notes connected with lectures fall 88

Cyclic cohomology is the theory of higher traces. Field k, algebra A associative nonunital. Definition of trace.

Define b and λ on multilinear functionals. Define cyclic cocycle.

Example of $C^{infty}(M)$, where a (continuous) trace = distribution = 0-dimensional current. Define k-dimensional current as a continuous linear functional on the space of k-forms, define the cochain assciated to a current, state that the cochain is a Hochschild cocycle and that it is a cyclic cocycle iff the current is closed.

Analytical example. S^1 is the unit circle, $H = L^2(S^1, d\theta/2\pi)$, e is the Hardy projector, $A = C^{infty}(S^1)$. Define the Toeplitz operator T_f for $f \epsilon A$ to be the operator $T_f = efe$ operating on eH.

$$\varphi(f,g) = Tr_{eH}(T_fTg - T_{fg}) - Tr_{eH}(T_gT_f - T_{gf})$$

$$= Tr_H(f[e,g])$$

One can show that $[e,f]$ is an operator with smooth kernel, hence it belongs to the ideal of trace class operators. Moreover

$$\varphi(f,g) = (1/2\pi i)\int f dg$$

Comments on the last equation being a kind of index theorem, since if $g = f^{-1}$, then one has one of the analytical expressions for the index in terms of a parametrix on one side, and a cohomological evaluation of it on the other.

Cochains on A. Set $C^p(A,V) = Hom(A^{\otimes p}, V)$. Define the operators b', λ, N , also introduce $C^p_\lambda(A,V)$. Connes double complex.

Cup product of cochains. If R is an algebra, then $C^\cdot(A,R)$ with the differential $\delta = -b'$ is a cochain algebra (DGA with differential of degree +1). Example. Let $f : A - - > R$ be a 1- cochain. Its curvature is the deviation of f from being an algebra homomorphism.

Trace on the algebra of cochains. Let $C^p_\lambda(A,V)$ be the space of cyclic p-cochains on A with values in the vector space V. Let $\tau : R - - > V$ be a trace on R and define

$$tr_\tau : C^p(A,R) - - > C^{p-1}_\lambda(A,V)$$

$$tr_\tau(f) = N(\tau f) = \tau(Nf)$$

Additional comments. Fundamental cyclic cocycle on a compact oriented manifold. Idea that it might be possible to construct the fundamental cyclic cohomology class of a quantum field theory without actually having to construct the quantum field theory itself. And one should be able to derive the topological consequences from the cyclic class.

In the actual lecture, I forgot to do the analytical example of the circle.

Start with the the Fredholm module over the circle. Purpose of the calculation? Introduce an example of an extension and a lifting into the extension.

Review the DGA $C^{\cdot}(A, R), \delta$ with the trace with values in $\Sigma C^{\cdot}_{\lambda}(A, R/[R, R])$. Construct the odd degree cyclic cocycles attached to an extension.

Let I be an ideal in the algebra R, and let $\tau : I^n - - > V$ be a linear map vanishing on $[R, I^n]$. Let $p : A - - > R$ be a linear map which is an algebra homomorphism modulo I in the sense that

$$(\delta p + p^2)(a_1, a_2) = p(a_1)p(a_2) - p(a_1 a_2)$$

has values in I. We wish to construct a cyclic cocycle of degree $2n - 1$ on A with values in V.

We can regard p as a 1-cochain with values in R and $\omega = \delta p + p^2$ as a 2-cochain with values in I. Then $\omega^n \,\epsilon\, C^{2n}(A, I^n)$. We have the Bianchi identity

$$\delta\omega = -p\omega - \omega p = -[p, \omega]$$

hence

$$\begin{aligned}\delta(\omega^n) &= \sum_{i=1}^{n} \omega^{i-1}(-p\omega + \omega p)\omega^{n-i} \\ &= -p\omega^n + \omega^n p\end{aligned}$$

Thus

$$-b\, tr_{tau}(\omega^n) = tr_\tau(\delta(\omega^n)) = tr_\tau(-[p, \omega^n]) = 0$$

Continue to give the formula for the cyclic cocycle. Also the simpler formula in the case when the trace τ vanishes on $[I, I^{n-1}]$.

Remark that by copying the proof that the cohomology class of the character forms is independent of the choice of connection, one proves that the cyclic cohomology classes of the above cocycle depends only on the homomorphism from A to R/I.

Discussion of the circle case. H, A, e, where e is the Hardy space projector. The Toeplitz operator is $p(f) = efe$ acting on eH. Lemma: $[e, g]$ is an operator with smooth kernel, in particular it is of trace class. Formula for $tr_H(f[e, g])$. Then one knows that p is a homomorphism modulo I on which a trace is given, hence one obtains a cyclic 1-cocycle. Evaluation of the cyclic cocycle.

Review: $H = L^2(S^1) \supset \overline{\mathbf{C}[\mathrm{z}]} = eH$ where e is the orthogonal projector on the Hardy space, $A = C^\infty(S^1)$.

Lemma: Let $f, g \in A$. Then $[g, e] \in \mathcal{L}^1(H)$ and

$$tr_H(f[e,g]) = \frac{1}{2\pi i}\int_{S^1} f\, dg$$

Let $R = \mathcal{L}(eH), I = \mathcal{L}^1(eH), \tau = tr_{eH} : I/[R,I] --> \mathbf{C}$. The linear map $\rho : A --> R$, $\rho(f) = efe$ satisfies

$$(\delta\rho + \rho^2)(f,g) = \rho(f)\rho(g) - \rho(fg) = efege - efge = ef[e,g]e \in I$$

Therefore we obtain the cyclic 1-cocycle $\varphi = N\tau(\delta\rho + \rho^2)$:

$$(*) \qquad \varphi(f,g) = tr_{eH}(\rho(f)\rho(g) - \rho(fg)) - tr_{eH}(\rho(g)\rho(f) - \rho(gf))$$

Claim: $\varphi(f,g) = tr_H(f[e,g])$

The proof uses two remarks. 1) If T is of trace class, then $tr_H(eT) = tr_H(eTe) = tr_{eH}(eTe)$. This is clear if one uses the block description of T with respect to the decomposition $H = eH \oplus (1-e)H$:

$$T = \text{ matrix with entries } eTe \text{ , etc.}$$

2) If D is a derivation, and $e^2 = e$, then $De\, e + e\, De = De$, so

$$e\, De = De(1-e) \textit{ and } De\, e = (1-e)De$$

In particular, $e[e,g] = [e,g](1-e)$, etc.

Then

$$tr_{eH}(\rho(f)\rho(g) - \rho(fg)) = tr_{eH}(ef[e,g]e) = tr_H(ef[e,g])$$

and

$$tr_{eH}(\rho(g)\rho(f) - \rho(gf)) = tr_{eH}(eg(e-1)fe) = tr_H(eg(e-1)f) =$$

$$= tr_H([e,g](e-1)f) = tr_H((e-1)f[e,g])$$

Subtracting these equations proves the claim.

Combining above two results gives a computation of the cyclic 1-cocycle as an integral

$$\varphi(f,g) = \frac{1}{2\pi i}\int_{S^1} f\, dg$$

We now show how this can viewed as a kind of index theorem. Recall the following.

Definition: An operator $P : H_0 - - > H_1$ is called a *Fredholm operator* if it is invertible modulo compact operators, that is, if there is an operator $Q : H_1 - - > H_0$ such that $1 - QP$ and $1 - PQ$ are compact. Such an operator Q is called a *parametrix* for P.

Proposition: If $(1 - QP)^n$ and $(1 - PQ)^n$ where $n \geq 1$ are of trace class, then

$$Ind(P) = tr_{H_0}((1 - QP)^n) - tr_{H_1}((1 - PQ)^n)$$

Let f be an invertible smooth function on the circle. Then the Toeplitz operator $P = \rho(f)$ is a Fredholm operator with parametrix $Q = \rho(f^{-1})$. Applying the above proposition we obtain the following evaluation of the index

$$Ind(\rho(f)) = -\varphi(f^{-1}, f) = -\frac{1}{2\pi i}\int f^{-1} df = \text{ winding number of } f$$

Motivation for $GNS(\rho)$.

Question: Given $\rho : A \to B$, $\rho(1) = 1$, can we find an $A \otimes B^{op}$-module E together with right B-module maps

$$(*) \qquad\qquad B \xrightarrow{i} E \xrightarrow{i^*} B$$

such that $i^* a i(b) = \rho(a)b$? The Gelfand-Neumark-Segal construction does this when A is a $*$-algebra, $B = \mathbf{C}$, and ρ is a positive linear functional. This construction yields a Hilbert space E on which A operates, where i, i^* are given by product and scalar product with a unit vector. The generalized Stinespring theorem extends the construction to a completely positive map between C^*-algebras (reference to Blackadar).

Given any (E, i, i^*) as above, the algebra $GNS(\rho)$ operates naturally on E by $(a, b, \alpha)(x) = ai(bi^*(\alpha x))$. It is clearly the minimal algebra of operators one can build with the given data, which is the reason we call it the GNS-algebra.

One can show easily that there is a 1-1 correspondence between (E, i, i^*) as above and factorizations of the canonical $A \otimes B^{op}$- module homomorphism

$$\tilde{\rho} : A \otimes B \to Hom(A, B)\ ,\ \tilde{\rho}(a \otimes b)(\alpha) = \rho(\alpha a)b$$

Hence there is a smallest choice for E, namely the image of $\tilde{\rho}$. In the C^*-algebra case the Hilbert module given by the generalized Stinespring theorem is a completion of this image.

Review the GNS-algebra. It is the semidirect product of A and the free A-bimodule $M = A \otimes B \otimes A$ equipped with the product

$$M \otimes_A M - \to M \quad , \quad (a, b, \alpha)(a', b', \alpha') = (a, b\rho(\alpha a')b', \alpha')$$

Denote it $C = GNS(\rho)$. Let $\hat{u} : A \to C$ be the inclusion of $\cancel{B}$ in the semi-direct product, let $\hat{e} = (1, 1, 1) \in M \subset C$, and let $\hat{v}$ be the map from B to C given by $(\hat{v})(b) = (1, b, 1)$. Then $\hat{e}$ is an idempotent in C, and $\hat{v}$ is an algebra isomorphism of B with $\hat{e}C\hat{e} = 1 \otimes B \otimes 1$. Furthermore we have $\hat{v}(\rho(a)) = \hat{e}\hat{u}(a)\hat{e}$. A

$(\hat{v})$

Proposition 1. Given (R, e, u, v) where R is an algebra, e is an idempotent in R, and $u : A \to R$ and $v : B \to eRe$ are algebra homomorphisms such that $v(\rho(a)) = eu(a)e$ *, there is a unique algebra homomorphism from $GNS(\rho)$ to R carrying $\hat{e}, \hat{u}, \hat{v}$ to e, u, v respectively.*

The algebra Ω_A of noncommutative differential forms over A.

Let A be a (unital) algebra, let $\bar{A} = A/k$, and write $\bar{a}$ for the image of a in $\bar{A}$. We consider the complex

$$A - - > A \otimes \bar{A} - - > A \otimes \bar{A}^{\otimes 2} - - >$$

with differential

$$d(a_0, \bar{a}_1, \ldots, \bar{a}_n) = (1, \bar{a}_0 \cdot \cdots, \bar{a}_n)$$

This complex is acyclic except for a k in degree zero.

Assertion: *There is a unique multiplication on this complex making it into a DGA such that left multiplication by $a \in A$ is (*)*

$$a(a_0, \cdots, \bar{a}_n) = (aa_0, \cdots, \bar{a}_n)$$

Furthermore in this DGA we have $a_0 da_1 \cdots da_n = (a_0, \bar{a}_1, \cdots, \bar{a}_n)$. Proof. We consider the algebra R of k-linear operators on this complex. It is a DGA with differential $d(u) = (-1)^p[d, u]$, if u is an operator of degree p. We identify A with the subalgebra of operators of the form (*) and we let Ω_A be the DG subalgebra of operators generated by these left-multiplication operators. Thus Ω_A is the algebra generated by the operators a, $da = [d, a]$ for $a \in A$. We note that Ω_A is spanned by the products $a_0 da_1 \cdots da_n$. In effect the subspace spanned by these products is stable under left multiplication by A and by d, hence it is stable under left multiplication by da for $a \in A$. This means it is a left ideal in Ω_A, and as it contains 1 it coincides with Ω_A.

Next we consider the map from Ω_A to the complex () which applies an operator to $1 \in A$. We have

$$da_n(1) = [d, a_n](1) = (1, \bar{a}_n)$$

$$a_0 da_1 \cdots da_n(1) = a_0[d, a_1] \cdots [d, a_n](1)$$
$$= a_0[d, a_1] \cdots [d, a_i](1, \bar{a}_{i+1}, \cdots, \bar{a}_n) = \cdots = (a_0, \bar{a}_1, \cdots, \bar{a}_n)$$

Since Ω_A is spanned by the products $a_0 da_1 \cdots da_n$, it is clear this map is an isomorphism.

Reorganize as follows. Define a map

$$\Phi : \bigotimes_{n \geq 0} A \otimes \bar{A}^{\otimes n} - - - - > \Omega_A \ , \ \Phi(a_0, \bar{a}_1, \cdots, \bar{a}_n) = a_0 da_1 \cdots da_n$$