1

module theory ~~of~~ for idempotent rings
Morita equivalence, theory of
Morita invariance of cyclic homology for h-unital rings, also for K_1.

how do I review this? you would love to have an outline, introduction in your mind.

first an outline of the main ideas. this is not ~~so~~ important.

you would like an overview of the paper

the first part concerns $\mathcal{M}(A)$

motivation:

to extend $R \longmapsto \mathrm{mod}(R)$ for R unital to nonunital rings A.

$\mathrm{mod}(\tilde{A})$ = cat of A-modules is too big

If A is unital with id elt e, then $\underline{\mathbb{Z}}$

$$\mathrm{mod}(\tilde{A}) = \mathrm{mod}(A) \times \mathrm{mod}(\tilde{A}/A)$$

$$M = eM \oplus (1-e)M$$

to cut $\tilde{A}$ down

Def firm A-module M by $A \otimes_A M \xrightarrow{\sim} M$

nil A-mod M by $A^n M = 0$ for some n.

Thm: $\{\text{firm } A\text{-mods}\} \hookrightarrow \mathrm{mod}(\tilde{A}) / \bigcup_{n \geq 0} \mathrm{mod}(\tilde{A}/A^n)$

is fully faithful, is equiv. when $A = A^2$.

go over proofs: $A \otimes_A M \longrightarrow M$ is nil iso.

$$a \sum a_i \otimes m_i = a \otimes \sum a_i m_i$$

$$\begin{tikzcd} A \otimes_A M \arrow[r] \arrow[d, "a \cdot"'] & M \arrow[d, "a \cdot"] \arrow[dl] \\ A \otimes_A M \arrow[r] & M \end{tikzcd} \qquad m \mapsto a \otimes m$$

If $AM = M$, then $A \otimes_A M$ is firm

$$A \otimes_A K \longrightarrow A \otimes_A A \otimes_A M \xrightarrow{\mu \otimes 1 = 1 \otimes \mu} A \otimes_A M \longrightarrow 0$$

2. M firm $\iff$ $\mathrm{Hom}_A(M, \cancel{-})$ inverts nil-isomorphisms.

$$\cancel{M_1 \to M_2} \qquad 0 \to N' \to N \to N'' \to 0$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(M,N') \arrow[r] & \mathrm{Hom}_A(M,N) \arrow[r] & \mathrm{Hom}_A(M,N'') \arrow[r] & {} \end{tikzcd}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{0}$ if $AN''=0$

if $AN'=0$ injective.

$$\begin{tikzcd} A\otimes_A M & M \arrow[d] \\ A\otimes_A N' \arrow[r] & A\otimes_A N'' \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} A\otimes_A N \arrow[r, "\sim"] \arrow[d] & A\otimes_A N'' \arrow[d] \\ & N' \end{tikzcd}$$

$\cancel{\mathrm{Ext}^1_A(M, N')}$

given $u : M \to N''$ get

$$\begin{tikzcd} A\otimes_A M \arrow[r, "1\otimes u"] \arrow[d, "\wr"'] & A\otimes_A N'' \arrow[d] & A\otimes_A N \arrow[l, "\sim"'] \arrow[d] \\ M \arrow[r, "u"] & N'' & N \arrow[l] \end{tikzcd}$$

Key props are

1. M firm $\iff$ $\mathrm{Hom}_A(M,-)$ inverts nil isos.
2. $M = AM \implies A\otimes_A M$ firm
   $\forall M$, $A\otimes_A A\otimes_A M$ is firm.

These imply: [redacted] equivalence

$$\mathrm{firm}(A) \longrightarrow \mathrm{mod}(\tilde{A})/\mathrm{mod}(\tilde{A}/A).$$

can I prove 1 in general.

$$0 \to N' \to N \to N'' \to 0 \qquad AN'=0.$$

$$A\otimes_A N' \to A\otimes_A N \to A\otimes_A N'' \to 0$$

3.

$$\begin{tikzcd}
N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0 \\
A\otimes_A N' \arrow[u, "0"] \arrow[r] & A\otimes_A N \arrow[u] \arrow[r] & A\otimes_A N'' \arrow[u] \arrow[r] \arrow[ul, dotted, "\exists\ \text{can.}"'] & 0
\end{tikzcd}$$

so

$$\begin{tikzcd}
N \arrow[r] & N'' \\
M \arrow[ur, "u"] & A\otimes_A N'' \arrow[u] \arrow[ul, dotted, "e"'] \\
A\otimes_A M \arrow[u, "s"] \arrow[ur, "1\otimes u"'] &
\end{tikzcd}$$

$\Leftarrow$ Assume $\mathrm{Hom}_A(M,-)$ res. nil-isos.

Then $A\otimes_A M \overset{s}{\leftrightarrows} M$

Then $A\otimes_A M = s(M)\oplus K$ where $AK=0$.

but ~~$A(A\otimes_A M)$~~ $M = AM \Rightarrow$ same for $A\otimes_A M$. $A\otimes_A M = A\otimes_A AM = A^2\otimes_A M = A(A\otimes_A M)$.

[redacted]. You do this for an ambient ring R, independence of R.

$\mathrm{firm}(\tilde{A}, A) \overset{\text{res.}}{\leftrightarrows} \mathrm{firm}(R, A)$

Why?

N

$$0 \to A \to R \to R/A \to 0$$

$$A\otimes_{\tilde{A}} M \xrightarrow{\sim} R\otimes_{\tilde{A}} M$$

$$0 \to A \to R \to R/A \to 0$$

4. Recall this independence of R proof.

and also in the h-unital case. So what am I actually doing?

Result. $\mathrm{Tor}_*^R(R/A, A)=0 \iff \mathrm{Tor}_*^{\tilde A}(\tilde A/A, A)=0.$

$$\begin{array}{c} R \\ \uparrow \\ \tilde A \end{array}$$

Why? $R/A \overset{L}{\otimes}_{\tilde A} A = R/A \overset{L}{\otimes}_R R \overset{L}{\otimes}_{\tilde A} A$

$$A \overset{L}{\otimes}_{\tilde A} A = A \overset{L}{\otimes}_{\tilde A} R \overset{L}{\otimes}_R A$$

which direction

Start with $A \otimes_{\tilde A} M \xrightarrow{\sim} M$. Then M automatically is an R-module such that $AM=M$.

$$A \otimes_{\tilde A} M \xrightarrow{\sim} A \otimes_R M$$

$$ar \otimes a'm = ara' \otimes m = a \otimes r(a'm)$$

easy case is that $A \otimes_{\tilde A} N \xrightarrow{\sim} A \otimes_R N$ is N is an R-module $\ni AN=N$, so $N \in \mathrm{firm}(R,A)$ $\Longrightarrow N \in \mathrm{firm}(\tilde A, A)$. Converse is that if $M \in \mathrm{firm}(\tilde A, A)$, then M has unique R-mod. st. ext. A-mod str. & same as above.

Try your h-unital case now.

$$\begin{array}{ccc} & & R \\ & \nearrow & \\ A & & \\ & \searrow & \\ & & \tilde A \end{array}$$

$$\tilde A/A \overset{L}{\otimes}_{\tilde A} A = \tilde A/A \overset{L}{\otimes}_{\tilde A} R \overset{L}{\otimes}_R A$$

$$\mathrm{Tor}_n^{\tilde A}(\tilde A/A, A) \Longleftarrow E^2_{pq} = \mathrm{Tor}_p^R\left(\mathrm{Tor}_q^{\tilde A}(\tilde A/A, R), A\right)$$

assume known that $R/A \overset{L}{\otimes}_R A = 0$

$\Longrightarrow$ $K \overset{L}{\otimes}_R A = 0$ for all A-nil R-modules

rest seems inductive.

5. A ideal in R, ~~...~~ M an R-module.

$$\mathrm{Tor}_p^R(R/A, M) = 0 \quad \forall p \le n$$

$$\iff \mathrm{Tor}_p^{\tilde{A}}(\mathbb{Z}, M) = 0 \quad \forall p \le n.$$

Lemma: $\mathrm{Tor}_p^R(R/A, M) = 0 \quad \forall p \le n$

$\iff \exists$ res. $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$

with F_n R-flat and $AF_p = F_p \quad \forall p \le n$.

Prf: $(\Leftarrow)$ can use this res. to compute Tor_p $p \le n$

$(\Rightarrow)$ induction on n.

If $n = 0$. $\mathrm{Tor}_0^R(R/A, M) = 0$ means $M = AM$

$\Rightarrow \exists\ F \twoheadrightarrow M$ with F firm flat.

$$0 \to M' \to F \to M \to 0$$

$$\to M'/AM' \to 0$$

Let P be a proj. resol. of M. Then

$\mathrm{Tor}_p^R(R/A, P) = 0$ for $p \le n$ means $H_p(P/AP) = 0$ $p \le n$.

P/AP projective over R/A acyclic for $p \le n$.

means ~~...~~ $\exists\ \bar{h}$ defined on P/AP such that

$[d, \bar{h}] = 1$ in degrees $\le n$. lift $\bar{h}$ to h on P

get $[d, h] = 1 - f$ where $f_p : P_p \to AP_p$ $p \le n$.

Now take $\varinjlim(P \xrightarrow{f} P \xrightarrow{f} \cdots) = F$. flat complex

~~equipped with~~ resolving M.

6. Where are we ??

So if $\mathrm{Tor}_p^R(R/A, A) = 0$ $\quad p \le n$, then

$\exists$ $F_n \to \cdots \to F_0 \to A \to 0$ $\quad F_p$ firm flat.

so it's also true over $\tilde{A}$.

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firm rings

firm ring

why not review firm cover of an idempotent ring

A idempotent call a hom. $B \xrightarrow{w} A$ a covering when $\underline{B = B^2}$, $\underline{w \text{ surj}}$, $B\,\mathrm{Ker}(w) = \mathrm{Ker}(w)\,B = 0$.

$$0 \to K \to B \to A \to 0$$

B naturally an A-bimodule

$$B \otimes_B B = B \otimes_A B \xrightarrow{\sim} B \otimes_A A$$

$$\downarrow$$

$$B \otimes_B B \xrightarrow{\sim} B \otimes_B A \simeq B \otimes_A A \simeq A \otimes_A A$$

$$\begin{tikzcd} B \otimes_B B \arrow[r, "\sim"] \arrow[d] & A \otimes_A A \arrow[d] \\ B \arrow[r] & A \end{tikzcd}$$

firm cover

define covering of $A = A^2$ to be idemp B equipped with surj hom $w: B \to A$ such that $BK = KB = 0$ where $K = \mathrm{Ker}(w)$. ~~except that~~

ex. $A \otimes_A A \xrightarrow{w} A$ $\qquad (a_1 \otimes a_2)(a_3 \otimes a_4) = a_1 a_2 \otimes a_3 a_4$

~~etc.~~

$$a_1 a_2 \sum a_t' \otimes a_t'' = a_1 a_2 \otimes \sum a_t' a_t''$$

7. ~~Define~~ Given cover $w : B \to A$ define

$A \otimes_A A \xrightarrow{h} B \qquad a_1 \otimes a_2 \xrightarrow{h} b_1 b_2 \qquad w(b_i) = a_i$.

well-defined surjective hom. $\qquad a_3 \otimes a_4 \xrightarrow{h} b_3 b_4$

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = a_1 a_2 \otimes a_3 a_4 \xrightarrow{h} b_1 b_2 b_3 b_4 .$$

Conclusion coverings of A are equiv. to subgroups of $\operatorname{Ker}\{A \otimes_A A \to A\} \simeq HB_2(A)$. ~~first Hochschild homology~~

If $B \to A$ covering why is $\mathcal{M}(B) \simeq \mathcal{M}(A)$.

more generally, we have equivalence on left mods when $B/I = A \qquad \underline{IB} = 0$.

$$B \otimes_B N \xrightarrow{\sim} N$$

$B^{(2)} \to B \to \tilde{B}$

theory of Morita equivalence.

$\mathcal{W} : A \to B \qquad \mathcal{M}(A) \underset{\mathcal{W}^*}{\overset{\mathcal{W}_*}{\rightleftarrows}} \mathcal{M}(B)$

$$M \longmapsto B^{(2)} \otimes_A M \xrightarrow{\sim} B \otimes_A M \xrightarrow{\wr} \tilde{B} \otimes_A M$$

$$A^{(2)} \otimes_A N \longleftarrow N$$

adjunction ~~map~~ isom.

$$\operatorname{Hom}_A(M, A^{(2)} \otimes_A N) \qquad \operatorname{Hom}_B(B \otimes_A M, N)$$

$$\xrightarrow{\sim} \operatorname{Hom}_A(M, N) = \operatorname{Hom}_B(\tilde{B} \otimes_A M, N)$$

$\alpha : \tilde{B} \otimes_A A^{(2)} \otimes_A N \longrightarrow N \qquad b \otimes a_1 \otimes a_2 \otimes n \mapsto b\,w(a_1 a_2) n$

$\beta : M = A^{(2)} \otimes_A M \longrightarrow A^{(2)} \otimes_A \tilde{B} \otimes_A M$

$$a_1 a_2 m \longmapsto a_1 \otimes a_2 \otimes 1 \otimes m$$

8. bimodules are $\tilde{B}\otimes_A A^{(2)}$, $A^{(2)}\otimes_A \tilde{B}$

when ~~is~~ w induces a meq.

assume $u_!$ fully faithful.

$$A^{(2)} \xrightarrow{\sim} A^{(2)}\otimes_A \tilde{B}\otimes_A A^{(2)}$$

equivalent to $A \xrightarrow{w} \tilde{B}$ nil isin wrt $A\otimes A^{op}$

want $ABA = A$ and $K = \mathrm{Ker}(w)$ to sat $AKA = 0$

You want α isin.

$$\tilde{B}\otimes_A A^{(2)}\otimes_A B^{(2)} \xrightarrow{\sim} B^{(2)}$$

$B\tilde{B}\bar{A}B = B^2$

$B\bar{A}B = B.$

conversely if $B\bar{A}B = B$, then what method? YES.

$$\underbrace{\tilde{B}\otimes_A A^{(2)}\otimes_A \tilde{B}}_{\text{ring}} \xrightarrow[\text{onto}]{} \tilde{B}$$

$b_1\otimes a_1\otimes a_2\otimes b_2 \mapsto b_1 w(a_1) \otimes w(a_2) b_2$

~~To make~~

~~$b\otimes a_1$~~

$$(b_1\otimes a_1\otimes a_2\otimes b_2)(b_3\otimes a_3\otimes a_4\otimes b_4)$$

too tricky

there's another idea.

$$\underbrace{A \longrightarrow \bar{A}}\subset B$$

$\bar{A} = A/I$ where $AIA = 0$.

~~Necessarily~~

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, two heads] & A/AI \arrow[d, two heads] \\ A/IA \arrow[r, two heads] & A/I \end{tikzcd}$$

A/I

kernel I/AI killed by A

kernel I/IA killed by A^{op}

left ideal

$\bar{A} \subset \bar{A}B$

$\cap$ $\cap$ right ideal

$B\bar{A} \subset B$

9. Left ideal case $A \subset B$ $BA = A$, $AB = B$

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix} \qquad \begin{aligned} \mathcal{M}(A) &\xrightarrow{\sim} \mathcal{M}(B) \\ M &\longmapsto A \otimes_A M = M \\ B \otimes_B N &\longleftarrow\!\shortmid\ N \end{aligned}$$

So when A is a left ideal in B gen. B ~~A~~ identity between firm A + firm B-modules

$\underbrace{A}\otimes_A M = M$ autom. B operates.

can you prove the equivalence in this case.

$w_!(M) = B \otimes_A M \xleftarrow{\sim} A \otimes_A M = M$ since $A \subset B$ is A^{op}-nil iso.

what I want to do is to identify things properly. Main thm. Given

$$\begin{tikzcd} & \mathcal{M}(A) \arrow[dl, "P\otimes_A -"'] \arrow[dr, "P'\otimes_A -"] & \\ \mathcal{M}(B) \arrow[rr, "w_!"'] \arrow[urr, phantom, "\overset{\theta}{\Nearrow}"] & & \mathcal{M}(B') \end{tikzcd} \qquad \theta : B' \otimes_B P \longrightarrow P'$$

Left ideal inc. $A \overset{w}{\subset} B$ $BA \subset A$, $B \subset AB$

In this case $\mathcal{M}(A) = \mathcal{M}(B)$ in a def. sense

and I need to check this agrees with $w_!$, w^*.

$w_!(M) = B \otimes_A M \longleftarrow A \otimes_A M$

No just that $w_!$ is an equiv.

general situation here is $A \xrightarrow{f} B$ where

A is a left B-module, f B-mod map ?

10. Adjunction of $w_!$, w^*

$$\mathrm{Hom}_A(M, A^{(2)}\otimes_A N) \xrightarrow{\sim} \mathrm{Hom}_A(M,N)$$
$$= \mathrm{Hom}_B(\tilde{B}\otimes_A M, N)$$
$$\xrightarrow{\sim} \mathrm{Hom}_B(B\otimes_A M, N)$$

---

go over main result.

two categories 1. firm dual pairs. $(P, Q, \phi : Q\otimes P \to A)$

2. $B, F : \mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$

map $(B, F) \longrightarrow (B', F')$ $\quad w : B \to B' \quad \theta : w_! F \xrightarrow{\sim} F'$.

Anyway ~~we have not~~ you need a functor

$(P, Q, \phi) \longmapsto B = P\otimes_A Q \;,\; F = P\otimes_A -$

$$\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \to \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix} \quad \text{obvious}$$

but then you have to construct

$\theta : \tilde{B}'\otimes_B P\otimes_A M \xrightarrow{\sim} P'\otimes_A M$

where does this come from? obvious

Start with $v : Q \longrightarrow Q'$ $\qquad v(aq) = a\,v(q)$

v is a B^{op}-nil isom. $\qquad v(qb) = v(q)\,w(b)$

$$v(q) = 0 \implies q\,p_1 q_1 = v(q)\,u(p_1)\,q_1 = 0 \text{ so } qB = 0$$

$$q'\,w(pq) = \underbrace{q'\,u(p)}_{\in A}\,v(q) = v(q'u(p)\,q)$$

$$\therefore\quad Q\otimes_B P\otimes_A M \xrightarrow{\sim} Q'\otimes_B P\otimes_A M$$

$$P'\otimes_A M = P'\otimes Q\otimes P\otimes M \xrightarrow{\sim} P'\otimes_A Q'\otimes_B P\otimes_A M = B'\otimes_B P\otimes_A M$$

11. $$p'qp \otimes m \longmapsto p'v(q) \otimes p \otimes m$$

$$b'\,[redacted]\,u(p) \otimes m \longleftarrow [redacted]_{b'} \otimes p \otimes m$$

Yes!!

other direction $\quad Q \overset{G}{\otimes}_B B^{(2)} \otimes_B N' \simeq Q' \overset{G'}{\otimes}_{B'} N'$

adjoint. maps. What is adjoint to $\theta : \underbrace{B' \otimes_B P \otimes_A}_{F} M \to \underbrace{P' \otimes_A}_{F'} M$?

meaning.
$$\mathrm{Hom}(Fx, Y) = \mathrm{Hom}(X, GY)$$
$$\uparrow$$
$$\mathrm{Hom}(F'x, Y) = \mathrm{Hom}(X, G'Y)$$

$\theta : F \to F'$ has $\theta^t : G' \to G$

$$G'Y \xrightarrow{\beta.G'} GFG'Y \xrightarrow{G.\theta.G'} GF'G'Y \xrightarrow{G.\alpha} GY$$

$$Q' \otimes_{B'} N' \qquad v(q)\overset{w}{(}b_1 b_2) b' u(p) q' \otimes n'$$
$$(q b_1 b_2 b_3 p) q' \otimes n'$$
$$\|$$
$$Q \otimes_B B^{(2)} \otimes_B B' \otimes_B P \otimes_A Q' \otimes_{B'} N' \qquad q \otimes b_1 \otimes b_2 \otimes \overset{w}{(}b_3) \otimes p \otimes q' \otimes n'$$
$$\downarrow$$
$$Q \otimes_B B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N' \qquad q \otimes b_1 \otimes b_2 \otimes w(b_3) u(p) \otimes q' \otimes n'$$
$$\|$$
$$Q \otimes_B B^{(2)} \otimes_B N' \qquad q \otimes b_1 \otimes b_2\, (b_3) \otimes u(p) q' n'$$

seems like $(q b_1 b_2 p) q' \otimes n' \longmapsto q \otimes b_1 \otimes b_2 \otimes u(p) q' n'$

$$v(q b_1 b_2) u(p) q' \otimes n \longleftarrow$$

12. $Q \xrightarrow{v} Q'$ is B^{op}-nil iso.

$P \longrightarrow P'$ is B nil iso.. ?

$B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$

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$\cancel{Q \otimes_B B^{(2)} \xrightarrow{\sim} Q' \otimes_B B^{(2)}}$

$Q \otimes_B B^{(2)} \otimes_B P$

$p \otimes q_1 p_1 q_2 p_2 \otimes m \qquad p q_1 \otimes p_1 q_2 \otimes u(p_2) \otimes m$

$P \longrightarrow P' \qquad P \otimes_A M \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A M$

$B/I = A \qquad M = Q \otimes_B P \otimes_A M \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B P' \otimes_A M$

$Q' \otimes_{B'} N' = Q' \otimes_{B} P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} Q \otimes B^{(2)} \otimes P' \otimes_A Q' \otimes_{B'} N'$

$\cancel{P \otimes_A A^{(2)} \otimes_A}$

$B \otimes_B P \otimes_A M$

$$\begin{tikzcd} B \otimes_B P \arrow[r] \arrow[d, "\cdot q p"'] & P \arrow[dl, "p'"] \\ B \otimes P & \end{tikzcd}$$

$p' q \otimes p$

$b \otimes p' \longmapsto b p'$

p'

$b p' q \otimes p$

$p' q \otimes p \longmapsto p'(qp)$

---

told $\qquad M = A^{(2)} \otimes_A M$

$$\begin{tikzcd} \overset{q_1}{Q} \otimes_B \overset{p_1}{P} \otimes_A \overset{q_2}{Q} \otimes_B \overset{p_2}{P} \arrow[r] \arrow[d] & A \otimes_A A \arrow[d] \arrow[dl, dashed, Rightarrow] \\ Q \otimes_B P \arrow[r] & A \end{tikzcd}$$

$(q_1 p_1) q_2 \otimes p_2 = q_1 \otimes p_1 (q_2 p_2)$

13. Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $QP = A$, $PQ = B$

$$\begin{array}{ccc} \mathcal{M}(A) & \simeq & \mathcal{M}(B) \\ M & \longmapsto & P \otimes_A M \\ Q \otimes_B N & \longleftarrow & N \end{array}$$

$$B \otimes_B P \longrightarrow P$$

$$\left(\sum b_i \otimes p_i\right) q p = \left(\sum b_i p_i\right) q \otimes p$$

~~PQP~~ $BP = PQP = PA$

$\Rightarrow (P/BP)A = 0.$

$$Q \otimes_B P \longrightarrow A$$

$$\sum (q_i \otimes p_i) q p = \left(\sum q_i p_i\right) q \otimes p$$

---

module theory so what?? Given an A-bimod map $Q \otimes P \longrightarrow A$ the image is an ideal QP such that $AQP = QPA = QP$. Then $QPQP = Q$

7/15 I have to work hard to get basic result. ~~started~~

~~[illegible]~~

What are your main results??

[description of meghoms: $w : A \to B$ is a meghom iff —

~~equivalence of categories~~

vaguely: description of firm rings meg to A

precisely two cats are equivalent

first cat consists of $(B, F : m(A) \xrightarrow{\sim} m(B))$

2nd ———— (P, Q, φ) dual pairs.

~~[illegible]~~

14. Organizing this stuff. Suppose you have

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

you have to prove that $B \xrightarrow{w} B'$ is a morphism.
Can prove w is a $B \otimes B^{op}$-mil isom.

$$w(b) = 0. \Rightarrow$$

$$0 = v(q)\, w(b)\, u(p) = v(qb)u(p) = qbp$$

$$\Rightarrow p_1 q b p q_1 = 0 \Rightarrow BbB = 0.$$

$$\underbrace{u(p_1)v(q_1)}_{\in A} p' \underbrace{q' \, u(p_2)}_{\in A} v(q_2) = u\big(p_1 v(q_1) p' q' u(p_2)\big) v(q_2)$$

$$= w\big(p_1 v(q_1) p' q' u(p_2) q_2\big)$$

$$\therefore \quad w(B)\, B'\, w(B) \subset w(B)$$

$$B'\, w(B)\, B' = B' \quad ?$$

$$\omega$$

OKAY.

$$p_1' q_1' \, p_2' q_2' \, p_3' q_3' \, p_4' q_4'$$

$$w(B)B'w(B) = \overbrace{u(P)v(Q)}^{\in A} \overbrace{P'Q'}^{A} u(P)v(Q)$$

$$\cancel{w(B)B'w(B)} \subset \cancel{\ldots} \quad u(PA'A)v(Q) \subset w(B). \quad v(Q)u(P)$$

$$B' = B'B'B' = B'P'\underbrace{A}_{QBP}Q'B'$$

$$B' = P'Q'P'Q' = P'AQ' = P'v(Q)u(P)Q'$$

$$B' = P'Q'P'Q' = P'AQ' = P'AAQ'$$

$$\cancel{= P'QPQP}$$

$$= P'v(Q)u(P)v(Q)u(P)Q'$$

$$\subseteq B'\, w(B)\, B'$$

15. So what next?!

$w(b) = 0$ $\qquad$ $\cancel{0 = v(Q)\,w(b)\,u(P)}$

$$BbB = \cancel{v(Q)\,u(P)\,b\,v(Q)\,u(P)}$$

$$\cancel{= u(P)\,v(Q)\,b\,u(P)\,v(Q)}$$

$$BbB = PQbPQ$$

$w : B \longrightarrow B'$ $\qquad$ $w(b) = 0$.

$p q^b p_1 q_1$ $\qquad$ $q b p_1 = v(qb)\,u(p_1)$

$\qquad\qquad = v(q)\,w(b)\,u(p_1)$.

$BbB = P\underbrace{QbP}_{\in A}Q$. But $QbP \subset Q'\,\underset{0}{\underset{\shortparallel}{w(b)}}\,P'$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad B' \otimes_B P \otimes_A M \qquad P' \otimes$$

Given

$$\begin{tikzcd}
 & \mathcal{M}(A) \arrow[dl, "P\otimes_A -"'] \arrow[dr, "P'\otimes_A -"] & \\
\mathcal{M}(B) \arrow[rr, "w_!"] & & \mathcal{M}(B')
\end{tikzcd}$$

$\theta : B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M$

$w_! F \xrightarrow{\sim} F'$ $\qquad$ $w_! \simeq F'G$

$\qquad\qquad\qquad$ $w^* \simeq FG'$

$\cancel{\mathrm{Hom}_{B'}(B' \otimes_B P}$?

$$\begin{tikzcd}
 & \mathcal{M}(A) \arrow[dl, "F"'] \arrow[dr, "F'"] & \\
\mathcal{M}(B) \arrow[rr, "w_!"] & & \mathcal{M}(B')
\end{tikzcd}$$

$\theta : w_! F \xrightarrow{\sim} F''$ $\quad$ given

idea: Take

$$\mathrm{Hom}_{B'}(B' \otimes_B P, P') = \mathrm{Hom}_B(P, B^{(2)} \otimes_B P')$$

$$= \mathrm{Hom}_B(P, P')$$

16. So what? $\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$ describes eq. F'.

$w_!$ des. by $\begin{pmatrix} B & B \otimes_B B' \\ B' \otimes_B B & B' \end{pmatrix}$

so $w_! F$ des. by $\begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix}$

now θ must set up an isom

$$\begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix} \simeq \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

But?

$$\begin{pmatrix} B & B \\ B & B \end{pmatrix} \longrightarrow \begin{pmatrix} B & B \otimes_B B' \\ B' \otimes_B B & B' \end{pmatrix}$$

so you have a canonical

Suppose you have $\theta : w_! F \xrightarrow{\sim} F'$, get

$w_! \xrightarrow{\sim} F'G$ ~~...~~. So you're in situation

of ~~$B' \otimes_B B$~~ an isom of Mcats

$$\begin{pmatrix} B & B \otimes_B B' \\ B' \otimes_B B & B' \end{pmatrix} \simeq \begin{pmatrix} B & P \otimes_A Q' \\ P' \otimes_A Q & B' \end{pmatrix}$$

now you have an obvious hom. of $\begin{pmatrix} B & B \\ B & B \end{pmatrix}$ into former. get

$$\begin{pmatrix} B & B \\ B & B \end{pmatrix} \longrightarrow \begin{pmatrix} B & P \otimes_A Q' \\ P' \otimes_A Q & B' \end{pmatrix}$$

$B' \otimes_B B \xrightarrow{\sim} P' \otimes_A Q$ yields $B' \otimes_B P \xrightarrow{\sim} P'$

17. Anyway, what happens? ~~OKAY~~.

You seem to have a map of pairs

$$(B, B, \mu) \longrightarrow (P' \otimes_A Q,\ P \otimes_A Q', \ldots)$$

But now translate back to A

$$(B \otimes_B P,\ Q \otimes_B B, \ldots) \longrightarrow (P' \otimes_A Q \otimes_B P,\ Q \otimes_B P \otimes_A Q', \ldots)$$

$$\parallel$$

$$(P',\ Q').$$

Thus I am after is equiv of cats ~~So you have~~ ~~the yield~~

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \rightsquigarrow m(A) \underset{G}{\overset{F}{\rightleftarrows}} m(B)$$

$$\downarrow$$

$$C' = \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix} \rightsquigarrow$$

$$\begin{tikzcd} & m(A) \arrow[dl, "F"'] \arrow[dr, "F'"] & \\ m(B) \arrow[rr, "w_!"'] \arrow[rr, phantom, bend left, "\theta\nearrow"] & & m(B') \end{tikzcd}$$

The only thing I should check is that, ~~in the~~ in the affine case, there is no simple argument ~~that~~ that constructs $\theta : w_! F \xrightarrow{\sim} F'$. Thus if you have $u : P \to P'$, $v : Q \to Q'$. how do you get

$B' \otimes_B P \xrightarrow{\sim} P'$? Answer:

$$\parallel$$

$$P' \otimes_A Q' \otimes_B P \qquad Q' \otimes_{B'} P' \cancel{\otimes_A} Q' \otimes_B P \xrightarrow{\sim} Q' \otimes_{B'} P'$$

$$\parallel$$

$$Q' \otimes_B P \xrightarrow{\sim} A$$

$$Q' \longleftarrow Q$$

~~[illegible]~~ $P \longrightarrow P'$ is a B-nil isom.

$$\Rightarrow Q \otimes_B P \xrightarrow{\sim} Q \otimes_B P'$$

18. $\quad P \to P'$ $\quad$ B-mod isom

$$\Longrightarrow \quad P \xrightarrow{\sim} B \otimes_B P' \quad \Longrightarrow \quad B' \otimes_B P \xrightarrow{\sim} B' \otimes_B B \otimes_B P' \xrightarrow{\sim} P'$$

$Q \longrightarrow Q'$ $\quad$ B^{op}-mod isom.

$$Q \otimes_B B \xrightarrow{\sim} Q' \otimes_B B$$

$P \to P'$ $\quad$ B-mod isom

$$\Longrightarrow \quad B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$$\Longrightarrow \quad B' \otimes_B \underbrace{B^{(2)} \otimes_B P \otimes_A M}_{P \otimes_A M} \xrightarrow{\sim} B' \otimes_B B^{(2)} \otimes_B P' \otimes_A M$$

M ferm $\cancel{\text{f}}$

$$\wr\wr \qquad P' \otimes_A M$$

~~Avail~~ $\quad Q \longrightarrow Q'$ $\quad$ B^{op} mod-isom

$$\Longrightarrow \quad Q \otimes_B B^{(2)} \xrightarrow{\sim} Q' \otimes_B B^{(2)}$$

$$\Longrightarrow \quad Q \otimes_B B^{(2)} \otimes_B B' \xrightarrow{\sim} Q' \otimes_B B^{(2)} \otimes_B B'$$

~~[illegible]~~ $\cancel{N' \in \mathcal{M}(B')}$

$$\cancel{Q \otimes_B B^{(2)} \otimes_B N' \xrightarrow{\sim} Q' \otimes_B B^{(2)} \otimes_B N'}$$

$$M = Q \otimes_B P \otimes_A M \xrightarrow{\sim} Q' \otimes_B P \otimes_A M$$

$$P' \otimes_A M \xrightarrow{\qquad\sim\qquad} \cancel{B}' \otimes_A Q' \otimes_B P \otimes_A M$$

~~several ways to pr~~

$P \xrightarrow{u} P'$ $\quad$ B-mod isom. $\quad$ $p \otimes q^*$ $\qquad$ $\phi(p') = \cancel{p \otimes p'}$

$= p(v(q)p')$

$$\begin{array}{ccc} P & \xrightarrow{u} & P' \\ {\scriptstyle p\otimes q^*}\downarrow & \swarrow{\scriptstyle \phi} & \downarrow{\scriptstyle p\otimes q^*} \\ P & \xrightarrow{u} & P' \end{array}$$

$\phi(u(p_1))$

$= p\, v(q) u(p_1) = pq p_1$ $\qquad \overset{u}{\longmapsto} \underbrace{u(p)\, v(q)}_{pq} p'$

19. $\therefore\ B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$

$$\begin{array}{ccc}
\Rightarrow B' \otimes_B B^{(2)} \otimes_B P \otimes_A M & \xrightarrow{\sim} & B' \otimes_B B^{(2)} \otimes_B P' \otimes_A M \\
\downarrow s & & \downarrow s \\
B' \otimes_B P \otimes_A M & & P' \otimes_A M
\end{array}$$

if M ferin ~~$P \otimes_A M$~~

here are $w, w^* = 1$.

$$b' \otimes b_1 b_2 p \otimes m \longmapsto b' \underbrace{v(b_1 b_2) v(p)}_{v(b_1 b_2 p)} \otimes m$$

$$\boxed{b' \otimes p \otimes m \longmapsto b' v(p) \otimes m.}$$

this is different from my argument:

$Q \xrightarrow{v} Q'$ $\quad B^{op}$-mil isom.

$$\begin{array}{ccc}
\Rightarrow\ P' \otimes_A Q \otimes_B P \otimes_A M & \xrightarrow{\sim} & P' \otimes_A Q' \otimes_B P \otimes_A M \\
s \downarrow & & \downarrow s \\
P' \otimes_A M & & B' \otimes_B P \otimes_A M
\end{array}$$

$$\boxed{p' \otimes q p m \longmapsto p' v(q) \otimes p \otimes m}$$

other isomorphism between $Q' \otimes_{B'} N'$ and $Q \otimes_B B^{(2)} \otimes_B N'$

~~$Q' \otimes_{B'} N' \longleftarrow Q' \otimes_{B'} P' \otimes_A Q' \otimes_{B'} P' \otimes_A Q' \otimes_{B'} N'$~~

$$Q' \otimes_{B'} N' \xleftarrow{\sim} (Q \otimes_B P) \otimes_A (Q \otimes_B P) \otimes_A (Q \otimes_B P) \otimes_A Q' \otimes_B N'$$

$$\downarrow$$

$$Q \otimes B \otimes B \otimes N'$$

$q_1 p_1 q_2 p_2 q_3 p_3 \otimes q' n' \longmapsto q_1 \otimes p_1 q_2 \otimes p_2 q_3 \otimes u(p) q' n'$

20. My old version uses $Q \otimes_B B \otimes_B B \otimes_B P \otimes_A M \xrightarrow{\sim} M$ for M firm

~~You construct~~ Steps: Start with $C \to C'$

get $\theta : B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M$ $\qquad w_! F \xrightarrow{\sim} F'$

and $\xi : Q \otimes_B B^{(2)} \otimes_B N' \xrightarrow{\sim} Q' \otimes_{B'} N'$ $\qquad G w^* \xrightarrow{\sim} G'$

compatible with the 'pairings'

$$\begin{tikzcd} w_! F G w^* \arrow[r] \arrow[d, "\theta \cdot \xi"] & 1 \arrow[d, equal] \\ F'G' \arrow[r, "\sim"] & 1 \end{tikzcd} \qquad \begin{tikzcd} G w^* w_! F \arrow[r] \arrow[d, "\xi \cdot \theta"] & 1 \arrow[d, equal] \\ GF \arrow[r, "\sim"] & 1 \end{tikzcd}$$

$$\begin{tikzcd} B' \otimes_B P \otimes_A Q \otimes_B B^{(2)} \otimes_B N' \arrow[d, "\theta \cdot \xi"] \\ P' \otimes_A Q' \otimes_{B'} N' \arrow[d] \\ N' \end{tikzcd}$$

$b' \otimes p \otimes q \otimes b_1 \otimes b_2 \otimes n' \mapsto b' \otimes pqb_1 \otimes b_2 \otimes n'$

$\downarrow$ $\qquad b' w(pqb_1 b_2) n'$

$b' u(p) \otimes v(qb_1 b_2) \otimes n'$

$\downarrow$

$b' u(p) v(qb_1 b_2) n'$

$b' w(pqb_1 b_2) n'$

$\cancel{\ }\ Q' \otimes_{B'} P' \otimes_A M$

$\uparrow \xi \otimes \theta$

$v(qb_1 b_2) \otimes u(p) w(b_3) \otimes m$

$\uparrow$

next page

$$\begin{array}{ccc}
Q \otimes_B B^{(2)} \otimes_B B' \otimes_B P \otimes_A M & & q\otimes b_1 \otimes b_2 \otimes w(b_3) \otimes p \otimes m \\
\uparrow s & & \uparrow \\
Q \otimes_B P \otimes_A M & & q \otimes b_1 b_2 b_3 p \otimes m \\
\downarrow s & & \downarrow \\
M & & (q b_1 b_2 b_3 p) m
\end{array}$$

$\xi \otimes \theta$

formula for

$$M \xrightarrow{\sim} A^{(2)} \otimes_A B' \otimes_A M$$

$$a_1 a_2 a_3 m \longmapsto a_1 \otimes a_2 \otimes w(a_3) \otimes m.$$

~~So you know~~ But you need inverse. The point is that $w(A) B w(A) = w(A)$. so that

$a_1 \otimes a_2 a_3$ YES!

doesn't change. The only interesting point is that one can calculate the transpose of θ and find it's ξ^{-1}.

$\theta : \tilde{F} \longrightarrow F'$ $\qquad G' \longrightarrow \tilde{G}\tilde{F}G' \longrightarrow \tilde{G}F'G' \longrightarrow \tilde{G}$

$\theta : B' \otimes_B P \otimes_A M \longrightarrow P' \otimes_A M$ $\qquad b' \otimes p \otimes m \longmapsto b' u(p) \otimes m$

$$\begin{array}{ccc}
Q' \otimes_{B'} N' & & q b_1 b_2 b_3 p\, q' \otimes n' \\
\downarrow & & \downarrow \\
Q \otimes_B B^{(2)} \otimes_B B' \otimes_B P \otimes_A Q' \otimes_{B'} N' & & q \otimes b_1 \otimes b_2 \otimes w(b_3) \otimes p \otimes q' \otimes n' \\
\downarrow & & \downarrow \\
Q \otimes_B B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N' & & q \otimes b_1 \otimes b_2 \otimes w(b_3) u(p) \otimes q' \otimes n'
\end{array}$$

22.

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d, mapsto] \\
Q \otimes_B B^{(2)} \otimes_B N' & q \otimes b_1 \otimes b_2 b_3 \otimes u(p) q' n'
\end{tikzcd}$$

So $\theta^t : \; q b_1 b_2 p q' \otimes n' \longmapsto q \otimes b_1 \otimes b_2 \otimes u(p) q' n'.$

Similarly for ξ^t $\qquad \xi : Q \otimes_B B^{(2)} \otimes_B N' \longrightarrow Q' \otimes_{B'} N'$

$G w_2^* \longrightarrow G' \qquad\qquad q \otimes b_1 \otimes b_2 \otimes n' \longmapsto v(q b_1 b_2) \otimes n'$

$F' \longrightarrow F' G w^* w_! F \longrightarrow F' G' w_! F \longrightarrow w_! F.$

$$\begin{tikzcd}
P' \otimes_A M \arrow[d] & p' \otimes q b_1 b_2 b_3 p m \arrow[d, mapsto] \\
P' \otimes_A Q \otimes_B B^{(2)} \otimes_B B' \otimes_B P \otimes_A M \arrow[d] & p' \otimes q \otimes b_1 \otimes b_2 \otimes w(b_3) \otimes p \otimes m \arrow[d, mapsto] \\
P' \otimes_A Q' \otimes_B B' \otimes_B P \otimes_A M \arrow[d] & p' \otimes v(q b_1 b_2) \otimes w(b_3) \otimes p \otimes m \arrow[d, mapsto] \\
B' \otimes_B P \otimes_A M & p' v(q b_1 b_2 b_3) \otimes u(p) \otimes m
\end{tikzcd}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$

$p' \otimes q p m \longmapsto p' v(q) \otimes p \otimes m$ ~~Hence proof enough~~

~~Anyway life goes on~~ ~~OKAY~~ Next?? ~~YES~~

This should take care of the functor from $\mathcal{C}^A$ to pairs (B, F).

23. Now the converses. Suppose given? Where have you used surjective pairings.

here used $B = PQ$.

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "P\otimes"'] & P' \arrow[d, "P\otimes"] \arrow[dl, "\phi"'] \\ {} & {} \end{tikzcd}$$

$u : P \to P'$ B-nil isom.

$$\phi_{B,q}(p') = p(v(q)p')$$

$$B^{(2)} \otimes_B P \otimes_A M \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A M$$

$Q \to Q'$ B^{op}-nil isom

$$\begin{array}{ccc} P' \otimes_A Q \otimes_B P \otimes_A M & \xrightarrow[\sim]{P' \otimes_A} & Q' \otimes_B P \otimes_A M \\ \| & & \| \\ P' \otimes_A M & \xrightarrow{\sim} & B' \otimes_B P \otimes_A M \end{array}$$

here have used $B' = P'Q'$.

$w : B \to B'$ $\quad$ meq hom.

$w(b)$ $\qquad p_1 q_1 b p_2 q_2$

$$q_1 b p_2 = v(q_1 b) u(p_2) = v(q_1) w(b) u(p_2) = 0$$

$$\underbrace{u(p_1) v(q_1) p'}_{a_1} \underbrace{q' u(p_2)}_{a_2} v(q_2) = u(p_1 a_1 a_2) v(q_2) = w(p_1 a_1 a_2 q_2).$$

$$B' = P'Q' = P'Q'P'Q' = P'AQ'$$

$$B' = P'Q'P'Q'P'Q' = P'AAQ'$$

$$= P' v(Q) \underbrace{u(P)\; v(Q)}_{w(B)} u(P) Q'$$

24. category. $B, F : \mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$

$$\mathrm{Hom}((B,F),(B',F')) = \left\{ (w,\theta) \;\middle|\; \begin{array}{l} w : B \to B' \text{ homom.} \\ \theta : w_! F \xrightarrow{\sim} F' \end{array} \right\}$$

$$B \xrightarrow{w_1} B' \xrightarrow{w_2} B''$$

$$\theta_1 : w_{1!} F \longrightarrow F' \qquad \theta_2 : w_{2!} F' \longrightarrow F''$$

$w_{2!}($

$$B \xrightarrow{w} B' \xrightarrow{w'} B''$$

$$\theta : w_! F \xrightarrow{\sim} F', \qquad \theta' : w'_! F' \xrightarrow{\sim} F''$$

$\cancel{w'_!(\theta)}$

$$\begin{tikzcd}
w'_! w_! F \arrow[r, "w'_!(\theta)"] & w'_! F' \arrow[r, "\theta'"] & F'' \\
(w'w)_! F \arrow[u, "\wr"]
\end{tikzcd}$$

$\cancel{w''_! w'_!}$

$$\begin{tikzcd}
w''_! w'_! w_! F \arrow[r, "w''_! w'_!(\theta)"] & w''_! w'_! F' \arrow[r, "w''_!(\theta')"] & w''_! F'' \arrow[r, "\theta''"] & F''' \\
(w''w')_! w_! F \arrow[u, "\wr"] \arrow[r, "(w''w')_!(\theta)"] & (w''w')_! F' \arrow[u, "\wr"] \arrow[urr] & & \\
(w''w'w)_! F \arrow[u, "\wr"] \arrow[urrr, bend right]
\end{tikzcd}$$

$w''w'$ tog with this is $(w'',\theta'')(w',\theta')$

$w''w'w$ tog with this is $((w'',\theta'')(w',\theta'))(w,\theta)$

$$(\theta'', w'')(\theta', w) = \cancel{\theta'' w''_!(\theta') c_{w''w}}$$

$$= (\theta''$$

$$(\theta', w')(\theta, w) = \theta' w'_!(\theta)\, c_{w',w}$$

25.

$$
\begin{tikzcd}
w''_! w'_! w_! F \arrow[r, "w''_! w'_! \theta"] \arrow[d, "\wr"] & w''_! w'_! F' \arrow[r, "w''_! \theta'"] & w''_! F'' \arrow[r, "\theta''"] & F''' \\
w''_! (w'w)_! F \arrow[d, "\wr"] \arrow[urr, bend left, "w'' \text{ of } (w',\theta')(w,\theta)"] & & & \\
(w''w'w)_! F \arrow[uurrr, bend left, "(w'',\theta'')((w',\theta')(w,\theta))"'] & & &
\end{tikzcd}
$$

So the only question is

$$
\begin{tikzcd}
(w''w'w)_! \arrow[r] \arrow[d] & (w''w')_! w_! \arrow[d] \\
w''_! (w'w)_! \arrow[r] & w''_! w'_! w_!
\end{tikzcd}
$$

and this should be the usual cocycle condition.

Continue: Consider B idemp rings + homs.
claim have a $\mathcal{M} \longrightarrow \mathcal{B}$. fibres are $\mathcal{M}(B)$.

$$
\begin{array}{ccc}
\mathcal{M}(A) & & \mathcal{M}(B) \\
A & \xrightarrow{w} & B
\end{array}
$$

$$\mathrm{Hom}_B(M, N)$$

$$\mathrm{Hom}_A(M, \underbrace{A^{(2)} \otimes_A N}_{w^* N}) = \mathrm{Hom}_B(M,N) = \mathrm{Hom}_B(\underbrace{B \otimes_A M}_{w_! M}, N)$$

So what? Assoc. $(\theta', w')(\theta, w) = (\theta' w'_!(\theta) c_{w',w}, w'w)$

$$(\theta'', w'')((\theta', w')(\theta, w)) = (\theta'' w''_!(\theta' w'_!(\theta) c_{w',w}) c_{w'', w'w}, w''w'w)$$

$$= (\theta'' w''_!(\theta') w''_! w'_!(\theta) w''_!(c_{w',w}) c_{w'',w'w}, \ldots)$$

$$((\theta'', w'')(\theta', w'))(\theta, w) = (\theta'' w''(\theta') c_{w'',w'}, w''w')(\theta, w)$$

$$= \theta'' w''_!(\theta') c_{w'',w'} (w''w')_!(\theta) c_{w''w',w} \quad w''w'w)$$

26. $w'_! w'_!(\theta)(c_{w'', w'} \cdot w'_!)$ So you need to find a suitable language.. How? I seem to have a crazy mess. Standard cocycle stuff.

Try to go back over category stuff. Structure on the cat of B, F.

Question. Consider cat of firm rings & morphisms. Take a component. What is the homotopy type?? ~~What is the~~ Over this component consider the 2-groupoid assoc. to A, B $\mathrm{Equiv}(m(A), m(B))$. get fibre cat in groupoids. To each A you have $m(A)$.

first: base cat $\overset{B}{=}$ component of firm rings + morphisms.

~~fibre~~ cat fibre $m(B)$ over B.

What does this "fibre" bundle tell you ~~about~~ B? actually its a g-fibration Fix a fibre say A look at corresp. principal bundle. this has $B \longmapsto \mathrm{Equiv}(m(A), m(B))$. acted on by self-equiv. of $m(A)$. ~~Anyway what are it~~

~~Fully faithful functor~~

you have cat of $B, F: m(A) \xrightarrow{\sim} m(B)$ equiv. pairs B, P where P is a firm (B,A)-bimodule which is invertible – not a cat?

$w: B \longrightarrow B$ pairs B, F

maps $(B, F) \xrightarrow{(\theta, w)} (B', F') \xrightarrow{(\theta', w')} (B'', F'')$

$(\theta', w')(\theta, w) = (\theta' w'_!(\theta) c_{w', w}, w'w)$

27.

$$(B,P) \xrightarrow{(\theta,w)} (B',P') \xrightarrow{(\theta',w')} (B'',P'')$$

$\theta : B' \otimes_B P \xrightarrow{\sim} P'$ $\qquad$ $\theta' : B'' \otimes_{B'} P' \xrightarrow{\sim} P''$

$w'_! w_!(P)$

$$\begin{tikzcd} B'' \otimes_{B'} B' \otimes_B P \arrow[r, "B''\otimes\theta", "\sim"'] & B'' \otimes_{B'} P' \arrow[r, "\theta'", "\sim"'] & P'' \\ B'' \otimes_B P \arrow[u, "\wr"] \arrow[urr, bend right] & & \end{tikzcd}$$

$(w'w)_!(F)$

yes! ~~Okay~~ $\qquad$ $Q \otimes \underline{P} \to A$

Given B, P where P invertible

$Q = A^{(2)} \otimes_A \mathrm{Hom}_B(P, B)$.

$$\mathrm{Hom}_B(P \otimes_A M, N) = \mathrm{Hom}_A(M, \mathrm{Hom}_B(P, N))$$

$$= \mathrm{Hom}_A(M, A^{(2)} \otimes_A \mathrm{Hom}_B(P, N))$$

$Q \otimes_B P =$

Start with $\theta : B' \otimes_B P \xrightarrow{\sim} P'$ $\qquad$ $\theta : w_! F \xrightarrow{\sim} F'$

$(\theta^t)^{-1} : G w^* \xrightarrow{\sim} G'$. Another way: Uniqueness of q-inv. says $\exists! \ \xi : Q \otimes_B B' \xrightarrow{\sim} Q'$ compatible with pairings i.e.

$$\begin{tikzcd} (Q \otimes_B B') \otimes_{B'} (B' \otimes_B P) \arrow[r, "\sim"] \arrow[d] & Q \otimes_B P = A \arrow[d, equal] \\ Q' \otimes_{B'} P' \arrow[r, "\sim"] & A \end{tikzcd}$$

28. and

$$
\begin{array}{ccccc}
(B' \otimes_B B) \otimes_A (Q \otimes_B B') & \xrightarrow{\sim} & B' \otimes_B B \otimes_B B' & \xrightarrow{\sim} & B' \\
\big\downarrow {\scriptstyle \theta \otimes q} & & & & \| \\
P' \otimes_A Q' & & = & & B'
\end{array}
$$

Define θ^t. $\quad \theta : w_! F \longrightarrow F' \quad \theta^t : G' \longrightarrow G w^*$

$$
\begin{array}{ll}
Q' \otimes_{B'} N' & q b_1 b_2 b_3 p q' \otimes n' \\
\big\downarrow \wr & \\
\cancel{Q \otimes_B P \otimes_A Q' \otimes_{B'} N'} & \\
Q \otimes_B B^{(2)} \otimes_B B' \otimes_B P \otimes_A Q' \otimes_{B'} N & q \otimes b_1 \otimes b_2 \otimes w(b_3) \otimes p \otimes q' \otimes n' \\
\big\downarrow \wr & \\
Q \otimes_B B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N' & \\
\big\downarrow \wr & \\
Q \otimes_B B^{(2)} \otimes_B N' & q \otimes b_1 \otimes b_2 b_3 \otimes u(p) q' n'
\end{array}
$$

$$
\begin{array}{lll}
Q' & (q b p) q' & q p q' \\
\big\downarrow & & \big\downarrow \wr \\
Q \otimes_B B' \otimes_B P \otimes_A Q' & q \otimes w(b) \otimes p \otimes q' & \\
\big\downarrow & & \\
Q \otimes_B B' & q \otimes u(bp) q' & q \otimes u(p) p'
\end{array}
$$

29. This is OKAY.

$$\begin{tikzcd}
Q' \arrow[d, "\wr"] & G' \arrow[d, "\wr"] & (gbp)g' \\
Q \otimes_B B' \otimes_B P \otimes_A Q' \arrow[d, "\wr"] & G w^* w_! F G' \arrow[d, "Gw^*.\theta.G'"] & g \otimes w(b) \otimes p \otimes g' \\
Q \otimes_B P' \otimes_A Q' \arrow[d] & G w^* F' G' \arrow[d, "\wr"] & g \otimes p \otimes g' \\
Q \otimes_B B' & G w^* & gb \otimes u(p) g'
\end{tikzcd}$$

What should be the conceptual proof. You have

$$\begin{tikzcd}
& \cdot \arrow[dl, "F"'] \arrow[dr, "F'"] & \\
\cdot \arrow[rr, "w_!"'] & & \cdot
\end{tikzcd} \qquad \theta : w_! F \xrightarrow{\sim} F'$$

you compute the M contexts belonging to $w_! F$

$$C_\perp = \begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix}$$

pairings

$$(Q \otimes_B B') \otimes_{B'} (B' \otimes_B P) = Q \otimes_B P \xrightarrow{\sim} A$$

$$(B' \otimes_B P) \otimes_A (Q \otimes_B B') \longrightarrow B' \otimes_B B \otimes_B B' = B'.$$

Get an isom then

$$\begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix} \simeq \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

So what else. Want zilch.

30. $\theta : w_! F \xrightarrow{\sim} F'$

$\tilde{u} : B' \otimes_B P \xrightarrow{\sim} P'$ $\qquad u : P \to P'$

$\tilde{u} : b' \otimes p \longmapsto b' u(p)$

$u(bp) = \tilde{u}(b \otimes p)$

$$\underbrace{P = B \otimes_B P \xrightarrow{w \otimes 1}}\; B' \otimes_B P \xrightarrow{\tilde{u}} P'.$$

$$\begin{array}{lll} G' & Q' & (gbp)g' \\ & \wr\| & \\ Gw^* w_! F G' & Q \otimes_B B' \otimes_B P \otimes_A Q' & g \otimes w(b) \otimes p \otimes g' \\ & \wr\| & \\ Gw^* F' G' & Q \otimes_B P' \otimes_A Q' & g \otimes u(bp) \otimes g' \\ & \| & \\ Gw^* & Q \otimes_B B' & gb \otimes u(p) g' \end{array}$$

So you get $Q' \sim Q \otimes_B B'$

$(gbp)g' \longmapsto g \otimes u(p) g'$

inverse has form $g \otimes b' \longmapsto v(g) b'$. Conclude then

$$(gp)g' \longmapsto g \otimes u(p) g' \longmapsto \boxed{v(g) u(p) g' = (gp) g'}$$

$v(g) u(p) \underbrace{g' p'}_{a} = gp \underbrace{g' p'}_{a}$

$v(g) u(p\cancel{a}) = gp\cancel{a}$

$$g_1 p_1 g \otimes b' \longmapsto g_1 p_1 v(g) b' \longmapsto \boxed{g_1 \otimes u(p_1) v(g) b' = g_1 p_1 g \otimes b'}$$

$g_1 \otimes u(p_1) v(g) b' = g_1 \otimes w(p_1 g) b'$

$w(pg_1) u(p_1) v(g) b' = w(pg_1) w(p_1 g) b'$

$u(p_1) v(g) b' = w(p_1 g) b'$ $\qquad b' = w(b)$

$u(p_1) v(g b) = w(p_1 g b)$

31.

$$\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix} = \begin{pmatrix} A \\ P' \end{pmatrix} \otimes_A \begin{pmatrix} A & Q' \end{pmatrix}$$

$$\begin{pmatrix} A & Q\otimes_B B' \\ B'\otimes_B P & B' \end{pmatrix} = \begin{pmatrix} A \\ B'\otimes_B P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q\otimes_B B' \end{pmatrix}$$ ~~YES~~

~~YES! What no~~

$$B'\otimes_B P \simeq P'$$

$$Q\otimes_{B'} B'\otimes_B P \otimes_A Q'$$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q\otimes_B B' \\ B'\otimes_B P & B' \end{pmatrix}$$

$$\begin{pmatrix} a & qb \\ bp & pq \end{pmatrix} \longmapsto \begin{pmatrix} a & q\otimes w(b) \\ w(b)\otimes p & w(b) \end{pmatrix}$$

$$w(b_1 p\, q\, b_2) \overset{?}{=} (w(b_1)\otimes p)(q\otimes w(b_2))$$
$$\| $$
$$w(b_1)\, w(pq)\, w(b_2)$$

$$q b_2\, b_1 p \overset{?}{=} (q\otimes w(b_2))(w(b_1)\otimes p)$$
$$\|$$
$$q\otimes w(b_2 b_1)\otimes p$$
$$\cap$$

$$A = Q\otimes_B B\otimes_B B\otimes P \twoheadrightarrow Q\otimes_B B'\otimes_B B'\otimes_B P$$

whole business is hard.

$B = AB \subset B^2$

$A = A^2 \subset BA \subset A$

$AB = B$

A A

B B

32. ~~Point to Alad~~ Have $Q\otimes_B P \xrightarrow{\sim} A$ and a morphism $w: B \to B'$ ~~[illegible]~~ of firm rings. Then get $(Q\otimes_B B', B'\otimes_B P, \ldots)$
with

$$\begin{tikzcd}
Q\otimes_B B'\otimes_{B'} B'\otimes_{B'} P = Q\otimes_B B'\otimes_B B'\otimes_B B\otimes P \\
Q\otimes_B P \arrow[u, "\wr"']
\end{tikzcd}$$

$$\begin{tikzcd}
Q\otimes_B B'\otimes_B B'\otimes_B P \arrow[d, "\wr"] \\
Q\otimes_B B'\otimes_B P \\
Q\otimes_B B\otimes_B P \arrow[u, "\wr"'] \arrow[d, "\wr"] \\
A
\end{tikzcd}$$

$\tilde{u}: B'\otimes_B P \xrightarrow{\sim} P'$ $\qquad w_! F \xrightarrow{\sim} F'$

$G' \xrightarrow{\sim} Gw^*$

$Q' \xrightarrow{\sim} Q\otimes_B B'$

$qpq' \longmapsto q\otimes u(p)q'$

q

$q\otimes b'_1\otimes b'_2\otimes p \qquad B\otimes_B B'\otimes_{B'} B'\otimes_B B$

direct firm version: $P \xrightarrow{u} P' \quad Q \xrightarrow{v} Q'$

$\ni v(q)u(p) = qp$. Then constructs compatible isomorphism

$B'\otimes_B P \xrightarrow{\sim} P' \qquad Q\otimes_B B' \cong Q'$

Proof: $u: P \longrightarrow P'$ is B-nil iso means that

$\Longrightarrow P = B\otimes_B P \xrightarrow{\sim} B\otimes_B P'$ $\qquad \mathcal{M}(B) \rightleftarrows \mathcal{M}(B')$

$\Rightarrow B'\otimes_B P \xrightarrow{\sim} P'$ by equivalence

33 Similarly $v : Q^* \longrightarrow Q'$ B^{op}-mil isom.

$$\Rightarrow Q = Q \otimes_B B \xrightarrow{\sim} Q' \otimes_B B$$

$$\Rightarrow Q \otimes_B B' \xrightarrow{\sim} Q' \otimes_B B \otimes_B B' = Q'$$

---

final step can either be done by

$$\tilde{u} : B' \otimes_B P \xrightarrow{\sim} P' \qquad \tilde{u}(b' \otimes p) = b' u(p)$$

$$u(bp) = \tilde{u}(w(b) \otimes p)$$

$$\tilde{v} : Q \otimes_B B' \xrightarrow{\sim} Q'$$

satisfying

$$\begin{array}{ccc} (B' \otimes_B P) \otimes_A (Q \otimes_B B') & \longrightarrow B' \otimes_B B \otimes_B B' \xrightarrow{\sim} & B' \\ \downarrow{\scriptstyle \tilde{u} \otimes \tilde{v}} & & \\ P' \otimes_A Q' & & \\ \shortparallel & & \\ B' & & \end{array}$$

$$\boxed{b_1' u(p) v(q) b_2' = b_1' w(pq) b_2'} \qquad b_i' = w(b_i)$$

$$\begin{array}{ccc} (Q \otimes_B B') \otimes_{B'} (B' \otimes_B P) & \xrightarrow{\sim} Q \otimes_B P & = A \\ \downarrow{\scriptstyle \tilde{u} \otimes \tilde{v}} & & \\ Q' \otimes_{B'} P' & & \\ \shortparallel & & \\ A & & \end{array}$$

$$\boxed{v(q) w(b_1) w(b_2) u(p) = q b_1 b_2 p}$$

$$\shortparallel$$

$$v(qb_1) u(b_2 p) = q b_1 b_2 p \quad \checkmark$$

37. $\begin{pmatrix} B & B \\ B & B \end{pmatrix} \longrightarrow \begin{pmatrix} B & B\otimes_B B' \\ B'\otimes_B B & B' \end{pmatrix} = \begin{pmatrix} B \\ B' \end{pmatrix} \otimes_B (B \otimes B')$

$$B\otimes_B B' = P\otimes_A Q\otimes_B B' = P\otimes_A Q'$$

$$B'\otimes_B B = B'\otimes_B P\otimes_A Q = P'\otimes_A Q$$

$$B'\otimes_B B' = P'\otimes_A Q'\otimes_B P'\otimes_A Q'$$

Suppose $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ M context. grading ~~...~~ wrt $\mathbb{Z}_2\times\mathbb{Z}_2$?

what about $C\otimes_C C$ should this be a M. context.

~~Make strictly firm~~ Assume $A = A^2 = QP$, $B = B^2 = PQ$

$$B = B^2 = PQPQ = PAQ$$

$\cancel{\begin{pmatrix} A & AQ \\ PA & B \end{pmatrix}}$

Suppose C is idempotent. then.

$$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q) = \begin{pmatrix} A\otimes_A A & A\otimes_A Q \\ P\otimes_A A & P\otimes_A Q \end{pmatrix}$$

should be strictly firm.

$$\cancel{P\otimes_A A\otimes_A Q \longrightarrow P\otimes_A \tilde{A}\otimes_A Q}$$

$$Q = AQ$$

$$0 \longrightarrow K \longrightarrow A\otimes_A Q \longrightarrow \tilde{Q} \longrightarrow 0$$

$$P\otimes_A K \longrightarrow P\otimes_A A\otimes_A Q \longrightarrow P\otimes_A \tilde{Q} \longrightarrow 0$$

$\shortparallel$

$PA\otimes_A K = 0.$

35. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s idem

$$P \otimes_A Q \xrightarrow{\sim} B^{(2)}$$

?

$$q_1 p_1 (q_2 \otimes p_2) = (q_1 \otimes p_1) q_2 p_2$$

$$\begin{tikzcd} \overset{q_1}{Q} \otimes_B \overset{p_1}{P} \otimes_A \overset{q_2}{Q} \otimes_B \overset{p_2}{P} \arrow[r] \arrow[d] & Q \otimes_B P \arrow[d] \arrow[dl, dashed, "\varphi"'] \\ A \otimes_A A \arrow[r] & A \end{tikzcd}$$

$$K \longrightarrow Q \otimes_B P \longrightarrow A \longrightarrow 0 \quad \text{homom.}$$

~~idempotent~~

$$\left(\sum q_i \otimes p_i\right) qp = \sum q_i p_i (q \otimes p)$$

---

useful result seems to be that to convert a s idem $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ to s firm ~~...~~ it ~~...~~ suffices to use tensor. rep.

$$A' = A \otimes_A A = Q \otimes_B P$$

$$P' = P \otimes_A A = B \otimes_B P, \quad Q' = A \otimes_A Q$$

$$\begin{tikzcd} A \otimes_A Q \otimes_B P \arrow[r, "\sim"] \arrow[d, "s"] & A \otimes A \\ Q \otimes_B P & \end{tikzcd}$$

$$\begin{tikzcd} A \otimes_A Q \otimes_B B \arrow[r, "\sim"] \arrow[d, "s"] & A \otimes_A Q \arrow[d] \\ Q \otimes_B B \arrow[r] & Q \end{tikzcd}$$

36.

$$\begin{pmatrix} A & AB \\ BA & B \end{pmatrix} \qquad \begin{aligned} &A = A^2 = AB^2A \\ &B = BA^2B \subseteq BAB \\ &\cancel{A \quad A^3 \subset ABA \subset A} \\ &A^4 \subset AB^2A \subset AB\cancel{B}A \subset A \end{aligned}$$

$$\cancel{\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad A = A^2 = QP \qquad B = B^2 = PQ}$$

7/18 Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with $\begin{aligned} A &= A^2 = QP \\ B &= B^2 = PQ \end{aligned}$

$w(b) = 0$

$$q_1 b p_2 = w(q_1 b) u(p_2) = v(q_1) w(b) u(p_2) = 0$$

OKAY getting clearer.

w: meghan tells us that $m(B) \underset{w^*}{\overset{w_!}{\rightleftarrows}} m(B')$

composite equiv. $m(A) \longrightarrow m(B) \longrightarrow m(B')$

described by Mcont. $\begin{pmatrix} A & B' \otimes_B Q \\ P \otimes_B B' & B' \end{pmatrix}$

pairings.

$$\cancel{(P \otimes_B B') \otimes_{B'} (B' \otimes_B Q) \xleftarrow{\sim} A \otimes_B B \otimes_B B \otimes_B Q \xrightarrow{\sim} A}$$

$$\cancel{p \otimes w(b_1) \otimes w(b_2) \otimes q \longleftarrow p \otimes b_1 \otimes b_2 \otimes q \longmapsto p b_1 b_2 q}$$

$$(B' \otimes_B P) \otimes_A (Q \otimes_B B') \xrightarrow{\sim} B'$$

$$b_1' \otimes p \otimes q \otimes b_2' \longmapsto b_1 w(pq) b_2'$$

$$(Q \otimes_B B') \otimes_{B'} (B' \otimes_B P) \sim A$$

$$q \otimes w(b_1) \otimes w(b_2) \otimes p \longleftarrow q b_1 b_2 p$$

37 Given (P, Q) and $w : B \to B'$ a morphism then ~~[illegible]~~ you get a dual pair $B' \otimes_B P$, $Q \otimes_B B'$ with pairing

So much to do.

$$\left(Q \otimes_B B'\right) \otimes_{B'} \left(B' \otimes_B P\right)$$
$$\|$$
$$Q \otimes_B B' \otimes_B P \xleftarrow{\sim} Q \otimes_B B \otimes_B P = A.$$

You can consider (P, Q) as a dual pair over B.

Have P, Q over B YES

Morita equiv.

The idea is to take $P \in \mathcal{M}(B)$, $Q \in \mathcal{M}(B^{op})$ together with ~~[illegible]~~ $\underbrace{P \otimes Q}_{\text{naturally of rt exact functor from}} \twoheadrightarrow B$.

~~Interesting~~ question. You know that $P \otimes_A -$ has right adjoint $N \mapsto A^{(2)} \otimes_A \operatorname{Hom}_B(P, N)$

$$\operatorname{Hom}_B(P \otimes_A M, N) = \operatorname{Hom}_A(M, \operatorname{Hom}_B(P, N))$$
$$= \operatorname{Hom}_A(M, A^{(2)} \otimes_A \operatorname{Hom}_B(P, N))$$

What does it mean that this functor ca

What would you like to know?? Basic idea is to consider $N \mapsto \operatorname{Hom}_B(P, N)$. This is a nasty functor of N but the image of ~~[illegible]~~ mult. by $a \in A$ might be nicer. This should relate to Roos' thm proof.

Anyway

$$\begin{pmatrix} B & B \\ B & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} B & B \otimes_B B' \\ B' \otimes_B B & B' \end{pmatrix}$$

Still after the logic ~~Feb~~

equiv. of cats.

Given $B, F : \mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$

complete F to $F, G,\ \varepsilon : FG \xrightarrow{\sim} 1,\ \eta : GF \xrightarrow{\sim} 1$

What we do have is an obvious functor from

~~Idea: Stick to Mcart with A fixed.~~

~~Consider the next result!!!~~

$B, \Phi : \mathcal{M}(A) \xleftrightarrow{\sim} \mathcal{M}(B)$

$$\begin{tikzcd} & \mathcal{M}(B) \arrow[d, "w_!"] \\ \mathcal{M}(A) \arrow[ur, "F"] \arrow[r, "F'"'] & \mathcal{M}(B') \end{tikzcd}$$

fibred category

~~fib~~ have fibred + cofibred cat. over firm rgs. + maghams.

Idea: Object over B is (B, F) a map $(B, F) \to (B', F')$ over $w : B \to B'$

is a natural ~~transf.~~ isom. $\theta : w_! F \xrightarrow{\sim} F'$

i.e. $\forall A$ a

---

so we have over the cat of idemp. rings B

this cof cat of firm modules.

$$\mathrm{Hom}(N, N')_w = \mathrm{Hom}_B(N, N') \simeq \mathrm{Hom}_{B'}(B' \otimes_B N, N')$$

$$\uparrow s$$

$$\Leftarrow \mathrm{Hom}_B(N, B \overset{(2)}{\otimes}_B N')$$

~~$\mathrm{Hom}_{\mathcal{M}(B)}$~~

a map $f : w_! N \longrightarrow N'$ same as a B-mod.

39 $w_! N \longrightarrow N'$ same as $f: N \longrightarrow N'$

satisfying $f(bn) = w(b) f(n)$.

$\theta : w_! N \longrightarrow N'$ $\qquad$ $\theta' : w'_! N'$

OKAY

$$M \longrightarrow M' \longrightarrow M''$$

$$B \xrightarrow{f} B' \xrightarrow{g} B''$$

$\theta : f_! M \longrightarrow M'$

$\xi : g_! M' \longrightarrow M''$

$$(M, B) \xrightarrow{(\theta, f)} (M', B') \xrightarrow{(\xi, g)} (M'', B'')$$

$$\cancel{(g, \xi)(f, \theta) = (gf,}$$

$$\cancel{(\theta, f)(\xi, g) = (\theta f_!(\xi), fg)}$$

$$(\xi, g)(\theta, f) = (\xi\, g_!(\theta)\, c_{g,f}, gf)$$

$$(gf)_! M \xrightarrow[\sim]{c_{g,f}} g_! f_! M \xrightarrow{g_!(\theta)} g_! M' \xrightarrow{\xi} M''$$

M

$$f_! M \xrightarrow{\theta} M'$$

$$g_! f_! M \xrightarrow{g_!(\theta)} g_! M' \xrightarrow{\xi} M''$$

$$\begin{tikzcd}
M & f_! M \arrow[d, "\theta"] & g_! f_! M \arrow[d, "g_!(\theta)"] \\
 & M' & g_! M' \arrow[d, "\xi"] \\
 & & M''
\end{tikzcd}$$

$$B \xrightarrow{f} B' \xrightarrow{g} B''$$

40 cofibred cat of modules.

$B \rightsquigarrow \mathcal{M}(B)$ functor

next want ~~F: $\mathcal{M}$~~ to replace $\mathcal{M}(B)$ by an object of a ~~$\mathcal{B}$~~ functor $F: \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ i.e. a family of objects $F(M)$ $\forall M \in \mathcal{M}(A)$. Leads to replacing $\mathcal{M}(B)$ by Hom$(\mathcal{M}(A), \mathcal{M}(B))$.

~~Leads to replacing~~ ~~Fibre over $\mathcal{M}(P$~~

Fibre over B is $\mathcal{M}(B)$.

$$\text{rtcartfun}(\mathcal{M}(A), \mathcal{M}(B)) = \text{firm}(B \otimes A^{op})$$

then have $B \rightsquigarrow \mathcal{M}(B \otimes A^{op})$

$$(B,P) \xrightarrow{(w,\alpha)} (B',P') \qquad \text{OKAY.}$$

it means a hom. $B \xrightarrow{w} B'$ and $B' \otimes_B P \to P'$.

criterion that P has an adjoint bimodule

i.e. Q [redacted] together with

$$A \xrightarrow{\alpha} Q \otimes_B P \qquad \underbrace{P \otimes_A Q \xrightarrow{\beta} B}_{Q \to A \otimes_A \mathrm{Hom}_B(P,B)} \quad \text{etc.}$$

${}_B P_A$

Return: ~~key idea~~ Basic idea: $B \rightsquigarrow \mathrm{Eq}(\mathcal{M}(A), \mathcal{M}(B))$ "functor" of B get a ~~...~~ category of (B,F). ~~Bimod. ...~~

cat of B, P

$$\begin{tikzcd} & \mathcal{M}(A) \arrow[dr, "F'"] \arrow[dl, shift left, "G"] & \\ \mathcal{M}(B) \arrow[ur, shift left, "F"] \arrow[rr, shift left, "w_!"] & & \mathcal{M}(B') \arrow[ul, bend right, "G'"'] \arrow[ll, shift left, "w^*"] \end{tikzcd}$$

$Gw^* \xrightarrow{\sim} G'$

$Q \otimes_B B' \simeq Q'$

$G'w_!$

$Q' \otimes_{B'} B \otimes_B B$

`\def\mapsto{\mapstochar\rightarrow}`

41. Given $(B; F, G) \xrightarrow{(w, \theta, \xi)} (B'; F', G') \longrightarrow (B''; F'', G'')$

$\theta : w_! F \xrightarrow{\sim} F'$

point is $(w_!, w^*)(F, G)$ yields Mcont

$$\begin{pmatrix} A & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

composite map

<u>Question</u>: Given ${}_B P_A$ can you tell when P is invertible. You can forget A first, and then ~~P~~ should generates $\mathcal{M}(B)$. Then go through Roos analysis. Put $R = \mathrm{Hom}_B(P, P)^{op}$ and you get $\mathrm{mod}(R) \twoheadrightarrow \mathcal{M}(B)$

$M \longmapsto P \otimes_B M$.

It seems you plan to remove the old $C \to C'$ version.

~~you make~~

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

[illegible] $PA \hookrightarrow P$

$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ assume $A = A^2 = QP$ $B = B^2 = PQ$

$C^2 = \begin{pmatrix} A & AQ + QB \\ PA + BP & B \end{pmatrix}$ But $AQ = QPQ = QB$ and $PA = PQP = BP$.

So C^2 is strictly idempotent i.e.

$C \otimes_C C^2$ [illegible]

$C = C^2 \iff P = [illegible]\ PA$ and $Q \overset{A}{=} Q$

42 Start with $C \ni A = A^2 = QP \qquad B = B^2 = PQ$

Replace C by C^2 $= \begin{pmatrix} A & \overset{Q'}{AQ = QB} \\ \underset{P'}{PA = BP} & B \end{pmatrix}$

Then $AQ' = AAQ = AQ = Q'$, $Q'B = \cancel{B}\, Q'$

sim $P'A = \cancel{B}\, P'$

$P'Q' = BPQB = B^3 = B$

$Q'P' = AQPA = A^3 = A.$

Suppose C strictly idemp.

$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q)$ is strict.

$P \otimes_A M \qquad \cancel{P \otimes_A A^{(2)} = PA \otimes_A A}$

corresp func bimods are $B^{(2)} \otimes_B P \otimes_A A^{(2)}$

$$P \otimes_A A = B(P \otimes_A A)$$

$$p \otimes q_1 p_1 a = p q_1 (p_1 \otimes a)$$

$$P \otimes_A A^{(2)} = B \otimes_B P \otimes_A A = B^{(2)} \otimes_B P.$$

---

meg hom.

$B \otimes_A A^{(2)} \qquad A^{(2)} \otimes_A B$

how do I do this.

43. Once $PA = P$ $AQ = Q$, then the dual pair is $\begin{pmatrix} A^{(2)} \\ P\otimes_A A \end{pmatrix}$ $\begin{pmatrix} A^{(2)} & A\otimes_A Q \end{pmatrix}$

so the sfirm m cont is

$$\begin{pmatrix} A^{(2)} \\ P\otimes_A A \end{pmatrix} \otimes_A \begin{pmatrix} A^{(2)} & A\otimes_A Q \end{pmatrix} = \begin{pmatrix} A^{(2)} & A\otimes_A Q \\ P\otimes_A A & P\otimes_A Q \end{pmatrix}$$

example $\begin{pmatrix} A & AB \\ BA & B \end{pmatrix}$ assume $A = A^2$, $ABA \subset A$, $B = BAB$.

of $\begin{pmatrix} A^{(2)} & A\otimes_A AB \\ BA\otimes_A A & BA\otimes_A AB \end{pmatrix}$

So what else is there?

Assume C ~~s idempotent~~

then
$$\begin{pmatrix} P\otimes_A A \leftarrow B\otimes_B P\otimes_A A \xrightarrow{\sim} B\otimes_B P \\ Q\otimes_B B \xleftarrow{\sim} A\otimes_A Q\otimes_B B \xrightarrow{\sim} A\otimes_A Q \\ P\otimes_A Q \xrightarrow{\sim} B^{(2)} \quad Q\otimes_B P \xrightarrow{\sim} A^{(2)}. \end{pmatrix}$$

It seems I felt that I needed to know that A, B, P, Q firm + surj pairings $\Longrightarrow$ sfirm.

Yes because given (P, Q) firm dual pair so that $Q\otimes P \to A$ is surjective how do you get that P, Q B firm and $Q\otimes_B P \xrightarrow{\sim} A$?

7/21 write out & meg homs theory of meg.

meghom.
$$\mathrm{Hom}_A(M, A^{(2)}\otimes_A N) \cong \mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_A(\tilde{B}\otimes_A M, N) \simeq \mathrm{Hom}_A(B\otimes_A M, N)$$

$w_!(M) = B\otimes_A M$ $\quad w^*(N) = A^{(2)}\otimes_A N$

$\alpha : B\otimes_A A^{(2)}\otimes_A M \longrightarrow M$ $\quad b\otimes a_1\otimes a_2\otimes m \mapsto b\,w(a_1a_2)m$

$\beta : M \longrightarrow A^{(2)}\otimes_A B\otimes_A M$ $\quad a_1a_2a_3m \mapsto a_1\otimes a_2\otimes w(a_3)\otimes m$

44. $\omega_!$ ff $\Leftrightarrow$ β isom. for $M = A^{(2)}$

$A^{(5)} \xrightarrow{\sim} A^{(2)} \otimes_A B \otimes_A A^{(2)}$ $\Leftrightarrow$ $A \xrightarrow{\omega} B$ $A \otimes A^{op}$-mil isoms

$A\,\mathrm{Ker}(\omega)\,A = 0$ and $\omega(A)\,B\,\omega(A) \subseteq \omega(A)$

ω equiv $\Leftrightarrow$ β also isom

$$B \otimes_A A^{(2)} \otimes_A B^{(2)} \xrightarrow{\sim} B^{(2)}$$

$\Rightarrow$ $B\,\omega(A)^2 B^2 = B^2$ i.e. $B\,\omega(A)\,B = B$

conv: assume this holds. Factor ω into $A \to \omega(A) \subset B$.

can suppose 1) ω surj. $B = A/I$ where $AIA = 0$.

2) $A \subset B$, $ABA \subset A$, $BAB = B$.

$A \longrightarrow A/$ $A \otimes_A N \xrightarrow{\sim} N$

four cases.

1) $w : A \longrightarrow A/I$ canon. surj, I ideal, $IA = 0$.

2) —— $AI = 0$.

3) $w : A \hookrightarrow B$ inclusion of subring s.t. $BA \subset A$, $AB = B$.

4) —— $AB \subset A$, $BA = B$.

case 1) $A \otimes_A M \xrightarrow{\sim} M$ $IA = 0 \Rightarrow$ $IM = 0$ so M autom. an A/I-module

$\Vert$

$A/I \otimes_A M = A/I \otimes_{A/I} M$ $\therefore\ m(A) = m(A/I)$ via rest. of scalars. w^* isom

2) $m(A^{op}) = m(A^{op}/I)$

3) $A \otimes_A M \xrightarrow{\sim} M$ B

needs to be worked out – variant of the mag arg.

45. $w : A \longrightarrow B$ factors

$$A \to A/AK \longrightarrow \bar{A} \subset \bar{A}B \subset$$

$$A \xrightarrow{(1)} A/AK \xrightarrow{(2)} A/K = \bar{A} \xhookrightarrow{(3)} \bar{A}B \xhookrightarrow{(4)} B\bar{A}B = B$$

$A \otimes_A M \xrightarrow{\sim} M \implies M$ has unique B-mod. st. such that $b(am) = (ba)m$

$$\begin{array}{ccc} B \otimes_B N & \xrightarrow{\sim} & N \\ \uparrow & & \uparrow \\ A \otimes_A B \otimes_B N & \xrightarrow{\sim} & A \otimes_A N \end{array}$$

$a \otimes b \longmapsto ab$

$A \otimes_B B \longrightarrow B \longrightarrow 0$

$(a \otimes b) a_1 b_1 = ab(a_1 \otimes b_1)$

~~$B \otimes_B A \otimes_A M \xrightarrow{\sim} B \otimes_B M$~~

M

$$\begin{array}{ccc} B \otimes_B A \otimes_A M & \xrightarrow{\sim} & B \otimes_B M \\ \downarrow & & \downarrow \\ A \otimes_A M & \xrightarrow{\sim} & M \end{array}$$

$B \otimes_B A \longrightarrow A \longrightarrow 0$

$(b \otimes a) a' = ba \otimes a'$

Take $(P_A, {}_AQ, \ Q \otimes P \longrightarrow A)$ given dual pair put $B = P \otimes_A Q$ etc. whence get Mcontext.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q \end{pmatrix}$$

need to see that B, P, Q B-given $Q \otimes_B P \xrightarrow{\sim} A$.

$a \otimes$

46.

$$\begin{array}{ccc} A\otimes_A B\otimes_B N & \xrightarrow{1\otimes\mu} & A\otimes_A N \\ \downarrow & & \downarrow \\ B\otimes_B N & \xrightarrow{\mu} & N \end{array}$$

$$\begin{array}{ccc} a\otimes b\otimes n & \mapsto & a\otimes bn \\ \downarrow & & \downarrow \\ ab\otimes n & \mapsto & abn \end{array}$$

Yes!! Lost again.

$$\begin{pmatrix} A & Q=B \\ P=A & B \end{pmatrix}$$

~~this way~~.

$A\subset B$ $\quad BA\subset A$

$AB=B$

for what ???

$$M \longmapsto A\otimes_A M = M$$

$$N = B\otimes_B N \longleftarrow N$$

M A-firm from $A\otimes_A M = M$ get

B action $b(am) = (ba)m$. Then

$$B\otimes_B A \longrightarrow A$$

A^{op} ~~ml~~-firm

$$B\otimes_B P \longrightarrow P$$

p'

$$B\otimes_B P \xrightarrow{p'q\otimes p} P$$

$(b\otimes p')\,\cancel{\otimes}\, qp = \cancel{(b p q\otimes p)}$

$b'pq\otimes p$

$b\otimes a b_1 q_1 = ba b_1\otimes q_1$

$b\otimes a q_1 = ba\otimes q_1$

$\cancel{B\otimes}$ $\quad \cancel{A\otimes_A N \to A\otimes_A}$

$\cancel{A}$ $\quad B\otimes_B M = B\otimes_B A\otimes_A M$

47. $A \subset R$ $\qquad \begin{pmatrix} A & R \\ A & R \end{pmatrix}$

$$\mathcal{M}(\tilde{A}, A) \qquad \mathcal{M}(R, A)$$

$$M \longmapsto A \otimes_{\tilde{A}} M$$

$$A \otimes_R N \longleftarrow\!\shortmid\; N$$

Start with $A \otimes_{\tilde{A}} M \xrightarrow{\sim} M$

get R action on M ✓ $\qquad r(am) = (ra)m$

is it firm $\quad A \otimes_A M \twoheadrightarrow A \otimes_R M$

$$ar \otimes_A a'm = ara' \otimes_A m = a \otimes ra'm$$

~~easy diss~~.

Suppose then that $\cancel{RA}$ $A \otimes_R N \longrightarrow N$.

~~Take $A \otimes_R$~~

$A \otimes_A M \xrightarrow{\sim} M \qquad b(am) = (ba)m$.

$$A \otimes_A M \twoheadrightarrow B \otimes_B M \longrightarrow M$$

$$b \otimes am$$
$$\shortparallel$$
$$ba \otimes m$$

$$\begin{tikzcd} B \otimes_B M \arrow[r, "\sim"] & M \\ A \otimes_A M \arrow[u] \arrow[ur, twoheadrightarrow] & \end{tikzcd}$$

$\cancel{B \otimes B}$

$$\begin{tikzcd} B \otimes_B A \otimes_A M \arrow[r] \arrow[d] & B \otimes_B M \arrow[d] \\ A \otimes_A M \arrow[r] & M \end{tikzcd}$$

48. $A \quad B \qquad QP = BA = A$

$A \quad B \qquad PQ = AB = B$

and then have

$$M \longmapsto A \otimes_A M = M$$

$$B \otimes_B N = N \longleftarrow\!\shortmid\; N$$

$M \in \mathcal{M}(A) \implies A \otimes_A M = M$ is B-module

why firm $\quad B \otimes_B A \longrightarrow BA = A$

because $\quad B \otimes_B P \longrightarrow P$ is A^{op}-mod isom.

need $\boxed{B \otimes_B A \longrightarrow A}$ is A^{op}-mod isomorphism

easier $\quad A \otimes_A M \twoheadrightarrow B \otimes_B M \longrightarrow M$ easy case

---

other $N \in \mathcal{M}(B)$ want N to be A-firm.

$$A \otimes_A N = A \otimes_A B \otimes_B N \xrightarrow{\sim} B \otimes_B N$$

~~need that $A \otimes_A B \longrightarrow B$ is B^{op}-mod isom.~~

---

rtcanfun.
$$\begin{tikzcd} F: \mathcal{M}(A) \arrow[r] & \mathcal{M}(B) \arrow[d] \\ \mathrm{mod}(\tilde{A}) \arrow[u] & \mathrm{mod}(\tilde{B}) \end{tikzcd}$$

$$M \longmapsto F(A^{(2)} \otimes_A M) = P \otimes_A M \qquad P = F(A^{(2)})$$

as $P \otimes_A -$ inverts A-mod isos $\quad P \otimes_A A \longrightarrow$

$$\mathcal{M}(A^{op}) \simeq \mathrm{rtcontfun}(\mathcal{M}(A), \mathcal{A}b)$$

$$P \longmapsto P \otimes_A -$$

49. a map in $\mathcal{M}(A)$ $M' \to M$ is inj. when kernel is nil.

$$B \otimes_B P \otimes_A A \xrightarrow{\sim} P$$

$$\Rightarrow \quad B^{(2)} \otimes_B P \otimes_A A^{(2)} \xrightarrow{\sim} B \otimes_B P \otimes_A A \xrightarrow{\sim} P$$

$$\Rightarrow \quad B \otimes_B P \xrightarrow{\sim} P \quad \text{and} \quad P \otimes_A A \xrightarrow{\sim} P.$$

$\therefore$ firm B, A-bimodule $=$ firm $(\tilde{B} \otimes \tilde{A}, B \otimes A)$

P flat. $\quad M' \to M$ inj in $\mathcal{M}(A)$

i.e. $0 \to K \to M' \to M$ ex. $AK = 0$

$$\Rightarrow \quad 0 \to P \otimes_A K \to P \otimes_A M' \to P \otimes_A M$$

$$P \otimes_A K = PA \otimes_A K = P \otimes_A AK = 0$$

P flat $\tilde{A}^{op}$
$\Rightarrow$ $P \otimes_A -$ exact $\mathcal{M}(A) \to \mathcal{A}b$.

next. $0 \to M' \to M$ in

P A^{op}-firm $\Rightarrow$ $P \otimes_A -$ inverts A-nullisoms.

$$0 \to M' \to M \to M'' \to 0$$

$AM' = 0$ $\quad P \otimes_A M' \to P \otimes_A M \to P \otimes_A M'' \to 0$

$$P \otimes_A M' = \underbrace{PA \otimes_A M'}$$

$AM'' = 0$.

$$\begin{tikzcd} A \otimes_A M' \arrow[r] \arrow[d] & A \otimes_A M \arrow[r] \arrow[d] \arrow[dl, dotted] & A \otimes_A M'' \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} P\otimes_A A\otimes_A M' \arrow[r] \arrow[d, "\wr"] & P\otimes_A A\otimes_A M \arrow[d, "\wr"] \arrow[dl, dotted] \\ P\otimes_A M' \arrow[r] & P\otimes_A M \end{tikzcd}$$

7/22 sections.

1. firm modules $\simeq$ modules / nil-modules
2. biloc. Serre subcats + Roos theorem.
3. independence of R.
4. firm bimodules + mcont functors.
5. adj functors assoc. to a ring homom.
6. morphism examples.

firm ~~xxxxxxx~~ rings

$\cancel{A}\, Q\otimes_B B = QB\otimes_B B = \overline{Q\otimes_B B}$

$q\otimes p'\otimes b$

normal. $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

$A^2 = QP$
$B^2 = PQ$

$\Rightarrow$ $PA = PQP = BP$
also $AQ = QB$.

$\therefore$ $C^2 = \begin{pmatrix} A & AQ = QB \\ PA = BP & B \end{pmatrix}$

s. idemp: $BPQB = B^3 = B$
$AQPA = A^3 = A$.

---

claim: If C s idem then the normalized mcont is

$$\begin{pmatrix} A\otimes_A A = Q\otimes_B P & A\otimes_A Q = Q\otimes_B B \\ P\otimes_A A = B\otimes_B P & B\otimes_B B = P\otimes_A Q \end{pmatrix}$$

51.

$$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q \end{pmatrix}$$

$$P' \otimes_A Q'$$ where $P' = P'A$, $Q' = AQ'$ and $Q'P' = A$.

This ↑ is a firm ring.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

~~$Q \otimes A$~~

$$P' = P \otimes_A A^{(2)}$$

$$Q' = \cancel{Q \otimes_B B^{(2)}}\ A^{(2)} \otimes_A Q$$

$$Q' \otimes P' \longrightarrow A^{(2)} \otimes_A Q \otimes P \otimes_A A^{(2)} \longrightarrow A^{(5)} \to A^{(2)}$$

$$P' \otimes Q' \longrightarrow P \otimes_A A^{(2)} \otimes_A A^{(2)} \otimes_A Q \to P \otimes_A Q \to B$$

$$P \otimes_A A^{(2)} \otimes Q \otimes_B B^{(2)} \longrightarrow B \otimes_B B^{(2)} \longrightarrow B^{(2)}$$

$$Q \otimes_B B^{(2)} \otimes P \otimes_A A^{(2)} \longrightarrow A \otimes_A A^{(2)} \longrightarrow A^{(2)}$$

Example. $Q \otimes P \longrightarrow A$ unital

$q_0 \otimes p_0 \longmapsto 1$

~~$(P, Q) = (A, A) \oplus ($~~

$$P = p_0 A \oplus P'$$

$$Q = A q_0 \oplus Q'$$

$$\sum q_i p_i = 1.$$

$$\begin{pmatrix} A & Q_0 \\ P_0 & P_0 \otimes_A Q_0 \end{pmatrix}$$

can be quite degenerate.

52. ~~[illegible]~~ $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ A

A firm P_A, ${}_AQ$ firm

$P = P \otimes_A A$ is B-firm sim Q_B firm.

$B = P \otimes_A Q$ ———— . $\therefore$ B-firm ring.

$Q \otimes_B P \longrightarrow A$ Know $Q \otimes_B \underbrace{P \otimes_A A}_{P} \xrightarrow{\sim} A$

Idea: $F : \mathrm{mod}(R) \longrightarrow \mathcal{Ab}$

right cont. $F(R) \otimes_R M \xrightarrow{\sim} F(M)$

$\underline{\mathrm{firm}(R^{op}, A^{op})}$ $=$ rtcontfun$(\mathcal{M}(R,A), \mathcal{Ab})$

$A \subset R$ $RA \subset A$

~~[illegible]~~ A ideal in R $\begin{pmatrix} \tilde{A} & R \\ A & R \end{pmatrix}$

F $\tilde{A}^{op}$ flat, $FA = A$.

~~$F \otimes_A N$ ———— $F \otimes$~~ ~~to see if M flat~~

~~[illegible]~~ M over $\tilde{A}$ $\longmapsto F' \otimes_A M$

$N \longmapsto F \otimes_A N \longrightarrow F \otimes_R N$

F A^{op} flat firm $F \otimes_A N \xrightarrow{\sim} F \otimes_R N$

$\therefore$ F A-flat ~~[illegible]~~ $(fa)r \otimes n = f(ar) \otimes n$

$\Longrightarrow F$ R-flat $= f \otimes arn$

assume F R-flat $0 \longrightarrow M' \longrightarrow M$ $= fa \otimes rn$

$\longrightarrow R \otimes_A M' \longrightarrow R \otimes_A M$

$$\sum r_i \otimes m_i' \longmapsto \sum r_i m_i' = 0.$$

$$a \cancel{a_1 a_2} \sum r_i \otimes m_i' = a_1 \otimes a_2 \sum_i r_i m_i' = 0.$$

$$\cancel{= \sum 1 \otimes a_i m_i}$$

$$a \sum r_i \otimes m_i' = \sum 1 \otimes a r_i m_i' = 0$$

$$0 \longrightarrow F \otimes_R K \longrightarrow F \otimes_A M' \longrightarrow F \otimes_A M$$

$$F \otimes_R K = 0$$

---

exact rt cont. = flat firm rt modules

$$P \rightsquigarrow P \otimes_R - : \text{firm}(R,A) \longrightarrow \text{Ab}$$

rt cont.
is exact. obvious from viewpt of $\text{firm}(R,A)$ as $\text{mod}(R)/\text{nil}$

Other direction Given F ~~functor~~ exact rt cont : $\text{firm}(R,A) \longrightarrow \text{Ab}$
compose with $A^{(2)} \otimes_R -$ to get $M \longmapsto F(A^{(2)} \otimes_R M)$

$$F(A^{(2)} \otimes (M)) = P \otimes_R M \qquad P = F(A^{(2)})$$

P must be firm. Why flat? $0 \to M' \to M$

$$\Rightarrow A^{(2)} \otimes M' \longrightarrow A^{(2)} \otimes M \quad \text{monic in } \mathcal{M}$$

~~firm modules~~ $P \otimes_A A \xrightarrow{\sim} P \Rightarrow P \otimes_A -$ inverts nil isos. $0 \to N' \to N \to N'' \to 0$

$$AN' = 0 \Longrightarrow P \otimes_A N' = PA \otimes_R N' = P \otimes_R AN' = 0$$

$$\Longrightarrow P \otimes_R N \xrightarrow{\sim} P \otimes_R N'' \qquad a(\sum a_i \otimes n_i)$$

$$AN'' = 0. \quad 0 \longrightarrow K \longrightarrow A \otimes_R N' \longrightarrow A \otimes_R N \longrightarrow 0$$

$AK = 0$

54.

$$\begin{tikzcd}
 & A\otimes_R N' \arrow[r] \arrow[d] & A\otimes_R N \arrow[r] \arrow[d] \arrow[dl, dashed] & A\otimes_R N'' \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

---

char. of firm module

$\mathrm{Tor}_p^R(\text{nil}, M) = 0 \qquad p = 0, 1.$

$\mathrm{Ext}_R^p(M, \text{nil}) = 0 \qquad p = 0, 1.$

$\mathrm{Hom}_R(M, -)$ inv. nil isos.

$- \otimes_R M$

Proof: $0 \to N' \to N \to N'' \to 0$

$$\mathrm{Hom}_R(P_\cdot, N) = \mathrm{Hom}_{R/I}(R/I \otimes_R P, N)$$

$$E_2^{pq} = \mathrm{Ext}_R^p(\mathrm{Tor}_q^R(R/I, M), N) \Rightarrow \mathrm{Ext}_R^{p+q}(M, N)$$

$\therefore \mathrm{Tor}_q^R(R/I, M) = 0 \quad q = 0, 1 \implies \mathrm{Ext}_R^i(M, N) = 0$ for $i = 0, 1$

Converse take N injective R/I-module

$$\mathrm{Hom}_{R/I}(\mathrm{Tor}_q^R(R/I, M), N) = \mathrm{Ext}_R^q(M, N)$$

Consider class of all $M \ni \mathrm{Ext}_R^i(M, N) = 0 \quad i = 0, 1.$

closed under extension

~~Tor~~

55. ~~For~~ Q a left R-module.

$$R/A \otimes_R Q = 0 \qquad Q = AQ$$

$$\Rightarrow \quad N \otimes_R Q = 0 \quad \text{all nil modules}$$

$\mathrm{Tor}_1^R(N, Q)$ ~~closed~~ right exact

vanishes for $N = R/A$ $\Rightarrow$ vanishes for all nil ~~RAN~~ -modules.

To see that $N \mapsto N \otimes_R Q$ inverts nil isos.

~~Also~~ enough to show for inj surj, cok ker killed by A

~~~~$0 \to N' \to N \to N$~~~~

$$0 \to N' \to N \to N'' \to 0$$

$$\mathrm{Tor}_1^R(N'', Q) \to \mathrm{Tor}_0^R(N', Q) \to \mathrm{Tor}_0^R(N, Q) \to N'' \otimes_R Q \to 0$$

converse $\quad A \to R$

$$0 \to \mathrm{Tor}_1^R(R/A, N) \to A \otimes_R N \to N \to \mathrm{Tor}_0^R(R/A, N) \to 0$$

---

0 ~~To~~ ~~Tor~~

vanishing Tor_i $i = 0, 1$ $\Rightarrow$ inv. nil isos.

$$0 \to L_1 \to L_0 \to N \to 0 \qquad (L_0 \text{ free})$$

$$0 \to \mathrm{Tor}_1 \to L_1 \otimes_R Q \xrightarrow{\sim} L_0 \otimes_R Q \to N \otimes_R Q \to 0$$

$$\begin{array}{ccccccc} & & A \otimes_R M & \to & M & & \\ & & \vdots & & \downarrow f & & \\ 0 \to & S' & \to & S & \to & S'' & \to 0 \end{array}$$

take $a \otimes m \mapsto a s$

where $s \mapsto f(m)$
~~~~

$$\begin{tikzcd}
 & A\otimes_R S' \arrow[r] \arrow[d, "o"'] & A\otimes_R N \arrow[r] \arrow[d] & A\otimes_R M \arrow[r] \arrow[d, "\cong"] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & S' \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S' \arrow[r] & S \arrow[r] & S'' \arrow[r] & 0
\end{tikzcd}$$

Is a M context a graded ring of some sort. $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded ring? NO.

Think of it as the dual pair $A \oplus P$, $A \oplus Q$

$(A,A) \oplus (P,Q)$.

~~idea Suppose A has a left ann. ideal~~ ?

Idea: This time you notice $w : A \to B$ such that $m(A)$ $m(B)$ are isom. cats. Two cases

$A \longrightarrow A/I$ where $IA = 0$ $\quad \begin{pmatrix} A & A/I \\ A & A/I \end{pmatrix}$

$A \subset B$ where $BA \subset A$, $AB = B$ $\quad \begin{pmatrix} A & B \\ A & B \end{pmatrix}$

Question: Given A is there a maximal such B?

Real question: Can you eliminate the nil ideals somehow?

First question clear, you take $P = A$ and look at possible Q. $Q \otimes A \longrightarrow A$.

Then $Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$ $\qquad B = P \otimes_A Q$

Life is hard.

Think a little $\quad m(A)$ generator A.

You will work things in such a way that $m(A) \underset{w^*}{\leftrightarrows} m(B)$ via a homom. $A \xrightarrow{w} B$. In other words B naturally acts on any firm A-module

Obvious first candidate is ~~$\mathrm{Hom}_{A^{op}}$~~ ? $\qquad M = A \otimes_A M$

57. $\mathrm{Hom}_{A^{op}}(A, A)$ acts on M.

~~But now is But this B is uni~~

$$A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A)$$

kernel K is largest such that $KA = 0$.

$$A \twoheadrightarrow A/K \hookrightarrow \mathrm{Hom}_{A^{op}}(A, A)$$

$f \in \mathrm{Hom}_{A^{op}}(A, A)$ $\qquad f(a(a')) = f(a)a'$

shows the image $A/K \subset \mathrm{Hom}_{A^{op}}(A, A)$ is a right ideal.

$$0 \to K \longrightarrow A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A) \longrightarrow C \longrightarrow 0$$

K, C should be A^{op}-nil, which agrees with $Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$.

Other case. ~~Take largest~~ Start with A look for $J \hookrightarrow Z \twoheadrightarrow A$ $\quad JZ = 0$ so that Z is a left A-module.

So you just take $(A, Q) \twoheadrightarrow (A, A)$.

$$m(Q) \qquad m(A)$$

A firm $\qquad Z$ a firm A-module $\quad f: Z \twoheadrightarrow A$ surj

$$z_1 z_2 = f(z_1) z_2$$

$$\begin{pmatrix} Z & A \\ Z & A \end{pmatrix}$$

By this construction you ~~do not~~ increase the left ann of A. What about right ann? $\quad \mathrm{Hom}_A(Z, Z)^{op}$

58. Proper picture: Look at the Ross cat $\mathcal{M}(A)$ and the generator A. [redacted] You want all [redacted] Morita contexts of the form

$$\begin{pmatrix} A & Q \\ A & B \end{pmatrix} \qquad \text{i.e. } P = A$$

$$\text{so} \quad B = P \otimes_A Q = A \otimes_A Q = Q$$

So you are choosing Q and $Q \otimes A \twoheadrightarrow A$. want all $Q \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$ which lead to $Q \otimes A \twoheadrightarrow A$. Among the choices are quotients of A i.e. $Q = A/I$ where $IA = 0$. and left ideals $\mathfrak{a} \subset A$ such that $\mathfrak{a} \otimes A \twoheadrightarrow A$ i.e. $\mathfrak{a}A = A$. YES!!!!

Back to writing!

~~Take $A \subset B$ right ideal: $\boxed{\begin{matrix} AB \subset A \\ BA = B \end{matrix}}$~~

~~Then $\mathcal{M}(A^{op}) = \mathcal{M}(B^{op})$.~~

~~$V \otimes_A A = V \implies V$ has unique B^{op}-mod str.~~

$A \subset B$ $\qquad \begin{matrix} BA \subset A \\ AB = B \end{matrix}$

$\mathcal{M}(A) = \mathcal{M}(B)$ $\qquad A \otimes_A M \xrightarrow{\sim} M \implies$ unique B-mod structure on M

$$A \otimes_A M \twoheadrightarrow B \otimes_B M \longrightarrow M$$

$$A \otimes_A B \longrightarrow B \qquad \text{nil } B^{op}. \qquad \begin{matrix} (a \otimes b) a_1 b_1 \\ \downarrow \\ a b a_1 \otimes b_1 \end{matrix}$$

$$A \otimes_A M \xrightarrow{\sim} M$$

59.

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

$$\mathcal{M}(A^{op}) \qquad \mathcal{M}(B^{op})$$

$$V \longmapsto V \otimes_A B$$

$$W \otimes_B A \longleftarrow\!\shortmid\; W$$

Idea is that

$$\begin{array}{ccc} \mathcal{M}(A^{op}) & & \mathcal{M}(B^{op}) \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{rtcontfun}(\mathcal{M}(A), ab) & = & \mathrm{rcf}(\mathcal{M}(B), ab) \end{array}$$

$$V \longmapsto (M \longmapsto V \otimes_A M)$$

$$(N \longmapsto V \otimes_A N) \longmapsto V \otimes_A B^{(2)}$$

$$W \longmapsto (N \longmapsto W \otimes_B N)$$

$$(M \longmapsto W \otimes_B \cancel{A \otimes_A} M) \longmapsto W \otimes_B A^{(2)}$$

$$\mathcal{M}(A^{op}) \qquad \mathcal{M}(B^{op})$$

$$\mathrm{rcf}(\mathcal{M}(A), ab) \xrightarrow{(\omega^{-1*})^t} \mathrm{rtf}(\mathcal{M}(B), ab)$$

$$(M \mapsto V \otimes_A M) \longmapsto (N \mapsto V \otimes_A A^{(2)} \otimes_A N)$$

$$V \otimes_A N$$

$$V \otimes_A A^{(2)} \otimes_A B^{(2)} = V \otimes_A B^{(2)} = \omega_!(V).$$

$$W \otimes_A A^{(2)}$$

$$W.$$

$$(M \mapsto W \otimes_B B \otimes_A M) \longleftarrow\!\shortmid\; (N \mapsto W \otimes_B N)$$

60. $A \xrightarrow{w} B$

$$\mathrm{Hom}_{m(A)}(M, w^*N) = \mathrm{Hom}_A(M,N) = \mathrm{Hom}_{m(B)}(w_!M, N)$$

$B^{(2)} \longrightarrow B$, $B \subset \tilde{B}$ B^{op}-nil isos., ~~&~~ hence A^{op}-nil isos

$$B^{(2)} \otimes_A M \xrightarrow{\sim} B \otimes_A M \xrightarrow{\sim} \tilde{B} \otimes_A M$$

$$\mathrm{Hom}_A(M, {}^{(2)}\otimes_A N) \xrightarrow{\sim} \mathrm{Hom}_A(M, N)$$
$$= \mathrm{Hom}_B(\tilde{B} \otimes_A M, N)$$
$$\xrightarrow{\sim} \mathrm{Hom}_B(B \otimes$$

~~[illegible]~~. $A \longrightarrow A/I = \bar{A} \subset B$

assume $IA = 0$, $B\bar{A} \subset B$, $\bar{A}B = B$.

Then have map $w : A \longrightarrow B$, a B-module structure on A such that w is a B-mod. map.

So we have a B-mod ~~structure~~ on A, a B-module map $w : A \to B$ such that $\bar{A}B = B$.

Also we have $\boxed{a_1 a_2 = w(a_1)\, a_2}$

firm $A^{(2)} \otimes_A B$ $\qquad B \otimes_A A^{(2)}$

Let's go back to see if you can rig up some nondegeneracy. Yesterday I analyzed

M-contexts $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\qquad B \otimes_A A \longrightarrow A$

$A \otimes_A B \longrightarrow B$

Point $\begin{pmatrix} A & Q \\ A & B = Q \end{pmatrix}$ $\qquad \phi : Q \otimes A \longrightarrow A$

$Q \longrightarrow A \otimes \mathrm{Hom}_{A^{op}}(A, A)$

61. Ultimately what's going on is that I have $M(A)$ and the generator A chosen, a choice of Q [illegible] somehow amounts to choosing enough maps $\xi : A \to A$?

Just what does this mean?

<u>Think abstractly</u>. You have M a Roos cat and some generator P.

Think abstractly. You are given M and $M' = \text{rtcontfirm}(M, ab)$. Given $Q \in M$ $P \in M'$ you can form $Q \otimes_{\mathbb{Z}} P \in \text{rtcontfirm}(M, M)$.

You want a surj $Q \otimes_{\mathbb{Z}} P \twoheadrightarrow \mathbf{1}$

$$0 \to \underbrace{\mathrm{Tor}_1^{\tilde{A}}(A, k)}_{\text{ind of } k} \to A \otimes_A A \to A \to A/A^2 \to 0$$

$A^{(2)}$

~~NO because k enters into~~

---

what happens? A idempotent k-algebra $M(A)$ is ind of k, so

$$A \otimes_{\mathbb{Z} \oplus A} A \longrightarrow A \otimes_{k \oplus A} A$$

must be an isomorphism

$$\text{firm}(\mathbb{Z} \oplus A, A) \xleftarrow[\text{iso.}]{} \text{firm}(k \oplus A, A)$$

$$A \otimes_{k \oplus A} A$$

Anyway firm covering.

$$\begin{array}{ccc} A \otimes_A A & \xrightarrow{\sim} & B \otimes_B B \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

62. Propose exploring the idea of making a firm ring non-degenerate. Start with A firm. You have this (left + right ann) ideal

$$0 \longrightarrow I \longrightarrow A \longrightarrow \mathrm{Hom}_{A^{op}}(A,A) \times \mathrm{Hom}_{A}(A,A)^{op}$$

general case: P, Q $\qquad P \otimes_A Q = B$.

$$P \otimes_A Q \to M(B) \subset \mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(Q,Q)$$

$$\begin{array}{cc} \| & \| \\ \mathrm{Hom}_{B^{op}}(B,B) & \mathrm{Hom}_B(B,B). \end{array}$$

Suppose we keep P, Q to avoid confusion, say we have the firm dual pair (P,Q) over A given. What do we know about

$$B \longrightarrow M(B)$$

Lots of things are nice. I know that the cokernel analogue of Calkin alg is B-nil; because $\bar{B}$ is an ideal in $M(B)$.

Idea. You want to go from P, Q to another firm dual pair $(P,Q) \longrightarrow (P',Q')$ so that $B \to B'$ detect part of the left-right ann. ideal. Thus if you have $b \neq 0$ $\quad Bb = bB = 0$ you would like to ~~construct~~ see if you can arrange that b is detected on some P' or Q'.

Try enlarging P

Hom(fp, flat)

 Go back over nuclear maps. What

$$\mathrm{Hom}_R(M, R) \otimes_R N \longrightarrow \mathrm{Hom}_R(M, N)$$

this is an isom. if N flat and M f.p.
I seem to recall using this to decide when an element of $P \otimes_R N$ is zero. What you want is to take $\xi \in P \otimes_A Q$, $\xi \neq 0$, and to find P' ~~You have~~ right A module and pairing $Q \otimes P' \longrightarrow A$ so that

$$P \otimes_A Q \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P', A)$$
$$\downarrow$$
$$\mathrm{Hom}_{A^{op}}(P', P)$$

takes ξ somewhere non-zero.

$P \otimes_A Q$ keeping Q fixed but enlarging P, i.e. universal case is

~~$Q = A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$~~

$P \qquad \mathrm{Hom}_A(Q, A) \otimes_A A$

$A \subset B \qquad BA \subset A \qquad AB = B$

$A \xrightarrow{w} B \qquad M \quad A\text{-flat} \quad AM = M.$

$\tilde{B} \otimes_A M \quad B\text{-flat}$

$$\tilde{B} \otimes_A M = \tilde{B} \otimes_A AM = \overbrace{\tilde{B}\, w(A)}^{\subset M} \otimes_A M$$

$$\tilde{A} \longrightarrow R \qquad \begin{pmatrix} \tilde{A} & R \\ A & R \end{pmatrix}$$

$$M \longmapsto A \otimes_{\tilde{A}} M$$

$$M = R \otimes_R M \longleftarrow N$$

M firm

$\mathrm{Hom}_A(M, -)$ inverts nil isos.
$\mathrm{Ext}^i_R(M, N) = 0$ N nil $i = 0, 1$

} formally equivalent

$- \otimes_R M$ inverts nil isos.

$\mathrm{Tor}^R_i(R/A, M) = 0$, $i = 0, 1$.

M is a cokernel of a map of firm flat modules.

$$0 \to N' \to N \to N'' \to 0$$

$AN'' = 0$ $\quad 0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \underset{\overset{\shortparallel}{0}}{\mathrm{Hom}(M, N'')} \to \mathrm{Ext}^1(M, N')$

since $AM = M$ and $AN'' = 0$.

~~Anyway, what next?~~

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0 \qquad N\text{-nil}$$

$$\mathrm{Hom}(M, M) \to \quad \to \mathrm{Ext}^1(M, N)$$

E any R/A-mod

$$\mathrm{Ext}^i_R(M, E) = H^i(\mathrm{Hom}_R(P_\cdot, E))$$
$$= H^i(\mathrm{Hom}_{R/A}(R/A \otimes_R P, E))$$
$$= \mathrm{Hom}_R(\mathrm{Tor}^R_i(R/A, M), E).$$

so this Ext vanishes for $i=0,1$ $\Rightarrow$ $\mathrm{Tor}_i = 0$ $\quad i = 0, 1$.

$$E_2^{pq} = \mathrm{Ext}^p_{R/A}\left(\mathrm{Tor}_q^R(R/A, M), N\right)$$

$$\Rightarrow \mathrm{Ext}^*_R(M, N).$$

If the Tor's vanish for $q=0,1$. then

$\mathrm{Ext}^i_R(M,N) = 0$ $i=0,1$ all R/A mod.

conversely if ~~these ex~~ N inj. R/A-mod.

$$\mathrm{Hom}_{R/A}\left(\mathrm{Tor}_i^R(R/A, M), N\right) = \mathrm{Ext}^i_R(M,N)$$

Let A be unital to fix the ideas.
Consider firm dual pairs (P,Q) over A where P, Q are f.g. free.
You want to understand the ring $P \otimes_A Q$
in terms of matrix rings over A. A basic
trick you have is to replace ~~a ring B by~~
~~any A by a~~ a ring B by an $A \twoheadrightarrow B$ where
A is flat on both sides and the ideal I
such that $AIA = 0$. The typical thing you
have is degenerate. You hope relate the
degenerate thing to a non degenerate one.
Start with A embed into $B = \begin{pmatrix} A \\ P_1 \end{pmatrix} \otimes \begin{pmatrix} A & Q_1 \end{pmatrix}$

7/28 0950 Stop wasting time

Start with A unital, (P,Q) firm dual pair
where P,Q are f.g. projective. ~~Then we can choose~~
To simplify suppose $(P,Q) = (A,A) \oplus (P_0, Q_0)$ where
the pairing $Q_0 \otimes P_0 \to A$ is arbitrary. You
should be able to embed (P_0, Q_0) into a
non degenerate pair. Method $Q_0 \to \mathrm{Hom}_{A^{op}}(P_0, A) = P_0^*$

66. You have

$$\begin{tikzcd} Q_0 \oplus P_0^* \arrow[r, "\begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}"] & Q_0 \oplus P_0^* \arrow[d, "(0\ \ 1)"] \\ Q_0 \arrow[u, hook, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[r, "\rho"] & P_0^* \end{tikzcd}$$

$$(0\ \ 1)\begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \rho .$$

Take $Q = Q_0 \oplus P_0^*$

$P = Q_0^* \oplus P_0$

usual pairings

~~[illegible]~~

$Q_0 \otimes Q_0^* \longrightarrow A$

$P_0^* \otimes P_0 \longrightarrow A$

ρ pairing $Q_0 \otimes P_0 \longrightarrow A$

any other pairing $\sigma : P_0^* \otimes Q_0^* \longrightarrow A$ such that $\begin{pmatrix} 1 & \sigma \\ \rho & 1 \end{pmatrix}$ remains non-degenerate.

So if I start with $\begin{pmatrix} A & Q_0 \\ P_0 & P_0 \otimes_A Q_0 \end{pmatrix}$

I can embed it in

$$\begin{pmatrix} A & Q_0 & P_0^* \\ P_0^* & * & * \\ P_0 & B_0 & * \end{pmatrix}$$

So what is the problem now? You've managed to embed $\begin{pmatrix} A & Q_0 \\ P_0 & B_0 \end{pmatrix}$ inside the non-deg. thing. But the question is whether you can embed the non deg. thing inside a matrix ring over the deg. alg. This is clear – it goes into the A part! Then what happens???

~~So what we do~~

$$A \subset M_n A$$
$$\cap \quad \not\cup \quad \cap$$
$$B \dashrightarrow M_n B$$

need some sort of homotopy for homomorphisms yielding the same Morita equivalence i.e. a theory of the Grassm.

e.g. ~~a theory of the Grassm~~ you want to be able to link different homomorphisms $\mathbb{C} \to M_n\mathbb{C}$.

A homom $\mathbb{C} \longrightarrow M_n\mathbb{C} = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ is an idempotent matrix. ~~These form a sp~~ I am interested in nzeghoms – means rank 1 projections. this space has homotopy type of $\mathbb{P}(\mathbb{C}^n)$. ~~Homotopy type~~ ~~$\mathbb{P}(\mathbb{C}^n)$~~ Algebraically what happens? higher rank

$M_p\mathbb{C} \longrightarrow M_n\mathbb{C}$ map of dual pairs

$(\mathbb{C}^p, \mathbb{C}^p) \longrightarrow (\mathbb{C}^n, \mathbb{C}^n)$ amounts to

$(W, W^*) \longrightarrow (V, V^*)$

$$W \xrightarrow{\alpha} V \qquad W^* \xrightarrow{\beta} V^*$$

comp. with duality

so that $\big(\alpha(w), \beta(\mu)\big) = (w, \mu)$

$\qquad\qquad \| \qquad\qquad\qquad \|$

$(\beta^{*}\alpha(w), \mu)$ $\qquad \beta^t\alpha = 1.$

$W \underset{\beta^t}{\overset{\alpha}{\rightleftarrows}} V.$ So what I learn is that the full subcat of nondeg dual pairs over A unital is the category of split injections between fg projectives. You know that Volodin's ~~says you get~~ "topology" must be applied to get the good K-theory thing.

 ~~[illegible]~~ First part.

~~Problem:~~

properties of firm modules.

M firm $\iff$ $\mathrm{Hom}(M,-)$ inv. nil iso.

$\Rightarrow$ suff to consider surj w kernel ~~nil~~ killed by A
inj cokernel ______ ______ .

$$0 \to N' \to N \to N'' \to 0$$

$$0 \to \mathrm{Hom}(M,N') \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N'') \to \mathrm{Ext}^1(M,N)$$

0 if $M=AM$, $AN''=0$.

If N' nil then inj, so you need to lift

$$\begin{tikzcd} & & M \arrow[dl, dotted] \arrow[d, "f"] & \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0 \end{tikzcd}$$

can assume $f = 1_M$

$$\begin{tikzcd} A\otimes_R N' \arrow[r] \arrow[d, "0"] & A\otimes_R N \arrow[r] \arrow[d] & A\otimes_R M \arrow[r] \arrow[d, "s"] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

$\Leftarrow$ $A\otimes_R M \to M$ nil iso

$\therefore$ $\exists$ section $\qquad$ $M = AM \Rightarrow A(A\otimes_R M) = A\otimes_R M$

$a_1(a_2\otimes m) = a_1\otimes a_2 m$

$\mathrm{Ext}^i(M,N) = 0 \quad i=0,1 \qquad N$ nil

$$0 \to \mathrm{Hom}(M,N') \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N'') \to \mathrm{Ext}^1(M,N')$$

$$0 \to \underset{\text{nil}}{N} \to E \to M \to 0$$

69. equiv. of (b) + (c) is formal

(c) $\Rightarrow$ (b) clear: $0 \to N' \xrightarrow{i} N \xrightarrow{j} N'' \to 0$

N' nil $\Rightarrow$ j_* isom on $\mathrm{Hom}_R(R,\ )$

N'' " i_*

(a) $\Rightarrow$ (c) M firm $\Leftrightarrow$ $\mathrm{Tor}_j^R(R/A, M) = 0$ $j = 0, 1$.

$$E_2^{pq} = \mathrm{Ext}^p_{R/A}\left(\mathrm{Tor}_q^R(R/A, M), N\right) \Rightarrow \mathrm{Ext}^{p+q}_R(M, N)$$

$= 0$ for $q = 0, 1$

$\mathcal{M}(A)$ abelian

(a) M firm

(b) $\mathrm{Hom}_R(M, -)$ inv. nil isos

(c) $\mathrm{Ext}^j_R(M, N) = 0$ $\forall N$ nil (resp. $AN = 0$) $\forall j = 0, 1$.

(d) $- \otimes_R M$ inv. A^{op}-nil isos. of R^{op}-modules

(e) $\mathrm{Tor}_j^R(W, M) = 0$ for $j = 0, 1$ all A^{op}-nil W.

(f) M cokernel of a map between firm flat modules.

$$\begin{tikzcd}
 & A \otimes_R N \arrow[r] \arrow[d, "0"] & A \otimes_R E \arrow[r] \arrow[d] & A \otimes_R M \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$AN = 0$

(a) M firm

(b) $\mathrm{Ext}^j_R(M, N) = 0$ for $j = 0, 1$ and all N sat $AN = 0$.

(b') same as (b) but for ~~all~~ any N nil module.

(c) $\mathrm{Hom}_R(M, -)$ inv. nil-isoms.

(a) $\Rightarrow$ (b) $M = AM \Rightarrow \mathrm{Hom}_R(M, N) = 0$ if $AN = 0$.

given $0 \to N \to E \to M \to 0$.

---

need $\otimes$ conditions

firm modules and $\otimes$.

70. ~~[illegible]~~ Back to A unital. I think I ought to be able to handle ~~[illegible]~~ left + right flat rings mneq to a unital ring.

Take $B = P \otimes_A Q$ $\quad P_A, {}_AQ$ f.g. proj.

to simplify assume $p_0 \in P$, $q_0 \in Q$ $\ni$ $q_0 p_0 = 1 \in A$.

Then get $A \longrightarrow B$ actually $(A, A) \longrightarrow (P, Q)$

I think I can embed $(P, Q) \hookrightarrow (A^n, A)^n$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{comm.}"] & M_n A \arrow[d, hook] \\ B \arrow[ur, hook] & M_n B \end{tikzcd}$$

$$B = \begin{pmatrix} A & 0 & Y \\ X & 0 & X \otimes_A Y \\ 0 & 0 & 0 \end{pmatrix} \subset M_3(A)$$

$$\begin{tikzcd} (A, A) \arrow[r, hook] \arrow[d] & (A, A)^n \arrow[d, hook] \\ (P, Q) \arrow[ur] & (P, Q)^n \end{tikzcd}$$

$M_3(B)$ $\qquad M_9(A)$

first thing to understand is the Suslin case

~~[illegible]~~ Rough idea is that we have an admissible mono $P \subset P$

Try something else

Start with P, Q have

can enlarge ~~[illegible]~~ $Q \subset Q_1$

$$\begin{tikzcd} Q \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{op}(P, A) \\ Q_1 \arrow[ur] & \end{tikzcd}$$

~~[illegible]~~ We have $B \subset B_1$ $\qquad P \otimes_A Q \subset P \otimes_A Q_1$

actually this looks pretty nice!

Start with any pairing

enlarge Q

$$\begin{tikzcd} Q \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{A^{op}}(P, A) \\ Q_1 \arrow[ur] & \end{tikzcd}$$

~~[illegible]~~

$Q_1^* \otimes_A Q_1$

71. What to do? ~~Etc.~~

$$\begin{tikzcd} P\otimes_A Q \arrow[d] & \subset & P\otimes_A Q_1 \arrow[dll] \\ P\otimes_A P^* & & Q_1^*\otimes Q_1 \arrow[u, phantom, "\cap"] \end{tikzcd}$$

$$(P,Q) \subset (P,Q_1) \subset (Q_1^*, Q_1)$$

Can assume Q_1/Q nice, then $Q_1 \twoheadrightarrow P^*$ says Q_1^*/P nice. But natural complements.

---

tensor product $\quad A^{(2)}\otimes_A M$ firm.

$A=A^2 \qquad 0 \to N' \to N \to N'' \to 0$

---

if $A=A^2$ ~~[illegible]~~ and λ

then 1) $AN'=0 \implies A\otimes_R N \xrightarrow{\sim} A\otimes_R N''$

2) $AN''=0 \implies A^{(2)}\otimes_R N \xrightarrow{\sim} A^{(2)}\otimes_R N''$

In particular

1) $A\otimes_R M \twoheadrightarrow M \implies$ ~~[illegible]~~ $A\otimes_R A\otimes_R M \xrightarrow{\sim} A\otimes_R M$

i.e. $A\otimes_R M$ is firm

2) $A\otimes_R M \to M$ nil isom $\implies A^{(2)}\otimes_R A\otimes_R M \to$

$\implies A^{(2)}\otimes_R M$ firm.

---

criterion

M firm, $\quad \mathrm{Tor}_j^R(R/A, M)=0 \qquad j=0,1.$

$-\otimes_R M$ inverts nil^{ok} isom

M cokernel of map of firm flats.

Suppose M flat and $AM=M$. Then

$$0 \to A \to R \to R/A \to 0$$

$$\implies 0 \to A\otimes_R M \xrightarrow{\sim} \cancel{R\otimes}M \to \underset{0}{\overset{\shortparallel}{M/AM}} \to 0$$