369 Jan 1, 98 In the real symplectic you need an analogue of S. Recall that for LC situation you have a real pol. Hilb space $H = H^+ \oplus H^-$ ~~and a $V \subset H$ a subquotient~~ another polarization $V \oplus V^\perp$ and comparing the two leads to ~~eigenvalues~~. spectral decomposition. Real symplectic case: Consider a polarized real symp. vector space, ~~standard thing is $\mathbb{C}^n$ with $\xi^t \eta = (\xi, \eta)$ +~~ i.e. complex Hilbert space H, say $\mathbb{C}^n$ with $\xi^* \eta = \sum \bar{\xi}_i \eta_i$. Polarization is I = mult by i. Consider another polarization J. J symplectic $J^2 = -1$. Siegel UHP description is appropriate to a real hyperbolic description.

$$i(ij)(-i) = ji = (ij)^{-1}$$

I want to describe a polar j via C.T. Suppose

$$ji = \frac{1+X}{1-X} \ ?$$

Wait. j, i generate quaternionic group. What are its irred representation?

Maybe better idea is to think of a polarization as an appropriate ~~splitting of~~ Hodge splitting of $V \otimes_{\mathbb{R}} \mathbb{C}$

~~Notation: V. The first~~

The first thing to do is to describe two complex structures IJ on a real vector space V by passing to $V \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}$. Then you get a representation of the dihedral group on $V_{\mathbb{C}}$ whence $F = i \otimes J$ $E = i \otimes I$. Then $FE = -1 \otimes JI$ so you look at the spectral decomp.

What should happen is that V_c will split into

Consider V a $\mathbb{C}$-v.s., $\mathbb{R}$-v.s. with of I sat $I^2 = -1$. $V_c = \mathbb{C} \otimes_{\mathbb{R}} V$ cx. vs. with $I^2 = -1$, so $V_c = V^+ \oplus V^-$

$F = -iI$ $\qquad -iI = 1$ means $I = i$

Given

370 V complex v.s. $=$ $\mathbb{R}$ v.s. with I sat $I^2=-1$

$V_c = \mathbb{C}\otimes_{\mathbb{R}} V = V_c^+ \oplus V_c^-$ where $I = \pm i$, $\varepsilon = -iI$

Let J be another complex st. on V, $J^2=-1$, $F = -iJ$.

Look at $g = F\varepsilon = (-iJ)(-iI) = -JI$. If $(g-1)^{-1}\exists$,

then $g = \dfrac{1+X}{1-X}$ where $X = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$ somehow

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Again: V real vector with 2 complex st. I, J : $I^2 = J^2 = -1$.

$V_c = V_c^+ \oplus V_c^-$ where $I = \pm i$

Better what is a complex on V? Given $I^2=-1$, get $V_c = V_c^+ \oplus V_c^-$ where $I = \pm i$, also $\sigma V_c^+ = V_c^-$ $\sigma =$ conj.

Conversely given linear F on V_c such that $\sigma F \sigma = -F$, then $\sigma i F \sigma^{-1} = iF$ so $J = iF$ is real gives a $\mathbb{C}$-st. on V.

Note that $V_c^+ \subset V \oplus iV$ is the graph of a map from V to itself. When does $x+iy \in V_c = V \oplus iV$ lies in V_c^+

i.e. $I(x+iy) = i(x+iy)$. $Ix + i(Iy) = -y + ix$

$Ix = -y$ and $Iy = x$. Thus $V_c^+ = \begin{pmatrix} I \\ 1 \end{pmatrix} V$, ~~$V_c^- = \begin{pmatrix} -1 \\ I \end{pmatrix} V$~~

$V_c = V \oplus iV = \{(x+iy) \mid x, y \in V\}$.

V_c^+ cons. of $x+iy$ such that $I(x+iy) = Ix + i(Iy)$

equals $i(x+iy) = -y + ix$, $V_c^+ = \begin{pmatrix} I \\ 1 \end{pmatrix} V = \begin{pmatrix} -1 \\ I \end{pmatrix} V$

V_c^- $I(x+iy) = Ix + i(Iy)$

$-i(x+iy) = y - ix$ $V_c^- = \begin{pmatrix} 1 \\ I \end{pmatrix} V = \begin{pmatrix} I \\ -1 \end{pmatrix} V$

Same will be true for J. Actually you now have two ways to describe J, namely relative to the splittings $V + iV$ and $V_c^+ \oplus V_c^-$

371 First description of an I. V real vector space: $V_c = V + iV$, and $V_c^+ \subset V_c$ complex subspace such that $V_c^+ \oplus \overline{V_c^+} = V_c$. Let $V_c^+ = \binom{1}{iT} V$ where $T : V \to V$ is $\mathbb{R}$ linear inv.

Start with a real vector space V form $V_c = \mathbb{C} \otimes_{\mathbb{R}} V = V \oplus iV$. Consider $W \subset V_c$ $\mathbb{C}$-subspace sat $W \oplus \overline{W} = V_c$. Have projections $V_c \xrightarrow{Im} V$. Claim

$Im : W \to V$
$Re : W \to V$ are invertible. Thus W is a graph $\binom{1}{iT} V$ where $T : V \to V$ is $\mathbb{R}$ linear. $\overline{W} = \binom{1}{-iT} V$ Then

$W \oplus \overline{W} \longrightarrow V \oplus iV$ given by $\begin{pmatrix} 1 & 1 \\ iT & -iT \end{pmatrix}$ T inv nec + suff.

So it seems that complex structures on V are described by invertible operators $T : V \to V$. Check dims. $\dim_{real} GL_{2n}(\mathbb{R}) = 4n^2$ $\dim_{real} GL_n(\mathbb{C}) = 2n^2$ Mistake

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V real vector space $V_c = \mathbb{C} \otimes_{\mathbb{R}} V = V \oplus iV$ $\dim_{\mathbb{C}} V_c = \dim_{\mathbb{R}} V$

Consider $\mathbb{C}$-subspace $W \subset V_c$ such that $W \oplus \overline{W} = V_c$.
$\Rightarrow \dim_{\mathbb{C}} V_c = \dim_{\mathbb{R}} V$ is even. Consider $Re, Im : V_c \to V$
$Re : W \to V$ must be an $\mathbb{R}$-linear isom.

Let $w = x + iy \in W$ $x, y \in V$. Ass $Re(w) = x = 0$

Then $iy \in W$, $-iy \in \overline{W}$ $\Rightarrow y \in W \cap \overline{W} = 0$.

So have real v.s isos $V \xleftarrow{Re} V_c \xrightarrow{Im} V$. So conclude

$\exists T : V \to V$ $\mathbb{R}$ linear $\ni$ $V_c = \binom{1}{T} V$. Now use V_c stable under i. $\exists I$ real on V $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \binom{1}{T} = \binom{1}{T} I$

$T($

$$\begin{pmatrix} -Tx \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ Tx \end{pmatrix} = \begin{pmatrix} Ix \\ TIx \end{pmatrix}$$

$\therefore \quad T = -I \quad$ and $\quad I^2 = -1.$

Nothing much so far has been learned | Graph

Start again. V real vector space of even dim. Then complex structures on V can be identified with certain subspaces W ~~splittings~~ of V_c in a trivial way. Namely, W such that $W \oplus \bar{W} = V_c$. But now you want to bring in a symplectic structure on V.

Notion of complex structure I compatible with $\Omega(v,v')$ namely $\Omega(Iv, Iv') = \Omega(v,v')$, this says $\Omega(v, Iv') = -\Omega(IIv, Iv') = \Omega(Iv, v') = \Omega(v', Iv)$

Thus $\Omega(v, Iv')$ is symmetric, i.e. a quadratic form on V invariant under I. We want positivity.

So given a real symp. sp (V, Ω) there's a class of complex structures on V, namely, those J preserving Ω such that $\Omega(v, Jv) > 0$ for $v \neq 0$.

~~old viewpoint.~~ This is an old viewpoint. But the new point is to complexify, replace J by corres. W. Then can ask what preserving Ω means, probably means W is max isot. subspace, and then positivity condition.

space of complex st. is $GL_{2n}(\mathbb{R})/U_n$ has r dim $4n^2 - 2n^2 = 2n^2$

$W \subset V \otimes_{\mathbb{R}} \mathbb{C} \ni W \oplus \bar{W} = V_c$ cont. in $Gr_n(\mathbb{C}^{2n})$ has r dim $2n^2$.

$\dim Sp_{2n} = \overset{a}{n^2} + \overset{b,\, c}{n(n+1)} = 2n^2 + n \qquad \dim U_n = n^2$

$\dim Sp_{2n}/U_n = n^2 + n \qquad$ complex symm. matrices.

Lagrangian subspaces of V_c

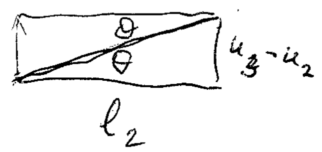
 ~~twisted symp. geom~~

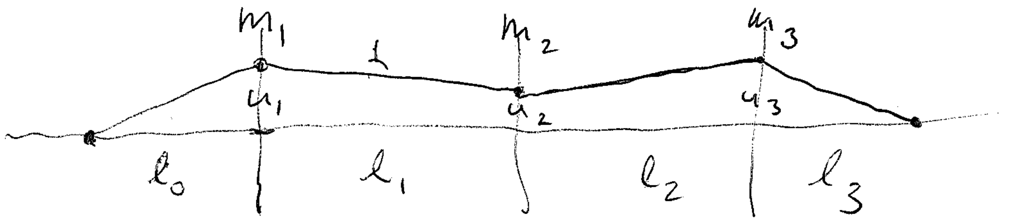
count Lagrangian subspaces in V^{2n}. One basis element at a time. $2n + (2n-1) + \cdots + (n+1) = \frac{2n+n+1}{2} n = \frac{3n^2+n}{2}$

remove GL_n to get $\frac{n^2+n}{2}$ again symm. ~~matrices~~

~~So where are things. Still need~~

Let's look at strings to see if there is something I can learn. Algebraically a string has point masses separated by lengths.





$T\sin\theta$
$\parallel$
$\tan\theta = \frac{u_3-u_2}{l_2}$

$$m_2\ddot{u}_2 = \frac{u_3-u_2}{l_2} + \frac{u_1-u_2}{l_1}$$

$$K.E. = \frac{1}{2}\sum_{i=1}^{3} m_i \dot{u}_i^2 , \qquad P.E. = \frac{1}{2}\sum_{i=0}^{3} \frac{(u_{i+1}-u_i)^2}{l_i}$$

maybe you should look at the Green's fcn.

Fix freq. ω $\qquad u_i = \hat{u}_i(\omega)\, e^{-i\omega t}$

$$-\omega^2 \hat{u}_i = \frac{u_{i-1}-u_i}{l_{i-1}} - \frac{u_i-u_{i+1}}{l_i}$$

The point was to get the solution to the right. What questions to ask? What's important is to understand response at a vertex.

forced harmonic oscillator. Go over what they say in an elementary physics book.

$$m\frac{d^2x}{dt^2} = -kx + F(t) - \underbrace{2\ell\frac{dx}{dt}}$$

Let's add a little resistance ~~Resist~~

$$m\frac{d^2x}{dt^2} + 2\ell\frac{dx}{dt} + kx = F(t). \qquad F(t) = \hat{F}e^{-i\omega t}$$

$$\left(-m\omega^2 - 2\ell i\omega + k\right)\hat{x} = \hat{F}$$

$$m\omega^2 + 2\ell i\omega - k = 0$$

$$\omega = \frac{-\ell i \pm \sqrt{km - \ell^2}}{m} = -\frac{\ell}{m}i \pm \sqrt{\frac{k}{m} - \left(\frac{\ell}{m}\right)^2}$$

so the steady state solution at freq. ω is

$$\hat{x} = \frac{\hat{F}}{-m\omega^2 - 2\ell i\omega + k}$$

A mode is

$$\underbrace{e^{-i\left(-\frac{\ell}{m}i \pm \sqrt{\frac{k}{m} - \left(\frac{\ell}{m}\right)^2}\right)t}}_{e^{-\frac{\ell}{m}t} \pm i\sqrt{\quad}}$$

So how can I analyze ~~the~~ a forced harmonic osc, i.e. how do you handle applied forces in a Hamiltonian or Lagrangian situation. Example: external force at a point of a string. The equations of motion are then

$$m_i\frac{d^2u_i}{dt^2} = \frac{u_{i-1} - u_i}{\ell_{i-1}} - \frac{u_i - u_{i+1}}{\ell_{i+1}} + \begin{cases} 0 & i \neq p \\ F(t) & i = p. \end{cases}$$

~~[illegible]~~ ~~the~~ eqns of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_i}\right) = \frac{\partial L}{\partial u_i} + F_i(t) \qquad i = 1, \dots, n$$

How do you connect this to the abstract harm. osc. picture i.e. symp. vector space + pos. def. q. form.

$$L = K.E. - \left(P.E. + \sum_i F_i(t)\, u_i\right)$$

$$H = \underbrace{\sum_i \frac{\partial L}{\partial \dot u_i}\, \dot u_i}_{2\,K.E.} - L$$

$$\qquad\qquad - K.E. + \left(P.E. + \sum F_i(t)\, u_i\right)$$

$$H = \sum \frac{p_i^2}{2m_i} + \sum_i \frac{1}{2}\frac{(u_i - u_{i-1})^2}{l_{i-1}} + \sum_i F_i(t)\, u_i$$

$$\dot q_i = \frac{\partial H}{\partial p_i} = \frac{1}{m_i} p_i \qquad \dot p_i = -\frac{\partial H}{\partial q_i} = -\frac{q_i - q_{i-1}}{l_{i-1}} - \frac{q_i - q_{i+1}}{l_i}$$

$$+ F_i(t).$$

So it looks like you add a time dependent linear function to the Hamiltonian. ~~We took~~ You took $\sum F_i(t)\, q_i$ i.e. a linear function of position. In the string case say ~~...~~ the applied force is acting at a single vertex. Keep on trying. ~~...~~

In general you have a phase space ~~and~~ a linear first order constant coeff. D.E. $\frac{dX}{dt} = AX$

You want to solve $\frac{dX}{dt} = AX + B(t)$

$$s\hat X - X(0) = A\hat X + \hat B$$

$$(s - A)\hat X = X(0) + \hat B$$

$$\hat X = \frac{X(0)}{s - A} + \frac{\hat B}{s - A} \qquad e^{tA}\int_0^t e^{-t'A} B(t')\, dt'$$

$$X = e^{tA} X(0) + \mathcal{L}^{-1}\left(\frac{\hat B}{s - A}\right)$$

$$\frac{dX}{dt} - AX = B(t)$$

$$\underbrace{e^{-tA}\frac{dX}{dt} - e^{-tA}AX}_{\frac{d}{dt}\left(e^{-tA}X\right)} = e^{-tA}B(t)$$

$$\frac{d}{dt}\left(e^{-tA}X\right) \implies e^{-tA}X = [redacted]\, X(0) + \int_0^t e^{-t'A}B(t')\,dt'$$

$$\boxed{X(t) = e^{tA}X(0) + \int_0^t e^{(t-t')A}B(t')\,dt'}$$

Now what does it mean to ~~connect~~ look at just one point?? Start by looking at all the position, i.e. We ~~[illegible]~~ consider the response to an arbitrary $F(t) = \left(F_i(t)\right)$ applied forces at the vertices. Then we have a 2nd order D.E.

$$m\ddot{q} + kq = F(t) \qquad m, k \text{ pos. def.}$$

Look at steady state response, get

$$\left(ms^2 + k\right)\hat{q} = \hat{F}$$

$$\hat{q} = \frac{1}{ms^2+k}\hat{F}$$

~~[illegible]~~

~~[illegible]~~ Now you apply a force at one point. Your applied force is zero at all other vertices.

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1) Describe A-parameters (A as in $G = KAK$) of a point of Sp_{2n}/U_n rel to basept.

2) Are there 2 symplectic structures naturally associated to a harmonic oscillator whose A-pars. give the frequencies.

37

3) D.E. relating applied force & response
Z_s = ratio of polys

4) applied force + response on phase space rather than config. space

Jan 2, 97 Let's calculate symplectic stuff

$V = \mathbb{R}^n \oplus \mathbb{R}^n$ $\qquad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$V_c = \mathbb{C}^n \oplus \mathbb{C}^n$ starting point should be a complex vector space with hermitian inner product $\langle\ \rangle$, real part is Hamiltonian, imag part is symp form

$V = \mathbb{R}^n \oplus \mathbb{R}^n \quad \begin{pmatrix} x \\ y \end{pmatrix} \longleftrightarrow x + Iy$

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle = \langle x+Iy, x'+Iy' \rangle$$

$$= \underbrace{x^t x' + y^t y'}_{\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} x' \\ y' \end{pmatrix}} + i\underbrace{(x^t y' - y^t x')}_{\Omega\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right)}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = (x^t\ y^t)\begin{pmatrix} y' \\ -x' \end{pmatrix} = x^t y' - y^t x'$$

$$\therefore\ \Omega(v, v') = v^t(-I)v'$$

Note $\Omega(v, Iv') = v^t v'$.

Extend I, Ω $\mathbb{C}$-linear to V_c. A graph $\begin{pmatrix} 1 \\ T \end{pmatrix}\mathbb{C}^n$ is isotropic wrt Ω iff

$$\begin{pmatrix} 1 \\ T \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ T \end{pmatrix} = (1\ \ T^t)\begin{pmatrix} T \\ -1 \end{pmatrix} = T - T^t = 0 \quad \text{i.e. } T \text{ symm.}$$

Examples are $\quad W = \begin{pmatrix} 1 \\ i \end{pmatrix}\mathbb{C}^n \qquad \overline{W} = \begin{pmatrix} 1 \\ -i \end{pmatrix}\mathbb{C}^n$

$$I\begin{pmatrix} x \\ ix \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ ix \end{pmatrix} = \begin{pmatrix} -ix \\ x \end{pmatrix} = -i\begin{pmatrix} x \\ ix \end{pmatrix}$$

$\therefore I = -i$ on W, $I = i$ on $\overline{W}$

$$\Omega\left(\begin{pmatrix} x \\ ix \end{pmatrix}, \begin{pmatrix} x' \\ -ix' \end{pmatrix}\right) = \begin{pmatrix} x \\ ix \end{pmatrix}^t (-i) \begin{pmatrix} x' \\ -ix' \end{pmatrix} = -i(2x^t x')$$

Next you take an isotropic subspace W giving a complex structure on V. Complex structures J on V corresp. to subspaces W of V^c such that $W \oplus \overline{W} = V^c$. Corresp. is $J = -i$ on W

$$W_J = \begin{pmatrix} 1 \\ iJ \end{pmatrix} \mathbb{C}^n \qquad J\begin{pmatrix} 1 \\ iJ \end{pmatrix} = \begin{pmatrix} J \\ -i \end{pmatrix} \mathbb{C}^n = \begin{pmatrix} 1 \\ -iJ^{-1} \end{pmatrix} \mathbb{C}^n$$

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start again. $V = \mathbb{R}^n \oplus \mathbb{R}^n \ni \begin{pmatrix} x \\ y \end{pmatrix} \longleftrightarrow x + Iy$

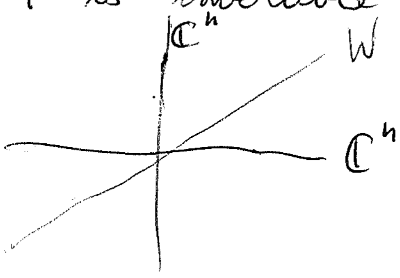
$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. hermitian form

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle = \langle x + Iy, x' + Iy' \rangle$$

$$= \underset{\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} x' \\ y' \end{pmatrix}}{\overset{\parallel}{(x^t x' + y^t y')}} + i \underbrace{(x^t y' - y^t x')}_{\Omega\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right)}$$

$$\Omega\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}^t \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{(-I)} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Aim to understand complex structures on $V = \mathbb{R}^n \oplus \mathbb{R}^n$ which are compat with Ω. A complex structure J on V same as a complex subspace W of V_c such that $W \oplus \overline{W} = V_c$. Compat with Ω should mean W is isotropic. ~~means~~ $W \oplus \overline{W} = V_c$ should mean $W = \begin{pmatrix} 1 \\ T \end{pmatrix} \mathbb{C}^n$ where T is invertible



Relate J to W_J. J op on $\mathbb{R}^n$. extend to $\mathbb{C}^n$ comm with σ.

$$\begin{pmatrix} 1 \\ iJ \end{pmatrix} \mathbb{C}^n$$

J real matrix, of square -1.

$$J\begin{pmatrix} x \\ iJx \end{pmatrix} = \begin{pmatrix} Jx \\ -ix \end{pmatrix}$$

379 $V = \mathbb{R}^n \oplus \mathbb{R}^n$ $\qquad V_c = \mathbb{C}^n \oplus \mathbb{C}^n$

misleading. First take $V = \mathbb{R}^{2n}$ $\quad V_c = \mathbb{C}^{2n}$. Then a complex st. J on V extends to $\mathbb{C}\otimes_{\mathbb{R}} V$ as $1\otimes J$ and $(\iota\otimes J)^2 = -1$. So $V_c = \underbrace{W}_{J=-i} \oplus \underbrace{\overline{W}}_{J=+i}$ Conversely given $W \ni V_c = W\oplus\overline{W}$, define $J = \begin{matrix} -i \text{ on } W \\ +i \text{ on } \overline{W}\end{matrix}$

Then $J\sigma = \sigma J$ etc. Next equip V with $\Omega\left(\binom{x}{y},\binom{x'}{y'}\right) = \binom{x}{y}^t \begin{pmatrix} 0 & +1 \\ -1 & 0\end{pmatrix}\binom{x'}{y'} = x^t y' - y^t x'$. Consider the proj

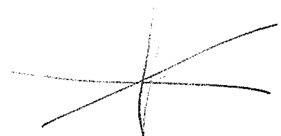
$W \longrightarrow \mathbb{C}^n$ <u>doesn't work</u>

~~$\downarrow \mathbb{C}^n$~~ begin again V real $2n$ diml v.s.

complex structures on V described by $GL_{2n}(\mathbb{R})/GL_n(\mathbb{C})$ has real dim $4n^2 - 2n^2 = 2n^2$. Homotopy type O_{2n}/U_n.

~~[illegible]~~ We have identified this with an open subset of $Gr_n(\mathbb{C}^{2n})$ consisting of $W \ni W\oplus\overline{W} = \mathbb{C}^{2n}$. Pick basepoint, i.e. a complex structure on V, whence get $V_c = W \oplus \overline{W}$ and other complex structure described by $T: W \to \overline{W}$ satisfying some indep. cond. ~~Also need~~ Suppose V equipped with $\langle , \rangle$ hermitian inner prod. It might help to identify V with $\{p_i, q_i\}$ and a polarization as a $[a_i, a_j^*] = \delta_{ij}$. ~~How does~~ The inner prod on V yields one on V_c. Basically we work in $V_c = V^+ \oplus V^-$ polarized and there is σ. get $\varepsilon = \iota I$. Also consider $F = \iota J$. $\quad F\varepsilon = -IJ$ preserved by σ Assume W^+ is close to V^+ whence $W^+ = \Gamma_T = \binom{1}{T}$ $\quad T: V^+ \to V^-$.

know $F = \dfrac{1+X}{1-X}$ $\quad X = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$?



 ~~So now~~ Perhaps you can get somewhere by interpreting V_c as span operator a_i, a_i^*

$$[a_i + c_{ik} a_k^*, a_j + c_{jk} a_k^*]$$

$$= c_{jl}\delta_{il} - c_{ik}\delta_{kj} = c_{ji} - c_{ij} = 0.$$

$$[a_i + c_{ik} a_k^*, a_j^* + \bar{c}_{jl} a_l] = \delta_{ij} - c_{ik}\delta_{kl}\bar{c}_{jl}$$

$$= \delta_{ij} - c_{ik}\bar{c}_{jk}$$

This will probably yield $\begin{cases} 1 - cc^* > 0 \\ 1 - c^*c > 0. \end{cases}$

Looks similar to the nondef. unitary case.

$W \subset V^+ \oplus V^-$ This ended with any contraction here have symmetry condition

$\|$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} V^+$ What are the parameters. There's an action of U_n. ~~So you have the space~~

W spanned by $a_i + c_{ik} a_k^*$. How to apply U_n.

Let $u_{ij} \in U_n$. Then $a_{ij}(a_j + c_{ik} a_k^*)$. Wait

U_n acts on V^+ and on V^- preserving the pairing

Thus if $g \in U_n$, ~~then say g say $a_i \mapsto a_{ij}$~~

say $a_i \longmapsto g_{\mu i} a_\mu$ $a_j^* \longmapsto g_{j\nu} a_\nu^*$

Then $[g_{\mu i} a_\mu, g_{j\nu} a_\nu^*] = g_{\mu i} g_{j\nu}\delta_{\mu\nu} = g_{j\mu} g_{\mu i} = \delta_{ji}$

~~So now~~ a_i suppose ~~$a_i \longmapsto g_{ij} a_j$~~

381 On $V = \mathbb{C}$ What's the problem?

$V_c = V^+ \oplus V^-$ Hilbert space structure clear, but you need σ. Basically I think of V^+ as consisting of destruction ops. a_i to take $[a + ca^*, a^* + \bar{c} a] = 1 - |c|^2 > 0$.

~~and then~~ Check dim's again. V^{2n} ~~Ham.t~~

		$n=1$
dim Hamiltonians	$\frac{(\cancel{2}n)(2n+1)}{\cancel{2}} = 2n^2+n$	3
dim polarizations	$= n(n+1) = n^2+n$	2
dim U_n	$= n^2$	1

space of polarizations is $\{c \in \mathbb{C} \mid |c| < 1\}$.

(x, y) Hamiltonian

$V = \mathbb{R}^n \oplus I\mathbb{R}^n$ $\qquad I = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

$V_c = \mathbb{C}^n \oplus I\mathbb{C}^n \simeq \mathbb{C}^{2n}$

$W = \begin{pmatrix}1\\ iI\end{pmatrix}\mathbb{C}^n \qquad I\begin{pmatrix}v\\ iIv\end{pmatrix} = \cancel{\begin{pmatrix}Iv\\ -iv\end{pmatrix}}\begin{pmatrix}-iv\\ Iv\end{pmatrix}$

$\bar{W} = \begin{pmatrix}1\\ -iI\end{pmatrix}\mathbb{C}^n$ $\qquad$ Try leaving I out.

$V = \mathbb{R}^n \oplus \mathbb{R}^n$ $\qquad I = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

$V_c = \mathbb{C}^n + \mathbb{C}^n$

$W = \begin{pmatrix}1\\ i\end{pmatrix}\mathbb{C}^n \qquad \bar{W} = \begin{pmatrix}1\\ -i\end{pmatrix}\mathbb{C}^n$

$\cancel{\Omega(v_1, v_2) = v_1^t\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}v_2}$

$$\Omega\left(\begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}x'\\ y'\end{pmatrix}\right) = \begin{pmatrix}x\\ y\end{pmatrix}^t\begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix}^t\begin{pmatrix}+y'\\ -x'\end{pmatrix} = x^t y' - y^t x'$$

$\begin{pmatrix}1\\ T\end{pmatrix}\mathbb{C}^n$ is isot when $\begin{pmatrix}1\\ T\end{pmatrix}^t\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ T\end{pmatrix} = \begin{pmatrix}1\\ T\end{pmatrix}^t\begin{pmatrix}-T\\ 1\end{pmatrix}$

$$= \quad (1 \;\; T^t)\begin{pmatrix}-T\\ 1\end{pmatrix} = -T + T^t = 0$$

$$\Omega\left(\begin{pmatrix}x\\ y\end{pmatrix}, -I\begin{pmatrix}x'\\ y'\end{pmatrix}\right) = (x^t \;\; y^t)\cancel{\begin{pmatrix}-y'\\ x'\end{pmatrix}}\begin{pmatrix}+x'\\ +y'\end{pmatrix} = +(x^t x' + y^t y')$$

You can do $SL_2(\mathbb{C})$ calculations using $\begin{pmatrix}x\\ y\end{pmatrix}$ $\quad x, y \in \mathbb{C}^n$.

So you want to get straight, V itself has hermitian scalar product

$$\left\langle \begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}x'\\ y'\end{pmatrix}\right\rangle = \langle x + Iy, x' + Iy'\rangle$$
$$= x^t x' + y^t y' + i(x^t y' - y^t x')$$

381 Suppose $a_i \longmapsto g_{i\mu} a_\mu$

$a_j^* \longmapsto h_{j\nu} a_\nu^*$

Then $g_{i\mu} \delta_{\mu\nu} h_{j\nu} = \boxed{\delta_{ij} = g_{i\nu} h_{j\nu}}$

$h^t = g^{-1} \qquad h = (g^{-1})^t = \bar{g}$.

$a_i + c_{ik} a_k^* \longmapsto g_{i\mu} a_\mu + c_{ik} h_{k\nu} a_\nu^*$

$\underbrace{\bar{g}_{i\lambda} g_{i\mu}}_{\delta_{\lambda\mu}} a_\mu + \bar{g}_{i\lambda} c_{ik} \bar{g}_{k\nu} a_\nu^*$

$a_\lambda + \bar{g}_{i\lambda} c_{ik} \bar{g}_{k\nu} a_\nu^*$

$\bar{g}^t c \bar{g}$

so we have symmetric $n \times n$ complex matrices c being acted on by $U_n \subset GL_n(\mathbb{C})$ for the action

$g(c) = g^t c g$

You have a symmetric bilinear form. Look on unit sphere. There's a max value which is >0

If you let U_1 act on $\mathbb{C}$ by $g^t g = g_{11}^2$ $e^{2i\theta}$ if $g = e^{i\theta}$

the only invariant is the norm.

Count dimensions. Possible c form $2\frac{n(n+1)}{2}$

Stabilizer of diagonal c with dist pos. entries?

$2\frac{n(n+1)}{2} = n^2 + n$ — n^2 dim U_n, n dim diagonal c.

$g^t c g = c$

$cg = (g^t)^{-1} c = \bar{g} c$

So apparently what happens is that a positive definite Hamiltonian determines a pair of polarizations in good cases. Generically equivalent?

~~Go~~ back to W^+ generated by a_i $\overline{W}$ by a_i^*

and W by $a_i + c_{ik} a_k^*$ $\qquad n=1$ $\qquad \dfrac{a + c a^*}{\sqrt{1-|c|^2}}$

$$\left[\frac{a+ca^*}{\sqrt{1-|c|^2}}, \frac{a^* + \bar{c}a}{\sqrt{1-|c|^2}}\right] = 1$$

Also can change $a \mapsto e^{i\theta} a$ $\qquad \dfrac{e^{i\theta}a + c e^{-i\theta} a^*}{\sqrt{1-|c|^2}}$

same subsp as $\dfrac{a + c e^{-2i\theta} a^*}{\sqrt{1-|c|^2}}$ ~~so~~ can assume $c \geq 0$.

So your parameters are $0 \leq c < 1$. $\qquad 0 \leq c_1 \leq c_2 \leq \cdots \leq c_n < 1$

$\dfrac{2n(2n+1)}{2}$ pos. def. quad. forms on $\mathbb{R}^{2n}$ $\qquad 2n^2+n$

$n(n+1)$ dim Sp_{2n}/U_n

n^2 dim U_n

n^2 dim pos ~~def~~ hermitian ops on $\mathbb{C}^n$

I go from a Hamiltonian to a polarization + pos. def. herm op

On the other [redacted] hand you consider a polarization + a tangent vector to it, ~~same as having the tangent vector is actually I~~

So it seems that two polarizations on a symplectic v.s. are i.e. a geodesic segment are slightly finer ~~polarization plus that~~ than a Hamiltonian

Need to go over this much more — maybe introduce s somehow?

383 Jan 3 ~~My Prof~~ V real symplectic. Given Hamiltonian H get skew adj op wrt H ~~whose~~ whose phase gives a complex structure on V making it a $\mathbb{C}$ Hilbert space

$$\Omega(v,v') \qquad H(v,v') = \tfrac{1}{2}\big(H(v+v') - H(v) - H(v')\big).$$

Define B by $H(v, Bv') = \Omega(v,v')$. Then B nonsingular

$$H(Bv', v) = H(v, Bv') = \Omega(v,v')$$

$$H(v', B^*v) \qquad -H(v', Bv) = -\Omega(v',v) \qquad \therefore B^* = -B.$$

$$I = \frac{B}{|B|} = \frac{B}{(B^*B)^{1/2}} = \frac{B}{(-B^2)^{1/2}} \qquad B = |B| I \text{ comm.} \qquad |B|>0 \text{ pos. def. symm.}$$

$$H(v, |B| I v') = \Omega(v,v')$$

$$H(v, |B| v') = -\Omega(v, Iv')$$

$$\Omega(Iv, Iv') = H(Iv, BIv') = H(v, (-I)BIv') = \Omega(v,v').$$

So I is symplectic

Properties: Ω

Def. Given (V,Ω) define pol. to be I such that $\Omega(Iv, Iv') = \Omega(v,v')$ ✓
$I^2 = -1$ ✓ and $\Omega(v, Iv) > 0 \quad v \neq 0$.

$$\Omega(v, Iv') = -\Omega(Iv', v) = -\Omega(I^2v', Iv) = \Omega(v', Iv)$$

Given (V,Ω) Consider $B \ni \Omega(v, Bv')$ is symmetric

$$\Omega(v,v') = \begin{pmatrix} x \\ y\end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} x' \\ y'\end{pmatrix} \qquad \Omega(v, Bv') = \begin{pmatrix} x \\ y\end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} B \begin{pmatrix} x' \\ y'\end{pmatrix}$$

$$\therefore \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} B = B^t \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \qquad \text{or } JB = -B^tJ$$

$$JB + B^tJ = 0$$

Lie (Span)

384 First point is the ~~isom.~~ isom. between quadratic forms on V and inf. symplectic transf given by $X \mapsto \Omega(v, Xv')$

$$\Omega(v, Xv') + \underbrace{\Omega(Xv, v')}_{-\Omega(v', Xv)} = 0 \iff \Omega(v, Xv') \text{ symm.}$$

Let symplectic group $Sp(V)$ act on these.

$$\Omega(gv, Xgv') = \Omega(v, (g^{-1}Xg)v'). \qquad G \text{ on } Lie(\mathfrak{g}).$$

Quadratic forms divide up according to signature, can ask to describe conjugacy classes. Focus on pos. def.

2nd point. If $X \ni \Omega(v, Xv) > 0$ then X is skew adjoint wrt this scalar prod:

$$\Omega((Xv), X(v')) + \Omega((v), X(Xv')) = 0$$

so we have $X = |X| J$ $\quad J^2 = -1. \quad |X| = (-X^2)^{1/2}$

$$H(v, v') = \Omega(v, Xv') = \Omega(|X|v, Jv')$$

$$H_J(v, v') = \Omega(v, Jv') = H($$

wait. let $H_J(v, v') = \Omega(v, \quad Jv')$

start with X $\qquad \Omega(Xv, v') + \Omega(v, Xv') = 0$

$$\Omega(v, Xv) > 0.$$

assertion: $\exists!$ fact. $X = |X| J$ inside $\mathfrak{sp}(V) \subset End(V)$

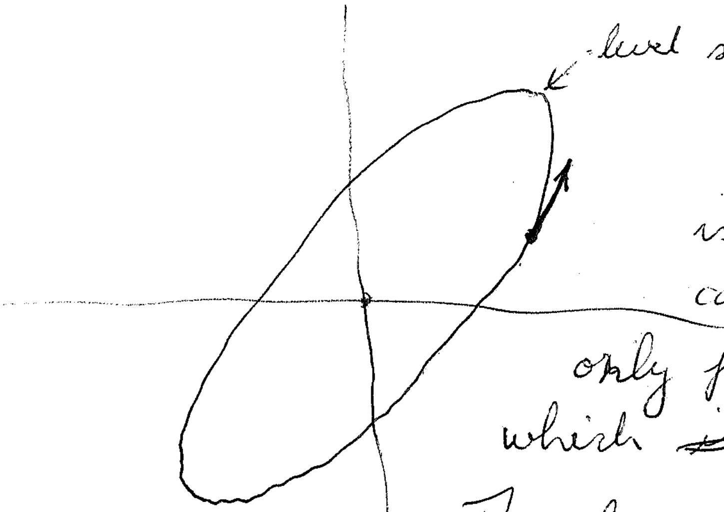
$|X|, J$ commute $\qquad J \in Sp(V)$ and $J^2 = -1$

properties of $|X|$? You don't even know about $SL_2(\mathbb{R})$. $\quad \frac{1}{2} tr(g) = \frac{\lambda + \lambda^{-1}}{2} \in \mathbb{R}$ $\quad$ ~~three~~ 5 cases

$$\frac{1}{2} tr(g) \quad < -1, \quad \underset{\text{uni } -1}{-1}, \quad \underset{\text{rotations}}{(-1, 1)}, \quad \underset{\text{uni } +1}{1}, \quad > 1.$$

My feeling is that $|X|$ can be arb. pos. def. herm.

 Get act together in the case of $SL_2(\mathbb{R})$.
Start with a q.form pos. def.



As far as the motion is concerned it's rotation counterclockwise, and the only parameter is the frequency which ~~is as all of~~ can be any $0<\omega<\infty$.

The frequency is related to the volume of $H \leq 1$. Relate this to the complex lines.

Suppose you let $W \subset V^+ \oplus V^-$

$$\begin{array}{ccc} V^+ & & V^- \\ \cup & & \cup \\ a & & a^* \end{array} \qquad [a,a^*]=1.$$

can change a to $e^{i\theta}a$, then $a^* \mapsto e^{-i\theta}a^*$

Let $W = \mathbb{C}(a+ca^*) \Rightarrow \bar{W} = \mathbb{C}(a^*+\bar{c}a)$

$$\left[\frac{a+ca^*}{\sqrt{1-|c|^2}}, \frac{a^*+\bar{c}a}{\sqrt{1-|c|^2}}\right] = 1$$

~~Since~~ So other polarizations are described exactly by c sat $|c|<1$. dim = 2.

Checking yesterday's counting. ~~Poss H ... $2n^2$~~

Possible H: $\frac{2n(2n+1)}{2} = 2n^2+n$

Possible J: $2\frac{n(n+1)}{2} = n^2+n$ open subset of symplectic Grass in $\mathbb{C}^{2n}$ (complex symm mat).

Have map $H \mapsto J$. The fibre ~~then~~ I thought to be all pos. quad. forms commuting with J, i.e. all pos. herm. op. on the Hilbert space $(V, J, H_J + i\Omega)$. The space of these should be open in all herm. ops. has real dim n^2.

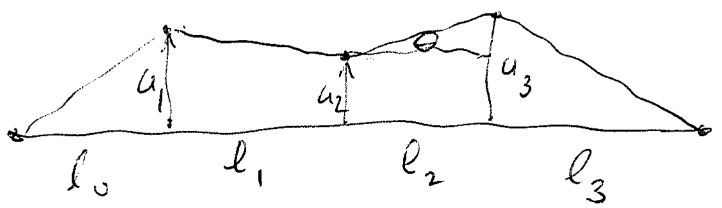
NO Problem. Problem comes from pairs of pols.

386 Nevertheless when I look at the U_n orbits of pols. W ~~different from~~ around the given one $V^+ \oplus \bar{V}$ I got n parameters $0 \leq c_1 \leq \cdots \leq c_n < 1$, which looks suspiciously like a set of frequencies.

Interpretation: Given your basepoint pol. $V^+ \oplus \bar{V}$, then another pol $W \oplus \bar{W}$ gives rise a possibly degenerate oscillator structure on V^+, namely, an orthogonal together with different frequencies for each of the summands, one of which can be zero.

Now time to return compressing harmonic oscillator, which should be very easy, and related to Gaussian calculations you do on the quantum level. So you have harm. osc. Go back to string

$$m_i \ddot{u}_i = \sin\theta = \tan\theta = \frac{u_{i+1}-u_i}{l_i} + \frac{u_{i-1}-u_i}{l_{i-1}}$$



$$T = K.E. = \sum_i \tfrac{1}{2} m_i \dot{u}_i^2 \qquad V = P.E. = \sum_i \frac{1}{2l_i}(u_{i+1}-u_i)^2$$

$$L = T - V \qquad \frac{\partial T}{\partial \dot{u}_i} = p_i = m_i \dot{u}_i \qquad \frac{\partial T}{\partial u_i} = -\frac{\partial V}{\partial u_i} = -\frac{u_i - u_{i-1}}{l_{i-1}} - \frac{u_i - u_{i+1}}{l_i}$$

The idea is to ~~pick a mass say m_i and~~ apply $F_i(t)$ to the ith mass

$$m_i \ddot{u}_i = \frac{u_{i+1}-u_i}{l_i} - \frac{u_i - u_{i-1}}{l_{i-1}} + F_i$$

What this does is to convert homog. linear DE to inhomog. one. But probably better to find appropriate Lagrangian. Change V by adding $-\sum_i F_i(t) u_i$ now have time dep. potential.

$$L = T - V + \sum_i F_i u_i$$

$$\frac{\partial L}{\partial \dot{u}_i} = \frac{\partial T}{\partial \dot{u}_i} = m_i \dot{u}_i = p_i \quad \text{same as before}$$

$$\frac{\partial L}{\partial u_i} = \frac{\partial V}{\partial u_i} + F_i = \frac{u_{i+1} - u_i}{\ell_i} - \frac{u_i - u_{i-1}}{\ell_{i-1}} + F_i$$

$$H = \sum p_i \dot{u}_i - L = 2T - T + V - \sum F_i u_i$$

$$H = T + V - \sum F_i(t) q_i \quad \text{where } u_i = q_i.$$

In effect what you've done is to alter H by a time dependent linear fn an H.

I want to set up carefully, although it should be pretty easy ultimately. So how do I proceed? In general you have $H = \underbrace{H_0}_{\text{inf symp.}\ \text{of}\ A} + \underbrace{J(t)}_{\text{linear}}$

$$\dot{\xi} = A\xi + B(t)$$

$$\xi(t) = \int_0^t e^{A(t-t')} B(t') dt' + e^{tA} \xi_0$$

$$s\hat{\xi} - \xi_0 = A\hat{\xi} + \hat{B}$$

$$\hat{\xi} = \frac{1}{s-A} \xi_0 + \frac{1}{s-A} \hat{B}$$

So this gives the motion, but you want the response.

Recall how you once understood the forced harmonic oscillator. The picture: Feynman path integral in time - this is a Gaussian integral but infinite dimensional - it describes f.d. harmon. osc. quantum mechanically. $\langle 0 | e^{a\gamma - a^*\bar{\gamma}} | 0 \rangle$. Let's leave these details alone for the moment, but mull over the ideas e.g. variational methods with quadratic forms.

388 Consider a forced harmonic oscillator. Think of ~~your masses on a~~ a discrete string. Get a phase space. Apply an external force to a single mass. Add $-F(t)q_i$ to Hamiltonians. Hamiltonian probably not the energy? What is

$$\frac{d}{dt}H(q,p,t) = \frac{\partial H}{\partial q}\dot q + \frac{\partial H}{\partial p}\dot p + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial q}\left(+\frac{\partial H}{\partial p}\right) + \left(\frac{\partial H}{\partial p}\right)\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}$$

What do we want? You take $F(t) = \mathrm{Re}(\hat F e^{-i\omega t})$ find the resulting

$q_i(t) = \mathrm{Re}(\hat q_i(\omega)e^{-i\omega t})$. First order DE on phase space. $\xi = \begin{pmatrix} q \\ p \end{pmatrix}$ satisfies $\dot\xi = X\xi + B(t)$

$$\dot q = \frac{p}{m}$$

$$\dot p = -\frac{\partial H}{\partial q} = -kq + F(t)$$

$$s\hat q = \frac{1}{m}\hat p \Longrightarrow \hat p = ms\hat q$$

$$s\hat p = -k\hat q + \hat F(s)$$

$$(ms^2+k)\hat q = \hat F(s) \qquad \boxed{\hat q = \frac{1}{ms^2+k}\hat F}$$

Now restrict $F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \\ F_i \\ 0 \\ 0 \end{pmatrix}$ and look only at the mass q_i. Get

$$\hat{q}_i = (0 \cdots 0 1 0 \cdots 0) \frac{1}{ms^2+k} \begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ \vdots\\ 0\end{pmatrix} \hat{F}_i$$

$$\|$$

$$\sum_{\text{finite}} A_\omega \underbrace{\frac{(1+\omega^2)s}{s^2+\omega^2}}_{\frac{(1+\omega^2)}{2}\left(\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right)}$$

so you end up with $\sum a_\omega \frac{(1+\omega^2)s}{s^2+\omega^2}$ for the response function. Rational function. General case

Now look at a general oscillator.

~~Here it seems there might be a problem connecting~~

Discuss the question to ask. You have an osc. phase space V + Hamiltonian function whence complex Hilbert space structure on V and pos. def. operator H such that time evolution is given by e^{-iHt}. What does an applied force mean? I guess it means?

Possibilities: Inhomogeneous term for Hamilton's equations

$$i\frac{\partial}{\partial t}\xi = H\xi + F(t) \qquad F(t)\in V.$$

whence

$$\xi = e^{-iHt}\xi_0 + \frac{1}{i}\int_0^t e^{-iH(t-t')}F(t')\,dt'$$

$$i(s\hat{\xi} - \xi_0) = H\hat{\xi} + \hat{F}$$

$$is\hat{\xi} - H\hat{\xi} = i\xi_0 + \hat{F}$$

$$\hat{\xi} = \frac{1}{is-H}\, i\xi_0 + \frac{1}{is-H}\hat{F}$$

$$\boxed{\hat{\xi} = \frac{1}{s+iH}\xi_0 + \frac{1}{s+iH}\frac{1}{i}\hat{F}}$$

Finally what? Forcing term = inhomogeneous term for the time evol. DE.

Back to the forcing term.

$$i\partial_t \xi = H\xi + F(t)$$

What means response. I guess it means the steady state solution: $\boxed{\hat{\xi} = \frac{1}{is - H}\hat{F}}$

Take $(q,p) \longrightarrow q_i$ a linear functional on V.

units, q cm $\qquad F$ $\text{gr cm}/\text{sec}^2$

$p = m\dot{q}$ $\quad$ gr cm/sec $\qquad Fds$ $\quad \text{gr cm}^2/\text{sec}^2$

$p\dot{q}$ $\quad \text{gr}(\text{cm}/\text{sec})^2$ energy units $\qquad \frac{i}{\hbar}Ht \qquad \hbar : \frac{\text{gr cm}^2}{\text{sec}}$

$pdq \quad \text{g cm}^2/\text{sec} = \frac{\text{gr cm}^2}{\text{sec}^2}\,\text{sec} \qquad \frac{p\,dq}{\hbar}$ dimensionless.

Take the linear functional $(q,p) \longmapsto q_i$ on phase space V.

Then change $H = \frac{p^2}{2m} + \frac{1}{2}q^t k q - F_i(t)q_i$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial q} = -kq + F_i(t)$$

What's happened is that the linear function $(q,p) \mapsto q_i$ has been converted to a vector $-\frac{\partial H}{\partial q_j} = F_i\delta_{ij}$. So what? You have $V \twoheadrightarrow W \qquad W^* \hookrightarrow V^* = V$

And I am trying to see what I get on W.

Continue. You want to understand what responses can be obtained from a harmonic oscillator. The oscillator itself is a ~~Hamiltonian~~ infl. symplectic transf of a certain type on a symplectic vector space V.

Thus harmonic oscillator is given by a pos. def form on a real symplectic V. Get an Hilbert space structure on V and a pos def herm. op H such that time evolution is given by e^{-itH}. I understand perfectly this picture I think. Now you want to understand a forced harmonic oscillator. Possible meanings: 1) make the time evolution DE $\partial_t \xi = X\xi$ inhomogeneous $(\partial_t - X)\xi = \eta(t)$, where $\eta(t)$ is a path in V

2) add to the Hamiltonian (pos. def form on V) a linear term depending on t. 1) and 2) ~~are basically~~ the same. You should emphasize the idea of $\boxed{\text{coupling}}$ the oscillator to something else.

You want to be able to handle coupling in full generality.

Let's try to bridge the gap between the case of all forcing and partial forcing.

All forcing means we consider $(\partial_t - X)\xi = \eta(t)$ for all paths $\eta(t)$ in V and the response is the graph of $(s-X)^{-1}$ from V to V.

Next consider force derived from a potential which is a time dependent linear function of position. Thus if $q_1, \dots, q_n$ are the position coords, then add $-\sum_i F_i(t) q_i$ to the Hamiltonian, DE is

$$s\hat{q} = \frac{\hat{p}}{m}$$

$$s\hat{p} = -k\hat{q} + \hat{F}$$

We eliminate $\hat{p}$ to get $(ms^2+k)\hat{q} = \hat{F}$, so the response seems to be the graph of ms^2+k from configuration space to its dual = momentum space. Then we can consider a quotient of configuration space and the corresp subspace of momentum space

 You are dealing with a q-form, i.e. a map ms^2+k : Config. space $\longrightarrow$ Momentum space

$$\underset{C}{\text{Config. space}} \qquad \underset{C^*}{\text{Momentum space}}$$

But if I want a quotient C/D of C and the corresp subspace $D^0 \subset C^*$, then I need

$$D^0 \overset{j^t}{\hookrightarrow} C^* \xrightarrow{(ms^2+k)^{-1}} C \overset{j}{\twoheadrightarrow} C/D$$

Is it possible to handle the harmonic oscillator itself this way. My idea is to limit the forcing term to a subspace L of V. What do I add to the Hamiltonian H? ~~Linear subspace of~~ Have $L \hookrightarrow V$. For each $v \in V$ I get a linear function on V. Get

$$L \hookrightarrow V \xrightarrow{\sim} V^* \longrightarrow L^*.$$

What sort of question should you be asking? ~~The Youre looking Corresponding to~~ Solving the equation $(s-X)\hat{q} = \hat{\eta} \in L$ is possible namely $\tilde{q} = (s-X)^{-1}\hat{\eta}$. The real issue maybe whether

$$L \hookrightarrow V \xrightarrow{(s-X)^{-1}} V \longrightarrow L^*$$

is an isom? $V = W \oplus W^*$ $\qquad V = W \oplus W^*$. Start with K.E. m $\quad$ P.E k $\quad$ positive quadratic forms on W

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \begin{matrix} k & 0 \\ 0 & m^{-1} \end{matrix}$$

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} kq' \\ m^{-1}p' \end{pmatrix} = q^t k q' + p^t m^{-1} p' = H$$

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q^t & p^t \end{pmatrix} \begin{pmatrix} p' \\ -q' \end{pmatrix} = q^t p' - p^t q' = \Omega$$

393 Jan 4. Program - ~~forced~~ response of a forced harmonic osc. You have seen that when the applied force depends only on position - better: phase space is the direct sum of ~~position~~ position (configuration) space and momentum space - I mean that the forcing term lies in momentum space, equiv. that ~~term added~~ addition to the Hamiltonian is a function of position. So when the forcing is restricted to a subspace L of ~~the~~ momentum space, then one gets a nice response map from L to L^*: $L \subset V_{mom} \xrightarrow{(ms^2+k)^{-1}} V_{pos} \to L^*$

Try again. $\boxed{V = V_{pos} \oplus V_{mom}} = W \oplus W^*$. kinetic & potential energy

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} k & \\ & m^{-1} \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = q^t k q' + p^t m^{-1} p' \qquad \text{symmetric + pos.}$$

In addition you have the symplectic form.

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = q^t p' - p^t q' \qquad \text{up to sign}$$

where $q^t p = p^t q$ is the canonical ~~pairings~~ pairing between W, W^*.

Hamilton DE. for $H = \frac{1}{2} q^t k q + \frac{1}{2} p^t m^{-1} p - F^t q$ are

$$\dot{q} = \frac{\partial H}{\partial p} = m^{-1} p \qquad \dot{p} = -\frac{\partial H}{\partial q} = -kq + F$$

Strange here is how k, m appear as maps $W \to W^*$.

So you have $L \subset W^* \xrightarrow{(ms^2+k)^{-1}} W \longrightarrow L^*$.

What happens when we ~~take allow~~ do not restrict F to ~~the~~ lie in a subspace L of momentum space W^*.

Then $H = \frac{1}{2} q^t k q + \frac{1}{2} p^t m^{-1} p - F^t q - G^t p$

$$\dot{q} = \frac{\partial H}{\partial p} = m^{-1} p - G \qquad \dot{p} = -\frac{\partial H}{\partial q} = -kq + F$$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} -G \\ F \end{pmatrix}$$

~~these are~~ this is the eqn of motion what does it mean?

 Aim: ~~[illegible]~~ (Grothendieck completeness-style of Cayley theorem. You want to obtain any oscillator in a universal way) ~~Can I obtain~~

Start with V symplectic, then $V \oplus V^*$ has a canonical complex structure. Wait - use the λ-ring pattern: For any A define universal λ-ring cogenerated by A, namely $W(A) = (1 + tA[[t]])$, then you define a λ-ring structure as a lifting $A \to W(A)$ compatible with λ-operations.

Try for an analogue: Basic operation takes W into $W \oplus W^*$ symplectic. A quad form on W gives embedding $W \xrightarrow{\binom{1}{k}} \begin{matrix} W \\ \oplus \\ W^* \end{matrix}$ as max isot subspace

Note that $W \oplus W^*$ also carries a canonical (hyperbolic) quadratic form, so $\begin{matrix} W \\ \oplus \\ W^* \end{matrix}$ carries a natural ~~Hamiltonian~~ oscillator which is probably degenerate, however it might impress ~~complex~~ to the operator structure on W.

What are we seeking?. For any vector space W, $W \oplus W^*$ has a canonical oscillator structure i.e. symp. + quad form. Suppose W already equipped with oscillator structure, it has a flow $W \underset{b}{\overset{a}{\rightrightarrows}} W^*$ a symp. b symm

whence $(as+b)^{-1} = (s + a^{-1}b)^{-1} a^{-1}$ response function

Now on $W \oplus W^*$ you have canonical oscillator

You have $\binom{1}{b} : W \longrightarrow \begin{matrix} W \\ \oplus \\ W^* \end{matrix}$ max. isotropic

On $W \oplus W^*$ you have bilinear forms given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \begin{matrix} W \\ \oplus \\ W^* \end{matrix} \longrightarrow \begin{matrix} W \\ \oplus \\ W^* \end{matrix}$$

$$As + B = \begin{pmatrix} 0 & 1+s \\ 1-s & 0 \end{pmatrix}$$

can try

$$W \xrightarrow{\binom{1}{b}} \begin{matrix} W \\ \oplus \\ W^* \end{matrix} \qquad \begin{matrix} W \\ \oplus \\ W^* \end{matrix} \xrightarrow{(a^* \ \ 1)} W^*$$

$$(a^* \ \ 1)\begin{pmatrix} 0 & (1\mp s)^{-1} \\ (1\mp s)^{-1} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} = (a^* \ \ 1)\begin{pmatrix} (1+s)^{-1} \\ (1-s)^{-1}b \end{pmatrix}$$

other idea is to use s on W s^{-1} on W^*.

For W a vector space $W \oplus W^*$ can

Aim? Organize logically response functions associated to harmonic oscillators. You think you understand what to do ~~this~~ in the case of a subspace of configuration space. Answer should be similar to LC circuits. Look carefully at LC circuits. $C^1 \oplus C_1$ comes with a simple structure namely

$$\begin{array}{ccc} C^{1,L} & \oplus & C^{1C} \\ \Big\downarrow{\scriptstyle (Ls)^{-1}} & & \Big\downarrow{\scriptstyle Cs} \\ C_1^L & \oplus & C_1^C \end{array}$$

C^1 itself is a real polarized Hilbert space and s is used to transform the inner product $s\|\xi_+\|^2 + s^{-1}\|\xi_-\|^2$ too confusing - go back to a ~~natural~~ Lagrangian type oscillator - go to

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organize response fns. assoc. to harmonic oscillators. You can handle case of ~~spring~~ Lagrangian oscillators ($T-V$ on $W \oplus W^*$) and a quotient ~~sub~~ space of W. From ~~an~~ Hamiltonian oscillator viewpoint with pos. def Ham. you have a symplectic space (V), isotropic subspace L and then response takes the form $L \hookrightarrow V \longrightarrow V \longrightarrow L^*$

396 Let's try another approach. Take simple oscillator, use a, a^* picture of QM. Hamiltonian is $\omega a^* a$, time evolution $e^{-itH} a^* e^{itH} = e^{-it\omega} a^*$

Now you have a typical time dep. forcing term namely

$$H(t) = \omega a^* a + J(t) a^* + \bar{J}(t) a. \quad \text{Yes.}$$

$a^*(t) = e^{-iH}$ It is not $e^{-iH_t t}$

Actually the puzzle for me is the difference between the above $H = \omega_0 a^* a + J a^* + \bar{J} a$ where J is a $\mathbb{C}$-number maybe depending on t and the case where is a quantum oscillator variable - $J = b$ $\bar{J} = b^*$, and then the situation is more interesting.. Complete square as usually. $\omega_0 a^* a + a^* b + b^* a =$

$$(\sqrt{\omega_0} a)^* \sqrt{\omega_0} a + (\sqrt{\omega_0} a)^* \frac{1}{\sqrt{\omega_0}} b + \left(\frac{1}{\sqrt{\omega_0}} b\right)^* (\sqrt{\omega_0} a) + \left(\frac{1}{\sqrt{\omega_0}} b\right)^* \left(\frac{1}{\sqrt{\omega_0}} b\right) - b^* \omega_0^{-1} b$$

$$\omega_0 a^* a + a^* b + b^* a = \left(\sqrt{\omega_0} a + \frac{1}{\sqrt{\omega_0}} b\right)^* \left(\sqrt{\omega_0} a + \frac{1}{\sqrt{\omega_0}} b\right) - b^* \omega_0^{-1} b$$

$$= \boxed{(a + \omega_0^{-1} b)^* \omega_0 (a + \omega_0^{-1} b) - b^* \omega_0^{-1} b}$$

This completing the square process does not see whether $[b^*, b] \neq 0$.

~~Let V be symplectic with~~

Question: Let V be a real symp. polarized v.s. Then we have the oscillator of frequency 1 on V. For a general operator we take a positive hermitian operator. Take a real subspace L of V. What can you say about $\partial_t + i$ on V relative to L. In fact I am interested in $s + i$ and

397 More generally $s+iH$ where H is positive self-adjoint. When L is isotropic, this means L and iL are perpendicular for the scalar product then things should be easy. Why? In any case we can first consider $L+iL$ the smallest complex subspace containing L.

Let's first study the case of a complex subspace. In other words you have a pos self adjoint operator H and a complex subspace. But I think you have understood this case namely. When we embed a Hilbert space into a polarized Hilbert space the polarization restricts to?

To what ~~extent~~ does a positive hermitian operator correspond to a polarization? This is the question that caused so much trouble yesterday. Recall the answer. Suppose V symplectic equipped with a polarization. Another polarization is described by $W \subset V^+ \oplus V^- = V_c$ ~~of the form~~ spanned by $a_i^* + c_{ij} a_j$ where $c_{ij} = c_{ji}$ and $c^*c < 1$. Use eigenspaces of c^*c to split V into orth complex lines, where c^*c is scalar $0 \le c^*c < 1$. Look at $SL(2,\mathbb{R}) = SU(1,1)$ case

Now an oscillator yield a complex Hilbert space V with positive self adjoint op. H. The equation of motion is $(\partial_t + iH)\xi = 0$ i.e. $\xi(t) = e^{-iHt}\xi_0$. I am now looking at the inhomog. equation $(\partial_t + iH)\xi = F$ where $F(t)$ is restricted to lie in some ~~real~~ subspace of V. I have some insight in the case when L is isotropic. Question: Given an oscillator and a max isotropic subspace of phase space,

~~[illegible]~~ Equivalent gadgets: ① real symplectic vector space equipped with positive definite scalar product. ② complex Hilbert space equipped with positive def. hermitian operator.

Given an oscillator and a vector $v \in V$, what is the response to ~~using~~ v as ~~a~~ an applied force? Ans:

$$\frac{1}{s+iH} v = \sum \frac{1}{s+i\omega} \pi_\omega v \qquad H = \sum \omega \pi_\omega$$

So if I view the response using some ~~[illegible]~~ linear functional on V. I get $\mathrm{Re}\langle v', \frac{1}{s+iH} v\rangle$

Now what happens in the case of Lagrangian type oscillator

$$\cancel{\mathrm{Re}\langle v', \frac{1}{s+iH} v\rangle}$$

$$\cancel{\mathrm{Re}\langle v', \frac{s-iH}{s^2+H^2} v\rangle}$$

Look at $$\langle v, \frac{1}{s+iH} v\rangle = \sum_\omega \frac{1}{s+i\omega} \|\pi_\omega v\|^2$$

$$\langle v', \frac{1}{s+iH} v\rangle = \sum_\omega \frac{1}{s+i\omega} \langle \pi_\omega v', \pi_\omega v\rangle$$

This is the most general type of response to the forcing vector v. Recall ~~that~~ V has both the scalar product $\mathrm{Re}\langle\, ,\rangle$ and skew symm. bilinear form $\mathrm{Im}\langle\, ,\rangle$.

Let's try to relate this Hilbert stuff to the case ~~[illegible]~~ $F_i q_i$ ~~[illegible]~~ This amounts to simply

~~[illegible]~~ $$\mathrm{Im}\langle v, \frac{1}{s+iH} v\rangle \qquad s-i\omega$$

$$\mathrm{Im}\langle v, \frac{i}{\omega - H} v\rangle = \mathrm{Re}\langle v, \frac{1}{\omega - H} v\rangle$$

399 Another error. $v \in V$ has to vary periodically in t. So $\mathrm{Re}(e^{-i\omega t}c)v = \frac{1}{2}(e^{-i\omega t}c + e^{i\omega t}\bar{c})v$.

~~[illegible]~~ To solve

$$\left(\partial_t + iH\right) u(t) = \tfrac{1}{2}(e^{-i\omega t}c + e^{i\omega t}\bar{c})v$$

u is some path in V periodic in t

$$u(t) = \tfrac{1}{2}(e^{-i\omega t}u + e^{i\omega t}\bar{u}) \qquad u_1, u_2 \in V_c$$

$$u_2 = \bar{u}_1$$

$$\left(\partial_t + iH\right) u(t) = \tfrac{1}{2}(-i\omega + iH)e^{-i\omega t}u + \tfrac{1}{2}(i\omega + iH)e^{i\omega t}\bar{u}$$

$$\therefore \left(-i\omega + iH\right)u = cv$$

$$\left(i\omega + iH\right)\bar{u} = \bar{c}v$$

Try again. The point you missed is that ~~because~~ a periodic ~~[illegible]~~ function in time with values in V of frequency ω

Start with V complex Hilb. space with pos s.c. of H. Forced oscillator. Take periodic function in V $\quad \mathrm{Re}(Fe^{-i\omega t})$ where $F \in V_c$. Let Response be $\mathrm{Re}(ue^{-i\omega t})$. You know that $V_c \simeq V^+ \oplus V^-$ where $V \subset V_c \to V^+$ is complex linear and the projection to V^- is antilinear.

$$\left(\partial_t + IH\right)\mathrm{Re}(ue^{-i\omega t}) = \mathrm{Re}(Fe^{-i\omega t})$$

The problem seems to be how to handle

As always start with the simple harm. oscillator phase space $= \mathbb{C}$ time evolution ~~mult~~ mult by $e^{-i\omega t}$ with $\omega > 0$. Solutions $e^{-i\omega t}z_0$ $\quad (\partial_t + i\omega)z(t) = 0$.

Now you wish to solve

$$(\partial_t + i\omega_0)\xi = F(t)$$

where $F(t)$ is a $\mathbb{C}$-valued periodic fn. with freq. ω

$$F(t) = Ae^{-i\omega t} + Be^{i\omega t}$$

So do $F(t) = e^{-i\omega t}$ first

Try $\xi(t) = Ce^{-i\omega t}$

$$(\partial_t + i\omega_0)\xi(t) = (-i\omega + i\omega_0)Ce^{-i\omega t} = e^{-i\omega t}$$

$$C = \frac{1}{-i\omega + i\omega_0} = \frac{i}{\omega - \omega_0}$$

Sim to get $F(t) = e^{i\omega t}$, try $\xi = De^{i\omega t}$

$$(\partial_t + i\omega_0)\xi = (i\omega + i\omega_0)De^{i\omega t} = e^{i\omega t}$$

$$\Rightarrow D = \frac{1}{i(\omega+\omega_0)} e^{i\omega t}$$

So

$$\boxed{\begin{aligned} F(t) &= Ae^{-i\omega t} + Be^{i\omega t} \Longrightarrow \\ \xi(t) &= \frac{Ai}{\omega-\omega_0}e^{-i\omega t} + \frac{B}{i(\omega+\omega_0)}e^{i\omega t} \end{aligned}}$$

Suppose $F(t) \in i\mathbb{R}$ i.e. $B = -\bar{A}$, then

$$\xi(t) = \frac{Ai}{\omega-\omega_0}e^{-i\omega t} - \frac{\bar{A}}{i(\omega+\omega_0)}e^{i\omega t}$$

$$= \frac{A(-i)}{\omega_0-\omega}e^{-i\omega t} + \frac{\bar{A}i}{\omega_0+\omega}e^{i\omega t}$$

401 Jan 5 Response of ~~an o~~ a harmonic oscillator. recall that an oscillator is a complex Hilbert space V equipped with a positive o.a of H. The symplectic form is $\Omega(v,v') = \mathrm{Im}\langle v, v'\rangle \overset{\text{!!}}{=} \mathrm{Re}\langle v, -iv'\rangle$. the Hamiltonian is $\frac{1}{2}\mathrm{Re}\langle v, Hv\rangle$.

Ham equation $\mathrm{Re}\langle v, Hv'\rangle = \mathrm{Re}\langle \dot v, -iv'\rangle$

$df(v) = \Omega(X_f, v)$. $\mathrm{Re}\langle Hv - i\dot v, v'\rangle = 0 \quad \forall v'$.

$\dot v = +iHv$ OK up to sign

Response involves solving $(\partial_t + iH)v(t) = F(t)$. where F is a time dep. vector in V. Use L.T.

$(s+iH)\hat v - v(0) = \hat F \qquad \hat v = \frac{1}{s+iH}\hat F + \frac{v(0)}{s+iH}$

$$v(t) = e^{-iHt}v(0) + \int_0^t e^{-iH(t-t')}F(t')dt'$$

But you want the steady state response at a given frequency, which means taking ~~$F(t) = e^{st}$ $\quad s = i\omega_0 + \varepsilon$~~

$$F(t) = \mathrm{Re}(Ae^{s_0t}) \qquad s_0 = -i\omega_0 + \varepsilon \qquad \varepsilon > 0$$

$$v(t) = \int_{-\infty}^t e^{-iH(t-t')}\mathrm{Re}(Ae^{s_0t'})\,dt'$$

$$\hat v = \frac{1}{s+iH}\mathrm{Re}\left(\frac{A}{s-s_0}\right)$$

Here you must work in V_c somehow, and this is where the problem arises. The i in $s+iH$ will conflict with the i on V_c. way to handle is to change mult by i on V to I.

$$\hat v = \frac{1}{s+IH}\frac{1}{2}\left(\frac{A}{s-s_0} + \frac{\bar A}{s-\bar s_0}\right)$$

So we solve $(\partial_t + IH)\,v(t) = Ae^{s_0 t}$

$$(s+IH)\,\hat v = A\,\frac{1}{s-s_0} \qquad s_0 = -i\omega_0$$

$$\hat v = \frac{1}{s+IH}\,\frac{A}{s-s_0}$$

NO, this is an IVP with IV $= 0$. You want $v(t) = Be^{s_0 t}$, then you get

$$(s_0+IH)\,B = A \qquad B = \frac{1}{s_0+IH}\,A$$

So apparently $V_c \xrightarrow{\simeq} V \oplus \bar V \ni (A, \bar A)$

$$IH \qquad i \qquad -i \qquad \frac{1}{s_0+iH}A\,,\ \frac{1}{\bar s_0-iH}\bar A$$

$$F = \left(Ae^{s_0 t},\ \bar A e^{\bar s_0 t}\right)$$

$$u = \left(\frac{1}{s_0+iH}Ae^{s_0 t},\ \frac{1}{\bar s_0-iH}\bar A e^{\bar s_0 t}\right)$$

In order to straighten this out you might take $V = \mathbb{R}^2$ with $I = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Apparently the response at frequency ω is the map $\frac{1}{s+I\omega_0}A \mapsfrom A$ from V to V.

Model. $H = \frac{p^2}{2m} + \frac{k}{2}q^2 - Fq \qquad \dot q = \frac{p}{m} \qquad \dot p = -kq + F$

say $m=k=1$.

$$\frac{\partial}{\partial p}(-Fq) = 0 \;\Big|\; \left(-\frac{\partial}{\partial q}\right)(-Fq) = F$$

so the vector $\begin{pmatrix}0\\ F\end{pmatrix}$ corresp. to the linear functional $(-F \quad 0)$ on phase space.

403 ~~Let's look~~ It seems clear that the full response ~~of the~~ is $F \longmapsto \frac{1}{s+iH} F$ really $(e^{st}F) \longmapsto \frac{1}{\partial_t + iH}(e^{st}F) = e^{st}\left(\frac{1}{s+iH}F\right)$. This is a ~~Clin~~ map from V to itself depending holom. on s. The problem arises when we try to restrict F to lie in a real subspace of V.

Start with a harmonic oscillator V Hilbert H positive self-adjoint. You understand response (at frequency ω) in term of $F \longmapsto \frac{1}{s+iH}F$ $\quad s=-i\omega$.
You have examples of how response looks when compressed to (real) isotropic subspace e.g. line.

Various questions. Suppose W is a max. iso subspace. Take iW to be complement. NO.
You want ~~the Hamiltonian~~ to take the complement to be the perpendicular of W wrt the Hamiltonian quadratic form. Then you should have a standard Lagrangian type oscillator. NO How do you know that the orthogonal complement is isotropic?
~~How do you know that the orthogonal complement~~

~~Ham. is $\langle v, Hv \rangle$~~ Look around a good case.
Take $W \oplus W^*$ $\qquad \dot{q} = m^{-1}p \qquad \dot{p} = -kq \qquad p \in W^*$, $q \in W$.
Hamiltonian fn is $\frac{1}{2}(p^t m^{-1} p + q^t k q)$. Take another max. isot. subspace $U = \begin{pmatrix} 1 \\ T \end{pmatrix} W \subset \begin{matrix} W \\ \oplus \\ W^* \end{matrix}$ where $T: W \to W^*$ is symm. $T^t = T$.

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = q^t p' - p^t q'$$

$$\begin{pmatrix} 1 \\ T \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & T^t \end{pmatrix} \begin{pmatrix} p' \\ -q' \end{pmatrix} = p' - T^t q'$$

So U^0 is $\begin{pmatrix}1\\+T^t\end{pmatrix} q'$

Check $\begin{pmatrix}q\\Tq\end{pmatrix}^t \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}q'\\+T^t q'\end{pmatrix} = \begin{pmatrix}q^t & q^tT^t\end{pmatrix}\begin{pmatrix}+T^t q'\\-q'\end{pmatrix}$

$= +q^tT^tq' - q^tT^tq' = 0.$

~~But~~ It checks. You want for $U^\perp$ for H

$$0 = q^t\begin{pmatrix}1\\T\end{pmatrix}^t \begin{pmatrix}k&0\\0&m^{-1}\end{pmatrix}\begin{pmatrix}q'\\p'\end{pmatrix} = \underbrace{q^tkq' + q^tTm^{-1}p'}_{q^t(kq'+Tm^{-1}p')}$$

for all $q \Rightarrow \quad kq' + Tm^{-1}p' = 0$

$\Longrightarrow \quad q' = -k^{-1}Tm^{-1}p'$

so $U^\perp = \begin{pmatrix}-k^{-1}Tm^{-1}\\1\end{pmatrix} \cancel{W}^{+}$

Is this max **isotropic** i.e. is $-k^{-1}Tm^{-1}$ symm.

So it seems that the choice of max. isot. subspace matters somehow.

How do I continue? ~~I have the~~

~~Question...~~ Return to discrete string. External force F_i at ~~q~~ i-th position leads to ~~exp.~~ inhomog. term in Hamilton's equations $\dot q = \frac{p}{m}$, $\dot p_j = (-kq)_j + F_i\delta_{ij}$, and ~~you take~~ for the response you take q_i. In general you restrict the external force to a subspace of V and view the response in a quotient space. How are these chosen?

Maybe a better question than response is to ask how are oscillators coupled. This brings to mind the idea that a periodic forcing term is the same

as coupling ~~to~~ a very <u>massive oscillator.</u>
So what form does this take in general.

$$\underbrace{a_i^*, a_i, \sum \omega_i a_i^* a_i}_{\text{basis oscillator}} \qquad b^*, b, \sqrt{\tfrac{k}{m}}\, b^* b$$

$m\omega^2 = k$
$\omega = \sqrt{\frac{k}{m}}$

What is coupling? You take the direct sum so you have a_i^*, b^*, a_i, b and then must extend the hamiltonian by $c_i a_i^* b + \bar{c}_i b^* a_i$
Here (c_i) is arbitrary, but we are free to rotate in each of the eigenspaces, which means that only the $|c_i|$ are important. We have to keep the matrix

$$\left(\begin{array}{c|c} \omega_i & c_i \varepsilon \\ \hline c_i \varepsilon & \omega_0 \end{array}\right)$$

positive definite
$\varepsilon \neq 0$.

fixed

So how do you carry out the analysis.
For small ε there should be an eigenvector with eigenvalue close to ω_0. In this way first order perturbation theory should yield response to any vector (c_i).

The above seems promising and very familiar. We have an oscillator of the form H_0 + perturbation.

$$H = \left(\begin{array}{c|c} B & \gamma \\ \hline \gamma^* & \omega_0 \end{array}\right)$$

$$(\omega - H)^{-1} = \begin{pmatrix} \omega - B & -\gamma \\ -\gamma^* & \omega - \omega_0 \end{pmatrix}^{-1}$$

You need the stuff again.

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix}\cdot\begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix}$$

$$\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & (d-ca^{-1}b)^{-1} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & (d-ca^{-1}b)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1} \\ 0 & (d-ca^{-1}b)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^{-1}+a^{-1}b(d-ca^{-1}b)^{-1}ca^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1} \\ -(d-ca^{-1}b)^{-1}ca^{-1} & (d-ca^{-1}b)^{-1} \end{pmatrix}$$

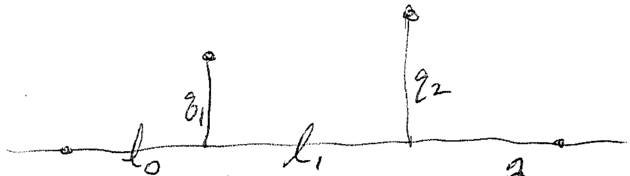
$$\begin{pmatrix} \omega-B & -\gamma \\ -\gamma^* & \omega-\omega_0 \end{pmatrix}^{-1} = \begin{pmatrix} & \\ & \dfrac{1}{\omega-\omega_0 - \gamma^*\frac{1}{\omega-B}\gamma} \end{pmatrix}$$

So we have $H = \left(\begin{array}{c|c} B & \gamma \\ \hline \gamma^* & \omega_0 \end{array}\right)$ perturbation of $\left(\begin{array}{c|c} B & 0 \\ \hline 0 & \omega_0 \end{array}\right)$

Significance. H describes things a massive oscillator of freq ω_0 coupled to an oscillator described by B. The fact that ω_0 is massive ~~small~~ means that γ is small.

There's a lot to understand, mostly how do I get response. Let's take an example. Coupled pendulums.

4.07 Where do I start?



$$H = \sum_1^2 \frac{p_i^2}{2m_i} + \sum^2 \frac{(q_i - q_{i+1})^2}{2 l_{i-1}}$$

let $l_1 \longrightarrow +\infty$ is one possibility

Another possibility it to let $m_2 \to \infty$, $l_2 \to 0$. ✓

$$H = \frac{p_1^2}{2m_1} + \frac{q_1^2}{2l_0} + \frac{(q_2-q_1)^2}{2l_1} + \frac{p_2^2}{2m_2} + \frac{q_2^2}{2l_2}$$

Somehow I would like to see the frequency of ~~the~~ the second mass stay constant ~~which~~ while its mass becomes very large.

$$H = \frac{p^2}{2m} + \frac{q^2}{2l} \qquad \dot q = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot p = -\frac{\partial H}{\partial q} = -\frac{q}{l}$$

$$\ddot q = \frac{\dot p}{m} = -\frac{q}{ml} \qquad \ddot q + \frac{1}{ml} q = 0$$

~~then~~ ~~Whatever~~ say $m_2 l_2 = \omega_0^2$ $\qquad \omega_0 = \sqrt{\frac{1}{ml}}$

$$\frac{p_2^2}{2m_2} + \frac{q_2^2}{2l_2} = \frac{p_2^2}{2m_2} + \frac{m_2 q_2^2}{2\omega_0^2}$$

Equations. $\frac{p_i}{m_i} = \dot q_i \qquad \dot p_1 = -\frac{\partial H}{\partial q_1} = -\frac{q_1}{l_0} + \frac{q_1 - q_2}{l_1}$

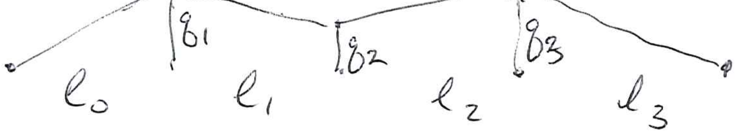
$$\boxed{\begin{aligned} -m_1 \ddot q_1 &= \frac{q_1}{l_0} + \frac{q_1 - q_2}{l_1} \\ -m_2 \ddot q_2 &= \frac{q_2}{l_2} + \frac{q_2 - q_1}{l_1} \end{aligned}}$$

let $m_2 \to \infty$, $l_2 \to 0$

such that $m_2 l_2 \to \frac{1}{\omega_0^2}$

$$-\ddot q_2 = \frac{q_2}{m_2 l_2} + \frac{q_2 - q_1}{m_2 l_1} \longrightarrow \boxed{\begin{aligned} -\ddot q_1 &= \frac{q_1}{m_1}\left(\frac{1}{l_0} + \frac{1}{l_1}\right) - \frac{q_2}{m_1 l_1} \\ -\ddot q_2 &= \omega_0^2 q_2 \end{aligned}}$$

408 Jan 6 Pursue yesterday's idea of ~~[illegible]~~ coupling to a massive harm. oscillator to study response. Consider discrete string



$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + \sum_{i=0}^{n+1} \frac{(q_{i+1}-q_i)^2}{2\, l_i}$$

First study $\quad \dfrac{p_1^2}{2m_1} + \dfrac{q_1^2}{2l_0} + \dfrac{p_2^2}{2m_2} + \dfrac{q_2^2}{2l_2} \quad q_1 q_2$

basic oscillator $\quad \dfrac{p^2}{2m} + \dfrac{kq^2}{2} + \dfrac{P^2}{2M} + \dfrac{KQ^2}{2}$

interaction $\quad - cqQ$

$$\dot q = \frac{p}{m}, \quad \dot Q = \frac{P}{M}, \quad \dot p = -kq + cQ, \quad \dot P = -KQ + q$$

$$\frac{\ddot q}{m} = -kq + cQ, \quad \frac{\ddot Q}{M} = -KQ + cq$$

$$\ddot q + \overbrace{mk}^{\omega_0^2}\, q = cmQ$$

$$\ddot Q + \underbrace{MK}_{\omega^2} Q = cMq$$

so you let $M \to \infty$ and you get a strange response.

The idea was that the massive oscillator would be unaffected by the position of the light one. You started with the q oscillator of frequency ω_0.

Try a variant where the interaction ~~couples~~ is $-cqP$

$$\frac{\ddot q}{m} = \dot p = -kq + cP \qquad \frac{\ddot Q}{M} = \dot P = -KQ + cq$$

$$\ddot q + mkq = cmP$$

$$\ddot Q + MKQ = cMq$$

$$\frac{p^2}{2m} + \frac{kq^2}{2} + \frac{P^2}{2M} + \frac{KQ^2}{2} - cqP$$

$$m\ddot{q} = \dot{p} = -kq + cP \qquad \dot{Q} = \frac{P}{M} - cq \qquad \dot{P} = -KQ$$

$$\ddot{Q} = \frac{\dot{P}}{M} - c\dot{q}$$

$$\frac{p^2}{2m} + \frac{kq^2}{2} + \frac{P^2}{2M} + \frac{KQ^2}{2} - cqQ$$

$$m\ddot{q} = \dot{p} = -\frac{\partial H}{\partial q} = -kq + cQ$$

$$M\ddot{Q} = \dot{P} = -\frac{\partial H}{\partial Q} = -KQ + cq$$

$$\boxed{\begin{aligned} \ddot{q} + \frac{k}{m}q &= \frac{c}{m}Q \\ \ddot{Q} + \frac{K}{M}Q &= \frac{c}{M}q \end{aligned}}$$

So if $M, K \to \infty$ such that $\frac{K}{M} = \omega^2$. Then you have the Q oscillator unaffected by the position of the q oscillator, and ~~motion~~ you have ~~forced~~ the q oscillator forced by the periodic motion of the Q oscillator.

Let's pass to creation & annihilation operators.

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} = \hbar\omega\left(a^* a + \frac{1}{2}\right) \qquad [a, a^*] = 1. \qquad [p, q] = \frac{\hbar}{i}$$

$$[sq + tip, sq - tip] = st\,2\hbar$$

$$\hbar\omega\,(sq - itp)(sq + itp) = \hbar\omega(s^2q^2 + t^2p^2)$$

$$st\hbar = \frac{1}{2}$$

$$\hbar\omega t^2 = \frac{1}{2m}$$

$$\hbar\omega s^2 = \frac{k}{2}$$

$$\hbar^2 \omega^2 (s^2 t^2) = \frac{k}{4m}$$

4/10

$m\ddot{q} = -kq$ Newton

$q = e^{st}$

$ms^2 + k = 0$

$s = \sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}\, i$

$\sqrt{\frac{k}{m}} = \omega$

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \qquad \frac{\partial L}{\partial q} = -kq$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Lagrange

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \qquad H = p\dot{q} - L = \dot{q}\frac{\partial T}{\partial \dot{q}} - (T - V) = T + V$$

$$= \frac{p^2}{2m} + \frac{1}{2}kq^2$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial q} = -kq.$$

Hamiton

Q.M. $[p, q] = \frac{\hbar}{i}$ $\quad |pq| = gr\frac{cm^2}{sec} = gr\left(\frac{cm}{sec}\right)^2 sec.$

$$p = \frac{\hbar}{i}\partial_x \qquad p\, e^{ikx} = \hbar k\, e^{ikx}$$

$\frac{\omega^2}{m} =$

$$\cancel{\frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{1}{2m}\left(p^2 + \omega^2 q^2\right)}$$

$$\cancel{= \frac{1}{2m}\underbrace{(\omega q - ip)}_{b^*}\underbrace{(\omega q + ip)}_{b}}$$

$$\frac{p^2}{2m} + \frac{1}{2}kq^2 =$$

$$= \left(-i\frac{1}{\sqrt{2m}}p + \sqrt{\frac{k}{2}}\,q\right)\left(i\frac{1}{\sqrt{2m}}p + \sqrt{\frac{k}{2}}\,q\right)$$

$$\frac{1}{\omega}\left(\frac{p^2}{2m} + \frac{1}{2}kq^2\right) = \left(\frac{1}{2\sqrt{mk}}p^2 + \frac{1}{2}\sqrt{mk}\,q^2\right)$$

ω = $\sqrt{\frac{m}{k}}$

411 Try again $H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kq^2$

$$\dot q = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot p = -\frac{\partial H}{\partial q} = -kq$$

$$p = m\dot q \qquad m\ddot q + kq = 0 \qquad \ddot q + \frac{k}{m}q = 0$$

$$q = e^{-i\omega t} q_0 \qquad \left(-\omega^2 + \frac{k}{m}\right) q_0 = 0 \qquad \boxed{\omega^2 = \frac{k}{m}}$$

Now quantize, interpret p, q as operators on complex Hilbert space E satisfying $[p,q] = \frac{\hbar}{i}$, e.g. $q =$ mult by x on $L^2(\mathbb{R})$, $p = \frac{\hbar}{i}\partial_x$ on $L^2(\mathbb{R})$. Then

$$H = \frac{1}{2m}\hbar^2(-\partial_x^2) + \frac{1}{2}k x^2 .$$

Let
$$b = \left(\frac{k}{2}\right)^{1/2} x + \frac{\hbar}{(2m)^{1/2}}\partial_x$$

$$b^* = \left(\frac{k}{2}\right)^{1/2} x - \frac{\hbar}{(2m)^{1/2}}\partial_x$$

$$[b, b^*] = \frac{\hbar}{(2m)^{1/2}}\left(\frac{k}{2}\right)^{1/2}[\partial_x, x]\cdot 2 = \hbar\left(\frac{k}{m}\right)^{1/2} = \hbar\omega$$

$$b\psi_g = 0 \qquad \partial_x\psi_g + \frac{(2m)^{1/2}}{\hbar}\frac{k^{1/2}}{2^{1/2}}\psi_g = 0$$

$$(mk)^{1/2}\hbar^{-1}$$

$$\psi_g = \exp\left(-\frac{1}{2}\frac{(mk)^{1/2}}{\hbar}x^2\right)$$

$[q, ip]$
$= -[ip, q]$
$= -[\hbar\partial_x, x] = -\hbar$

$$b = \left(\frac{k}{2}\right)^{1/2} q + (2m)^{-1/2} ip$$

$$b^* = \left(\frac{k}{2}\right)^{1/2} q - (2m)^{-1/2} ip$$

$$b^*b = \frac{k}{2}q^2 + (2m)^{-1}p^2 + \left(\frac{k}{2}\right)^{1/2}(2m)^{-1/2}\left([q, ip]\right) - \frac{\hbar\omega}{2}$$

so $H = b^*b + \frac{\hbar\omega}{2}$ ground level $\frac{1}{2}\hbar\omega$

$$b(b^*\psi_g) = \underbrace{[b, b^*]}_{\hbar\omega}\psi_g - \cancel{b^*b\psi_g}$$

$$\therefore \; b(b^*\psi_g) = \hbar\omega\,\psi_g$$

$$H(b^*\psi_g) = \left(b^*b + \tfrac{1}{2}\hbar\omega\right)b^*\psi_g$$

$$= b^*\underbrace{b\,b^*\psi_g} + \tfrac{1}{2}\hbar\omega\, b^*\psi_g$$

$$= \hbar\omega\, b^*\psi_g + \tfrac{1}{2}\hbar\omega\, b^*\psi_g = \left(\tfrac{3}{2}\hbar\omega\right)b^*\psi_g$$

In general $\therefore$ $H(b^{*n}\psi_g) = (n+\frac{1}{2})\hbar\omega\,(b^{*n}\psi_g)$ $\quad n \geq 0$

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and so $a = (\hbar\omega)^{-1/2}\, b$

$$[a, a^*] = (\hbar\omega)^{-1}[b, b^*] = 1. \qquad h \;\frac{gr\,cm^2}{sec}$$

and $\hbar\omega\, a^*a = b^*b$

$$\hbar\omega\left(a^*a + \tfrac{1}{2}\right) = b^*b + \tfrac{1}{2}\hbar\omega = H_{osc.}$$

$$= \frac{p^2}{2m} + \frac{k}{2}q^2$$

$$\cancel{F = \frac{G m_1 m_2}{r^2}}$$

$$\cancel{gr\frac{cm^3}{s^2}\frac{1}{gr^2} = G\frac{cm^3}{gr\, s^2}} \qquad \cancel{G\frac{cm^3 gr^{-1}}{s^2}\left(c\frac{cm}{sec}\right)^{-1} = Gc^{-1}\;\ldots\;\frac{cm^2}{gr}}$$

$$G = \frac{Fr^2}{m_1 m_2} = \frac{\frac{gr\,cm}{s^2}\,cm^2}{gr^2} = \frac{cm^3}{gr\,sec^2}$$

$$\frac{G}{c^3} = \frac{sec}{gr} \qquad \frac{Gh}{c^3} = cm^2 \qquad \left(\frac{Gh}{c^3}\right)^{1/2} \text{ Planck length}$$

413 ~~Go back to gilch~~

Review: Two harmonic osc. coupled

$$\frac{p^2}{2m} + \frac{1}{2}kq^2 + \frac{P^2}{2M} + \frac{1}{2}KQ^2 - cqQ$$

leads to the D. equation

$$m\ddot{q} + kq = cQ \qquad \ddot{q} + \underbrace{\frac{k}{m}}_{\omega_0^2} q = \frac{c}{m}Q$$

$$M\ddot{Q} + KQ = cq \qquad \ddot{Q} + \underbrace{\frac{K}{M}}_{\omega^2} Q = \frac{c}{M}q$$

So if you let $K, M \to \infty$ such that $\frac{K}{M} = \omega$, then you get $\ddot{Q} + \omega^2 Q = 0$ and $\ddot{q} + \omega_0^2 q = \frac{c}{m}Q$ which means that q is responding to the periodic force $\frac{c}{m}Q$ of period ω.

Now you want to study this in the quantum case. ~~Recall~~ Actually you are really studying the classical problem, but somehow using the complexification of phase spaces, or complex functions on phase space. For $\frac{p^2}{2m} + \frac{1}{2}kq^2$ you use what?

$$[p,q] = \frac{\hbar}{i}$$

$$[sq + itp, sq - itp] = 2st\hbar$$

$$\hbar\omega(sq - itp)(sq + itp) = \hbar\omega(s^2q^2 + t^2p^2) = \frac{p^2}{2m} + \frac{k}{2}q^2$$

$$\hbar\omega s^2 = \frac{k}{2} \qquad \hbar\omega t^2 = \frac{1}{2m} \qquad \hbar^2\omega^2 s^2 t^2 = \frac{k}{4m} \checkmark$$

$$s = \left(\frac{k}{2\hbar\omega}\right)^{1/2} \qquad t = \left(\frac{1}{2\hbar\omega m}\right)^{1/2}$$

$$\omega m = \left(\frac{k}{m}\right)^{1/2} m = (km)^{1/2}$$

$$\frac{k}{\omega} = k\left(\frac{m}{k}\right)^{1/2} = (km)^{1/2}$$

so ~~$s = t = \left(\frac{(km)^{1/2}}{2\hbar}\right)^{1/2}$~~

$$s^2 = t^2 = \frac{(km)^{1/2}}{2\hbar}$$

$$s = \left(\frac{(km)^{1/2}}{2\hbar}\right)^{1/2} \qquad t = \left(\frac{1}{2\hbar(km)^{1/2}}\right)^{1/2}$$

$$a = \left((km)^{1/4} q + i (km)^{-1/4} p \right) (2\hbar)^{-1/2}$$
$$a^* = \left(\quad - \quad \right) \quad ''$$

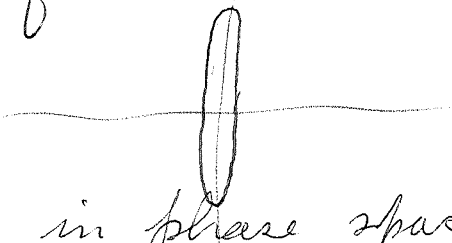
$$[a, a^*] = (-2)[q, ip](2\hbar)^{-1} = 2\overbrace{[ip, q]}^{\hbar}(2\hbar)^{-1} = 1$$

$$\hbar\omega\, a^* a = \hbar\omega \frac{1}{2\hbar}\left((km)^{1/2} q^2 + (km)^{-1/2} p^2 \right)$$
$$= \frac{1}{2}\left(\underbrace{\left(\frac{k}{m}\right)^{1/2} (km)^{1/2}}_{k} q^2 + \underbrace{(km)^{-1/2}\left(\frac{k}{m}\right)^{1/2}}_{m^{-1}} p^2 \right)$$

So now try taking a ~~similar~~ thing with capitals. So do the same thing. But next you need the interaction $-cqQ$. In general this interaction it seems ~~would~~ be any pairing between the phase spaces, thus four real constants, coefficients of qQ, qP, pQ, pP. It's natural to use the basis aA, aA^*, a^*A, a^*A^*. When you looked earlier you assumed only A^*a and a^*A occurred, but this seems rather special. It means probably that the complex structure on the combined phase space is preserved by the interaction.

So already we have even with coupling to a simple ~~extra~~ massive harmonic oscillator a complicated (potentially) situation.

4/15 ~~Idea~~ Question - What is a massive harmonic oscillator. $\omega = \sqrt{\frac{k}{m}}$ so it means both k, m are large. Thus if we fix the energy $\frac{p^2}{2m} + \frac{k}{2} q^2$ this means p is large and q is small. Think of the motion on a fixed energy ellipse - picture



q

In the large m limit the particle hardly moves, so the motion in phase space resemble a vertical oscillation

Coupling a simple osc. to a multiple one.

It looks as if this approach - coupling oscillators then taking a large Mass limit might be more complicated than analyzing a forced oscillator. But perhaps you can achieve some understanding. Meaning of large M limit. Take a symplectic plane - possible Hamiltonians are quadratic forms. Fix frequency, then dealing with complex structures, i.e. points in the UHP and the natural limits are maximal isotropic subspaces.

OK we have a picture now - We take a fixed oscillator

define harmonic oscillator.

1) Lagrangian version: real v.s. with 2 pos. def. q. forms.

m, k $\qquad L = T - V = \frac{1}{2}\dot{q}^t m \dot{q} - \frac{1}{2} q^t k q$

$m\ddot{q} + kq = 0 \qquad \frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \frac{\partial L}{\partial q} = -kq$

$(-m\omega^2 + k)\hat{q} = 0 \qquad q = e^{-i\omega t}\hat{q}$

$(\omega^2 - m^{-1}k)\hat{q} = 0$ says $m^{-1}k$ has ω^2 as eigenvalue. Does $m^{-1}k$ have enough real pos. eigenv?

$(v, v') = v^t m v'$ | Then $(v, m^{-1}k v') = v^t k v' = v'^t k v = (v', m^{-1}k v)$

2) symplectic vector equipped with pos. def. q. form.
3) complex Hilbert space equipped with pos. def. s.a. op.

Prop: Splitting into irreducibles which are complex lines. canonical orthogonal splitting into harmonic oscillators of a ~~single~~ pure frequency.

Question: Lagrangian description is always possible? Note that any Lagrangian oscillator splits canonically into pure frequency ones, so can assume pure frequency.

Note 2) arises from 1) iff can find complementary max. isot. subspace orthog. for the Hamiltonian. Suppose pure frequency. Then L isotropic $\Longleftrightarrow$ $L \perp iL$. Seems that poss. isot. subspaces are U_n/O_n has dim $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Next I want to understand what might be meant by "large mass" limits of an oscillator with fixed frequency ω. I keep the symplectic v.s. fixed. So you are looking at different polarizations of the symp. v.s. V. Fix a basepoint, i.e. have complex Hilbert space, ~~complex structure have description~~ better, you have $V_c = V^+ \oplus V^-$, then another polar. given by graph of symmetric c $c^*c < 1$. Action of U_n should lead to a compactification. Because up to U_n action c is diagonal with entries $0 \le c_1 \le c_2 \le \cdots \le c_n < 1$. So you take $U_n \times \Delta(n)$ a quotient of this for the compactification.

The details are not clear, but the ult. picture ends with a splitting of V into 2 planes and a real line chosen for those $c_n = 1$.