141 Goal: $A \twoheadrightarrow A/I = B$ $IA = 0$, A left flat, B h-unital $\Rightarrow K_*A \cong K_*B$. B is h-unital iff ~~B~~ it has a resolution by firm flat B modules, ~~same~~ same as a firm flat res. of A-modules, so B h-unital $\Leftrightarrow \mathbb{Z} \overset{\mathbb{L}}{\otimes}_{\tilde{A}} B = 0 \Leftrightarrow I \overset{\mathbb{L}}{\otimes}_{\tilde{A}}$. Concentrate.

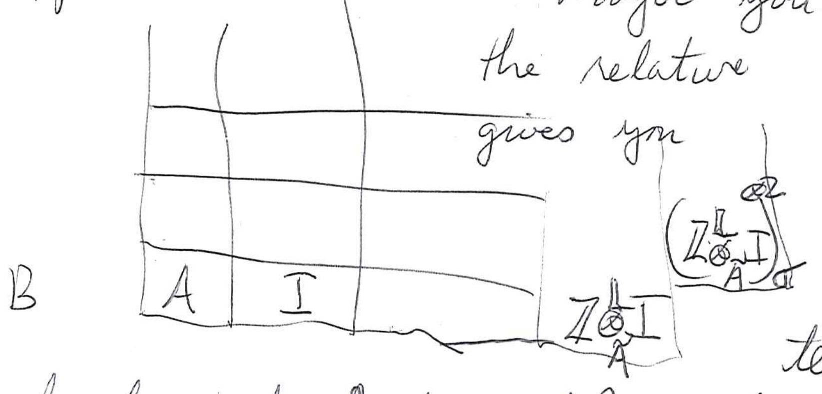
$$0 \to I \to A \to B \to 0$$

exact sequence of B-modules, where A is flat. ~~Thus~~ Then B h-unital iff B is h-unitary B-module iff I is an h-unitary B-module.

Let's analyze this carefully $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

Somehow I feel that the crux of the problem concerns the homology of $GL(B)$ acting on $M(I)$, where I is a B-module. What's our motivation? At one point you studied cyclic homology of an alg extension $A/I = B$ using the DGA $I \to A$ resolution of B. And this led to a spectral sequence involving $[I \overset{\mathbb{L}}{\otimes}_{\tilde{A}}]^{(n)}$. In the present situation A acts as 0 on the right. So $I \overset{\mathbb{L}}{\otimes}_{\tilde{A}} = \mathbb{Z} \overset{\mathbb{L}}{\otimes}_{\tilde{A}} A$ and $[I \overset{\mathbb{L}}{\otimes}_{\tilde{A}}]^{(n)} = (\mathbb{Z} \overset{\mathbb{L}}{\otimes}_{\tilde{A}} I)^{\otimes n}$. In the spectral sequence then you expect

Maybe you want to look at the relative $HC(A \twoheadrightarrow B)$ which gives you



So it seems clear that the leading term is $\mathbb{Z} \overset{\mathbb{L}}{\otimes}_{\tilde{A}} I$.

So how do I proceed? Somehow the problem will be to invoke? ~~$gl(A) \to gl(B)$~~

$$0 \to gl(I) \to gl(A) \to gl(B) \to 0$$

 Then you have all this invariant theory conn. with $\Lambda\mathfrak{gl}$. Important is the grading the degree in I. So our problem is to understand something about $H_*(GL(B), H_*(GL(I)))$ when I is a B-module regarded as a bimodule with B acting trivially on the right. How now brown cow.

Suppose I is a flat B-module. Should this homology vanish. What's important is the relative homology of $GL(A) \to GL(B)$ whose leading term should be $\mathbf{Z} \overset{L}{\otimes}_{\tilde{B}} I$. Why. Naively. You need

Note: It seems the relative homology of $\mathfrak{gl}(A) \longrightarrow \mathfrak{gl}(B)$ is zero when I is a flat B-module. Does this use h-unitality of B? Take $B \oplus I$. Should it be true that $K_*(B \oplus I) = K_*(B)$ for I firm flat over B?

Q. Does Mun. of K for h-unital rings follow from Suslin's results? Same way of lifting excision?

03/05/97 I need to study how K_* behaves for extensions $0 \to I \to A \to B \to 0$ such that $IA = 0$. This is the same as a B-mod ~~map f: A~~ surjection $f: A \twoheadrightarrow B$. Have a M cont. $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

Have dual pair $(\tilde{B}, A, A \otimes \tilde{B} \longrightarrow B)$ $\ni$ $\tilde{B} \otimes_B A = A$.

$a\ \tilde{b} \mapsto f(a)\tilde{b}$

$$0 \to I \to \tilde{A} \to \tilde{B} \longrightarrow 0$$

Note $I^2 = 0$

so this is a square-zero extension where right mult by $\tilde{B}$ on I is thru ε. Thus I is a unitary bimodule over $\tilde{B}$

143 Classify extensions by $H^2(\tilde{B}, I)$. As to how much is clear? I see an analogy with group extensions. It should be possible to see that

$$H^2(\tilde{B}, I) = \mathrm{Ext}^1_{\tilde{B}}(B, I) \overset{?}{=} \mathrm{Ext}^2_{\tilde{B}}(\mathbb{Z}, I)$$

$f : B \otimes B \to I$

$$\cancel{x} f(y,z) - f(xy,z) + f(x,yz) = 0$$

$D : B \to I$ $\qquad D(xy) = \cancel{Dxy} + xDy$?

$\mathrm{Der}(B, I) = \mathrm{Hom}_B(B, I)$.

~~The question is the Given $I \subset A_*$~~

So I want to understand K_* for ring extensions

$$0 \to I \to A \to B \to 0 \quad \text{st.} \quad IA = 0$$

same as B module extensions of B by I, same as square zero unital ~~alg~~ ring extensions of $\tilde{B}$ by the bimodule I

We get $\quad 0 \to M(I) \to GL(\tilde{A}) \to GL(\tilde{B}) \to 0$

Note that $GL(\mathbb{Z})$ operates ~~here~~. This is a gp extn.

I need to study how K_* behaves for ring extensions

$$0 \to I \to A \to B \to 0$$

such that $IA = 0$. Note $I^2 = 0$ so this is a square zero extn ~~...~~ of B by the B-bimodule I, where $I \cdot B = 0$. These extensions form a category equivalent to cat of B-modules A equipped with a map onto B. A, B are M. eq.

$$\begin{pmatrix} \tilde{B} \otimes_{\tilde{B}} A & \tilde{B} \\ A & \tilde{B} \end{pmatrix} \qquad A \otimes_A \tilde{B} = B$$

$$A \otimes_A \tilde{B} \rightrightarrows B$$

$$a \otimes \tilde{b} \longmapsto f(a)\tilde{b}$$

$$= 1 \otimes f(a)\tilde{b}$$

$$\left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline A & \tilde{B} \end{array}\right)$$

$$\tilde{B} \otimes_B A = A$$

$$A \otimes_A \tilde{B} = B.$$

 I don't understand what happens.

In any case you need to examine

$A \twoheadrightarrow B \quad \rightsquigarrow \quad K_*(A).$

Review why B left or right flat $\overset{+\text{firm}}{\Longrightarrow}$

$$K_*(B) = K_*\begin{pmatrix} 0 & 0 \\ B & B \end{pmatrix} = K_*\begin{pmatrix} 0 & B \\ 0 & B \end{pmatrix}$$

In general for a firm M. cont. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ one has

A is A flat $\iff$ $P = P \otimes_A A$ is B-flat

Q is A-flat $\iff$ $B = P \otimes_A Q$ is B-flat

A is A^{op}-flat $\iff$ $Q = A \otimes_A Q$ is B^{op}-flat

P is A^{op}-flat $\iff$ $B = P \otimes_A Q$ is B^{op}-flat..

$$A = \begin{pmatrix} 0 & 0 \\ B & B \end{pmatrix} = \begin{pmatrix} 0 \\ B \end{pmatrix} \otimes_B \begin{pmatrix} B & B \end{pmatrix} \qquad \left\langle (b_1 \ \ b_2) \,\middle|\, \begin{pmatrix} 0 \\ b \end{pmatrix} \right\rangle = b_2 b$$

A is A-flat $\iff P = \begin{pmatrix} 0 \\ B \end{pmatrix}$ is B flat $\iff B$ is B flat $\iff Q = (B \ \ B)$ is A-flat

A is A^{op}-flat $\iff Q = (B \ \ B)$ is B flat $\iff B$ is B flat.

Similarly for

$$A = \begin{pmatrix} 0 & B \\ 0 & B \end{pmatrix} = (0 \ \ B) \otimes_B \begin{pmatrix} B \\ B \end{pmatrix} \qquad \left\langle \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \,\middle|\, b \right\rangle = b_2 b$$

A is A flat $\iff P = (0 \ \ B)$ is B-flat $\iff B$ is B-flat

A is A^{op} flat $\iff Q = \begin{pmatrix} B \\ B \end{pmatrix}$ is B^{op}-flat $\iff B$ is B^{op}-flat.

I know in this case that $K_*(A) \cong K_*(B)$.

The interesting case is when A is A flat?

Idea. Can look at $C = \begin{pmatrix} A & B \\ A & B \end{pmatrix}$

Start again. Consider for any B-module surjection $A \xrightarrow{f} B$ the ring A with $a_1 a_2 = f(a_1) a_2$, so we have a square zero extension

$$0 \longrightarrow I \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

where I is a B-bimodule such that $IB = 0$. Look at $K_*(A)$. ~~Slight generalization: $f: A \to \mathrm{Hom}_{B^{op}}(\tilde{B}, B)$?~~

Slight generalization: dual pair $(\tilde{B}, A, A \otimes \tilde{B} \longrightarrow B)$, but this pairing is same as B-map $A \longrightarrow \mathrm{Hom}_{B^{op}}(\tilde{B}, B) = B$, so we replace surjectivity of f by requiring $f(A)B = B$.

To simplify consider B-module extensions of B: $0 \longrightarrow I \longrightarrow A \xrightarrow{f} B \longrightarrow 0$, ~~regard f as a ring homomorphism~~ regard as a type of square zero extn of rings, and consider the relative K_*. Know $K_0(A) \xrightarrow{\sim} K_0(B)$. What kind of games can I play? Derived functor game. If A is a left flat B-module, then the result might be independent of A.

$$\begin{pmatrix} A & \tilde{B} \\ A & B \end{pmatrix}$$

$$A \underset{A}{\otimes} \tilde{B} \longrightarrow B$$
$$a \otimes \tilde{b} \longmapsto f(a)\tilde{b}$$
$$\longleftarrow\!\shortmid\, b$$

$$\left(\tilde{B}, A, \underset{\mathbb{Z}}{A \otimes \tilde{B}} \longrightarrow B\right)$$
$$a \otimes \tilde{b} \longmapsto f(a)\tilde{b}$$

~~$A \otimes_A \tilde{B} \to B$~~

certainly we have $Q \otimes_B P = \tilde{B} \otimes_B A = A$

but $P \otimes_A Q = A \otimes_A \tilde{B} \longrightarrow B$, $a \otimes \tilde{b} \longmapsto f(a)\tilde{b}$.

$f(a_1)\tilde{b}_1 f(a_2)\tilde{b}_2 =$ ~~[illegible]~~

$(a_1 \otimes \tilde{b}_1)(a_2 \otimes \tilde{b}_2) = a_1 \otimes \tilde{b}_1 f(a_2)\tilde{b}_2$

146 I am very confused. ~~[illegible]~~

For any $A \xrightarrow{f} B$ B-mod surj consider $K_*(A)$

If A and B are left flat, then $K_*(A) \xrightarrow{\sim} K_*(B)$.

~~[illegible]~~ This is OKAY if things are idempotent.

Why true? $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ everything's left flat

Point $B = A \otimes_A B$ and A A-flat $\Rightarrow$ ~~[illegible]~~ is B flat $P \otimes_A A = A$

so B acts on B

Assume $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ firm and A and B are left flat rings. Then $P \otimes_A A = A \otimes_A A = A$ is B-flat as A has right action on A which is left B-flat, so get $K_*(A) = K_*(Q \otimes_B P) \longrightarrow K_*(B)$. Now B

[redacted] $B \otimes_B A$

is B-flat $\Rightarrow$ $Q \otimes_B B = B \otimes_B B = B$ is A-flat and we get $K_*(B) = K_*(P \otimes_A Q) \longrightarrow K_*(A)$ defined by the right action of B on B

Look at $K_1(A) = H_1(GL(\tilde{A}))/H_1(GL(\mathbb{Z}))$

I want to find some sly methods for handling this stuff. [redacted] Look at K_1

$K_1(\tilde{A}) =$

I wonder if you are following a dead end. Should you be using finite sets and partial orderings — Volodin style.

Look at cyclic homology $0 \to I \to A \to B \to 0$

~~[illegible]~~ DGA $C_\lambda(I \to A)$ leads to $\mathrm{Fib}(C_\lambda(A) \to C_\lambda(B))$ having a filt. with quotients $[I \otimes^L_{\tilde{A}}]^{(n)}_\sigma = (\mathbb{Z} \otimes^L_{\tilde{A}} I)^{\otimes n}_\sigma$

147 so if I is a flat A-module, then it would seem that these cyclic tensor products ~~are~~ reduce to $(I/AI)^{\otimes n}_{\sigma}$. Let's examine low degrees.

$$\begin{tikzcd} I/[A,I] \arrow[r] & A/[A,A] \arrow[r] & B/[B,B] \arrow[r] & 0 \\ & HC_1(A) \arrow[r] & HC_1(B) \arrow[ull, bend left] & \end{tikzcd}$$

So the real puzzle. How to get a real understanding

$$0 \to I \to A \to B \to 0$$

$$0 \to \boxed{\mathfrak{gl}(I) \to \mathfrak{gl}(A)} \to \mathfrak{gl}(B) \to 0$$

$$\mathfrak{gl}(I \to A) \to \mathfrak{gl}(B)$$

$$S\Sigma\,\cancel{\mathfrak{gl}}(I \to A) \longrightarrow S\Sigma\,\mathfrak{gl}(B)$$

use invariant theory

$$\Lambda^p(\overset{\mathfrak{g}}{\mathfrak{gl}(k)} \otimes_k B)$$

$$\|$$

$$(\mathfrak{g}^{\otimes p} \otimes B^{\otimes p})_{\Sigma_p}$$

then take $\mathfrak{g}$ coinvariants, + primitive

~~But what actually happens~~ What about the nontrivial ~~other~~ representations of $\mathfrak{g}$?

Important seems to be first you embed B in $\tilde{B}$ then take $\mathfrak{gl}(\tilde{B})$ and then reduce wrt the reductive subalg $\mathfrak{g}$. This amounts to relative Lie alg homology $(\mathfrak{gl}(\tilde{B}), \mathfrak{g})$. One knows this is the same as the Lie homology of $\mathfrak{gl}(\tilde{B})$. So it seems that one is actually working with the unital square zero ring extension $0 \to I \to \tilde{A} \to \tilde{B} \to 0$. Question: When is the Lie alg homology of $\mathfrak{gl}(B)$ the same as the relative L. alg hom. of $(\mathfrak{gl}(\tilde{B}), \mathfrak{g})$? This should be the Lie version of excision, hyp. should be B H-unital. Is there some way I should be able to handle this?

So now there arises the question whether on Lie algebra homology you can see things like h-unital. You want $H_*(gl(B)) \xrightarrow{\sim} H_*(gl(\tilde{B}), gl(k))$. This is what Suslin understands.

Problem. Defining $K_*(A)$ internally.

Working with $C_\lambda(A)$ amounts to the relative Lie homology of $(gl(\tilde{A}), gl(k))$. Hanlon said that if you want Lie homology of $gl(A)$ for A non-unital, then it involves the bar homology of A. $gl(\tilde{A}) = gl(k) \oplus gl(A)$ semi-direct product. Exact sequence

$$0 \to gl(A) \to gl(\tilde{A}) \to gl(k) \to 0$$

HS
$$H_p(gl(k), H_q(gl(A))) \Longrightarrow H_*(gl(\tilde{A}))$$
$$\shortparallel$$
$$H_p(gl(k)) \otimes H_q(gl(A))_{gl(k)}$$

So you are not going to learn about $H_*(gl(A))$

What can we do?

$$\Lambda^p\, gl(A) = \Lambda^p(g \otimes A) = (g^{\otimes p} \otimes A^{\otimes p})$$

Is there a way to obtain the [redacted] part corresp to an irred rep of [redacted] $g = gl(k)$.

$$g^{\otimes p} = V^{\otimes p} \otimes [redacted] (V^*)^{\otimes p}$$

Take an irred rep of. The idea is to tensor with $g^{\otimes k}$ and look at the result as a rep of Σ_k

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Look at $C = \Lambda\, gl(A)$ the complex of Lie chains on $gl(A)$. Conjugation action by $g = gl(k)$. ~~Then have~~ Since

149 ~~Back to power operations.~~ $\mathfrak{g}$ reductive we have

$$C = \bigoplus W_i \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(W_i, C)$$

where W_i ranges over the irred reps of $\mathfrak{g}$. But now you want to use the fact that the irred repns. of $\mathfrak{g} = V \otimes V^*$ are given somehow by repns. of symmetric groups. $\mathfrak{g} = \mathrm{End}(V) \longrightarrow \mathrm{End}_{\Sigma_n}(V^{\otimes n})$. What is the relation? Double commutant thm. Inside $\mathrm{End}(V^{\otimes n})$ you have the image of $k[\Sigma_n]$ and its centralizer ~~is the image~~ should be $(\mathrm{End}(V)^{\otimes n})^{\Sigma_n}$ which is the image of $U(\mathfrak{g})$. $\therefore$ Double comm. thm. says $\mathrm{End}(V^{\otimes n})^{\mathfrak{g}} = k[\Sigma_n]$

$$V^{\otimes n} = \bigoplus W_i \otimes_{\mathbb{C}} Q_i$$

There are some other things known. If $\dim(V) \geq n$, so that $k[\Sigma_n] \hookrightarrow \mathrm{End}(V^{\otimes n})$, then each irred rep of Σ_n must occur in $V^{\otimes n}$. So what do you want?? Go back to $\Lambda^p(\mathfrak{g} \otimes A) = (\mathfrak{g}^{\otimes p} \otimes A^{\otimes p})^{\Sigma_p}$. Now $\mathfrak{g}$ acts on this and we need somehow to describe non-trivial reps. of $\mathfrak{g}$ occurring in this complex. You want to take W_χ an irred rep of $\mathfrak{g}$ and form $\mathrm{Hom}_{\mathfrak{g}}(W_\chi, C_\cdot)$. W_χ should occur in V Jean's music

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Yesterday I looked at Lie hom. of $\mathfrak{gl}(A)$, A nonunital, a problem understood by Hanlon. $\mathfrak{g} = \mathfrak{gl}(k)$ acts by conj. on $\mathfrak{gl}(A) = \mathfrak{g} \otimes A$, and one can split the Lie chain cx which is $\Lambda(\mathfrak{g} \otimes A)$ ignoring d into $\mathfrak{g}$-invariant subcomplexes according to the irred reps. of $\mathfrak{g}$. The trivial rep of $\mathfrak{g}$ yields $(\Lambda(\mathfrak{g} \otimes A))_{\mathfrak{g}}$ which reduces to the cyclic complex $C_\lambda(A)$. To handle a general component, you need to use the description of irreducibles $\mathfrak{g}$-modules.

If you do all this multilinear algebra, then you should find the result that the Lie homology of $gl(A)$ is given by the cyclic complex iff A is h-unital.

Let's go back and write things up.

The main construction

Given a unitary dual pair over $\tilde{A}$: $(P, Q, Q\otimes_{\mathbb{Z}} P \to \tilde{A})$ such that P is A^{op}-flat, one has a canonical tr_P map $K_*(P\otimes_A Q) \longrightarrow K_*(\tilde{A})$.

Properties:

naturality $(P, Q, \langle\ \rangle) \to (P', Q', \langle\ \rangle')$

$$\begin{tikzcd} K_*(P\otimes_A Q) \arrow[r] \arrow[d] & K_*(\tilde{A}) \\ K_*(P'\otimes_A Q') \arrow[ur] & \end{tikzcd}$$

commutes.

this naturality property means functorial in P keeping Q fixed and functorial in Q keeping P fixed.

~~[illegible]~~

Use K_* compat with filtered $\varinjlim$'s + fact that flat means P filtered limit of f. free A^{op}-module.

If P f.g. free i.e. $\tilde{A}^n$, then have homom.

$$(P, Q) \longrightarrow (P, \check{P}) \qquad \check{P} = \mathrm{Hom}_{A^{op}}(P, \tilde{A})$$

$$P\otimes_A Q \longrightarrow P\otimes_A \check{P} \simeq M_n A$$

Put another way for $P \in \mathcal{P}(A^{op})$ one has canon. map

$$K_*(\mathrm{End}_{A^{op}}(P)) \longrightarrow K_*(\tilde{A}).$$

$$P\otimes_A Q \longrightarrow \mathrm{End}_{A^{op}}(P)$$

this makes naturality in Q clear.

So what do we learn, ~~...~~

So what is the point actually? Given $(P, Q, Q \otimes P \to A)$, then we get $Q \to \mathrm{Hom}_{A^{op}}(P, A)$ and $P \otimes_A Q \to P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$. When P is finite free, then $P \otimes_A \mathrm{Hom}_{A^{op}}(P, A) = M_n A$.

So we get $K_*(P \otimes_A Q) \to K_*(A)$, actually a little bit more namely a map into a possible K-theory defined using $GL(A)$. What about a map $(P,Q) \to (P', Q)$. Again you factor $P \to P'$ ~~two cases are direct~~ into $P \hookrightarrow P_1 \twoheadrightarrow P'$ and you have

$$P \otimes_A Q \hookrightarrow P_1 \otimes_A Q \twoheadrightarrow P_1/P \otimes_A Q$$

$$K \otimes_A Q \hookrightarrow P_1 \otimes_A Q \to P' \otimes_A Q$$

Somehow these rings are affine. Better is to ~~look at~~ compare the action of $B = P \otimes_A Q$ on P and P'. In the first case you have $0 \to P \to P_1 \to P_1/P \to 0$ and the B action on P_1/P is trivial $\left(\begin{array}{c|c} * & * \\ \hline 0 & 0 \end{array}\right)$

In second case action on subrep is trivial

$\therefore$ $\left(\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array}\right)$ which flips to $\left(\begin{array}{c|c} * & 0 \\ \hline * & 0 \end{array}\right)$. ~~YES~~!!

Now is there something we can say when P not f. free.

The ~~problem here~~ point is ~~that you would like~~ to weaken P to be pseudo free? $P = A^n$ instead of $P = \tilde{A}^n$.

Is there some way to

Suppose the homology of A

Get the proof in a good form.

Consider A firm and A^{op} flat. $\begin{pmatrix} A & Q=A \\ P=A & B=A \end{pmatrix}$

Given $P \longrightarrow A$ A^{op} module map with $P \simeq \tilde{A}^n$. To calculate $K_*(P \otimes_A A) \longrightarrow K_*(A)$. really. $P \otimes_A A \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$. Good problem: Assume A either left or right flat and idempotent, show that $BGL(A)^+ \longrightarrow \text{fibre}\left(BGL(\tilde{A})^+ \to BGL(\mathbb{Z})^+\right)$ is a h.eq. Keep on trying.

Go back to $P \longrightarrow A$ A^{op}-map with P free fin. Then get $P \otimes_A A$ acting on the A^{op}-module P, so

$$P \otimes_A A \longrightarrow P \otimes_{A^{op}} \mathrm{Hom}_{A^{op}}(P, A)$$

See, I am somehow approximating A by $P \otimes_A A \simeq A^n$ with a funny mult. Then I have two representations of $P \otimes_A A$ ~~over~~ in $\mathcal{P}(A^{op})$ namely $\tilde{A}$ via the hom. $P \otimes_A A \to A \subset \tilde{A}$ and P via the action on $P = \tilde{A}^n$ which gives a hom $P \otimes_A A \longrightarrow M_n(A)$. This argument shows that the ~~~~ trace map $K_*(A) \longrightarrow K_*(A)$ assoc. to $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ is the identity.

$K_*(A \otimes_A A) \nearrow$

~~At some point you have to decide~~

Given A flat, say right flat. Use $\begin{pmatrix} A & Q=A \\ P=A & P\otimes_A Q=A \end{pmatrix}$ get $K_*(P \otimes_{\tilde{A}} Q) \longrightarrow K_*(A)$ which should be the identity. So start by replacing P by a f. free module $\tilde{A}^n$ approx. $\tilde{A}^n \xrightarrow{\alpha_i} A$. Then get $P \otimes_A Q = \tilde{A}^n \otimes_A A = A^n$ with prod. ~~$(a_i a_i') = (\sum \alpha_i a_i') a_i''$~~

$\vec{a}_1 \vec{a}_2 = \vec{a}_1 \langle \alpha, \vec{a}_2 \rangle$ In general $(p_1 q_1)(p_2 q_2) = p_1 \langle q_1, p_2 \rangle q_2$

$= p_1 q_1 f(p_2) q_2 = p q_1 f(p_2 q_2)$.

So $P = A^n$ is a ring $p_1 p_2 = p_1 f(p_2) = p_1 \langle \alpha, p_2 \rangle$
and $f: P \to A$ is a homom. Also have homom.

$$P = P \otimes_A A \longrightarrow \mathrm{End}_{A^{op}}(P, P) \qquad p \longmapsto (p' \mapsto \underset{\substack{\| \\ pf(p')}}{pp'})$$

$$p \otimes a \longmapsto (p' \mapsto \underset{\substack{\| \\ p\,a\,f(p') = \cancel{\ \ }}}{p\langle a, p' \rangle})$$

Then you have two homos. from P to matrices over A, which you should be able to relate via exact sequences.
Yes, the idea involves the map of dual pairs.

$$\Big(P,\ A,\ \langle a, p\rangle = \underset{\substack{\| \\ a\langle \alpha, p\rangle}}{af(p)}\Big) \longrightarrow \Big(\tilde{A},\ A,\ \underset{\langle a_1, \tilde{a}_2\rangle = a_1 \tilde{a}_2}{A \otimes_{\mathbb{Z}} \tilde{A} \to A}\Big)$$

$$\begin{tikzcd}
P \otimes_A A \arrow[r] \arrow[d] & \tilde{A} \otimes_A A \arrow[d] \\
M_n(A) & M_1(A)
\end{tikzcd}$$

$$\begin{tikzcd}
P \otimes_A A \arrow[r, hook] \arrow[d] & (P \oplus \tilde{A}) \otimes_A A \arrow[r, two heads] & \tilde{A} \otimes_A A \\
M_n(A) & & \\
 & M_{n+1}(A) &
\end{tikzcd}$$

Actually these are really the affine groups it seems i.e. you add a line or column. And that's interesting because it's a consequence of ~~taking~~ Q, the left module in the dual pair, to be A.

Maybe you even learn something, namely, you can now take P to be A^n.

~~[scribble]~~ Are there any implications of this argument when A is not $_{\wedge}^{\text{right}}$ flat? ~~[scribble]~~ Suppose we have a map $\underset{\overset{\shortparallel}{P}}{A^n} \xrightarrow{f} A$ ~~[scribble]~~ over A^{op}. Then $P \otimes_A A = P$ equipped with ~~[scribble]~~ $p_1 p_2 = p_1 f(p_2)$ over A^{op}. This is acted on $_{\wedge}^{\text{on}}$ the left $_{\wedge}$ by P so we get $P \to M_n(A)$?

$$p \circ p' = p\, f(p'), \quad p' \longmapsto |p\rangle\langle f|p'\rangle$$

is this a matrix, i.e. given $a \in A$ and $f \in \mathrm{Hom}_{A^{op}}(A, A)$ is $af : a' \longmapsto a f(a')$ mult by an element of a.

I need insight, say from cyclic homology.

Critical case: B h-unital, choose $A \twoheadrightarrow B$ surj B-module map where A is flat firm over B.

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

A is B-flat $\iff$ $Q \otimes_B A = B \otimes_B A = A$ is A-flat. It's in this situation that you must link $K_*(B)$ to $K_*(A)$. Here's an idea to try. A is constructed as an ind. limit of free B-modules. So can you ~~make A~~ start with ?

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\pi"] & AF \arrow[dr] & \subset F \arrow[d, "\pi"] \arrow[r] & F \arrow[r] & \cdots \\ M & = & M & & \end{tikzcd}$$

Do this starting with $B \;(=B^2)$ namely choose a free B-module F with surj $F \overset{\pi}{\twoheadrightarrow} B$. Then you have the dual pair

$B, F,\ F \otimes B \longrightarrow B$ $\qquad B \otimes_B F = BF$

$\xi \otimes b \longmapsto \pi(\xi) b$

$$A = \varinjlim F_n = \varinjlim A \otimes_A F_n$$

<u>extremely concrete</u>

semi-simplicial approach. Start with B h-unital then $\exists$ firm flat $^{B\text{-mod}}$ resolution of B which can be converted by Dold-Kan to a semi-simp. mod over B

$$A_2 \overset{\leftleftarrows}{\Rrightarrow} A_1 \overset{\leftarrow}{\rightrightarrows} A_0 \to B \to 0.$$

then get ~~s.s. group resolves B~~ s.s. ring $A_\bullet$ of square-zero extns. of B.

$$GL(\tilde A_2) \rightrightarrows GL(\tilde A_1) \rightrightarrows GL(\tilde A_0) \twoheadrightarrow GL(\tilde B)$$

clearly exact because the kernels to $GL(B)$ are just matrices. Then $\exists$ spec sequence – apply $B = \overline{W}$ vertically, get double s. set, apply $\mathbb{Z}$. Point is that $H_*(GL(\tilde A_n))$ ind. of n. Since the $\tilde A_n$ are flat and H-uni. so you get $H_*(GL(\tilde A_n)) \xrightarrow{\sim} H_*(GL(B))$ so $K_*(\tilde A_n) = K_*(B)$.

What happens when B not h-unital.

03/07/97. Can you show that if B is left flat, then the map $BGL(B) \to \text{Fibre}\{BGL(\tilde B)^+ \to BGL(\mathbb{Z})^+\}$ induces an isom on homology?

$$1 \to GL(B) \to GL(\tilde B) \to GL(\mathbb{Z}) \to 1$$

$$\begin{array}{ccccc} B(GL(B)) & \to & BGL(\tilde B) & \to & BGL(\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Fib} & \to & BGL(\tilde B)^+ & \to & BGL(\mathbb{Z})^+ \end{array} \qquad \text{fib}$$

$$\begin{array}{ccc} E^2_{pq} = H_p(BGL(\mathbb{Z}), H_q(BGL(B))) & \Rightarrow & H_*(BGL(\tilde B)) \\ \downarrow & & \downarrow\wr \\ H_p(BGL(\mathbb{Z})^+, H_q(\text{Fib})) & \Rightarrow & H_*(BGL(\tilde B)^+) \end{array}$$

Use Comparison thm. Induction will prob. yield

$$H_0(BGL(\mathbb{Z}), H_q(BGL(B))) \xrightarrow{\sim} H_0(BGL(\mathbb{Z})^+, H_q(\text{Fib}))$$

Thus you need $GL(\mathbb{Z})$ to act trivially on $H_*(BGL(B))$

Continue with your flat res.

Suppose $\mathbb{Z} \overset{L}{\otimes}_{\tilde B} B$ begins in degree n,

e.g. B idemp. $\qquad n = 2$

same as B having a firm $\qquad n = 3$.

156 $M = AM \iff \exists \quad F_0 \twoheadrightarrow M$

$M \xleftarrow{\sim} A \otimes_A M \iff \exists \quad F_1 \to F_0 \to M \to 0$ exact.

$\exists\ 0 \to K \to F_{p-1} \to \cdots \to F_0 \to M \to 0 \implies \mathrm{Tor}_n^{\tilde{A}}(\mathbb{Z}, M) = 0$ for $n < p$.

($K = M_p$, M_{p-1} over F_{p-1})

$$\mathrm{Tor}_p^{\tilde{A}}(\mathbb{Z}, M) \xrightarrow{\sim} \mathrm{Tor}_{p-1}^{\tilde{A}}(\mathbb{Z}, M_1)$$
$$\vdots$$
$$\xrightarrow{\sim} \mathrm{Tor}_1^{\tilde{A}}(\mathbb{Z}, M_{p-1}) \xrightarrow{\sim} M_p/AM_p$$

$$0 \to M_p \to F_{p-1} \to \cdots \to F_0 \to M \to 0$$

$$\mathrm{Tor}_p^{\tilde{A}}(\mathbb{Z}, M) \to \mathrm{Tor}_{p-1}^{\tilde{A}}(\mathbb{Z}, M_1) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{Tor}_1^{\tilde{A}}(\mathbb{Z}, M_{p-1}) \xrightarrow{\sim} M_p/AM_p$$

First case: ~~not so much~~.

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

$$GL(\tilde{A}) \qquad GL(\tilde{B})$$

B firm $\iff I = BI$.

Have simplicial ring $A_n = A \times_B A \times_B \cdots \times_B A$ ($n+1$) res. B.

Have simp gp $GL(\tilde{A}_n)$ resolving $GL(\tilde{B})$.

We have a choice between having A_* flat and having A_* a resolution of B. If we take the former then the homology of $B(G_*)$ should be the $H(BGL(A_p))$ for any p, the good $K_*(B)$

$$\begin{array}{ccccccc} \cdots & G_2 & \rightrightarrows & G_1 & \rightrightarrows & G_0 \\ & \downarrow & & \downarrow & & \downarrow \\ & Q & = & Q & = & Q \end{array}$$

either $G_\cdot \to Q$ is a q-iso. and $H_*(G_p)$ differs from $K_*(\tilde{A}_0)$

157 Start with B idempotent and construct a firm flat resolution ~~$A_1 \to A_0 \to B$~~ $\to F_1 \to F_0 \to B \to 0$ modulo nil modules. Convert to a simplicial res.

$$\begin{array}{ccccc} \cdots & A_2 & \rightrightarrows & A_1 & \rightrightarrows & A_0 \\ & \downarrow & & \downarrow & & \downarrow \\ & B & = & B & = & B \end{array}$$

What exactly do we know?? For each n one has

$$0 \longrightarrow I_n \longrightarrow A_n \xrightarrow{\pi} B \longrightarrow 0$$

an extension of B-modules with A_n flat firm. So A_n is a ring: $aa' = \pi(a)a'$. Let $G_n = GL(\tilde{A}_n)$. $\{G_n\}$ is a simplicial group.

$$1 \longrightarrow GL(I_n) \longrightarrow G_n \longrightarrow GL(\tilde{B}) \longrightarrow 1$$

First suppose B h-unitary whence F_* is a resolution of B. In general $GL(I_*)$ is the simp. abelian group of matrices with entries in I_*. ~~When~~ F_* resolves $B \Leftrightarrow I_*$ is acyclic $\Leftrightarrow GL(I_n)$ acyclic $\Leftrightarrow G_*$ resolves $GL(\tilde{B})$.

Consider the double simp cx. $\mathbb{Z}[\overline{W}_q(G_p)]$.

$$G_2 \times G_2 \quad G_1 \times G_1 \quad G_0 \times G_0 \longrightarrow GL(\tilde{B}) \times GL(\tilde{B})$$

$$G_2 \quad G_1 \quad G_0 \longrightarrow GL(\tilde{B})$$

applying $\mathbb{Z}[-]$ and taking horiz homology ~~gives~~ yields

$$H^h_q = \begin{cases} 0 & q > 0 \\ \mathbb{Z}[GL(\tilde{B})^{q+1}] & \end{cases}$$

S.S. degenerates yielding total homology is $H_*(GL(\tilde{B}))$ group homol.

 In other direction we ~~..~~ get

$$H^{\vee}_q(G_p)$$

get spec. sequence $E_2 = H_p(H_q(G_*))$

But we know that $n \mapsto H_q(G_n)$ is a constant fun.

$\therefore$ find $A_0 \to B$ induces $H_*(GL(\tilde{A}_0)) \xrightarrow{\sim} H_*(GL(\tilde{B}))$.

Now suppose that ~~...~~ F_* is a simp. res. mod nil modules of B. ~~The second is~~ Then each A_n is flat so the second spectral sequence degenerates yielding $H_*(GL(\tilde{A}_n))$ $\forall n$. Next what happens is ~~that we~~ ~~...~~ the rows are no longer acyclic. In fact what do we know?

$$\begin{array}{ccc} \mathbb{Z}[G_2^2] \rightrightarrows \mathbb{Z}[G_1^2] \rightrightarrows \mathbb{Z}[G_0^2] & \longrightarrow & \mathbb{Z}[GL(\tilde{B})^2] \\ & & \downarrow\uparrow \\ \mathbb{Z}[G_2] \rightrightarrows \mathbb{Z}[G_1] \rightrightarrows \mathbb{Z}[G_0] & \dashrightarrow & \mathbb{Z}[GL(\tilde{B})] \end{array}$$

It seems that the pth row is the $(p+1)$th tensor ~~...~~ product of the simp. ab. group $\mathbb{Z}[G_*]$. Add the aug to $\mathbb{Z}[GL(\tilde{B})^{p+1}]$. So we have a complex $\mathbb{Z}[G_*]$ with $H_0(\mathbb{Z}[G_*]) = \mathbb{Z}[GL(\tilde{B})]$. Look at the first non vanishing homology group. You have a simplicial group G_* with $\pi_0(G_*) = GL(\tilde{B})$ and ~~...~~ with ~~$\pi_i(G_*) = \pi_i(I_*)$~~ Look at π_1

$$1 \longrightarrow M(I_*) \longrightarrow G_* \longrightarrow GL(\tilde{B}) \longrightarrow 1$$

This is unstable a bit. If B is firm, then $\pi_0(I_*) = 0$

$$0 \longrightarrow I_0 \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

B ferin iff $I_0 = BI_0$ in which case F_1 maps onto I_0.

We have to be a little careful because of the simplicial stuff. You start with $F_0 \twoheadrightarrow B$ then choose F_1

First take $\longrightarrow F_1 \longrightarrow \overset{A_0}{\overset{\parallel}{F_0}} \longrightarrow B \to 0$ $\quad$ F_i ferin flat /B exact mod B-nil mods.

$$\cdots A_2 \rightrightarrows \overset{F_0\oplus F_1}{\overset{\parallel}{A_1}} \rightrightarrows \overset{F_0}{\overset{\parallel}{A_0}} \to B \quad \text{the semi. s. res.}$$

$$G_n = GL(\tilde{A}_n) \qquad 0 \to I_n \to A_n \to B \to 0$$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I_0 \oplus F_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & I_0 \arrow[d] \\
F_0 \oplus F_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F_0 \arrow[d] \\
B \arrow[r, equal] \arrow[d] & B \arrow[d] \\
0 & 0
\end{tikzcd}$$

YES

$F_0 \oplus F$

In any case $\quad 0 \longrightarrow I_n \to A_n \longrightarrow B \longrightarrow 0$

Any we have this complex.

$$\longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

whose homology starts in degree n. One can think of $F_\cdot$ as a DG ring where right mult is nonzero only on F_0. Corresp. simp. ring.

$$\longrightarrow A_n \longrightarrow \cdots \longrightarrow \overset{I_1}{A_1} \longrightarrow \overset{I_0}{A_0} \to B \longrightarrow 0$$

$$\longrightarrow B = B$$

simp. group $G_n = GL(\tilde{A}_n)$

$$
\begin{array}{ccccc}
M(I_n) & & M(I_1) & M(I_0) \\
 & & \downarrow & \downarrow \\
G_n & G_2 & G_1 & G_0 \\
 & & \downarrow & \downarrow \\
 & = & GL(B) = & GL(\tilde{B})
\end{array}
$$

So $M(I_*)$ is a simplicial abelian gp.
Now I need to calculate ~~this~~ the homology of the simp. gp. G. Made of

$$
\begin{array}{ccccccc}
 & & G^3 & & & & \\
 & & |||| & & & & \\
G_1^2 & \rightrightarrows & G^2 & \longrightarrow & GL(\tilde{B})^2 & \to & 1 \\
||| & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \\
G_n \quad G_1 & \rightrightarrows & G_0 & \longrightarrow & GL(\tilde{B}) & \to & 1 \\
 & & \downarrow\downarrow & & & & \\
 & & pt & & & &
\end{array}
$$

Basic ~~[illegible]~~ you have something like a top group with ~~[illegible]~~ $\pi_0 = GL(\tilde{B})$ and $\pi_n =$ ~~[illegible]~~ $\pi_n M(I)$
$= M(\pi_n I)$.

Enough to drive you nuts!

$$
\begin{array}{ccc}
I_n & I_1 \rightrightarrows & I_0 \\
\downarrow & \downarrow & \downarrow \\
A_n & A_1 \rightrightarrows & A_0 \\
\downarrow & & \downarrow \\
B & & B \\
 & & \downarrow \\
 & & 0
\end{array}
$$

$$
\begin{array}{c}
M(I) \\
\downarrow \\
GL(\tilde{A}) \\
\downarrow \\
GL(\tilde{B})
\end{array}
$$

You have a top gp G with $\pi_0 G = GL(\tilde{B})$
and $\pi_i G = 0$, $0 < i < n$
$\pi_n G = M(\pi_n I)$

$$\to F_n \to \qquad \to F_1 \to \overset{M = I_0}{F_0} \to \overset{M}{B} \to 0$$

$M_n \quad F_{n-1} \quad M_{n+1} \quad F_1 \xrightarrow{M_1} F_0 \to M_0 \to 0$

$\mathrm{Tor}_n^{\tilde{B}}(\mathbb{Z}, M_0) \xrightarrow{\sim} \mathrm{Tor}_{n-1}^{\tilde{B}}(\mathbb{Z}, M_1)$

$\xrightarrow{\sim} \mathrm{Tor}_1^{\tilde{B}}(\mathbb{Z}, M_{n-1}) \xrightarrow{\sim} \mathrm{Tor}_0^{\tilde{B}}(\mathbb{Z}, M_n)$

$H_n(F_\cdot) = \mathrm{Ker}(F_n \to F_{n-1}) / \mathrm{Im}(F_{n+1} \to F_n) = M_{n+1}/BM_{n+1}$

161 It seems that ~~Tor~~

$$0 \to M_n \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to B \to 0$$

$$\mathrm{Tor}^{\tilde B}_n(\mathbb{Z}, B) \xrightarrow{\sim} M_n/BM_n = H_{n-1}(F)$$

so if the bar homology $\mathrm{Tor}^{\tilde B}_*(\mathbb{Z}, B)$ begins in degree n, this is $H_{n-1}(F)$. e.g. if B is ferm where the bar homology begins in degree 2, and this is $H_1(F)$ can be $\neq 0$.

Change n to $n+1$. Then bar homology begins in degree $n+1$ ~~$\mathrm{Tor}^{\tilde B}_k(B,B)=0$~~

i.e. $\mathrm{Tor}^{\tilde B}_{\le n}(\mathbb{Z}, B) = 0 \iff H_k(F) = 0$ for $0 \le k < n$?

$$0 \to M_0 \to F_0 \to B \to 0$$

Assume $\mathrm{Tor}^{\tilde B}_0(\mathbb{Z}, B) = 0$ i.e. $B = B^2$

$$\mathrm{Tor}^{\tilde B}_{n+1}(\mathbb{Z}, B) \xrightarrow{\sim} M_{n+1}/BM_{n+1} = \frac{\mathrm{Ker}(F_n \to F_{n-1})}{B(\qquad)} = H_n(F).$$

So going back to $G = GL(\tilde A_*)$ top. gp

with $\pi_0(G) = GL(\tilde B)$ $\pi_i(G) = 0$ $0 < i < n$

and $\pi_n(G) = M(H_n(F)) = M\left(\mathrm{Tor}^{\tilde B}_n(\mathbb{Z}, B)\right)$.

If this is true, then ~~the~~ what do we find about $H_*(\bar W(G))$?

The point I guess is that the simp. gp. G leads to a fibring

$$B(G^{(e)}) \longrightarrow BG \longrightarrow B(\pi_0 G) = GL(\tilde B)$$

$G^{(e)}$ = identity component = $M(I_*)$ starts in degree n with nth bar homology

$\rightsquigarrow H_*\left(GL(\tilde B), M\left(\mathrm{Tor}^{\tilde B}_n(\mathbb{Z}, B)\right)\right)$ is some sort of obstruction.

~~What do we know?~~

Real problem: Show that A flat $\Longrightarrow GL(\mathbb{Z})$ acts trivially on $H_*(BGL(A))$. Can show following

$$H_*(GL(\tilde{A})) = \varinjlim H_*(GL(P \otimes_A A))$$ ~~where P~~

filtered colimit of finite free [redacted] $\tilde{A}^{op}$-modules equipped with $P \to A$. ~~We also found~~

$$P \otimes_A A \longrightarrow P \otimes_A \operatorname{Hom}_{A^{op}}(P, A) = M_n(A) \text{ if}$$

Suppose $A \in \mathcal{P}(\tilde{A}^{op})$. Then we have a rep of $\tilde{A}$ by left mult. on $A \in \mathcal{P}(\tilde{A}^{op})$. Get

$$\tilde{A} \longrightarrow \operatorname{End}_{A^{op}}(P)$$

What is the point? The point may be that since $A \in \mathcal{P}(A^{op})$ when one chooses $A \rightleftarrows \tilde{A}^n$ to calculate

$$K_*(\tilde{A}) \longrightarrow K_*(\operatorname{End}_{A^{op}}(A)) \longrightarrow K_*(\tilde{A})$$

one actually factors thru

$$\begin{tikzcd}
GL(\tilde{A}) \arrow[r] & GL(\operatorname{End}_{A^{op}}(A)) \arrow[r, hook] \arrow[dr] & GL(\tilde{A}) \\
 & & GL(A) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Problem: Assume $A \in \mathcal{P}(A^{op})$ i.e. f.g. proj over $\tilde{A}^{op}$ and $A^2 = A$. Recall $P \mapsto P \otimes_A A$, $\mathcal{P}(\tilde{A}^{op}) \to \mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op})$ ~~is an idempotent~~ induces a map $K_*(\tilde{A}) \longrightarrow K_*(\tilde{A})$ which is idempotent.

Basic properties: $U \in \mathcal{P}(\tilde{A}^{op})$

$$0 \longrightarrow U \otimes_A A \longrightarrow U \longrightarrow U/AU \longrightarrow 0$$
$$\parallel$$
$$0 \longrightarrow \bar{U} \otimes_{\mathbb{Z}} A \longrightarrow \bar{U} \otimes_{\mathbb{Z}} \tilde{A} \longrightarrow U/AU \longrightarrow 0$$

Shanuel gives an isom:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \bar{U}\otimes_{\mathbb{Z}} A \arrow[r, equal] \arrow[d] & \bar{U}\otimes_{\mathbb{Z}} A \arrow[d] & \\
0 \arrow[r] & U\otimes_A A \arrow[r] \arrow[d, equal] & \Phi(U) \arrow[r] \arrow[d] & \bar{U}\otimes_{\mathbb{Z}} \tilde{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U\otimes_A A \arrow[r] & U \arrow[r] \arrow[d] & \bar{U} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

This tells me that

Th

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0$$

~~This business may be hard. I know that~~

I know things about $BGL(\tilde{A})^+$ ~~the triangular~~ the triviality of affine group homology. I want to prove these results for $BGL(A)^+$ when A flat.

Assume $A \in \mathcal{P}(A^{op})$. ~~Suppose we start with~~ ~~Suppose we~~ We have $A \in \mathcal{P}(A^{op})$ i.e. $\exists$

$$A \underset{x}{\overset{y}{\rightleftarrows}} \tilde{A}^n \qquad yx = 1.$$

Then functors defined on $\mathcal{P}(\tilde{A}^{op})$

$$U \longmapsto U \otimes_A A \quad ?$$

I get a homom. $\tilde{A} \longrightarrow \mathrm{End}_{A^{op}}(\tilde{A}^n) = M_n(\tilde{A})$

$$\tilde{a} \longmapsto x\tilde{a}y$$

This gives me ~~a~~ $\tilde{A} \longrightarrow M_n(A) \subset M_n(\tilde{A})$ homom, and it apparently means that

164 What ~~[illegible]~~ do I do? ~~Make amends~~

Consider $P \longrightarrow A$ $\quad P$ f.-free $\tilde{A}^{op}$,

$$\begin{tikzcd} P \otimes_A A \arrow[r] \arrow[d] & \tilde{A} \otimes_A A \arrow[d] \\ \mathrm{End}_{A^{op}}(P) & A \end{tikzcd}$$

$P \to A$ $\quad P = \tilde{A}^n$

$$\begin{tikzcd} P \otimes_A A \arrow[r] \arrow[d] & \tilde{A} \otimes_A A \arrow[d] \\ P \otimes_A \mathrm{Hom}_{A^{op}}(P,A) & \tilde{A} \otimes_A \mathrm{Hom}_A(\tilde{A},A) \end{tikzcd}$$

$$P \otimes_A \mathrm{Hom}_{A^{op}}(P,A) = M_n A \qquad \tilde{A} \otimes_A \mathrm{Hom}_A(\tilde{A},A) = M_1 A$$

What I am trying is to calculate the map $K_*(A) \to K_*(A)$ associated to $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$. You take as definition

flat A^{op}

$$K_*(A) = \cancel{K_*(\ldots)}\ \mathrm{Ker}\{K_*(\tilde{A}) \to K_*(\mathbb{Z})\}$$

so it's a functor of a non-unital ring. Then you assume

$$\varinjlim_i K_*(P_i \otimes_A A) = K_*(A) \qquad \text{OKAY}$$

define

$$\begin{tikzcd} K_*(A) & K_*(P_i \otimes_A A) \arrow[l, mapsto] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Commutes}"] & K_*(A) \arrow[d] \\ & K_*(\mathrm{End}_{A^{op}}(P_i)) \arrow[r, equal] & K_*(\tilde{A}) \end{tikzcd}$$

$$K_*(\mathrm{End}_{A^{op}}(P_i)) = K_*(\tilde{A})$$

What happens is that we seem to have a proof that ?

Tomorrow you try to get $H_*(\underline{GL(A)}) \twoheadrightarrow H_*(\text{fibre } BGL(\tilde{A})^+ \to BGL(\mathbb{Z}))$

03/08/97

Problem: Given two flat firm rings A, B which are m.eq., ~~[illegible]~~ is it true that $H_*(BGL(A)) \simeq H_*(BGL(B))$??

The case to look at carefully should be $A \in \mathcal{P}(A^{op})$

$B = \mathrm{Hom}_{A^{op}}(A,A) = A \otimes_A \mathrm{Hom}_{A^{op}}(A,\tilde{A})$

In ~~[illegible]~~ general I would expect that the arguments which I gave for $K_*(A) \underset{\text{defn}}{=} K_*(\tilde{A})/K_*(\mathbb{Z})$ should go through for $H_*(BGL(A))$ provided one has Suslin's thms on the affine groups i.e.

$A \hookrightarrow \begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}$ and $A \hookrightarrow \begin{pmatrix} A & 0 \\ A & 0 \end{pmatrix}$ induce isos on $H_*(BGL(-))$.

Then I can use the simplicial group trick to handle the h-unital cases, and probably also calculate the obstruction.

$B, A, \quad A\otimes B \to B$
$a \otimes b \mapsto f(a)b$

See what happens for $A \in P(A^{op})$. $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

$B = \mathrm{Hom}_{A^{op}}(A,A) = A \otimes_A \mathrm{Hom}_{A^{op}}(A, \tilde{A}) = A \otimes_A B$, and $B \otimes_B A = A$ because B is unital. But in this case A amounts to a flat unitary B-module with B-map $A \xrightarrow{f} B$ WRONG such that $f(A)B = B$. So we can reduce by $\varinjlim$'s to the case of $A = B^n$, and the result is clear.

Next suppose A right flat and meq a unital ring B.

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. Then ~~we have~~ A arises from the form dual pair $(Q, P, \; P \otimes Q \twoheadrightarrow B)$ over B

Wait: Go back to $A \in P(A^{op})$ $\quad \begin{pmatrix} A & B \\ A & B \end{pmatrix}$

A is a B-module equipped with $A \otimes B \to B$, a B-bimodule map, i.e. $A \xrightarrow{f} \mathrm{Hom}_{B^{op}}(B,B) = B$, i.e. $A \otimes B \to B$ has the form $a \otimes b \mapsto f(a)b$, where f is a B-module map. B is B^{op} flat $\Rightarrow$ $B \otimes_B P = A$ is A^{op} flat (in fact finite proj.). Thus A can be any B-module equipped with a B-map $f: A \to B$ whose image (which is a left ideal) gen. B as an ideal. ~~So the only filtere~~ By filtered ind limits can suppose A is finitely presented B-module.

166 Special case to be well understood.

$A \in \mathcal{P}(A^{op}) \qquad B = \mathrm{Hom}_{A^{op}}(A, A)$

Put another way ~~B~~ B is a unital ring, A is a unitary B-module equipped with a B-map $f : \cancel{A} \to \cancel{B}$ such that $f(A)B = B$. I now need to understand $H_*(BGL(A))$. Now I know that ~~...~~ $A \in \mathcal{P}(A^{op})$ in particular A is right flat. ~~...~~

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix} \qquad \begin{pmatrix} B, A, & A \otimes B \to B \\ & a \otimes b \mapsto f(a)b \end{pmatrix}$$ I know that

A is A-flat $\iff$ $P \otimes_A A = A$ is B-flat. Examine this case first.

But ~~...~~ because B left acts on $A \in \mathcal{P}(A^{op})$ we should get a ~~map~~ homom. $\cancel{B} \to M_n(A)$. To simplify suppose $A \subset B$ is a left ideal such that $AB = B$, say $yx = 1$ $y \in A$, $x \in B$. Then we have

$$A \underset{x\cdot}{\overset{y\cdot}{\rightleftarrows}} A$$

Given $b \in B$, then have $b\cdot$ on A, send b to $xby\cdot$. This is a homom. $\phi : B \to A$. Now ask about compositions

$$A \overset{}{\underset{f}{\hookrightarrow}} B \xrightarrow{\phi} A \underset{f}{\hookrightarrow} B.$$

~~...~~ It's probably true for the old arg. that $f\phi : B \to B$ induces the identity on $H_*(GL(B))$, also that $\phi f : A \to A$ induces the identity on $H_*(GL(\tilde{A}))$.

$A \in P(A^{op})$ $\qquad B = \mathrm{Hom}_{A^{op}}(A, A)$ $\qquad \begin{pmatrix} A & B \\ A & B \end{pmatrix}$

form dual pair over B
B is unital $\qquad \left(B,\ A,\ \begin{array}{l} A \otimes B \to B \\ a \otimes b \mapsto f(a)b \end{array}\right)$. Then

$\exists \quad \sum y_i \otimes x_i \in A \otimes_{\mathbb{Z}} B \quad \ni \quad \sum f(y_i) x_i = 1.$

To simplify suppose $f : A \to B$ is inj, whence A is a left ideal in B generating B as ideal. Also suppose $y \otimes x \in A \otimes B$, $yx = 1$. Then one has homos.

$$\begin{array}{ccccccc} A & \xhookrightarrow{f} & B & \xrightarrow{\phi} & A & \hookrightarrow & B \\ & & b & & xby & & \end{array}$$

aim to show $\phi f : \begin{array}{l} A \longrightarrow A \\ a \longmapsto xay \end{array}$, $\quad f\phi : \begin{array}{l} B \longrightarrow B \\ b \longmapsto xby \end{array}$

induce isos on $H_*(BGL(-))$.

What is going on here? You have $A \in P(A^{op})$ so A is a direct summand of a free module

$$0 \longrightarrow A \underset{x}{\overset{y}{\rightleftarrows}} \tilde{A} \longrightarrow \tilde{A}/xA \longrightarrow 0$$

The hom. $A \xrightarrow{f} B$ corresp. to A left acting on itself. $B \longrightarrow \tilde{A}$ corres to B acting on A transported to the summand $xy\tilde{A}$ of $\tilde{A}$. So what happens?

Thus we need to be able to compare the two A actions on $\tilde{A}$, namely left mult. by a and left mult by xay.

 One repn. is $\tilde{\tilde{A}}$, the other is $\tilde{A} \otimes_A \tilde{\tilde{A}}$

Take $P = A$. Then you have $P \otimes_A A$ left acting on P over A^{op}, and you have it acting on $\tilde{A}$ through the homom. $P \otimes_A A \to \tilde{A} \otimes_A \overset{A}{\overset{\parallel}{A}}$

Fund. Idea is ~~for~~ that $P \xrightarrow{f} \tilde{A}$ is a nil isom for action of $B = P \otimes_A A$.

$$P \xrightarrow{\Gamma_f} P \oplus \tilde{A} \xrightarrow{pr_2} \tilde{A}$$

$$U \otimes_B P \hookrightarrow U \otimes_B (P \oplus \tilde{A}) \twoheadrightarrow U \otimes_B \tilde{A}$$

I'm thinking of functors from $\mathcal{P}(\tilde{B}^{op})$ to ?

You have $P \otimes_A A$ acting on P and on $\tilde{A}$

Here P and $\tilde{A}$ are in $\mathcal{P}(\tilde{A}^{op})$. You representations

---

Start again. $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$. ~~The key point is~~

You assume $A \in \mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op})$. You have exact sequence of functors of $U \in \mathcal{P}(\tilde{A}^{op})$

$$0 \to U \otimes_A A \to U \to U/UA \to 0$$

$0 \to (U/$[redacted]$UA) \otimes_{\mathbb{Z}} A \to (U/UA) \otimes_{\mathbb{Z}} \tilde{A} \to \overset{\parallel}{(U/UA)} \to 0$

169 which you can use to ~~differ~~ relate the functors $U\otimes_A A$, U, $U/AU\otimes_{\mathbb{Z}} A$, $U/AU\otimes_{\mathbb{Z}}\tilde{A}$ from $\mathcal{P}(\tilde{A}^{op})$ to itself.

But now I really want to understand [redacted] $H_*(BGL(A))$. ~~We have the functor~~

We have the functor $U \longmapsto U\otimes_A A$. In the end I need to relate $H_*(BGL(A))$ with $H_*(BGL(\tilde{A}))$.

[redacted] Take

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

Assume A right flat, so $A = \varinjlim P_i\otimes_A A$ where $\varinjlim_i P_i = A$, $P_i \in \mathcal{P}(\tilde{A}^{op})$. Now $P\otimes_A A$ acts on P

$P \quad \tilde{A}$

$P \quad A$

$(P, \tilde{A}, \tilde{A}\otimes P \to A)$
$\tilde{a}\otimes p \mapsto \tilde{a}f(p)$

$$P\otimes_A \tilde{A} \longrightarrow P\otimes_A \mathrm{Hom}_{A^{op}}(P,A)$$

~~This makes~~ Have homom. $P \to \mathrm{Hom}_{A^{op}}(P, PA)$

So we have

$$P \longrightarrow \mathrm{Hom}_{A^{op}}(P, PA) = P\otimes_A A\otimes \mathrm{Hom}_{A^{op}}(P,\tilde{A})$$

$$\cap$$

$$P\otimes_A \mathrm{Hom}_{A^{op}}(P,\tilde{A})$$

I also have $P \to A$ $\quad p\otimes q$

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "f"] & P' \arrow[d, "f"] \arrow[dl, "\phi"'] \\ P \arrow[r, "u"] & P' \end{tikzcd}$$

$$\begin{array}{ccc} p\otimes q & \longmapsto & (p' \longmapsto p\langle q, p'\rangle) \\ \downarrow & & \\ p\otimes v(q) & \longmapsto & (p' \longmapsto p\langle v(q), p'\rangle) \end{array}$$

~~Hence~~ $\phi_{p\otimes q}(p') = p\langle q, p'\rangle$

$p_1 \longmapsto u(p_1) \quad p'$

$p\langle q, u(p_1)\rangle$
$\quad \|$
$(p\otimes q)\, p_1$

$(u(p)\otimes q)\, p'$
$\quad \|$
$p\langle q, p'\rangle \mapsto u(p)\langle q, p'\rangle$

$$
\begin{tikzcd}
B = P\otimes_A Q \arrow[r] \arrow[dr] \arrow[dd] & \mathrm{End}(P) \\
& \begin{pmatrix} \mathrm{End}(P) & \mathrm{Hom}(P'/u(P), P) \\ 0 & 0 \end{pmatrix} \arrow[u] \arrow[d, phantom, "\cap"] \\
B' \arrow[r] & \mathrm{End}(P \oplus P'/u(P'))
\end{tikzcd}
$$

Suppose $u : P \to P'$ com. with $u\otimes 1 : P\otimes_A Q \to P'\otimes_A Q$

$$P\otimes_A Q \longrightarrow \mathrm{Hom}_{A^{op}}(P, P)$$

How to think? ~~You need to handle. You have~~

Given a repn of B on $P \in \mathcal{P}(\tilde{A}^{op})$ one has a map $K_*(\tilde{B}) \xrightarrow[\cdot[P]]{} K_*(\tilde{A})$.

Moreover an exact sequence of repns. $0 \to P_1 \to P_2 \to P_3 \to 0$ yields $(\cdot[P_1]) + (\cdot[P_3]) = (\cdot[P_2])$. Also if B kills P/PA then get

$$
\begin{tikzcd}
K_*(B) \arrow[r, dashed] \arrow[d] & K_*(A) \arrow[d] \\
K_*(\tilde{B}) \arrow[r] \arrow[d] & K_*(\tilde{A}) \arrow[d] \\
K_*(\mathbb{Z}) \arrow[r, "n"] & K_*(\mathbb{Z})
\end{tikzcd}
\qquad
\begin{aligned} n &= \text{rank } P \\ &= \text{rank}_{\mathbb{Z}}\, P/\cancel{PA} \end{aligned}
$$

So now give $P \xrightarrow{u} P'$ you factor

$$P \longrightarrow \underbrace{P\oplus P'}_{P_1} \longrightarrow P'$$

$$0 \to P \to P_1 \longrightarrow P_2 \longrightarrow 0 \qquad \text{exact sequence}$$

of repns of B on $\mathcal{P}(\tilde{A}^{op})$. $\therefore$ Now $BP_2 = 0$ so we have

$$
\begin{tikzcd}
K_*(\tilde{B}) \arrow[rr, "{\cdot[P_2]}"] \arrow[dr] & & K_*(\tilde{A}) \\
& K_*(\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
$$

mult. by class of P_2 in $K_0(\tilde{A})$

$\therefore \cdot[P_2]$ is zero on $K_*(B)$.

 So review the steps.

consider dual pair $(P, Q, \ Q\otimes_{\mathbb{Z}} P \to A)$ over A. If $P \in \mathcal{P}(\tilde{A}^{op})$,

then have homom. $P \otimes_A Q \longrightarrow \mathrm{Hom}_{A^{op}}(P, P) \to \mathrm{Hom}_{\mathbb{Z}}(P/PA, P/PA)$

$$K_*(\widetilde{P\otimes_A Q}) \longrightarrow K_*(\tilde{A})$$ compatible with augmentation

$$\therefore \quad K_*(P\otimes_A Q) \longrightarrow K_*(A)$$

Preliminaries: $\mathrm{Rep}(\tilde{B}, \mathcal{P}(\tilde{A}^{op}))$

$$K_0(\mathrm{Rep}(\tilde{B}, \mathcal{P}(\tilde{A}^{op}))) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_*(\tilde{B}), K_*(\tilde{A})).$$

if P is (B,A) bimodule with $P \in \mathcal{P}(\tilde{A}^{op})$

then $U \longmapsto U \otimes_B P$, $\quad \mathcal{P}(\tilde{B}^{op}) \longrightarrow \mathcal{P}(\tilde{A}^{op})$

additive, induces

$$\begin{tikzcd}
K_*(\mathcal{P}(\tilde{B}^{op})) \arrow[r] \arrow[d, equal] & K_*(\mathcal{P}(\tilde{A}^{op})) \arrow[d, equal] \\
K_*(\tilde{B}) \arrow[r, "\phi^P"] \arrow[d] & K_*(\tilde{A}) \arrow[d, "\varepsilon"] \\
K_*(\mathbb{Z}) \arrow[r, "\text{mult by rank of } P/PA \text{ over } \mathbb{Z}"] & K_*(\mathbb{Z})
\end{tikzcd}$$

$$U \longmapsto U \otimes_B P \longmapsto U \otimes_B P/PA = (U/UB) \otimes_{\mathbb{Z}} (P/PA)$$

~~Another basic property~~

$B = P \otimes_A Q$ acts on $P \in \mathcal{P}(\tilde{A}^{op})$

get $K_*(P\otimes_A Q) \xrightarrow{\phi^P} K_*(A)$

Claim this functorial in (P, Q) $\qquad (u, v) : (P, Q) \longrightarrow (P', Q')$

$$(P, Q) \longrightarrow (P', Q) \longrightarrow (P', Q')$$

$$\begin{tikzcd}
B_1 = P'\otimes_A Q \arrow[d] \arrow[dr] & \\
B_2 = P'\otimes_A Q' \arrow[r] & \mathrm{Hom}_{A^{op}}(P',P')
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{P}(B_1^{op}) \arrow[d, "-\otimes_{B_1}\tilde{B}_2"'] \arrow[dr, "-\otimes_{B_1}P'"] & \\
\mathcal{P}(B_2^{op}) \arrow[r, "-\otimes_{B_2}P'"'] & \mathcal{P}(\tilde{A}^{op})
\end{tikzcd}$$

$$U\otimes_{B_1}\tilde{B}_2\otimes_{B_2}P' = U\otimes_{B_1}P'$$

$$\begin{tikzcd}
K_*(\tilde{B}_1) \arrow[d] \arrow[dr, "\phi^{P'}"] & \\
K_*(\tilde{B}_2) \arrow[r, "\phi^{P'}"'] & K_*(\tilde{A})
\end{tikzcd}$$

$u: P \longrightarrow P'$ $\quad u\otimes 1$: $B = P\otimes_A Q$, $B' = P'\otimes_A Q$

$$\begin{tikzcd}
\mathcal{P}(\tilde{B}^{op}) \arrow[r, "\otimes_B P"] \arrow[d, "-\otimes_B\tilde{B}'"'] \arrow[dr, "-\otimes_B P'"] & \mathcal{P}(\tilde{A}^{op}) \\
\mathcal{P}(\tilde{B}'^{op}) \arrow[r, "\otimes_{B'}P'"'] & \mathcal{P}(\tilde{A}^{op})
\end{tikzcd}$$

$$\begin{tikzcd}
K_*(\tilde{B}) \arrow[r, "\phi^P"] \arrow[d] \arrow[dr, "\phi^{P'}"] & K_*(\tilde{A}) \\
K_*(\tilde{B}') \arrow[r, "\phi^{P'}"'] & K_*(\tilde{A})
\end{tikzcd}$$

Fact $\quad u: P \longrightarrow P'$ is a B-nil isom.

Can factor: Two cases $\quad P \hookrightarrow P' \twoheadrightarrow P_0$

$\qquad\qquad\qquad\qquad\quad P_0 \hookrightarrow P \twoheadrightarrow P'$

$$\phi^{P'} = \phi^P + \phi^{P_0} \qquad \phi^{P_0} = 0 \text{ on } K_0(B).$$

$$U \longmapsto U\otimes_B P_0 = (U/UB)\otimes_{\mathbb{Z}} P_0$$

173 Use the fact that P flat ~~is~~ is a filtered ind limit of <u>f.g. projectives.</u>

03/10/97 For lecture

~~[illegible]~~: Review of $K_*(\tilde{A})$

$$K_*(\tilde{A}) = \pi_*\left(BGL(\tilde{A}^\bullet)^+\right)$$

functorality property: P a (B,A)-bimodule such that $P \in \mathcal{P}(\tilde{A}^{op})$, get $K_*(\tilde{B}) \xrightarrow{\phi_P} K_*(\tilde{A})$

e.g. a homom. $B \longrightarrow A$

And now comes a big effort to show that $\pi_* BGL(A)^+ = K_*(A)$ for A flat ~~[illegible]~~ then h-unital.

The idea I had is to adopt the unital ring <u>proof</u> of the affine group result. Lets begin Let's go over things until we understand them. The ~~[illegible]~~ Affine group $GL_n(A) \ltimes A^n$, for A unital, describes autos of $0 \to A^n \longrightarrow A^{n+1} \longrightarrow A \to 0$ inducing the identity on A.

$$\left(\begin{array}{c|c} \alpha & \beta \\ \hline 0 & 1 \end{array}\right)$$

So there is one affine group for each $n \geq 0$, Objects are exact sequences $0 \longrightarrow V \longrightarrow E \longrightarrow V_0 \longrightarrow 0$ where V_0 is fixed. monoidal operation $E_1 \times_{V_0} E_2$. key point is that $E \times_V E \simeq E \times_V \cancel{(V \oplus C)}\ \bar{E}$, $\bar{E}$ the split extension. So ~~we win~~ in the K-theory. $E = \bar{E}$. Now we have problems. What should be done?

174 A place to start might be the proof that $\begin{pmatrix} A & \tilde{A} \\ A & \tilde{A} \end{pmatrix}$ and A flat yields the identity map on $K_*(A)$. $\boxed{\text{YES}}$.

Start with $P \to A$ $\quad P \simeq \tilde{A}^n$ then

Start in general with $\begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix}$ $\quad P = \tilde{A}^n$

Want $H_*(GL(P\otimes_A Q)) \longrightarrow H_*(GL(A))$.

$$\begin{array}{ccc} P\otimes_A Q & \longrightarrow & \mathrm{Hom}_{A^{op}}(P, P) \\ \downarrow & & \| \\ M_n(A) & \subset & M_n(\tilde{A}) \end{array}$$

So you see the problem: You will have $B \to M_n(A)$ hence $GL_k(B) \to GL_{kn}(A)$. But things are defined at least. But there's an ordering problem, identifying $GL_k(M_n(A))$ with $GL_{kn}(A)$.

There are delicate issues here.

Special case where $A \in \mathcal{P}(A^{op})$, $B = \mathrm{Hom}_{A^{op}}(A, A)$. Such an A arises from a unitary B-mod. map $f: A \to B$ such that $f(A)B = B$.

Dual pair $(B, A, A\otimes_{\mathbb{Z}} B \to B)$. For any such A
$$a\otimes b \mapsto f(a)b$$
we know A is right flat $\quad \begin{pmatrix} A & B \\ A & B \end{pmatrix}$

B is rt flat $\Longrightarrow$ $B\otimes_B P = A$ $\quad$ is A^{op} flat.

175 So now I would like to prove that $GL(A) \to GL(B)$ induces an isomorphism on homology. First case would be $A \twoheadrightarrow B$, then $A = B \oplus N$ where it follows from the unital theory I think.

Let's examine this. $A \xrightarrow{f} B$ given B-linear. A is ring: $a_1 a_2 = f(a_1) a_2$, f is a homom. If B unital, f onto then can lift B into A, i.e. choose $e \in A$ $f(e) = 1$. Then

$$B \oplus N \xrightarrow{\sim} A \qquad N = \mathrm{Ker}(f).$$
$$b + n \longmapsto be + n$$

and $(b,n)(b_1,n_1) = b(b_1,n_1) = (bb_1, bn_1)$. So I'm now interested in $B \oplus N$ where N is an unitary B-module viewed as a B-bimodule w usual left action and 0 right action. Claim $GL(B) \to GL(B \oplus N)$ homology iso.

$$GL(B) \to GL(B\oplus N) = GL(B) \ltimes M(N)$$

You should be able to prove this somehow. You expect to use a category of extensions $0 \to B^n \to E \to N \to 0$ "Whitney sum" given by fibre product over N.

$$\begin{tikzcd}
B^n \arrow[r] & E \times_N E \arrow[r] \arrow[d, "f"'] & E \arrow[l, bend right, dashed] \arrow[d, "f"] \arrow[dl] \\
B^n & E \arrow[r] & N
\end{tikzcd}$$

 Basic idea. For each $n \in \mathbb{N}$ you have group $GL_n(B) \ltimes N^n$

$$\left(\begin{array}{c|c} GL_n(B) & \underset{\mathrm{Hom}_B(B^n, N)}{N^n} \\ \hline & \end{array}\right)$$

$$\mathrm{Hom}_B(B^n, B^n)^{op} = M_n(B)$$

So you want to consider extns.

$$0 \longrightarrow N \longrightarrow E \longrightarrow B^n \longrightarrow 0$$

What I learn this way I think is that

$$\varinjlim_n H_*\left(GL_n(B) \ltimes N^n\right) \xleftarrow{\sim} H_*(GL(B))$$

Ultimately you will argue that the two homom.

$$\begin{tikzcd} \mathrm{Aut}(E_n) \arrow[r] \arrow[dr] & \mathrm{Aut}(E_n \perp E_n) \\ & \mathrm{Aut}(E_n \perp \bar{E}_n) \end{tikzcd}$$

are conjugate?

Actually it might help to use the action of GL on $\widetilde{GL}$

There might be an alternative.

$$M_n B \subset \left(\begin{array}{c|c} M_n B & B^n \\ \hline 0 & 0 \end{array}\right) \leftarrow \subset \left(\begin{array}{c|c} M_n B & M_n B \\ \hline 0 & 0 \end{array}\right)$$

Another point. A is the same thing as a ring with left identity. $A = Ae \oplus A(1-e)$

 03/~~0~~2/97 So what's going on?

I consider $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ with B unital, i.e.

$f: A \to B$ a B-module map, ~~with~~ A unitary, such that ~~B~~ $f(A)B = B$. Thus f is a surjection onto a left ideal generating B. Then f is a homomorphism when A has prod $a_1 a_2 = f(a_1)a_2$. To show $f: GL(A) \to GL(B)$ homology isom. We are considering the map of dual pairs ~~Q~~ over B ~~[illegible]~~

$$\left(B, A, \underset{a\otimes b \mapsto f(a)b}{A\otimes B \to B}\right) \xrightarrow{(\cancel{1}, f)} \left(\cancel{A}, \cancel{B}, \underset{b_1 \otimes b_2 \mapsto b_1 b_2}{B\otimes B \to B}\right).$$

It is natural to factor $f: A \to B$ into $A \xrightarrow{\binom{1}{f}} A \oplus B \xrightarrow{pr_2} B$.

Then we are looking at ~~GLB~~

~~What~~ Does M inv ~~hold~~ for the Volodin model?

I should carefully go over my arguments so as to see what I need to establish M inv. Write them up!! Especially trans.

Basic construction. Given dual pair (P, Q) over A with $P \in \mathcal{P}(\tilde{A}^{op})$

Given dual pair $(\tilde{A}^n, Q)$ over A get homom.

$$\tilde{A}^n \otimes_A Q \longrightarrow \tilde{A}^n \otimes_A \mathrm{Hom}_{A^{op}}(\tilde{A}^n, A) = M_n(A)$$

Now ask about maps, i.e. given $u: \tilde{A}^p \xrightarrow{(a_{ij})} \tilde{A}^n$ have map $(u, 1): (\tilde{A}^p, Q) \longrightarrow (\tilde{A}^n, Q)$. Point: You factor u $\tilde{A}^p \longrightarrow \tilde{A}^p \oplus \tilde{A}^n \longrightarrow \tilde{A}^n$. The surjection case is ~~easy~~ simpler

How do you ~~propose~~ to analyze this? You are constructing a system of matrix like rings. For each n you have $(\tilde{A}^n, A^n)$ dual pair. And then for each $\tilde{A}^p \longrightarrow \tilde{A}^n$ you have some link. So what happens? You eventually factor this map. So really you are looking at

$$\begin{tikzcd} & \tilde{A}^p \oplus \tilde{A}^n \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \tilde{A}^p & & \tilde{A}^n \end{tikzcd}$$

and all possible sections of pr_1. What sort of condition arises? Actually, what am I trying to do? You have a category somewhere - the objects are $\tilde{A}^n$ $n \in \mathbb{N}$, and to each object $\tilde{A}^n$ you assign the ring $M_n(A) = \tilde{A}^n \otimes_A A^n$. Maybe the object is the dual pair $(\tilde{A}^n, A^n)$.

$$P \rightsquigarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$

You want a "space" X which receives a map $BGL_n(A) \longrightarrow X$ for each n. For each pair $p,$ [redacted] n there is going to be some link between these maps. What is it that you want to know?

What do you need? There's a compatibility between. You need compat with maps of dual pairs. I think you can assume Q doesn't change. The result

The first condition

First review factoring. $f : U \longrightarrow V$

$$U \xrightarrow{\binom{1}{f}} U \oplus V \xrightarrow{(0\ 1)} V$$

$$U \xrightarrow{\binom{1}{0}} U \oplus V \xrightarrow{(f\ 1)} V$$

simplest case $\quad (P_0 \oplus P, P^*) \xrightarrow{pr_2, 1} (P, P^*)$

$\langle P^*, P_0 \rangle = 0$

Then we have homoms.

$$\begin{tikzcd}
(P \oplus P_0) \otimes_A P^* = P \otimes_A P^* \oplus P_0 \otimes_A P^* \arrow[r] \arrow[d, hook] & P \otimes_A P^* \\
(P \oplus P_0) \otimes_A (P^* \oplus P_0^*) &
\end{tikzcd}$$

$$\begin{pmatrix} P \otimes_A P^* & P \otimes_A P_0^* \\ P_0 \otimes_A P^* & P_0 \otimes_A P_0^* \end{pmatrix} \hookleftarrow \begin{pmatrix} P \otimes_A P^* & 0 \\ P_0 \otimes_A P^* & 0 \end{pmatrix} \twoheadrightarrow P \otimes_A P^*$$

The condition is simply that if Γ = endo ring of $\quad 0 \longrightarrow P_0 \longrightarrow P \oplus P_0 \longrightarrow P \longrightarrow 0$ inducing 0 on P_0, then the two maps $\Gamma \rightrightarrows GL(A)$ have the same effect

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inj. $\quad 0 \longrightarrow P \xrightarrow{u} P \oplus P_0 \longrightarrow P_0 \longrightarrow 0$

$$\begin{tikzcd}
P \otimes_A (P^* \oplus P_0^*) \arrow[r, hook] \arrow[d] & (P \oplus P_0) \otimes_A (P \oplus P_0)^* \\
P \otimes_A P^* &
\end{tikzcd}$$

183 So anything else?

Problem: Find ~~[illegible]~~ a good way to glue together ~~[illegible]~~ the spaces BGL_n, $n \geq 0$ so as to incorporate these affine ~~group~~ conditions. ~~[illegible]~~ ~~[illegible]~~ Is there an analogue of ~~[illegible]~~ the Q category? ~~[illegible]~~

Philosophy: I think you are trying to construct an analogue of ~~[illegible]~~ appropriate monoidal quotient of $\mathcal{P}(\tilde{A})$ ~~[illegible]~~ by the action of $\mathcal{P}(\mathbb{Z})$.

Try for Q-category. Recall that ~~[illegible]~~ a map in the Q category has the form $i_* j^*$ where

$$\begin{array}{ccc} W & \xrightarrow{i} & V_2 \\ \downarrow{\scriptstyle j} & & \\ V_1 & & \end{array}$$

j adm. surj, i adm. injection. ~~[illegible]~~ ~~[illegible]~~ One has something similar, namely exact sequences of representations

$$0 \longrightarrow P \xrightarrow{i} P' \longrightarrow P_0 \longrightarrow 0$$

$$0 \longrightarrow P_0 \longrightarrow P \xrightarrow{j} P' \longrightarrow 0$$

where the action is trivial on P_0.

Get specific. injective to

injective type $\quad 0 \to P \xrightarrow{\iota} P' \to P_0 \to 0$, $\quad P' = P \oplus P_0$

$$\begin{array}{ccc} \mathrm{Aut}(P) \ltimes \mathrm{Hom}(P_0, P) & \hookrightarrow & \mathrm{Aut}(P') \\ \downarrow & & \\ \mathrm{Aut}(P) & & \end{array}$$

$$\begin{array}{ccc} \begin{pmatrix} P \otimes_A P^* & P \otimes P_0^* \\ 0 & 0 \end{pmatrix} & \hookrightarrow & \begin{pmatrix} P \otimes P^* & P \otimes P_0^* \\ P_0 \otimes P^* & P_0 \otimes P_0^* \end{pmatrix} \\ \downarrow & & \\ P \otimes_A P^* & & \end{array}$$

surjective type $\quad 0 \to P_0 \to P' \to P \to 0$, $\quad P' = P \oplus P_0$

$$\begin{array}{ccc} \begin{pmatrix} P \otimes_A P^* & 0 \\ P_0 \otimes_A P^* & 0 \end{pmatrix} & \subset & \begin{pmatrix} P \otimes_A P^* & P \otimes_A P_0^* \\ P_0 \otimes_A P^* & P_0 \otimes_A P_0^* \end{pmatrix} \\ \downarrow & & \\ P \otimes P^* & & \end{array}$$

What might be a map?? ~~[illegible]~~

What is the key idea? What should be an <u>injective type map from</u> P to P' ???

too hard. Given B choose a flat

What am I trying to do? I think it's true that

185 You need to get your arguments so clean that you know what should be true. You want to show that ~~[illegible]~~ two firm ~~[illegible]~~ left flat rings which are meq have the same $H_*(GL(-))$. ~~[illegible]~~ Suppose this true in the stronger form that when

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Let's work out the details of trans.

Basic construction: ~~[illegible]~~ Given $(P, Q, Q\otimes_{\mathbb{Z}} P \to A)$

P ~~[illegible]~~ flat A^{op}-module, get canonical

$$\tau_P : K_*(P\otimes_A Q) \longrightarrow K_*(A).$$

$$\begin{pmatrix} A & P^* & P^*\otimes_B Q^* \\ P & B & Q^* \\ Q\otimes_B P & Q & \end{pmatrix}$$

Clarify transitivity:

Given (P, P^*) over A, P ~~[illegible]~~ A^{op}-flat

(Q, Q^*) over $B = P\otimes_A P^*$ where Q is B^{op}-flat

get $\left(Q\otimes_B P,\ P^*\otimes_B Q^*,\ (P^*\otimes_B Q^*)\otimes(Q\otimes_B P) \to P^*\otimes_B B\otimes_B P \to P^*\otimes_B P \to A\right)$

with $Q\otimes_B P$ A^{op} flat

Claim

$$\begin{tikzcd}
K_*\left((Q\otimes_B P)\otimes_A (P^*\otimes_B Q^*)\right) \arrow[r, "\tau_{Q\otimes_B P}"] \arrow[d] & K_*(A) \\
K_*(Q\otimes_B Q^*) \arrow[r, "\tau_Q"] & K_*(B) \arrow[u, "\tau_P"']
\end{tikzcd}$$

How to prove? ~~[illegible]~~ Reduces to case $Q = \tilde{B}^n$ and $Q^* = \mathrm{Hom}_{B^{op}}(Q, B)$

Suppose Q is also left A flat. Can suppose P, Q f. free right resp left ~~[illegible]~~ over $\tilde{A}$. ~~Suppose P, Q~~

$$\begin{tikzcd}
 & P\otimes_A Q \arrow[dl] \arrow[r] & Q^*\otimes_A Q \\
P\otimes_A P^* & &
\end{tikzcd}$$

 Consider $(P, Q, Q\otimes_{\mathbb{Z}} P \to A)$ where $P \in \mathcal{P}(\tilde{A}^{op})$ and $Q \in \mathcal{P}(\tilde{A})$. To show

$$\tau_P, \tau_Q : K_*(P\otimes_A Q) \longrightarrow K_*(A)$$

coincide.

$$\begin{array}{ccccc} \mathrm{Hom}_{A^{op}}(P, PA) & \leftarrow & P\otimes_A Q & \longrightarrow & \mathrm{Hom}_A(Q, AQ)^{op} \\ \| & & & & \| \\ M_n(A) & & & & M_{n'}(A) \end{array}$$

Roughly one should be the contragredient repn of the other.

Do the unital case first. $P \in \mathcal{P}(A^{op})$, $Q \in \mathcal{P}(A)$, $Q\otimes P \longrightarrow A$ given. The point is that we have a map of dual pairs.

Consider the case where $Q = P^\vee = \mathrm{Hom}_{A^{op}}(P, A)$. Then

$$\begin{array}{ccc} P\otimes_A Q & \xrightarrow{\sim} & \mathrm{Hom}_{A}(Q, Q)^{op} \\ \wr\| & & \\ \mathrm{Hom}_{A^{op}}(P, P) & & \end{array}$$

So now consider A nonunital case. We have a map of dual pairs $(P, Q) \longrightarrow (P, P^*)$ where $P^* = \mathrm{Hom}_{A^{op}}(P, A) = A\otimes_A P^\vee$

$$\begin{array}{ccc} K_*(P\otimes_A Q) & \xrightarrow{\tau_Q} & \\ \downarrow & & K_*(A) \\ K_*(P\otimes_A P^*) & \xrightarrow{\tau_{P^*}} & \end{array}$$

Point is that $\quad P\otimes_A P^* \longrightarrow$

187 What is so confusing.

You assume $P \simeq \tilde{A}^n_{\text{right}}$ $\quad Q \simeq \tilde{A}^p_{\text{left}}$. ~~Then you~~

~~If you have~~ In your case ~~you take~~ $K_*(A) \overset{\text{def}}{=} \operatorname{Ker}\{K_*(\tilde{A}) \to K_*(\mathbb{Z})\}$. Then you have the trace maps τ_P, τ_Q are induced by

$$\begin{tikzcd}
\mathrm{End}_{A^{op}}(P) \arrow[d, phantom, "\parallel"] & P\otimes_A Q \arrow[l] \arrow[r] \arrow[d] & \mathrm{End}_A(Q)^{op} \arrow[d, phantom, "\parallel"] \\
M_n(\tilde{A}) & P\otimes_A \check{P} \arrow[dr] & M_p(\tilde{A}) \\
& & \mathrm{End}_A(\check{P})^{op}
\end{tikzcd}$$

But we know that

$$\begin{tikzcd}
& K_*(P\otimes_A Q) \arrow[dl, "\tau_P"'] \arrow[r] \arrow[d] \arrow[rr, bend left, "\tau_Q"] & K_*(\mathrm{End}_A(Q)^{op}) \arrow[r] & K_*(\tilde{A}) \arrow[d, phantom, "\parallel"] \\
K_*(\tilde{A}) & K_*(P\otimes_A \check{P}) \arrow[l, "\tau_P"] \arrow[dl] \arrow[r] \arrow[rr, bend right, "\tau_{\check{P}}"'] & K_*(\mathrm{End}_A(\check{P})^{op}) \arrow[r] & K_*(\tilde{A}) \\
K_*(\mathrm{End}_{A^{op}}(P)) \arrow[u] & & &
\end{tikzcd}$$

commutes.

But suppose you want to prove this for $H_*(BGL(-))$.

$$\begin{tikzcd}
& P\otimes_A Q \arrow[dl, "\tau_P"'] \arrow[d] \arrow[r, "\tau_Q"] & \mathrm{Hom}_A(Q,A)\otimes_A Q = M_p(A) \\
M_n(A) = & P\otimes_A A \otimes_A \check{P} \arrow[r, "\tau_{\check{P}}"] & \mathrm{Hom}_A(\check{P},A)\otimes_A \check{P} = M_n(A)
\end{tikzcd}$$

It seems OK. You have

$$\begin{tikzcd}
(P, Q) \arrow[r] \arrow[d] & (P, A\otimes_A \check{P}) \\
(\check{Q}\otimes_A A, Q) &
\end{tikzcd}$$

This is too confusing ~~...~~

188 Try again $(P, Q, Q\otimes P \to A)$ $\quad P \in \mathcal{P}(\tilde{A}^{op})$, $Q \in \mathcal{P}(\tilde{A})$

2 homos from $P \otimes_A Q$ to matrices over A:

$$\begin{array}{ccc} P\otimes_A Q & \longrightarrow & P \otimes_A A \otimes_A \check{P} = M_n(A) \\ \downarrow & & \\ \check{Q} \otimes_A A \otimes_A Q = M_p(A) & & \end{array}$$

$P = \tilde{A}^n$ right

$Q = \tilde{A}^p$ left.

03/13/97 Get details straight. BANK CD

dual pair (P, Q) over A $\quad P \simeq \tilde{A}^n$

$$P^* = \mathrm{Hom}_{A^{op}}(P, A) = A \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(P, \tilde{A})}_{P^\vee}$$

canon $(P, Q) \longrightarrow (P, P^*)$

$$P \otimes_A Q \longrightarrow P \otimes_A P^* = M_n(A)$$

$$GL(P \otimes_A Q) \longrightarrow GL(A)$$

Let's discuss elementary considerations

Idea: Use only maps $\tilde{A}^n \longrightarrow \tilde{A}^m$ given by matrices over A. Thus when you construct a flat firm cover of $M = AM$ you do it as a limit $P \xrightarrow{f} AP \subset P \xrightarrow{f} AP \subset P$.

Can this help? ~~So try this. How can~~

~~Start with~~

Start with $B = B^2$ let $\overset{\to B}{F}$ be a free B-mod mapping onto B. Let $F \xrightarrow{g} F$ be a map over 1_B * $gF \subset BF$.

Let $A = \varinjlim (F \xrightarrow{g} F \longrightarrow \cdots)$ $\quad A$ is a firm flat B-mod mapping onto B. $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

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$A = \varinjlim F \qquad 0 \to I \to F \to B \to 0$

$GL(A) = \varinjlim GL(F)$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "g"] & F \arrow[r] \arrow[d, "g"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r] & F \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

$P = \tilde{A}^n \qquad P^* = \mathrm{Hom}_{A^{op}}(P, A) \qquad P \otimes P^* = M_n(\tilde{A})$

given (P, Q) dual pair get $(P, Q) \to (P, P^*)$

hence ring homom. $P \otimes_A Q \longrightarrow M_n(\tilde{A})$.

whence $GL(P \otimes_A Q) \longrightarrow GL(M_n(\tilde{A})) = GL(A)$

I now need to worry about a map of dual pairs $P \xrightarrow{f} P'$. Factor $P \xrightarrow{\binom{1}{f}} P \oplus P' \longrightarrow P'$

$\binom{1}{0} \qquad (f \ \ 1)$

$P \oplus P' \xleftarrow{\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}} P \oplus P'$ This auto of $P \oplus P'$ belongs to $E(A)$ if $f : P \to P'A$

If I restrict to such f, then on homology something won't matter

~~$P \otimes_A P^* \to P \otimes_A (P \oplus P_0)^* \to (P \oplus P_0) \otimes_A (P \oplus P_0)^*$~~

injective case $P \longrightarrow P \oplus P_0 \qquad (P, P^* \oplus P_0^*) \to (P \oplus P_0, P^* \oplus P_0^*)$

~~$P \otimes_A P^* \to P \otimes (P \oplus P_0)^*$~~

$P \otimes (P^* \oplus P_0^*$

~~inj type~~ $P \xrightarrow{\binom{1}{f}} P \oplus P_0 \qquad Q = P^* \oplus P_0^*$

$$\begin{array}{ccc} P \otimes_A (P \oplus P_0)^* & \hookrightarrow & (P \oplus P_0) \otimes_A (P \oplus P_0)^* \\ \downarrow & & \\ P \otimes_A P^* & & \end{array}$$

$$\begin{array}{ccc} M_{n,n+k}(A) & \hookrightarrow & M_{n+k}(A) \\ \downarrow & & \\ M_n(A) & & \end{array}$$

surj type $P \oplus P_0 \longrightarrow P \qquad Q = P^*$

$$\begin{array}{ccc} (P \oplus P_0) \otimes_A P^* & \hookrightarrow & (P \oplus P_0) \otimes_A (P \oplus P^0)^* \\ \downarrow & & \\ P \otimes_A P^* & & \end{array}$$

$$\begin{array}{ccc} M_{n+k,n}(A) & \hookrightarrow & M_{n+k}(A) \\ \downarrow & & \\ M_n(A) & & \end{array}$$

It's not much clearer except that I see that I can handle maps $f : P \to P'$ which are 0 modulo ~~[illegible]~~ A.

Let's try to see what's true about $H_*(GL(A))$ when A is flat.

Fix a ring B, say idempotent, and consider ~~[illegible] A, B B-module surjections f : A → B~~ ring extensions $0 \to I \to A \to B \to 0$ such that $IA = 0$. I want to study the relation of $H_*(GL(-))$ for A, B. Such an extension is equiv. to an A-mod surjection $f : A \to B$, the product on A being $a_1 a_2 = f(a_1) a_2$. Also $I^2 = f(I) I = 0$.

191 ~~Ubbb~~ You better go over what can be done with the functor $K_*(A) = \mathrm{Ker}(K_*(\tilde{A}) \to K_*(\mathbb{Z}))$.

Assuming $B = B^2$, can construct a firm flat (modulo) res. of B over B^{op}

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

Convert to s.s. complex by Dold-Kan:

$$\cdots A_2 \rightrightarrows A_1 \rightrightarrows A_0 \longrightarrow B$$

Apply GL to get s.s. gp.

$$G_2 \rightrightarrows G_1 \rightrightarrows G_0 \qquad G_n = GL(A_n)$$

Because the A_i are left flat rings and $A_i \to B$ are all meq homos. all simp. arrows ~~are those~~ are meq homs between left rings, ∴ should induce isos on $H_*(GL(-))$

Consider double complex. $\mathbb{Z}[\overline{W}(G)]$. Columns compute group homology. ~~Components GL~~

$$\pi_0(G_*) = GL(\pi_0(A_*)) = GL(B^{(2)})$$

∴ ~~$H_*(GL(A_0))$~~

Spectral sequence abuts to $H_*(GL(A_n))$ any n

$$E^1_{pq} = H_p(\mathbb{Z}[GL(A_*)^q])$$

Think of $GL(A_*)$ as a top group G with $\pi_0(GL(A_*)) = GL(B^{(2)})$. ~~I have some sp. sequence~~

If B not firm you go no further. You are getting $H_p(BGL(B^{(2)})) = H_1(BGL(A_0))$ for the abutment in degree 1.

192 $A_2 \substack{\to\\\to\\\to} A_1 \rightrightarrows A_0 \to B$ $\qquad G_n = GL(A_n)$

$B\mathbb{G}$:

$G_2^2 \quad G_1^2 \quad G_0^2$

$G_2 \quad G_1 \quad G_0$

pt pt pt

If B is h-unital, then I know that $\{A_n\}$ resolves B. so $B\mathbb{G} \to BGL(B)$ is a heq. On the other hand, assuming $H_*(GL(-))$ agrees for flat Mog rings, we should know that $H_*(BGL(A_0)) \xrightarrow{\sim} H_*(B\mathbb{G})$. Now weaken B h-unital to having a flat firm resolution of length n.

$$F_{n+1} \to F_n \to F_{n-1} \cdots\cdots \to F_0 \to B \to 0$$

exact.

$n=0$ B idemp
$n=1$ B firm.

So besides $\pi_0 \mathbb{G} = GL(B)$ (assume $n \geq 1$)

the next htpy gp is $\pi_n \mathbb{G} = M(\mathrm{Tor}^{\tilde{B}}(\mathbb{Z};B)$

Get integers straight

$\exists$ $\quad F_n \to F_{n-1} \to \cdots \to F_0 \to B \to 0$ exact

with F_i firm flat for $0 < i \leq n \iff \mathrm{Tor}_i^{\tilde{B}}(\mathbb{Z},B)=0$, $0 \leq i \leq n$

Then, if $K_n = \mathrm{Ker}(F_n \to F_{n-1})$, we have $K_n/BK_n = \mathrm{Tor}_{n+1}^{\tilde{B}}(\mathbb{Z},B)$.

so if we choose F_{n+1} to map onto BK_n we have the homology $H_{n+1}(F) = \mathrm{Tor}_{n+1}^{\tilde{B}}(\mathbb{Z},B)$

Replace $n+1$ by n.

Assume $\mathrm{Tor}_i^{\tilde{B}}(\mathbb{Z},B) = 0 \quad i<n$
$\qquad\qquad\qquad\qquad \neq 0 \quad i=n.$

Then get $\to F_n \to \{F_{n-1} \to F_{n-2} \cdots \to F_0\} \to B \to 0$ exact in degrees $< n$

exact.

193 Try again

$$\begin{array}{ccccccccccc} \to & F_n & \to & \overset{\text{ex}}{F_{n-1}} & \to & \cdots & \to & \overset{\text{ex}}{F_0} & \to & B & \to 0 \\ & & & \| & & & & \| & & & \\ 0 \to & Z_{n-1} & \to & F_{n-1} & \to & \cdots & & F_0 & \to & B & \end{array}$$

$$Z_{n-1}/BZ_{n-1} = \mathrm{Tor}_n^{\breve{B}}(\mathbb{Z}, B)$$

$$F_n \twoheadrightarrow Z_{n-1}$$

Consider a firm flat complex $F.$ with $H_i(F_\bullet) = \begin{cases} B & i=0 \\ 0 & 0<i<n \\ X & i=n \end{cases}$

$$F_{n+1} \to BZ_n \to 0$$
$$\cap$$
$$0 \to Z_n \to F_n \to F_{n-1} \to \cdots \to F_0 \to B \to 0$$

$$\mathrm{Tor}_{n+1}^{\tilde{B}}(\mathbb{Z}, B) \xrightarrow{\sim} \mathrm{Tor}_n^{\breve{B}}(\mathbb{Z}, Z_0) \xrightarrow{\sim} \mathrm{Tor}_0^{\breve{B}}(\mathbb{Z}, Z_n) = Z_n/BZ_n$$

$$X = H_n(F) = \mathrm{Tor}_{n+1}^{\tilde{B}}(\mathbb{Z}, B) \quad \text{so w}$$

<u>Conclusion</u>: $\mathrm{Tor}_i^{\vec{B}}(\mathbb{Z}, B) = 0$ for $i \leq n$ $\iff$

$\exists$ firm flat resolution $F_n \to \cdots \to F_0 \to B \to 0$.

If $F_{n+1} \to F_n \to \cdots$ such that $H_n(F)$ nil,
then $H_n(F) = \mathrm{Tor}_{n+1}^{\vec{Z}}(\mathbb{Z}, B)$.

Return to $A_{n+1} \overset{\to}{\underset{\to}{\to}} A_n \quad \rightrightarrows A_0 \to B \to 0$

$\uparrow$

here $\pi_n = \mathrm{Tor}_{n+1}^{\tilde{B}}(\mathbb{Z}, B)$

so $$\pi_i(G) = \begin{cases} GL(B^{(2)}) & i=0 \\ 0 & 0<i<n \\ \mathrm{Matr}(\mathrm{Tor}_{n+1}^{\breve{B}}(\mathbb{Z}, B)) & i=n. \end{cases}$$

So G analogue of top grp with $\pi_0 = GL(B)$, $\pi_i = 0$ for $0 < i < n$, and $\pi_n \neq 0$.

194 What is the homology of BG?

$$1 \longrightarrow \widetilde{BG} \longrightarrow BG_0 \longrightarrow B\pi_0 \longrightarrow 1$$

$$E^2_{pq} = H_p(B\pi_0, H_q(\widetilde{BG})) \Rightarrow H_{p+q}(BG)$$

and you get

$$H_q(\widetilde{BG}) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & 0<q\le n \\ (\mathrm{M}(\ker\alpha_{n+1})) & q=n+1 \end{cases}$$

~~This is a tool, namely,~~

$$\begin{pmatrix} A & \\ & B \end{pmatrix}$$

Let's go over it all again. B idempotent, $\exists$ complex $F_\cdot$ of flat firm B-modules with aug to B exact mod nil module.

$$\cdots\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

B-module complex, make simp.

$$\substack{\longrightarrow\\\longrightarrow\\\longrightarrow} A_1 \rightrightarrows A_0 \longrightarrow B \longrightarrow 0$$

apply GL get s gp $GL(A_*)$

There is something going on here which linearizes in some way the functor $H_*(BGL(-))$.

03/14/97 1410

Question: Do I have enough to handle $BGL(A)^+ \sim$ fibre $BGL(\tilde{A})^+ \longrightarrow BGL(\mathbb{Z})^+$ for A flat? The hope is to use some sort of induction as follows. The ind hypotheses would be that ~~all homology~~ one has M inv of K_* for flat rings in degrees $<n$. Then ~~~~ via simplicial resolutions you might be able to ~~~~ analyze an extension $A_0 \to B$ of flat rings.

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		*	*	*
$H_3(GL(B))$	$H_3(GL(A_0))$	*	*	*
$H_2(GL(B))$	$H_2(GL(A_0))$	0	0	0
$H_1(GL(B))$	$H_1(GL(A_0))$	0	0	0
$\mathbb{Z}$	$\mathbb{Z}$	0	0	0

$\uparrow$

abutment

need to understand K_1.

What happens at the beginning? You should try to play the different themes together.

Begin with B flat and firm.

Form $A \twoheadrightarrow B$ A flat firm

Begin with B firm left flat, let $f: A \to B$ be a surjection of B-modules with A firm flat B-mod. Then get $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$, actually a firm dual pair $(B, A, A \otimes B \to B)$ and $A \otimes_A B \xrightarrow{\sim} B$, $B \otimes_B A \xrightarrow{\sim} A$

$a \otimes b \mapsto f(a)b$

To show $H_*(GL(A)) \xrightarrow{\sim} H_*(GL(B))$.

Now I have to use the techniques I've found. Take

$P \simeq \tilde{B}^n \longrightarrow A$ $\qquad \left(B, \tilde{B}^n, \tilde{B}^n \otimes_{\mathbb{Z}} B \to B \right.$

$(\tilde{b}_i) \longmapsto \sum \tilde{b}_i a_i$ $\qquad \left. \tilde{b}_i \otimes b \mapsto \sum \tilde{b}_i f(a_i) b \right)$

Then you are going to get $\tilde{B} \otimes_B P \longrightarrow \tilde{B} \otimes_B A$

$$\begin{array}{ccc} \tilde{B} \otimes_B P & \longrightarrow & \tilde{B} \otimes_B A \\ \| & & \| \\ P & \longrightarrow & A \end{array}$$

a ring homom. Actually what are you doing?

You want to define a map $H_*(GL(B)) \to H_*(GL(A))$

196 using the fact that B is left ~~A~~-flat. ?

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

A is ~~left~~ A-flat $\Longleftrightarrow$ $A \otimes_A A = A$ is B flat

B left B-flat $\Longrightarrow$ $B \otimes_B B = B$ is A flat

Because <u>B is A-flat</u>, and B acts on the right we expect to define this map by approx. ~~So start~~ So we approx B by $\tilde{A}^n \longrightarrow B$. So we have map of dual pairs over A: $(\ , \tilde{A}^n,$?

$$B = A \otimes_A B \longleftarrow B_? = A \otimes_A \tilde{A}^n$$

$$\left(A, B, \begin{matrix} B \otimes A \to A \\ b \otimes a \mapsto ba \end{matrix}\right) \longleftarrow \left(A, \tilde{A}^n, \begin{matrix} \tilde{A}^n \otimes A \longrightarrow A \\ (\tilde{a}_i) \otimes a \mapsto (\sum \tilde{a}_i b_i) a \end{matrix}\right)$$

$$\sum \tilde{a}_i b_i = f(\tilde{a}_i) b_i$$

$$\downarrow$$

$$(A^n, \tilde{A}^n, \mu)$$

$$\begin{array}{ccc} B = A \otimes_A B & \longleftarrow & A \otimes_A \tilde{A}^n \\ & & \downarrow \\ & & A^n \otimes_A \tilde{A}^n = M_n(A). \end{array}$$

~~[illegible]~~ So you have an approximation to B mapping to matrices over A. Now you have this homom $A \longrightarrow B$. Look carefully at what you need? You have $H_1(GL(A)) \longrightarrow H_1(GL(B))$

197. ~~So~~ look carefully. You have $A \longrightarrow B$ given, you need the inverse map on K-theory. This you get because B is B-flat, hence A-flat. So if you approx $Q \xrightarrow{u} A$ Q free, you get

$$\begin{array}{ccc}
\left(\tilde{B},\, Q,\, Q\otimes\tilde{B} \longrightarrow B\right) & \longrightarrow & Q^*\otimes_B Q \\
\quad q\otimes b \longmapsto \underbrace{f'(q)}_{f(u(q))}b & & \\
\big\downarrow & & \\
\left(\tilde{B},\, A,\, A\otimes\tilde{B} \longrightarrow B\right) & & \\
\quad a\otimes b \longmapsto f(a)b & &
\end{array}$$

homos.

$$\begin{array}{ccc}
Q = \tilde{B}\otimes_B Q & \longrightarrow & A = \tilde{B}\otimes_B A \\
\big\downarrow & & \\
Q^*\otimes_B Q & &
\end{array}$$

Now suppose $Q \xrightarrow{u} A$ ~~given~~ chosen so as to yield, given

$$\begin{array}{ccc}
H_*(GL(Q)) & \longrightarrow & H_*(GL(A)) \\
\big\downarrow & & \\
H_*(GL(M_n(B))) & & \\
\| & & \\
H_*(GL(B)) & &
\end{array}$$

We now apply the homom. $A \xrightarrow{f} B$, and try to compare

$$\begin{array}{ccccc}
GL(Q) & \xrightarrow{u} & GL(A) & \xrightarrow{f} & GL(B) \\
\big\downarrow & & & & \\
GL(M_n(B)) & & & &
\end{array}$$

So they are obviously not the same because of n. e.g.

$$\begin{tikzcd} GL_1(\mathbb{Q}) \arrow[r] \arrow[d] & GL_1(A) \arrow[r] & GL_1(B). \\ GL_n(B) & & \end{tikzcd}$$

And I already know what happens I think. You have this map $Q \xrightarrow{u} A \xrightarrow{f} B$.

OKAY let's work this out carefully. You have a ring B a map $Q = \tilde{B}^n \xrightarrow{u} B$, $u(b_i) = \sum_i b_i x_i$ where $x_i \in B$ fixed. Then you wish to compare the Q^{op} action on $\tilde{B}^n$ and on $\tilde{B}$.

$$q q' = \Big(\sum_i q_i x_i\Big) q'$$

$$u(qq') = \sum_j \Big(\sum_i q_i x_i\Big) q'_j x_j = u(q)\, u(q').$$

u is a Q^{op}-nil isom.

$$\begin{tikzcd} & & \tilde{B}^{n} \arrow[d, "\binom{1}{0}"] \arrow[dr, "-u"] & & \\ 0 \arrow[r] & \tilde{B}^n \arrow[r, "\binom{1}{u}"] \arrow[dr, "u"'] & \tilde{B}^{n+1} \arrow[r, "(-u\ 1)"] \arrow[d, "(0\ 1)"] & \tilde{B} \arrow[r] & 0 \\ & & \tilde{B} \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

How does this help?

it seems that B acts here by

$$(-u_1 \cdots -u_n\ 1)$$

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ u_1 & \cdots & u_n \end{pmatrix}$$

199 03/15/97 ~~...~~ I have to get control over this stuff. The problem is to show that ~~...~~ if $0 \to I \to A \to B \to 0$ is an extension such that $IA = 0$, ~~...~~ and ~~...~~ A, B are left flat, then ~~...~~ $GL(A) \to GL(B)$ is a homology isom. I ~~...~~ should be able to drop A being left flat and still construct a ^lifting^ map $H_*(GL(B)) \to H_*(GL(A))$. Why? by choosing $A' \twoheadrightarrow A$ with A' left flat.

~~...~~ The important thing here is that B is A-flat and this should follow from B being left flat.

~~Let's change notation. Let A be ... flat~~

Go over the basic construction: (P, Q) over A P is A^{op}-flat. Claim have canonical map

$$K_*(P \otimes_A Q) \longrightarrow K_*(A)$$

Can suppose P f. free $\tilde{A}^{op}$-module $\tilde{A}^n$.

$$P \otimes_A Q \longrightarrow P \otimes_A P^* = M_n(A)$$

induce K_* " $\qquad K_*(M_n(A)) = K_*(A)$.

naturality. ~~...~~ $(u, 1) : (P, Q) \longrightarrow (P', Q)$ $\qquad$ can assume $u(P) \subset P'A$

$$\begin{tikzcd}
 & P \arrow[d, "\binom{1}{0}"] \arrow[dr, bend left, "-u"] & \\
P \arrow[r, "\binom{1}{u}"] \arrow[dr, bend right, "u"'] & P \oplus P' \arrow[r, "(-u\ 1)"] \arrow[d, "(0\ 1)"] & P' \\
 & P' &
\end{tikzcd}$$

I need to ask what I must know.

Let's begin with A idempotent, and let's try to understand well what we need. To do what? To define a trace maps

$$K_*(P \otimes_A Q) \longrightarrow K_*(A)$$

with the good properties. I think the important

To define trace maps $K_*(P\otimes_A Q) \to K_*(A)$ when (P,Q) is a dual pair over A and $P = \tilde{A}^n$ for some n. We need to a trace map

$$\begin{array}{c} K_*(\tilde{A}^n \otimes_A (\tilde{A}^n)^*) \longrightarrow K_*(A) \\ \| \\ M_n(A) \end{array}$$

for each $\underline{\underline{n \geq 0}}$. $n=0$ is trivial. Exactly <u>what do I need</u> to make them consistent?

$\forall P \quad (=\tilde{A}^n)$ have $P \otimes_A P^*$.

I need to analyze a map $u: P \longrightarrow P'$. You need comm of

$$\begin{tikzcd} P \otimes_A P'^* \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes u^t"'] \arrow[d, phantom, shift left=4ex, "\text{comm. by defn.}"] \arrow[dr, dotted] & P' \otimes_A P'^* \arrow[d, dotted] \\ P \otimes_A P^* \arrow[r, dotted] & M(A) \end{tikzcd}$$

Your method consists of factorization $P \longrightarrow P \oplus P' \longrightarrow P'$

The effect of u is confusing me.

The important point is that u is a nil osm for the ring $P \otimes_A P'^*$.

I have to think carefully about the kind of u's. I think I can restrict to $u: P \longrightarrow P'$ which ~~are~~ map P into $P'A$, because these are the kinds of

201. maps that are needed to obtain a firm flat cover. But notice that such a u cannot be assumed inj or surj. This is an angle I didn't use before. So I'm not going to try for as general inj. + surj u, so factoring may be out!!!! **OKAY.** diagram

$$
\begin{tikzcd}
 & P \arrow[d, "\binom{1}{0}"] \arrow[dr, bend left, "-u"] & \\
P \arrow[r, "\binom{1}{u}"] \arrow[dr, bend right, "u"'] & P \oplus P' \arrow[r, "(-u\ \ 1)"] \arrow[d, "(0\ \ 1)"] & P' \\
 & P' &
\end{tikzcd}
$$

Somehow this diagram must be used. The important point should be that $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in \mathrm{Aut}\begin{pmatrix} P \\ \oplus \\ P' \end{pmatrix}$ is in $GL_{n+n'}(A)$. Maybe you need Vasersteins lemma with and x, y.

$$
\begin{tikzcd}
 & P' \arrow[d, "\binom{y}{1}"] \arrow[dr, bend left, "1-xy"] & \\
P \arrow[r, "\binom{1}{x}"] \arrow[dr, bend right, "1-yx"'] & P \oplus P' \arrow[r, "(-x\ \ 1)"] \arrow[d, "(1\ \ -y)"] & P' \\
 & P &
\end{tikzcd}
$$

seems different.

 Try again. You have $u : P \longrightarrow P'$ a map and you want to compare

$$\begin{tikzcd}
B \simeq P \otimes_A P'^* \arrow[r, "u \otimes 1"] \arrow[d, "1 \otimes u^*"] & P' \otimes_A P'^* \simeq M_{n'}(A) \\
P \otimes_A P^* \simeq M_n(A) &
\end{tikzcd}$$

$$(p_1 \otimes \lambda_1')(p_2 \otimes \lambda_2') = p_1 \langle \lambda_1', u(p_2) \rangle \otimes \lambda_2'$$

$$(p_1 \otimes \lambda_1' u)(p_2 \otimes \lambda_2' u) = p_1 \langle \lambda_1' u, p_2 \rangle \otimes \lambda_2' u$$

You have some kind of correspondence between $M_n(A)$ and $M_{n'}(A)$ given by $M_{nn'}(A)$. In fact it's very clear, namely $u \in M_{n'n}(A)$ and the correspondence is our friend

$$\begin{array}{ccc} v & \longmapsto & uv \in M_{n'n'} \\ \downarrow & & \\ vu \in M_{nn} & & \end{array}$$

$\bigoplus M$

203 03/16/97 ~~[illegible]~~ Yesterday you reached something like ~~[illegible]~~ the Hochschild complex. You have objects $n \in \mathbb{N}$ and for each ... You have the sequence of objects

First for a ring A you can form

$\{(x_0, \ldots, x_n) \mid 1 - x_0 \cdots x_n \text{ inv}\}$ $\quad \{(x_0, x_1) \mid 1 - x_0 x_1 \text{ inv}\}$ $\quad \{x_0 \mid 1 - x_0 \text{ inv.}\}$

This appears to be a pre cyclic set ~~[illegible]~~

$$A \otimes A \otimes A \rightrightarrows A \otimes A \rightrightarrows A$$

$$(x_0, x_1, x_2) \longmapsto \begin{array}{l} x_0 x_1, x_2 \\ x_0, x_1 x_2 \\ x_2 x_0, x_1 \end{array}$$

Better check well-defined.

$$\begin{array}{ccc} P \otimes_A P'^* & \xrightarrow{u \otimes 1} & P' \otimes P'^* \\ \downarrow{\scriptstyle 1 \otimes u^*} & & \\ P \otimes_A P^* & & \end{array} \qquad u \in P' \otimes_A P^*$$

Once u is fixed, you get a ring structure on $P \otimes_A P'^*$ namely $(p_1 \otimes \lambda_1')(p_2 \otimes \lambda_2') = p_1 \langle \lambda_1', u(p_2) \rangle \otimes \lambda_2'$

and you can consider invertible elements $1 - v \in P \otimes_A P'^*$.

~~Now these are elements of the form $1 - uv$~~

$$(u \otimes 1)(1 - v) = 1 - (u \otimes 1) v = 1 - uv \ ; \ P' \longrightarrow P$$

$$(u \otimes 1)(p \otimes \lambda') = u(p) \otimes \lambda' = u \circ (p \otimes \lambda')$$

Similarly $\quad (1 \otimes u^*)(p \otimes \lambda') = p \otimes \lambda' u = (p \otimes \lambda') \circ u$

$$(p_1 \otimes \lambda_1' u)(p_2 \otimes \lambda_2' u) = p_1 \langle \lambda_1' u, p_2 \rangle \otimes \lambda_2' u$$
$$= \left(p_1 \langle \lambda_1', u p_2 \rangle \otimes \lambda_2'\right) \circ u$$

Seems clear. At level 0 you have

You don't have much time to get this into shape. ~~You have~~ The structure should come from the Hochschild complex of a ring w many objects. Let X be the set of objects. For each $x, y \in X$ you have A_{xy} and for each x, y, z you have $A_{xy} \otimes A_{yz} \to A_{xz}$ associative. The thing you are after is a complex of the form

$$\bigoplus_{x,y,z} A_{xy} \otimes A_{yz} \otimes A_{zx} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \bigoplus_{x,y} A_{xy} \otimes A_{yx} \rightrightarrows \bigoplus A_{xx}$$

So basically you have a ring $A = \bigoplus_{x,y} A_{xy}$ with a matrix decomposition relative to set X. There's an obvious trace map in the present case from the Hochschild complex of the ring with matrix decomposition to the Hochschild complex of the total ring A.

Where does K-theory enter? One actually produces an invertible subset. Inside

$A_{xy} \otimes A_{yz} \otimes A_{zx}$ you consider $\alpha_0 \otimes \alpha_1 \otimes \alpha_2$, or should I look at tuples $(\alpha_0, \alpha_1, \alpha_2)$, such that $1 - \alpha_0\alpha_1\alpha_2 \in A_{xx}$ is invertible.

~~Interesting idea. Take~~

How to obtain K-theory?

Given x look at $\alpha_0 \in A_{xx}$ such that $1-\alpha_0$ invertible

Given x, y consider $(\alpha_0, \alpha_1) \in A_{xy} \times A_{yx}$ such that $1 - \alpha_0\alpha_1$ invertible in A_{xx}, whence

205 of course $1-\alpha_1\alpha_0 \in A_{yy}$ is invertible.

What can I say about the structure?

You have (α_0, α_1) YES.

What sort of structure? Each x gives G_{xx}.

a pair x, y gives a

But what about bar homology?

$$\rightrightarrows A_{xz}\otimes A_{zy} \longrightarrow A_{xy}$$

1545 let's try to write up something that will clarify the situation.

Let's begin with the basic const. You have a dual pair P, Q over A such that P is A^{op} flat. To construct a canonical trace map

$$K_*(P\otimes_A Q) \longrightarrow K_*(A)$$

Use the fact that P is a filtered ind limit of finite free $\tilde{A}^{op}$ modules. This result admit strengthening. If P firm flat you have Wodzicki's refinement

$$\begin{tikzcd}
\tilde{A}^n \arrow[r, "a_\cdot"] \arrow[dr, "0"'] & \tilde{A}^p \arrow[r, "\exists a'"] \arrow[d, "x_\cdot"] & \tilde{A}^q \arrow[d, "y_\cdot"] \\
& P \arrow[r, equal] & P
\end{tikzcd}$$

Thus given $xa = 0$ in A $\quad \sum_{i=1}^{p} x_i a_{ij} = 0, \ 1\le j\le n$

$\exists$ a', y such that $x = ya'$ and $a'a = 0$.

I think this means that P can be written as a filtered $\underrightarrow{\lim}$ of a system in $P(\tilde{A}^{op})$ such that the transition maps are 0 modulo A.

This fact should be useful.

So we can ~~restrict~~ to P of the form $\tilde{A}^n$ and we want naturality wrt maps $P \xrightarrow{u} P'$ of such modules ~~[illegible]~~ which are zero modulo A.

i.e. $u \in P' \otimes_A P^*$ $\qquad P^* = A \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(P, \tilde{A})}_{\tilde{P}^*}$

~~[illegible]~~ Basic Construction: Have

$$P \otimes_A Q \longrightarrow P \otimes_A P^* = M_n(A)$$

which induces $K_*(P \otimes_A Q) \longrightarrow K_*(M_n(A)) \xrightarrow[\text{canon}]{} K_*(A)$

want naturality in the dual pair. ~~to for~~

$$P \xrightarrow{u} P', \quad Q \xrightarrow{v} Q'$$

$$P \otimes_A Q \longrightarrow P' \otimes_A Q' \longrightarrow P' \otimes_A P'^*$$

Can take $Q' = P'^*$. Then must check

$$P \otimes_A Q \longrightarrow P' \otimes_A P'^*$$

I think you can take both $Q = Q' = P'^*$

Good diagram

$$\begin{array}{ccc} P \otimes_A Q & \longrightarrow & P' \otimes_A P'^* \\ \downarrow & & \\ P \otimes_A P^* & & \end{array}$$

$$\begin{array}{ccc} P \otimes_A Q & \longrightarrow & P' \otimes_A Q' \\ \downarrow & & \downarrow \\ P \otimes_A P'^* & \longrightarrow & P' \otimes_A P'^* \\ \downarrow & & \searrow \\ P \otimes_A P^* & & M_\ell(A) \\ \searrow & \sim & \\ M_k(A) & & \end{array}$$

~~I need to ...~~

So the critical ~~thing~~ case you need to treat is for any $u : P \to P'$ in $\mathcal{P}(\tilde{A}^{op})$ and the map $(P, P'^*) \xrightarrow{(u,1)} (P', P'^*)$. You then need comm of

$$\begin{tikzcd}
P \otimes_A P'^* \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes u^*"] & P' \otimes_A P'^* \arrow[d, squiggly] \\
P \otimes_A P^* \arrow[r, squiggly] & M(A)
\end{tikzcd}$$

Notice that you haven't used $u : P \to P'$ has image in $\mathcal{P}A$.

Review: You have for $(P, Q, Q\otimes P \to A)$ $P \in \mathcal{P}(\tilde{A}^{op})$ a homom. $P \otimes_A Q \to P \otimes_A P^* \hookrightarrow M(A)$. What I need is some feeling for wh

~~...~~ Basic construction takes $(P, Q, Q\otimes P \to A)$ $P \in \mathcal{P}(\tilde{A}^{op})$ and assigns the homom.

$$P \otimes_A Q \longrightarrow P \otimes_A P^*$$

The gadget I'm after ~~should~~ be gen. by the groups $GL(P \otimes_A P^*)$ and I need to put in <u>relations</u> to get naturality. Given ~~...~~ $(u, v) : (P, Q) \to (P', Q')$

Case ass. $V = 1$ for Q'^*.

$$\begin{tikzcd}
P \otimes_A Q \arrow[r] \arrow[d] & P' \otimes_A Q' \arrow[d] \\
P \otimes_A P'^* \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes u^*"] & P' \otimes_A P'^* \\
P \otimes_A P^* &
\end{tikzcd}$$

You have the idea to <u>factor u</u>.

$$\begin{tikzcd}
 & {} \arrow[d] & \\
P \arrow[r, "\binom{1}{u}"] \arrow[dr, "u"'] & P\oplus P' \arrow[r, "(-u\ \ 1)"] \arrow[d, "(0\ \ 1)"] & P' \\
 & P' &
\end{tikzcd}$$

What are you doing? Instead of just $P\otimes_A P^*$ and $P\otimes_A P'^*$ we now have to consider $(P\oplus P')\otimes_A (P^*\oplus P'^*)$ So how does this affect things? Before you had any $u\in P'\otimes_A P^*$ and you used u to make $P\otimes_A P'^*$ into a ring.

$$\begin{tikzcd}
 & {} \arrow[d] \\
P\otimes_A (P\oplus P')^* \arrow[r, hook, "\binom{1}{u}\otimes 1"] \arrow[d, "1\otimes\binom{1}{u}^*"] & (P\oplus P')\otimes_A (P\oplus P')^* \\
P\otimes P^* &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (P\oplus P')\otimes P'^* \arrow[r, two heads, "(0\ 1)\otimes 1"] \arrow[d, hook, "1\otimes(0\ 1)^*"] & P'\otimes_A P'^* \\
P\otimes (P\oplus P')^* \arrow[r, hook, "\binom{1}{u}\otimes 1"] \arrow[d, "1\otimes\binom{1}{u}^*"] & (P\oplus P')\otimes_A (P\oplus P')^* & \\
P\otimes_A P^* & &
\end{tikzcd}$$

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Review end of yesterday's work.

The basic construction assigns to a dual pair $(P, Q, Q\otimes_{\mathbb{Z}} P \to A)$ over A with $P \in \mathcal{P}(\tilde{A}^{op})$ the homom. $P\otimes_A Q \longrightarrow P\otimes_A P^*$. The gadget X I'm looking for is generated by the groups $GL(P\otimes_A P^*)$ and I need to put in the relations necessary for naturality. ~~This means~~ For a map $(P,Q)\to(P',Q')$

$$
\begin{tikzcd}
P\otimes_A Q \arrow[r] \arrow[d] & P'\otimes_A Q' \arrow[d] \\
P\otimes_A P'^* \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes u^*"] & P'\otimes_A P'^* \\
P\otimes_A P^* &
\end{tikzcd}
$$

naturality reduces to the case $Q = Q' = P'^*$. We want

$$
\begin{tikzcd}
GL(P\otimes_A P'^*) \arrow[r] \arrow[d] & GL(P'\otimes_A P'^*) \arrow[d, squiggly] \\
GL(P\otimes_A P^*) \arrow[r, squiggly] & X
\end{tikzcd}
$$

to commute up to homotopy for each $u: P \to P'$.

Factoring u $\quad P \xrightarrow{\binom{1}{u}} P\oplus P' \xrightarrow{(0\ 1)} P'$ $\quad$ leads to

$$
\begin{tikzcd}
& \binom{P}{P'}\otimes_A P'^* \arrow[r, two heads, "(0\ 1)\otimes 1"] \arrow[d, hook, "1\otimes\binom{0}{1}"] & P'\otimes_A P'^* \\
P\otimes_A (P^*\ P'^*) \arrow[r, hook, "\binom{1}{u}\otimes 1"] \arrow[d, "1\otimes(1\ u^*)"'] & \binom{P}{P'}\otimes_A (P^*\ P'^*) & \\
P\otimes_A P^* & &
\end{tikzcd}
$$

$$
\begin{pmatrix} 0 & P\otimes_A P'^* \\ 0 & P'\otimes_A P'^* \end{pmatrix} \longrightarrow\!\!\!\!\rightarrow P'\otimes_A P'^*
$$

$$\cap$$

$$
\begin{pmatrix} P\otimes_A P^* & P\otimes_A P'^* \\ 0 & 0 \end{pmatrix} \subset \left(\begin{array}{c|c} & P\otimes_A P'^* \\ \hline & \end{array}\right)
$$

$$\downarrow$$

$$P\otimes_A P^*$$

210 Dennis trace – review. Basic idea

$G \longrightarrow A^\times$ a homom. Idea: ~~[illegible]~~

Trying to probe the cyclic homology type of A using group rings. ~~Be~~ a ~~[illegible]~~ ring (unital) homom.

$k[G] \longrightarrow A$ is equivalent to a gp hom $G \longrightarrow A^\times$.

More generally you consider unital rings meq to A i.e. $P \in \mathcal{P}(A^{op})$ and a hom. $G \to End_{A^{op}}(P)$.

Then you get $k[G] \longrightarrow$ [redacted] $P \otimes_A P^\vee$ and ~~[illegible]~~ hence a map from the cyclic homology type of $k[G]$ to the cyclic homology type of A. There ~~[illegible]~~ might be interesting possibilities in the non unital context.

At some point use the Burghelea result on cyclic homology of group rings, namely, the identity conjugacy class leads to the group homology tensored with $k[u]$ sitting as a summand of the cyclic homology.

$$\begin{array}{ccc} HC^- & \longrightarrow & HP \\ \downarrow & & \downarrow \\ HH & \longrightarrow & HC \end{array}$$

group ring

~~[illegible]~~

~~YES!!!!!! OKAY!!! NOOO NOOO~~

$$k[G] \longrightarrow M_n(A)$$

$$H_*(G) \longrightarrow Tor_*^{k[G]\otimes k[G]}(k[G], k[G]) \longrightarrow Tor_*^{A\otimes A}(A, A)$$

mechanism of Dennis trace

cyclic set giving $EG \times^G G^c$

$$G\times G\times G \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G\times G \;\substack{\longrightarrow\\ \longrightarrow}\; G$$

b complex $\qquad (x_0, x_1) \qquad x_0x_1$

x_1x_0

211 To get the homology of the group ring you fix the product to be 1. So if you want the Dennis trace it is very simple. But now how do I ~~[illegible]~~ correlate this construction with what I am doing now?

I take a gp homom. $G \longrightarrow GL_n(A)$ equiv. $k[G] \longrightarrow M_n(A)$. Then I can map BG into the Hochschild cx of $M_n(A)$

BG appears as the cyclic subset of $EG \times_G G^2$

$$G \times G \times G \; \overset{d_0}{\underset{d_2}{\overset{d_1}{\rightrightarrows}}} \; G \times G \; \overset{d_0}{\underset{d_0}{\rightrightarrows}} \; G$$

Consider $g_0, g_1, \ldots, g_n \ni g_0 g_1 \cdots g_n = 1$.

So what am I doing? Replacing $M_n(A)$ by A. So inside the Hochschild complex of A I have ~~a~~ cyclic ~~subset~~ consisting of $(g_0, \ldots, g_n)$ such that $g_0 \cdots g_n = 1$. So how does this relate to ~~[illegible]~~ non-unital game. Why not examine the unital case?

~~I can form~~ For A unital you can ~~[illegible]~~ look for finite "cycles" ~~sequences~~ $(g_0, \ldots, g_n)$ in A s.t. $g_0 \cdots g_n = 1$. For A non-unital what happens? ~~Is~~ there an analogue? ~~[illegible]~~ What would you like? ~~You need~~ ~~[illegible]~~ It seems completely different. Instead of $(g_0, \ldots, g_n)$ such that $g_0 \cdots g_n = 1$, you consider $(x_0, \ldots, x_n)$ " " $(1 - x_0 \cdots x_n)^{-1}$ exists.

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Dennis trace review: Suppose A unital.

Consider representations of groups over A, i.e. group homos $G \to GL_n(A)$, equivalently a (unital) ring homos $k[G] \xrightarrow{\rho} M_n(A)$. Such a ρ induces a map from cyclic hom. types from $k[G]$ to A. Apply Burghelea. The former splits according to the conjugacy classes of G. Focus on identity coset. Key result is that you get a divisible S module for the cyclic homology, whose Hochschild homology is $H_*(BG)$

$$\begin{tikzcd} HC^-_*(k[G])_e \arrow[r] \arrow[d] & HP_*(k[G])_e \arrow[d] \\ HH_*(k[G])_e \arrow[r] \arrow[d, phantom, "//"] & HC_*(k[G])_e \\ H_*(BG) & \end{tikzcd} \qquad \begin{tikzcd} HC^-_*(A) \arrow[r] \arrow[d] & HP_*(A) \arrow[d] \\ HH(A) \arrow[r] & HC_*(A) \end{tikzcd}$$

In fact there is probably a canonical map $H_*(BG) \longrightarrow HC^-_*(k[G])_e$ which extends to an isom $k[[u]]\hat{\otimes} H_*(BG) \xrightarrow{\sim} HC^-_*(k[G])_e$.

Easy to understand the Dennis trace $H_*(BG) \longrightarrow HH(A)$

$EG \times^G G^c$ realized by $\overbrace{\cdots G^3}^{\text{cyclic set}} \rightrightarrows G\times G \rightrightarrows G$

with usual faces $(g_0, g_1, g_2) \mapsto \begin{cases} (g_0g_1, g_2) \\ (g_0, g_1g_2) \\ (g_2g_0, g_1) \end{cases}$

Induces map $k[EG \times^G G] \longrightarrow$ Hochschild cx of $A = k(G)$ of cyclic objects. [redacted] Put another way Given A look at the cyclic set of sequences of invertibles $g_0, \ldots, g_n$ in A with these faces.

$\mathcal{B}$ is the sub cyclic set cons. of $(g_0, \ldots, g_n)$ + $g_0 \cdots g_n = 1$.

So the Dennis trace map is the map $H_*(BGL_r) \to HH_*(A)$ induced by the map of cyclic sets

$$\begin{array}{c} \{(g_0, \ldots, g_n) \in GL_r(A)^{n+1} \mid g_0 \cdots g_n = 1\} \\ \downarrow \\ (g_0 \otimes \cdots \otimes g_n) \in M_r(A)^{\otimes n+1} \\ \downarrow \mathrm{tr} \\ \mathrm{tr}(g_0 \otimes \cdots \otimes g_n) \in A^{\otimes n+1} \end{array}$$

What about the non unital D.T. map?

$$\overline{k[G]} \longrightarrow P \otimes_A Q \qquad k[G] \longrightarrow \widetilde{P \otimes_A Q}$$

unital homom.

A good question may be what changes are needed to handle the non unital case. true Hoch complex has bar complex included.

~~Somehow it seems that~~

Proposal: Find characteristic classes of representations of groups on dual pairs (P, Q) over A with P flat, eventually $P \in \mathcal{P}(\tilde{A}^{op})$. This means assembling $H_*(G)$, $G = (P \otimes_A Q)^*$ for all these pairs.

First construct Dennis trace

217 1140 So what's going on?

nonunital Dennis trace map. Given dual pair $(P, Q, \langle\ \rangle : Q \otimes_{\mathbb{Z}} P \to A)$. We have ring $P \otimes_A Q$ and can consider $(\underbrace{P \otimes_A Q}_{B})^{\times} = \{1 - \sum p_i \otimes q_i \mid 1 - \sum q_i p_i \text{ invertible}\}$.

As before have map of cyclic sets from $k[BG]$ to, better

$$\begin{array}{ccccc} (G^3)_e & \substack{\to\\ \to\\ \to} & (G^2)_e & \rightrightarrows & (G)_e \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{B} \otimes \tilde{B} \otimes \tilde{B} & \substack{\to\\ \to\\ \to} & \tilde{B} \otimes \tilde{B} & \rightrightarrows & \tilde{B} \end{array}$$

Do I have a trace map to Hochschild cx of A? Guess? true Hoch cx ~~has~~ bar complex as ~~quotient~~

Actually what do you do for $A = \tilde{A} \otimes_A A$. I guess what works is to go into $\tilde{B}$ then normalize).

Actually the true Hoch cx is produced by the cyclic bicomplex of Connes + Tsygan. So what? $g_0 dg_1 \cdots dg_n \in \Omega(\tilde{B})$. So what about from $B = P \otimes_A Q$ to A?

It seems that we have a trace map NO

$$B^{\otimes n} = (P \otimes_A Q) \otimes (\quad) \otimes \cdots \otimes (\quad)$$

Did I even get the trace map straight for matrices? $A \otimes \overset{n}{\cdots} \otimes A \to A \otimes_S \overset{n}{\cdots} \otimes_S A \otimes_S$

This seems OKAY.

$$(P \otimes_A A \otimes_A \check{P}) \otimes (P \otimes_A A \otimes_A \check{P})$$

$$(V \otimes A \otimes \check{V}) \otimes$$

$$\longrightarrow \tilde B\otimes B\otimes \tilde B \longrightarrow \tilde B\otimes\tilde B \longrightarrow \tilde B \to 0$$

You have to understand $M \overset{L}{\otimes}_B$ for a B-bimodule M. ~~First~~ Let $E \to M$ be a flat B-bimodule resolution. Need to assume $\tilde B\otimes\tilde B$ is left and right flat, e.g. if $\tilde B$ flat over k. Let $F \to \tilde B$ be a ~~flat~~ bimod. res. of $\tilde B$. Then

$$M\otimes_B F\otimes_B \longleftarrow E\otimes_B F\otimes_B \longrightarrow E\otimes_B \tilde B\otimes_B = E\otimes_B$$

quis because can suppose $F = \tilde B\otimes\tilde B$

quis because can suppose $E = \tilde B\otimes\tilde B$

~~so~~ for now take $\tilde B = k[G]$ $\quad M = k$

Use $F \longrightarrow \tilde B$, $E \longrightarrow \mathbb{Z}$, Then

$$k\otimes_B F\otimes_B \simeq E\otimes_B$$
$$\|$$
$$k\otimes_B F\otimes_B k$$

Take F:

$$\xrightarrow{b'} \tilde B\otimes B\otimes\tilde B \xrightarrow{b'} \tilde B\otimes\tilde B$$

Then $k\overset{L}{\otimes}_B = k\otimes_B F\otimes_B$: $\quad B^{\otimes 2} \longrightarrow B \longrightarrow k$

so there should be little problem!

Anyway ~~we agree that~~ it is clear that

$$B\overset{L}{\otimes}_B \longrightarrow \tilde B\overset{L}{\otimes}_B \longrightarrow k\overset{L}{\otimes}_B \longrightarrow$$

amounts to the 2 columns of the double cx.

Now what about Dennis trace

215 It seems you are using P free so as to get a left action on P, effectively writing $P = V \otimes \tilde{A}$.

13:52. We've been through this before - if you want to compare Hoch cxs for A and $B = P \otimes_A Q$ you ~~[illegible]~~ use a bicomplex

$$B \overset{L}{\otimes}_B \xleftarrow{\sim} P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B = Q \overset{L}{\otimes}_B P \overset{L}{\otimes}_A \longrightarrow A \overset{L}{\otimes}_A$$

if P is A^{op} flat
and $P \overset{\sharp}{\otimes}_A Q = B$

so you use something like

$$P \otimes A^{\otimes i} \otimes Q \otimes B^{\otimes j}$$

now ask whether there's a good way to handle the Dennis trace map. Where does h-unital enter? What sort of things happen. Here A, B are ~~[illegible]~~ nonunital. Let's understand the nature of the argument. You have $k[G] \longrightarrow \widetilde{P \otimes_A Q} = \tilde{B}$.

You ~~[illegible]~~ get

$$\begin{array}{ccccc} k[G] \overset{L}{\otimes}_{k[G]} & \longrightarrow & \tilde{B} \overset{L}{\otimes}_B & \longleftarrow & B \overset{L}{\otimes}_B \\ \downarrow & & \downarrow & & \\ \mathbb{Z} \overset{L}{\otimes}_{k[G]} & \longrightarrow & \mathbb{Z} \overset{L}{\otimes}_B & & \end{array}$$

0 because $k[G]$ unital.

~~It seems like there is always a map from~~

15:26 Treat these problems. How to handle? Start again. You have $k[G] \longrightarrow \widetilde{P \otimes_A Q} = \tilde{B}$

Wait: Take $\tilde{B} = k[G]$. have Δ

$$B \overset{L}{\otimes}_B \longrightarrow \tilde{B} \overset{L}{\otimes}_B \longrightarrow \mathbb{Z} \overset{L}{\otimes}_B$$

 So first you have to worry about $B^{\times}$

But the point somehow is that? You have $\mathbb{Z}[G] \longrightarrow \tilde{B}$, a map of augmented rings. What am I going to do.

You have $0 \to B \to \tilde{B} \to \mathbb{Z} \to 0$

exact seq of B-bimodules, hence

$$0 \to B \overset{L}{\underset{B}{\otimes}} \longrightarrow \tilde{B} \overset{L}{\underset{B}{\otimes}} \longrightarrow \mathbb{Z} \overset{L}{\underset{B}{\otimes}} \longrightarrow 0$$

visualize using columns of cyc bic.

$$\begin{tikzcd}
B^{\otimes 3} \arrow[d, "b"] & B^{\otimes 3} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] \\
B \otimes B \arrow[d, "b"] & B^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] \\
B & B \arrow[l, "1-\lambda"']
\end{tikzcd}$$

You now take $B = \overline{\mathbb{Z}[G]} = I[G]$. So life goes on slowly. But there is a Dennis trace map which should go from $H_*(BG) \longrightarrow HH_*(\mathbb{Z}[G])$. This I sort of understand, namely BG is simp. set of $(g_0, \dots, g_n)$ $\ni$ $g_0 \cdots g_n = 1$, and this simplex goes to $g_0 \otimes \bar{g}_1 \otimes \cdots \otimes \bar{g}_n \in \tilde{B} \otimes B^{\otimes n}$ = quotient of $\tilde{B}^{\otimes n+1}$ by degenerate

This might be very interesting

Summarize: We have this simplicial (cyclic) model for BG ~~namely~~ consisting of $(g_0, \dots, g_n)$ $\ni$ $g_0 \cdots g_n = 1$

Standard model $(g_1, \dots, g_n)$ $\qquad G^2 \overset{\to}{\underset{\to}{\to}} G$

$d_0(g_1, g_2) = g_2 \qquad d_1(g_1, g_2) = g_1 g_2$

$d_2(g_1, g_2) = g_1$

Now use (g_0, g_1, g_2) $\quad g_0 g_1 g_2 = 1.$

$$(g_2^{-1}g_1^{-1}, g_1, g_2) \begin{array}{l} \xrightarrow{d_0} (g_2^{-1}, g_2) \\ \xrightarrow{d_1} (g_2^{-1}g_1^{-1}, g_1 g_2) \\ \xrightarrow{d_2} (g_1^{-1}, g_1) \end{array}$$

$$(g_3^{-1}g_2^{-1}g_1^{-1}, g_1, g_2, g_3) \begin{array}{l} \xrightarrow{d_0} (g_3^{-1}g_2^{-1}, g_2, g_3) \\ \xrightarrow{d_1} (g_3^{-1}(g_1g_2)^{-1}, g_1g_2, g_3) \\ \xrightarrow{d_2} ((g_2g_3)^{-1}g_1^{-1}, g_1, g_2g_3) \\ \xrightarrow{d_3} ((g_1g_2)^{-1}, g_1, g_2) \end{array}$$

deleting commas and crossover. So a n-simplex in BG $(g_1, g_2, \dots, g_n)$ goes to $(g_1 g_2 \cdots g_n)^{-1} \otimes \bar{g}_1 \otimes \cdots \otimes \bar{g}_n$ in $\tilde{B} \otimes B^{\otimes n}$

the image of this element in the b' complex is $\bar{g}_1 \otimes \bar{g}_2 \otimes \cdots \otimes \bar{g}_n$. Does this sound reasonable? Is it a map of complexes? Go back to BG

$$\mathbb{Z}[G^3] \equiv \mathbb{Z}[G^2] \equiv \mathbb{Z}[G] \rightrightarrows pt$$

It seems likely that the normalization is

$$\longrightarrow I[G] \otimes I[G] \quad I[G]$$

here $I(G) = \mathbb{Z}[G]/\mathbb{Z}$. Keep on trying!!

$$d(g_1, g_2, g_3) = (g_2, g_3) - (g_1g_2, g_3) + (g_1, g_2g_3) - (g_1, g_2)$$

$$d(g_1, g_2) = (g_2) - (g_1g_2) + (g_1)$$

But what you want is $I[G]^{\otimes n}$ inside $\mathbb{Z}[G]^{\otimes n}$ somehow.

219 to look at

$\mathbb{Z}[G^3]$ $\qquad$ $\mathbb{Z}[G^2]$ $\qquad$ $\mathbb{Z}[G]$

$$(g_1,g_2)-(1,g_2)-(g_1,1)+(1,1) \qquad\qquad g_1-1$$

What I want to do is to embed

$$\begin{tikzcd}
I[G]^{\otimes 3} \arrow[r, "b'"] \arrow[d, phantom, "\cap"] & I[G]\otimes I[G] \arrow[r, "-b'"] \arrow[d] & I[G] \arrow[r, "0"] \arrow[d] & \mathbb{Z} \\
\mathbb{Z}[G]^{\otimes 3} \arrow[r, "d"] & \mathbb{Z}[G]^{\otimes 2} \arrow[r, "d"] & \mathbb{Z}[G] \arrow[r, "0"] & \mathbb{Z}
\end{tikzcd}$$

$$\bar g_1\otimes \bar g_2 \longmapsto (g_1-1)(g_2-1) = g_1g_2-g_1-g_2+1$$

$$d(g_1,g_2) = g_2 - g_1g_2 + g_1$$

might use derivation. $\delta g = g-1$, $\delta(g_1g_2) = g_1\delta(g_2)+\delta g_1$

$$\begin{tikzcd}
\bar g_1\otimes\bar g_2 \arrow[r, mapsto] \arrow[dd] & -\bar g_1\bar g_2 \arrow[d] \\
 & -g_1g_2+g_1+g_2-1 \arrow[d, phantom, "\|"] \\
(g_1,g_2)-(1,g_2)-(g_1,1)+(1,1) \arrow[r, mapsto] & (g_2-g_1g_2+g_1)-(g_2\cancel{-g_2}+1)-(1\cancel{-g_1+g_1})+(1-1+1)
\end{tikzcd}$$

$$\bar g_1\otimes\bar g_2\otimes\bar g_3 \longmapsto -\bar g_1\bar g_2\otimes\bar g_3 + \bar g_1\otimes\bar g_2\bar g_3$$

Look for a more intelligent method. Go back. Begin with the model $(g_0,\dots,g_n)$ $\quad g_0\cdots g_n=1$ for BG sitting inside the b complex for $\tilde B$. Then use the explicit heq to the norm. cx $\tilde B^{\otimes n+1} \longrightarrow \tilde B\otimes\bar B^{\otimes n}$

$(g_1\cdots g_n)^{-1}\otimes g_1\otimes\cdots\otimes g_n$

Why do you care?

 ~~[illegible]~~ Actually what you care about is the composition

$$\mathbb{Z}[BG] \longrightarrow (\tilde{B}^{\otimes *+1}, b) \twoheadrightarrow (\tilde{B}\otimes B^{\otimes *}, b) \hookrightarrow (B^{\otimes *}, -b')$$

and this should the normalization of the simplicial abelian gp $\mathbb{Z}[BG]$. ~~[illegible]~~ The first point is to identify BG (in degree n is G^n with faces deleting commas, degeneracies inserting 1's ∋

$$d_i(g_1, \dots, g_n) = \begin{pmatrix} g_2, \dots, g_n \\ g_1g_2, \dots g_n \\ g_1, \dots, g_{n-1}g_n \\ g_1, \dots, g_{n-1} \end{pmatrix}$$

with the subsimplicial set of $EG \times^G G$ consisting of $(g_0, \dots, g_n)$ in deg. n ∋ $g_0 \cdots g_n = 1$, faces delete commas and crossover, degeneracies ~~[illegible]~~ insert 1's. after g_0 until after g_n.

In degree n $\mathbb{Z}[BG] = \mathbb{Z}[G^n]$

$$(g_1, \dots, g_n) \longmapsto (g_1 \cdots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_n$$

$$\longmapsto (g_1 \cdots g_n)^{-1} \otimes \bar{g}_1 \otimes \cdots \otimes \bar{g}_n$$

$$\longmapsto \bar{g}_1 \otimes \cdots \otimes \bar{g}_n$$

$\bar{g}_1 = \delta g_1 = g_1 - 1$

$-b'(\bar{g}_1 \otimes \bar{g}_2) = -(g_1 - 1)(g_2 - 1)$ ~~[illegible]~~ $g_1 \delta g_2 + \delta g_1$

$$(g_1, g_2) \xmapsto{d} g_2 - g_1g_2 + g_1 \longmapsto \delta(g_2) - \delta(g_1g_2) + \delta(g_1)$$

$$= \delta(g_2) - g_1\delta g_2$$

$$= -(\delta g_1)(\delta g_2)$$

221 It seems as if we have lifted the complex $(B^{\otimes *}, -b')$ yielding $H_*(BG)$

You have this map of complex

$$\mathbb{Z}[BG] \longrightarrow (\tilde{B}^{\otimes *+1}, b) \longrightarrow (\tilde{B}\otimes B^{\otimes *}, b) \longrightarrow (B^{\otimes *}, -b')$$

$$(g_1,\dots,g_n) \longmapsto (g_1\cdots g_n)^{-1}\otimes g_1\otimes\cdots\otimes g_n \qquad\qquad \bar{g}_1\otimes\cdots\otimes\bar{g}_n$$

This composition is probably the projection on the normalized quotient complex. This is clear if the degeneracies in BG insert 1's. Yes. So what it means is that the above composition is a homotopy equivalence. So you find that the homology $H_*(BG)$ is a summand of the Hochschild homology $HH_*(\mathbb{Z}[G])$ — this is something you know!

This is confusing because we have linked bar homology of $I[G]$ with something. I want mainly to go from $\mathbb{Z}[G] \longrightarrow P\otimes_A Q$ to a map $H_*(BG) \longrightarrow HH_*(A)$? So how do I get to the bottom of this??? Try ~~$HH_*(A)$~~

03/19/97 Go over what you learned yesterday

Dennis trace. Consider a unital ring homom. $k[G]\to A$, equiv. a group hom $G \to A^\times = GL_1(A)$.

map of cyclic sets

$$\begin{array}{ccc} EG\times^G G^c & = & \{(g_0,\dots,g_n)\} \\ \downarrow & & \downarrow \\ A^{\otimes *+1}, b & & g_0\otimes\cdots\otimes g_n \end{array}$$

$BG \subset EG\times^G G^c = \{(g_0,\dots,g_n) \mid g_0\cdots g_n = 1\}$.

$BG_n = G^n$ $\quad d_i(g_1,\dots,g_n) = \begin{cases} (g_2,\dots,g_n) & i=0 \\ (g_1,\dots,g_ig_{i+1},\dots) & 0<i<n \\ (g_1,\dots,g_{n-1}) & i=n \end{cases} \qquad s_i(g_1,\dots,g_n) = (1,g_1,\dots,g_n)$ $\;\vdots\;$ $(g_1,\dots,g_n,1)$

isom. of two models $(g_0, \ldots, g_n) \mapsto (g_1, \ldots, g_n)$

$g_0 = (g_1 \cdots g_n)^{-1}$.

Suppose $A = \tilde{B}$

$$\mathbb{Z}[BG] \longrightarrow (\tilde{B}^{\otimes *+1}, b) \longrightarrow (\tilde{B} \otimes \tilde{B}^{\otimes *}, b) \longrightarrow (B^{\otimes *}, -b')$$

$\mathbb{Z}[G^n]$ $\quad$ $\mathbb{Z}[G^{n+1}]$

$(g_1, \ldots, g_n) \quad (g_1 \cdots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_n \longmapsto \bar{g}_1 \otimes \cdots \otimes \bar{g}_n$

normalized chain complex of $\mathbb{Z}[BG]$

What does this mean? Have

$$0 \to (B^{\otimes *+1}, b) \longrightarrow (\tilde{B} \otimes B^{\otimes *}, b) \longrightarrow (B^{\otimes *}, -b') \longrightarrow 0$$

so it seems that this splits canonically. But this should mean that

$$H_*(\mathbb{Z}[G], \overline{\mathbb{Z}[G]}) \simeq \bigoplus_{\langle x \rangle \neq 1} H_*(BG_x)$$

This is no surprise, but it would seem that the cx $(B^{\otimes *+1}, b)$ which computes $B \overset{L}{\otimes}_B$ is slightly removed from the group homology. Puzzle.

Anyway consider next $\mathbb{Z}[G] \longrightarrow \widetilde{P \otimes_A Q}$.

Idea: get map on Hoch homology, then you want to use M inv. of Hochschild to get $H_*(BG) \longrightarrow HH(A)$.

Philosophy is simple, but there seem to be technical problems. Start with A non unital, consider a dual pair (P, Q) over A with $P \in \mathcal{P}(\tilde{A}^{op})$ and a rep $\mathbb{Z}[G] \longrightarrow \widetilde{P \otimes_A Q}$. I can suppose $Q = P^* = {}^{A\otimes}\mathrm{Hom}_{A^{op}}(P, \tilde{A})$.

Put $B = P \otimes_A Q$. Have map

$$B \overset{L}{\otimes}_B \simeq P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B = Q \overset{L}{\otimes}_B P \overset{L}{\otimes}_A \longrightarrow A \overset{L}{\otimes}_A$$

and map $\bar{\mathbb{Z}}[G] \overset{L}{\otimes}_{\bar{\mathbb{Z}}[G]} \longrightarrow B \overset{L}{\otimes}_B$

223 So how do I proceed next?

Try to handle it: given $\mathbb{Z}[G] \longrightarrow P \otimes_A Q$ to map $\mathbb{Z}[BG]$ to $A \overset{\mathbb{L}}{\otimes}_A$. ~~You get nothing~~

~~Take $P \in \mathcal{P}(\tilde{A}^{op})$, put~~ ~~Can suppose~~

Suppose $P \in \mathcal{P}(\tilde{A}^{op})$, $Q = \check{P}$, choose non-unital embedding $P \otimes_A Q \hookrightarrow M_n(\tilde{A})$ - usual business of embedding in a trivial bundle.

There are problems continuing. What should I do? Scan

I have two approaches to Morita invariance of HH say.

Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ firm right flat $\boxed{\text{WD}}$

then $A \overset{\mathbb{L}}{\otimes}_A \xleftarrow{\sim} Q \overset{\mathbb{L}}{\otimes}_B P \overset{\mathbb{L}}{\otimes}_A = P \overset{\mathbb{L}}{\otimes}_A Q \overset{\mathbb{L}}{\otimes}_B \xrightarrow{\sim} B \overset{\mathbb{L}}{\otimes}_B$

double cx $\quad Q \otimes B^{\otimes i} \otimes P \otimes A^{\otimes j}$ $\quad$ gluing $\quad$ die drying

There are things here I find puzzling.

~~K-theory viewpoint~~ Maybe your approach to K-theory is biased, namely gluing $(P \otimes_A Q)^\times$ for appropriate dual pairs over A.

~~$(P \otimes_A Q)^\times = \mathrm{Hom}_{\text{aug rings}}$~~

$$\mathrm{Hom}_{gps}(G, B^\times) = \mathrm{Hom}_{\text{aug rings}}(\mathbb{Z}[G], \tilde{B})$$

$$= \mathrm{Hom}_{\text{rings}}(\mathbb{Z}[G], B)$$

So you are probing non unital rings by <u>non unital group rings.</u>

Perhaps what is happening is that when you probe B via nonunital group rings you detect <u>bar homology of B</u>

I am rapidly getting the impression that I should work more on the details using $K_*(A) = \mathrm{Ker}\{K_*(\tilde{A}) \to K_*(\mathbb{Z})\}$.

Now struggle for a few hours with details.

Review basic construction of trace map

$$K_*(P \otimes_A Q) \longrightarrow K_*(A)$$

assoc. ~~to~~ a dual pair (P, Q) over A with P A^{op}-flat.

suffices to define in a natural way for dual pairs with $P \in \mathcal{P}(\tilde{A}^{op})$.

canonical map $(P, Q) \longrightarrow (P, P^*)$ where

$$P^* = \mathrm{Hom}_{A^{op}}(P, A) = A \otimes_A \check{P}$$

$$P \otimes_A Q \longrightarrow P \otimes P^* = P \otimes_A A \otimes_A \check{P} \subset P \otimes_A \check{P} = \mathrm{End}_{A^{op}}(P)$$

ring homom. Better to think of $P \otimes_A Q$ acting on P

$$K_*(\widetilde{P \otimes_A Q}) \longrightarrow K_*(\tilde{A})$$

induces $K_*(P \otimes_A Q) \longrightarrow K_*(A)$ since

$P \otimes_A \mathbb{Z} = P/PA$ is a trivial rep of $P \otimes_A Q$.

naturality $(u, v) : (P, Q) \longrightarrow (P', Q')$ map of dual pairs.

note cons. functorial in Q with P fixed. hence reduces to $Q = Q' = (P')^*$. $v = \mathrm{id}$

 reduce to $P \hookrightarrow P\oplus P' \longrightarrow P'$

I must do this carefully.

General case: $u : P \longrightarrow P'$ $\quad u \in P'\otimes_A P^*$

$B_u = P\otimes_A P'^*$ $\quad (p_1\otimes\lambda_1')(p_2\otimes\lambda_2') = p_1\langle\lambda_1', u(p_2)\rangle\otimes\lambda_2'$

$$\begin{tikzcd}
P\otimes_A P'^* \arrow[r, "u\otimes 1"] \arrow[d, "1\otimes u^*"] & P'\otimes_A P'^* \\
P\otimes_A P^* &
\end{tikzcd}$$

Note $v \in P\otimes_A P'^*$ such that $1+v$ invertible

$\Longrightarrow (u\otimes 1)v = \underline{uv : P' \to P'}$ $\ni$ $1+uv$ invertible on P'

But suppose you factor u

Two cases: $P \hookrightarrow P\oplus P'$

$$\begin{tikzcd}
 & \begin{pmatrix}P\\P'\end{pmatrix}\otimes_A P'^* \arrow[r, two heads, "(0\ 1)\otimes 1"] \arrow[d, "1\otimes\binom{0}{\varphi}"] & P'\otimes_A P'^* \\
P\otimes_A(P^*\ P'^*) \arrow[r, hook, "\binom{1}{u}\otimes 1"] \arrow[d, "1\otimes(1\ u^*)"] & \begin{pmatrix}P\\P'\end{pmatrix}\otimes_A(P^*\ P'^*) & \\
P\otimes_A P^* & &
\end{tikzcd}$$

The real puzzle is whether one can assume $u : P \to P'$ is zero modulo A. It seems that ~~the general case is~~ to construct the trace for a general flat firm P you can write P as ind. limit of free fg modules transitions are matrices $/A$, 226 03/20/97 CD due today

Let's try to do a little. Take B flat firm, $A = B \oplus I$, I is a B-module regarded as B-bimodule with $IB = 0$.

$$\begin{pmatrix} B\oplus I & \tilde B \\ B\oplus I & B \end{pmatrix}$$

Now assume $I = B$ ~~[illegible]~~ then we have homo $A = 1$. NO.

Better A is a B-module equipped with B-map $f : A \longrightarrow B$.

$$\begin{pmatrix} A = \tilde B \otimes_B A & \tilde B \\ A & B \end{pmatrix}$$

dual pair over B

$$\cancel{\left(\tilde A, \vec B, \tilde B \otimes A \to B\right) \quad \tilde b \otimes a \mapsto \tilde b f(a)}$$

$$\left(\tilde B, A, A\otimes \tilde B \longrightarrow B\right)$$
$$a \otimes \tilde b \longmapsto f(a)\tilde b$$

Now suppose $A = B \oplus I$ $f = pr_1 : A \longrightarrow B$. Then have homos. $A \longrightarrow B < A$, actually maps of dual pairs $(\tilde B, A) \xrightarrow{(1,f)} (\tilde B, B) \xrightarrow{(1,i)} (\tilde B, A)$

$\langle a, \tilde b\rangle = f(a)\tilde b$ $\quad \langle b, \tilde b\rangle = b\tilde b$ $\quad \langle ib, \tilde b\rangle \overset{def}{=} f(ib)\tilde b = b\tilde b = \langle b, \tilde b\rangle$.

I want to show that the ring homom $A \xrightarrow{f} B < A$ induces the identity on $H_*(GL(A))$. I think I know that this is true for $K_*(A) = \mathrm{Ker}\{K_*(\tilde A) \to K_*(\mathbb{Z})\}$.

Digress to ask whether, instead of looking at $B \oplus_1 B_0$ with B left flat, it might be better to work with $B \oplus_0 B_1$ and B left flat. Or by symmetry $B\oplus_1 B_0$ with B right flat. Given

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

B is B^{op}-flat $\Leftrightarrow$ $B \otimes_B P = A$ $\;A^{op}$-fl

B is B-flat $\Leftrightarrow$ $Q \otimes_B B = B$ is A-fl

227 ass $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ form

B is B^{op}-flat $\iff$ $B \otimes_B P = A$ is A^{op}-flat

B is B-flat $\iff$ $Q \otimes_B B = B$ is A-flat

~~A is A-flat $\iff$ $P \otimes_A A = A$ is A-flat~~

A is A^{op}-flat $\iff$ $A \otimes_A Q = A \otimes_A B = B$ is B^{op}-flat

I want to take $A = B$

I persist stupidly in trying to show flat rings have Morita inv. for $H_*(BGL(-))$. What you should be doing is everything you can concerning the construction you know works. For example, what about ~~associative~~ rings which are both left and right flat, what about dual pairs (P,Q) over A where P is A^{op} flat and Q is A-flat. Show the two trace maps. Can assume $P \in \mathcal{P}(\tilde{A}^{op})$ and $Q \in \mathcal{P}(A)$. ~~Then~~ you have $Q \longrightarrow \mathrm{Hom}_{A^{op}}(P, A) = A \otimes_A \check{P}$ and $P \longrightarrow \mathrm{Hom}_A(Q, A) = \check{Q} \otimes_A A$. Then we can look at

$$\begin{tikzcd} & P \otimes_A Q \arrow[dl] \arrow[dr] & \\ P \otimes_A P^* & & Q^* \otimes_A Q = \check{Q} \otimes_A A \otimes_A Q \end{tikzcd}$$

So you need to know whether the representations of $B = P \otimes_A Q$ ~~and~~ on P and on $\check{Q}$ are equiv. This ought to follow from functoriality wrt $P \longrightarrow \check{Q}^* \subset \check{Q}$.

Review: $P \xrightarrow{u} P'$ $\qquad P \xrightarrow{\binom{1}{u}} P \oplus P' \xrightarrow{(0\ 1)} P'$

Factor:

$$\begin{tikzcd}
 & \binom{P}{P'} \otimes_A P'^* \arrow[r, "(0\ 1)\otimes 1"] \arrow[d, "\wr"', "1\otimes(0\ 1)"] & P' \otimes_A P'^* \\
P \otimes_A (P^*\ P'^*) \arrow[r, hook, "\binom{1}{u}\otimes 1"] \arrow[d, "\wr"] & \binom{P}{P'} \otimes_A (P^*\ P'^*) & \\
P \otimes_A P^* & &
\end{tikzcd}$$

Let's review the implication that ~~A flat ⇒ A⊕~~ Morita invariance $\Rightarrow$ K triviality of affine extns. Assume A, B left flat, $f: A \twoheadrightarrow A/I = B$. Two cases $IA = 0$. Then A is B-module mapping onto B and $a_1 a_2 = f(a_1) a_2$. Here $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ and A is

A flat $\Leftrightarrow$ $P \otimes_A A = A \otimes_A A = A$ is B flat.

<u>Note</u> A is A^{op}-flat $\Longrightarrow$ $A \otimes_A Q = A \otimes_A B = B$ is B^{op}-flat

True for any $f: A \twoheadrightarrow B$. This is m

So you want to prove that for B a right flat ring, and any B-module map $f: A \to B$ s.t. $f(A)B = B$ that $GL(A) \longrightarrow GL(B)$ is a homology iso. This seems much simpler than what I was trying to do.

Review: Consider a left Morita equiv. $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$, say given by a firm dual pair ~~(B, A, A⊗B→B)~~ $(B, A, A\otimes B \twoheadrightarrow B)$. Then $A \simeq B\otimes_B A$ is a ring with $(\cancel{b}a_1)(b_2a_2) = \cancel{b}\langle a_1, b_2\rangle a_2$. To simplify suppose $\langle a, b\rangle = f(a)b$ where $f: A \to B$ is a B-mod. map. Then $a_1(b_2a_2) = f(a_1)b_2a_2$

i.e. $\underline{a_1a_2 = f(a_1)a_2}$.

Suppose given $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ left Morita equiv.

$\mathcal{M}(A) \ni M \longmapsto P\otimes_A M = A\otimes_A M = M$, $\mathcal{M}(B) \ni N \mapsto Q\otimes_B N = B\otimes_B N = N$

Then A is A^{op}-flat $\iff A\otimes_A Q = A\otimes_A B = B$ is B^{op}-flat. So therefore, once B is B^{op} flat then so any A left Morita equiv. to B. Now

Let try to really understand this, especially in terms of matrix equations.

Assume $f: A \to B$ gives the pairing.

Can you show A is A^{op} flat if B is B^{op} flat by equations

U is A^{op} flat means

$$\begin{tikzcd} \tilde{A}^{n'} \arrow[r, "a\cdot"] \arrow[dr, "0 = ua"'] & \tilde{A}^{n} \arrow[r, "a'\cdot"] \arrow[d, "u\cdot"] & \tilde{A}^{p} \\ & U & U \arrow[u, "u'\cdot"'] \end{tikzcd}$$

$\exists\, a', u' \ni a'a = 0$ and $u = u'a'$

 Take the case $A \longrightarrow A/I = B$ where $IA = 0$

Thus if you have

$$\begin{array}{ccc} \tilde{A}^{n'} & \xrightarrow{a_0} & \tilde{A}^{n} \\ & & \downarrow{\scriptstyle u_0} \\ & & A \end{array}$$

$$\begin{array}{ccc} A^{n'} & \xrightarrow{a_0} & A^{n} \\ & \searrow{\scriptstyle 0} & \downarrow{\scriptstyle u_0} \\ & & A \end{array}$$

apply the hom $f: A \to B$

you get $f(u_0)f(a_0)$ ~~$f(a_0)f(u_0)$~~ $= 0$

so can factor

$f(u_0) =$

The idea should be that if A is a filtered ind limit

So we have ~~…~~ $(B, A) \longrightarrow (B, B)$ and we assume B is right flat. ~~…~~ Can write B as filtered ind limit of $P_i = \bar{B}^{n_i}$ transition maps from B.

$$M(P_i \otimes_B I) \hookrightarrow GL(P_i \otimes_B A) \twoheadrightarrow GL(P_i \otimes_B B)$$

But OK we have ~~BA~~ $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

B is right flat/B

A is right fl/A.

The fact that A is A^{op} flat and B ~~…~~ left acts on A ~~…~~ might lead to a K map $B \rightsquigarrow A$

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suppose B right flat firm, ~~ABBB~~

$$0 \longrightarrow I \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

exact in $\mathcal{M}(B)$. General case is firm dual pair (B,A).

Then $A = B \otimes_B A$ is right flat. $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

$f : A \longrightarrow B$ is a ring hom of right flat rings.

To prove $GL(A) \longrightarrow GL(B)$ induces isom. on $H_*(B(-))$

$$\phi \longrightarrow M(I) \longrightarrow GL(A) \longrightarrow GL(B) \longrightarrow \phi$$

group extension. $E^2_{pq} = H_p(GL(B), H_q(M(I))) \Rightarrow H_*(GL(A))$

Observe that A can be $B \oplus I$ with I any firm B-module, e.g. B. You have a lot of freedom here, namely, any firm ring left Morita equivalent to B, any firm B-mod A equipped with $f : A \to B$. gen. B.

Do I understand the unital case. B unital A and unit B mod. This is the case $A \in \mathcal{P}(A^{op})$, $B = \mathrm{End}_{A^{op}}(A,A)$, that I studied – Davydov.

Have $A \to B$ homom. ${}_A B_B$ bimodule

have ${}_B A_A$ inducing $K_*(B) \to K_*(A)$.

$U \in \mathcal{P}(\tilde{A}^{op})$ $\quad U \longmapsto U \otimes_A B \qquad GL_n(A) \to GL_n(B)$

$V \in \mathcal{P}(B) \quad V \longmapsto V \otimes_B A \qquad GL_n(B) \to \underline{GL_n(A)}$

$B^n \qquad A^n \qquad$ No. ?

$$U \mapsto U \otimes_A B \longmapsto U \otimes_A B \otimes_B A = U \otimes_A A$$

$$A \in \mathcal{P}(A^{op}) \qquad B = \mathrm{Hom}_{A^{op}}(A,A)$$

$$A \xrightarrow{f} B \qquad f(A)B = B.$$

For example if $A = B \oplus B$ and $f = pr_1$

Functors $\quad \mathcal{M}(A^{op}) \qquad \mathcal{M}(B^{op})$

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix} \qquad U \longmapsto U \otimes_A B$$

$$V \otimes_B A \longleftarrow V$$

One composition is the identity, the other is $U \longmapsto U \otimes_A A$ which is A^{op}-nil isom. to the identity. And I guess we know how to handle this. ~~Namely~~ ~~P^n~~ ~~P'~~

I am comparing $U \mapsto U \otimes_A P = U \otimes_A A$ with $U \longrightarrow U \otimes_A P' = U \otimes_A \tilde{A} = U$. ~~We have this~~

~~map~~ $\quad P \otimes_A Q \longrightarrow P' \otimes_A Q$

$$\pi$$

~~You have~~

$$A \in \mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op}) \qquad A \underset{x}{\overset{y}{\rightleftarrows}} \tilde{A}^n \qquad yx = 1$$

$\mathcal{P}(\tilde{A}^{op}) \ni U \longmapsto U \otimes_A A$. I want to see the effect on matrices. So we get $\tilde{A} \longrightarrow M_n(A) \subset M_n(\tilde{A})$

$$\tilde{a} \longmapsto x\tilde{a}y$$

and I need to compare this homom. with the identity. So how to proceed? Factor $A \subset \tilde{A}$ into

$$A \longrightarrow A \oplus \tilde{A} \longrightarrow \tilde{A}$$

233 Suppose $A = B \oplus I$ I a unitary B module, B unital. Note that A has a left identity: $A = eA$ $B = eAe$

This implies that $A \overset{e}{\underset{\longrightarrow}{\longleftarrow}} \tilde{A}$ is a summand of the right A-module $\tilde{A}$. ~~We need to~~ ~~compare~~ In this situation besides the homom $A \xrightarrow{f} B$ we also have a homom. $B \hookrightarrow A$. The second is in general replaced by the bimodule ${}_B A_A$. Wait: a homom. $B \hookrightarrow A$ yields $V \mapsto V \otimes_B \tilde{A}$. But here we have $V \longmapsto V \otimes_B A$ which is defined ~~and~~ more generally. Point is that B is the multiplier alg.

So what? You have

03/23/97 Consider $A = B \oplus I$ B unitary, I a unitary B-module. Can I find a simple proof that $GL(A) \longrightarrow GL(B)$ induces isom. on H_*. This is the simplest case, but more generally one can look at any unitary firm dual pair (B, A) over B, ~~i.e. a~~ equiv. a B-module map $A \xrightarrow{f} B$ $\ni$ $f(A)B = B$. Such an A is right flat, in fact, in $\mathcal{P}(A^{op})$.