$$

So for $|\lambda| > 1$ $\quad v^+ = \lambda(1-p_X)(\lambda - up_X)^{-1}v^-$

$|\lambda| < 1$ $\quad v^- = \cancel{(1-p_{uX})(\lambda - u p_{uX})^{-1} v^+}\ (1-up_{u^*})(1-\lambda p_{u^*})^{-1}v^+$

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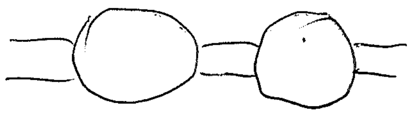
Let's look at ~~the~~ ladder.



Suppose you are after the Green's function $G$ such that $(\lambda - u)G = \delta_0^+$

So $Y$ has basis all $\delta_n^{\pm}$ except $\delta_0^+$. So

$$Y = (\delta_0^+)^{\perp} \qquad X = Y \cap u^{-1}(Y) = (\delta_0^+)^{\perp} \cap (u^{-1}\delta_0^+)^{\perp}$$

Anyway look at what happens when we take $Y$ to be spanned by $(\delta_n^{\pm}, n \geq 0)$. Then $X = (\delta_0^-)^{\perp}$ $\quad uX = (\delta_0^+)^{\perp}$. So $v^+$ mult. of $\delta_0^-$ and $v^-$ is a mult of $\delta_0^+$.

$$\begin{tikzcd} \delta_0^+ \arrow[r] \arrow[d] & \circ \\ 0 & \delta_1^- \arrow[l] \arrow[u] \end{tikzcd}$$

You should have

$aX + V^+$
$\shortparallel$
$V^- \oplus bX$

For $|\lambda| < 1$ the projection onto what do you have?

$$Y = aX \oplus V^+ = V^- \oplus bX$$

$$\lambda a x + v^+ = v^- + bx$$

$$(\lambda a - b)x = -v^+ + v^-$$

$\qquad\qquad\qquad\qquad\qquad\quad \shortparallel \qquad\quad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad \delta_0^- \qquad c(\lambda)\delta_0^+ \qquad |\lambda| < 1.$

859 So it seems that for $|\lambda|<1$ we get $c(\lambda)$ analytic for $|\lambda|<1$ such that

Do the formulas again.

$$H = Y^\perp \oplus aX \oplus V^+ \qquad \ni \xi = \xi^- + ax_1 + v^+$$
$$= Y^+ \oplus b\cancel{X} \oplus V^- \qquad = \xi^- + bx + v^-$$

$$\lambda\xi = \lambda\xi^- + \lambda b x_0 + \lambda v^-$$
$$u\xi = u\xi^- + bx_1 + uv^+$$

get $\quad \xi - \xi^- = \lambda a x + v^+ = bx + v^-$

or $\quad (\lambda a - b)x = -v^+ + v^-$

$$(\lambda b^* a - 1)\, x = -b^* v^+ \qquad x = (1-\lambda b^* a)^{-1} b^* v^+$$

$$v^- = v^+ + (\lambda a - b)\, b^* (1-\lambda a b^*)^{-1} v^+$$
$$= (1 - \cancel{\lambda a b^*} + \cancel{\lambda a b^*} - bb^*)(1-\lambda a b^*)^{-1} v^+$$

$$\boxed{v^- = (1-bb^*)(1-\lambda a b^*)^{-1} v^+} \qquad |\lambda|<1.$$

$$v^- = (1 - \underbrace{upu^*}_{pr_{uX}})(1 - \lambda \underbrace{pu^*}_{u^{-1}pr_{uX}})\, v^+$$

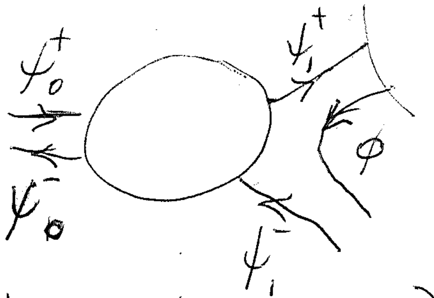
So important is that for $|\lambda|<1$ you have $v^- = S(\lambda)v^+$, i.e. solving

$$(\lambda - u)x = -v^+ + v^- \qquad \text{for } |\lambda|<1$$

leads to simply $\quad v^- = S(\lambda)v^+$. Now take $Y$ spanned by $\delta_n^\pm$ $n\geq 0$, ~~$v^+ = \delta_0^-$, $v^- = c\delta_0^+$.~~

now $\quad v^- = \varphi_0^+ \delta_0^+ \qquad v^+ = \varphi_0^- \delta_0^- \qquad$ so $\boxed{\dfrac{\varphi_0^+}{\varphi_0^-} = S(\lambda)}$

860 Check the signs out with




$$\begin{pmatrix} \psi_0^- \\ \psi_0^+ \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda a} & \frac{b}{a} \\ \frac{\bar b}{a} & \frac{\lambda}{a} \end{pmatrix} \begin{pmatrix} \psi_1^- \\ \psi_1^+ \end{pmatrix}$$

$$\begin{pmatrix} \psi_0^+ \\ \psi_0^- \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{a} & \frac{\bar b}{a} \\ \frac{b}{a} & \frac{1}{\lambda a} \end{pmatrix} \begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix}$$

$$S\phi = \psi_1^+$$
$$S\psi_1^- = \phi$$
$$S^2 = \frac{\psi_1^+}{\psi_1^-}$$

$$S = \frac{\lambda S^2 + \bar b}{b S^2 + \lambda^{-1}}$$

$$bS^3 + \lambda^{-1} S = \lambda S^2 + \bar b$$ set $\lambda = 0$

to get $S(0) = 0$. Put $S(\lambda) = \lambda \omega$, then

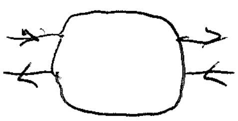
$$b\lambda^3 \omega^3 + \omega = \lambda^3 \omega^2 + \bar b$$

$$\lambda^3 (b\omega^3 - \omega^2) = \bar b - \omega$$

$$\lambda^3 = \frac{\bar b - \omega}{b\omega^3 - \omega^2} = \frac{\omega - \bar b}{\omega^2(1 - b\omega)}$$ OK it checks.

---

Go back to line. So you have $u$ and again



Consider $Y = \text{span } \delta_n^\pm \quad n \geq 0$

$X = (\delta_0^-)^\perp$, $uX = (\delta_0^+)^\perp$, $V^+ = \mathbb{C}\delta_0^-$, $V^- = \mathbb{C}\delta_0^+$

eigenvector equation says $(\lambda - u)x = -v^+ + v^-$, here

$x$ described by $\sum_{n \geq 0} \psi_n^+ \delta_n^+ + \psi_{n+1}^- \delta_{n+1}^-$ $\quad v^+ = \phi_0^- \delta_0^-$, $v^- = \phi_0^+ \delta_0^+$

wait: $\xi = \xi^- + \lambda x + v^+$
$\quad = \xi^- + u(x) + v^-$

86 | What's important is that

$$\xi^+ = \lambda x + v^+ = u(x) + v^-$$

$\xi^+$ is the restriction of [illegible] $\xi$ ~~such that~~ satisfying $(\lambda - u)\xi \in Y^\perp$. So

Suppose you start with $\xi = \psi = \sum_{n\in\mathbb{Z}} \psi_n^\pm \delta_n^\pm$ & $(\lambda - u)\psi = 0$. How to understand this?

Write $\xi = \underline{\xi^-} + \lambda x + v^+$

$$\sum_{n<0} \psi_n^\pm \delta_n^\pm + \underbrace{\sum_{n\geq 0} \psi_n^+ \delta_n^+ + \sum_{n\geq 1} \psi_n^- \delta_n^-}_{\lambda x} + \underbrace{\psi_0^- \delta_0^-}_{v^+}$$

$$\xi = \xi^- + u(x) + v^-$$

$$= \sum_{n<0} \psi_n^\pm \delta_n^\pm + \underbrace{\sum_{n\geq 1} \psi_n^+ \delta_n^+ + \sum_{n\geq 0} \psi_n^- \delta_n^-}_{u(x)} + \underbrace{\psi_0^+ \delta_0^+}_{v^-}$$

If $|\lambda| < 1$, then $v^- = S(\lambda) v^+$ so that

$$S(\lambda)\psi_0^- = \psi_0^+ \qquad \begin{pmatrix} \psi_0^+ \\ \psi_0^- \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{a} & \frac{\bar b}{a} \\ \frac{b}{a} & \frac{1}{\lambda a} \end{pmatrix} \begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix}$$

$$S = \frac{\lambda S + \bar b}{bS + \lambda^{-1}}$$

$$bS^2 + \lambda^{-1} S = \lambda S + \bar b$$

$S(\lambda)$ anal. at $\lambda = 0$ $\Rightarrow S(0) = 0$.

Put $S(\lambda) = \lambda w$

$$\lambda^2 b w^2 + w = \lambda^2 w + \bar b$$

$$\lambda^2 (bw^2 - w) = \bar b - w$$

$$\lambda^2 = \frac{\bar b - w}{bw^2 - w} = \frac{1}{w}\,\frac{w - \bar b}{1 - bw}$$

862 Is it possible to get a cont. fraction expansion for $S(\lambda)$? First observation is that

$$x^2 = kx + 1$$

$$x = k + \frac{1}{x} = k + \cfrac{1}{k + \cfrac{1}{k+}}$$

So if you have ~~$\lambda + \lambda^{-1}$ ... , then~~

~~you get~~

$$\cancel{bS + \lambda^{-1} = \lambda + \frac{\bar{b}}{S}}$$

$$\cancel{S = \frac{\lambda - \lambda^{-1}}{b} + \frac{\bar{b}}{bS}}$$

$$\cancel{\lambda S = \frac{\lambda^2 - 1}{b} + \frac{\bar{b}\lambda}{bS}}$$

$$s(\lambda) = \begin{pmatrix} \lambda & \bar{b} \\ b & \lambda^{-1} \end{pmatrix} S \qquad T_\lambda = \begin{pmatrix} \frac{\lambda}{a} & \frac{\bar{b}}{a} \\ \frac{b}{a} & \frac{\lambda^{-1}}{a} \end{pmatrix}$$

$$\lambda^2 b\omega^2 + \omega = \lambda^2\omega + \bar{b}$$

$$\lambda^2 b\omega + (1-\lambda^2) = \frac{\bar{b}}{\omega} \qquad \frac{1}{a^2} - \frac{|b|^2}{a^2} = \frac{a^2}{a^2} = 1$$

$$\lambda^2 b\omega = \lambda^2 - 1 + \frac{\bar{b}}{\omega}$$

$$\omega = \frac{\lambda^2 - 1}{\lambda^2 b} + \frac{\bar{b}}{\lambda^2 b\omega}$$

$$(1-\lambda^2)\omega = \bar{b} - \lambda^2 b\omega^2$$

$$S = \frac{\lambda^2 S + \bar{b}}{bS + \lambda^{-1}} \qquad \lambda\omega = \frac{\lambda^2\omega + \bar{b}}{b\lambda\omega + \lambda^{-1}}$$

$$\omega = \frac{\lambda^2\omega + \bar{b}}{b\lambda^2\omega + 1} = b^{-1} \qquad \overline{b\lambda^2 \omega + 1}$$

863 $S = \dfrac{\lambda S + \bar{b}}{bS + \lambda^{-1}}$ $\qquad \lambda w = \dfrac{\lambda^2 w + \bar{b}}{b\lambda w + \lambda^{-1}}$

$$w = \frac{\lambda^2 w + \bar{b}}{b(\lambda^2 w + b^{-1})} = \frac{1}{b}\left(1 + \frac{\bar{b} - b^{-1}}{\lambda^2 w + b^{-1}}\right)$$

This ~~is~~ is a ~~damn~~ puzzle. You have

$$S = \begin{pmatrix} \lambda & \bar{b} \\ b & \lambda^{-1} \end{pmatrix} S$$

probably there is a unique $S(\lambda)$ analytic for $|\lambda| < 1$ satisfying this equation.

$$w = \frac{\lambda^2 w + \bar{b}}{b(\lambda^2 w) + 1} = \bar{b} + \frac{a^2 \lambda^2 w}{1 + b(\lambda^2 w)}$$

$$w = \bar{b} + \frac{a^2 \lambda^2}{b\lambda^2 + \frac{1}{w}}$$

$$\lambda w = \bar{b}\lambda + \frac{a^2\lambda^2}{b\lambda + \frac{1}{\lambda w}} \qquad S = \bar{b}\lambda + \frac{a^2\lambda^2}{b\lambda + \frac{1}{S}}$$

$$\begin{pmatrix} 1 & \bar{b}\lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^2\lambda^2 & 0 \\ b\lambda & 1 \end{pmatrix} \doteq \begin{pmatrix} \lambda^2 & \bar{b}\lambda \\ b\lambda & 1 \end{pmatrix}$$

March 18, 1998

$$T_\lambda = \begin{pmatrix} \frac{\lambda}{a} & \frac{\bar{b}}{a} \\ \frac{b}{a} & \frac{\lambda^{-1}}{a} \end{pmatrix} = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}\begin{pmatrix} \frac{1}{a} & \frac{\bar{b}}{a} \\ \frac{b}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$$

$$\frac{\lambda S + \bar{b}}{bS + \lambda^{-1}} = \qquad S \longmapsto \lambda S \longmapsto \frac{\lambda S + \bar{b}}{b\lambda S + 1} \longmapsto \lambda\frac{\lambda S + \bar{b}}{b\lambda S + 1}$$

864 Notice that my coupling of 2 ports yields a function of $\lambda^2$ essentially

This brings up my efforts ~ 20 years ago namely to construct the Hilbert space corresponding to the "continued fraction" expansion of S. Inverse spectral problem.

Let's try to understand. Basic object is a partial unitary of rank 1:

Example. Let's start with $0 < t < 1$ and

$$S = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} (S)$$

$$S = \frac{\lambda S + t}{t\lambda S + 1} \qquad t\lambda S^2 + S = \lambda S + t$$

$$t\lambda S^2 + (1-\lambda)S - t = 0$$

$$\cancel{S = \frac{\lambda - 1 \pm \sqrt{(1-\lambda)^2 + 4t^2}}{2t\lambda}} \quad ?$$

$$\cancel{S = \frac{1-\lambda^{-1} \pm \sqrt{(1-\lambda^{-1})^2 + \frac{4t^2}{\lambda^2}}}{2t}}$$

$$\lambda(tS^2 - S) = t - S$$

$$\lambda = \frac{1}{S}\,\frac{S-t}{1-tS}$$

Consider the map $S \mapsto \lambda = \frac{1}{S}\frac{S-t}{1-tS} = \frac{1-tS^{-1}}{1-tS}$ degree 0 from $S^1$ to $S^1$. degree 2 from $S^2$ to $S^2$

In general there as two $S$ values for each $\lambda$.

$$S^2 + \frac{(1-\lambda)}{t\lambda} S - \frac{1}{\lambda} = 0$$

865 $$\frac{d\lambda}{ds} = -\frac{1}{s^2}\frac{s-t}{1-ts} + \frac{1}{s(1-ts)} - \frac{s-t}{s(1-ts)^2}(-t)$$

$$= \cancel{\frac{1}{s^2(1-ts)^2}\left\{\begin{array}{l} -(1-ts) + s(1-ts) + s(s-t)t \\ -1+ts+s-\cancel{ts^2}+\cancel{ts^2}-t^2s \end{array}\right\}}$$

$$\cancel{-1+(t+1)s-t^2s}$$

$$\cancel{= -(1-s)(1-ts)}$$

$$= \frac{1}{s^2(1-ts)^2}\left[\begin{array}{l} -(s-t)(1-ts) + s(1-ts) + s(s-t)t \\ -(\cancel{s}-t-ts^2+t^2s) + \cancel{s}-\cancel{ts^2} + \cancel{ts^2}-t^2s \end{array}\right]$$

$$+t+ts^2-t^2s-t^2s$$

$$\cancel{\quad}$$

$$t - 2t^2s + ts^2$$

$$= t(1-2ts+s^2)$$

ram. points are: $s^2-2ts+1=0$ $\quad s = t \pm \sqrt{t^2-1}$

~~the~~ two points on the circle with Re part $t$.

Alternative - look for $\lambda$ values. For what $\lambda$ are roots of $(t\lambda)s^2 + (1-\lambda)s - t = 0$ dist.

$$(1-\lambda)^2 - 4(t\lambda)(-t) = 0$$

$$(1-\lambda)^2 + 4t^2\lambda = 0 \qquad \left(\frac{\lambda^{1/2}-\lambda^{-1/2}}{2}\right)^2 + t^2 = 0$$

$$\lambda^2 + (4t^2-2)\lambda + 1 = 0. \qquad \frac{\lambda^{1/2}-\lambda^{-1/2}}{2} = it$$

?

$$\lambda = -2t^2+1 \pm \sqrt{(2t^2-1)^2-1}$$

$$\lambda = \frac{1-ts^{-1}}{1-ts} = \frac{1-t^2+t\sqrt{t^2-1}}{1-t^2-t\sqrt{t^2-1}} = \frac{\left((1-t^2)+t\sqrt{t^2-1}\right)^2}{(1-t^2)^2-t^2(t^2-1)}$$

866 $\quad S = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} S$

what about $\quad S' = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} S'$

$S' = \lambda S$

i.e. $\lambda^{-1} S' = \dfrac{S'+t}{tS'+1}$

$$tS'^2 + S' = \lambda S' + \lambda t$$

$$tS'^2 + (1-\lambda)S' - \lambda t = 0$$

$$S'^2 + \frac{1-\lambda}{t} S' - \lambda = 0.$$

$$S' = -\left(\frac{1-\lambda}{2t}\right) \pm \sqrt{\left(\frac{1-\lambda}{2t}\right)^2 + \lambda}$$

$$2tS' = -1 + \lambda \pm \sqrt{1 - 2\lambda + \lambda^2 + 4t^2\lambda}$$

analytic in $\lambda$ provided $\quad \lambda^2 + (4t^2-2)\lambda + 1 \neq 0$

Convergent series for $\quad \left|\lambda^2 + (4t^2-2)\lambda\right| < 1.$

Wait $0 < t < 1 \Rightarrow -1 < 2t^2 - 1 < 1$

so singularities are the conj points on the unit circle $\quad \lambda = -(2t^2-1) \pm \sqrt{(2t^2-1)^2 - 1}$

---

How to analyze this

I'm trying to compare

$S = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} S \quad$ with

$S_1^{\circ} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} S_1^{\circ}$

867

$$S = \frac{\lambda S + t}{t\lambda S + 1} \qquad t\lambda S^2 + S = \lambda S + t$$

$$\lambda(tS^2 - S) = t - S$$

$$\lambda = \frac{S-t}{S-tS^2} = \frac{S-t}{S(1-tS)} = \frac{1-tS^{-1}}{1-tS}$$

$$S_1 = \lambda\frac{S_1+t}{tS_1+1} \qquad \lambda = S_1\frac{1+tS_1}{S_1+t} = \frac{1+tS_1}{1+tS_1^{-1}}$$

$$t\lambda S^2 + (1-\lambda)S - t = 0 \qquad tS_1^2 + (1-\lambda)S_1 - \lambda t = 0$$

$$S^2 + \frac{1-\lambda}{t\lambda}S - \frac{1}{\lambda} = 0 \qquad S_1^2 + \left(\frac{1-\lambda}{t}\right)S_1 - \lambda = 0$$

$$S = \frac{-(1-\lambda) \pm \sqrt{(1-\lambda)^2 + 4t^2\lambda}}{2t\lambda} \qquad S_1 = \frac{-(1-\lambda) \pm \sqrt{(1-\lambda)^2 + 4t^2\lambda}}{2t}$$

$\therefore S_1 = \lambda S.$ Both are analytic except where $(1-\lambda)^2 + 4t^2\lambda = 0$

$$\lambda^2 + (4t^2 - 2)\lambda + 1 = 0$$

$$\lambda = -(2t^2-1) \pm \sqrt{(2t^2-1)^2 - 1}$$

$$= -(2t^2-1) \pm \sqrt{4(t^4-t^2)}$$

$$\boxed{\lambda = -(2t^2-1) \pm 2t\sqrt{t^2-1}}$$

~~So $\lambda$ should be~~ At this point I ~~...~~ should find the ~~...~~ range of $S(\lambda)$ for $|\lambda| < 1$. ~~...~~ Take radial limits to define $S(\lambda)$ for $|\lambda| = 1$. You have the formula above

Say $S_1(\lambda)$ there are two roots $\forall \lambda$

Describe branches

$$S_1^2 + \left(\frac{1-\lambda}{t}\right) S_1 - \lambda = 0$$

product of two roots is $-\lambda$.

$$\left(i\lambda^{-1/2} S_1\right)^2 + \left(\frac{\lambda^{-1/2}-\lambda^{1/2}}{-it}\right)\left(i\lambda^{-1/2} S_1\right) + 1 = 0$$

~~So you find that nothin~~

How to analyze

$$S_1^2 + \left(\frac{1-\lambda}{t}\right) S_1 - \lambda = 0$$

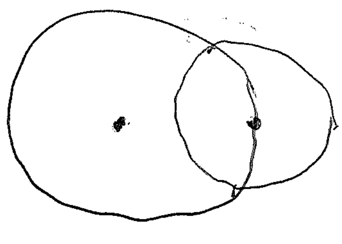
for $|\lambda| = 1$. Introduce $\mu^2 = -\lambda$

$$S_1^2 + \left(\frac{1+\mu^2}{t}\right) S_1 + \mu^2 = 0 \qquad S_1 = \mu S_2$$

$$\mu^2 S_2^2 + \left(\frac{1+\mu^2}{t}\right)\mu S_2 + \mu^2 = 0$$

$$S_2^2 + \left(\frac{\mu^{-1}+\mu}{t}\right) S_2 + 1 = 0$$

So you find that for $|\lambda| = 1$, then the two values for $S_1(\lambda)$ has product $-\lambda$ so when $\left|\frac{1-\lambda}{t}\right| < 2$ the two roots ~~are purely~~ are on the unit circle and otherwise one is inside and the outside. ~~One in~~



869 $$S = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} S$$

$$S = \frac{\lambda S + t}{t\lambda S + 1} \qquad t\lambda S^2 + (1-\lambda)S - t = 0$$

$$S^2 + \frac{1-\lambda}{t\lambda} S - \frac{1}{\lambda} = 0$$

Given $|\lambda| = 1$, the two roots have product $-\lambda^{-1}$, so ~~[illegible]~~ both are on the unit circle when $\left|\frac{1-\lambda}{t\lambda}\right| = \left|\frac{1-\lambda}{t}\right| \leq 2$ and otherwise one is inside and the other outside ~~the~~ $S^1$.

no: $(1-\lambda)^2 + 4t^2\lambda = 0$

$$\lambda^2 + (4t^2 - 2)\lambda + 1 = 0$$

$$\lambda = 2t^2 - 1 \pm \sqrt{(2t^2-1)^2 - 1}$$

$$S = \frac{-(1-\lambda) \pm \sqrt{(1-\lambda)^2 + 4t^2\lambda}}{2t\lambda}$$

$$\left(\frac{1-\lambda}{2t\lambda}\right)^2 = \frac{1}{\lambda}$$

$$S = t \pm \sqrt{t^2 - 1}$$

$$S^2 = 2t^2 - 1 \pm 2t\sqrt{t^2-1}$$

diag.

$$\lambda(t^2 S^2 - S) = t - S$$

$$\lambda = \frac{S - t}{S(1 - tS)} = \frac{1 - tS^{-1}}{1 - tS}$$

$$\frac{d\lambda}{dS} = -\frac{1}{S^2}\frac{S-t}{(1-tS)} + \frac{1}{S(1-tS)} - \frac{(S-t)(-t)}{S\,(1-tS)^2}$$

$$= \frac{1}{S^2(1-tS)^2}\left[-(S-t)(1-tS) + \cancel{S(1-tS)} + tS(S-t)\right]$$

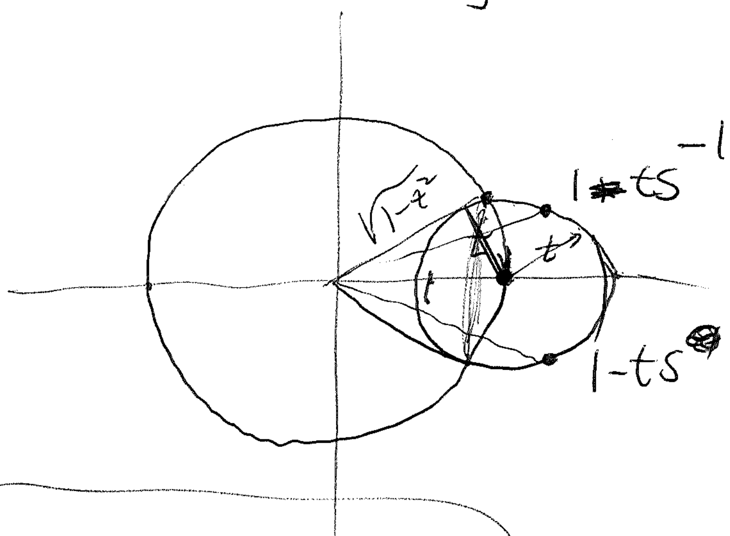
$$t - t^2S + tS^2 - t^2S$$

$$t - 2t^2S + tS^2$$

$$= \frac{t}{S^2(1-tS)^2}(S^2 - 2tS + 1) \qquad S = t \pm \sqrt{t^2-1}$$

870 So look at $S \mapsto \lambda = \frac{1-tS^{-1}}{1-tS}$

S

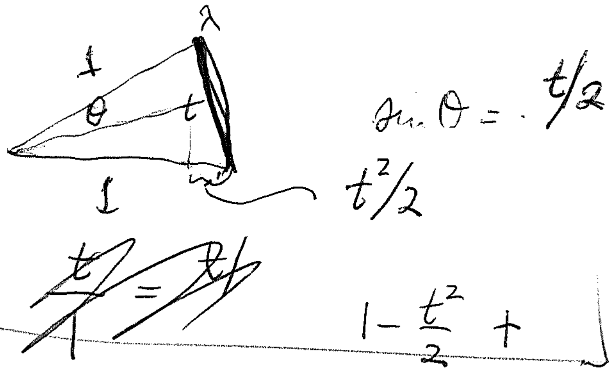



separate question

Consider $L^2(S^1)$ an

a unitary of $u$ = mult.

by $\frac{z-h}{1-\bar{h}z} = \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix}(z)$




Now look at the partial unitary defined on $H^2(S^1)$

Problem: ~~Schwartz~~ Given $S(\lambda)$ bdd by $|\ | \leq 1$ anal. for $|\lambda|<1$.

then you get cont. frac.

$$S(\lambda) = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 0 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & h_2 \\ \bar{h}_2 & 1 \end{pmatrix} \cdot S_n(\lambda)$$

Find partial unitary Hilb. space. Case 1. $S(\lambda) \in \mathbb{C}(\lambda)$ and $|S(\lambda)|=1$

on $|\lambda|=1$. So what method. Something like

$H^2/SH^2$. Partial unitary. Yes.

~~[illegible]~~ ~~Yes! OK~~

Filtration?

You have a clutching function no.

871 March 19.

Problem is to find Hilbert space picture corresponding to the partial fraction ~~decomposition~~ expansion

$$S(z) = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots$$

of an analytic $S(z)$ in $|z|<1$ sat. $|S(z)| \le 1$. What I actually have constructed is ~~too big~~

$$\begin{pmatrix} \psi_0^+ \\ \psi_0^- \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\lambda}{a} & \frac{\bar{b}}{a} \\ \frac{b}{a} & \frac{1}{\lambda a} \end{pmatrix}}\begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix}$$

$$\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}\begin{pmatrix} \frac{1}{a} & \frac{\bar{b}}{a} \\ \frac{b}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$$

So we get

$$S(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \bar{b}_0 \\ b_0 & 1 \end{pmatrix}\begin{pmatrix} \lambda^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \bar{b}_1 \\ b_1 & 1 \end{pmatrix} \cdots$$

~~So we have to compress further.~~ This should be any odd $S(\lambda)$.

Consider $S(z)$ when the expansion is finite. Process stops when $|h_n| = |S_n(0)| = 1$, whence

$$S(z) = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}(h_n)$$

with $|h_0|, \ldots, |h_{n-1}| < 1$ and $|h_n| = 1$. ~~Such an S is the scattering operator for meromorphic~~ Clutching function. How does it arise? Divisor in $|z|<1$ yields outgoing subspace $SH^2$ in $H^2 = H^+$

$$\begin{matrix} H^+ & SH^+ \\ zH^+ & zSH^+ \end{matrix}$$

$\mathcal{O}(-1)\otimes X \Rightarrow \mathcal{O}\otimes Y$

It's hopeless.

 ~~[illegible]~~

Back to $H, u, Y, \quad X = Y \cap u^{-1}(Y)$

$$H = Y^\perp \oplus X \oplus V^+$$
$$= Y^\perp \oplus uX \oplus V^-$$

Let $\xi \in H$ sat. $(\lambda - u)\xi \in Y^\perp$. We have

$$\xi = \xi^- + x_1 + v^+$$
$$\xi = \xi^- + u(x_2) + v^-$$
$$u(\xi) = u(\xi^-) + u(x_1) + u(v^+)$$

$$\underbrace{\lambda\xi - u(\xi)}_{\in Y^\perp} = \underbrace{u(\lambda x_2 - x_1)}_{\in uX} + \underset{\in Y^\perp}{\lambda\xi^-} + \underset{V^-}{\lambda v^-} - \underset{\underset{u(X)}{\perp}}{u(\xi^-)}_{\cap\, u(Y^\perp)} - \underset{\underset{uX}{\perp}}{u(v^+)}_{\cap\, uV^+}$$

$\therefore \; u(\lambda x_2 - x_1) = 0 \qquad x_1 = \lambda x_2.$

$\therefore$ have $\quad \xi = \xi^- + \lambda x + v^+ = \xi^- + u(x) + v^-.$

yielding $\quad (\lambda - u)(x) = -v^+ + v^-$

---

I've gone from $H, u, Y$ to a partial unitary on $Y: X \xrightarrow{\sim} Y$

Note that $\lambda\xi - u(\xi) \in Y^\perp$ can be relaxed to

$$\lambda\xi - u(\xi) \in (uX)^\perp = (uY \cap Y)^\perp \supseteq uY^\perp + Y^\perp$$

so it clear that $\xi^-$ is irrelevant.

Observe $\quad u : \underbrace{Y^\perp \oplus V^+}_{X^\perp} \xrightarrow{\sim} \underbrace{Y^\perp \oplus V^-}_{uX^\perp}$

873 Starting from $H, u, Y$ you get a partial unitary $X \underset{u}{\hookrightarrow} Y$. Conversely suppose given a partial unitary $X \underset{b}{\overset{a}{\rightrightarrows}} Y$. Then have $Y = aX \oplus V^+ = bX \oplus V^-$ so to extend to a unitary you need $Y^\perp$ together with an isomorphism $Y^\perp \oplus V^+ \simeq Y^\perp \oplus V^-$.

Can the possible extensions be organized in the GNS spirit? So what?

~~Stable isomorphisms. Let us take~~

What are possible stable isos. $V^+ \xrightarrow{\sim} V^-$?

If $V^- = 0$, then need $Z = Y^\perp$ + isom $Z \simeq V^+ \oplus Z$

Then get isom. $s : Z \to Z$ and

$$Z = V^+ \oplus sZ = \overline{\left(V^+ \oplus sV^+ \oplus s^2 V^+ \oplus \cdots\right)} \oplus \bigcap_{n \geq 0} s^n Z$$

What about the scattering game.

$$aX \oplus V^+ \oplus zV^+ \oplus \cdots$$

$$V^- \oplus bX \oplus zV^+ \oplus \cdots$$

---

Start again.

Given $H, u, Y$ get partial unitary

$Y = aX \oplus V^+ = bX \oplus V^-$ and eigenv. equation $(a\lambda - b)(x) = -v^+ + v^-$ s.t. ~~which has~~

1) $\forall \lambda, |\lambda| > 1, \forall v^-, \exists !$ solution

2) $\forall \lambda \ |\lambda| < 1, \ \forall v^+, \exists !$ solution $S(\lambda) v^+$ anal in $\lambda$ $\| \ \| \leq \|v^+\|$

Also get solution of eigenvector equation from any $\xi$ in $H$ $\ni (\lambda - u)\xi \in Y^\perp + u(Y^\perp)$ (can assume $\xi \in Y$). Solutions of eigenv. equation same as $\xi \in Y \ni (\lambda - u)(\xi) \in Y^\perp + u(Y)^\perp$

874 ~~[illegible]~~

~~Conversely you start with $Y = aX \oplus V^+ = bX \oplus V^-$~~
~~and then you show that~~

Can you reconstruct the partial unitary from $S(\lambda) : V^+ \to V^-$ for $|\lambda| < 1$?

First case: Assume $S(\lambda)$ meromorphic in $\lambda$ and $|S(\lambda)| = 1$ where $|\lambda| = 1$. We know then ?

First a general construction starting from $Y = aX \oplus V^+ = bX \oplus V^-$. Take $Y^\perp = \underbrace{\overset{(2)}{\bigoplus_{n\geq 1}} z^n V^+}_{H^+} \oplus \underbrace{\overset{(2)}{\bigoplus_{n\geq 1}} z^{-n} V^-}_{H^-}$ and $H = Y^\perp \oplus Y$.

$$\oplus\ z^{-2}V^- \oplus z^{-1}V^- \oplus\ aX \oplus V^+ \oplus\ zV^+ \oplus z^2V^+$$

$$\overset{ba^{-1}}{\cdots\to}$$

$$\oplus\ z^{-1}V^{-1} \oplus\ V^- \oplus bX\ \oplus zV^+ \oplus z^2V^+$$

define $u =$ mult. by $z$ on $(aX)^\perp$ and $ba^{-1}$ on $aX$. Note that $H$ is generated by $Y$. Observe that $H = H^- \oplus Y \oplus H^+$. Also

$u(H^+) \subset H^+$, $u(Y \oplus H^+) \subset Y \oplus H^+$

$u(H^- \oplus Y) \supset H^- \oplus Y$, $u(H^-) \supset H^-$. By general theory we have a contraction operator $\gamma$ on $Y$ such that $H$ is the corresponding dilation. critical thing is that $pr_Y(u^n y) = \gamma^n y \quad n \geq 0$

$= (\gamma^*)^{-n} y \quad n \leq 0$

Keep on going

---

Assume $Y$ f.d. Can you show $S$ unitary.

$$(\lambda a - b)x = -v^+ + v^-$$

$S(\lambda)v^+$

$$|\lambda|^2 \|x\|^2 + \|v^+\|^2 = \|x\|^2 + \|v^-\|^2$$

$\|v^+\|^2 = (1 - |\lambda|^2)\|x\|^2 + \|v^-\|^2 \geq \|v^-\|^2.$

§75 Assume $S(\lambda)$ given rational function of $\lambda$ analytic for $|\lambda| \leq 1$, $|S(\lambda)| = 0$ if $|\lambda| = 1$.

Then get
$$\begin{tikzcd} H^2 \arrow[r, "aX"] & \check{S} H^2 \\ zH^2 \arrow[u, "V^-"] \arrow[r, "uaX"] & SzH^2 \arrow[u, "V^+"'] \end{tikzcd}$$

$$Y = H^2 \ominus SzH^2 \qquad X = H^2 \ominus SH^2$$

$$\xi = \xi^- + ax_1 + v^+$$
$$\lambda\xi = \lambda\xi^- + \lambda b x_2 + \lambda v^-$$
$$u(\xi) = u(\xi^-) + b x_{\cancel{1}} + b v^+ \qquad \Rightarrow \lambda x_2 = x_1$$

$$\boxed{\lambda a x_{2} + v^+ = b x_2 + v^-}$$

$\xi$: $\quad z^{-1} v^-_{-1} + a x + v^+_0 + z v^+_1$

$u(\xi)$ $\quad z^{-1} v^-_{-2} + v^-_{-1} + b x + \cancel{z}(v^+_0)$

$$\boxed{\lambda(ax + v^+_0) = v^-_{-1} + bx}$$

$$\xi = ax + v^+_0 = \lambda^{-1} v^-_{-1} + b\lambda^{-1} x$$

---

try again

suppose $\xi = \xi^- + ax_1 + v^+ = \xi^- + v^- + bx_2$

such that $(\lambda - u)(\xi) \perp uX \quad \Rightarrow x_1 = \lambda x_2$

$$\lambda a x_2 + v^+ = v^- + bx_2$$

876 Continue. Given $S(\lambda)$ analytic for $|\lambda| \leq 1$ and $|S(\lambda)| = 1$ for $|\lambda| = 1$. Then $S(\lambda)$ is a rational function of $\lambda$. ~~and~~ ~~we know~~ $S(z)$ is unitary commuting with $z$ on $L^2(S^1)$, $SH^2 \subset H^2$, so you can put $Y = H^2 \ominus zSH^2$, $aX = H^2 \ominus SH^2$ then $bX = zH^2 \ominus zSH^2$. $V^- = \mathbb{C}1$ and $V^+ = S\mathbb{C}$. WAIT

$$aX \oplus V^+ = V^- \oplus bX$$

Go back to $\quad (\lambda a - b)x = -v^+ + v^-$

$|\lambda| < 1 \quad (\lambda b^* a - 1)x = -b^* v^+$

Problem is that mult. by $S$ goes from $V^- = \mathbb{C}1$ to $V^+ = \mathbb{C}S$, whereas the eigenvector equation $(\lambda a - b)(x) = -v^+ + v^-$ for $|\lambda| < 1$ yields a map from $V^+$ to $V^-$. So how can we handle this?

~~At the moment you have simply~~

Go over the above. Namely suppose given $Y = aX \oplus V^+ = V^- \oplus bX$ and you form

$$H = \cdots \oplus z^{-2}V^- \oplus z^{-1}V^- \oplus aX \oplus V^+ \oplus zV^+ \oplus \cdots$$

with the evident unitary $u$. Eigenvector equation

$$\lambda a x + v^+ = v^- + bx$$

$$\lambda^2 z^{-2} v^- + \lambda z^{-1} v^- + \underset{\substack{\| \\ v^- + bx}}{\lambda a x + v^+} + \lambda^{-1} z(v^+) + \lambda^{-2} z^2(v^+)$$

$S(\lambda) v^+$

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$$\left(1-\lambda \bar{z}^{-1}\right)^{-1} S(\lambda) v^+ + bx$$
$$= \lambda a x + \left(1-\lambda^{-1} z\right) v^+$$

$$(\lambda a - b) x = \left[\left(1-\lambda z^{-1}\right)^{-1} S(\lambda) - \left(1-\lambda^{-1} z\right)^{-1}\right] v^+$$

$$(\lambda a - b) x = S(\lambda) v^+ - v^+$$

---

You want to start with $S(\lambda)$ and reconstruct the partial unitary. Somehow your assuming $S(\lambda)^* S(\lambda) = 1$ restricts the argument.

The situation: Given $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\quad Y = aX \oplus V^+ = V^- \oplus bX$

you ~~construct~~ solve the eigenvector equation

$$(\lambda a - b) x = -v^+ + v^-$$

for $\begin{pmatrix} |\lambda| < 1 \\ \forall v^+ \end{pmatrix}$ and $\begin{pmatrix} |\lambda| > 1 \\ \forall v^- \end{pmatrix}$

Now you would like to reverse the process, somehow from $S(\lambda) : V^+ \longrightarrow V^-$ $\quad \forall |\lambda| < 1$

and $S(\lambda)^{-1} : V^- \longrightarrow V^+$ $\quad \forall |\lambda| > 1$

reconstruct the partial unitary.

This does not look easy. Your missing the bound states! Maybe you should try to understand the general case:

$$H^2_-(S^1, V^-) \oplus aX \oplus H^2_+(S^1, V^+)$$

$$\overset{(2)}{\bigoplus_{n<0}} z^n V^- \qquad \overset{(2)}{\bigoplus_{n\geq 0}} z^n V^+$$

This is some sort of universal construction

not symm. Instead use $Y$.

$$\bigoplus_{n\le -1}^{(2)} z^n V^- \oplus Y \oplus \bigoplus_{n\ge 1}^{(2)} z^n V^+$$

$$L^2(S^1,V^-) \underset{S(z)\ \ |z|<1 \text{ anal. extn.}}{\overset{S(\cancel{z})^*\ \ \text{anal ext. } |z|>1}{\rightleftarrows}} L^2(S^1,V^+)$$

One thing we know is that $S(z)$ is in $L^\infty$ from Hilb. space theory. What do you do for $S^*S = SS^* = 1$. ~~Starting~~ from $S(z)$. Refer everything to $L^2(S^1,V^+)$. Inside here you will find the image of $V^-$ which will have powers $z^n$ $n\le 0$. $V^-$ in $L^2(S^1,V^-)$ will go to a subspace $W \subset L^2(S^1,V^+)$ such that $W \perp z^n W$ all $n\neq 0$, $W = T(z)V^+$ where $T$ anal for $|z|\geq 1$ unitary boundary values.

Probably $T(z) = S(z)^{-1}$ so $V^- = S(z)^{-1}V^+$ ?

$$z^{-1}V^- \oplus Y \oplus zV^+ \oplus$$

So how might I handle this? You have $\dots z^{-1}V^- \oplus Y \oplus zV^+ \dots$ Recall one can assume $\overline{aX+bX} = Y$ because can split off $(aX+bX)^\perp = V^+ \cap V^-$

$$\oplus\, z^{-1}V^- \oplus aX \oplus V^+ \oplus zV^+$$
$$V^- \oplus bX \oplus$$

Somehow you are very confused.

879 To simplify suppose $V^+, V^-$ one-diml.

form $H = \cdots \oplus z^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{V^- \oplus bX} \oplus zV^+$

Consider eigenvectors w eigenvalue $\lambda$. Know these have form $\cdots + \lambda z^{-1}v^- + \lambda ax + v^+ + \lambda^{-1}zv^+ +$

$$\lambda ax + v^+ = v^- + bx$$

Thus if we fix $\lambda$ and look at a corresp eigenvector $\xi$, then

$\xi_n \sim \lambda^n z^{-n} v^-$ $\quad n \leq 0$

$\xi_n \sim \lambda^n z^{-n} v^+$ $\quad n \geq 0.$

and $S(\lambda)$ should set up the correspondence. Which way should $S$ go? I would like to think $S(\lambda): V^- \to V^+$. If so then

$$(\lambda a - b)x = -v^- + v^+ = S(\lambda)v^- - v^-$$

$S(\lambda)$ defined this way should be analytic for $|\lambda| > 1$

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March 20 Begin with $aX \oplus V^+ = V^- \oplus bX$ and try to find $u^n(V^-)$

$$V^- = aa^*V^- + (1-aa^*)V^-$$

$$u(V^-) = ba^*V^- + z(1-aa^*)V^-$$

$$= aa^*ba^*V^- + (1-aa^*)ba^*V^- + z(1-aa^*)V^-$$

$$u^2(V^-) = (ba^*)^2V^- + \pi[z\,ba^* + z^2]V^-$$

$$u^3(V^-) = (ba^*)^3V^- + \pi[z(ba^*)^2 + z^2(ba^*) + z^3]V^-$$

 The scattering operator should be

$$\pi \sum_{n\geq 0} (z^{-1}ba^*)^n V^- = (1-aa^*)(1-z^{-1}ba^*)^{-1} V^-$$

Summary.

$$H, u, Y \rightsquigarrow \begin{array}{l} Y = aX + V^+ \\ \quad = bX + V^- \end{array} \longrightarrow (\lambda a - b)x = -v^+ + v^-$$

for $|\lambda| > 1$. $\quad v^+ = (1-aa^*)(1-\lambda^{-1}ba^*)^{-1}V^- = S^-V^-$

$$|\lambda|^2\|x\|^2 + \|v^+\|^2 = \|x\|^2 + \|V^-\|^2$$

$$\|v^-\|^2 = (|\lambda|^2 - 1)\|x\|^2 + \|V^+\|^2$$

so $\|v^+\|^2 \leq \|v^-\|^2$

for $|\lambda| < 1$ $\quad v^- = (1-bb^*)(1-\lambda ab^*)^{-1} v^+ = S^+V^+$

and $\|v^-\|^2 \leq \|v^+\|^2$.

Now $\begin{array}{l} S^-(\lambda) \text{ for } |\lambda| > 1 \\ S^+(\lambda) \text{ for } |\lambda| < 1 \end{array}$ have $L^\infty$ boundary values on $|\lambda| = 1$.

which are necessarily contraction operators: $\| \ \| \leq 1$.

$$S^+(\lambda) = (1-bb^*)(1-\lambda ab^*)^{-1}(1-aa^*)$$

$$S^+(\lambda)^* = (1-aa^*)(1-\underset{\underset{\lambda^{-1} \text{ when } |\lambda|=1.}{\|}}{\bar\lambda} ba^*)^{-1}(1-bb^*)$$

so the ~~boundary values~~ $L^\infty$ operators are adjoint

Problem: How to reconstruct $Y = aX + V^+ = bX + V^-$ from $S(\lambda)$. ~~(or what happens?)~~

~~Now start with the $S^+(\lambda)$ an $L^\infty$ function of $\lambda$ with values in maps~~

Reconstruct. You are given $V^\pm$ and $S(\lambda) : V^- \to V^+$ an $L^\infty$ function of $\lambda \in S^1$ with $\|S(\lambda)\| \leq 1$. S can be identified with an operator $L^2(S^1, V^-) \longrightarrow L^2(S^1, V^+)$ commuting with z-mult, having $\|S\| \le 1$.

You want to dilate somehow. Your $H$ (w. $u$) should contain $L^2(S^1, V^-)$ and $L^2(S^1, V^+)$, and $L^2(S^1, V^-) \hookrightarrow H \xrightarrow{p} L^2(S^1, V^+)$ should be $S$. There should be an obvious procedure for dilating $\gamma : V^- \longrightarrow V^+$ of norm $\le 1$.

Concentrate. Look for $V^- \xrightarrow{j} H \xleftarrow{k} V^+$ $\ni$ $\gamma = k^* j$ $\quad j^* j = 1$, $k^* k = 1$. Then

$$\| j v_1 + k v_2 \|^2 = \|v_1\|^2 + \|v_2\|^2 + (v_1, \overset{\gamma}{j^* k}\, v_2) + (v_2, \gamma v_1)$$

$$= \|v_1 + \gamma v_2\|^2 + \|v_2\|^2 - \|\gamma v_2\|^2$$

$$= \|v_2 + \gamma^* v_1\|^2 + \|v_1\|^2 - \|\gamma^* v_1\|^2$$

The point is there's a canonical way to ~~treat~~ a contraction $\gamma : V^- \longrightarrow V^+$, and if we do this for $S : L^2(S^1, V^-) \longrightarrow L^2(S^1, V^+)$ ~~or even $S(\lambda) : V^- \to V^+$ for each $\lambda$, then we get~~ a canonical Hilbert space $H$ with $u$. I think you can also view this $H$ as obtained from the individual $S(\lambda) : V^- \to V^+$, i.e. you get a family of Hilbert spaces $V_\lambda$ over $S^1$

So given $S(\lambda) : V^- \to V^+$ $L^\infty$ you get $H, u$.

Next use analyticity of $S(\lambda)$ in $|\lambda| > 1$. This should give the half spaces you want. Also if you know $S(\lambda)$ unit on $S^1$ then $\underbrace{L^2(S^1, V^-) \xrightarrow{\sim} H \xleftarrow{\sim} L^2(S^1, V^+)}_{S}$

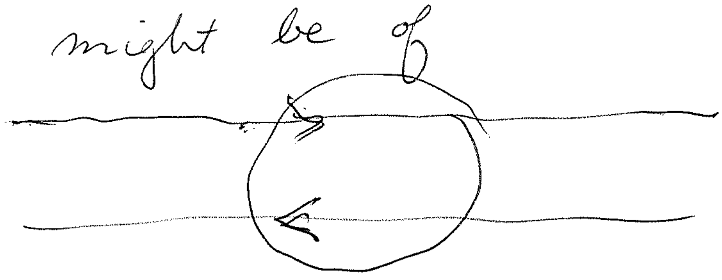
 You need a "formula" for $Y$.

You're given $S(\lambda): V^- \longrightarrow V^+$ which you use to construct $H$ containing $L^2(S^1, V^-)$ and $L^2(S^1, V^+)$. Then $H_-^2(S^1, V^-) = \bigoplus_{n<0} z^n V^-$ gives $H^- \subset H \ni uH^- \supset H^-$ and $H_+^2(S^1, V^+) = \bigoplus_{n\geq 0} z^n V^+$ gives $H^+ \ni uH^+ \subset H^+$. Analyticity of $S(\lambda)$ for $|\lambda|>1$ should imply $uH^-$, $H^+$ are $\perp$. Can define $Y$ by $H^- \oplus Y \oplus H^+$.

The idea is to rescale - replace $S(\lambda)$ by $S(\lambda/r)$ and let $r \uparrow 1$. Thus you assume first $S(\lambda)$ is analytic for $|\lambda| > r$. In this case $H$ should essentially be $L^2(S^1, V^+ \oplus V^-)$ and things should be easy to describe. ~~Same trans. coef.~~

Picture might be of

$\frac{\log n}{n}$



with transmission + reflection.

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$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t} \qquad s\Gamma(s) = \int_0^\infty e^{-t} d(t^s)$$

$$= \underbrace{\left[e^{-t}t^s\right]_0^\infty}_{0} + \underbrace{\int_0^\infty e^{-t} t^s dt}_{\Gamma(s+1)}$$

$$\Gamma(s)\zeta(s) = \sum_{n=1}^\infty \int_0^\infty e^{-nt} t^s \frac{dt}{t}$$

$$= \int_0^\infty \frac{e^{-t}}{1-e^{-t}} t^s \frac{dt}{t} = \int_0^\infty \frac{1}{e^t - 1} t^s \frac{dt}{t} \qquad \mathrm{Re}(s) > 1$$

$$= \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^s \frac{dt}{t} \qquad 0 < \mathrm{Re}(s) < 1$$

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$$\frac{1}{e^t-1} - \frac{1}{t} = \frac{1}{t+\frac{t^2}{2}} - \frac{1}{t} =$$

$$\frac{1}{t}\left(\frac{1}{1+\frac{t}{2}+\frac{t^2}{6}} - 1\right) = \frac{1}{t}\left(\cancel{1} - \left(\frac{t}{2}+\frac{t^2}{6}\right) + \left(\frac{t}{2}+\frac{t^2}{6}\right)^2\right)$$

$$= \frac{1}{t}\left(-\frac{t}{2} + t^2\left(-\frac{1}{6}+\frac{1}{4}\right)\right)$$

$$= -\frac{1}{2} + t\left(\frac{1}{12}\right) + O(t^2)$$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t^2} t^{s}\, 2\frac{dt}{t}$$

---

Start with $S(\lambda) : V^- \longrightarrow V^+$ anal. $|\lambda| > 1$.

Form $\sqrt{1-S^*S}$ $S^*$

$S$ $\sqrt{1-SS^*}$

Basically $H$ = completion of $L^2(S^1, V^-) \oplus L^2(S^1, V^+)$
with $\| j\xi_1 + k\xi_2\|^2 = \|\xi_1\|^2 + \|\xi_2\|^2 + (S\xi_1, \xi_2) + (\xi_2, S\xi_1)$

$S(\lambda)$ analytic for $|\lambda| \geq 1$. means that

$H^2(S^1, V^-)$

---

Analyze basic construction with a contraction $\gamma$.
$\gamma : V^+ \longrightarrow V^-$ $\quad \binom{1}{\gamma} V^+$ is pos. subspace
for $\|v^+\|^2 - \|v^-\|^2$. ~~Corresp. subspace~~ What is the
orthogonal wrt pseudo hermitian product

$$0 = \left\langle \binom{1}{\gamma} v^+, \binom{x}{-y} \right\rangle = (v^+, x) - (\gamma v^+, y)$$

$$= (v^+, x - \gamma^* y) \quad \therefore \quad x = \gamma^* y \quad \binom{\gamma^*}{1} V^-$$

884 Given polarized $H^+ \oplus H^-$ then another polarization $V^+ + V^-$ has the form $V^+ = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} H^+$, $V^- = \begin{pmatrix} \gamma^* \\ 1 \end{pmatrix} H^-$, where $\gamma : H^+ \to H^-$ is a contraction. We have a pseudo unitary mapping the original pol. to the other one.

$$\begin{pmatrix} 1 & \gamma^* \\ \gamma & 1 \end{pmatrix}$$

$$H^+ \oplus H^- = V^+ \oplus V^-$$

$$\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \begin{pmatrix} (1-\gamma^*\gamma)^{-1/2} & \gamma^* \\ \gamma(\ )^{-1/2} & (1-\gamma\gamma^*)^{-1/2} \end{pmatrix} \begin{pmatrix} h^+ \\ h^- \end{pmatrix}$$

$$v^+ = (1-\gamma^*\gamma)^{-1/2} h^+ + \gamma^*(1-\gamma\gamma^*)^{-1/2} h^-$$

$$v^- = \gamma(1-\gamma^*\gamma)^{-1/2} h^+ + (1-\gamma\gamma^*)^{-1/2} h^-$$

$$\cancel{\|v^+\|^2 = (h^+, (1-\gamma^*\gamma) h^+)}$$

$$\begin{pmatrix} (1-\gamma^*\gamma)^{-1/2} & \gamma^*(1-\gamma\gamma^*)^{-1/2} \\ \gamma(1-\gamma^*\gamma)^{-1/2} & (1-\gamma\gamma^*)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} (1-\gamma^*\gamma)^{-1/2} & \gamma^*(1-\gamma\gamma^*)^{-1/2} \\ \gamma(1-\gamma^*\gamma)^{-1/2} & (1-\gamma\gamma^*)^{-1/2} \end{pmatrix}$$

$$\begin{pmatrix} (1-\gamma^*\gamma)^{-1} - \gamma^*\gamma(1-\gamma^*\gamma)^{-1} & \gamma^*(1-\gamma\gamma^*)^{-1} - \gamma^*(1-\gamma\gamma^*)^{-1} \\ & \end{pmatrix}$$

What are you learning? A contraction op.

You've learned something about a contraction $\gamma : H^+ \longrightarrow H^-$ namely ? It gives you a Hilbert space completion of $H^+ \oplus H^-$

$$\| g\xi_1 + k\xi_2 \|^2 = \|\xi_1\|^2 + \|\xi_2\|^2 + (\gamma\xi_1, \xi_2) + (\xi_2, \gamma\xi_1)$$

$$= \|\xi_2 + \gamma\xi_1\|^2 + \left(\|\xi_1\|^2 - \|\gamma\xi_1\|^2\right)$$

$$= \|\xi_1 + \gamma^*\xi_2\|^2 + \left(\|\xi_2\|^2 - \|\gamma^*\xi_2\|^2\right)$$

Call this Hilbert space $Y$, then you have

$$Y = H^+ \oplus \overbrace{\text{completion of } H^- \text{ with norm } \|\xi_2\|^2 - \|\gamma^*\xi_2\|^2}^{V^-}$$

$$\simeq H^- \oplus \underbrace{\text{———— } H^+ \text{ ———— } \|\xi_1\|^2 - \|\gamma\xi_1\|^2}_{V^+}$$

$$\boxed{Y = H^+ \oplus V^- \simeq H^- \oplus V^+}$$ "unitary" picture

But there is also the pseudo-~~unitary~~ picture when $\|\gamma\| < 1$, namely an isomorphism

$$H^+ \oplus H^- \simeq V^+ \oplus V^-$$

preserving pseudo-herm. product.

$$\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \Big( 1$$

Not clear enough. Anyway let's go on ~~to~~ $\gamma = S(\lambda) : V^- \longrightarrow V^+$ contraction $\forall \lambda$.

Suppose that. $|S(\lambda)| \leq 1-\varepsilon$ for $|\lambda| = 1$. Then for each $\lambda$ you get a pol. of $V^+ \oplus V^-$