72. tensor product criteria

A unital f. dual pair $(A, A) \oplus (X, Y)$ $\qquad B = \begin{pmatrix} A & Y \\ X & X \otimes_A Y \end{pmatrix}$

In fact take X, Y r+l. ideals in A. $\quad XYXY \subset XY.$

B is ~~some sort of~~ a ring generated by an idempotent.

We have the Davydov proof as an example for motivation.

$$0 \longrightarrow Re \otimes_{eRe} eR \longrightarrow R \longrightarrow R/\overbrace{ReR}^{B} \longrightarrow 0$$

$$0 \longrightarrow Re \otimes_{eRe} eR \longrightarrow P \longrightarrow R/B \otimes_R P \longrightarrow 0$$

$R/B \otimes_R Re = Re/ReRe =$ ~~eRe/eR~~ $Re/Re = 0$

$A = eBe \qquad B = Be \otimes_A eB \qquad$ I will assume

Be, eB are f.g. proj over A.

go back to earlier notation. You have $B = \begin{pmatrix} A & Y \\ X & X \otimes_A Y \end{pmatrix}$

with $Y \longrightarrow \operatorname{Hom}_{A^{op}}(X, A)$ arbitrary. Then you factor

$$\begin{tikzcd} Y \arrow[r] \arrow[d, phantom, "\cap"] & \operatorname{Hom}_{A^{op}}(X, A) \\ Y_1 \arrow[ur, two heads] & \end{tikzcd}$$

~~then~~ $Y_1 \twoheadrightarrow X^* \Rightarrow X \hookrightarrow Y_1^*$

So you have $B = \begin{pmatrix} A & Y \\ X & X \otimes_A Y \end{pmatrix} \subset \begin{pmatrix} A & Y_1 = X_1^* \\ X_1 & \operatorname{End}_{A^{op}}(X_1) \end{pmatrix}$

But now you find

$$\begin{tikzcd} Q_1 \arrow[r, "\sim"] \arrow[d, phantom, "\cup"] & \operatorname{Hom}_A(P_1, A) \arrow[d, two heads] \\ Q \arrow[r] & \operatorname{Hom}_{A^{op}}(P, A) \arrow[dl] \\ A \arrow[u] & \end{tikzcd}$$

$$P \otimes_A Q \subset P_1 \otimes_A Q_1$$

$$A \subset \overset{\shortparallel}{B} \subset B_1 = M_n(A)$$

$$\begin{tikzcd} A \arrow[d, phantom, "\cap"] & M_n(A) \arrow[d, phantom, "\cap"] \\ B \arrow[ur] & M_n(B) \end{tikzcd}$$

73. ~~So what we have is~~ [illegible]
need to find zilch!!! **OKAY** so what?
So basically you have (P, Q) more or less arbitrary, embed into ~~$P \oplus Q$~~ $\left(P \oplus Q^*, P^* \oplus Q\right)$.
Suppose you work with maps instead of pairings

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A \\
Q \arrow[r] & P^* \arrow[u]
\end{tikzcd}$$

any map "dilates" to an isomorphism. There are other things you need to know.

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d] & A \\
Q \arrow[r, "\phi"] \arrow[d] & P^* \arrow[u] \\
A^n \arrow[r, "1"] & A^n \arrow[u] \\
Q^n \arrow[r, "\phi^n"] \arrow[u] & P^{*n} \arrow[u]
\end{tikzcd}$$

So the critical question seems to involve the space of maps from ϕ to ϕ^n. i.e. from B to $M_n(B)$. Is there some Volodin type, [illegible] elementary path, ~~[illegible]~~ between homomorphisms.

~~So~~ What are elementary moves for a map $(A, A) \longrightarrow (P, Q)$? Such a homom. is a pair $p \in P$, $q \in Q$ such that $qp = 1$. It seems that the ~~basic~~ elementary moves would be to change p by $\delta p \in q^\perp$ or q by $\delta q \in p^\perp$. Here you use nondegeneracy to split $P = pA \oplus q^\perp$

~~It seems I need to understand.~~

74. Consider $\begin{pmatrix} A & Y \\ X & X\otimes_A Y \end{pmatrix} = \begin{pmatrix} A \\ X \end{pmatrix} \otimes_A (A \ \ Y)$. When is this h-unital? iff $X \overset{L}{\otimes}_A Y = X \otimes_A Y$. assume A unital

Ind of pairing $Y \otimes X \to A$. Can you deform pairings? Can you deform pairings? is unclear. If X is f.free then any Y works, any Y equipped with $Y \to X^*$.

Go on to Davydov. What happens.

$$B = \underset{A+X}{Re} \otimes_{\underset{A}{eRe}} \underset{A+Y}{eR}$$

$$0 \longrightarrow B \longrightarrow \tilde{B} \longrightarrow Z \longrightarrow 0$$

---

char. of A-torsion R-modules M

1) $\forall \ M' < M \qquad \mathrm{Hom}_R(R/A, M/M') \neq 0$

2) $\mathrm{Hom}_R(M, I) = 0$ all inj $I \ni \mathrm{Hom}_R(R/A, I) = 0$

3) $\forall F$ firm flat right mod $F \otimes_R M = 0$

4) $\forall \ m \in M$ and sequence $a_1, a_2, \dots$ in A $\exists n \ni \ a_n a_{n-1} \cdots a_1 m = 0$.

0) $M \in$ smallest Serre subcat closed under $\oplus$'s containing R/A.

3) $\Rightarrow$ 4) Take $F = \varinjlim \left(R \xrightarrow{a_1 \cdot} R \xrightarrow{a_2 \cdot} R \longrightarrow \cdots \right)$ let $v \in F$ image of 1. Then $F \otimes_R M = 0$

2) $\Rightarrow$ 3) $\mathrm{Hom}_{\mathbb{Z}}(F \otimes_R M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_R\big(M, \underbrace{\mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})}_{\text{inj}}\big)$

75. $\forall M' < M \quad \mathrm{Hom}_R(R/A, M/M') \neq 0$.

$\Downarrow$ $\forall N$, $\mathrm{Hom}_R(R/A, N) = 0 \Longrightarrow \mathrm{Hom}_R(M, N) = 0$

$\Downarrow$ $\forall I$ inj $\mathrm{Hom}_R(R/A, I) = 0 \Longrightarrow \mathrm{Hom}_R(M, I) = 0$

If ${}_A N = 0$ $\quad N \neq 0$

pick $n \neq 0$ then $An \neq 0$ $\exists a_1$ $a_1 n \neq 0$

$\exists a_2$

~~these form~~ 2), 3) define Serre subcats closed under $\oplus$'s.

Let S be Serre subcat closed under $\oplus$'s containing R/A.

contains all R/A modules. ~~[illegible]~~ $\forall M$ ~~[illegible]~~ $\exists$ largest $M' \subseteq M$ with $M' \in S$. $\therefore$ If M has 1) concludes $M \in S$. ~~Let M~~

$$\mathrm{firm}(R^{op}, A^{op}) \xrightarrow{\sim} \mathrm{rtcontHom}\left(\mathrm{mod}(R)/\mathrm{tors}(R,A), \mathrm{Ab}\right)$$

$$P \longmapsto P \otimes_R -$$

$$F(R) \longleftarrow\!\shortmid \; F$$

$F(M) \cong F(R) \otimes_R M$ so $F(R)$ is firm.

~~So what goes on. Suppose A unital, consider~~
~~$B = \begin{pmatrix} A & Y \\ X & X \otimes_A Y \end{pmatrix}$. want B h-unital: $X \otimes_A Y \simeq X \otimes_A Y$.~~

~~special case to be handled $X = A$, Y arbitrary with a map $Y \xrightarrow{f} X^* = A$.~~

~~$B = \begin{pmatrix} A & X \\ A & X \end{pmatrix}$ $\quad X \xrightarrow{f} A$ arb A-mod map~~

~~$A \oplus Y \longrightarrow Y$~~

76. $\begin{pmatrix} A \\ A \end{pmatrix} \otimes_A (A \quad 0)$

Go back to $R = (ReR)^\sim$ where $eR \in \mathcal{P}(A)$

$A = eRe$. What about

$$1 \longrightarrow GL_n(B) \longrightarrow GL_n(R) \longrightarrow GL_n(\mathbb{Z}) \longrightarrow 1$$

$$0 \longrightarrow B^n \longrightarrow R^n \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

Now $GL_n(B)$ is the group of autos. of R^n trivial on $\mathbb{Z}^n$. So what do we see? Equivalence of categories between mod(A) and $m(B)$

$$\begin{tikzcd} \mathrm{mod}(A) \arrow[r, shift left, "Re\otimes_A -"] & m(B) \arrow[l, shift left, "e-"] \end{tikzcd}$$

so $\mathrm{Hom}_B(B^n, B^n) = \mathrm{Hom}_A(eB^n, eB^n)$

$$= M_n(\mathrm{Hom}_A(eB, eB))$$

Is eB a faithful proj.

$\underbrace{eR}_{Q} \otimes_R \underbrace{Re}_{P} = eRe$??

You're assuming $\begin{pmatrix} eRe & eR \\ Re & ReR \end{pmatrix} = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

We know $Q \otimes P \longrightarrow A$ is surjective so Q is a generator for mod(A). But we have add. assumption that $Q \in \mathcal{P}(A)$. Changing A you reach critical case when $Q = eR = A$, and $P = eR$ is

77. $A \oplus X$ X arb.

$$Re \otimes_A eR = P \otimes_A Q = B$$

Take $Q = eR = A$. Then you get ~~a homom.~~ functors

$$\mathcal{P}(R) \longrightarrow \mathcal{P}(A) \longrightarrow \mathcal{P}(R)$$

$$E \longmapsto eE \longmapsto Re \otimes_A eE = BE.$$

$$GL_n(R) \longrightarrow GL_n(A) \longmapsto Aut_R(Re^n)$$

$$Aut_R(R^n) \longrightarrow Aut_A(Q^n) \xrightarrow{\sim} Aut_B(B^n)$$

$P = Re$ arbitrary right A-module.
automatically f.g. projective over R.

Start again. $C = \begin{pmatrix} A & eR \\ Re & B \end{pmatrix}$ if you work with left modules then

$R = \vec{B}$

you want eR to be in $\mathcal{P}(A)$. Get

$$\begin{tikzcd} \mathcal{P}(R) \arrow[r] & \mathcal{P}(A) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\cap"] & \text{finin proj } B\text{-mods} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & mod(R) \\ & mod(A) & \text{finin}(B) = m(B) & \end{tikzcd}$$

$$A \longmapsto Re \otimes_A A = Re$$

$B \in m(B)$ corresponds to $Q = eR \in mod(A)$
Since $Q = eR \in \mathcal{P}(A)$ by assumption
we have B projective in $m(B)$. In fact

78. Q summand of A^k $\Longrightarrow$ $B = P\otimes_A Q$ summand of $P^k = Re^k$ so B ~~[illegible]~~ $\in \mathcal{P}(\tilde{B})$. Thus we seem to have functors

$$\mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \xrightarrow{\sim} \text{firm}\,\mathcal{P}(B) \subset \mathcal{P}(\tilde{B})$$

$$\tilde{B} \longmapsto e\tilde{B} = Q \longmapsto P\otimes_A Q = B$$

~~[illegible]~~ So the situation seems to be this. You have $B = P\otimes_A Q$, where $Q \in \mathcal{P}(A)$, and of course one assumes $Q\otimes P \twoheadrightarrow A$ so that Q is a generator for $\mathcal{P}(A)$. You find $B \in \mathcal{P}(\tilde{B}) \cap \mathcal{M}(B)$. Why? Q summ of A^k $\Longrightarrow$ B summand of P^k

$$\mathrm{Hom}_B(P, N) = \mathrm{Hom}_A(A, Q\otimes_B N)$$

$$\overset{\shortparallel}{\mathrm{Hom}_{\mathcal{M}(B)}(P, N)} \qquad \cancel{=} \qquad ?$$

~~[illegible]~~ the point is that $Q \in \mathcal{P}(A) \cap \mathcal{M}(A)$ so $P\otimes_A Q \in \mathcal{P}(B) \cap \mathcal{M}(B)$.

$B = P\otimes_A Q$ A unital (P, Q) firm dual pair

$P \in \mathcal{P}(A) \Longrightarrow B \in \mathcal{P}(\tilde{B}) \cap \mathcal{M}(B)$

Thus have functor $\mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(\tilde{B})$

$$L \longmapsto B\otimes_{\tilde{B}} L = BL$$

which actually factors

$$\mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \longrightarrow \mathcal{P}(\tilde{B}) \cap \mathcal{M}(B)$$

$$L \longmapsto Q\otimes_B L \longmapsto P\otimes_A Q\otimes_B L = B\otimes_B L$$

Notice that because Q is a s.proj generator for mod(A), have Morita equivalence of A with $\mathrm{End}_A(Q)^{op} = A'$

79 $\begin{pmatrix} A & Q \\ Q^* & \underset{\overset{\shortparallel}{A'}}{Q^* \otimes_A Q} \end{pmatrix}$. So should be able to replace A by A'.

$$\begin{tikzcd} \mathrm{mod}(A') & \mathrm{mod}(A) & \mathcal{M}(B) \\ A' \arrow[r, mapsto] & Q \arrow[r, mapsto] & B \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{mod}(A'^{op}) & \mathrm{mod}(A^{op}) & \mathcal{M}(B^{op}) \\ A' \arrow[r, mapsto] & Q^* \arrow[r, mapsto] & Q^* \otimes_A Q = A' \end{tikzcd}$$

so it seems that B is a right A' module and A' is a right B-module

$B = P \otimes_A Q \longrightarrow Q^* \otimes_A Q = A'$

I've seen this before ~~$A' \to Q^* \to A'$~~

$\begin{pmatrix} A' & A' \\ B & B \end{pmatrix}$

The upshot is that I can assume $Q = A$.

$\begin{pmatrix} A & A \\ P & B \end{pmatrix}$ A unital

$A \otimes P \twoheadrightarrow A$

So we have $f \in \mathrm{Hom}_{A^{op}}(P, A)$ such that $Af(P) = A$. Then $B \rightsquigarrow M_n(B)$ can assume $f : P \twoheadrightarrow A$.

Begin with A unital (Q, P) such that $Q \in \mathcal{P}(A)$ and $Q \otimes P \twoheadrightarrow A$. Want to show $K_* A$ and $K_* B$, $B = P \otimes_A Q$ are canon. isom. Can replace (Q, P) by $(Q, P)^n$ whence you have $A \to B$. Next Q must be proj. gen. of mod(A)

80 So ~~should~~ have a map $\begin{pmatrix} A & Q \\ Q^* & Q^*\otimes_A Q \end{pmatrix}$ between

A and $A' = Q^*\otimes_A Q = \mathrm{Hom}_A(Q,Q)^{op}$. This leads to a map

$$\begin{pmatrix} A' & Q^* & Q^*\otimes_A Q = A' \\ Q & A & Q \\ \underset{\overset{\parallel}{B}}{P\otimes_A Q} & P & B \end{pmatrix}$$

pairing is

$$(Q^*\otimes_A Q)\otimes_B (P\otimes_A Q) \xrightarrow{\sim} Q^*\otimes_A Q$$

$$A' \otimes_B B \xrightarrow{\sim} A'$$

So we've reached the ~~situation~~ $\begin{pmatrix} A' & A' \\ P' & B \end{pmatrix}$ with

$A'\otimes P' \twoheadrightarrow A'$. So this means we have ~~#~~ a right A'-map $P'\to A'$ ~~whose~~ such that the left ideal of A' gen. by the image is A'.

$$P' \longrightarrow A' \quad \text{is}$$

$$P\otimes_A Q \longrightarrow Q^*\otimes_A Q$$

---

critical case B right ideal in A unital such that $AB = A$. Good question:

---

critical case A unital, B ~~left~~ right ideal in A $AB = A$, $BA = B$

$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ But how does this compare with our earlier picture? ~~[illegible]~~

$$\begin{tikzcd} \mathrm{mod}(\tilde B) \arrow[r] & \mathrm{mod}(A) \arrow[r, "\text{equiv.}"] & m(B) \subset \mathrm{mod}(\tilde B) \\ L & A\otimes_B L & B\otimes_A A\otimes_B L = B\otimes_B L \end{tikzcd}$$

~~$GL_n(\tilde B) \longrightarrow GL_n(A)$~~

$$\mathrm{Aut}_B(\tilde B^n) \longrightarrow \mathrm{Aut}_A(A^n) \longrightarrow \mathrm{Aut}_B(B^n)$$

$$\tilde B \longmapsto A \longmapsto B\otimes_A A = B$$

81.

$$\begin{array}{ccccccc} \mathrm{mod}(\tilde{B}) & \longrightarrow & \mathrm{mod}(A) & \longrightarrow & \mathcal{M}(B) & \subset & \mathrm{mod}(\tilde{B}) \\ L & \longmapsto & A\otimes_B L & \longmapsto & B\otimes_B L & & \\ & & M & \longmapsto & B\otimes_A L & & \end{array}$$

these are functors, the second is an equivalence

The functor $\mathrm{mod}(\tilde{B}) \longrightarrow \mathcal{M}(B)$ is the natural

$$L \longmapsto B\otimes_B L$$

functor to the quotient category since B is firm

B firm because $B = \overset{P\otimes_A Q}{B\otimes_A A}$ B, A are firm mods (A and $AB = A$.

So you have

$$\mathrm{Aut}_{\tilde{B}}(\tilde{B}^n) \longrightarrow \mathrm{Aut}_A(A^n) \xrightarrow{\sim} \mathrm{Aut}_{\tilde{B}}(B^n)$$

maybe these maps are compatible with ⊕.

The real question is what is $\mathrm{Aut}_{\tilde{B}}(B^n)$? e.g. what is $\mathrm{Aut}_{\tilde{B}}(B)$? Since you are dealing with left modules you have ${}_{\tilde{B}}B_A$ i.e. a hom. $A^{op} \longrightarrow \mathrm{End}_{\tilde{B}}(B)$ which is an isom. I don't know much about B as a $\tilde{B}$ module, except I know that

$$\left(\begin{array}{ll} A \text{ should be f.g. proj } \tilde{B}^{op}\text{-module} \\ B \text{ ———————————— } \tilde{B}\text{-module.} \end{array} \right.$$

Another point is that A might be a kind of multiplier algebra. $A = \mathrm{Hom}_B(B, B)^{op}$.

$$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$$

mult alg of B is endos of the pair $(P=B,\ Q=A)$.

$$M(B) \subset \mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(Q,Q)^{op}$$

$$A = M(A) \subset \mathrm{Hom}_{B^{op}}(\underset{A,\,A}{Q,\,Q}) \times \mathrm{Hom}_B(\underset{B,\,B}{P,\,P})$$

82 Start again with ~~[illegible]~~.

~~Assume~~ $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ afirm A unital

Assume $Q \in \mathcal{P}(A)$. Since $Q \otimes P \twoheadrightarrow A$ we know Q is a gen. for mod(A). Up to ~~an~~ ordinary unital Morita equivalence can suppose $Q = A$. Then we have

72
3.95
75.95

$$\begin{pmatrix} A & A \\ P & B \end{pmatrix}$$

6

where P is ~~a right A~~ an A^{op}-module together with ~~[illegible]~~ an A^{op}-map $P \xrightarrow{f} A$ such that $A \otimes P \to A$ ~~this~~ is surjective. Then Suslin's excision should tell us that the problem reduces to the right ideal $f(P)$.

So we consider $\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ where $B \subset A$

8

$BA = B$ (B right ideal generating A).
$AB = A$

get More ~~[illegible]~~ specific: suppose $\exists\ y \in A$ $x \in B$ such that $yx = 1$. So A unital ring with elt ~~[illegible]~~ $x, y \ni yx = 1$. $B = xA$

$$\begin{pmatrix} A & Ay \\ xA & xA \otimes_A Ay \end{pmatrix}$$

$xyxy = xy$

$Ay = Ayxy \subset Axy \subset Ay$

$$\begin{pmatrix} A & Ae \\ eA & eA \otimes_A Ae \end{pmatrix}$$

$$eA \otimes_A Ae \; \| \; eAe$$

$xAy = xyxAyxy$
$\subset xyAxy$
$\subset xAy.$

$$\begin{pmatrix} A \\ eA \end{pmatrix} \otimes_A (A \quad Ae \oplus Ae^{\perp})$$

null pairing

$$(A, A) \oplus (eA, Ae) \oplus (\cancel{A}, Ae^{\perp})$$

8/3

$$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$$

If you have $yx = 1$
$x \in B$ $y \in A$

then you get

$$\begin{tikzcd}
(A, A) \arrow[r] & (B, A) \\
(a_1, a_2) \arrow[r, mapsto] \arrow[d, mapsto] & (xa_1, a_2y) \arrow[d, mapsto] \\
a_2a_1 & a_2yxa_1 = a_2a_1
\end{tikzcd}$$

and then you have a homom. $A \longrightarrow B$

$a_1a_2 \longmapsto xa_1a_2y$

$a \longmapsto xay$

$$(xa_1y)(xa_2y) = xa_1a_2y$$

Have I really assumed $B \subset A$?

In general I deal with dual pairs, so I have A unital, B an A^{op} module with a map $f : B \longrightarrow A$ of A^{op}-modules such that $A \otimes B \longrightarrow A$, $a \otimes b \longmapsto af(b)$ is surjective.

Then a map $(A, A) \longrightarrow (B, A)$ is of form $(a_1, a_2) \longmapsto (xa_1, a_2y)$ such that

$$a_2a_1 = a_2yf(x)a_1$$

So all I need is an A^{op}-mod map $f : B \longrightarrow A$ an $x \in B$, $y \in A$ ∋ $yf(x) = 1$. Then I get a hom

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & B \arrow[d, "\shortparallel"] \\
A \otimes_A A \arrow[r] & B \otimes_A A
\end{tikzcd}$$

$a_1a_2 \longmapsto xa_1a_2y$

$a \longmapsto xay$

is ring homom. from A to B, but you also have ring hom $B \xrightarrow{f} A$ ~~$b_1a_1 \cdot b_2a_2 = b_1$~~

$b_1 \overset{\otimes}{} a_1 \cdot b_2 \overset{\otimes}{} a_2 = b_1 \overset{\otimes}{} a_1 f(b_2)a_2$ $\quad b_1 \cdot b_2 = b_1 f(b_2)$ yes.

So you have two homs.

84 two homoms.

$$\begin{tikzcd} A \arrow[r, "g"] & B \arrow[r, "f"] & A \arrow[r, "g"] & B \\ a \arrow[r, mapsto] & xay & f(x)\,ay & xf(x)ayy \end{tikzcd}$$

Let's look at the case where $f : B \hookrightarrow A$ so B is a right ideal in A. It should be clear ~~[illegible]~~ using $yx = 1$, $x \in B$, $y \in A$ that $A \in \mathcal{P}(\tilde{B}^{op})$, $B \in \mathcal{P}(B)$. Here $\mathcal{P}(B)$ means $\mathcal{P}(\tilde{B}) \cap \mathcal{M}(B) \subset \text{mod}(\tilde{B})$, and ~~this~~ such a module should be the image of an idempotent matrix over B. Let $L \in \mathcal{P}(\tilde{B}) \cap \mathcal{M}(B)$, so L is fgproj over B and $L = BL$.

$$\text{Hom}_B(L, \tilde{B}) \otimes_B L \xrightarrow{\sim} \text{Hom}_B(L, L)$$

$$\sum_i \lambda_i \otimes v_i \longmapsto 1$$

$$\text{Hom}_B(L, B) \xrightarrow{\sim} \text{Hom}_B(L, \tilde{B})$$

~~$\sum_{ij} \lambda_i \otimes b_{ij} v_j$~~

$$L \xhookrightarrow{(\lambda_i)} B^n \subset \tilde{B}^n \xrightarrow{(v_i)} L$$

Then $\tilde{B}^n \xrightarrow{(v_i)} L \xhookrightarrow{(\lambda_i)} B^n \subset \tilde{B}^n$ is an idemp matrix with image L.

So we have $B \hookrightarrow A$ inc. of rt ideal $\ni AB = A$, spec. have $y \in A$, $x \in B \ni yx = 1$.

$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ ~~Statement~~ is that $A \in \mathcal{P}(B^{op})$, $B \in \mathcal{P}(B)$ are dual $A \xrightarrow{\sim} \text{Hom}_B(B, \tilde{B})$

$$B \xrightarrow{\sim} \text{Hom}_{B^{op}}(A, B)$$

$$A \otimes_B B \longrightarrow$$

$A \in \mathcal{P}(B^{op})$?

$$\begin{tikzcd} A \arrow[r, hook, "x\cdot"] & B \arrow[r, two heads, "y\cdot"] & A \end{tikzcd}$$

$B \in \mathcal{P}(B)$

$$\begin{tikzcd} B \arrow[r, hook, "\cdot y"] & B \arrow[r, two heads, "\cdot x"] & B \\ b \arrow[r, mapsto] & by \arrow[r, mapsto] & byx = b. \end{tikzcd}$$

Thus $B = Bx$

What you want to understand is?

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Hom}_B(B,B) \arrow[dl] \\ A \arrow[r] & \mathrm{Hom}_B(B,B) \end{tikzcd}$$

$a \mapsto (b \mapsto ba)$, u

yxa

$y\,u(x) \longmapsto (b \mapsto by\,u(x)$
$\qquad\qquad \overset{\shortparallel}{u(byx)} = u(b))$

$B \longrightarrow \mathrm{Hom}_{B^{op}}(A,B)$ $b \mapsto (a \mapsto ba)$, v

b
$\downarrow$
byx

B

$v(y)x \longmapsto (a \mapsto v(y)xa$
$\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad v(yxa)$

~~Anyway what~~? Review situation

$$\begin{pmatrix} A & eR \\ Re & B \end{pmatrix}$$

$\mathrm{mod}(R) \longrightarrow \mathrm{mod}(A) \xrightarrow{\sim} m(B) \subset \mathrm{mod}(R)$
$L \longmapsto eL \longmapsto Re \otimes_A eL = BL$

$0 \to Re\otimes_A eL \to L \to L/BL \to 0$

assume $eR \in \mathcal{P}(A)$. then get ~~$Re\otimes_A eL$~~

$\mathcal{P}(R) \longrightarrow \mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(R).$

86 You want to prove that ~~$K_*(R)$~~

$$K_*(R) = K_*(A) \oplus K_*(R/B).$$

I know that

assumes $eR \in P(A)$

$$P(A) \subset P(R) \longrightarrow P(A)$$

$$M \longmapsto Re\otimes_A M \longmapsto A\otimes_A M = M$$

is the identity. Also resolution gives a map

$$K_*(R/B) \longrightarrow K_*(R).$$

Idea: In the case $R = \tilde{B}$ you have besides

$$0 \longrightarrow BL \longrightarrow L \longrightarrow L/BL \longrightarrow 0$$

the sequence

$$0 \longrightarrow B\otimes_{\mathbb{Z}} L/BL \longrightarrow \tilde{B}\otimes_{\mathbb{Z}} L/BL \longrightarrow L/BL \longrightarrow 0$$

so you can apply $\boxed{\text{Shanuel's lemma}}$. ~~Delicate~~

~~[illegible] So what can I do?? Answer!~~

So how to proceed? ~~Take a coherent~~

$R/B \otimes$

You have functors

$$P(R) \longrightarrow P(B) \subset P(R) \qquad R = \tilde{B}$$

$$L \longmapsto BL \longmapsto BL$$

~~[illegible] composition is the identity.~~

What you need to understand is the effect of $L \mapsto BL$ on $K_*(\tilde{B}) = K_*(\mathbb{Z}) \oplus K_*(B)$

$1 \longrightarrow GL(B) \longrightarrow GL(\tilde{B}) \longrightarrow GL(\mathbb{Z}) \longrightarrow 1$. We need to know about? $GL_n(B) =$

$GL_n(B)$ = group of autos of $\tilde{B}^n$ inducing 1 on $\mathbb{Z}^n$

~~$\mathrm{Aut}_B(B^n) = GL_n(\mathrm{Hom}_B(B,B))$~~ $\qquad GL_n(A) = \mathrm{Aut}_B(Be^n)$

$\mathrm{Hom}_R(Re, Re) = A$

87 looks nontrivial $\mathrm{Hom}_B(B,B) = \mathrm{Hom}_A(\underset{\overset{\uparrow}{\text{a generator for } P(A)}}{eR}, eR).$

Then I shifted to

$A' = \mathrm{Hom}_A(eR, eR) \longleftarrow R \supset B$

This yields a meg $\begin{pmatrix} A' & A' \\ B & B \end{pmatrix}$

instead of $\begin{pmatrix} A & eR \\ Re & B \end{pmatrix}$

$B \otimes_{A'} A' = B$

$A' \otimes_B B = A'$

So shift to A unital, B a rt A-module ~~say a right~~ with $f : B \to A$ an A^{op} ~~module~~ map, say $B \hookrightarrow A$ right ideal $\ni$ $AB = A$. ~~Right B ideal~~

---

example $B \subset A$ right ideal $BA = B$, gen $AB = A$

A unital $y \in A$, $x \in B$ $yx = 1$. Then $(xy)^2 = xy$ in A and we can compare

$$\begin{pmatrix} eAe & eA \\ Ae & AeA \end{pmatrix} \qquad \begin{pmatrix} A & A \\ B & B \end{pmatrix}$$

~~Need Toeplitz alg.~~ $a \longmapsto xay$ is a hom.

$(xa_1y)(xa_2y) = x\,a_1a_2\,y$. Is a homom from A to B $e \in \boxed{xy} \in BA = B$. $\therefore eA \subset B$

and

$xAy = eAe$

$xAy = xyxAyxy \subset \underset{\overset{\|}{eAe}}{\overset{xyAxy}{\cancel{eAe}}} \subset xAy$

$Ayxy \subset Ae = Axy \subset Ay$

88 Go back to the K-theory. You have

Wait the fact that $x, y \; \exists \quad y \in A, x \in B$ such that $yx = 1$ means that this M context has the form considered before.

So take A as before $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$?

Notation A as originally $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

$$\left(\begin{matrix} \begin{pmatrix} A' & Q^* \\ Q & A \end{pmatrix} & \begin{matrix} A' \\ Q \end{matrix} \\ \begin{matrix} B^* & P \end{matrix} & B \end{matrix}\right)$$

NO.

$\begin{pmatrix} A' & A' \\ B & B \end{pmatrix}$ $\quad y \in A', x \in B \qquad yx = 1.$

$A = eA'e \qquad e = xy \in B \subset A'.$

So I will have $A \longrightarrow A' \qquad A = eA'e \subset A'.$

$$\begin{matrix} A & eA' & eA' \\ A'e & A' & A' \\ Be & B & B \end{matrix}$$

$Bxyxy\,A' = BxyA' \subset B$

$BeA' = BA'eA'$
$= BA' = B.$

This might not be very important. The critical thing is what to do. Let's start again.

Go back to A unital, B right ideal in A $y \in A$, $x \in B$ such that $yx = 1$. Let's work out the relation between the K-theories. We need to compare $\mathcal{P}(A) \simeq \mathcal{P}(B)$ and $\mathcal{P}(\tilde{B})$. Actually the point seems to be to compare the K-theory of the categories $\mathcal{P}(B)$ and $\mathcal{P}(\tilde{B})$. I think we have

89 an inclusion (fully faithful functor) $\mathcal{P}(B) \subset \mathcal{P}(\tilde{B})$.

Claim B is a f.g. projective $\tilde{B}$ module – this is the end of R business. The fact that?

$BB = BAB$
$= BA = B$

Check: Assume $B < A$ unital

$BA = B$
$AB = A$
$yx = 1$

Claim then that $B \in \mathcal{P}(B)$, $A \in \mathcal{P}(B^{op})$.

$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ Show $B^2 = B$ $yx = 1$ $\underbrace{by}_{\in B}\underbrace{x}_{\cap B} = b$.

~~So what?~~

$$\begin{tikzcd} B \arrow[r, "\cdot y"] & B \subset \tilde{B} \arrow[r, "\cdot x"] & B \\ b \arrow[r, mapsto] & by \arrow[r, mapsto] & byx = b. \end{tikzcd}$$

$\therefore$ $B \in \mathcal{P}(B)$ OKAY.

Next

$$\begin{tikzcd} A \arrow[r, "x\cdot"] & B \subset \tilde{B} \arrow[r, "y\cdot"] & A \\ a \arrow[r, mapsto] & xa \arrow[r, mapsto] & yxa = a. \end{tikzcd}$$

so that $A \in \mathcal{P}(B^{op})$. ~~Notice that~~

So $B \in \mathcal{P}(B)$. ~~image of $\cdot xy$~~

~~$B \neq \tilde{B}xy$~~ ~~$xy \in B$ idempotent?~~

~~and b~~

$$\tilde{B} \xrightarrow{\cdot x} B \xrightarrow{\cdot y} B \subset \tilde{B}$$

Try again

$$\underbrace{B \xrightarrow{\cdot y} \tilde{B} \xrightarrow{\cdot x} B}\ \underbrace{\subset \xrightarrow{\cdot y} \tilde{B} \xrightarrow{\cdot x} B}$$

Thus $xy \in B$ is idempotent and

$Bxy \simeq By \simeq B$

$$\begin{tikzcd} b \arrow[r, mapsto] & bxy \\ B \arrow[r] & Bxy \arrow[d] \\ & {} \end{tikzcd}$$

$$Bxy \subseteq By \subseteq By\cdot xy \subset Bxy$$

$$\begin{tikzcd}
B \arrow[r, "\cdot xy"] & Bxy \arrow[r, "\cdot x"] & Bx \\
b \arrow[r, mapsto] & bxy ?
\end{tikzcd}$$

?

What am I doing?

$B \subset A$

$x \quad y$

$yx = 1. \Rightarrow (xy)^2 = xy$ in B

$ex = x \qquad ye = y.$

$$\begin{tikzcd}
& B & \\
B \arrow[r, "\cdot y"] \arrow[rr, bend right, "1"'] & By \arrow[u, phantom, "\cup"] \arrow[r, "\cdot x"] & B
\end{tikzcd}$$

$$\therefore \quad B \xrightarrow[\sim]{\cdot y} By \xrightarrow[\sim]{\cdot x} B$$

So what's really going on?

~~Bxy~~

$$\begin{tikzcd}
B \arrow[r, "\cdot y", "\sim"'] \arrow[rrr, bend right, "1"'] & By \arrow[r, phantom, "\subset"] & \tilde{B} \arrow[r, two heads, "\cdot x"] & B
\end{tikzcd}$$

$$\tilde{B} \overset{\cdot x}{\twoheadrightarrow} B \xrightarrow[\sim]{\cdot y} By \subset \tilde{B} \quad \Rightarrow$$

$$\boxed{\begin{tikzcd}
\tilde{B}e \arrow[d, equal] & \\
\tilde{B}xy \arrow[r, phantom, "="] \arrow[rd, "\cdot x"'] & By \\
& B \arrow[u, "\cdot y"', "\simeq"]
\end{tikzcd}}$$

~~$A \xrightarrow{x\cdot} xA \subset \tilde{B} \xrightarrow{y\cdot} A$~~

$$\begin{tikzcd}
A \arrow[r, "x\cdot", "\sim"'] \arrow[rrrr, bend right, "1"'] & xA \arrow[r, phantom, "\subset"] & B \arrow[r, phantom, "\subset"] & \tilde{B} \arrow[r, two heads, "y\cdot"] & A
\end{tikzcd}$$

$$\tilde{B} \overset{y\cdot}{\twoheadrightarrow} A \xrightarrow[\sim]{x\cdot} xA \subset B \subset \bar{B}$$

$$\therefore \; xy\tilde{B} = xA$$

$$\boxed{\begin{aligned} \tilde{B}e &= By \\ e\tilde{B} &= xA \end{aligned}}$$

91 Anyway, what's the point?

~~Go back to [illegible]!!~~

You have $P(B) \subset P(\tilde{B})$

$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ $\quad P(A) \xrightarrow{\sim} P(B)$ Karoubi subcat gen. by $B \in P(\tilde{B})$.

$$\operatorname{mod}(A) \qquad m(B)$$

$$M \longmapsto B \otimes_A M$$

$$A \longmapsto B \otimes_A A = B$$

You have to really understand why $B \in P(\tilde{B})$.

$$\begin{tikzcd} B \arrow[r, "\cdot y", "\sim"'] \arrow[rrrr, bend right, "1"'] & By \subset B \subset \tilde{B} \arrow[rrr, "\cdot x"] & & & B \end{tikzcd}$$

$$B \xrightarrow[\sim]{\cdot y} By \xrightarrow[\sim]{\cdot x} B$$

$Be = Bxy,$ $\quad Byxy \subset Bxy \subset By$

$\quad = By \qquad \qquad \overset{\shortparallel}{By}$

$$\begin{tikzcd} & b \arrow[l, mapsto] & B \arrow[dl, "s"] \arrow[d, "s"] \\ (b)xy & & \tilde{B}e = Bxy = By \end{tikzcd}$$

So $B \in P(\tilde{B})$.

$$byxy \longleftarrow\!\shortmid \; by$$

So what is going on? You have res. thm.

$$P(A) \longrightarrow P(B) \subset P(\tilde{B})$$

$$\overset{\psi}{A} \longmapsto B$$

$\operatorname{Hom}_B(B, B) = A^{op}$ so in an obvious way $GL_n(A) =$ automorphisms of $B^n \in P(B)$. But the real question is how ~~to~~ to link this to

92 $GL_n(\tilde{B})$ or $GL_n(B) = \mathrm{Ker}\{GL_n(\tilde{B}) \to GL_n(\mathbb{Z})\}$.

Suslin's excision result ??|??|??|?? tells us that

K_*

What is the real problem. The key point? You have $K_* B$ defined ~~[illegible]~~ $GL_n(B) = \mathrm{Ker}\{GL_n(\tilde{B}) \to GL_n(\mathbb{Z})\}$. You have the group $GL_n(B)$ acting on $\tilde{B}^n \in \mathcal{P}(\tilde{B})$. Functor

$$\begin{pmatrix} A & A \\ B & B \end{pmatrix} \qquad \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \simeq \mathcal{P}(B) \subset \mathcal{P}(\tilde{B}).$$

$$\tilde{B}^n \longmapsto A^n \longmapsto B^n$$

~~[illegible]~~ You need to embed B as a summand of a free $\tilde{B}$-module. $B \xrightarrow{\cdot y} \tilde{B} \xrightarrow{\cdot x} B$. Easy

$$By \oplus \mathrm{Ker}(\tilde{B} \xrightarrow{\cdot x} B) = \tilde{B}$$

~~Special case: $B = $ [illegible] $\times A$~~ ~~YES~~

$$\tilde{B}xy \oplus \tilde{B}(1-xy) = \tilde{B}$$

So what's the issue? You have a group $GL_n(B)$ acting on ~~$\tilde{B}xy \oplus \tilde{B}(1-xy)$~~ the exact sequence

$$0 \longrightarrow B^n \longrightarrow \tilde{B}^n \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

in $\mathcal{P}'(\tilde{B})$. Since the action on $\mathbb{Z}^n$ is trivial you expect the representations on B^n and $\tilde{B}^n$ should be equivalent. ~~Thus things proceed cleanly~~ But you also have

$$\begin{tikzcd} 0 \arrow[r] & B^n \arrow[r, "\cdot y"] & \tilde{B}^n \arrow[r] & (\tilde{B}/By)^n \arrow[l, bend right, "\cdot(1-xy)"'] \arrow[r] & 0 \end{tikzcd}$$

split exact sequence of repns of $GL_n(B)$.

$$\begin{tikzcd} 0 \arrow[r] & B^n \arrow[r, "\cdot y"] & B^n \arrow[r] & (B/By)^n \arrow[l, bend right, "\cdot(1-xy)"'] \arrow[r] & 0 \end{tikzcd}$$

seems to imply some sort of triviality for the representation $(B/By)^n$.

93. A $yx = 1$ set $B = xA = eA$

$\Big(xAy$ Toeplitz alg $k[x,y]/(yx-1)$.

$A \simeq k[x] \otimes k[y]$ $\quad x = z$, $y = z^*$ $\quad z^*z = 1$

$B = xA = xk[x] \otimes k[y]$

Any~~how~~way $\qquad e = xy$

We have a homom. $A \longrightarrow B \subset A$

$a \longmapsto xay$

$$\begin{pmatrix} A & Ay \\ xA & xAy \end{pmatrix} \subset \begin{pmatrix} A & A \\ B & B \end{pmatrix} \subset \begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

Suppose $B = xA$. Then

$$\begin{pmatrix} A & A \\ xA & xA \end{pmatrix} = \begin{pmatrix} A & Ay \oplus A(1-xy) \\ xA & xAy \oplus xA(1-xy) \end{pmatrix}$$

should be OKAY.

~~Not so~~

$$\begin{pmatrix} A & A \\ B & B \end{pmatrix} = \begin{pmatrix} A & Ay \oplus A(1-xy) \\ B & By \oplus B(1-xy) \end{pmatrix}$$

$$\overset{\shortparallel}{B} = xA \oplus (1-xy)B$$

Q

$$\begin{array}{c|cc} A & Ay & A(1-xy) \\ \hline xA & xAy & xA(1-xy) \\ (1-xy)B & (1-xy)By & (1-xy)B(1-xy) \end{array}$$

$k[x]\,\cancel{k[x]}(1-xy)\,\cancel{k[x]}\,k[y]$

94. ~~[illegible]~~ YES!! ~~[illegible]~~

$$\begin{tikzcd} xA \arrow[r, phantom, "\subset"] & B \arrow[r, phantom, "\subset"] \arrow[d, equal] & A \\ & xA \oplus (1-xy)B & \end{tikzcd}$$

What you have to do is to relate

$$\begin{tikzcd} GL(B) \arrow[r] & GL(\tilde{B}) \\ GL(B) \arrow[r] & GL(A) \arrow[u] \end{tikzcd}$$

to

inclusion $B \subset \tilde{B}$

$B \subset A \xrightarrow{} B$, $a \longmapsto xay$

This seems to be the issue, namely you have two homos $B \longrightarrow B$: the identity and $b \mapsto xby$, and you need an argument to show they induce the same map on $BGL(B)^+$.

---

what's important. A unital, $B \subset A$ $BA = B$ rt ideal, $AB = B$

setting $\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ ~~$B \otimes_B \ldots B \otimes_B B$~~

module viewpoint

$$\mathcal{M}(B) \subset \mathrm{mod}(\tilde{B}) \longrightarrow \mathrm{mod}(A) \xrightarrow{\sim} \mathcal{M}(B) \subset \mathrm{mod}(\tilde{B})$$

$$N \longmapsto A \otimes_B N, \quad M \longmapsto B \otimes_A M$$

$$\mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \longmapsto \mathcal{P}(B) \subset \mathcal{P}(\tilde{B})$$

$$L \longmapsto A \otimes_B L \longmapsto B \otimes_A A \otimes_B L = B \otimes_B L$$

my problem is to relate K_* of $\mathcal{P}(B)$ to $K_* B \overset{\text{def}}{=} \mathrm{Ker}\left(K_* \tilde{B} \to K_* \mathbb{Z}\right)$. Now I have

$$\mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A)$$

$$V \longmapsto B \otimes_A V \longmapsto A \otimes_B B \otimes_A V = V.$$

~~what seems important is to have ~~ yielding

$$\begin{tikzcd} K_*(A) \arrow[r] \arrow[rr, bend right, "1"'] & K_*(\tilde{B}) \arrow[r] & K_*(A) \end{tikzcd}$$

95. ~~non-unital rings.~~ $\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ $B \subset A$ $1 \in A$, $BA = B$, $AB = A$

$y \in A$, $x \in B$ $yx = 1$.

$$\begin{tikzcd} A \arrow[r, "\phi"] & B \arrow[r, hook] & A \\ a & xay & \end{tikzcd}$$

Mult. ring of B

Question: Is $\phi_* : K_*(A) \circlearrowleft$ the identity? ~~Mult. ring~~

$$\begin{pmatrix} A & Ay \\ xA & xAy \end{pmatrix}$$

OKAY because

$$\begin{tikzcd} \begin{pmatrix} A & Ay \\ xA & xAy \end{pmatrix} \arrow[r, hook] & \begin{pmatrix} A & A \\ A & A \end{pmatrix} \arrow[r, "?"] & \begin{pmatrix} A & Ay \\ xA & xAy \end{pmatrix} \end{tikzcd}$$

$$\begin{tikzcd} (A, A) \arrow[r, "\sim"] \arrow[rr, bend left, "(x\cdot\,,\,\cdot y)"] & (xA, Ay) \arrow[r] & (A, A) \arrow[r, "(x\cdot\,,\,\cdot y)"] & (A, A) \\ & & (a_1, a_2) \arrow[r, mapsto] & (xa_1, a_2y) \arrow[d, mapsto] \\ & & & a_2y\,xa_1 = a_2a_1 \end{tikzcd}$$

So here's an ~~interesting~~ situation, namely a nonunital homom. $A \xrightarrow{\phi} A$ which should be isom to the identity on mod(A).

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & \cdot y \\ x\cdot & \phi \end{pmatrix}} \begin{pmatrix} A & A \\ A & A \end{pmatrix}.$$

So what? ~~mod(A)~~

$$\begin{tikzcd} & \mathrm{mod}(A) \arrow[dl, "1"'] \arrow[dr, "1"] & \\ \mathrm{mod}(A) \arrow[rr, "\phi_!"] & & \mathrm{mod}(A) \end{tikzcd}$$

$A \underset{\phi\,A}{\otimes} M$ M

$$\psi : A \times M \longrightarrow M$$
$$(a, m) \mapsto axm$$

$A \underset{\phi\,A}{\otimes} M$

$\|$

$\psi(axa_1y, m)$

$= axa_1yxm = axa_1m$

$= \psi(a, [illegible]a_1, m)$

96. So $\phi : A_\phi \otimes_A M \longrightarrow M$

$a \otimes m \longmapsto axm$

m

$y \otimes axm$
$\shortparallel$
$y\,x(ax)y \otimes m$
$\shortparallel$
$a \otimes m$

[redacted] $y \otimes m \longmapsto$ [redacted] yxm

[redacted]

$eA = xyA = xA$

$eAe = xyAxy = xAy$

$Ae = Axy = Ay$

You get $GL(A) \longrightarrow GL(A)$

We have a nonunital hom. $A \longrightarrow A$, $a \longmapsto xay$

~~∃ similar arg. case of~~

[redacted] $yx = 1$.

and this induces $GL_n(A) \longrightarrow GL_n(A)$ in same way.
If $(1+a)(1+a') = 1$ i.e. $a + a' + aa' = 0$
then $\phi(a) + \phi(a') + \phi(a)\phi(a') = 0$ so we send
$1+a$ to $1+\phi(a)$. In terms of autos.

$A \xrightarrow{1+a} A$ [redacted] OKAY what next?

~~So where do I go from here? You have ⟨A⟩ ⊂ A~~

~~nonunital~~

Mult. ring of B: $\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ $P = B$ $Q = A$

$M \subset \mathrm{Hom}_{A^{op}}(B,B) \times \mathrm{Hom}_A(A,A)^{op}$
$(f \quad , \quad a)$

$a'f(b) = (a'a)b$
$\forall a', b$
Put $a' = 1$ get
$f(b) = ab$.

OKAY

So the multiplier ring of B is apparently the subring of $a \in A$ such that $aB \subset B$. Life is hard.

Let's start with $B \subset A$ unital $BA = B$ [redacted] $y \in A$ $x \in B$ $yx = 1$

$A \longrightarrow B \subset A$ nonunital homom. image of 1
$a \longmapsto xay$ is $xy = e$.

Interpretation. You have

$$\mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \simeq \mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A)$$

$$L \qquad A \otimes_B L \longmapsto B \otimes_B L \longleftarrow\!\!\longrightarrow A \otimes_A B \otimes_B L$$

97 $\mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A)$

$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ $V \longmapsto B\otimes_A V \longmapsto A\otimes_B B\otimes_A V = V$

$A = \mathrm{Hom}_B(B,B)^{op}$, $B \in \mathcal{P}(B)$,

you choose $B \underset{\cdot x}{\overset{\cdot y}{\rightleftarrows}} \tilde{B}$

Go over again why $\phi : A \longrightarrow A$ $a \mapsto xay$ has the same effect as 1 on $GL(A)$, roughly

Idea: $A \longrightarrow eAe \subset A$. $e = \phi(1)$. I am not clear about this.

$eA = xA$ look at this as [redacted]

~~mod(A)~~ Very def. homomorphism.

~~Take A, take B.~~

$\mathcal{P}(A') \xrightarrow{eAe\otimes_{A'}} \mathcal{P}(eAe)$ Ae

$L \longmapsto Ae \otimes_{eAe} L$

The idea is that given $A' \xrightarrow{\phi} A$ nonunital hom. of unital rgs then you have

$A' \longrightarrow eAe$

$\mathcal{P}(A') \longrightarrow \mathcal{P}(eAe) \subset \mathcal{P}(A)$

$\uparrow$ Kar. full subcat generated by Ae

So now return to $A' = A$ $\phi(a) = xay$. Then you take $V \in \mathcal{P}(A)$ to $Ae \otimes_{\phi A} V \in \mathcal{P}(A)$. Maybe the point is that you look at $Ae = Ay$ as a left A right ϕ. You get [redacted]

So you look at $\cdot xay$ on Ay

0 on $A(1-xy)$

So get $\underset{1+a}{GL_n(A)} \longrightarrow \underset{1+xay}{GL_n(A)}$. This should be correct $(1-xy)+x(1+a)y$

but it should somehow be equivalent to the identity.

$yx = 1.$ $\quad e = xy \quad Ae = Axy = Ay$

$\phi : A \longrightarrow A$

$a \longmapsto xay$

$\mathrm{mod}(A) \longrightarrow \mathrm{mod}(A)$

$A \longmapsto Ay$

$M \longmapsto Ay \otimes_A M$

$a'y \otimes am$
$= a'\cancel{yx}ay \otimes m$

$$\begin{tikzcd} Ay \otimes_A M \arrow[r] & M \\ a'y \otimes m \arrow[r, mapsto] & a'm \\ y \otimes m & m \arrow[l, mapsto] \\ y\otimes m \arrow[r, mapsto] & m \end{tikzcd}$$

$y \otimes a'm \longleftarrow\!\shortmid a'm$

$yxa'y \otimes m$.

So this ~~seems to~~ says that $\phi_!(M) \xrightarrow{\sim} M$

What about $g \in GL_n(A) = \mathrm{Aut}_A(A^n)$. So I start with G acting on A^n. Then I get G acting on $Ay \otimes_A A^n = (Ay)^n$. Normally the way you ~~think~~ get a matrix rep. is by ~~direct~~ split embedding

$$Ay \underset{\cdot y}{\overset{\cdot x}{\rightleftarrows}} A$$

$g = 1 + \alpha$ on A^n

$1 \otimes g = 1 \otimes (1+\alpha)$ on $Ay \otimes_A A^n$

$\phi(1+\alpha)$ on Ay^n
$= x(1+\alpha)y$

and then you add $1-xy$ on $A(1-xy)$.

$1 - xy + x(1+\alpha)y = 1 + x\alpha y$. But what's missing?? What's taking place: The effect of ϕ is

$$\mathrm{Aut}(A^n) \longrightarrow \mathrm{Aut}(Ay^n) \subset \mathrm{Aut}(A^n)$$

But $\mathrm{Aut}(A^n) \longrightarrow \mathrm{Aut}(Ay^n) \subset \mathrm{Aut}(A^n)$.

Something funny happens. No. all you are saying is that $A \simeq Ay$, $Ay \oplus A(1-xy) = A$

~~So you have the situation where things are better~~

You are in a situation where $A \oplus Z \simeq A$

$$A \oplus A(1-xy) \longrightarrow A$$

$$(a_1, \quad a_2(1-xy)) \longmapsto a_1 y + a_2(1-xy)$$

$$(ax, \quad a(1-xy)) \longleftarrow a$$

So what is the ultimate reason?

~~I feel like~~ From the viewpt of ~~GL~~ $G = GL_n(A)$

You have

~~$BGL_n(A) \to BGL_n(A)$?~~

~~You have $Aut(A^n) \longrightarrow$~~

Look at repns — you have G acting on A^n the extension of scalars functor via ϕ

$$\begin{tikzcd} G \arrow[r, "\xi"] \arrow[dr, "\phi_* \xi"'] & Aut(A^n) \arrow[d, "\phi"] \\ & Aut(A^n) \end{tikzcd}$$

$B = B^2$

$b = \boxed{by}\boxed{x}$

claim is $\phi_* \xi \simeq \xi \oplus$ trivial

Review: $B \subset A \ni 1 \qquad BA = A \qquad AB = A \qquad y \in A,\ x \in B,\ yx = 1$

$$P(\tilde{B}) \longrightarrow P(A) \xrightarrow{\sim} P(B) \subset P(\tilde{B}) \longrightarrow P(A)$$

$A \quad A$

$$L \longmapsto A \otimes_B L$$

$$V \longmapsto B \otimes_A V \xmapsto{\quad A \quad} A \otimes_B B \otimes_A V = V.$$

B ~~$A \otimes_B B$~~

$B \otimes_A A = B$

So where are we? ~~Nowhere~~. The module categories are well-understood. I understand ~~behavior~~ the Morita equivalence aspects, but now need to discuss K-theory. BGL^+. ~~First we have~~

Because A unital $\quad B \in P(B) \quad A \in P(B^{op})$ and these are dual $\quad A \xrightarrow{\sim} Hom_B(B,B)^{op} \qquad B \xrightarrow{\sim} Hom_{B^{op}}(A,B)$ need to see this

$$\begin{tikzcd} B \arrow[r, "\cdot y"] \arrow[rr, bend right, "1"'] & \tilde{B} \arrow[r, "\cdot x"] & B \end{tikzcd}$$

shows B summand of $\tilde{B}$ as left B-mods

$$\begin{tikzcd} A \arrow[r, "x\cdot"] & \tilde{B} \arrow[r, "y\cdot"] & A \arrow[r, "x\cdot"] & \tilde{B} \end{tikzcd}$$

show A summand of $\tilde{B}$ as B^{op}-mods.

$$\begin{tikzcd} \mathrm{Hom}_{B^{op}}(A,B) & \mathrm{Hom}_{B^{op}}(\tilde{B},B) \arrow[l, two heads] \arrow[r, leftrightarrow] \arrow[d, phantom, "\shortparallel"] & \mathrm{Hom}_{B^{op}}(A,B) & \mathrm{Hom}_{B^{op}}(\tilde{B},B) \arrow[l, two heads] \arrow[d, phantom, "\shortparallel"] \\ & B & & B \arrow[dl, two heads, "\cdot x"] \\ & & B \arrow[ul, "\cdot y"] \arrow[uu] & \end{tikzcd}$$

Problem. You understand the Morita equiv. aspects. You now need to handle the K-theory. How do you handle $K_1 B$ for example, or $K_0 B$? Try $K_0 B$?

Is it clear that $K_0 A = K_0 B$?

$$K_0(\tilde{B}) \longrightarrow K_0(A) \longrightarrow K_0(\tilde{B})$$

$$[V] \longmapsto [B \otimes_A V]$$

$$[L] \longmapsto [A \otimes_B L]$$

$$[L] \longmapsto [B \otimes_B L]$$

exact sequences

$$0 \longrightarrow B \otimes_B L \longrightarrow L \longrightarrow L/BL \longrightarrow 0$$

$$0 \longrightarrow B \otimes_{\mathbb{Z}} L/BL \longrightarrow \tilde{B} \otimes_{\mathbb{Z}} L/BL \longrightarrow L/BL \longrightarrow 0$$

By Shanuel get isom.

$$L \oplus (\tilde{B} \otimes_{\mathbb{Z}} L/BL) \simeq B \otimes_B L \oplus (\tilde{B} \otimes_{\mathbb{Z}} L/BL)$$

and you win.

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & B\otimes\bar{L} \arrow[r, phantom, "="] \arrow[d] & B\otimes\bar{L} \arrow[d] & \\ 0 \arrow[r] & BL \arrow[r] \arrow[d, phantom, "\shortparallel"] & F \arrow[r] \arrow[d] & \tilde{B}\otimes\bar{L} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & BL \arrow[r] & L \arrow[r] \arrow[d] & \bar{L} \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

You actually get equiv of complexes

This ought to work, namely, if $B \in \mathcal{P}(\tilde{B})$ then you have representations of $GL_n(\tilde{B})$, putting $L = \tilde{B}^n$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & B\otimes_{\mathbb{Z}} \bar{L} \arrow[d] & & \\
0 \arrow[r] & B\otimes_B L \arrow[r] & F \arrow[r] \arrow[d] & \tilde{B}\otimes_{\mathbb{Z}} \bar{L} \arrow[r] & 0 \\
 & & L \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

These are exact sequences in $\mathcal{P}(\tilde{B})$. So you win, maybe. So what's the next point?

The preceding has to be translated into something more transparent. Yes. What I would like ultimately is some argument with ring homos. The idea is that working with rings ring A should be better that ____________ the matrix rings $M_n A$.

Argument: Start again

$B < A^{\ni 1}$, $BA = B$, $y \in A$, $x \in B$, $yx = 1$.

$$\mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) < \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A)$$

$$L \longmapsto A\otimes_B L \longmapsto B\otimes_A A\otimes_B L = B\otimes_B L$$

$$V \longmapsto B\otimes_A V \longmapsto \overbrace{A\otimes_B B}^{A}\otimes_A V = V.$$

A A

B B

A unital $\Longrightarrow$ $B \in \mathcal{P}(B)$, $\underline{A \in \mathcal{P}(B^{op})}$ are dual

$\Rightarrow A$ is B^{op}-finin

What happens for K_0?

$$\begin{tikzcd}
K_0(\tilde{B}) \arrow[r] & K_0(A) \arrow[r, "\sim"] \arrow[rrr, bend left, "1"] & K_0(\mathcal{P}(B)) \arrow[r] & K_0(\tilde{B}) \arrow[r] & K_0(A)
\end{tikzcd}$$

Suppose $L \in \mathcal{P}(\tilde{B})$. Then $L \longmapsto B \otimes_B L \in \mathcal{P}(\tilde{B})$

But you have

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & B \otimes_{\mathbb{Z}} \bar{L} \arrow[d] & & \\ 0 \arrow[r] & B \otimes_B L \arrow[r] & F \arrow[r] \arrow[d] & \tilde{B} \otimes_{\mathbb{Z}} \bar{L} \arrow[r] & 0 \\ & & L \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

$$\therefore \quad [L] - [B \otimes_B L] = \left([\tilde{B}] - [B]\right) \cdot \mathrm{rank}_{\mathbb{Z}}(\bar{L})$$

$$\begin{tikzcd} K_0(B) \arrow[r, hook] & K_0(\tilde{B}) \arrow[rr, "1 - {[B]}\cdot"] \arrow[dr, two heads, "r"'] & & K_0'(\tilde{B}) \\ & & K_0(\mathbb{Z}) \arrow[ur, "\cdot({[\tilde{B}]} - {[B]})"'] & \end{tikzcd}$$

$$1 = [B]\cdot + r([\tilde{B}] - B)$$

$$K_0(\tilde{B}) = \mathbb{Z}(\underbrace{[\tilde{B}] - B}_{\text{has } r=1}) \oplus K_0(B)$$

~~Next need something~~ ~~Next try K_1~~

This seems like a very general argument.

---

try K_1. Take

$$\mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(B) \subset \mathcal{P}(\tilde{B})$$

$L \qquad B \otimes_B L$

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & B \arrow[r, phantom, "="] \arrow[d] & B \arrow[d] & \\ 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \tilde{B} \times_{\mathbb{Z}} \tilde{B} \arrow[r] \arrow[d] & \tilde{B} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & \tilde{B} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

103. Simplest

~~$0 \to B\otimes_B L \to B\otimes_B L \to L \to 0$~~

$B \xrightarrow{0} B$

$\tilde B \to \mathbb{Z} \hookrightarrow \tilde B$

$L \quad \bar L \quad \tilde B \otimes_{\mathbb Z} \bar L$

$$\begin{tikzcd}
 & & & B\otimes_{\mathbb Z}\bar L \arrow[d] & & \\
 & & {} \arrow[r] \arrow[d] & \tilde B\otimes_{\mathbb Z}\bar L \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & BL \arrow[r] & L \arrow[r] \arrow[d] & \bar L \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & B\otimes_{\mathbb Z}\tilde B/B \arrow[dl] & \\
 & & F \arrow[r] \arrow[d] & \tilde B\otimes_{\mathbb Z}(\tilde B/B) \arrow[d] & & \\
0 \arrow[r] & B \arrow[r] & \tilde B \arrow[r] & \tilde B/B \arrow[r] & 0 &
\end{tikzcd}$$

F is a B-bimodule, $F = \tilde B \times_{\mathbb Z} \tilde B$ left mult diagonal right mult. obvious on first component zero on second.

Then have

$B_0 \otimes_B L = B \otimes_{\mathbb Z} \bar L$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & B_0 \arrow[d, "\binom{0}{1}"] & & \\
0 \arrow[r] & B \arrow[r, "\binom{1}{0}"] & F \arrow[r, "pr_2"] \arrow[d, "pr_1"] & \tilde B_0 \arrow[r] & 0 \\
 & & \tilde B \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Look on this primarily as a diagram in $\mathcal{P}(\tilde B)$, but it's actually a diagram~~s~~ of representations of $\tilde B$

$\tilde{B} \oplus B \quad \in \mathcal{P}(\tilde{B})$

$$\mathrm{Hom}_B(\tilde{B} \oplus B, \tilde{B} \oplus B) \simeq \begin{pmatrix} A & A \\ B & \tilde{B} \end{pmatrix}$$

So what does all this mean? You have a module $F \in \mathcal{P}(\tilde{B})$ and two exact sequences in $\mathcal{P}(\tilde{B})$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & B \arrow[d] & & \\
0 \arrow[r] & B \arrow[r] & F \arrow[r] \arrow[d] & \tilde{B} \arrow[r] & 0 \\
 & & \tilde{B} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

and we have a homom $\tilde{B} \longrightarrow \mathrm{Hom}_B(F,F)^{op}$. How to get at this?

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & B \otimes_{\mathbb{Z}} \bar{L} \arrow[r, equal] \arrow[d] & B \otimes_{\mathbb{Z}} \bar{L} \arrow[d] & \\
0 \arrow[r] & BL \arrow[r] \arrow[d, equal] & F \otimes_B L \arrow[r] \arrow[d] & \tilde{B} \otimes_{\mathbb{Z}} \bar{L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BL \arrow[r] & L \arrow[r] \arrow[d] & \bar{L} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

~~YES OK do it.~~

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \tilde{B} \\
\tilde{B} &
\end{tikzcd}$$

So in the cat of repns of $\tilde{B}$ in $\mathcal{P}(\tilde{B})$ we have exact seq.

$$\boxed{0 \longrightarrow B \longrightarrow F \longrightarrow \tilde{B}_\varepsilon \longrightarrow 0}$$

$$0 \longrightarrow B_\varepsilon \longrightarrow F \longrightarrow \tilde{B} \longrightarrow 0$$

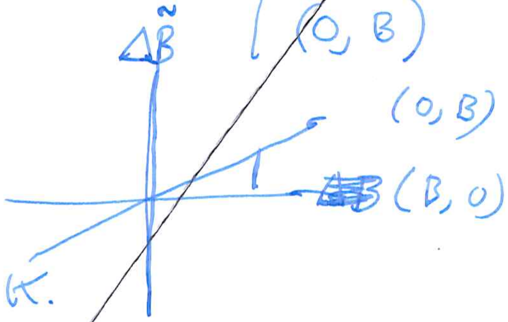
neither split. ~~However the~~ as bimodules sequences. They

105 do split in $\mathcal{P}(\tilde{B})$,

$F \subset \tilde{B} \times \tilde{B}$ so $\Delta \tilde{B}$

$$\begin{tikzcd}
 & B \arrow[r, equal] \arrow[d] & B \arrow[d] \\
B \arrow[r] \arrow[d, no head] & F \arrow[r] \arrow[d] & \tilde{B} \arrow[l, bend right, "\Delta"'] \arrow[d] \arrow[dl] \\
B \arrow[r] & \tilde{B} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

$$F = \Delta\tilde{B} \oplus \begin{cases} (B, 0) \\ (0, B) \end{cases}$$



graph of a map $B \to \tilde{B}$ OK.
There are many splittings

---

to Analyze this proof. Put $F = \tilde{B} \times_{\mathbb{Z}} \tilde{B}$

left B-module structure is $b(\tilde{b}_1, \tilde{b}_2) = (b\tilde{b}_1, b\tilde{b}_2)$

right ———————— $(\tilde{b}_1, \tilde{b}_2)b = (\tilde{b}_1 b, 0)$

Then we have bimodule exact sequences

$$0 \longrightarrow B \xrightarrow{\overset{\binom{1}{0}}{in_1}} F \xrightarrow{\overset{(0\ 1)}{pr_2}} \tilde{B}_\varepsilon \longrightarrow 0$$

$$0 \longrightarrow B_\varepsilon \xrightarrow{in_2} F \xrightarrow{pr_1} \tilde{B} \longrightarrow 0$$

where $\tilde{B}_\varepsilon$ means $b(\tilde{b}) = b\tilde{b}$, $\tilde{b}b = 0$.

I want to calculate $\mathrm{Hom}_B(F, F)$. Choose one exact sequence and split it as B-modules. Means you have to pick an elt of F such that either pr_1 or pr_2 is 1, so it has to be $1 + b_1$, $1 + b_2$. Simplest seems to be to use $(1, 1)$, then you lift via the diagonal map.

106 ~~splitting~~ $F = \Delta\tilde{B} \oplus \begin{matrix}(B,0)\\ \text{or}\\ (0,B)\end{matrix}$.

so then you need the right action of B.

$(x,x)b = (xb,0)$ ~~[illegible]~~

$(x,0)b = (xb,0)$

$$((x,0)+(y,y))b = (xb,0)+(yb,0)$$
$$= ((x+y)b,0)$$

$$B\oplus\tilde{B} \cdot \begin{pmatrix} \cancel{A} & \cancel{A} \\ B & \tilde{B} \end{pmatrix} \longrightarrow B\oplus\tilde{B}$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} (x+y)b & 0 \end{pmatrix}$$

---

other splitting $F = \cancel{A}\ (0,B)\oplus\Delta\tilde{B}$

$$((0,x)+(y,y))b = (y,x+y)b = (yb\ \ 0)$$
$$= (yb\ \ yb) - (0\ \ yb)$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix} = \begin{pmatrix} -yb & yb \end{pmatrix}$$

$$(x+y,y) = (x+y,x+y)-(0,x)$$
$$\xmapsto{b} ((x+y)b,0)$$

---

so what does this calc. mean?

Try for the meaning of life? F is a B-bimodule such that ~~[illegible]~~ as B-module its in $P(\tilde{B})$, so it defines a map $K_*(\tilde{B}) \to K_*(\tilde{B})$

But we have the two twined exact sequences

$$0 \longrightarrow B_\varepsilon \longrightarrow F \longrightarrow \tilde{B} \longrightarrow 0$$
$$0 \longrightarrow B^\varepsilon \longrightarrow F \longrightarrow \tilde{B}_\varepsilon \longrightarrow 0$$

$\Longrightarrow$

$$\begin{tikzcd}
 & (x-y,0)+(y\ y) & (x-y\quad y) & (u\quad v) \arrow[d, mapsto] \\
 & (B,0)\oplus\Delta\tilde{B} \arrow[r, phantom, "\simeq"] & B\oplus\tilde{B} \arrow[d, "S"] & (-u\quad u+v) \\
(x,y)\in F \arrow[uur, mapsto] \arrow[ur, "\sim"] \arrow[dr, "\sim"'] \arrow[ddr, mapsto] & & & \\
 & (0,B)\oplus\Delta\tilde{B} \arrow[r, phantom, "\simeq"] & B\oplus\tilde{B} & \\
 & (0\ y-x)+(\cancel{x}\ \cancel{x}) \arrow[r, phantom, "\simeq"] & (y-x\quad x) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (xb\quad 0) \\
(x,y)b=(xb\quad 0) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & (-xb\quad xb)
\end{tikzcd}$$

~~$(u\quad v)\longmapsto(u+v,\ v)\xmapsto{\cdot b}((u+v)b,\ 0)=(u\quad v)\begin{pmatrix} b & 0\\ b & 0\end{pmatrix}$~~

~~$(u'\quad v')\longmapsto(v'\quad u'+v')\xmapsto{\cdot b}(v'b\quad 0)=(u'\quad v')\begin{pmatrix} \cancel{0} & 0\\ b & 0\end{pmatrix}$~~

$$(u\quad v)\longmapsto(u+v,\ v)\xmapsto{\cdot b}((u+v)b\quad 0)\longmapsto((u+v)b\quad 0)$$

$$=(u\quad v)\begin{pmatrix} b & 0\\ b & 0\end{pmatrix}$$

$$(u'\quad v')\longmapsto(v',\ u'+v')\xmapsto{\cdot b}(\cancel{b}v'b,\ 0)\mapsto(-bv'\quad bv')=(u'\quad v')\begin{pmatrix} 0 & 0\\ -b & b\end{pmatrix}$$

~~$(u'\quad v')\mapsto(-u'\quad u'+v')=(u'\quad v')\begin{pmatrix} -1 & b\\ 0 & 1\end{pmatrix}$ $(-u\quad u+v)=(u\quad v)\begin{pmatrix} -1 & 1\\ & 1\end{pmatrix}$~~

$$(u'\quad v')=(-u\quad u+v)=(u\quad v)\begin{pmatrix} -1 & 1\\ 0 & 1\end{pmatrix}$$

inv. matrix is $\begin{pmatrix} -1 & 1\\ 0 & 1\end{pmatrix}$

$$\begin{pmatrix} -1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix} b & 0\\ b & 0\end{pmatrix}\begin{pmatrix} -1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix} -1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix} -b & b\\ -b & b\end{pmatrix}=\begin{pmatrix} 0 & 0\\ -b & b\end{pmatrix}$$

Is there a significance to this? So this means in K-theory exact

$$0\to B\otimes_{\mathbb{Z}}\bar{L}\to F\otimes_{B}L\to L\to 0$$

$$0\to B\otimes_{B}L\to F\otimes_{B}L\to\tilde{B}\otimes_{\mathbb{Z}}\bar{L}\to 0$$

Yes.

108. homos. You have two homos.

$$B \longrightarrow \mathrm{Hom}_B(B\oplus\tilde{B}, B\oplus\tilde{B}) = \begin{pmatrix} A & A \\ B & \tilde{B} \end{pmatrix}$$

$$b \longmapsto \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}$$ which are

conjugate. Somehow what's important is that

~~[illegible]~~ Situation B an idempotent ring such that $B \in \mathcal{P}(B)$, $A = \mathrm{Hom}_B(B,B)$. Can $B \to A$ be non-injective? Take A unital, $Q = A$, $P \xrightarrow{f} A$ ~~right~~ A^{op} map such that $A \otimes P \twoheadrightarrow A$. Then

$$\begin{pmatrix} A & A \\ P & P\otimes_A A = B \end{pmatrix}$$

A unital $\Longrightarrow$ $B \in \mathcal{P}(B)$ $A \in \mathcal{P}(B^{op})$ and they are dual

this really seems OKAY.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

A unital $Q \in \mathcal{P}(A)$ by surj of $Q \otimes P \to A$

Q must be a generator for $\mathcal{P}(A)$.

$Q \in \mathcal{P}(A) \Longrightarrow B = P \otimes_A Q \in \mathcal{P}(B)$. B is a generator of $\mathcal{P}(B)$.

$$\mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \simeq \mathcal{P}(B) \subset \mathcal{P}(\tilde{B})$$

$$L \qquad Q \otimes_B L \qquad P \otimes_A Q \otimes_B L = B \otimes_B L$$

$$V \longmapsto P \otimes_A V$$

~~Also~~ So it seems I get the ~~[illegible]~~ isom. when Q is flat over A unital

What else is known. $\begin{pmatrix} A & A \\ B & \tilde{B} \end{pmatrix} = \begin{pmatrix} A & A \\ B & B \end{pmatrix}^{\sim}$ should be Morita equivalent to $\tilde{B}$ as unital rings. The idea being use the generator $B \oplus \tilde{B}$ for $\mathcal{P}(\tilde{B})$

109 Try to translate to homomorphisms.

B idempotent $B \in \mathcal{P}(B)$ $A = \mathrm{Hom}_B(B,B)$

~~Write up summary~~.

$$\begin{tikzcd}
\mathcal{P}(\tilde{B}) \arrow[r] & \mathcal{P}(A) \arrow[r, shift left] & \mathcal{P}(B) \arrow[l, shift left] \\
L & A\otimes_B L & B\otimes_B L
\end{tikzcd}$$

so A is not really very important.

$B = B^2$, $B \in \mathcal{P}(B)$ then you have

$$\begin{tikzcd}
 & \mathcal{P}(\tilde{B}) \arrow[r] & \mathcal{P}(B) \arrow[r, phantom, "\subset"] & \mathcal{P}(\tilde{B}) \arrow[dd, "\otimes"] & \\
 & L \arrow[r, mapsto] & B\otimes_B L & & \\
 & & B\otimes_{\mathbb{Z}} L \arrow[u] \arrow[r, equal] \arrow[d] & B\otimes_{\mathbb{Z}} L \arrow[d] & \\
0 \arrow[r] & BL \arrow[r] \arrow[d, equal] & F(L) \arrow[r] \arrow[d] & \tilde{B}\otimes_{\mathbb{Z}} \bar{L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BL \arrow[r] & L \arrow[r] \arrow[d] & \bar{L} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

$b(x,y) = (bx, by)$

$(x,y)b = (bx, 0)$

$F(L) = F\otimes_B L$ $F = \tilde{B} \times_{\mathbb{Z}} \tilde{B}$

$F = (B,0) \oplus \Delta\tilde{B} = (0,B) \oplus \Delta\tilde{B}$

$(x,y) = (x-y,0)+(y,y) \longleftrightarrow (x-y \;\; y) = (u \;\; v)$

~~$(x \;\; y) = (u+v \;\; v)$~~

$\downarrow \cdot b$ $\downarrow \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}$

$(xb,0) \longleftrightarrow (xb \;\; 0) = (ub+vb \;\; 0)$

$(x,y) = (0,y-x)+(x,x) \Leftrightarrow (y-x \;\; x)$

$(v', u'+v') = (0,u')+(v',v') \Leftrightarrow (u' \;\; v')$

$\downarrow \cdot b$ $\downarrow \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}$

$(v'b, 0) \overset{?}{\longrightarrow} (-v'b \;\; v'b)$

$$(u' \; v') \longmapsto (v', u'+v') \longmapsto (-u' \; u'+v') = (u' \; v')\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} -b & b \\ -b & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} -b & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & -b \\ 0 & 0 \end{pmatrix}$$

What's left?? $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ assume B left flat ($\Leftrightarrow$ $Q \otimes_B B = Q$ is A-flat)

Then can assume $Q \in \mathcal{P}(A) \Rightarrow P \otimes_A Q = B \in \mathcal{P}(B)$. and the [redacted] rest is clear with luck. [redacted]
~~The rest is clear with~~ Now must get to work.

$$\begin{tikzcd} \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[d] \\ \begin{pmatrix} A & AF_\alpha \\ F_\alpha^* A & F_\alpha^* \otimes_A A \otimes_A F_\alpha \end{pmatrix} \end{tikzcd}$$

$$Q = \varinjlim AF_\alpha = \lim F_\alpha \qquad F_\alpha \simeq \tilde{A}^{n_\alpha}$$

$$\begin{tikzcd} P \arrow[r] & \mathrm{Hom}_A(Q, \bar{A}) \arrow[r] & \mathrm{Hom}_A(F_\alpha, A) \arrow[d, equal] \\ & & F_\alpha^* A \end{tikzcd}$$

$$Q \otimes P \longrightarrow A$$

Let I be the image. Then

$I = AI = IA$ What's bothering you is the ~~restriction~~ to bimodules of the form $Q \otimes P$. [redacted] Possible gen. might go as follows. Take a set of generators Q_i and then P_i and $Q_i \otimes P_i \to A$ such that $\sum_i Q_i \otimes P_i \to A$ is surjective. OKAY. Not much so far!!

111. Let's consider the generalized right ideal situation: idem ring A, firm A^{op}-module P together with A^{op}-map $P \xrightarrow{f} A$ s.t. $A \otimes P \xrightarrow{1 \otimes f} A \otimes A \xrightarrow{\mu} A$ is onto. ~~Then have homos. $B \longrightarrow A$~~ Put $B = P$ with $b_1 b_2 = b_1 f(b_2)$. Then $f : B \to A$ is a homom. Also have $A \longrightarrow \mathrm{Hom}_B(B, B)$

$$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$$

$$(bb_1)a = b f(b_1) a = b f(b_1 a) = b(b_1 a)$$

~~$f(b) = 0$ $ba_1 f(b_1)$~~

results: $m(A)$ $m(B)$

$$M \longmapsto B \otimes_A M$$

$$A \otimes_B N \longleftarrow\!\shortmid N$$

$$a(ba_1) = a f(b) a_1 = a f(ba_1)$$

$\therefore ab = a f(b).$

$m(A^{op})$ $m(B^{op})$

$$M' \longmapsto M' \otimes_A A = M'$$

$$N' = N' \otimes_B B \longleftarrow\!\shortmid N'$$

So what happens? How do we analyze things? any ideas. $K_0' A$ is Morita invariant - uses perfect complexes of $\tilde{A}$ modules acyclic mod A.

In above situation, we have $B \longrightarrow A$ a homom and $A \longrightarrow \mathrm{Hom}_B(B, B)^{op}$ same sort of representation. Yes!!!

Possibly use complexes.

$m' = m(A^{op})$ suppose I pick $Q \in m$ flat, gen. fact that it's gen. say $\exists\ Q \otimes P \twoheadrightarrow A$

Here's how you might define $K_*(M)$ for $M \sim \mathrm{mod}(A)$ A unital. namely you use $\mathcal{P}(A)$.

Take a Roos cat M ∋ ∃ small proj gen. Introduce $\mathcal{P}$ = full subcat of small projectives. Then $K_*(\mathcal{P})$ defined, this gives an intrinsic definition. Above arguments tell us that if we choose P, Q firm dual pair - transverse $P \otimes^L_A Q = P \otimes_A Q$, then $K_*(B) = K_*(\mathcal{P})$. Curious thing is that to $\mathcal{P}$ $\subset M$ belongs $\mathcal{P}^\vee \subset M^\vee$.

Intrinsic definition of cyclic stuff.

Given B put $A = \mathrm{Hom}_B(B,B)$ is always unital

$\mathrm{Hom}_B(B,B)$ $\qquad$ $\mathrm{Hom}_B(B,B)$

B $\qquad$ B

$$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$$

Given B put $A = \mathrm{Hom}_B(B,B)^{op}$. have right action of A on B. ${}_B B_A$. Do we have $A \otimes_B B \longrightarrow A$

$f : B \longrightarrow A$ $\qquad$ $f(b) = (b' \mapsto b'b)$

$b'(ba) = (b'b)a$

$f(ba) = (b' \mapsto (b'b)a) = f(b)a$

pairing $\qquad A \otimes B \longrightarrow A$

$a \otimes b \longmapsto af(b) = a(b' \mapsto b'b)$?

$= (b' \mapsto b'ab)$

$=$

$$B \xrightarrow{f} A$$

$$b \longmapsto (b' \mapsto b'b)$$

---

so end up with

$\mathrm{Hom}_B(B,B) \otimes_B B$

$$A \otimes B \longrightarrow A \otimes A \longrightarrow A$$

$a \otimes b$ $\qquad$ $a \otimes (b'' \mapsto b''b)$

$(b' \mapsto b'a) \otimes (b'' \mapsto b''b)$

$b' \mapsto b'ab$

$A \quad Q$

P

$$Q \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$

not usually isos.

$$P \longrightarrow \mathrm{Hom}_A(Q, A) \otimes_A A$$

What you would like is a

IDEAS finish M inv. for K_* in the case of h-unital rings meq a unital ring. This is the case where the Roos cat $\mathcal{M}$ has a small projective generator. So you have an intrinsic cat $\mathcal{P} \subset \mathcal{M}$ of small projectives, also a dual cat $\check{\mathcal{P}} \subset \mathcal{M}^\vee$

Use of $yx = 1$ - is a connection with Toeplitz algs

$$\mathrm{Hom}_{\mathcal{Q}}(M, N) = \varinjlim \mathrm{Hom}_R(M', N/N')$$

limit over directed set of $M' \subset M$ M/M' nil N' nil. But M firm $\Rightarrow$ $A^n M = M$ so M/M' nil $A^n M \subset M'$ $\therefore M' = M$ and $N \to N/N'$ nil isom. $\mathrm{Hom}(M,$

Once you have for M firm.

$$\begin{tikzcd} \mathrm{Hom}_R(M, N) \arrow[r, "\sim"] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{Q}}(j^*M, j^*N) \\ \mathrm{Hom}_R(M, A^{(2)} \otimes_R N) & \end{tikzcd}$$

Point:

$$\begin{tikzcd} \mathrm{mod}(R) \arrow[r, "j^*"] \arrow[d, shift right, "\varepsilon^*"'] & \mathcal{Q} \\ \mathrm{firm} \arrow[u, shift right, "\varepsilon"'] \arrow[ur, "\sim", "\phi"'] & \end{tikzcd}$$

$A^{(2)} \otimes_R N$

$$\mathrm{Hom}_R(\varepsilon(M), N) = \mathrm{Hom}_R(M, \varepsilon^* N)$$

via the equivalence ϕ j^* becomes ε^* so ε becomes left adj.

$$(\phi^{-1} j^*)(N) = \varepsilon^* N = A^{(2)} \otimes_R N.$$

$\mathrm{Ext}^j_R(N,M)=0$ $j=0,1$ for N in some class

$\Longleftrightarrow$ $\mathrm{Hom}_R(-,M)$ inverts maps with kernel + cokernel in this class.

Pf: $(\Longrightarrow)$ $0\to N'\to N\to N''\to 0$

$$0\to \mathrm{Hom}_R(N'',M)\to \mathrm{Hom}_R(N,M)\to \mathrm{Hom}_R(N',M)\to \mathrm{Ext}^1_R(N'',M)$$

$(\Longleftarrow)$ $0\to N\to N\oplus M\xrightarrow{pr_1} M\to 0$

$$\mathrm{Hom}_R(N\oplus M,M)\xleftarrow{\sim}\mathrm{Hom}_R(M,M))$$

$\Rightarrow$ $\mathrm{Hom}_R(N,M)=0$.

also $0\to M\to E\to N\to 0$ rep

$$\mathrm{Hom}_R(E,M)\xrightarrow{\sim}\mathrm{Hom}_R(M,M)\to \mathrm{Ext}^1_R(N,M)$$

$$\mathrm{Hom}_2(j^*N,j^*M)=\varinjlim \mathrm{Hom}_R(N',M/M')$$

~~M' torsion M~~

$M\to M/M'$

$\mathrm{Hom}_R(M,M)\xleftarrow{\sim}\mathrm{Hom}_R(M/M',M)$ $\Rightarrow$ $M'=0$.

$M\xrightarrow{\sim}\mathrm{Hom}_R(A,M)$

$\mathrm{Ext}^j_R(R/A,M)=0$ $j=0,1$.

$\{N \mid \mathrm{Ext}^j_R(N,M)=0 \quad j=0,1\}$ closed under extensions

$$0\to \mathrm{Hom}_R(N'',M)\to \mathrm{Hom}_R(N,M)\to \mathrm{Hom}_R(N',M)$$
$$\to \mathrm{Ext}^1_R(\quad)\to \mathrm{Ext}^1_R(N,M)\to \mathrm{Ext}^1_R(N',M)$$

II 5

$$0 \to M \to Q^0 \to Q^1 \to Q^2 \to \cdots$$

inj resol. Q^0, Q^1 torsion-free

$\mathrm{Hom}_R(-, M)$ ~~left~~ exact resp cokernels $\oplus$.

If it kills R/A it kills all of mod(R/A) then $\mathrm{Ext}^1_R(-, M)$ is same.

---

to show $\mathrm{Hom}_R(-, M)$ inverts nil isos.

$$0 \to N' \to N \to N'' \to 0$$

if $AN'' = 0$, then $\mathrm{Hom}_R(N'', M) = 0$ $\quad AN'' = 0$ as ${}_AM = 0$

if $AN' = 0$, then

~~$0 \to N' \to N \to N'' \to 0$~~

~~$\mathrm{Hom}_R(A, N') \to \mathrm{Hom}_R(A, N) \to \mathrm{Hom}_R(A, N'')$~~

$\mathrm{Ext}^1_R(N, M)$ rep by

$$\begin{tikzcd}
 & 0 \arrow[r] & M \arrow[r] \arrow[d, "\cong"'] & E \arrow[r] \arrow[d] \arrow[dl, dashed] & N \arrow[r] \arrow[d, "0"] & 0 \\
 & 0 \arrow[r] & \mathrm{Hom}_R(A, M) \arrow[r] & \mathrm{Hom}_R(A, E) \arrow[r] & \mathrm{Hom}_R(A, N) &
\end{tikzcd}$$

$\mathcal{S}$ class in mod(R) $\qquad$ $\mathcal{S}$-isom. means Ker + Coker $\in$ some $S \in \mathcal{S}$

$$0 \to N' \to N \to N'' \to 0$$

$$0 \to (M, N') \to (M, N) \to (M, N'') \to \mathrm{Ext}^1(M, N')$$

$$(M, N \oplus M) \xrightarrow{\sim} (M, M) \to (M, N) = 0$$

$$0 \to N \to E \to M \to 0$$

$$(M, E) \xrightarrow{\sim} \overset{1}{(M, M)} \to \overset{x}{\mathrm{Ext}^1(M, N)}$$

116. ~~[illegible]~~

a) $A\otimes_R M \xrightarrow{\sim} M$

b) $\mathrm{Ext}^j_R(M,N)=0$ $\quad j=0,1 \quad$ N nil

b') $\quad N \in \mathrm{mod}(R/A)$.

c) $\mathrm{Hom}_R(M,-)$ inverts nil isos.

~~c) ⇒ a) obvs~~ $\quad$ a) ⇒ b) $\quad \mathrm{Hom}_R(M,N)=0$ $\quad AM=M$, $AN=0$.

$\mathrm{Ext}^1_R(M,N)$

$$\begin{tikzcd}
 & A\otimes_R N \arrow[r] \arrow[d, "0"'] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d, "\wr"] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

none of this is important really.

dual versions

a) $M \xrightarrow{\sim} \mathrm{Hom}_R(A,M)$

b) $\mathrm{Ext}^j_R(N,M)=0$ $\quad j=0,1 \quad N \in$ ~~mod~~ nil (R/A)
(resp' version)

~~b') $\quad N \in \mathrm{nil}(R/A)$~~

c) $\mathrm{Hom}(-,M)$ inverts nil isos.
(resp version)

know b), c) equiv.

~~a.~~ c) ⇒ a. obvious

a) ⇒ b). $\quad \mathrm{Hom}_R(N,M)=0$ $\quad$ if $\quad AN=0$, ${}_AM=0$

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\wr"'] & E \arrow[r] \arrow[d] \arrow[dl, dashed] & N \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(A,M) \arrow[r] & \mathrm{Hom}_R(A,E) \arrow[r] & \mathrm{Hom}_R(A,N) &
\end{tikzcd}$$

What else do I need?

Fence of f. flat.

$\mathrm{Hom}_R(A^{(2)}, M)$ closed.

$\mathrm{Hom}_R(A,-)$ inverts injections nil cokernel.
since $A=A^2$.

117. $0 \to K \to M \longrightarrow \mathrm{Hom}_R(A,M)$

$\Longrightarrow \quad \mathrm{Hom}_R(A,M) \hookrightarrow \underline{\mathrm{Hom}_R(A^{(2)},M)}$

$N \longmapsto \mathrm{Hom}_R(N, \mathrm{Hom}_R(A^{(2)}, M))$

$\|$

$\mathrm{Hom}_R(A^{(2)} \otimes_R N, M)$ inverts nil iso in N

Suppose ~~M firm. How to see~~

(a) M firm

(b) M cokernel of $F_1 \to F_0$ of firm flat modules

(c) $- \otimes_R M$ inverts nil isos of right modules.

(a) ⇒ (c′) $\mathrm{Tor}^R_j(N,M)$ $j = 0,1$ N nil

~~clear~~ any for $j=0$. $N \mapsto N \otimes_R M$

rtcont kills R/A ∴ all R/A modules

then consider $\mathrm{Tor}^R_1(-,M)$ rtcont on mod(R/A).

$$\mathrm{Ext}^p_{R/A}(N, \mathrm{Ext}^q_R(R/A, M)) \Longrightarrow \mathrm{Ext}^{p+q}_R(N,M)$$

$$\mathrm{Tor}^{R/A}_p(N, \mathrm{Tor}^R_q(R/A, M)) \Longrightarrow \mathrm{Tor}^R_*(N,M)$$

$\mathrm{Tor}^R_j(N,M)$, $j = 0,1$ all $N \ni NA = 0$.

$N \to 0$ nill isom

$\Longrightarrow$ $N \otimes_R M \xrightarrow{\sim} 0$ isom

$0 \longrightarrow N' \longrightarrow \overset{\text{free}}{F} \longrightarrow N \longrightarrow 0 \Longrightarrow 0 \to \mathrm{Tor}^R_1(N,M) \to N' \otimes_R M \xrightarrow{\sim} F \otimes_R M$

118 new idea for getting started

R, $\mathrm{mod}(R)$, A

define nil module, nil-isom.

example $\mu : A \otimes_R M \longrightarrow M$

define firm module

possible properties of firm modules

$\mathrm{Hom}_R(M, -)$ inverts nil isoms.

$\mathrm{Ext}^j_R(M, N) = 0 \quad j = 0, 1$ and N nil

M cokernel of a map between firm flat modules.

$\mathcal{S}$ class of objects of $\mathcal{A}$ abelian M obj

ⓐ $\mathrm{Ext}^j_{\mathcal{A}}(M, N) = 0$ for $j = 0, 1$ and all N in $\mathcal{S}$

ⓑ $\mathrm{Hom}_{\mathcal{A}}(M, -)$ inverts maps with kernel + cokernel isom to objects of $\mathcal{S}$.

formal thm. first condition says $M \in {}^{\perp}\mathcal{S}$

Pf: ⓐ ⇒ ⓑ

$$0 \to N' \to N \to N'' \to 0$$

$$0 \to \mathcal{A}(M, N') \longrightarrow \mathcal{A}(M, N) \longrightarrow \mathcal{A}(M, N'') \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(M, N')$$

ⓑ ⇒ ⓐ $N \oplus M \longrightarrow M \quad \Rightarrow \quad \mathcal{A}(M, N) = 0$

$x \in \mathrm{Ext}^1_{\mathcal{A}}(M, N) \qquad 0 \to N \to E \to M \to 0$

$$\Rightarrow \quad \mathcal{A}(M, E) \longrightarrow \mathcal{A}(M, M) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(M, N')$$
$$1 \longmapsto x$$

In the firm case you find nothing

---

interesting $A \otimes_R N \longrightarrow N$

dual map $\mu' : N \longrightarrow \mathrm{Hom}_R(A, N)$
$n \longmapsto (a' \longmapsto a'n)$

$\Rightarrow f : a' \longmapsto f(a')$

$af : a' \longmapsto f(a'a) = a' f(a)$

$\therefore af = \mu'(f(a))$

$$\mathrm{Hom}_R(M, N) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_R(A, N)) = \mathrm{Hom}_R(A \otimes_R M, N)$$

The above seems clear.

See what I need!!

$\mathrm{Tor}_j^R(R/A, M) = 0 \qquad j = 0, 1$

$\mathrm{Tor}_j^R(W, M) = 0 \qquad j = 0, 1 \qquad WA = 0$

$\mathrm{Tor}_j^R(W, M) = 0 \qquad j = 0, 1 \qquad WA^n = 0$ some n.

$- \otimes_R M$ inverts nil isos.

$$0 \to W' \to W \to W'' \to 0$$

$$\mathrm{Tor}_1^R(W'', M) \to W' \otimes_R M \to W \otimes_R M \to W'' \otimes_R M \to 0$$

Conv. suppose $- \otimes_R M$ inv. nil isos.

W nil $\Longrightarrow$ $W \to 0$ is nil ism $\Longrightarrow$ $W \otimes_R M \xrightarrow{\sim} 0$.

~~$0 \to M' \to P \to M \to 0$~~ P proj.

~~$0 \to \mathrm{Tor}_1^R(W, M) \to$~~ Given W nil choose

$0 \to W' \to P \to W \to 0$ P proj.

$$0 \to \mathrm{Tor}_1^R(W, M) \to W' \otimes_R M \xrightarrow{\sim} P \otimes_R M$$

M closed $\Longleftrightarrow$ $M \xrightarrow{\sim} \mathrm{Hom}_R(A, M)$ $\qquad \mathrm{Ext}_R^j(R/A, M) = 0 \quad j = 0, 1$

$\mathrm{Ext}_R^j(N, M) = 0 \qquad AN = 0.$

$$0 \to N' \to N \to N'' \to 0$$

$$0 \to \mathrm{Hom}_R(N'', M) \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N', M) \to \mathrm{Ext}_R^1(N'', M)$$

120 Conversely assume M ~~[illegible]~~ [illegible] to show $\mathrm{Ext}^1(M,N) = 0$

$$\begin{tikzcd}
A\otimes_R N \arrow[r] \arrow[d] & A\otimes_R E \arrow[r] \arrow[d] & A\otimes_R M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

[redacted]

$$\begin{tikzcd}
A\otimes_A M' \arrow[r] \arrow[d] & A\otimes_R M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M
\end{tikzcd}$$

$AM = M$

~~$A\otimes_R M$~~ $M' = A\otimes_R M$

$a \otimes a_2 m$

$\shortparallel$

$aa_2 \otimes m$

$$A\otimes_R A\otimes_A M \longrightarrow A\otimes_R M$$

$a_1 \otimes a_2 \otimes m \qquad a_1 a_2 \otimes m = a_1 \otimes a_2 m$

---

have severe problems getting first chapter done there are too many things to straighten out.

concentrate on dual picture

A-closed modules $(R/A)^\perp$ $\quad \mathrm{Ext}^j_R(R/A, M) = 0$.

$j = 0, 1$.

$$M \xrightarrow{\sim} \mathrm{Hom}_R(A, M)$$

~~Hom_a~~

$$\begin{tikzcd}
a \arrow[r, shift right, "j^*"'] & a/s \arrow[l, shift right, "j_!"']
\end{tikzcd}$$

$\mathrm{Hom}_a(M,N) \to \mathrm{Hom}_a(j_!(j^*M), N) = \mathrm{Hom}_{a/s}(j^*M, j^*N)$

$\Rightarrow$ LHS inverts nil isos in N $\therefore$ $j_!(j^*M) \in {}^\perp s$

$j_!(j^*M) \longrightarrow M$ ~~[illegible]~~

If M is ~~[illegible]~~ $\in {}^\perp s$ Why??

~~Hom_a~~ Hom $j_!$

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend right, "j^*"'] & \mathcal{A}/\mathcal{S} \arrow[l, bend right, "j_!"'] \end{tikzcd}$$

$$\mathrm{Hom}_{\mathcal{A}}(j_!(j^*M), N) = \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(j^*M, j^*N).$$

$\Rightarrow$ LHS inverts $\mathcal{S}$-isos. in N, $\therefore\ j_!(j^*M) \in {}^{\perp}\mathcal{S}$

$$\Rightarrow \quad \mathrm{Hom}_{\mathcal{A}}(j_!(j^*M), N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(j^*j_!(j^*M), j^*N)$$

Assume $\mathcal{S}^{\perp} \xrightarrow{i} \mathcal{A}/\mathcal{S}$ equivalence

Then $\forall M$ $\exists$ $M^{\#} \in \mathcal{S}^{\perp}$ + $j^*(M^{\#}) \xrightarrow{\sim} j^*(M)$

same as a map $M^{\#} \longrightarrow M$ which must be an $\mathcal{S}$-isom. $\therefore$ $\mathrm{Hom}(N, M^{\#}) \xrightarrow{\sim} \mathrm{Hom}(N, M)$ $\forall N \in \mathcal{S}^{\perp}$

showing $M \mapsto M^{\#}$ rt adjoint to inclusion i

---

suppose $j_!$ i.e. $\forall\ j^*M = X$ $\exists$ $j_!$ ~~(j^*M)~~ X

$$\mathrm{Hom}_{\mathcal{A}}(j_!X, M) = \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(X, j^*M)$$

$\Rightarrow$ $j_!X \in \mathcal{S}^{\perp}$ together with $X \to j^*j_!X$

or $j_!j^*M \to M$

But

$$\begin{array}{l} \mathrm{Hom}_{\mathcal{A}}(j_!X, M) = \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(X, j^*M) \\ \quad\quad \wr\!\downarrow \\ \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(j^*j_!X, j^*M) \end{array}$$

$\leftarrow$ ~~should be induced by $X \to j^*j_!X$~~

2T1 CHAR 5362 9/27/94

2T2 ADJ 4913 9/26/94

122 $\forall X$ have $j_! X$ ∋

$$\begin{tikzcd}
\mathrm{Hom}_a(j_! X, M) \arrow[r, phantom, "="] \arrow[d, "\simeq"'] & \mathrm{Hom}_{a/s}(X, j^* M) \\
\mathrm{Hom}_{a/s}(j^* j_! X, j^* M) \arrow[ur, "\beta_X^*"] &
\end{tikzcd}$$

$\longrightarrow$ given by $X \xrightarrow{\beta} j^* j_! X$

$\longleftarrow$ given by $j_! j^* M \xrightarrow{\alpha} M$.

Conclude $\beta : X \xrightarrow{\sim} j^* j_! X$ isom.

Also have

$$\begin{tikzcd}
j^* M \arrow[r, "\beta \cdot j^*", "\simeq"'] \arrow[rr, bend right, "1"'] & j^* j_! j^* M \arrow[r, "j^* \cdot \alpha"] & j^* M
\end{tikzcd}$$

$\therefore j^* \alpha$ isom.

$\overset{j^*}{\Longrightarrow}$ α nil isom.

$\mathrm{Hom}_a(M, N) \cong \mathrm{Hom}_{a/s}(j^* M, j^* N)$

$\Longrightarrow$ ${}^{\perp}S \hookrightarrow a \xrightarrow{j^*} a/s$ is fully faithful

Prop. ${}^{\perp}S \hookrightarrow a \xrightarrow{j^*} a/s$ is an equivalence of cats if $\forall M$ in a $\exists$ S-isom $\varepsilon_M : M^{\#} \longrightarrow M$ ~~such that~~ ~~M is~~ $M^{\#}$ is in ${}^{\perp}S$.

Note then that

$\mathrm{Hom}_a(M^{\#}, N) = \mathrm{Hom}_{a/s}(j^*(M^{\#}), j^* N) = \mathrm{Hom}_{a/s}(j^* M, j^* N)$

follows that $j^* : a \longrightarrow a/s$ admits a left adjoint $j_* : j^* M \longmapsto M^{\#}$. Also

$\mathrm{Hom}_{{}^{\perp}S}(L, M^{\#}) = \mathrm{Hom}_a(L, M)$

etc.

122 a suppose $j^*(M)$ injective

$$N \mapsto \mathrm{Hom}_{m}(j^*N, j^*M)$$

$\|$ ⇐ if M closed

$$\mathrm{Hom}_{m}(N, M) \quad |$$

M torsion-free iff $_{(A)}M = 0$.

$\mathcal{T}$ = smallest Serre subcat closed under ⊕'s in mod(R) containing R/A. (∴ mod(R/A))

M t.f. ⇔ only $\mathcal{T}$-subm. is 0.

$_{(A)}M = 0$ ⇐

conversely if $_{(A)}M = 0$, then M embeds in Q injective & $_{(A)}Q = 0$. Note that $\{N \mid \mathrm{Hom}_R(N,Q) = 0\}$ is a Serre subcat closed under ⊕'s. Thus it contains $\mathcal{T}$.

~~[illegible] Fix a module~~ Let F be a flat right module, consider $\{M \mid F \otimes_R M = 0\}$. Could this be tors(R, A) for some A?

R comm. noetherian, then Take $R = \mathbb{Z}$. The torsion theories should be determined by the indecomposable torsion-free injectives. e.g.

~~$\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M = 0 \Longleftrightarrow$~~

$\mathbb{Z} \subset S^{-1}\mathbb{Z} \subset \mathbb{Q}$ S is a set of primes.

$A = \mathbb{Z}m \quad m \geq 1.$ $\mathbb{Z}[\frac{1}{m}]$ finite set of primes.

123 Take $R = \mathbb{Z}$, $A = \mathbb{Z}d$, $d \geq 1$.

~~Look at~~ A-torsion modules : each elt killed by a power of d.

ignore modules
each elt
supp $\mathbb{Z}d$

A-solid modules : $\mathbb{Z}[d^{-1}]$ modules.
modules from complement of closed set corresp to $\mathbb{Z}d = A$.

But now take all torsion modules ~~these will not arise from~~ as Serre subcat.

---

there are some things to be sorted out.

$\mathcal{T} = \mathrm{tors}(R,A)$ is smallest S. subcat closed under $\oplus$'s in mod(R) containing R/A, hence nil(R/A).

$\forall M$ $\exists$ largest submodule M_t which is in $\mathcal{T}$ Then ${}_{(A)}(M/M_t) = 0$, in fact $\mathrm{Hom}_R(N, M/M_t) = 0$ for any $N \in \mathcal{T}$.

How to see ${}_{(A)}M = 0 \implies \mathrm{Hom}_R(N, M) = 0 \quad \forall N \in \mathcal{T}$

~~Let Consider~~ ~~$\mathbb{Z}M_t$~~ Assume false. Suppose ${}_{(A)}M = 0$. ~~and $\exists N \not\subseteq M$~~ How do you see $M \in \mathcal{T}$ $M \neq 0 \implies {}_{(A)}M \neq 0$.

Answer : consider $N \ni \mathrm{Ext}^j_R(N, M) = 0$. $j = 0, 1$.

i.e. ${}^{\perp}\{M\}$.

$$0 \to N' \to N \to N'' \to 0$$

$N', N'' \in {}^{\perp}M \implies N \in M^{\perp}$

$N'', N \in {}^{\perp}M \implies N' \in M^{\perp}$

124 basic point $_{(A)}M = 0 \Rightarrow \exists M \hookrightarrow Q$ inj $\ni {}_{(A)}Q = 0$.

~~without inj.~~

~~first cons. $N \ni \mathrm{Hom}_R(N, M) = 0$.~~

~~closed under extensions.~~

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & M' \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

~~1) $\forall M' < M$ $_{(A)}(M/M') \neq 0$.~~

ex. of A left T-nilp- but no right

want all ~~$a_4 \cdots a_4 a_3 a_2 a_1$~~ $a_1 a_2 \cdots a_n \cdots$ to be event. zero

but some sequence the other way to be always $\neq$ zero.

$x_1, \ldots, x_n, \ldots$ gener.

relations. Want $x_i x_j = 0$ if $i \leq j$

Take any monomial $x_{i_1} \cdots x_{i_k}$ is zero unless $i_1 > \cdots > i_k$. Rings has basis of such decreasing monomials. Then if you take any a_1 look at largest x_m ~~contained~~ occuring in a_1. Then $a_1 \in \sum_{j \leq m} x_j R$

---

straighten out what to say

M torsion

1) $\forall M' < M$ we have $_{(A)}(M/M') \neq 0$.

2) $\mathrm{Hom}_R(M, N) = 0$ $\forall$ torsion-free N

3) $W \otimes_R M = 0$ $\forall$ rt R-modules $\ni W = WA$.

4) T-nilpotence condition

125

idea. $M \in \mathcal{T} \Leftrightarrow \exists$ ordinal number β and increasing filtration $F_\beta M$, $\beta \leq \alpha$ such that $F_0 M = 0$, $F_\alpha M = M$, $F_{\beta+1} M / F_\beta M$ killed by A and $F_\beta = \bigcup_{\beta'<\beta} F_{\beta'}$ if β ord.

Easily seen that the class of M in mod(R) for which such a filtr exists is Serre sub-cat closed under direct sums. Check quotients + sub.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M'_\alpha \arrow[r] \arrow[d] & M_\alpha \arrow[r] \arrow[d] & M''_\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M'' \arrow[r] & 0 \\
 & & {} & &
\end{tikzcd}$$

$$M''_\alpha = \frac{M_\alpha + M'}{M'} \subset M/M'$$

Is it clear that $\bigcup_{\gamma<\alpha} M'_\gamma + M$

now take $\varinjlim_{\gamma<\alpha}$ you get

$$\begin{tikzcd}
0 \arrow[r] & \bigcup_{\gamma<\alpha} M'_\gamma \arrow[r] \arrow[d] & \bigcup_{\gamma<\alpha} M_\gamma \arrow[r] \arrow[d, "\wr"] & \bigcup_{\gamma<\alpha} M''_\gamma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_\alpha \arrow[r] & M_\alpha \arrow[r] & M''_\alpha \arrow[r] & 0 \ ?
\end{tikzcd}$$

$$M''_\alpha = M_\alpha + M'/M' \subset M/M'$$

$$\bigcup_{\alpha<\beta} M''_\alpha = \bigcup_{\alpha<\beta} (M_\alpha + M')/M' \supset M_\gamma$$

$$\bigcup M'/M_\alpha \cap M'$$

126 Given M_α $\alpha \leq \gamma$

Put $M'_\alpha = M_\alpha \cap M'$. Then

$$\bigcup_{\alpha<\beta} M'_\alpha = \bigcup_{\alpha<\beta}\left(M_\alpha \cap M'\right) = \left(\bigcup_{\alpha<\beta} M_\alpha\right) \cap M' = \overset{\overset{M'_\beta}{\|}}{M_\beta \cap M'}$$

$$\bigcup_{\alpha<\beta} M''_\alpha = \bigcup_{\alpha<\beta}\left(M_\alpha + M'/M'\right) \subset M/M'$$

$$= \left(\bigcup_{\alpha<\beta} M_\alpha\right) + M'/M' \subset M/M'$$

$$= M''_\beta.$$ OKAY.

OKAY. this implies any tors module has a $\overset{\neq 0}{\text{submod}}$ killed by A. So torsion free $\Leftrightarrow$ ${}_AM = 0$.

M torsion $\Rightarrow$

1) $\forall M' < M$, ${}_A(M/M') \neq 0$.

2) $\mathrm{Hom}_R(M, N) = 0$ $\forall N \ni {}_AN = 0$.

3) $W \otimes_R M = 0$ $\forall W$, $WA = W$.

4) T-nilpotence

how do you see T-nilp. $\Rightarrow$ torsion.

not torsion $\Rightarrow$ some quotient.

$$\boxed{M \text{ torsion-free} \Leftrightarrow {}_AM = 0}$$ $\Leftarrow$ uses inj hull

what to do about trivial ones. ~~Anyway~~

Any module has a largest torsion submodule

127 ~~M torsion~~ TFAE for a module M

1) M is torsion

2) for every quotient module $M/M' \neq 0$ one has ${}_A(M/M') \neq 0$.

3) $\exists$ ordinal α_0 and weakly inc. filt M_α for $\alpha \leq \alpha_0$

$$M_0 = 0, \quad M_{\alpha_0} = M, \quad A(M_{\alpha+1}/M_\alpha) = 0, \quad \alpha < \alpha_0$$

4) $X \otimes_R M = 0$ for every right module X s.t. $XA = X$ (resp. for every X firm flat right module)

5) T-nilp. cond.

T-nilp can be used to define torsion mod.

4) $\Rightarrow$ 5) $\Rightarrow$ 2) $\Rightarrow$ 4) ~~What else~~

---

smallest Serre category closed under $\oplus$'s cont. R/A. Take any

---

So what next. rt cont fun.

$$\begin{tikzcd} \mathrm{mod}(R) \arrow[r, shift left, "j^*"] & m^t \arrow[l, shift left, "j_*"] \end{tikzcd}$$

j^* resps colims
rt exact
resps $\oplus$'s

Recall condition R/A is flat, equivalent to $(1+a)$ being filtering. To recall first do R/A proj.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & R \arrow[l, bend right, "\cdot e"'] \arrow[r] & R/A \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

$re' \longleftarrow$

$r \longmapsto r$

$1 - e' \in A$

$R(1-e') = A \Rightarrow a(1-e') = a$ $A = Re$.

right identity $\exists e \ni ae = a \ \forall a$.

condition was $\forall a_1, \ldots, a_n \ \exists a \ni a_i a = a_i$. Enough to show $S = \{(1+a)\}$ is filtering:

128. cat has one obj. and a map $1+a$ $\forall a \in A$.

$$\begin{tikzcd} \cdot \arrow[r, shift left, "1+a_1"] \arrow[r, shift right, "1+a_2"'] & \cdot \end{tikzcd}$$

$(\cancel{1}+a_1)(1+a) = (\cancel{1}+a_2)(1+a)$

so you get condition $\forall a_1\ \exists a \ni a_1(1-a) = 0$.

Can you distinguish such rings by some property of $\mathfrak{M}(A)$. No because, ~~$\mathfrak{M}(A)$ is~~ any A as above is left flat and there are ~~that~~ idemp. rings neg unital rings such that A is not left flat. So what??

Important example is Ae ~~Ae~~ with $e^2=e$.

~~Sticky to unital rings.~~ Consider a $P \otimes_A Q$ A unital and ask what it means for these to be a right identity? You want $e \in P \otimes_A Q$ such that $\xi e = \xi$ for all ξ. Is there a reasonable way to construct these? Let's try with $P = A$. Then we want to construct Q together with $Q \xrightarrow{f} A$ image is left ideal generates A. ~~That case~~ Case considered recently $Q = B \subset A$ left ideal generating A. For example $B = Ae$ where $AeA = A$. Suppose $e \in P \otimes_A Q$ $e = p_i \otimes \xi_i$ such that $(\xi p_i)\xi_i = \xi$ $\forall \xi$. Says that

$$Q \xrightarrow{\cdot p_i} A^n \xrightarrow{\cdot \xi_i} Q \quad \text{so} \quad Q \in \mathcal{P}(A).$$ ~~together~~

So I learn that $Q \in \mathcal{P}(A)$. $(\cdot p_i) \in \operatorname{Hom}_A(Q, A)$ generate, so we have $P \twoheadrightarrow \operatorname{Hom}_A(Q, A)$.

$$Q \otimes P \twoheadrightarrow Q \otimes \operatorname{Hom}_A(Q, A) \longrightarrow A$$

$$P \otimes_A Q \twoheadrightarrow \mathrm{Hom}_A(Q,A) \otimes_A Q = \mathrm{End}_A(Q)$$

Conclude what?

Given a finite set of q_i can we enlarge P appropriately. ~~Bunch~~

$$0 \to A \to R \to R/A \to 0$$

~~$0 \to A \otimes_R A \to A \to A/A^2 \to 0$~~

$$\begin{tikzcd} 0 \arrow[r] & R/A \otimes_R A \arrow[r, "0"] \arrow[d, equal] & R/A \arrow[r] & R/A \otimes_R R/A \arrow[r] & 0 \\ & A/A^2 . & & & \end{tikzcd}$$

Start with a finite set of q_μ. To construct $b = \sum p_i \otimes q_i$ after enlarging P, Q possibley so that $q_\mu b = q_\mu$. Thus $\sum_i \underbrace{(q_\mu p_i)}_{\in A} q_i = q_\mu$. So we first need to write each q_μ as $\sum_i a_{\mu i} q_i = q_\mu$.

To simplify assume there is a single q_μ. Since $Q =$ ~~AQ~~ $= QPQ$ ~~can write $q_\mu =$~~ we can ~~find~~ write $\sum_i q_i' p_i q_i = q_\mu$. ~~Hard~~

Wait: Write $q_\mu(1-b) = 0$. ~~Thus $q_\mu \otimes Q$~~ $\sum A q_\mu (1-b) = 0$, so if we set $Q' = \sum A q_\mu$ Then $Q' \subset Q' b$

$$\begin{tikzcd} \sum A q_\mu \arrow[r, "1"] \arrow[d, phantom, "\cap" rotate=90] & & \sum_i A q_\mu \arrow[d, phantom, "\cap" rotate=90] \\ Q \arrow[r, "\cdot p_i"] & A^n \arrow[r, "\cdot q_i"] & Q \end{tikzcd}$$

130. Take $P = Q = A$ initially and $q_\mu \in A$.

So take $\sum_\mu A q_\mu$.

Basic: If R/A is R-flat, then

$$0 \longrightarrow A \longrightarrow R \longrightarrow R/A \longrightarrow 0 \quad \text{exact}$$

$$\Longrightarrow \quad 0 \longrightarrow A \otimes_R M \longrightarrow M \longrightarrow M/AM \longrightarrow 0 \quad \text{is exact}$$

so $AM = M \Longrightarrow M$ is firm

Now for modules such that $AM = M$ we know that

$$A \otimes_A M \xrightarrow{\sim} A \otimes_R M.$$

~~So a rather interesting point is that if~~

~~Interesting!~~ Is the converse true?

Suppose $AM = M \Longrightarrow A \otimes_R M \xrightarrow{\sim} M$

Is it true that $A \otimes_R M \hookrightarrow M$ for all M?

Take $M = R/A$. Then $A/A^2 = A \otimes_R R/A \longrightarrow R/A$ is zero.

Assume $A = A^2$. Consider

$$\begin{tikzcd}
R^p \arrow[r, "\cdot x"] \arrow[dr, "0"'] & R^q \arrow[r, dashed, "\cdot x'"] \arrow[d, "\cdot m"] & R^s \arrow[dl, dashed, "\cdot m'"] \\
 & M &
\end{tikzcd}$$

$xm = 0$

$\Rightarrow \exists\ m = x'm'$

$xx' = 0.$

~~$0 \to AM \to M \to M/AM \to 0$~~

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & A \otimes_R AM \arrow[r, "\sim"] \arrow[d] & AM \arrow[r] \arrow[d] & 0 & \\
 & A \otimes_R M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/AM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{0}{\cancel{A \otimes_R M/AM}} \arrow[r] \arrow[d] & M/AM \arrow[r, "\sim"] \arrow[d] & M/AM \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

131 Idempotent ring A equiv. between firm and n cnf. $M = AM$ and ${}_AM = 0$.

$$\begin{tikzcd}
0 \arrow[r] & {}_AM \arrow[r] & M \arrow[r] & M/{}_AM \arrow[r] & 0 \\
\overset{0}{A \otimes_{A} ({}_AM)} & A \otimes_R M \arrow[u, "S"] \arrow[r, "\sim"] & A \otimes_R (M/{}_AM) \arrow[u, "S"] \arrow[r] & 0
\end{tikzcd}$$

R/A projective $\Longleftrightarrow$ $\exists e \in A, \forall a$ $a(1-e) = 0$

A left ideal $\qquad$ A has right identity

$Ae \subset Re \subset A \subset Ae$ $\qquad$ $A \subset Ae \subset Re \subset A$

R/A flat $\Longleftrightarrow$ $\forall a_1, \ldots, a_n \; \exists a$ $\quad a_j(1-a) = 0$.

$$\Longrightarrow \quad \begin{tikzcd}
R^q \arrow[r, "\cdot a_j"] & R \arrow[r, "\cdot x_i'"] \arrow[d, "\cdot \bar{1}"] & R^s \arrow[dl, "\bar{r}_i"] \\
& R/A &
\end{tikzcd}$$

$a_j x_i' = 0 \quad \forall j, i$

$\bar{1} = \sum x_i' \bar{r}_i$ $\qquad$ ~~$a = \sum x_i'$~~

$1 - a = \sum x_i' r_i$

so that part true for a left ideal

Now $0 \to A \to R \to R/A \to 0$ $\qquad$ here A must be a right ideal.

R/A is R^{op}flat $\Rightarrow$ $0 \to A \otimes_R M \to M \to M/AM \to 0$

$\Rightarrow$ $A \otimes_R M \xrightarrow{\sim} AM$ for all M.

But you want to take $M = R/A$ to get $A = A^2$.

So A must be two sided at this point.

~~Conversely ass. $A = A^2$ and $A \otimes_R M \xrightarrow{\sim} AM$ $\forall M$.~~

So R/A R^{op}flat $\Longrightarrow$ $A = A^2$ and ($AM = M \Rightarrow M$ firm)

Conversely

$$\begin{tikzcd}
& A \otimes_R AM \arrow[r] \arrow[d, "S"] & A \otimes_R M \arrow[r] \arrow[d] & \overset{0}{\cancel{A \otimes_R (M/AM)}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & AM \arrow[r] & M \arrow[r] & M/AM \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $A \otimes_R M \hookrightarrow M$

TTF ~~stuff~~ two torsion theories

$\mathcal{S}$ $\mathcal{T}$ such that $\mathcal{S}$ = $\mathcal{T}$-free modules

$\Rightarrow$ $\mathcal{S}$ closed under $\prod$ $\therefore$ $\mathcal{S} = \text{mod}(R/A)$ where $A = A^2$. Then $\mathcal{T}$ ~~torsion theory is~~ consists of M such that $\text{Hom}(M, N) = 0$ for all $N \in \mathcal{S}$. $\therefore$ $\mathcal{T} = \{M / M = AM\}$. This closed under subobj.

$$0 \to K \to A \otimes_R M \to AM \to 0$$

$\Rightarrow K = AK = 0$.

M firm $K = {}_A M$

$$0 \to K \to M \to M/K \to 0$$

So ~~is~~ every firm module is nondeg: ${}_A M = 0$.

Conversely. ~~given $M = AM \Rightarrow M$ firm iff ${}_A M = 0$.~~

assume ${}_A M = 0$ for all firm M.

Given $M = AM$, then ~~$M = AM$. Then~~ $A \otimes_A M$ is firm and $A \otimes_A M \to AM$ has kernel killed by A. ~~Local left iden~~

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

~~A local l.i.d.~~ $\Rightarrow {}_A Q = 0$.

Situation

$$\text{Mult}(B) \subset \text{Hom}_{A^{op}}(P, P) \times \text{Hom}_A(Q, Q)^{op}$$

$$\qquad\qquad \text{Hom}_{B^{op}}(B, B) \times \text{Hom}_B(B, B)^{op}$$

<u>I</u> want B to embed in its mult. ring.

necessary cond. ~~$\sum (p' \, q_i) p_i = p'$ So you need~~

p' $\qquad \sum p_i (q_i p') = p'$ So you need enough elements in $Qp' \subset A$

133 ~~Take~~ Start with P an A^{op}-mod e.g. A. Pick some $p' \in P$. ~~The you can look at Th~~

∞ ref.

If you want $\sum_i p_i q_i p' = p'$ Then you need $P(Qp')$ to contain p'. In other words you need the left ideal $\mathfrak{a} = Qp' \subset A$ to be big enough so that $P\mathfrak{a} \ni p'$. Once you are given $p' \in P$ you can look for new elements of Q in $\mathrm{Hom}_A(P, A)$. P right A-mod. $p' \in P$.

I think Q can always be taken to be free $\tilde{A}^{(I)}$

$a = \sum a'a''$

You want to construct $P \longrightarrow A$.

$$\begin{tikzcd}
\tilde{A} \arrow[r, "a_1"] \arrow[d, "a_0"] \arrow[rrr, bend left, "p_0"] & \tilde{A} \arrow[r, dotted] \arrow[d, "b_1"] \arrow[rr, bend left, "p_1"] & \cdots & P \arrow[d, "q_0"] \\
A \arrow[r, equal] & A & & A
\end{tikzcd}$$

in this case we construct $qp = a$

I want to start with an a_0.. $p_1 q_1 = p$

~~Maybe take~~ $p_0, p_1, p_2, \ldots\ldots$

p' want $p_i q_i p' = p_0$ if possible

Construct

Let's try for a ~~counter~~ example $A \subset R$ max valuation ring rank 1 nondiscrete A. What's the mult. ring. $\mathrm{Hom}_A(A, A)$

$A = \bigcup_{\varepsilon > 0} Rz^{\varepsilon}$ $\mathrm{Hom}_R(A, A) = \varprojlim_{\varepsilon > 0} Az^{-\varepsilon} = R$.

134. So R is the center. A firm ~~quotient~~ field K is firm. Interesting firm module is K/A?
Note that $K/R = A(K/R)$. Thus

$$A \otimes_R (K/R) = A \otimes_R K / A \otimes_R R = K/A.$$

is firm. And it has a non-trivial annihilator.
Other firm modules A/Az^ε any $\varepsilon > 0$.
Which are flat? A is, K also. ~~So to what~~.

$$0 \longrightarrow R \xrightarrow{z^\varepsilon} R \longrightarrow R/Rz^\varepsilon \longrightarrow 0$$

If M flat then $M \xrightarrow{z^\varepsilon} M$ is inj. Thus A/Az^ε K/A not flat. Flat should be equivalent to torsion-free, since every fg ideal is principal.
~~So what is next???? Anyway~~

Do I understand flat modules? You need two flat firm module P_A ${}_AQ$ and pairing $Q \otimes P \twoheadrightarrow A$.
~~A should be~~ You would like now to see whether something can be done. You want to start with
Wait $\operatorname{Hom}_R(K,K) = \varprojlim \operatorname{Hom}_R(Rz^{-\varepsilon}, K)$

$$= \varprojlim Kz^\varepsilon = K.$$

$M = AM$ means $M = \bigcup_{\varepsilon > 0} z^\varepsilon M$

So assume $M = \bigcup_{\varepsilon > 0} z^\varepsilon M$

So what do I know about P, just that any element p' can be divided a bit. i.e. have $p' = z^\varepsilon p_1'$ for some $\varepsilon > 0$. So I ask whether it's possible for such a thing to have local left idents.
Start with p_1' divide $p_1' = z^{\varepsilon_1} p_2'$. But now I want

135. R valuation ring max ideal A

fg ideals Rz^{ε} $\varepsilon \geq 0.$ $\varepsilon \in \bigcup \mathbb{Z}\frac{1}{2^n}$

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ start with p' nonzero in P

assume B has local left identities. Then $\forall\ p_j'$ finite we can find $b = \sum p_i q_i$ such that $p_j' = \sum_i p_i(q_i p_j')$ $\forall j$.

Look at $\sum p_j' R \subset P$. I know P is firm flat, so torsion free any every element can be divided by some z^{ε}. Let's work intrinsically

~~Take the~~ Look at $\left(\sum_j p_j' R\right) \otimes_R K \cap P \subset P \otimes_R K$

To simplify look at a single $p' \neq 0$. Then

Look at $p'K \cap P$. What is the meaning of

$p' = \sum p_i q_i p'$ $b = \sum_i p_i \otimes q_i \in P \otimes_A Q$

$p_i \otimes_A q_i = \tilde{p}_k r_{ki} \otimes q_i = \tilde{p}_k \otimes r_{ki} q_i$

You can consider $\sum p_i R \subset P$, assume p_i a basis, then ~~make~~ assume $p_1 \in p'R$ $p' \in p_1 R$.

What's the general picture?

$$\sum p_j' R \subset \sum p_i A$$

So can assume $p_j' \in p_j R$.

Back to one $p' = p_1 z^{\varepsilon}$. $p' = \sum p_i q_i p'$

$q_1 p' = z^{\varepsilon}$ $q_i p' = 0.$