136. What are the constraints on q_1?

The idea is to see what happens as you make $p' = p_1 z^{\varepsilon}$, approach p_1 i.e. make $\varepsilon \downarrow 0$. You need

$$p' = p_1 q_1 p' \quad \text{i.e.} \quad p_1 z^{\varepsilon} = p_1 q_1 p_1 z^{\varepsilon}$$

$\Longrightarrow \; p_1 = p_1 q_1 p_1 \; \Longrightarrow \; q_1 p_1 = 1$. This seems to be the contradiction you want.

Check. Take p' non-zero in P assume $\exists \; \sum p_i q_i \in B$ such that $p' = \sum p_i q_i p'$. Then you have $0 \neq p' \in \underbrace{\sum p_i R}_{\text{free } R \text{ module}} \subset P$

can modify p_i and arrange p_i to be a basis for $\sum p_i R$ and for $p' = p_1 z^{\varepsilon}$ Then $p' = p_1 q_1 p'$ $\quad p_1 z^{\varepsilon} = p_1 q_1 p_1 z^{\varepsilon}$

$\therefore \; q_1 p_1 = 1$. which is impossible.

Suppose $a_1, \ldots, a_n$ $\ni$ $(1-a) a_j = 0$

Then $\sum_j a_j \tilde{A} \subset A$ ~~such~~ sats

$a_1, \ldots, a_n \qquad (1-a) a_j = 0 \qquad \forall_j$:

$\mathfrak{a} = \sum \tilde{A} a_j$ left ideal $\qquad a_j = a a_j$

$\therefore \; a_j \in A a_j \quad \therefore \; \tilde{A} a_j \subset A a_j$

so you see that $\tilde{A} a_j$ cyclic $\tilde{A}$ module M $\ni$ $M = AM$, so you get simple nonnil modules.

So independence of R. How do I handle this. Go over this.

$\tilde{A} \longrightarrow R$

If $M = AM$ then $r(am) = (ra)m$ shows the R-module structure on M is determined by the A-module structure. [redacted] $A \otimes_{\tilde{A}} R \to A$

[redacted]

$A \otimes_{\tilde{A}} M \xrightarrow{\sim} A \otimes_R M$

$ar \otimes_{\tilde{A}} m \overset{?}{=} a \otimes_{\tilde{A}} rm \qquad m = a'm$

A ~~left~~ left ideal

$A \otimes_{\tilde{A}} M$

flatness: $V \mapsto V \otimes_{\tilde{A}} M$

assume M flat over A, then for $W \in \mathrm{mod}(R^{op})$

$\cancel{W \otimes_R M = W \otimes_R A \otimes_{\tilde{A}} M}$

$\cancel{W \mapsto W \otimes_{\tilde{A}} M = W \otimes_{\tilde{A}} A \otimes_R}$

M flat over A then $R \otimes_{\tilde{A}} M$ flat over R

$\uparrow \cong$

$\tilde{A} \otimes_{\tilde{A}} M$

$W' \hookrightarrow W$

$\begin{pmatrix} \tilde{A} & R \\ A & R \end{pmatrix}$

$AR \otimes_R N \xrightarrow{\sim} N$

$W \otimes_R$

[redacted] A^{op} nil

$\cancel{\mathrm{Tor}_1^{\tilde{A}}(R, V'')}$

$V' \hookrightarrow V \qquad \cancel{\longrightarrow R \otimes_{\tilde{A}} V' \longrightarrow R \otimes_A V}$

$\mathrm{Tor}_1^{\tilde{A}}(V'', R) \longrightarrow V' \otimes_{\tilde{A}} R \longrightarrow V \otimes_{\tilde{A}} R$

A^{op} nil because ~~mult by~~ right mult by a on R factors through $\tilde{A}$.

$$\begin{tikzcd} R \arrow[rr, "\cdot a"] \arrow[dr] & & R \\ & \tilde{A} \arrow[ur] & \end{tikzcd}$$

Start with $a \in A$ write it $a = a'a''$

$$\begin{tikzcd}
R \arrow[r, ".a'"] \arrow[d, ".a"'] & R^* \arrow[r] \arrow[d, ".a''"] & R^* \arrow[r] \arrow[d] & {} & {} \arrow[r] & Q \arrow[d, ".p"] \\
A \arrow[r, phantom, "="] & A \arrow[r, phantom, "="] & A & & & A
\end{tikzcd}$$

this is incredibly simple. this is the basic construction and it allows you to take any finite subset $a_1, \dots, a_n$ and write it $a_i = q_i p$. Maybe the correct way to say this is any matrix a can be factored $a = qp$

$$\begin{tikzcd}
R^* \arrow[dr, "q"] \arrow[d, ".a"'] & \\
A^* & Q \arrow[l, "p"]
\end{tikzcd}$$

$a_{ij} = q_i p_j$

where Q is firm flat

What might be the significance of this?

~~Here was the idea~~ This sort of argument yields a Morita context $\begin{pmatrix} A & Q \\ \tilde{A} & Q \end{pmatrix}$ with $B = Q$ left B-flat. But you want something more, namely, non degeneracy. You would like B to act faithfully on B.

I want to improve this construction.

Try the following. Suppose you start with A and construct $Q \overset{p}{\twoheadrightarrow} A$ with Q firm flat. Then you have moved to a ring Q which is left flat, but has a ~~degenerate~~ ~~ideal~~. ~~$q_1 q_2 = f(q_1) q_2$.~~ nonzero right nil ideal.

$I < Q$ $\qquad IQ = 0$.

Suppose then we have A left flat but ~~nonzero elts a~~ $\{a \mid aA = 0\} \neq 0$. The idea is to enlarge P. We have $\{a \mid aQ = 0\} \neq 0$.

the hope is that we can enlarge P.

$$P \longrightarrow \mathrm{Hom}_A(Q, A) \otimes_A A$$

So what?? Non-trivial.

$g(\lambda p) = (g\mu)p$

$\mathrm{Hom}_{A^{op}}(P, P) \times \mathrm{Hom}_A(Q, Q)$

$a_1(\mu a_2) = (a_1\mu)a_2$

Start again. The basic construction

$$\begin{tikzcd} R^z \arrow[r, "a_1"] \arrow[d, "a"'] & R^z \arrow[r, "a_2"] \arrow[d] & R^z \arrow[r] \arrow[d] & \cdots\cdots \arrow[r] & Q \arrow[d, "\cdot p"] \\ A \arrow[r, equal] & A \arrow[r, equal] & A \arrow[r, equal] & \cdots\cdots \arrow[r, equal] & A \end{tikzcd}$$

$a = gp$

so what happens

You are proposing to construct a Q as a inductive limit of [illegible] But then it's [illegible] harder to construct P. Probably what you want to do is to ~~[illegible]~~ start with a bad element a. A and $A \longrightarrow M(A) \subset \mathrm{Hom}_{A^{op}}(A, A) \times \mathrm{Hom}_A(A, A)^{op}$ λ ρ

Start with [illegible]

You idea before was to reach a ring B with local left identities $\forall p' \ \exists b,\ bp' = p'$. $\Rightarrow \forall b' \ \exists b \ \ bb' = b'$

$\Rightarrow \forall b' \neq 0 \ \exists b \ \ bb' \neq 0$

$\Rightarrow \ B \longrightarrow \mathrm{Hom}_{B^{op}}(B, B)$

$b' \longmapsto (b \longmapsto bb')$

Local left identities means $B \otimes_B N \xrightarrow{\sim} BN$ for all B-modules N, equiv. firm modules have only the trivial nil submodules.

This idea basically correct

Can we ask that $\forall p' \ \exists b$ $bp' \neq 0$.

So you need $0 \longrightarrow K \longrightarrow P \xrightarrow{b\cdot} P$

$$0 \longrightarrow K \otimes_A Q \longrightarrow P \otimes_A Q \xrightarrow{b\cdot} P \otimes_A Q$$

140. Very close to what we need.

What you need to formulate is a process that will improve A. So if you have $Aa = 0$, $a \neq 0$ then you want to construct P, Q ~~so as~~ to improve this situation.

How do I interpret the conditions $Aa = 0, a \neq 0$?

Either A is a degenerate right module, or a is a bad element of the left module A.

In the P, Q situation what do you want at the end? $\forall b' \neq 0 \exists b \ni bb' \neq 0$. At the end you want

$$B \longrightarrow \mathrm{Hom}_B(B, B)$$

bad elt of B as left mod

nonzero elt of ${}_B B$

$B \xrightarrow{\cdot b'} B$ is zero

same as $Q \xrightarrow{\cdot b'} Q$ is zero.

You need

$\forall b' \; \exists b \quad bb' = b'$

$\forall b' \neq 0 \; \exists b \quad bb' \neq 0$.

~~Anyways, you need a lot of things~~

Maybe the idea should be this. You want to eliminate $Aa = 0$, $a \neq 0$ i.e. a kernel of

$$A \longrightarrow \mathrm{Hom}_A(A, A) \qquad a \mapsto \cdot a$$

Thus you maybe want

$$B \longrightarrow \mathrm{Hom}_B(B, B) = \mathrm{Hom}_A(Q, Q)$$

to be injective, i.e. you want B faithful on Q

$b' \neq 0 \Rightarrow \exists \xi$ with $\xi b' \neq 0$. From this viewpoint the loc. left ident. seems ~~strange~~ wrong.

Maybe it isn't wrong. $b' \neq 0 \Rightarrow$?

141. $b' \neq 0 \Rightarrow \cdot b'$ on Q non zero

$\Rightarrow \cdot b$ on $P \otimes_A Q = B$ ——

so what's the issue? say you start with $Q_0 = A$. and you have a problem element a' i.e. $Aa' = 0$. $Aa' \subset Q_0 P_0$.

Suppose you keep $P_0 = A$ and you try to enlarge Q_0. You might look for $q \in \mathrm{Hom}_{A^{op}}(P_0, A)$ such that $qa' \neq 0$.

$$\underbrace{A}_{Q_0 \in P_0} a' = 0 \qquad \text{ask for } \phi \in \mathrm{Hom}_{A^{op}}(\overset{P_0}{\overset{\parallel}{A}}, A)$$

such that $\phi(a') \neq 0$. So increase $Q_0 = A$ to? ~~Doesn't work~~ Assume $Q_0 = A$ can be increased to

$$Q \longrightarrow \mathrm{Hom}_{A^{op}}(\overset{P_0}{\overset{\parallel}{A}}, A)$$

So you get a m'cont. $\begin{pmatrix} A & Q \\ A & Q = B \end{pmatrix}$

typically $Q = A \oplus X$ X left module

mult in Q? $(a_1 + x_1)(a_2 + x_2) = a_1 a_2 + a_1 x_2$

$$\begin{array}{ccc} A & \subset & Q \\ & \searrow \swarrow & \\ & \mathrm{Hom}_{A^{op}}(A, A) & \end{array}$$

$(a_1 q_1)(a_2 q_2) = a_1 q_1(a_2) q_2$

Is it true that A is a left ideal in Q.

Start

142. Start again. Suppose you have $a' \in A$ such that $\underset{\substack{\| \\ Q_0}}{A}\underset{\substack{\cap \\ P_0=A}}{a'} = 0$. Suppose you increase $Q_0 = A$.

i.e.
$$\begin{tikzcd} A \arrow[r, phantom, "\subset"] \arrow[d] & Q \arrow[dl, dashed] \\ \mathrm{Hom}_{A^{op}}(A,A) & \end{tikzcd}$$
with Q A-firm

Then $\begin{pmatrix} A & Q \\ A & A\otimes_A Q = Q \end{pmatrix}$

so you have $B = Q$ with mult. $a_1 g_1 a_2 g_2 = a_1 g_1(a_2) g_2$

so it seems that A is a left ideal in Q.

$QA \subseteq A$, $AQ = Q$. You would like this increased Q to satisfy $Qa' \neq 0$.

The ~~next~~ first question to ask is whether this can work at all. ~~You need~~ The universal firm module Q^u with a pairing $Q \overset{u}{\otimes} A \to A$ is $Q^u = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ so you need to know whether $Q^u a'$ can be $\neq 0$. This means you need a $\phi \in \mathrm{Hom}_{A^{op}}(A,A)$ such that $A\phi(a') \neq 0$.

Certainly this might be possible ~~in some cases~~ if you are lucky.

Look at $I = \{a' \mid Aa' = 0\}$, this is an ideal in A. $AI = 0$, $IA \subset I$

If $\phi \in \mathrm{Hom}_{A^{op}}(A,A)$ then $0 = \phi(Aa') = \phi(A)a'$

$AI = 0 \implies \phi(A)I = 0$

$IA \subset I \implies \phi(I)A \subset \phi(I)$.

Next it could be true

$$0 \longrightarrow I \longrightarrow A \longrightarrow \mathrm{Hom}_A(A,A)$$
$$a' \longmapsto (a \mapsto aa')$$

CONCLUSION: $I = \{a' \mid Aa' = 0\}$. There can exist right module maps $\phi \in \mathrm{Hom}_{A^{op}}(A,A)$ such that $\phi(I) \neq 0$, but we have to look for such ϕ outside of the image of

$$A \xrightarrow{\lambda} \mathrm{Hom}_{A^{op}}(A,A) \qquad \text{since} \quad a.a' = 0.$$

$$a \longmapsto (\lambda_a : a' \mapsto aa')$$

$$(\phi\,\lambda_a)(a') = \phi(aa') = \phi(a)a'$$
$$= \lambda_{\phi(a)}(a')$$

$$\boxed{\lambda(A) \subset \mathrm{Hom}_{A^{op}}(A,A) \quad \text{is a left ideal.}}$$

$L \subset A$ left ideal $\Rightarrow A \cdot L \subset L \Rightarrow (A/L) \cdot L = 0$

$L \subset A$ right L-nil scan.

UPSHOT: If you want to get rid of $I = \{a' \mid Aa' = 0\}$ you can take $Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ and then cut I down to $\{a' \mid a\phi(a') = 0$ for all $a \in A$ and $\phi \in \mathrm{Hom}_{A^{op}}(A,A)\}$

This is what you can accomplish just by enlarging A. ~~Next what is zilch~~

R valuation ring $\quad A = \bigcup_{\varepsilon > 0} z^{\varepsilon} R$. Then

$A = A^2$ and A is R-flat.

$R \longrightarrow R/zR$ hom. ~~R/zR is R-flat~~

$R/zR \otimes_R A = A/zA$ is R/zR-flat

so A/zA is flat firm wrt A/zR

144. $0 \to Rz \to A/Az \to A/Rz \to 0$

So you have this funny biflat ring ~~A/Rz~~ or A/Az

You construct P, Q $\quad Q \otimes P \twoheadrightarrow A/Az$. What is the multiplier ring? $\quad \mathrm{Hom}_A(\cancel{A/Az}\ A/Rz, A/Rz)$

$$\mathrm{Hom}_R(A,A) = \mathrm{Hom}_R\Big(\bigcup_{\varepsilon>0} Rz^{\varepsilon}, A\Big)$$

$$= \bigcap_{\varepsilon>0} Az^{-\varepsilon} = R$$

$$\mathrm{Hom}_A(A/Rz, A/Rz) = \varinjlim \underbrace{\mathrm{Hom}_R(Rz^{\varepsilon}/Rz, A/Rz)}_{\mathrm{Ker}\ Az^{-\varepsilon}/Rz^{1-\varepsilon} \xrightarrow{z^{\varepsilon}} A/Rz}$$

Can you use injectives?

Can you describe injectives over A/Rz?

---

finish ind of R. $\quad \tilde{A} \to R$

$A \otimes_{\tilde{A}} M \xrightarrow{\sim} A \otimes_R M \quad$ if $M = AM$ or if $A = A^2$

$a_1 a_2 \otimes_{\tilde{A}} rm = a_1 \otimes_{\tilde{A}} a_2 rm = a_1 a_2 r \otimes_{\tilde{A}} m$

$ar \otimes a_1 m = ara_1 \otimes m = \otimes ra_1 m$

So M (R,A)-firm $\Leftrightarrow$ M $(\tilde{A},A)$-firm.

---

mod$(\tilde{A})$ $\qquad$ mod(R) $\qquad\qquad A^n M = 0$

$M \longmapsto A \otimes_{\tilde{A}} M$, $R \otimes_{\tilde{A}} M \qquad A^{n+1} R \otimes_{\tilde{A}} M$

$N \longleftarrow N$

mod$(\tilde{A}) \to$ mod$(R) \to \mathcal{M}(R,A)$ $\qquad$ exact kills nil mods.

$0 \to M' \to M \to M'' \to 0$ $\qquad \underbrace{\mathrm{Tor}_1^{\tilde{A}}(R, M'')}_{A\text{-nil}} \to R \otimes_{\tilde{A}} M' \to R \otimes_{\tilde{A}} M$

145. Q is A-firm $\Longrightarrow$ Q is a module over $\mathrm{Hom}_{A^{op}}(A,A)$

M A-firm $\qquad A \subset R \longrightarrow \mathrm{Hom}_{A^{op}}(A,A)$

Proof. $\mathrm{mod}(\tilde{A}) \qquad \mathrm{mod}(R)$

$$A \otimes_{\tilde{A}} N \xrightarrow{\sim} A \otimes_R N$$

$$A \otimes_{\tilde{A}} M \xrightarrow{\sim} \tilde{A} \otimes_{\tilde{A}} M \longrightarrow R \otimes_{\tilde{A}} M$$

Think over valuation ring R with princ. ideals Rz^{ε}, $\varepsilon \geq 0$
$A = \bigcup_{\varepsilon > 0} Rz^{\varepsilon}$ max. ideal. $R' = R/Rz$, $A' = A/\cancel{R}z$

You need to go over multipliers and center

$$M(B) = \{ (l_\mu, r_\mu) \in \mathrm{Hom}_{B^{op}}(B,B) \times \mathrm{Hom}_B(B,B)^{op} \mid (b_1\mu)b_2 = b_1(\mu b_2) \}$$

$$l((b_1 b_2)r) = l(b_1(b_2 r)) = (l b_1)(b_2 r)$$

centralizer of B. Assume μ commutes with all $\mu_{b'}$. Then

$$\mu\mu_{b'} = \mu_{b'}\mu \quad \begin{cases} \mu(b'b) = b'(\mu b) \\ (b\mu)b' = (bb')\mu \end{cases} \qquad \therefore \mu_{e^{\varepsilon}}$$

$$\mu\mu_{b'} = \mu_{b'}\mu \quad \begin{cases} \mu(b'b) = b'(\mu b) = (b'\mu)b \\ (b\mu)b' = (bb')\mu \qquad (b'b)\mu \end{cases}$$

$\mu^l = \mu^r$ so $\mu \in \mathrm{Hom}_{B\text{-bimod}}(B,B)$.

So the basic question is whether ~~any truncated~~ A-
$\exists\, P, Q \Rightarrow P \otimes_A Q$ no nil elements.
What is the rough idea? Unclear.

146 $W \otimes_{\tilde{A}} M \xrightarrow{\sim} W \otimes_R M$ if $AM = M$, any R-mod W.

$w \otimes_{\tilde{A}} r(am) = wra \otimes_{\tilde{A}} m = wr \otimes_{\tilde{A}} am$

Then M is $\tilde{A}$-flat $\Longrightarrow$ M is R-flat.

$0 \to V' \to V \to V'' \to 0$ exact over $\tilde{A}$.

$\mathrm{Tor}_1^{\tilde{A}}(V'', R) \to V' \otimes_{\tilde{A}} R \to V \otimes_{\tilde{A}} R$

If M is R-flat then get V'

Go over proof. $\tilde{A} \to R$

$W \in \mathrm{mod}(R^{op})$ $\quad M \in \mathrm{mod}(R)$ $\quad AM = M$

$\Longrightarrow W \otimes_{\tilde{A}} M \xrightarrow{\sim} W \otimes_R M$.

~~In Part. $W = A$.~~ If $A \otimes_{\tilde{A}} M \xrightarrow{\sim} M$, then there is a ! R-module structure extending the A-mod. structure

$r(am) = (ra)m$. $\quad$ M firm then ! action of $\mathrm{Hom}_{A^{op}}(A, A)$ on M. such that $\phi(am) = \phi(a)m$

$$\begin{tikzcd}
\mathrm{End}_{\tilde{A}^{op}}(A) \arrow[r] & \mathrm{End}_{\mathbb{Z}}(A \otimes_{\tilde{A}} M) = \mathrm{End}_{\mathbb{Z}}(M). \\
A \arrow[u] \arrow[ur]
\end{tikzcd}$$

For each $\phi \in \mathrm{End}_{\tilde{A}^{op}}(A)$ define $\tilde{\phi}(m)$ so that

$\tilde{\phi}(am) = \phi(a)m$. Claim ! hom.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{End}_{\tilde{A}^{op}}(A) \arrow[dl, dashed, "! \text{ if it } \exists"] \\
\mathrm{End}_{\mathbb{Z}}(M)
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{End}_{\tilde{A}^{op}}(A) \arrow[d] \\
\mathrm{End}_{\mathbb{Z}}(M) \arrow[r, "\sim"] & \mathrm{End}_{\mathbb{Z}}(A \otimes_{\tilde{A}} M)
\end{tikzcd}$$

147

$$\begin{array}{ccc} a & A \xrightarrow{\quad} \mathrm{End}_{A^{op}}(A) & a \longmapsto (a' \mapsto aa') \\ \downarrow & \downarrow \qquad\qquad \downarrow & \phi \\ (m \mapsto am) & \mathrm{End}_{\mathbb{Z}}(M) \xrightarrow{\sim} \mathrm{End}_{\mathbb{Z}}(A \otimes_A M) & \downarrow \\ & & \phi \otimes 1 \end{array}$$

?

$$\begin{array}{ccc} A \otimes_A M & \xrightarrow{1 \otimes f} & A \otimes_A M \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

$a \longmapsto$

$(a' \otimes m \mapsto aa' \otimes m)$

$(m \mapsto am) \longmapsto a' \otimes m \mapsto a' \otimes am$

What are you doing wrong?

$a \longmapsto (a' \mapsto aa')$

$\downarrow$

$(a' \otimes m \mapsto aa' \otimes m)$

$f \longmapsto \mu^{-1} f \mu$

$a' \otimes m \xmapsto{\mu} a'm \xmapsto{f = a\cdot} aa'm \xleftarrow{\mu} aa' \otimes m$

$a'm \xleftarrow{\mu} a' \otimes m \longmapsto aa' \otimes m \xmapsto{\mu} aa'm$

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{End}_{\tilde{A}^{op}}(A) \\ & \searrow & \swarrow \text{ unique if it exists} \\ & \mathrm{End}_{\mathbb{Z}}(M) & \end{array}$$

because M spanned by am and

$\phi(am) = (\phi \lambda_a)(m)$

$= \lambda_{\phi(a)} m = \phi(a) m.$

---

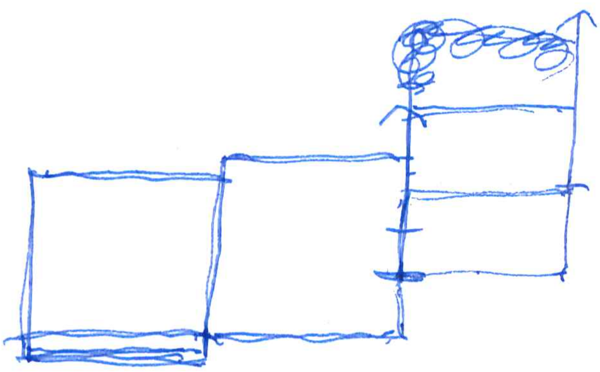
fun time. return to ring $A = m/mz$ $\quad m = \bigcup_{\varepsilon > 0} Rz^{\varepsilon}$

Is it possible to ~~understand~~ understand possible $P, Q,$ $Q \otimes P \to A$ firm dual pairs. Let's compute multipliers. R/mz has both top and bottom element.

$$\mathrm{Hom}_{R/Rz}(m/Rz, m/Rz) = \varprojlim_{\varepsilon \downarrow 0} \mathrm{Hom}_R(Rz^{\varepsilon}, m/Rz)$$

$$= \varprojlim_{\varepsilon \downarrow 0} (m/Rz) z^{-\varepsilon} = \varprojlim m z^{-\varepsilon}/Rz^{1-\varepsilon}$$

K8 $\xrightarrow{z^{1/8}} m/Rz \xrightarrow{z^{1/4}} m/Rz \xrightarrow{z^{1/2}} m/Rz$



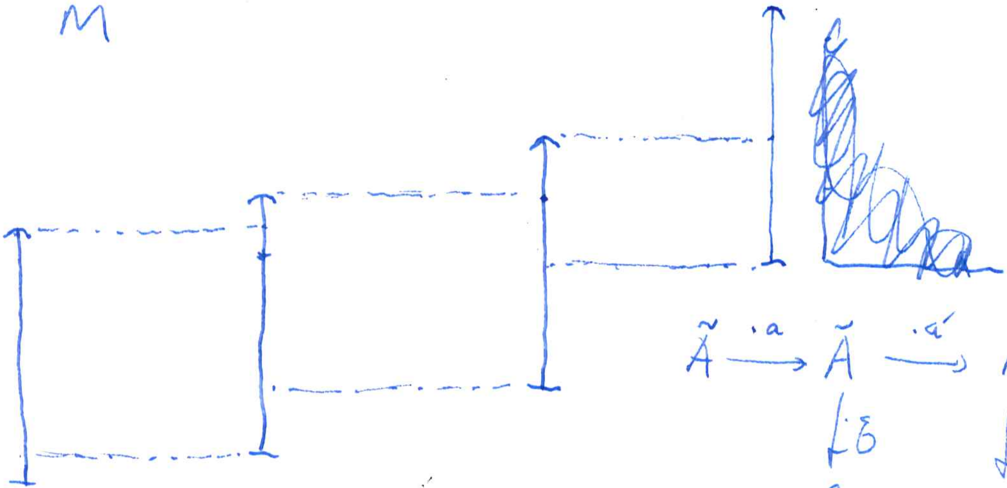
so it seems you get an inverse limit

$R \longrightarrow A \subset \tilde{A} \longrightarrow R$

$\xrightarrow{x} R^p \xrightarrow{y} R^q \qquad \xrightarrow{z^{1/4}} m/Rz \xrightarrow{z^{1/2}} m/Rz$

$\downarrow m \qquad \downarrow m'$

$M \qquad M$



$\tilde{A} \xrightarrow{\cdot a} \tilde{A} \xrightarrow{\cdot a'} \tilde{A}^{\Delta}$

$\downarrow \varepsilon \qquad \downarrow \varepsilon'$

Q

what is in

Let's go over things from the begining. Consider a truncated valuation ring R/Rz.

Basically I want ~~to~~ a counterexample to any firm A being meq to one with faithful left reg. repn. Question: Do $\exists$ flat firm $Q \neq 0$ with ${}_A\hat{Q} = 0$?

Consider R valuation ring value group $\bigcup 2^{-n}\mathbb{Z}$

$$R/Rz \xrightarrow{z^{t_1}} R/Rz \xrightarrow{z^{t_2}} R/Rz \longrightarrow \cdots$$

here the $t_i > 0$. and $\sum t_i < \infty$

149. ~~...~~ What is the limit of

$$\begin{array}{ccccccc} R & \xrightarrow{z^{t_1}} & R & \xrightarrow{z^{t_2}} & R & \xrightarrow{z^{t_3}} & \cdots\cdots \\ \cap & & \downarrow z^{-t_1} & & \downarrow z^{-t_1-t_2} & & \\ K & = & K & = & K & & \end{array}$$

so the limit is $\bigcup_n R z^{-(t_1+t_2+\dots+t_n)}$

powers z^ε for $\varepsilon \geq -(t_1+\dots+t_n)$ same n

$\Rightarrow \varepsilon > -\sum t_n$

$$\bigcup_{\varepsilon > -\sum t_n} m z^\varepsilon \Big/ \bigcup_{\varepsilon > -\Sigma} m z^{\varepsilon+1}$$ seems OK

---

concentrate. So you are looking at

$$F = \bigcup_{\varepsilon > t} R z^\varepsilon$$ which is a flat R-module

not isomorphic in general to $\bigcup_{\varepsilon > 0} R z^\varepsilon = m$

This seems correct. But in any case you can ~~consider~~ compare F/Fz and m/mz. We have $0 \neq z \in m/mz$ but let $x \in F$ satisfy $mx \in Fz$.

~~So K itself~~ K consists of $\sum_{\varepsilon \in S} c_\varepsilon z^\varepsilon$ where S is a discrete subset (bdd below) of $\bigcup 2^{-n}\mathbb{Z}$. S sequence tending to $+\infty$.

$x \in F$ means $x = \sum_{\varepsilon_n > t} c_n z^{\varepsilon_n}$ $\varepsilon_n \uparrow \infty$

$mx \in Fz$ means $\forall n, k \ni c_n \neq 0$ $\varepsilon_n + 2^{-k} > t+1$

$\therefore \varepsilon_n \geq t+1$. but since $t \notin \bigcup 2^{-n}\mathbb{Z}$ $\varepsilon_n > t+1$ $\therefore x \in Fz$

6960.73
1230.00
5730.73

So can arrange $\otimes_A Q = 0$.
Next you need enough maps $Q \to A$. ~~Need to be~~ $\mathrm{Hom}_{R/R_{\mathbb{Z}}}(Q, A)$. The dual^P should be based on $-t$. But then $P_{z^\varepsilon > -t} \otimes_A Q_{z^\varepsilon > t}$ will have a bottom element.

Try to do P, Q together What is going on?

Here's an idea. Consider $K = \bigcup R z^\varepsilon$, $\varepsilon \in \mathbb{Z}[\frac{1}{2}]$

Look at lattices inside here. = R submodule bdd above. If L lattice, consider $\Delta = \{\varepsilon \mid z^\varepsilon \in L\}$. Like a Dedekind cut. Δ closed under $+\varepsilon$ $\forall \varepsilon > 0$.
Ordered abel. gp.
When are these lattices isomorphic? It's like $\mathbb{R}/\mathbb{Z}[\frac{1}{2}]$. So how to handle this.

Various questions. Why not see what you can do when you stick to $L \subset K^d$. Stick to $L \subset K^d$? Suppose Q flat form A-mod. Does it have a rank? Injectives?
So what to do?

R valuation ring complete.

injectives. K, K/R, K/m inj hull of R/mz^ε

Can classify ideals in R.
These should be all the ind. injectives over R

$$0 \to R/m \to K/m \to K/R \to 0$$

$$0 \to R \to K \to K/R \to 0$$

$$0 \to m \to K \to K/m \to 0$$

151 How to use this information. I'm really interested in injectives over R/zR

$$\mathrm{Hom}_{R/I}\big(M, \mathrm{Hom}_R(R/I, Q)\big) = \mathrm{Hom}_R\big(\underbrace{R/I \otimes_{R/I} M}_{= M}, Q\big)$$

It's probably too ~~simple~~ naive to expect that K/R, K/m are injective$/R$.

Anyway you should look at

You might classify R lattices in K^2.

Intersect with flag $0 \subset K \subset K^2$ you end up with two real numbers for the lines.

What about extensions?

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$$

$$\mathrm{Ext}^1_R(L'', L') = \mathrm{Ext}^1_R\left(\varinjlim Rz^{\varepsilon}, L'\right)$$

What you would really like to know is that

there might be something to make this nondeg.

Maybe its impossible - why? You want a Morita context. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$, whence $Q \otimes_B P \xrightarrow{\sim} A$

and $P \otimes_A Q = B$. Now

$$\mathrm{Hom}_R(L_t, L_s) = \varprojlim_{\varepsilon > t} \mathrm{Hom}_R(Rz^{\varepsilon}, L_s)$$

$$= \bigcap_{\varepsilon > t} z^{-\varepsilon} L_s = L_{\geq s-t}$$

$\varepsilon \downarrow t$

$-\varepsilon \uparrow -t$

~~outline first~~

make a path through these results. 1C – 1F

1C about $\mathcal{a}/\mathcal{S}$, ${}^{\perp}\mathcal{S}$, colocalizing

$$E_2^{pq} = \mathrm{Ext}^p_{R/A}\left(\mathrm{Tor}^R_q(R/A, M), N\right) \Rightarrow \mathrm{Ext}^*_R(M, N)$$

$AM = M \Rightarrow$

How much characterizing of ~~~~ ~~~~ new idea last night is that the quasi-inverse functor to $j^*\iota$ is $j^*M \mapsto M_\#$

Prop. $j^*\iota : {}^{\perp}\mathcal{S} \to \mathcal{a}/\mathcal{S}$ is an equiv. iff $\forall M$ $\exists$ $\mathcal{S}$-isom $M_\# \to M$ with $M_\# \in \mathcal{S}^{\perp}$. In this case the quasi-inverse functor is $j^*M \mapsto M_\#$.

Assume $j^*\iota$ is an equiv. Can you see that $\mathcal{S}$ is closed under $\prod$'s. Let $N_i \in \mathcal{S}$.

Look at ~~$j^*(\prod N_i)$~~ $(\prod N_i)_\#$

$$\mathrm{Hom}_{{}^{\perp}\mathcal{S}}\left(M, (\prod N_i)_\#\right)$$

$$= \mathrm{Hom}_{\mathcal{a}}(M, \prod N_i) = \prod_i \mathrm{Hom}_{\mathcal{a}}(M, N_i)$$

$$= 0.$$

153 1c at end slightly rough (bullets)

$\Rightarrow$ 1f remains – bilocalizing stuff.

$\Rightarrow$ 1h

1i to be done.

1c \def

147 ~~8~~

1d 98 (2)}

117 \mod

~~1f~~ 40-53

h 11-14, 140, 140-44 145-47

i ~~8~~ 29, 13-21 85-90

$\mathfrak{T}(R,A)$ $\mathfrak{T}(R,A)$ $\mathfrak{M}(R,A)$?

1f.

$$\mathrm{Hom}_B(M,-) = \mathrm{Hom}_{\mathfrak{M}}(j^*M, j^*N)$$

$$\begin{array}{ccc} \mathrm{mod}(\tilde{A}) & \rightleftarrows & \mathrm{mod}(R) \\ \downarrow & & \downarrow \\ \mathfrak{M}(\tilde{A},A) & \longleftarrow & \mathfrak{M}(R,A) \end{array} \qquad \begin{pmatrix} \tilde{A} & R \\ A & R \end{pmatrix}$$

$$M \longmapsto \left(A \otimes_{\tilde{A}} M \to R \otimes_A M \right) \quad \text{invert mod isos.}$$

$$N \longleftarrow N$$

$M \longmapsto j^*(A \otimes_{\tilde{A}} M)$ is exact.

$0 \to M' \to M \to M'' \to 0$

$\mathrm{Tor}_1^{\tilde{A}}(A, M'') \to A \otimes_{\tilde{A}} M' \to A \otimes_{\tilde{A}} M$

A-nil since mult $A \xrightarrow{a\cdot} A$, $\tilde{A} \xrightarrow{a\cdot} A$

$N \longleftrightarrow N$ nil rim.

$N \longmapsto A \otimes_{\tilde{A}} N$

$M \longmapsto A \otimes_{\tilde{A}} M$

$A \otimes_{\tilde{A}} M$

---

alg $\quad \alpha \mapsto j^* M_\alpha \qquad j^*\left(\varinjlim_\alpha j_! j^* M_\alpha\right) \xleftarrow{\sim} \varinjlim_\alpha j^* M_\alpha$

$$\mathrm{Hom}_{\mathcal{M}}(j^*M, j^*N) = \mathrm{Hom}_R(j_! j^* M, N)$$

---

problem: existence of ~~[illegible]~~ $\varinjlim$ | resp. by j^*

---

suppose $j_! j^* M$ proj.

$1 \xrightarrow{\sim} j^* j_!$

$\mathrm{Mod} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{M}^{\Xi}$

$\mathcal{C}$

$\sim$

$\mathrm{Hom}_{\mathcal{M}}(X_\alpha, \quad)$

155 Sheaf theory case.

$$\mathrm{Mod} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{M}$$

$$\mathrm{Hom}_{\mathcal{M}}\Big(j^*M, j^*\big(\underset{\alpha}{\underleftarrow{\lim}}\, j_* X_\alpha\big)\Big) = \mathrm{Hom}_R\big(M, \underset{\alpha}{\underleftarrow{\lim}}\, j_* X_\alpha\big)$$

$$= \underset{\alpha}{\underleftarrow{\lim}}\, \mathrm{Hom}_R(M, j_* X_\alpha)$$

$$= \underset{\alpha}{\underleftarrow{\lim}}\, \mathrm{Hom}_{\mathcal{M}}(j^*M, X_\alpha)$$

Proof:
$$\mathrm{Hom}_{\mathcal{M}}\Big(\underset{\substack{\parallel \\ j^*j_*X}}{X}, j^*\big(\underbrace{\underset{\alpha}{\underleftarrow{\lim}}\, j_* X_\alpha}_{\text{closed}}\big)\Big)$$

$$\parallel$$

$$\mathrm{Hom}_{\mathcal{C}}\big(j_*X, \underset{\alpha}{\underleftarrow{\lim}}\, j_* X_\alpha\big)$$

$$\parallel$$

$$\underleftarrow{\lim}\, \mathrm{Hom}_{\mathcal{C}}(j_*X, j_*X_\alpha)$$

what does this proof amount to?
Start with $\alpha \mapsto X_\alpha$ you lift to $\alpha \mapsto j_*X_\alpha$ in $\mathcal{C} \subset \mathrm{Mod}$ and take $\underset{\alpha}{\underleftarrow{\lim}}\, j_*X_\alpha$

156 existence of $\varinjlim$'s in $\mathcal{M}$.

first method: Take $\alpha \mapsto X_\alpha$ a sys. in $\mathcal{M}$.

$$\mathrm{Hom}_{\mathcal{M}}(X_\alpha, X) = \mathrm{Hom}_{\mathcal{F}}(j_! X_\alpha, j_! X)$$
$$= \mathrm{Hom}_R(j_! X_\alpha, j_! X)$$

$$\varprojlim_\alpha \mathrm{Hom}_{\mathcal{M}}(X_\alpha, X) = \mathrm{Hom}_R(\varinjlim_\alpha j_! X_\alpha, j_! X)$$

$$\boxed{\begin{aligned} \varprojlim_\alpha \mathrm{Hom}_{\mathcal{M}}(X_\alpha, \overbrace{X}^{j^*(j_* X)}) &= \varprojlim_\alpha \mathrm{Hom}_R(j_! X_\alpha, j_* X) \\ &= \mathrm{Hom}_R(\varinjlim_\alpha j_! X_\alpha, j_* X) \\ &= \mathrm{Hom}_{\mathcal{M}}(j^*(\varinjlim_\alpha j_! X_\alpha), X). \end{aligned}}$$

$$\varprojlim_\alpha \mathrm{Hom}_{\mathcal{M}}(X_\alpha, X) = \varprojlim_\alpha \mathrm{Hom}_{\mathcal{F}}(j_! X_\alpha, j_! X)$$
$$= \mathrm{Hom}_{\mathcal{F}}(\varinjlim_\alpha j_! X_\alpha, j_! X)$$
$$= \mathrm{Hom}_{\mathcal{M}}(j^*(\varinjlim_\alpha j_! X_\alpha), X).$$

Check that $\varinjlim_\alpha j_! X_\alpha$ firm.

$$\mathrm{Hom}_R(\varinjlim_\alpha j_! X_\alpha, N) = \varprojlim_\alpha \mathrm{Hom}_R(j_! X_\alpha, N)$$
$$= \varprojlim_\alpha \mathrm{Hom}_{\mathcal{M}}(X_\alpha, \underbrace{j^* N}_{\text{univ. nil ison}})$$

157 need to understand better why $\exists j_!$

$\Rightarrow \mathcal{M} \simeq \mathcal{F}$ is closed under $\varinjlim$'s.

NO. Point is that always $\mathcal{F}$ is closed under $\varinjlim$'s + incl. fun. $\mathcal{F} \to \mathcal{M}$ respects $\varinjlim$'s.

last part:
$$\mathrm{Hom}_{\mathcal{M}}\left(j^*(\varinjlim M_\alpha), j^*N\right)$$
$$\simeq \mathrm{Hom}_R\left(\varinjlim M_\alpha, N\right)$$
$$= \varprojlim_\alpha \mathrm{Hom}_R(M_\alpha, N)$$
$$= \varprojlim_\alpha \mathrm{Hom}_{\mathcal{M}}(j^*M_\alpha, j^*N)$$

This assumes known that $\varinjlim M_\alpha \in \mathcal{F}$.

This clear as $\mathrm{Hom}_R\big(\underline{\varinjlim_\alpha M_\alpha}, -\big) = \varprojlim_\alpha \underbrace{\mathrm{Hom}_R(M_\alpha, -)}_{\text{inverts nil-isos.}}$

$A = \bigcap \ell \qquad R/A \hookrightarrow \prod R/\ell \qquad \therefore R/A \in \mathcal{S}$

$M \in \mathcal{S} \qquad m \in M \Rightarrow R/A \twoheadrightarrow R/\ell \hookrightarrow M$

$\Rightarrow AM = 0.$

34-38 $\qquad \{P_n\}_{n\geq 0} \qquad (P_n)_{n\geq 0}$

~~Iff~~

$$0 \to \ell \to A \xrightarrow{a \mapsto am} M \to 0$$

M simple $AM \neq 0 \Rightarrow \exists m \quad Am = M. \quad A/\ell \xrightarrow{\sim} M$

Want ℓ maximal ideal in A not containing A^2.

M simple A-module

$AM \neq 0$ $M = AM$ and ${}_A M = 0$

If $m \neq 0$ then $Am \neq 0$ so $Am = M$

so $A/\ell \xrightarrow{\sim} M$ $m_0 \neq 0$.

$a \longmapsto am_0$

$\exists$ a such that $am_0 = m_0$

Take $a_0 \notin \ell$ must find a

such that $aa_0 - a_0 \in \ell$

If M simple ~~non~~ non-nil, then $\forall m \neq 0$
$\exists a \ni (1-a)m = 0$.

Let ℓ be maximal left ideal in A.
When is A/ℓ non-nil? When $A(A/\ell) \neq 0$
i.e. $A^2 \not\subset \ell$. So

~~a am~~ $aM = 0 \implies aR$

M simple non-nil. $\{a \mid aM = 0\}$

$aM \neq 0$ ~~$\implies \exists\ RaM \neq 0 \implies \exists$~~

~~$\implies RaM = M \implies \exists$~~

$aM \neq 0 \implies \exists m \quad am \neq 0$

$\implies Ram = M \implies \exists r \quad ram = m$

$\implies 1 - ra$ has no left inverse

$\implies \exists \mathfrak{m} \quad R(1-ra) \subset \mathfrak{m}$

159 so $R/\mathfrak{m} = M$ simple $\exists\, m \neq 0$ in M

$\ni (1-ra)m = 0 \Rightarrow am \neq 0$.

$\exists$ simple non-nil M such that $aM \neq 0$

$\iff \exists r$ such that $R(1-ra) < R$.

$\forall$ simple nonnil M we have $aM = 0$

$\iff \forall r \quad R(1-ra) = R. \iff \forall r \quad (1-ra)^{-1}\,\exists$.

But $\quad x(1-ra) = 1 \Longrightarrow x = 1 + xra$

$\Longrightarrow Rx = R$.

I want to keep this simple.

$\underbrace{J(R)\cap A} = J(A)$. Variant:

$\{a \mid aM = 0$ for all simple non-nil R-mods.$\}$.

If l max left ideal in A, $l \not\supset A^2$ so that A/l is simple non-nil A-module does it follow $Rl \subset l$.

$$\begin{tikzcd} A^2 \arrow[dr] \arrow[d, phantom, "\cap"] & \\ A \arrow[r] & \underbrace{A/l = M} \end{tikzcd}$$

has R-action $r(am) = (ra)m$.

$$\begin{tikzcd} a_1a_2 \arrow[r, mapsto] \arrow[d, "r"] & a_1a_2 + l = a_1(a_2+l) \arrow[d, "r"] \\ ra_1a_2 \arrow[r, mapsto] & (ra_1)(a_2+l) \end{tikzcd}$$

160 ℓ modular if $\exists\, e \in \ell \ni a(1-e) \in \ell \;\; \forall a$

M strictly cyclic $\iff M = Am$ for some $m \in M$.

Then $M \xleftarrow{\sim} A/\ell \qquad \ell = \{a \mid am = 0\}$.

$am \leftarrow\!\shortmid a+\ell$

$\exists\, e \in A$ such that $m = em$ i.e.

~~[illegible]~~ this means. $1-e \in \ell$

and implies $A(1-e) \subset \ell$.

Conversely, if ℓ left ideal and $\exists\, e \in \ell$ such that $A(1-e) \subset \ell$. Then

A/ℓ has dist. elt. $\bar{e} = e + \ell$

~~and $A(e+\ell) = A/\ell$~~

$a - ae \in \ell \implies$ every elt of A/ℓ of form $a\bar{e}$.

---

closed modules. $M \xrightarrow{\sim} \underbrace{\mathrm{Hom}_{\tilde{A}}(A, M)}$ naturally a module over $\mathrm{Hom}_{\tilde{A}}(A,A)^{op}$

$(rf)(a) = f(ar)$

Suppose M an R-mod. $\ni {}_AM = 0$.

Then $\mathrm{Hom}_R(A, M) \longrightarrow \mathrm{Hom}_{\tilde{A}}(A, M)$

should be an isom. $f($

161 Why? Let $f: A \to M$ be $\tilde{A}$-linear.

~~Then ...~~ Point.

$$1 \longrightarrow R \otimes_{\tilde{A}} A \longrightarrow A \longrightarrow 0$$

$$\begin{array}{ccc} W \otimes_{\tilde{A}} M & \longrightarrow & W \otimes_R M \\ \wr\wr & & \| \\ W \otimes_R R \otimes_{\tilde{A}} M & \longrightarrow & W \otimes_R M \end{array}$$

$$R \otimes_{\tilde{A}} \tilde{A} \longrightarrow \tilde{A}$$

$\boxed{\text{YES!}}$

$$R \otimes_{\tilde{A}} \check{A} \to \check{A}$$

$\cancel{R \otimes_{\tilde{A}} \tilde{A}}$

$$R \otimes_{\tilde{A}} R \longrightarrow R$$

$$k = \sum r_i' \otimes r_i''$$

$$ka = \sum r_i' \otimes r_i'' a$$

$$= \sum r_i' r_i'' a \otimes 1$$

Then $\mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Hom}_{\tilde{A}}(N, M)$

$$\| $$

$$\searrow \quad \mathrm{Hom}_R(N, \mathrm{Hom}_{\tilde{A}}(R, M))$$

~~Hom$_R$~~ $\mathrm{Hom}_{\tilde{A}}(R, M) \to \mathrm{Hom}$

Do I feel stupid.

Suppose M R-mod A-closed

162 Look at

$$\begin{tikzcd} \mathrm{Hom}_R(A,M) \arrow[r] \arrow[dr] & \mathrm{Hom}_{\tilde{A}}(A,M) \arrow[d, equal] \\ & \mathrm{Hom}_R(R\otimes_{\tilde{A}} A, M) \end{tikzcd}$$

So it's OK because kernel of $R\otimes_{\tilde{A}} A \to A$ should be killed by A.

$$R\otimes_{\tilde{A}} N \longrightarrow N$$

$$a\sum r_i\otimes n_i = \cancel{\;} \sum 1 a r_i \otimes n_i$$

$$= 1\otimes \sum a r_i n_i$$

Then claim

$$\boxed{\mathrm{Hom}_R(N,M) \xrightarrow{\sim} \underset{\mathrm{Hom}_R(R\otimes_{\tilde{A}} N, M)}{\overset{\|}{\mathrm{Hom}_{\tilde{A}}(N,M)}}} \quad \text{if } {}_A M = 0$$

~~Pf~~ Given $f: N \longrightarrow M$ $\quad$ A-linear

Observe that $\quad (af)(rn) = f(arn) = arf(n)$

$$a f(rn) = f(arn) = ar f(n)$$

$$\Longrightarrow f(rn) = r f(n) \quad \text{since } {}_A M = 0.$$

~~Pf~~ M inj over $A \Rightarrow$ M inj over R

Take $0 \to N' \to N \to N'' \to 0$ over $\tilde{A}$

want

$$\begin{tikzcd} \mathrm{Hom}_{\tilde{A}}(N,M) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\tilde{A}}(N',M) \arrow[d, equal] & \mathrm{Tor}_1^{\tilde{A}}(R,N) \arrow[d] \\ \mathrm{Hom}_R(R\otimes_{\tilde{A}} N, M) \arrow[r] & \mathrm{Hom}_R(R\otimes_{\tilde{A}} N', M) \arrow[r] & \mathrm{Hom}_R(\ \ ,M) \end{tikzcd}$$

16~~4~~3 What else

$$\begin{array}{ccc} \mathrm{mod}(\tilde{A}) & \longleftarrow & \mathrm{mod}(R) \\ \downarrow & & \downarrow \\ \mathcal{M}(\tilde{A}, A) & \longleftarrow & \mathcal{M}(R, A) \end{array}$$

$$j^*N \longleftarrow\!\shortmid\; j^*N$$

$$j^*M \longmapsto j^*(A \otimes_{\tilde{A}} M)$$

$$0 \to M' \to M \to M'' \to 0$$

$$\mathrm{Tor}_1^{\tilde{A}}(A, M'') \to \underset{A \otimes_A M'}{\cancel{\mathrm{Tor}(A, M')}} \to A \otimes_A M$$

why composites are the identity

$$j^*N \longmapsto j^*N \longmapsto j^*(A \otimes_{\tilde{A}} N)$$

$$A \otimes_{\tilde{A}} N \longrightarrow N$$

$$\underline{j^*M \longmapsto j^*(A \otimes_{\tilde{A}} M)}$$

$$\begin{array}{ccc} M & \longrightarrow & R \otimes_{\tilde{A}} M \\ a\cdot \downarrow & \swarrow & \downarrow a\cdot \\ M & \longrightarrow & R \otimes_{\tilde{A}} M \end{array}$$

$r \otimes m$

$(ra)m \longmapsto 1 \otimes ram$

$$F \xrightarrow{F.\beta} FGF \xrightarrow{\alpha.F} F$$

$$(\alpha.F)(F.\beta) = 1$$

$$(G.\beta)(\beta.G) = 1$$

$$\mathrm{Mod}(R) \xrightarrow{L^*} \mathcal{M}(R, A) \underset{1}{\overset{FG}{\rightrightarrows}} m(R, A)$$

 ~~[illegible]~~ ℓ modular i.e. A/ℓ

strictly cyclic.

left ideal in $\mathbb{Z} \oplus A$

get logic straight.

$J(R) =$ ann. of all simple R-mods

$=$ largest ℓ ideal $\ell \ni$ ~~[illegible]~~
all elts of $1+\ell$ invertible,

$J(\tilde{A}) \subset A$ | define $J(A) = J(\tilde{A})$

Result. A ideal in R unital $\Rightarrow J(R) \cap A = J(A)$.

~~[illegible]~~ This is all confused by ~~[illegible]~~ unital stuff

A unital $\tilde{A} \simeq A \times \mathbb{Z}$

Important points. non-nil

A non unital - simple A-modules of form
A/ℓ ℓ maximal and $\ell \not\supset A^2$

$\Rightarrow$ ~~[illegible]~~ $M = Am$
A/ℓ strictly cyclic

$\Rightarrow \exists\ m \in A/\ell$ ~~[illegible]~~ $Am = A/\ell$

~~$\Leftrightarrow \exists\ a \in A,\ \ell \in A,\ [illegible]$~~

$\Rightarrow \exists\ e \in A$ $Ae + \ell = A$

M simple non-nil $\Leftrightarrow \forall m \neq 0$ $Am = M$

16~~5~~ M an A module

M simple non-nil $\iff$ $\forall m \neq 0 \quad Am = M$

$\implies$ Am either 0 or M

so $Am < M \implies Am = 0 \implies {}_AM \neq 0$

$\therefore \ Am = M \quad \forall m \neq 0.$

If $Am = M$ ~~then~~ $\forall m \neq 0$, then $\forall m \neq 0$

$\exists \ e \in A \ni \quad em = m \qquad$ so if $\ell = \{a \mid am = 0\}$

$A/\ell \xrightarrow{\sim} M \qquad a \mapsto am \qquad a(1-e) \mapsto am - aem = 0$

$A(1-e) \subset \ell. \qquad$ Very confusing.

Must think in terms of simple modules

$J(A)$ annihilator of all simple A-modules.
simple objects of $Mod(\tilde{A})$.

same as ann. of all A-non-nil simple A-modules

If R-unital $J(R)$ = ann. of all simple unitary R-modules.

$A \cap J(R) = J(A).$