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transitivity - ~~Given A unital~~

Given rings A, B, C bimodules $_BP_A$, $_CQ_B$ such that $P \in \mathcal{P}(\tilde{A}^{op})$, $Q \in \mathcal{P}(\tilde{B}^{op})$. Then we have functors

$$\mathcal{P}(\tilde{C}^{op}) \longrightarrow \mathcal{P}(\tilde{B}^{op}) \longrightarrow \mathcal{P}(\tilde{A}^{op})$$

$$W \longmapsto W \otimes_C Q, \quad V \longmapsto V \otimes_B P$$

with composition $W \longmapsto W \otimes_C (Q \otimes_B P)$ the functor assoc. to the C, A-bimodule $Q \otimes_B P$ which lies in $\mathcal{P}(\tilde{A}^{op})$. ∴ commutatively:

$$\begin{tikzcd} K_* \tilde{C} \arrow[rr, "{[- \otimes_C Q]}"] \arrow[dr, "{[-(Q \otimes_B P)]}"'] & & K_* \tilde{B} \arrow[dl, "{[- \otimes_B P]}"] \\ & K_* \tilde{A} & \end{tikzcd}$$

This is the elementary step I must use.

generalization from $_BP_A$ in $\mathcal{P}(\tilde{A}^{op})$ to just A^{op} flat

Given P flat A^{op}-module and a homom. $B \to \mathcal{N}_A(P)$ where $\mathcal{N}_A(P) = P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})$. $\quad \mathrm{Nuc}_{A^{op}}(P,P)$

Given E a flat A^{op}-module. Consider $\mathcal{P}(\tilde{A}^{op})/E$ category of fg proj $\tilde{A}^{op}$-mods P equipped with a homom. $P \to E$.
E flat $\Leftrightarrow$ $\mathcal{P}(\tilde{A}^{op})/E$ is filtering.

$$\mathcal{N}_A(E,E) = \varinjlim P \otimes_A \mathrm{Hom}_{A^{op}}(E, \tilde{A}) = \varinjlim_P \mathrm{Nuc}_{A^{op}}(E, P)$$

For each object P have homom.

$$\underbrace{P \otimes_A \mathrm{Hom}_{A^{op}}(E, \tilde{A})} \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A}) = \mathrm{Hom}_{A^{op}}(P,P)$$

whence $K_*(\qquad) \longrightarrow K_*(\tilde{A})$

Given
$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & P' \arrow[dl] \\ & E & \end{tikzcd}$$
factor $P \to P \oplus P' \to P'$

67 For each $P \to E$ in $\mathcal{P}(\tilde{A}^{op})/E$ get a ring

$P \otimes_A \check{E}$, $\check{E} = \mathrm{Hom}_{A^{op}}(E, \tilde{A})$, and a rep of $P \otimes_A \check{E}$ on P which is in $\mathcal{P}(\tilde{A}^{op})$, hence a homom.

$$K_*(P \otimes_A \check{E}) \subset K_*((P \otimes_A \check{E})^{\sim}) \longrightarrow K_*(\tilde{A}).$$

Claim $\forall\ P \xrightarrow{u} P'$ over E

$$\begin{tikzcd} K_*(P \otimes_A \check{E}) \arrow[r] \arrow[d] & K_*(\tilde{A}) \\ K_*(P' \otimes_A \check{E}) \arrow[ur] & \end{tikzcd}$$

comm.

Can factor $P \xrightarrow{\Gamma_u} P \oplus P' \xrightarrow{pr_2} P'$, two cases: ~~...~~ u direct injection whence $0 \to P \xrightarrow{u} P' \to P'' \to 0$ ~~...~~ ~~...~~ exact sequence of repns of $\bar{P} \otimes_A \check{E}$ in $\mathcal{P}(\tilde{A}^{op})$ such that the action is zero on P''. ~~...~~ Question of triviality of the affine rings.

Let's consider next ~~...~~ the trans. issue

$$\begin{matrix} A & \\ E & B \\ F & C \end{matrix}$$

$$\begin{tikzcd}[column sep=small] A & & E^* & & \\ & B \arrow[d] & & F^* \arrow[d] & \\ E & E \otimes_A E^* & & (E \otimes_A E^*)^{\sim} \otimes_B F^* & \\ & & C \arrow[d] & & \\ F \arrow[d] & & F \otimes_B F^* \arrow[d] & & \\ F \otimes_B (E \otimes_A E^*)^{\sim} & & F \otimes_B (E \otimes_A E^*)^{\sim} \otimes_B F^* & & \end{tikzcd}$$

Missing idea yesterday: You need B to act on a dual pair over A, not just on the module P. In fact more. You have been asking for $B \to P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})$. But really you have more generally P_A, ${}_A P^*$, and the pairing $P^* \otimes P \longrightarrow A$ (or $\tilde{A}$?). Then can form the ring $P \otimes_A P^*$. It might help to think of $(P, P^*, \langle\ \rangle : P^* \otimes P \to A)$ as having a multiplier ring $\{(\lambda, \rho) \in \mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(P^*, P^*)^{op}$

$$\mid \langle p^*\rho, p\rangle = \langle p^*, \lambda p\rangle\}.$$

You can always write this down. Check it gives a subring of . Suppose $\langle p^*\rho_i, p\rangle = \langle p^*, \lambda_i p\rangle$ for $\forall p^*, p$ & $i = 1, 2$. i.e. $(\lambda_i, \rho_i) \in$ Mult. Then

$$\langle p^*\rho_1\rho_2, p\rangle = \langle p^*\rho_1, \lambda_2 p\rangle = \langle p^*, \lambda_1\lambda_2 p\rangle$$

showing that $(\lambda_1, \rho_1)(\lambda_2, \rho_2) \in$ Mult.

Notice that there is a canonical homom.

$$P \otimes_A P^* \longrightarrow \text{Mult}$$

$$p_0 \otimes p_0^* \longmapsto (\cdot p_0 \otimes p_0^* \cdot\ ,\ \cdot p_0 \otimes p_0^*)$$

$$(p \overset{\lambda}{\mapsto} p_0\langle p_0^*, p\rangle,\ p^* \overset{\mu}{\mapsto} \langle p^*, p_0\rangle p_0^*)$$

Check $\langle \langle p^*, p_0\rangle p_0^*, p\rangle \overset{?}{=} \langle p^*, p_0\langle p_0^*, p\rangle\rangle$

$$\langle \langle p^*, p_0\rangle p_0^*, p\rangle = \langle p^*, p_0\rangle\langle p_0^*, p\rangle = \langle p^*, p_0\langle p_0^*, p\rangle\rangle$$

$M = P \otimes_A P^*$ is an R-bimodule

R ———————— map.

$$(\lambda, \rho)(p_0 \otimes p_0^* \cdot\ ,\ \cdot p_0 \otimes p_0^*) = (p \mapsto \lambda(p_0\langle p_0^*, p\rangle),\ p^* \mapsto \langle p^*\rho, p_0\rangle p_0^*)$$

$$= (p \mapsto \lambda(p_0)\langle p_0^*, p\rangle,\ p^* \mapsto \langle p^*, \lambda(p_0)\rangle p_0^*)$$

$$= (\lambda(p_0) \otimes p_0^* \cdot\ ,\ \cdot \lambda(p_0) \otimes p_0^*)$$

check axioms. $M \otimes_R M \longrightarrow M$?

$$\left((p_1 \otimes q_1)(\lambda, \rho)\right)(p_2 \otimes q_2) = (p_1 \otimes q_1)\left((\lambda, \rho)(p_2 \otimes q_2)\right)$$

$$(p_1 \otimes q_1 \rho)(p_2 \otimes q_2) \qquad (p_1 \otimes q_1)(\lambda p_2 \otimes q_2)$$

$$p_1 \otimes \underline{\langle q_1 \rho, p_2 \rangle} q_2 \qquad p_1 \otimes \underline{\langle q_1, \lambda p_2 \rangle} q_2$$

This preceding is not the right thing to check

M is an R-module with $f : M \to R$ a bimod map.

so you ~~should~~ can define $m_1 * m_2 = f(m_1) m_2$
$m_1 \cdot m_2 = m_1 f(m_2)$.

You need these two things to be the same

$$p_1 \otimes q_1, \; p_2 \otimes q_2 \longmapsto (p_1 \otimes q_1) p_2 \otimes q_2 = p_1 \langle q_1, p_2 \rangle \otimes q_2$$

$$\longmapsto p_1 \otimes q_1 (p_2 \otimes q_2) = p_1 \otimes \langle q_1, p_2 \rangle q_2$$

Back to transitivity - ~~tensor product~~.

$(A, E, E^*, \langle\ \rangle)$ If E flat over A^{op}, then $\exists$ canon.

$$K_*(E \otimes_A E^*) \longrightarrow K_*(A).$$

$(B, F, F^*, \langle\ \rangle)$, $B \longrightarrow E \otimes_A E^*$,

assume F flat over B^{op}, can compose

$$K_*(F \otimes_B F^*) \longrightarrow K_*(B) \longrightarrow K_*(E \otimes_A E^*) \longrightarrow K_*(A).$$

on the other hand have dual pair over A.

$F \otimes_B E$, $E^* \otimes_B F^*$,

70 $\quad (E^* \otimes_B F^*) \otimes (F \otimes_B E) \longrightarrow E^* \otimes_B B \otimes_B E \longrightarrow A$

I should check in general that the pairing

$E^* \otimes E \longrightarrow A$ is over ~~B~~ $E \otimes_A E^*$

$\langle \xi(\varphi_1, \xi_1), p \rangle \stackrel{?}{=} \langle \xi, (\varphi_1 \xi_1) p \rangle$

$\langle \xi, p_1 \rangle \xi_1 \stackrel{?}{=} \langle \xi, p_1 \rangle \langle \xi_1, p \rangle$

~~So we have~~ Now F B^{op}-flat, E A^{op}-flat

$\Rightarrow$ $F \otimes_B E$ is A^{op}-flat. So we have

$$K_*\big((F \otimes_B E) \otimes_A (E^* \otimes_B F^*)\big) \longrightarrow K_*(A)$$

$\| \,?$

Idea maybe $\quad B \longrightarrow B' = E \otimes_A E^*$

$F \otimes_B \tilde{B}' \otimes_{B'} E \qquad E^* \otimes_{B'} \tilde{B}' \otimes_{B'} F^* \quad ?$

$A \quad E^* \quad E^*$

$E \quad B = E \otimes_A E^* = B' \quad F^*$

$F \otimes_B E \quad F$

$A \quad E^*$

$B \to E \otimes_A E^* \qquad F^* \to \widetilde{E \otimes_A E^*} \otimes_B F^*$

E

$F \to F \otimes_B \widetilde{E \otimes_A E^*} \qquad F \otimes_B F^* \to F \otimes_B \tilde{B}' \otimes_{B'} \tilde{B}' \otimes_B F^*$

$= F \otimes_B \tilde{B}' \otimes_B F^*$

$(\tilde{B}' \otimes_B F^*) \otimes (F \otimes_B \tilde{B}')$

$\downarrow$

$\tilde{B}' \otimes_B B \otimes_B \tilde{B}'$

$\downarrow$

B'.

so what seems to happen is the fact that I can enlarge the dual pair

$(B, F, F^*, F^* \otimes F \to B)$

to $(B', F \otimes_B \tilde{B}', \tilde{B}' \otimes_B F^*, \tilde{B}' \otimes_B F^* \otimes_F F \otimes_B \tilde{B}' \longrightarrow B')$

and then the corresp ring is

$$(F \otimes_B \tilde{B}') \otimes_{B'} (\tilde{B}' \otimes_B F^*) = F \otimes_B \tilde{B}' \otimes_B F^*$$

which should be the target of an obvious map from $F \otimes_B F^*$

Start again

$$\begin{tikzcd} A & E^* & & \\ E & E \otimes_A E^* & & \\ & & B \arrow[ul] & F^* \\ & & F & F \otimes_B F^* \\ & & & & C \arrow[ul] \end{tikzcd}$$

I have a dual pair over A

$$\left(A,\ F \otimes_B E,\ E^* \otimes_B F^*,\ (E^* \otimes_B F^*) \otimes (F \otimes_B E)\right)$$

$$\begin{array}{c} (E^* \otimes_B F^*) \otimes (F \otimes_B E) \\ \downarrow \\ E^* \otimes_B B \otimes_B E \\ \downarrow \\ E^* \otimes_B E \\ \downarrow \\ A \end{array}$$

any

~~and [illegible] hm.~~

[illegible]

$$C \to F \otimes_B F^* = F \otimes_B B \otimes_B F^* \longrightarrow (F \otimes_B E) \otimes_A (E^* \otimes_B F^*)$$

There's a problem here even if $B = E \otimes_A E^*$ since the ring C is $F \otimes_B F^*$ in general, and the composite triple over its A is $(A, F \otimes_B E, E^* \otimes_B F^*$ and ~~this~~ ring is $F \otimes_B \underbrace{E \otimes E^*}_{A} \otimes_B F^* = F \otimes_B B \otimes_B F^*$.

72 This may ~~have been~~ be the same problem as before.

$$(F \otimes_B E) \otimes_A \operatorname{Hom}_{A^{op}}(F \otimes_B E, A)$$
$$\|$$
$$(F \otimes_B E) \otimes_A \operatorname{Hom}_{B^{op}}(F, \operatorname{Hom}_{A^{op}}(E, A))$$
$$\downarrow$$
$$F \otimes_B \operatorname{Hom}_{B^{op}}(F, \underbrace{E \otimes_A \operatorname{Hom}_{A^{op}}(E, A)}_{B})$$

trans. problem

$$\begin{array}{ccc} A & E^* & \\ E & B = E \otimes_A E^* & F^* \\ & F & C = F \otimes_B F^* \end{array}$$

$$(F \otimes_B E) \otimes_A (E^* \otimes_B \cancel{F}^*) = F \otimes_B B \otimes_B F^*$$

simplest to replace $\cancel{F}^*$ by $B \otimes_B F^* = E \otimes_A E^* \otimes_B F^*$

do trans.

Now let's take interesting cases. Let $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ be a s firm Morita context everything right flat, i.e. A, B are right flat, whence P, Q are also. NO how about

$$\begin{pmatrix} A & E^* & E^* \otimes_B F^* \\ E & B & F^* \\ F \otimes_B E & F & C \end{pmatrix}$$

E A^{op} flat $\Longrightarrow$ $E \otimes_A E^* = B$ is B^{op} flat $\Longrightarrow$ $B \otimes_B F^* = F^*$ is C^{op} flat.

F B^{op} flt $\Longrightarrow$ $F \otimes_B E$ is A^{op} flt $\Longrightarrow$ $F \otimes_B F^* = C$ is C^{op} flat.

$$F\otimes_B E\otimes_A E^*\otimes_B F^* = F\otimes_B B\otimes_B F^* \overset{?}{=} F\otimes_B F^* = C.$$

This is hard.

$$\begin{matrix} A & E^* \\ E_D & B_D \\ & F \end{matrix}$$

Let's try discussing special cases. Recall if $B \longrightarrow E\otimes_A E^*$, then you replace F by $F\otimes_B \widetilde{E\otimes_A E^*}$. Thus if $F = \tilde{B}^n$, then $F\otimes_B \widetilde{E\otimes_A E^*} = \widetilde{E\otimes_A E^*}^{\,n}$

and $F\otimes_B E = E^n$

Another point. Go back to $\begin{pmatrix} A & E^* \\ E & E\otimes_A E^* \end{pmatrix}$

you approx E by $\tilde{A}^n$

Another attempt.

Consider $(A, E, E^*, E^*\otimes E \xrightarrow{\langle\,,\rangle} A)$ with E A^{op} flat

Put $B = E\otimes_A E^*$ and consider $(B, F, F^*, F^*\otimes F \xrightarrow{\langle\,\rangle} B$ w. F B^{op} flat

First yields $K_*(B) \longrightarrow K_*(A)$

2nd — $K_*(F\otimes_B F^*) \longrightarrow K_*(B)$

Can form the composite

$$\left(A, F\otimes_B E, E^*\otimes_B F^*, (E^*\otimes_B F^*)\otimes(F\otimes_B E) \to E^*\otimes_B B\otimes_B E \to A\right)$$

note $F\otimes_B E$ is A^{op}-flat so get

$$\begin{tikzcd} K_*\left((F\otimes_B E)\otimes_A (E^*\otimes_B F^*)\right) \arrow[r] & K_*(A) \\ K_*(F\otimes_B F^*) \arrow[u, "\text{But } \exists \text{ homom ind.}"] \arrow[r] & K_*(B) \arrow[u] \end{tikzcd}$$

Claim commutative

Write $F = \varinjlim F_i$ filtered colimit
F_i free f.n. $\tilde{B}^{op}$-modules

Reduces us to the case where $F = \tilde{B}^k$. In this case we have a map of dual pairs

$$B, \tilde{B}^k, F^* \longrightarrow B, \tilde{B}^k, B^k$$

You have from $F^* \otimes F \longrightarrow B$ a map

$$F^* \longrightarrow \underset{\substack{\| \\ B\check{F}}}{\mathrm{Hom}_{B^{op}}}(\underset{\substack{\| \\ \tilde{B}^k}}{F}, B) = B^k.$$

$$\left(F \otimes_B E,\ E^* \otimes_B F^*\right) \longrightarrow \left(F \otimes_B E,\ E^* \otimes_B B\check{F}\right)$$

In general we should know that your trace is compatible with homomorphisms.

$$\begin{tikzcd}
K_*(F \otimes_B E \otimes_A E^* \otimes_B F^*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{comm.}"] & K_*(F \otimes_B E \otimes_A E^* \otimes_B B\check{F}) \arrow[r, "="] \arrow[d] & K_*(A) \\
K_*(F \otimes_B F^*) \arrow[r] & \underset{M_k(B)}{K_*(F \otimes_B B\check{F})} \arrow[r, "\text{obvious}"'] & K_*(B)
\end{tikzcd}$$

$M_k(B \otimes_B B)$

It's confusing, but it really ought to be trivial.

Now look at the critical case $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ where $A = A^2$ is A^{op} flat. You now need to see you get the identity. You should know the map $K_*(A) \to K_*(A)$ is idempotent. So pick $F \to A$ $F \in \mathcal{P}(\tilde{A}^{op})$ consider $\begin{pmatrix} A & A \\ F & FA \end{pmatrix}$

The construction is simple: $A = \varinjlim F_i$ so $B_i = F_i \otimes_A A$

$$K_*(A) = K_*(A \otimes_A A) = \varinjlim K_*(F_i \otimes_A A)$$
$$\longrightarrow \varinjlim K_*(F_i \otimes_A \check{F}_i)$$

75 Let F finite free $\tilde{A}^{op}$-module with a map $F \xrightarrow{u} A$. Form ring $F \otimes_A A = FA$

$$(f_1 a_1)(f_2 a_2) = f_1 a_1 u(f_2) a_2 = f_1 a_1\, u(f_2 a_2)$$

Call this ring B $\begin{pmatrix} \tilde{A} & \tilde{A} \\ F & F \end{pmatrix}$ $F \xrightarrow{u} \tilde{A}$ A^{op} map so $f_1 f_2 = f_1 u(f_2)$ makes F a ring.

$B = F \otimes_A A = FA$. Now we have two repns of B in $\mathcal{P}(\tilde{A}^{op})$, the one from the hemon, $B \longrightarrow A$ the other from the B-action on F, namely

$$(F \otimes_A A) \otimes_A F \longrightarrow F$$
$$f_1 \otimes a \otimes f_2 \longmapsto f_1\, a u(f_2)$$

You want to see these are K-equivalent
Special case of why I did before?

$$\begin{array}{ccc} P & \xrightarrow{u} & P' \\ \uparrow & & \uparrow \\ \text{rep of} & & \text{rep of} \\ B = P \otimes_A Q & & B' = P' \otimes_A Q \end{array}$$

a B-nil isom.

$$\begin{pmatrix} A & A \\ F & F \otimes_A A \end{pmatrix} \rightarrow \begin{pmatrix} A & A \\ \tilde{A} & A \end{pmatrix}$$

~~So we have~~ This time you have

$$\begin{array}{ccc} F & \xrightarrow{u} & \tilde{A} \\ \uparrow & & \uparrow \\ \text{rep of} & & \text{rep of} \\ B = F \otimes_A A & \xrightarrow{w} & B' = \tilde{A} \otimes_A A = A \\ f \otimes a & \longmapsto & u(f) a \end{array}$$

$$(fa) f' = f a u(f') \qquad w(fa) = u(f) a$$

$$w(f_1 a_1)\, w(f_2 a_2) = u(f_1) a_1\, u(f_2) a_2$$
$$\parallel$$
$$w(f_1 a_1 u(f_2) a_2) = u(f_1) a_1 u(f_2) a_2$$

Seems OKAY. Suppose $A \in \mathcal{P}(\tilde{A}^{op})$. Then I take $F = A$ and u the identity. I have

$$0 \longrightarrow A \xrightarrow{u} \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0$$

and I factor u.

$$0 \longrightarrow A \xrightarrow{\Gamma_u} A \oplus \tilde{A} \xrightarrow{pr_2} \tilde{A} \longrightarrow 0$$

I have A acting by left multiplication on the objects

I have to be more careful. But still if $A \in \mathcal{P}(\tilde{A}^{op})$, then I have two representations of A on objects of $\mathcal{P}(\tilde{A}^{op})$, namely A and $\tilde{A}$, so I get two homs.
$K_*(A) \longrightarrow K_*(\tilde{A})$.

On the other hand given $P \xrightarrow{u} P'$ I get

$w: P \otimes_A Q \longrightarrow P' \otimes_A Q$

$$\begin{tikzcd} P \arrow[r, "u"] & P' \\ \text{rep of } B \arrow[u] & \text{rep of } B' \arrow[u] \end{tikzcd} \quad \text{in } \mathcal{P}(\tilde{A}^{op})$$

So I look at both P and P' as repns of B

$$\begin{array}{ccc} P & \xrightarrow{u} & P' \\ \| & & \| \\ A & \hookrightarrow & \tilde{A} \end{array} \qquad Q = A \qquad \begin{array}{c} Q \otimes P \longrightarrow A \\ A \otimes \tilde{A} \xrightarrow{mult} A \end{array}$$

$$\begin{array}{ccc} B = P \otimes_A Q & \xrightarrow{u \otimes 1} & P' \otimes_A Q = B' \\ \| & & \| \\ A \otimes_A A & \xrightarrow{w = \mu} & A \end{array}$$

Now you want to factor $A \hookrightarrow \tilde{A}$ into

$$\underset{P}{A} \xrightarrow{\binom{1}{u}} \underset{P''}{A \oplus \tilde{A}} \xrightarrow{pr_2} \underset{P'}{\tilde{A}}$$

we have to be careful how $B \overset{P \otimes_A Q}{=}$ acts. This inv. $\langle Q, P'' \rangle$

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Check yesterday's calculation that $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ with A flat leads to the identity map on $K_* A$.

Write $P = A$ as filtered inductive limit $P = \varinjlim F_i$

F_i f free $\tilde{A}^{op}$-module. ~~Consider~~ Then consider $\begin{pmatrix} A & Q = A \\ F_i & F_i \otimes_A Q \end{pmatrix}$

You have $F_i \otimes_A Q \longrightarrow \underbrace{F_i \otimes_A \mathrm{Hom}_{A^{op}}(F_i, A)}_{\text{matrix ring over } A.} \in \underbrace{\mathrm{Hom}_{A^{op}}(F_i, F_i)}_{\text{matrix ring over } \tilde{A}}$

The map is
$$K_*(A) = \varinjlim K(F_i \otimes_A Q) \longrightarrow K_i(A).$$

So I have to compare the two hom. effect on K_* of

$$A = P \otimes_A Q \hookleftarrow F_i \otimes_A Q \longrightarrow \mathrm{Hom}_{A^{op}}(F_i, F_i) = F_i \otimes_A \check{F}_i$$

So take $\underset{f \text{ free}}{F} \xrightarrow{u} A$ $\qquad F \otimes_A Q = F A \xrightarrow{u} A^2 = A$

$F \otimes_A A$

F is a ring with $\{f_1 f_2 = f_1 u(f_2)\}$

Wait: $F \otimes_A A \subset F \otimes_A \tilde{A} \simeq F$ $\qquad \begin{matrix} A & \tilde{A} \\ F & F \otimes_A \tilde{A} = F \end{matrix}$

$(f_1 \otimes \tilde{a}_1)(f_2 \otimes \tilde{a}_2) = f_1 \otimes \tilde{a}_1 u(f_2) \tilde{a}_2$

F is a ring acting on itself, this is rep.
$F \otimes_A Q \longrightarrow \mathrm{End}_{A^{op}}(F)$, and $u : F \to A$ is a homom. giving the other repn. I want to compare these two repns. and show they yield the same map $K_*(F) \longrightarrow K_*(\tilde{A})$. In fact I wanted to check the case $F = A \in \mathcal{P}(\tilde{A}^{op})$.

~~Yesterday's idea: $A \in \mathcal{P}(A^{op})$ assume. Take $F = A$~~
~~$u : F \to A$ the identity. Then the~~

Review $\quad u: F \longrightarrow A \qquad F \in \mathcal{P}(\tilde{A}^{op})$

$$B = F \otimes_A A \subset F \otimes_A \tilde{A} = F \qquad \text{ring} \quad f_1 f_2 = f_1 u(f_2)$$

~~Thus~~ We have two repns of F namely $F \xrightarrow{u} A \subset \tilde{A}$ and $F \longrightarrow \mathrm{End}_{A^{op}}(F) \qquad f \mapsto (f' \mapsto ff' = fu(f'))$

F acts on itself by left mult and on $\tilde{A}$ via u. $\quad F \to \tilde{A}$

The idea is that u is a nil using, so when I factor $\quad F \xrightarrow{\binom{1}{u}} F \oplus \tilde{A} \xrightarrow{pr_2} \tilde{A} \quad$ I get exact seq.

$$0 \longrightarrow F \xrightarrow{\binom{1}{0}} F \oplus \tilde{A} \xrightarrow[pr_2]{(0\ 1)} \tilde{A} \longrightarrow 0$$

$$0 \longrightarrow F \xrightarrow{\binom{1}{u}} F \oplus \tilde{A} \xrightarrow{(-u\ 1)} \tilde{A} \longrightarrow 0$$

You have to keep track of ~~the~~ the left F action.

When you mult by $F = F \otimes_A \tilde{A}$ on the left you

$F \oplus \tilde{A}$ is a ring with $(f_1, \tilde{a}_1)(f_2, \tilde{a}_2) = (f_1, \tilde{a}_1)\tilde{a}_2 = f_1\tilde{a}_2, \tilde{a}_1\tilde{a}_2$

So $\binom{1}{u}$ $\qquad (f_1, u(f_1)) \cdot (f_2, \tilde{a}_2) = (f_1\tilde{a}_2, u(f_1)\tilde{a}_2)$

$$f_1 \cdot (f_2, \tilde{a}_2) = f_1\tilde{a}_2, u(f_1)\tilde{a}_2$$

Thus $F \oplus \tilde{A}$ as F-module is 0 action on first factor and u action on the second.

$$\begin{pmatrix} f_1 \\ a_1 \end{pmatrix} \cdot \begin{pmatrix} f_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ a_1 \end{pmatrix} a_2 \qquad \text{product in } F \oplus \tilde{A}$$

$$\begin{pmatrix} f \\ u(f) \end{pmatrix} \cdot \begin{pmatrix} f_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} f \\ u(f) \end{pmatrix} a_2 = \begin{pmatrix} fa_2 \\ u(fa_2) \end{pmatrix} \qquad \begin{pmatrix} ff_1 \\ u(ff_1) \end{pmatrix}$$

$$\shortparallel$$

$$f \longmapsto \begin{pmatrix} 0 & f \\ 0 & u(f) \end{pmatrix} \qquad \begin{pmatrix} 0 & f \\ 0 & u(f) \end{pmatrix} \begin{pmatrix} f_1 \\ u(f_1) \end{pmatrix} = \begin{pmatrix} fu(f_1) \\ u(f)u(f_1) \end{pmatrix}$$

$F = A$. On $A \oplus \tilde{A}$ you have the action

$$a \longmapsto \begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix} \qquad 0\,\text{rep} \oplus \text{reg rep.}$$

$$\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} = 0\begin{pmatrix} -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} a$$

---

define dual pair over A to be $(P, Q, \langle\ \rangle : Q \otimes_{\mathbb{Z}} P \to A)$
left mod, rt mod, A-bimod. map.
assoc. to a d.p. is rg $B = P \otimes_A Q$ $\quad (p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \otimes \langle q_1, p_2 \rangle q_2$

~~[illegible]~~.

$$B \longrightarrow \operatorname{Hom}_{A^{op}}(P, P) \times \operatorname{Hom}_A(Q, Q)^{op}$$

$$(p \otimes q)p_1 = p\langle q, p_1 \rangle$$

$$q_1(p \otimes q) = \langle q_1, p \rangle q$$

$$\langle q_1, (p \otimes q)p_1 \rangle = \langle q_1, p\langle q, p_1 \rangle\rangle$$

$$\langle q_1(p \otimes q), p_1 \rangle = \langle q_1, p \rangle \langle q, p_1 \rangle$$

A, B

$\therefore$ ${}_B P_A$, ${}_B Q_A$ bimodules

$$\langle\ \rangle : Q \otimes_B P \longrightarrow A$$

$$P \otimes_A Q \xrightarrow{\sim} B$$

$$(p \otimes q) \otimes p_1 = p\langle q, p_1 \rangle$$

$$p \otimes \langle q, p_1 \rangle$$

~~[illegible]~~

$$\begin{tikzcd}
Q \otimes P \otimes Q \arrow[r] \arrow[d] & Q \otimes B \arrow[d] \\
A \otimes Q \arrow[r] & Q
\end{tikzcd}$$

$$\begin{tikzcd}
q_1 \otimes p \otimes q \arrow[r, mapsto] \arrow[d] & q_1 \otimes p \otimes q \arrow[d] \\
\langle q_1, p \rangle \otimes q & \langle q_1, p \rangle q
\end{tikzcd}$$

$\therefore$ get $\begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$

$\mathcal{M}(A) \simeq \mathcal{M}(B)$ A, B firm.

$F, G, \ \varepsilon : FG \xrightarrow{\sim} 1, \ \eta : GF \xrightarrow{\sim} 1$ $\varepsilon.F = F.\eta$, $G.\varepsilon = \varepsilon.F$

$P \otimes_A -$, $Q \otimes_B -$, $P \otimes_A Q \xrightarrow{\sim} B$, $Q \otimes_B P \xrightarrow{\sim} A$

$\| \qquad \|$

$F(A) \qquad F(B)$ get s. firm M. context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $A \otimes_A A \xrightarrow{\sim} A$, $Q \otimes_B P \xrightarrow{\sim} A$

firm dual pair $(P, Q, \langle\ \rangle : Q \otimes P \to A)$

firm surj.

Prop. A idempotent. Then a firm ring B meq to A has the form $P \otimes_A Q$ ~~where~~ for ~~some~~ a unique firm d.p. over A.

Converse $\begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$ Then $QP = A$ $PQ = B$

$P \otimes_A Q = B$ so $\mathcal{M}(A) \simeq \mathcal{M}(B)$

But more is true because P A^{op}-firm $\Rightarrow P \otimes_A Q = B$ is B^{op}-firm so B is B-firm ~~so $Q \otimes_B B$ is A firm~~

$P \otimes_A - : \mathcal{M}(A) \to \mathcal{M}(B)$ $\therefore$ $P \otimes_A A^{(2)}$ is B-firm

$-\otimes_A Q : \mathcal{M}(A^{op}) \to \mathcal{M}(B^{op})$ $\overset{\|}{P}$ $b = P \otimes_A Q$ is B-firm.

$A^{(2)} \mapsto A^{(2)} \otimes_A Q \xrightarrow{\sim} Q$

$Q \otimes_B P = A^{(2)} = A$ we've seen.

Go back to multipliers

$R = \mathrm{Mult}(P, Q, \langle,\rangle) = \{ \phi \in \mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(Q,Q)^{op} \mid \langle q\phi, p\rangle = \langle q, \phi p\rangle \}$

$P \otimes_A Q \xrightarrow{d} R$

$\|$

$R = \mathrm{Mult}(B) = \{ \phi \in \mathrm{Hom}_{B^{op}}(B,B) \times \mathrm{Hom}_B(B,B)^{op} \mid \langle b_1\phi, b_2\rangle = \langle b_1, \phi b_2\rangle$

 ~~[illegible]~~ Assume B comm. firm

Is $B^{(2)}$ comm.? $\quad (b_1\otimes b_2)(b_3\otimes b_4) = b_1b_2\otimes b_3b_4$
$= b_1b_2b_3\otimes b_4$

$$(b_1\otimes b_2)(b_3\otimes b_4) = b_1\otimes b_2b_3b_4 = b_1\otimes b_3b_4b_2 = b_1b_3b_4\otimes b_2$$
$$= b_3b_4b_1\otimes b_2 = b_3\otimes b_4b_1b_2 = (b_3\otimes b_4)(b_1\otimes b_2).$$

$\Lambda = \mathrm{Hom}_{B,B}(B,B)$ = endos of identity firm in general

$= \mathrm{Hom}_{B^{op}}(B,B) = \mathrm{Hom}_B(B,B) = \mathrm{Mult}(B)$, when $[B,B]=0$.

$$\begin{array}{ccc} P\otimes_A Q & \longrightarrow & \mathrm{Mult}(P,Q,\langle\,\rangle) \\ \| & & \| \\ B & \longrightarrow & \mathrm{Mult}(B) \end{array}$$

$\mathrm{Hom}_{B,A^{op}}(P,P)$
$\|$

In general I have an isom $\mathrm{Hom}_{B,B^{op}}(B,B) = \mathrm{Hom}_{A,A^{op}}(A,A)$

$$\mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}_{B^{op}}(Q,Q)$$
$$\mathrm{Hom}_A(A,A) = \mathrm{Hom}_B(P,P)$$

YES

$$\begin{array}{ccccc} \mathrm{Hom}_A(A,A) & = & \mathrm{Hom}_B(P,P) & & \\ \cup & & \cup & & \\ \mathrm{Hom}_{A,A^{op}}(A,A) & = & \mathrm{Hom}_{B,A^{op}}(P,P) & = & \mathrm{Hom}_{B,B^{op}}(B,B) \\ & & \cap & & \cap \\ & & \mathrm{Hom}_{A^{op}}(P,P) & = & \mathrm{Hom}_{B^{op}}(B,B) \end{array}$$

Implications of A and B comm. are?
We have a commutative unital ring Λ. ~~[illegible]~~
Basically we have $\mathcal{M} = \mathcal{M}(A)$ ~~[illegible]~~

$\mathcal{M} = \mathcal{M}(A)$ take generator A $R = \mathrm{Hom}_A(A,A)^{op}$

operators ~~on~~ the right, whence

$$\begin{array}{ccc} \mathrm{Mod}(R) & \longrightarrow & \mathcal{M}(A) \\ \rotatebox{90}{$\in$} & & \\ N & \longmapsto & A \otimes_R N \end{array} \qquad \begin{pmatrix} A & A \\ R & R \end{pmatrix}$$

I think we get an equiv. $\mathrm{Mod}(R)/\mathrm{Mod}(R/RA) \xrightarrow{\sim} \mathcal{M}(A)$

Suppose A comm. Then $\mathrm{Hom}_A(A,A) = \mathrm{Hom}_{A^{op}}(A,A)$ is comm. (assuming $A = A^2$) and $R\bar{A} = \bar{A}R = \bar{A}$.

~~Cor. Next,~~ Now play the same game with the generator Q $R = \mathrm{Hom}_A(Q,Q)^{op}$

$$\begin{array}{ccc} \mathrm{Mod}(R) & \longrightarrow & \mathcal{M}(A) \\ N & \longmapsto & Q \otimes_R N \end{array} \qquad \begin{pmatrix} A & Q \\ \mathrm{Hom}_A(Q,A) & R \end{pmatrix}$$

$$\mathrm{Mod}(R)/\mathrm{Mod}(R/\mathrm{Hom}_A(Q,A)Q) \xrightarrow{\sim} \mathcal{M}(A)$$

Still very confusing. But we're assuming that $\mathrm{Hom}_A(Q,Q)^{op} = \mathrm{Hom}_B(B,B)$ is commutative. So R is commutative.

New viewpt. A firm comm. ring $\Rightarrow \mathcal{M}(A) = \mathcal{M}(A^{op})$ there's no diff between left and right module

$$\mathrm{Hom}_{A,A^{op}}(A,A) \qquad \begin{array}{l} \mathrm{Hom}_{A^{op}}(A,A) \searrow \\ \qquad\qquad\qquad \mathrm{Mult}(A) \\ \mathrm{Hom}_A(A,A) \nearrow_{\sim} \end{array}$$

Let ϕ be a multiplier:

$$\begin{array}{ccc} (a_1\phi)a_2 & = & a_1(\phi a_2) \\ \| & & \| \\ a_2(a_1\phi) & & (\phi a_2)a_1 \\ \| & & \| \\ (a_2a_1)\phi & & \phi(a_2a_1) \end{array}$$

$\therefore \phi a = a\phi$ $\forall a$ when $A = A^2$.

$$\begin{tikzcd} & \mathrm{Hom}_{A^{op}}(A,A) \arrow[dr, hook] & \\ \mathrm{Hom}_{A,A^{op}}(A,A) \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Hom}_{\mathbb{Z}}(A,A) \\ & \mathrm{Hom}_A(A,A) \arrow[ur, hook] & \end{tikzcd}$$

$$\phi(a_1 a_2) = a_1 \phi(a_2)$$

$$\phi(a_1 a_2) = \phi(a_2 a_1) = a_2 \phi(a_1) = \phi(a_1) a_2$$

So when A comm. have just $\mathcal{M} = \mathcal{M}(A) = \mathcal{M}(A^{op})$ the cat of firm A-modules. $R = \mathrm{Hom}_A(A,A)$

$$\mathrm{Mod}(R) \xrightarrow{F} \mathcal{M}$$
$$N \longmapsto A \otimes_R N$$

We know this functor is exact and we have

$$F(N) = 0 \iff A \otimes_R N = 0 \iff \bar{A}N = 0$$

Wait: We have $\quad A \xrightarrow{f} R \quad$ R-module map

(in gen. $\quad A \otimes_A A \xrightarrow{f} \mathrm{Mult}(A) \quad$ R-bimodule)

So the image $\bar{A}$ of A in R is an ideal $\bar{A} = A/\mathrm{ann}(A)$

$$A \otimes_R N = 0 \implies \bar{A}N = 0.$$

$$A \otimes_R N = A^2 \otimes_R N = A \otimes_R \bar{A}N$$

Now pick another generator Q for $\mathcal{M}$. and assume that $\mathrm{Hom}_A(Q,Q)$ is comm. In this case we should get. More gen. consider $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s.f. with B comm. It should be true that

$$\mathrm{Hom}_{A,A^{op}}(A,A) = \mathrm{Hom}_{B,A^{op}}(P,P) = \mathrm{Hom}_{B,B^{op}}(B,B)$$

Point is that $R = Hom_A(P, P)$

$Mod(R)/Mod(R/\bar{A})$

The idea is that the generator $A \leadsto R$
then gen. $Q \leadsto R$ so we have

$$Mod(R)/Mod(R/\bar{A}) \xrightarrow{\sim} \mathcal{M} \longleftarrow Mod(R)/Mod(R/\bar{B})$$

$$Q \otimes_R N \longleftarrow N$$

$$N \longmapsto A \otimes_R N.$$

Can I see that $\bar{A} = \bar{B}$. Answer should be
that $Q \otimes_R N = 0 \implies P \otimes_A Q \otimes_R N = 0 \implies B \otimes_R N = 0$
$\iff \bar{B}N = 0$.

So we have this commutative unitary ring R
and R modules P, Q together with products
$P \otimes_R Q \to R$ and $Q \otimes_R P \to R$ making a
Morita context $\begin{pmatrix} R & Q \\ P & R \end{pmatrix}$ except that we
have $QP = \bar{A}$ and $PQ = \bar{B}$. But then

$\bar{B} = PQPQ = \underline{P\bar{A}Q} \doteq \bar{A}PQ = \bar{A}\bar{B}$?

What do I have to do tomorrow.

Dual pair over A ~~$Q, P, \dots$~~ P_A, ${}_AQ$, $Q \otimes P \xrightarrow{\langle,\rangle} A$
firm dual pair ($A = A^2$) P, Q firm $\langle,\rangle$ surjective

$\begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix}$ $\ni$ $QP = A$, $PQ = B$
$\implies \mathcal{M}(A) \simeq \mathcal{M}(B)$ also if
$P_A \in \mathcal{M}(A^{op}) \implies B = P \otimes_A Q \in \mathcal{M}(B^{op})$ $\therefore B$ firm

$\cancel{\ast}\quad A^{(2)} \in \mathcal{M}(A) \implies P\otimes_A A^{(2)} = P \in \mathcal{M}(B)$

$\therefore$ P firm bimod.

$\cancel{\ast}\quad A^{(2)} \in \mathcal{M}(A^{op}) \implies A^{(2)}\otimes_A Q = Q \in \mathcal{M}(B^{op})$

---

① cat of firm rings B equipped with $F : \mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$

② cat of firm dual pairs $(P, Q, \langle,\rangle)$ over A.

②′ cat of ofirm M cont $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ A fixed

②′ $\longrightarrow$ ①

$$\begin{tikzcd}
\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[r, mapsto] \arrow[d, "\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}"] & (B,\ F = P\otimes_A -) \\
\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix} \arrow[r, mapsto] & (B',\ F' = P'\otimes_A -)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{M}(A) \arrow[d, "P\otimes_A -"'] \arrow[dr, "P'\otimes_A -"] & \\
\mathcal{M}(B) \arrow[r, "w_! = B'\otimes_B -"'] & \mathcal{M}(B')
\end{tikzcd}$$

$\cancel{\ast}\quad w_! \xleftarrow{\sim} P' \otimes_A Q \otimes_B -$

$B'\otimes_B N \xleftarrow{\sim} P'\otimes_A Q'\otimes_B N \xleftarrow{\sim} P'\otimes_A Q\otimes_B N$

$p'v(q)\otimes n \longleftarrow\!\shortmid\ p'\otimes q\otimes n$

$B'\otimes_B P\otimes_A M \xleftarrow{\sim} P'\otimes_A Q\otimes_B P\otimes_A M \xrightarrow{\sim} P'\otimes_A M$

$p'v(q)\otimes p\otimes m \longleftarrow\!\shortmid\ p'qp\otimes m$

$b'\otimes p\otimes m \longmapsto b'u(p)\otimes m$

This sets up the functor, Why ess. surj? clear

$$\mathcal{M}(A) \xrightarrow{(F = P\otimes_A -)} \mathcal{M}(B)$$

Fully faithful. Given B, F $\quad B', F'$

$$\begin{tikzcd}
& \mathcal{M}(A) \arrow[dl, "F"'] \arrow[dr, "F'"] & \\
\mathcal{M}(B) \arrow[rr, "w_!"'] \arrow[urr, phantom, "\cong"] & & \mathcal{M}(B')
\end{tikzcd}$$

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

$\mathrm{Hom}((B,F),(B',F')) = \{(\omega,\theta) \mid \theta : \omega_! F \simeq F'\}$ same as $\omega_! \simeq F'G$

same as $\omega_! \simeq F'G$, $\omega^* \simeq FG'$

$$\left(\quad\right) \simeq \left(\quad\right)$$

$$\mathrm{MegHom}(B,B') \longleftarrow \mathrm{Hom}((B,F),(B',F'))$$

[redacted] So how to straighten this out.

Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$, $\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$

(these are same as (B,F) and (B',F')).

a map $(B,F) \xrightarrow{w,\theta} (B,F')$

same ~~as as~~ as $\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}$. Why

(B,F) enlarges !ly to (B,F,G,ε,η) yielding $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

(B',F') ——— $(B',F',G',\varepsilon',\eta')$ $\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$

$\theta : w_! \simeq F'G$ induces $\xi : w^* \simeq FG'$

~~[illegible]~~

$\therefore$ w,θ same as (w,θ,ξ) where

θ,ξ isom. $(w_!, w^*)(F,G) \xrightarrow{\sim} (F',G')$

$$\begin{pmatrix} B & B \otimes_B B' \\ B' \otimes_B B & B' \end{pmatrix} \simeq \begin{pmatrix} B & P \otimes_A Q' \\ P' \otimes_A Q & B' \end{pmatrix}$$

87 $(w_!, w^*)(F, G) : \begin{pmatrix} A & Q\otimes_B B' \\ B'\otimes_B P & B' \end{pmatrix} \xrightarrow[\sim]{\begin{pmatrix} 1 & \tilde{v} \\ \tilde{u} & 1 \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$

thus have compatible isos $B'\otimes_B P \simeq P'$, $Q\otimes_B B' \simeq Q'$

$$\mathrm{Hom}_{B'}(B'\otimes_B P, P') \leftarrow \mathrm{Hom}_B(P, P')$$

$$P = B\otimes_B P \xrightarrow{w\otimes 1} B'\otimes_B P \longrightarrow P'$$

$$\tilde{u}(b'\otimes p) = b'u(p)$$
$$u(bp) = \tilde{u}(w(b)\otimes p)$$

02/25/97 ~~[illegible]~~

A firm fixed

cat of firm dual pairs $\{P, Q, \langle\ \rangle : Q\otimes P \to A\}$

cat of s. firm Mcont. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with A fixed

cat of firm rings B equipped with $\mathcal{M}(A) \simeq \mathcal{M}(B)$, F, G, ε, η

map $(B, \Phi) \longrightarrow (B', \Phi')$ is a meghom $w : B \to B'$ together with

isom $(w_!, w^*, \alpha, \beta^{-1})\circ\Phi \xrightarrow{\sim} \Phi'$.

how does functor f work? given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$

we have isos $\left[\begin{array}{l} \tilde{u} : B'\otimes_B P \xrightarrow{\sim} P' \\ \tilde{v} : Q\otimes_B B' \xrightarrow{\sim} Q' \end{array}\right.$ We know w meghom.

$w_! F : M \longmapsto B'\otimes_B P\otimes_A M$ firm bimo $B'\otimes_B P$

$Gw^* : N' \longmapsto Q \otimes_{B'} \cancel{B'} \otimes_B N'$ " " $Q\otimes_B B'$

$\varepsilon : w_! F G w^* \xrightarrow{\sim} 1$ $\quad B'\otimes_B P \otimes_A Q \otimes_B B' \simeq B'\otimes_B B \otimes_B B' \xrightarrow[\sim]{\alpha} B'$

$\eta : Gw^* w_! F \xrightarrow{\sim} 1$ $\quad Q\otimes_B \underbrace{B' \otimes_{B'} B'}_{B'} \otimes_B P \overset{\beta}{\xleftarrow{\sim}} Q\otimes_B B\otimes_B B\otimes_B P \longrightarrow A$

$$\begin{pmatrix} B & B\otimes_B B' \\ B'\otimes_B B & B' \end{pmatrix}$$

$$(B\otimes_B B')\otimes_{B'}(B'\otimes_B B)$$
$$\uparrow s$$
$$B^{(4)}$$
$$\uparrow$$
$$B$$

Anyway what can we do?

What's the way to state this ????? I think it's easiest to go from u, v, w to the isom. Two poss. for the isom.

$B' \otimes_B P \xrightarrow{\sim} P'$, $Q \otimes_B B' \xrightarrow{\sim} Q'$

$P \xrightarrow{\sim} B \otimes_{B'} P'$, $Q \xrightarrow{\sim} Q' \otimes_{B'} B$ immediate from P firm u a B-rel isom.

get isom $F \xrightarrow{\sim} w^* F'$, $G \xrightarrow{\sim} G' w_!$

$$Q' \otimes_{B'} B' \otimes_B B$$

$$(F, G, \varepsilon, \eta) \xrightarrow{\sim} (w^*, w_!, \beta^{-1}, \alpha)(F', G', \varepsilon', \eta')$$

Fully faithful. Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$ two objects we have functorality

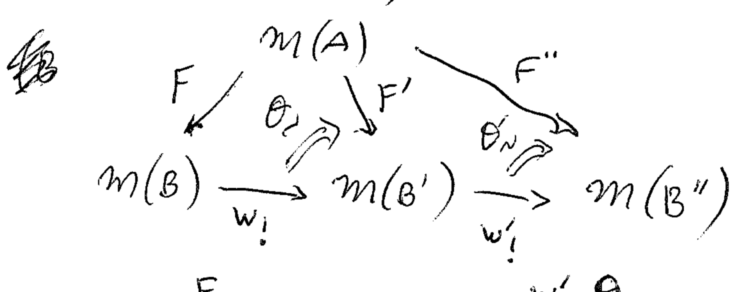
$$\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} \rightsquigarrow w, \text{ and the isom } \theta : w_! F \xrightarrow{\sim} F'.$$

The problem is to show bijectivity. You first have to clarify the categories.

cat of firm rings equipped with a meg to A.

objects: (B, ξ) B firm ring, $F : m(A) \to m(B)$. a fully faithful functor.

map $(B, F) \longrightarrow (B', F')$ is $w : B \to B'$ meg homom $+$ $\theta : w_! F \xrightarrow{\sim} F'$



$$B'' \otimes_{B'} B' \otimes_B B \xrightarrow{\sim} B'' \otimes_B B$$

$$(w'w)_! F = w'_! w_! F \xrightarrow[\sim]{w'_! \cdot \theta} w'_! F' \xrightarrow[\sim]{\theta'} F''$$

check assoc. comp.

89 ~~Next step~~!! Real step. Given $w: B \to B'$ a Morita homom of firm rings. Fix a Morita $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ between firm rings A, B. Consider $\{(w,\theta) \mid W: A \to B$ and $\theta: w_! \simeq P\otimes_A -\}$

Claim this set is ~~same as~~ bij to $\{$ ~~...~~
$$\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : \begin{pmatrix} A & A \\ A & A \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \}.$$

Given (w,θ) then $w_!$ is an eq. ~~so~~ get $\begin{pmatrix} A & A\otimes_A B^* \\ B^*\otimes_A A & B^* \end{pmatrix}$ Morita corresp. to the ~~functor~~ pair of quasi-inv. fcns $(w_!, w^*, \alpha, \beta^{-1})$.

to $\theta: w_! \xrightarrow{\sim} P\otimes_A -$ corresp a unique $\xi: w_* \xrightarrow{\sim} Q\otimes_B -$ compatible with ε, η. Yes. Thus get isom. from θ

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix} \xrightarrow{\text{canon.}} \begin{pmatrix} A & A\otimes_A B \\ B\otimes_A A & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

$$A = A\otimes_A A \xrightarrow{w\otimes 1} B\otimes_A A$$

A comm. $\quad \mathrm{Hom}_A(A,A) = \mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}_{A,A^{op}}(A,A)$

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s firm $\quad B$ comm. First there is the general argument that the centers are the same

$$\begin{array}{ccc} \mathrm{Hom}_{B,A^{op}}(P,P) & = & \mathrm{Hom}_{B,B^{op}}(B,B) \\ \| & & \| \\ \mathrm{Hom}_{A,A^{op}}(A,A) & = & \mathrm{Hom}_{A,B^{op}}(Q,Q) \end{array}$$

Call this common ring R. In general have

$$\mathrm{Hom}_{A^{op}}(P,P) \overset{\langle,\rangle}{\times} \mathrm{Hom}_A(Q,Q) = \mathrm{Hom}_{B^{op}}(B,B) \overset{\langle,\rangle}{\times} \mathrm{Hom}_B(B,B)$$

Your

$$\langle q\phi'', p\rangle = \langle q, \phi' p\rangle$$

$$\mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(Q,Q)^{op} \qquad (\phi', \phi'')$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\mathrm{Hom}_{A,A^{op}}(Q\otimes P, A) \qquad q\otimes p \mapsto \langle q\phi'', p\rangle - \langle q, \phi' p\rangle$$

$$\| ?$$

$$\mathrm{Hom}_{B,B^{op}}(B\otimes B, B)$$

There's problem some slick von Neumann type argument.

~~[illegible]~~

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s form A, B comm. $\qquad \mathrm{Hom}_{A,B^{op}}(Q,Q)$

Put $R = \mathrm{Hom}_{A,A^{op}}(A,A) = \mathrm{Hom}_{B,A^{op}}(P,P) = \mathrm{Hom}_{B,B^{op}}(B,B)$.

~~[illegible]~~ Ring of Endos of the identity functor

A comm. $\Rightarrow$ $R = \mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}_A(A,A) = \mathrm{Mult}(A)$.

So the A-action on $M \in \mathfrak{m}(A)$ extends uniquely to R

$r(am) = (ra)m$ When you deal with comm. rings have $\otimes$.

$$M\otimes_A N \xrightarrow{\sim} M\otimes_R N \quad \text{if } M = AMA \text{ or } N = AN.$$

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ ~~[illegible]~~

$B = P\otimes_A Q = P\otimes_R Q$. What $\overset{\text{sort of}}{\text{situation}}$

$P\otimes_R Q \longrightarrow$

We have $A \to R$ and $B \to R$

One idea: Look at $\mathrm{Hom}_R(Q,A) = \mathrm{Hom}_A(Q,A)$

$$f(r(aq)) = f((ra)q) = (ra)f(q) = r(af(q)) = rf(aq)$$

91 Is it possible that ~~[illegible]~~ P, Q become line bundles in the solid picture? Let's move to the solid picture $A \rightsquigarrow \mathrm{Hom}_A(A,A) = R$

P

$$R = \mathrm{Hom}_A(A,A) = \mathrm{Hom}_A(Q \otimes_R P$$

02/26/97 ~~A commutative firm, $R = \mathrm{Hom}_A(A,A)$~~

Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ a firm Morita context we have

$$\begin{array}{ccc} \mathrm{Hom}_{A,A^{op}}(A,A) & = & \mathrm{Hom}_{A,B^{op}}(Q,Q) \\ \| & & \| \\ \mathrm{Hom}_{B,A^{op}}(P,P) & = & \mathrm{Hom}_{B,B^{op}}(B,B) \end{array}$$

This is a commutative ring = endos of id functor on $\mathcal{M}(A)$. Denote it R. ~~[illegible]~~ Everything takes place over R, the cat $\mathcal{M}(A)$ is R-linear.

Assume A comm. Then $\mathrm{Hom}_A(A,A) = \mathrm{Hom}_{A,A^{op}}(A,A) = \mathrm{Hom}_{A^{op}}(A,A)$ $= \mathrm{Mult}(A)$. Have ~~[illegible]~~ homom $f: A \to R$ $a \mapsto \bar{a}$ $\bar{a}(a_1) = aa_1$ $= a_1a$ $r\bar{a} = \overline{ra}$. $\bar{a} = 0 \Leftrightarrow aA = 0$. $\mathcal{M}(A) = \mathrm{mod}(R)/\mathrm{mod}_R(\bar{A})$

Have ~~$B \simeq P \otimes_A Q \sim B \simeq Q \otimes_A P$~~,

$$\begin{array}{ccccccc} B\otimes_B & = & P\otimes_A Q\otimes_B & \simeq & Q\otimes_B P\otimes_A & \simeq & A\otimes_A \\ \| & & & & & & \| \\ P\otimes_R Q & & & & & & A \end{array}$$

The trace $B\otimes_B \longrightarrow A$ is R-linear

When A, B commutative we have

$$R = \mathrm{Hom}_A$$

Try the following idea. You have since $R = \mathrm{Hom}_A(A,A)$ a functor

$$\begin{aligned} \mathrm{Mod}(R) &\longrightarrow \mathcal{M}(A) \\ W &\longmapsto A\otimes_R W \end{aligned}$$

which is exact since A is a generator for $\mathcal{M}(A)$

The "kernel" is the Serre subcat $\mathrm{Mod}(R/\bar{A})$. You also have

$$\begin{aligned} \mathrm{Mod}(R) &\longrightarrow \mathcal{M}(B) \quad \text{exact} \\ W &\longmapsto B\otimes_R W \end{aligned}$$

with kernel $\mathrm{Mod}(R/\bar{B})$. ~~These functors are related by the M-eq.~~

No.

Have

$$\begin{aligned} \mathcal{M}(A) &\simeq \mathcal{M}(B) \\ M &\longmapsto P\otimes_A M \\ Q\otimes_B N &\longleftarrow N \end{aligned}$$

So if we take ~~the~~ $W \mapsto B\otimes_R W \longmapsto Q\otimes_B B\otimes_R W = Q\otimes_R W$ the 2nd functor can be ~~understood~~ identified with the functor

$$\begin{aligned} \mathrm{Mod}(R) &\longrightarrow \mathcal{M}(A) \\ W &\longmapsto Q\otimes_R W \end{aligned}$$

assoc. to the generator Q of $\mathcal{M}(A)$. So I have these two generators of $\mathcal{M}(A)$, namely A and Q both with endo ring R. ~~These have~~ giving rise to exact functors ~~Mo~~ $\mathrm{Mod}(R) \rightrightarrows \mathcal{M}(A)$ $W \mapsto A\otimes_R W,\ Q\otimes_R W$. ~~But~~ The kernels should be the same, why? Because!!

Comm. rings A comm. firm $R = \mathrm{Hom}_A(A,A)$ its mult. ring $\bar{A} = \mathrm{Im}\{A \xrightarrow{\text{can}} R\}$. Then $\mathcal{M}(A) = \mathcal{M}(R,\bar{A})$ R-mods M satisfying $\bar{A}\otimes_R M \xrightarrow{\sim} M$. We have

$$\begin{aligned} \mathrm{Mod}(R)/\mathrm{Mod}(R/A) &\simeq \mathcal{M}(A) \\ W &\longmapsto A\otimes_R W = A\otimes_A W \end{aligned}$$

since $AA = A$.

Now suppose given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ firm with B comm.

$$R = \mathrm{Hom}_{A,A^{op}}(A,A) = \mathrm{Hom}_{B,A^{op}}(P,P) = \mathrm{Hom}_{B,B^{op}}(B,B)$$

in general

$$R = \mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}_{B^{op}}(Q,Q)$$

So we have

$$\mathrm{Hom}_B(P,P) = \mathrm{Hom}_A(Q\otimes_B P,\, Q\otimes_B P) = \mathrm{Hom}_A(A,A)$$

$$\cancel{\mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{A^{op}}(P,P) & = & \mathrm{Hom}_{B^{op}}(B,B) \\
\| & & \| \\
\mathrm{Hom}_{B,A^{op}}(P,P) & = & \mathrm{Hom}_{B,B^{op}}(B,B) \\
 & & \|
\end{array}$$

$$\cancel{\mathrm{Hom}_A(P,P) = \mathrm{Hom}_B(B,B)}$$

$$\begin{array}{ccccc}
\mathrm{Hom}_B(P,P) & = & \mathrm{Hom}_A(Q\otimes_B P,\, Q\otimes_B P) & = & \mathrm{Hom}_A(A,A) \\
\| & & & & \| \\
\mathrm{Hom}_{B,A^{op}}(P,P) & = & & & \mathrm{Hom}_{A,A^{op}}(A,A)
\end{array}$$

seems OKAY. So in fact all of A, ~~AP~~ P, Q, B have R as End ring for either side module structure.

So ~~[illegible]~~ the problem we have is ~~[illegible]~~

Go back to $\mathrm{Mod}(R)$ inside here we have the modules ~~[illegible]~~ A, P, Q, B. We have isos.

$$Q\otimes_R P \xrightarrow{\sim} A \qquad\qquad P\otimes_R Q \xrightarrow{\sim} B$$

But the point should be that each of A, P, Q, B yield exact functors

$$\begin{array}{ccccc}
\mathrm{Mod}(R) & \longrightarrow & \mathcal{M}(A) & & \\
W & \longmapsto & A\otimes_R W & & Q\otimes_R W \\
 & & P\otimes_R W & & B\otimes_R W
\end{array}$$

$$\bar{A} \qquad R \qquad \bar{B}$$

So we have everything inside Mod(R).

~~We have~~ We start with $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ form with A, B comm.

Then get $R =$ ~~[illegible]~~ ${}_A A_A$

$\mathrm{End}(X)$ where $X = {}_A A, A_A, P_A, {}_B P$ etc.

Also have homs $A \xrightarrow{\phi} R \xleftarrow{\psi} B$ giving ideals $\bar{A}, \bar{B} \subset R$

I think we know that ~~[illegible]~~ $\phi(a) =$

left mult by a on A, Q [redacted]

$=$ right ——— a on A, P. Sim. $\psi(b) =$

left mult by b on B, P.

$=$ rt ——— b on B, Q. ~~These we have~~

~~Nothing~~ ~~so far you see that left mult~~

~~It seems we have a coupled [illegible]~~

So everything is an R-module.

$m(A)$ is the full subcat of $\mathrm{Mod}(R)$ cons. of M such that $\bar{A} \otimes_R M \xrightarrow{\sim} M$, sim for $m(B)$. Then $P \otimes_R -$, and $Q \otimes_R -$ go between these full subcats.

Note $\bar{A} Q = QPQ = Q\bar{B} = \bar{B} Q$

~~$\bar{A} = \bar{A}QP = \bar{B}QP = \bar{B}\bar{A}$~~

Thus $\bar{A} Q P = \bar{B} Q P \subset \bar{B}$

$\bar{A} = \bar{A}\bar{A}$ ($\bar{A}\bar{A} = \bar{A}QP$)

Sim. $\bar{B} \subset \bar{A}$. $\therefore$ $\bar{A} = \bar{B}$.

alternative: ~~[illegible]~~

$\mathrm{Mod}(R) \to m(A) \to$

$W \mapsto A \otimes_R W$

~~$m(A) \xleftarrow{Q\otimes A} m(B) \xleftarrow{\otimes_R B} \mathrm{Mod}(R)$~~

two generators A, Q of $m(A)$

$$\begin{tikzcd}
\mathrm{Mod}(R) \arrow[r, "A\otimes_R W"] \arrow[dr, "Q\otimes_R W"] \arrow[dd, "B\otimes_R W"'] & m(A) \\
& m(A) \arrow[dl, "P\otimes_R -", "\simeq"'] \\
m(B) &
\end{tikzcd}$$

4 functors $W \to \begin{pmatrix} A \\ P \\ Q \\ B \end{pmatrix} \otimes_R W$

from $Mod(R)$ to itself. ~~Also Note that~~ ~~If added supposed they~~ Note compositions

$$W \longmapsto P\otimes_R W \longmapsto Q\otimes_R P\otimes_R W \xrightarrow{\sim} A\otimes_R W$$

I'm interested in those W such that $Q\otimes_R W = 0$.

Same as those $W \ni B\otimes_R W = 0$ i.e. $W \in Mod(R/\bar{B})$

But $P\otimes_R Q\otimes_R W \simeq Q\otimes_R P\otimes_R W \simeq A\otimes_R W$

$\quad\quad \| $

$\quad B\otimes_R W$ This is the variant of the additive

isom $A \simeq P\otimes_R Q \simeq Q\otimes_R R \xrightarrow{\sim} B$. YES.

Thus $A\otimes_R W = 0 \iff B\otimes_R W = 0$ so

$Mod(R/\bar{A}) = Mod(R/\bar{B})$ so $\bar{A} = \bar{B} \subset R$ and

so $A = \bar{A}^{(2)} = \bar{B}^{(2)} = B$ as rings. YES.

Now I have to return to

$\frac{.742}{87.333} = .85$ $\quad$ $15 \times 1.4 = 21$ $\quad$ $13 \times 3.35 = 43.55$ $\quad$ 64.55 $\quad$ 1.75, .0166, 1.7333

2.90, .033, 2.866, .03

So let's go over the theorem I discussed in lecture: cat of firm dual pairs over A equipped with a meg to A $\simeq$ cat of firm rings, and meg homos.

A fixed, basic functor

$\{ (P,Q,\langle,\rangle) \longmapsto B = P\otimes_A Q$ and meg given by $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. $\}$

given $(P,Q,\langle\,\rangle)$ $\quad\quad \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

$\downarrow (u,v) \quad\quad\quad\quad \downarrow$

$(P',Q',\langle\rangle) \quad\quad \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$

we know that ω is a meg and that one has

$(\omega_!, \omega^*)(F,G) \xrightarrow{\sim} (F',G')$

 Instead examine the question of Morita inv. for h-unital rings. This is where you bring in Suslin's result, which I really need to learn anyway. Go back to the case of a field. Stability. You were at one time (1974) able to prove $H_i(GL_n(F)) \xrightarrow{\sim}$ [redacted] $H_i(GL_{n+1}(F))$ for $i < n$. I recall using buildings and various Lusztig type complexes. Also trying to prove Matsumoto's thm. describing $K_2 F$ as $F^* \otimes_{\mathbb{Z}} F^* /$ symbol relations. All this stuff I don't know. Now it becomes important. How do I get started?

It's possible but [redacted] would be surprising if your flatness [redacted] methods yield something.

What I know. Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ firm ~~and~~ (say) everything right flat ~~then~~ $K_*(A) = K_*(B)$. This includes K_0. I recall also ~~trying to~~ thinking about **deducing** Morita invariance of K_0 from the result for K_1. Anyway start with K_1 yes.

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need serious effort on Suslin's thm. as well as finishing up the paper. Let's spend 1/2 hour on a review of Morita invariance for K_* of flat firm rings.

Basic construction: Given a dual pair $(P, Q, \langle\ \rangle)$ over A such that P is A^{op}-flat we can define a canon. map

$$K_*(P \otimes_A Q) \longrightarrow K_*(A).$$

Suppose first $P \in \mathcal{P}(\tilde{A}^{op})$. Then have hom.

$$P \otimes_A Q \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P, A) \subset \mathrm{Hom}_{A^{op}}(P, P)$$

$$\begin{array}{ccc} K_*(P \otimes_A Q) & \longrightarrow & K_*(\mathrm{Hom}_{A^{op}}(P,P)) \\ \downarrow \exists & & \downarrow \\ K_*(A) & \subset & K_*(\tilde{A}) \end{array}$$

Need to check functorial in the dual pair

$$\begin{tikzcd} K_*(P\otimes_A Q) \arrow[d] \arrow[r] & K_*(A) \\ K_*(P'\otimes_A Q') \arrow[ur] & \end{tikzcd}$$ commutes?

$$\begin{matrix} P\otimes_A Q & P\otimes_A Q' \\ P'\otimes_A Q & P'\otimes_A Q' \end{matrix}$$

$$\begin{tikzcd} P\otimes_A Q \arrow[r] & P\otimes_A Q' \arrow[r] \arrow[d] & P\otimes_A \mathrm{Hom}_{A^{op}}(P',A) \\ & P'\otimes_A Q' \arrow[r] & \cancel{[illegible]} \end{tikzcd}$$

~~[illegible] forget~~

$B = P\otimes_A Q$ acts on P $\qquad$ $u: P\to P'$ is a
$B' = P'\otimes_A Q'$ " " P' $\qquad$ B-nil isom.

first point is that $K_*(P'\otimes_A Q$

$$\begin{tikzcd} K_*(P\otimes_A Q) \arrow[r, "\text{comm.}"'] \arrow[d] & K_*(\mathrm{Hom}_A(P,P)) \\ K_*(P\otimes_A Q') \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r] \arrow[d] & \mathrm{Hom}_{A^{op}}(P,A) \\ Q' \arrow[r] \arrow[ur] & \mathrm{Hom}_{A^{op}}(P',A) \arrow[u] \end{tikzcd}$$

can suppose $Q = Q' = \mathrm{Hom}_{A^{op}}(P,A)$. Then factor

So it's obviously commutative for a map with P fixed.
So now fix Q and look at a map $P \to P'$:

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & P' \arrow[d] \\ & \mathrm{Hom}_A(Q,A) \end{tikzcd}$$

Then can factor (i) into split injection. Enough to consider these case $P \xrightarrow{(i)} P\oplus P' \xrightarrow{pr_2} P' \longrightarrow \mathrm{Hom}_A(Q,A)$

We have $B = P\otimes_A Q$ acting on P and on P' which are in $\mathcal{P}(\tilde{A}^{op})$. Moreover, there is a B-nil isom $P \xrightarrow{u} P'$. First case is u inj $P'/uP \in \mathcal{P}(\tilde{A}^{op})$

Then
$$\mathrm{End}_{\tilde{A}^{op}}(P') = \begin{pmatrix} \mathrm{End}_{A^{op}}(P) & \mathrm{Hom}_{A^{op}}(P'/uP, P) \\ \mathrm{Hom}_{A^{op}}(P, P'/uP) & \mathrm{End}_{A^{op}}(P'/uP) \end{pmatrix}$$
split
$P' = P \oplus P'/u(P')$

The map $B \longrightarrow \mathrm{End}_{\tilde{A}^{op}}(P')$ has image in $\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \subset$ the unital subring $\begin{pmatrix} * & \cancel{*} \\ 0 & \cancel{*} \end{pmatrix}$ which has the same K-theory as the quotient ring $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. So we have

$$
\begin{tikzcd}
K_*(B) \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & K_*\begin{pmatrix} * & * \\ & * \end{pmatrix} \arrow[r] \arrow[d, "\wr"] & K_*(\mathrm{End}_{\tilde{A}^{op}}(P')) . \\
& K_*\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} &
\end{tikzcd}
$$

etc.

Other case is $P \longrightarrow P'$ surjective. Then $P = K \oplus P'$ $\quad \mathrm{End}_{A^{op}}(P) = \begin{pmatrix} \mathrm{End}(K) & \mathrm{Hom}(P',K) \\ \mathrm{Hom}(K,P') & \mathrm{End}(P') \end{pmatrix}$

and $B \longrightarrow \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \subset \mathrm{End}_{A^{op}}(P)$

So the upshot is that on the cat of dual pairs $(P, Q, \langle\ \rangle : Q \otimes P \to A)$ such that $P \in \mathcal{P}(\tilde{A}^{op})$ we have a natural transf of functors $K_*(P \otimes_A Q) \longrightarrow K_*(A)$. Then we can extend to P A^{op}-flat by taking filtered inductive limits. Now discuss transitivity

Suppose given $(P, P^*, P^* \otimes P \to A)$ with $P \in \mathcal{P}(\tilde{A}^{op})$ whence $K_*(P \otimes_A P^*) \longrightarrow K_*(A)$.

Suppose given $B \longrightarrow P \otimes_A P^*$ and $(Q, Q^*, Q^* \otimes Q \to B)$ with $Q \in \mathcal{P}(\tilde{B}^{op})$, whence $K_*(Q \otimes_B Q^*) \xrightarrow{K_*} K_*(B) \to K_*(P \otimes_A P^*) \to K_*(A)$

~~Consider~~ Set $B' = P \otimes_A P^*$ and replace Q, Q^* by

~~and~~ $(Q \otimes_B \tilde{B}', \tilde{B}' \otimes_B Q^*, \tilde{B}' \otimes_B Q^* \otimes Q \otimes_B \tilde{B}' \to \tilde{B}' \otimes_B B \otimes_B \tilde{B}' \to B')$

99 have $(Q\otimes_B \tilde{B}')\otimes_{B'}(\tilde{B}'\otimes_B Q^*) = Q\otimes_B \tilde{B}'\otimes_B Q^* \longleftarrow Q\otimes_B Q^*$

~~and you [redacted] nowhere~~

No, try again.

~~[illegible]~~

$(P, Q, Q\otimes P \xrightarrow{\langle\,\rangle} A)$

suppose $B \longrightarrow P\otimes_A Q = B'$

~~$(P' = P\otimes_B \tilde{B}', \; Q' = \tilde{B}'\otimes_B Q), \; Q'\otimes P' = \tilde{B}'\otimes_B Q\otimes P\otimes_B \tilde{B}' \to \tilde{B}'\otimes_B B\otimes_B \tilde{B}' \to B'$~~

~~$P'\otimes_{B'} Q' = P\otimes_B \tilde{B}'\otimes_{B'} \tilde{B}'\otimes_B Q = P\otimes_B \tilde{B}'\otimes_B Q$~~

$P\otimes$

$$\begin{matrix} A & P^* \\ P & P\otimes_A P^* \end{matrix}$$

suppose we have $B \to B'$ and $(Q, Q^*, Q^*\otimes Q \to B)$ a dual pair over B. Then

$Q\otimes_B \tilde{B}', \; \tilde{B}'\otimes_B Q^*, \; (\tilde{B}'\otimes_B Q^*)\otimes(Q\otimes_B \tilde{B}') \to \tilde{B}'\otimes_B B\otimes_B \tilde{B}' \to B')$

~~Claim should be that an~~

$$Q\otimes_B \tilde{B}' \otimes_{B'} \tilde{B}'\otimes_B Q^* = Q\otimes_B \tilde{B}'\otimes_B Q^*$$

$$\begin{tikzcd} K_*(Q\otimes_B Q^*) \arrow[r] \arrow[d] & K_*(B) \arrow[d] & \text{assoc. to} \\ K_*(Q\otimes_B \tilde{B}'\otimes_B Q^*) \arrow[r] & K_*(B') & \text{assoc to} \end{tikzcd}$$

Claim this commutes.

Proof. Can suppose $Q \in \mathcal{P}(\tilde{B}^{op})$

$$\begin{tikzcd}
K_*(Q \otimes_B Q^*) \arrow[r] \arrow[d] & K_*\left(Q \otimes_B \mathrm{Hom}_{B^{op}}(Q,B)\right) \arrow[r] \arrow[d, "\text{matrices under } w"] \arrow[dr, phantom, "\checkmark"] & K_*(B) \arrow[d, "w_*"] \\
K_*\left((Q \otimes_B \tilde{B}') \otimes_{B'} (\tilde{B}' \otimes_B Q^*)\right) \arrow[r] & K_*\left(Q \otimes_B \tilde{B}' \otimes_{B'} \mathrm{Hom}_{B'^{op}}(Q \otimes_B \tilde{B}', B')\right) \arrow[r] & K_*(B')
\end{tikzcd}$$

$$Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, B')$$

If Q is a fin gen free $\tilde{B}^{op}$-module, then we have a homom. from $Q \otimes_B Q^*$ to matrices over B, hence a canon. map $K_*(Q \otimes_B Q^*) \longrightarrow K_*(B)$. Claim compatible with extension of scalars: i.e. given $B \to B'$, then have dual pair $(Q \otimes_B \tilde{B}', \tilde{B}' \otimes_B Q^*,$ [redacted]$)$ $\langle b_1' \otimes q_1^*, q_2 \otimes b_2' \rangle = b_1' \, w(\langle q_1^*, q_2 \rangle) \, b_2'$). Then

$$\begin{tikzcd}
Q \otimes_B Q^* \arrow[r] \arrow[d] & Q \otimes_B \mathrm{Hom}_{B^{op}}(Q^*, B) \arrow[d] \\
(Q \otimes_B \tilde{B}') \otimes_{B'} (B' \otimes_B Q^*) \arrow[r] & Q \otimes_B \tilde{B}' \otimes_{B'} \mathrm{Hom}_{B'^{op}}(\tilde{B}' \otimes_B Q^*, B')
\end{tikzcd}$$

$= \text{matrices over } B$

$$\begin{aligned}
&\parallel \\
&Q \otimes_B \mathrm{Hom}_{B^{op}}(Q^*, B') \\
&\parallel \\
&Q \otimes_B B' \otimes_{B'} \mathrm{Hom}_{B^{op}}(Q^*, \tilde{B})
\end{aligned}$$

matrices over B'

pretty clear. So where are we now? Take a

Trans. $\quad A \quad (P, P^*, P^* \otimes P \xrightarrow{\langle\,\rangle} A)$

get $\quad K_*(P \otimes_A P^*) \longrightarrow K_*(A)$

Now suppose given $B \to P \otimes_A P^*$ and $(Q, Q^*, Q^* \otimes Q \to B)$

Form $\left(Q \otimes_B P, \; P^* \otimes_B Q^*, \; (P^* \otimes_B Q^*) \otimes_A (Q \otimes_B P) \to P^* \otimes_B B \otimes_B P \to A\right)$

101

$$
\begin{array}{c}
(Q \otimes_B P) \otimes_A (P^* \otimes_B Q^*) \\
\big\downarrow \wr \\
Q \otimes_B (B') \otimes_B Q^* \qquad \text{matrices over } B' \\
\big\downarrow \wr \\
Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, B')
\end{array}
$$

What do you need?

You have 3 rings A, B, C with

$B \longrightarrow$ matrix ring over A

$C \longrightarrow$ ——————— B

Then you get a homom $C \longrightarrow$ matrix ring over A.

$$B \longrightarrow P \otimes_A P^* \qquad\qquad C \longrightarrow Q \otimes_B Q^*$$

Then

$$
\begin{array}{ccc}
C & \longrightarrow & Q \otimes_B Q^* & & \\
 & & \uparrow & & \\
 & & Q \otimes_B B \otimes_B Q^* & \longrightarrow & (Q \otimes_B P) \otimes_A (P^* \otimes_B Q^*)
\end{array}
$$

So I have to be a little careful. Suppose $Q \in \mathcal{P}(\tilde{B}^{op})$ ~~That~~ There should be no problem if Q is a firm flat B^{op}-module. Then $Q = Q \otimes_B B = \varinjlim Q_i \otimes_B B$ with $Q_i \in \mathcal{P}(\tilde{B}^{op})$. Another point however is that

$$Q^* \longrightarrow \mathrm{Hom}_{B^{op}}(Q, B) \longrightarrow \mathrm{Hom}_{B^{op}}(Q_i, B) = B \otimes_B \mathrm{Hom}_{B^{op}}(Q_i, \tilde{B})$$

so that $C = \varinjlim Q_i \otimes_B Q^*$ where

$$Q_i \otimes_B Q^* \longrightarrow Q_i \otimes_B B \otimes_B \mathrm{Hom}_{B^{op}}(Q_i, \tilde{B}) \longrightarrow (Q_i \otimes_B P) \otimes_A \big(P^* \otimes_B \overbrace{\mathrm{Hom}_{B^{op}}(Q_i, \tilde{B})}^{\check{Q}_i}\big)$$

pairing $(P^* \otimes_B \check{Q}_i) \otimes_A (Q_i \otimes_B P) \longrightarrow P^* \otimes_B \tilde{B} \otimes_B P \longrightarrow P \otimes_B P^* \longrightarrow A$

 What's the preferred form?

Given $(P, P^*, P^* \otimes P \longrightarrow \tilde{A})$ a unitary dual pair over $\tilde{A}$ s.t. P is $\tilde{A}^{op}$-flat.

get $K_*(P \otimes_A P^*) \longrightarrow K_*(\tilde{A})$.

$$\begin{gathered} \parallel \\ \underbrace{\varinjlim K_*(P_i \otimes_A P^*)}_{} \end{gathered}$$

$\forall i$ $\exists$ homom. can $P_i \otimes_A P^* \longrightarrow P_i \otimes_A \check{P}_i$

$K_*(P_i \otimes_A P^*) \longrightarrow K_*(P_i \otimes_A \check{P}_i) \longrightarrow K_*(\tilde{A})$.

note that if $\langle P^*, P \rangle \subset A$, then get $K_*(P \otimes_A P^*) \to K_*(A)$.

Now given A, B, C and $B \to P \otimes_A P^*$, $C \to Q \otimes_B Q^*$ with P A^{op}-flat, Q B^{op}-flat, get

$$\begin{tikzcd} C \arrow[r] & Q \otimes_B Q^* \\ & Q \otimes_B B \otimes_B Q^* \arrow[u] \arrow[r] & (Q \otimes_B P) \otimes_A (P^* \otimes_B Q^*) \end{tikzcd}$$

There's a problem here. One can suppose that $Q \in \mathcal{P}(\tilde{B}^{op})$, say Q is a f. free B^{op}-mod $\tilde{B}^n$. Am I assuming that $Q^* \otimes Q \xrightarrow{\langle\,,\,\rangle} B$, or $\tilde{B}$?

~~Supposedly A, P and B are~~

Critical case A, P flat, $B = P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$

Critical case A, $P = \tilde{A}^m$, $B = P \otimes_A A \otimes_A \check{P} = M_m(A)$.

$Q = \tilde{B}^n$ $\quad C = Q \otimes_B B \otimes_B \check{Q} = M_n(B)$. Then

$C = M_n(M_m A) = M_{nm}(A)$. Let's see what alterations to make. If I take $B = P \otimes_A \check{P} = \mathrm{Hom}_{A^{op}}(P, P) = M_m(\tilde{A})$ then I get $K_*(B) \xrightarrow{\sim} K_*(\tilde{A})$, and sim:

$B = M_m(\tilde{A})$ $\qquad C = M_n(\tilde{B})$

$K_*(C) = K_*(\tilde{B}) = K_*(\mathbb{Z}) \oplus K_*(B) = K_*(\mathbb{Z}) \oplus$

unital case

$$\left(\begin{array}{l} A \text{ unital}, \quad P = A^m, \quad B = M_m(A), \quad Q = B^n \\ C = M_n(B) = M_{nm}(A). \qquad K_*(C) = K_*(B) = K_*(A). \end{array} \right.$$

Now you want to extend to P flat by $\varinjlim$'s. HI.
~~So~~ what does this mean? $P = \varinjlim P_i$ and then we have a choice $P^* = \mathrm{Hom}_{A^{op}}(P, A)$ or $\underbrace{\mathrm{Hom}_{A^{op}}(P, \tilde{A})}_{\check{P}_i}$.

Then $\mathbf{B} = P \otimes_A P^* = \varinjlim \underbrace{P_i \otimes_A P^*}$

$$\downarrow$$

$$P_i \otimes_A \check{P}_i \longrightarrow M_k \tilde{A}$$

The Largest poss. is for $K_*(B) \longrightarrow K_*(\tilde{A})$

~~So we can have~~ In general B is non-unital

$$K_*\left(P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})\right) \longrightarrow K_*(\tilde{A})$$

So suppose we have

$$K_*\left(Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})\right) \longrightarrow K_*(\tilde{B})$$

$$K_*\left([redacted]\, P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})\right) \longrightarrow K_*(\tilde{A})$$

General construction seems to produce

$$K_*\left(P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})\right) \longrightarrow K_*(\tilde{A}) \qquad P \ \tilde{A}^{op}\text{-flat}.$$

$$\|$$

$$\varinjlim \underbrace{K_*\left(P_i \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})\right)}$$

$$\downarrow$$

$$K_*\left(P_i \otimes_A \mathrm{Hom}_{A^{op}}(P_i, \tilde{A})\right) \longrightarrow K_*(\tilde{A}).$$

It seems to be a perfectly unital construction ~~of an~~
We will have $K_*\left(Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})\right) \longrightarrow K_*(\tilde{B})$

$$\longrightarrow K_*\left(P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})\right) \longrightarrow K_*(\tilde{A}).$$

 Is comp. defined

$$B = P\otimes_A \overbrace{\mathrm{Hom}_{A^{op}}(P,\tilde{A})}^{P^*} \qquad C = Q\otimes_B \overbrace{\mathrm{Hom}_{B^{op}}(Q,\tilde{B})}^{Q^*}$$

$$\begin{tikzcd}
K_*\left(\left(Q\otimes_B P\right)\otimes_A\left(P^*\otimes_B Q^*\right)\right) \arrow[r] \arrow[d] & K_*(\tilde{A}) \\
K_*(\tilde{C}) = K_*\left(Q\otimes_B \tilde{B}\otimes_B Q^*\right) &
\end{tikzcd}$$

I still encounter a problem

How to get this clear?

~~App~~ Apparently there is a problem, namely I can compose.

$$B = P\otimes_A P^* \qquad C = Q\otimes_B Q^*$$

$$\begin{tikzcd}
\left(Q\otimes_B P\right)\otimes_A\left(P^*\otimes_B Q^*\right) \arrow[d, equal] & 0 \arrow[d] \\
Q\otimes_B B\otimes_B Q^* \arrow[d] & Q\otimes_B B \arrow[d] \\
Q\otimes_B Q^* \arrow[d] & Q\otimes_B \tilde{B} \arrow[d] \\
Q\otimes_B \mathbb{Z}\otimes_B Q^* & Q\otimes_B \mathbb{Z} \arrow[d] \\
 & 0
\end{tikzcd}$$

I that the essential case is $Q = \tilde{B}^n$.

I'm still confused. ~~Explain why no pre~~

Explain what's wrong with

B is a matrix ring over A	$B = M_n(A)$
C ——— B	$C = M_k(B)$
$\therefore$ C ——— A	$C = M_{kn}(A)$

This should be the basic ~~transitivity~~ principle.

Take $P = \tilde{A}^n$. $\quad B = P \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(P, A)}_{P^*}$

$Q = \tilde{B}^k$, $\quad C = Q \otimes_B \underbrace{\mathrm{Hom}_{B^{op}}(Q, B)}_{Q^*}$

Then how do I see that $C = (Q \otimes_B P) \otimes_A \mathrm{Hom}_{A^{op}}(Q \otimes_B P, A)$?

$$\mathrm{Hom}_{A^{op}}(Q \otimes_B P, A) = \mathrm{Hom}_{B^{op}}(Q, P^*) = P^* \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})$$

$$C = Q \otimes_B B \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})$$
$$= (Q \otimes_B P) \otimes_A (\underbrace{P^* \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})}_{\mathrm{Hom}_{B^{op}}(Q, P^*)})$$

So what seems to happen is when $Q \in \mathcal{P}(\tilde{B}^{op})$, then
$Q^* = B \otimes_B \check{Q}$ $\qquad \check{Q} = \mathrm{Hom}_B(Q, \tilde{B})$. Then

$$C = Q \otimes_B Q^* = Q \otimes_B B \otimes_B \check{Q} = (Q \otimes_B P) \otimes_A (P^* \otimes_B \check{Q})$$

So what do you need to formulate??

P A^{op}-flat, $B = P \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(P, A)}_{P^*}$, Q is B^{op} flat

$C = Q \otimes_B \underbrace{\mathrm{Hom}_{B^{op}}(Q, B)}_{Q^*}$. Then maps $K_*(B) \to K_*(A)$ and $K_*(C) \to K_*(B)$ are defined. Change C to $C' = Q \otimes_B B \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})$. Then we have

$$C' = (Q \otimes_B P) \otimes_A (P^* \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})) \longrightarrow (Q \otimes_B P) \otimes_A \mathrm{Hom}_{B^{op}}(Q, P^*)$$
$$\mathrm{Hom}_{B^{op}}(Q, P^*) = \mathrm{Hom}_{A^{op}}(Q \otimes_B P, A)$$

So $K_*(C') \to K_*(A)$ is defined. Main claim is that

$$\begin{tikzcd}
K_*(C') \arrow[d] \arrow[drr] & & \\
K_*(C) \arrow[r] & K_*(B) \arrow[r] & K_*(A)
\end{tikzcd}$$

commutes.

reduce to the case $Q \in \mathcal{P}(\tilde{B}^{op})$

Idea: ~~Given~~ A, $(P, P^*, \langle\ \rangle : P^*\otimes P \to A)$ over A, $P \in \mathcal{P}(\vec{A}^{op})$

$B \to P\otimes_A P^*$, $(Q, Q^* = B\otimes_B \check{Q})$ over B, $C \to Q\otimes_B Q^*$,
$Q \in \mathcal{P}(\vec{B}^{op})$.

(Note – you would like any $Q^* \to \mathrm{Hom}_{B^{op}}(Q, B)$
but you assume $Q^* \to B\otimes_B \underbrace{\mathrm{Hom}_{B^{op}}(Q, \vec{B})}_{\check{Q}}$)

Then $K_*(C) \to K_*(B) \to K_*(A)$ is the induced map $K_*(C) \to K_*(A)$ assoc to

$$C \to Q\otimes_B Q^* \to Q\otimes_B B\otimes_B \check{Q} \to (Q\otimes_B P)\otimes_A (P^*\otimes_B \check{Q})$$

$$\begin{pmatrix} A & P^* & \\ P & B & Q^* \\ Q\otimes_B P & Q & C \end{pmatrix}$$

So if Q^* is finm, then it factors.

Try to straighten this out! $(A, P, P^*, P^*\otimes P \to A)$ P A^{op} flat
get $K_*(P\otimes_A P^*) \to K_*(A)$. So $K_*(B) \to K_*(A)$
for $B \to P\otimes_A P^*$. Sim. given $(B, Q, Q^*, Q^*\otimes Q \to B)$ Q B^{op} flat
get $K_*(Q\otimes_B Q^*) \to K_*(B)$ hence $K_*(C) \to K_*(B)$ for $C \to Q\otimes_B Q^*$. So suppose given all this. +
TO compute $K_*(C) \to K_*(B) \to K_*(A)$.

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & Q\otimes_B Q^* \\ & (Q\otimes_B P)\otimes_A (P^*\otimes_B Q^*) \arrow[u] \end{tikzcd}$$

Assume given lifting. So actually you wish to prove comm. of

$$\begin{tikzcd} K_*((Q\otimes_B P)\otimes_A (P^*\otimes_B Q^*)) \arrow[r] \arrow[dd] & K_*(A) \\ & K_*(P\otimes_A P^*) \arrow[u] \\ K_*(Q\otimes_B Q^*) \arrow[r] & K_*(B) \arrow[u] \end{tikzcd}$$

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Given ~~P flat over A^{op}, Q flat over B^{op}~~

rings A, B dual pairs $(P, P^*, \langle\ \rangle)$ $(Q, Q^*, \langle\ \rangle)$ over A, B resp.
s.t. P is A^{op} flat, Q is B^{op} flat; given $B \longrightarrow P \otimes_A P^*$

~~Start with~~ Start with ~~A, P, B, Q~~ rings A, B
P an A^{op}-mod flat Q an B^{op}-mod flat, put P

How to get started.

$$B = P \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(P, A)}_{P^*}$$

If P A^{op}-flat, then one has

$$K_*(B) \longrightarrow K_*(A)$$

Let Q be B^{op}-flat. Have homom.

$$\begin{array}{c} (Q \otimes_B P) \otimes_A \mathrm{Hom}_{A^{op}}(Q \otimes_B P, A) \\ \downarrow \alpha \\ Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, B) \end{array}$$

Claim

$$\begin{tikzcd} K_*\big((Q \otimes_B P) \otimes_A \mathrm{Hom}_{A^{op}}(Q \otimes_B P, A)\big) \arrow[r, "tr"] \arrow[d, "\alpha_*"] & K_*(A) & \\ K_*\big(Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, B)\big) \arrow[r, "tr"] & K_*(B) \arrow[u] \arrow[r] & K_*(P \otimes_A P^*) \arrow[ul, "tr"'] \end{tikzcd}$$

commutes. To prove this you want to reduce to $Q = \tilde{B}^n$.
You have to generalize the result, because you don't want to move on the star side.

Introduce $X = \mathrm{Hom}_{A^{op}}(Q \otimes_B P, A)$ so there's a pairing

$$X \otimes Q \otimes_B P \longrightarrow A$$

B-linear

$$\begin{pmatrix} A & P^* & X \\ P & B & Q^* \\ Q \otimes_B P & Q & C \end{pmatrix}$$

$$\begin{tikzcd}
K_*((Q \otimes_{\mu B} P) \otimes_A X) \arrow[rr] \arrow[d] & & K_*(A) \\
K_*(Q \otimes_{\mu B} Q^*) \arrow[r] & K_*(B) \arrow[r] & K_*(P \otimes_A P^*) \arrow[u]
\end{tikzcd}$$

puts you to sleep

Tentative formulation

$$\begin{tikzcd}
K_*((Q \otimes_B P) \otimes_A \mathrm{Hom}_{A^{op}}(Q \otimes_B P, A)) \arrow[r] \arrow[d] & K_*(A) \\
K_*(Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, B)) &
\end{tikzcd}$$

$$\begin{tikzcd}
Q \otimes_B P \otimes_A \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P, A)) \arrow[r, "tr"] \arrow[d] & A \\
Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)) \arrow[r, "tr"] & P \otimes_A \mathrm{Hom}_{A^{op}}(P, A) \arrow[u, "tr"']
\end{tikzcd}$$

here you need $B = P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$

How to set this up? What's the idea. I feel you should be able to handle $B \to P \otimes_A P^*$. General setup is a dual pair $(P, P^*, \langle\ \rangle : P^* \otimes P \to A)$ w P A^{op}-flat then $B \to P \otimes_A P^*$ and a dual pair $(Q, Q^*, \langle\ \rangle : Q^* \otimes Q \to B)$. What kind of C can you handle. C must operate on Q/B and on $Q \otimes_B P / A$.

$$\begin{tikzcd}
(Q \otimes_B P) \otimes_A (P^* \otimes_B Q^*) \arrow[d, "NO"] & \\
Q \otimes_B Q^* \arrow[r, "tr"] & B
\end{tikzcd}$$

$$\begin{tikzcd} Q\otimes_B B\otimes_B Q^* \arrow[r] \arrow[d] & Q\otimes_B Q^* \arrow[r, "tr"] & B \arrow[d] \\ & & P\otimes_A P^* \arrow[d, "tr"] \\ (Q\otimes_B P)\otimes_A (P^*\otimes_B Q^*) \arrow[rr, "tr"] & & A \end{tikzcd}$$

Would like defined on $Q\otimes_B Q^*$, but then I need

$$Q\otimes_B Q^* \longrightarrow (Q\otimes_B P)\otimes_A \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P,A))$$

need

$$\begin{tikzcd} Q^* \arrow[r, "?"] \arrow[d] & P\otimes_A \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P,A)) \arrow[d] \\ \mathrm{Hom}_{B^{op}}(Q,B) \arrow[r] & \mathrm{Hom}_{B^{op}}(Q, P\otimes_A \mathrm{Hom}_{A^{op}}(P,A)) \end{tikzcd}$$

I think there is a slight improvement possible where $B\otimes_B Q^*$ becomes $B\otimes H$

$K_*(P\otimes_A \check{P}) \longrightarrow K_*(\tilde{A})$. $\qquad \check{P} = \mathrm{Hom}_{A^{op}}(P,\tilde{A})$

At the moment you can do this

$$\begin{tikzcd} Q\otimes_B (B\otimes_B \check{Q}) \arrow[d] \\ Q\otimes_B P\otimes_A \check{P}\otimes_B \check{Q} \arrow[d] \\ (Q\otimes_B P)\otimes_A \mathrm{Hom}_{A^{op}}(Q\otimes_B P, \tilde{A}) \arrow[d, "tr"] \\ \tilde{A} \end{tikzcd}$$

what I would like is to get Q^* or $B\otimes_B \check{Q}$ bigger.

$$\begin{tikzcd} P\otimes_A \mathrm{Hom}_{B^{op}}(Q, \overbrace{\mathrm{Hom}_{A^{op}}(P,\tilde{A})}^{\check{P}}) \arrow[d] \\ \mathrm{Hom}_{B^{op}}(Q, P\otimes_A \check{P}) \end{tikzcd}$$

So the problem is to lift an element of

Say this differently. You want to

cover the maximum of

$$Q\otimes_B \mathrm{Hom}_{B^{op}}(Q,B) \quad \text{or} \quad Q\otimes_B \mathrm{Hom}_{B^{op}}(Q,\tilde{B})$$

but can handle only something mapping to

$$(Q\otimes_B P)\otimes_A \mathrm{Hom}_{B^{op}}\left(Q, \mathrm{Hom}_{A^{op}}\left(P, \begin{smallmatrix}A\\ \tilde{A}\end{smallmatrix}\right)\right)$$

$$\begin{tikzcd}
P\otimes_A \mathrm{Hom}_{B^{op}}\left(Q, \mathrm{Hom}_{A^{op}}(P,\tilde{A})\right) \arrow[d] \\
\mathrm{Hom}_{B^{op}}\left(Q, P\otimes_A \mathrm{Hom}_{A^{op}}(P,\tilde{A})\right)
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_B P)\otimes_A \mathrm{Hom}_{A^{op}}(Q\otimes_B P, \tilde{A}) & Q\otimes_B \mathrm{Hom}_{B^{op}}(Q,\tilde{B}) \\
(Q_\mu\otimes_B P)\otimes_A \mathrm{Hom}_{A^{op}}(Q\otimes_B P, \tilde{A}) \arrow[u] \arrow[d] & Q_\mu\otimes_B \mathrm{Hom}_{B^{op}}(Q,\tilde{B}) \arrow[u] \arrow[d] \\
(Q_\mu\otimes_B P)\otimes_A \mathrm{Hom}_{A^{op}}(Q_\mu\otimes_B P, \tilde{A}) \arrow[d, equal] & Q_\mu\otimes_B \underbrace{\mathrm{Hom}_{B^{op}}(Q_\mu,\tilde{B})}_{\text{matrices over } \tilde{B}} \\
Q_\mu\otimes_B \underbrace{\left(P\otimes_A \mathrm{Hom}_{A^{op}}(P,\tilde{A})\right)}_{\text{matrices over } B}\otimes_B \mathrm{Hom}_{B^{op}}(Q_\mu,\tilde{B}) \arrow[ur] &
\end{tikzcd}$$

Let's ~~try to~~ get to the bottom of this.

Start with $P \in \mathcal{P}(\tilde{A}^{op})$, $B \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A}) = \mathrm{Hom}_{A^{op}}(P,P)$

Do unital case first. $P \in \mathcal{P}(A^{op})$, $B \to \mathrm{End}_{A^{op}}(P)$

$Q \in \mathcal{P}(B^{op})$. Then

$$Q \otimes_B P \otimes_A \mathrm{Hom}_{A^{op}}(Q \otimes_B P, \tilde{A}) = \mathrm{End}_{A^{op}}(Q \otimes_B P)$$
$$\|$$
$$Q \otimes_B P \otimes_A \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P, \tilde{A}))$$
$$\|$$
$$Q \otimes_B \big(P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})\big) \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})$$
$$\uparrow$$
$$Q \otimes_B B \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B}) = \mathrm{End}_{B^{op}}(Q)$$

Thus if A unital, $P \in \mathcal{P}(A^{op})$, $B = \mathrm{End}_{A^{op}}(P)$, $Q \in \mathcal{P}(B^{op})$, then $Q \otimes_B P \in \mathcal{P}(A^{op})$ and

$$\mathrm{End}_{A^{op}}(Q \otimes_B P) = \mathrm{End}_{B^{op}}(Q).$$

Now suppose A^{B} nonunital, $P \in \mathcal{P}(\tilde{A}^{op})$, $B \longrightarrow \mathrm{End}_{A^{op}}(P)$
$Q \in \mathcal{P}(\tilde{B}^{op})$

$$Q \otimes_B P \otimes_A \mathrm{Hom}_{A^{op}}(Q \otimes_B P, \tilde{A}) = \mathrm{End}_{A^{op}}(Q \otimes_B P)$$
$$\|$$
$$Q \otimes_B \big(P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})\big) \otimes \mathrm{Hom}_B(Q, \tilde{B})$$
$$\uparrow$$
$$Q \otimes_B \tilde{B} \otimes_B \mathrm{Hom}_B(Q, \tilde{B}) = \mathrm{End}_{B^{op}}(Q)$$

Now suppose that P is flat over A^{op}, $Q \in \mathcal{P}(B^{op})$

$$Q \otimes_B P \otimes_A \operatorname{Hom}_{A^{op}}(Q \otimes_B P, \tilde{A}) \qquad \text{f r ops on } Q\otimes_B P \text{ over } A$$

$$\|$$

$$Q \otimes_B \big(P \otimes_A \operatorname{Hom}_{A^{op}}(P, \tilde{A})\big) \otimes \operatorname{Hom}_{B^{op}}(Q, \tilde{B}) \qquad \text{matrices over } P\otimes_A \operatorname{Hom}_{A^{op}}(P,\tilde{A})$$

$$\uparrow$$

$$Q \otimes_B \quad B \quad \otimes \operatorname{Hom}_{B^{op}}(Q, \tilde{B}) \qquad \text{matrices over } B.$$

What happens it seems is that ~~the endos of~~ Q $\operatorname{Hom}_B(Q,Q)$ = matrices over $\tilde{B}$ do not act as finite rank operators on $Q \otimes_B P$ over A. e.g if $Q = \tilde{B}$, then $\tilde{B}$ does not ~~operate~~ act by f.r. ops on ~~B~~ P.

<u>Critical case</u>: P flat over $\tilde{A}^{op}$ ($\tilde{A}$ can be any unital ring) and P unitary over $\tilde{A}$). ~~Then~~

$$B \longrightarrow P \otimes_A \operatorname{Hom}_{A^{op}}(P, \tilde{A}), \qquad Q \in \mathcal{P}(\tilde{B}^{op})$$

$$Q \otimes_B \operatorname{Hom}_{B^{op}}(Q, \tilde{B}) \qquad \text{matrices over } \tilde{B} \quad (\text{if } Q \text{ free})$$

$$\cup$$

$$Q \otimes_B B \otimes_B \operatorname{Hom}_{B^{op}}(Q, \tilde{B}) = Q \otimes_B \operatorname{Hom}_{B^{op}}(Q, B)$$

$$\downarrow$$

$$Q \otimes_B P \otimes_A \operatorname{Hom}_{A^{op}}(P, \tilde{A}) \otimes_B \operatorname{Hom}_{B^{op}}(Q, \tilde{B})$$

$$\|$$

$$Q \otimes_B P \otimes_A \operatorname{Hom}_{A^{op}}(Q \otimes_B P, \tilde{A})) \qquad \text{f. rank ops on } Q \otimes_B P \text{ over } A.$$

So we learn that
P is flat over A^{op} but not f.projective, then for $Q \in P(\tilde{B}^{op})$ we can not expect
$End_{B^{op}}(Q) = Q \otimes_B Hom_{B^{op}}(Q, \tilde{B})$ to act as finite rank operators on $Q \otimes_B P$ over A. Here $B \to P \otimes_A Hom_{A^{op}}(P, \tilde{A})$

So how do I formulate transitivity?

Dual pair $(P, P^*, \langle\,;\,\rangle)$ over A with P A^{op}-flat.
Then get $K_*(P \otimes_A P^*) \longrightarrow K_*\left(\begin{smallmatrix} A \\ \tilde{A} \end{smallmatrix}\right)$ depending on whether $\langle\,,\,\rangle : P^* \otimes_A P \to \begin{smallmatrix} A \\ \tilde{A} \end{smallmatrix}$.

Next have $B \longrightarrow P \otimes_A P^*$ and a dual pair (Q, Q^*) over B with Q B^{op}-flat. Then $Q \otimes_B P$ is A^{op}-flat and we have $\big(Q \otimes_B P,\ P^* \otimes_B Q^*,$

$$(P^* \otimes_B Q^*) \otimes_{\mathbb{Z}} (Q \otimes_B P) \longrightarrow P^* \otimes_B \tilde{B} \otimes_B P \longrightarrow \tilde{A}\big).$$

Then

$$\begin{tikzcd}
K_*(Q \otimes_B P \otimes_A P^* \otimes_B Q^*) \arrow[r] & K_*(\tilde{A}) \\
K_*(Q \otimes_B (B \otimes_B Q^*)) \arrow[u] \arrow[r] & K_*(B)
\end{tikzcd}$$

$$\begin{tikzcd}
K_*(Q \otimes_B (B \otimes_B Q^*)) \arrow[r] \arrow[d, "tr^Q"] & K_*(Q \otimes_B P \otimes_A P^* \otimes_B Q^*) \arrow[r, "tr^{Q \otimes_B P}"] \arrow[d, "tr^Q"] & K_*(A) \arrow[d, equal] \\
K_*(B) \arrow[r] & K_*(P \otimes_A P^*) \arrow[r, "tr^P"] & K_*(A)
\end{tikzcd}$$

Basically you have to map to the dual pair
Q, $P \otimes_A Hom_{A^{op}}(Q \otimes_B P, \tilde{A})$
over B.

want to land in $\quad Q\otimes_B P \otimes_A \mathrm{Hom}_{A^{op}}(Q\otimes_B P, \tilde{A})$

You have the dual pair $\left(Q,\ P\otimes_A \mathrm{Hom}_{A^{op}}(Q\otimes_B P, \tilde{A}),\ \langle\, ,\, \rangle \right)$ over B

where the pairing is $\qquad$ NO

$$P\otimes_A \overbrace{\mathrm{Hom}_{A^{op}}(Q\otimes_B P, \tilde{A})}^{\mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P,\tilde{A}))} \otimes_{\mathbb{Z}} Q$$

$$\Vert$$

$$P\otimes_A \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P,\tilde{A})) \otimes_{\mathbb{Z}} Q$$

$$\downarrow$$

$$B' = P\otimes_A \mathrm{Hom}_{A^{op}}(P,\tilde{A}) \qquad ? \qquad \text{defeated again}$$

This maybe would allow one to handle $B = B'$.

<u>Dual pair</u> $\left(Q\otimes_B \tilde{B}',\ P\otimes_A \mathrm{Hom}_{A^{op}}(\overbrace{Q\otimes_B P}^{Q\otimes_B \tilde{B}'\otimes_{B'} P}, \tilde{A}) \right)$ over B'.

Maybe this works. Start with P flat over A^{op} and put $B = P\otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(P,\tilde{A})}_{\check{P}}$. Take Q flat over B^{op} put $Q^* = P\otimes_A \mathrm{Hom}_{A^{op}}(Q\otimes_B P, \tilde{A}) = P\otimes_A (Q\otimes_B P)^\vee$. Then we have a pairing

$$Q^*\otimes Q = P\otimes_A \underbrace{(Q\otimes_B P)^\vee}_{\mathrm{Hom}_{B^{op}}(Q, P^\vee)} \otimes Q \longrightarrow P\otimes_A P^\vee = B.$$

so it seems we have a dual pair Q, Q^* over B whence

$$\begin{array}{ccc} K_*(Q\otimes_B Q^*) & \longrightarrow & K_*(B) = K_*(P\otimes_A \check{P}) \\ \Vert & & \swarrow \\ K_*((Q\otimes_B P)\otimes_A (Q\otimes_B P)^\vee) & \longrightarrow & K_*(A) \end{array} \qquad \text{TRICKY!!}$$

 This might work but does it do any good.

This particular Q^* namely $P \otimes_A (Q \otimes_B P)^\vee \longleftarrow P \otimes_A \check{P} \otimes_B \check{Q}$.

It looks like one is being forced roughly to replace $Q \otimes_B Q^\vee$ with $Q \otimes_B P \otimes_A \check{P} \otimes_B \check{Q}$ again. Slightly different version of $B \otimes_B \check{Q} = B \otimes_B \mathrm{Hom}_{B^{op}}(Q, \tilde{B})$.

See if you can formulate the result, at least ~~xxx~~ write up something.

General construction: Given P A^{op}-flat construct

$$K_*(P \otimes_A \check{P}) \longrightarrow K_*(\tilde{A}) \qquad \check{P} = \mathrm{Hom}_{A^{op}}(P, \tilde{A})$$

Let $B = P \otimes_A \check{P}$, let Q be B^{op}-flat. Then $Q \otimes_B P$ is A^{op}-flat, whence one has

whence $K_*(Q \otimes_B \check{Q}) \to K_*(\check{B})$

$$K_*(Q \otimes_B P \otimes_A (Q \otimes_B P)^\vee) \longrightarrow K_*(\tilde{A})$$

~~Hence~~ One has a ring homom. $Q \otimes_B P \otimes_A (Q \otimes_B P)^\vee \longrightarrow Q \otimes_B \check{Q}$ whence

$$* \quad K_*(Q \otimes_B P \otimes_A (Q \otimes_B P)^\vee) \longrightarrow K_*(Q \otimes_B \check{Q}) \longrightarrow K_*(\tilde{B})$$

claim that ~~$Q \otimes_B P \otimes_A (Q \otimes_B P)^\vee \longrightarrow Q \otimes_B \check{Q}$~~

the pairing $\quad P \otimes_A (Q \otimes_B P)^\vee \otimes_{\mathbb{Z}} Q \longrightarrow \check{Q} \otimes_{\mathbb{Z}} Q \longrightarrow \tilde{B}$

has image in B. There's a map $P \otimes_A (Q \otimes_B P)^\vee \longrightarrow Q^\vee$ which factors thru $\mathrm{Hom}_{B^{op}}(Q, B)$, Thus $*$ should factor thru $K_*(B)$. Then we should have commutatively

$$\begin{tikzcd}
K_*(Q \otimes_B P \otimes_A (Q \otimes_B P)^\vee) \arrow[r, "tr^{Q \otimes P}"] \arrow[d] & K_*(\tilde{A}) \\
K_*(Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, B)) \arrow[r, "tr^Q"] & K_*(B) = K_*(P \otimes_A P^\vee) \arrow[u, "tr^P"']
\end{tikzcd}$$

 Maybe I should try to understand better the limiting process. Suppose ~~[illegible]~~ given unital dual pair over $\tilde{A}$, $(P, Q, Q\otimes P \to \tilde{A})$. Next suppose P flat, whence ~~[illegible]~~ $P = \varinjlim P_i$ filtered ind limit of f.g proj A^{op}-modules. To define

$$\begin{array}{ccc} K_*(P\otimes_A Q) & \longrightarrow & K_*(\tilde{A}) \\ \| & & \\ \varinjlim K_*(P_i \otimes_A Q) & & \end{array}$$

for each i you get $P_i \otimes_A Q \longrightarrow P_i \otimes_A \check{P}_i = \mathrm{End}_{A^{op}}(P_i, P_i)$, where $\check{P}_i = \mathrm{Hom}_{A^{op}}(P_i, \tilde{A})$,
inducing $K_*(P_i \otimes_A Q) \to K_*(P_i \otimes_A \check{P}_i) \to K_*(\tilde{A})$. Problem is to show that

$$\begin{tikzcd} P_i \arrow[r] \arrow[dr] & P_j \arrow[d] \\ & \mathrm{Hom}_{A^*}(Q, \tilde{A}) \end{tikzcd} \quad \text{that} \quad \begin{tikzcd} K_*(P_i \otimes_A Q) \arrow[r] \arrow[dr] & K_*(P_j \otimes_A Q) \arrow[d] \\ & K_*(A) \end{tikzcd}$$

commutes. But I know how to do this. But think it out carefully for further insight. Important cases are are direct injection and surjection

~~$P' \to P \to P'' \to 0$~~

~~$P\otimes_A Q \to P''\otimes Q$~~

One has $P_i \hookrightarrow P_j$ in general inducing homom. $P_i \otimes_A Q \to P_j \otimes_A Q$ and your rep of $P_i \otimes_A Q$ on P_i and on P_j.

In the end you try to compare $B = P \otimes_A Q$ actions on P and on P' where you have this map $P \to P'$ compatible with actions which is a B-nil iso. This is not enough by itself, but you factor $P \to P\oplus P' \to P'$ and it's true for each map separately.

117 What sort of lemma? Given $P \to P' \to \mathrm{Hom}_A(Q,\tilde{A})$, then

$$\begin{tikzcd} K_*(P\otimes_A Q) \arrow[r] \arrow[dr, "(P)"'] & K_*(P'\otimes_A Q) \arrow[d, "(P')"] \\ & K_*(\tilde{A}) \end{tikzcd}$$

$$\begin{tikzcd} P\otimes_A Q \arrow[r, hook] \arrow[d] & P''\otimes_A Q \arrow[r, two heads] \arrow[d] & P'\otimes_A Q \arrow[d] \\ \mathrm{End}(P) & \mathrm{End}(P'') & \mathrm{End}(P') \end{tikzcd}$$

Take universal cases:

$$\begin{tikzcd} B = P\otimes_A \check{P}'' \arrow[r, hook] \arrow[d] & P''\otimes_A \check{P}'' \\ P\otimes_A \check{P} & \end{tikzcd} \qquad \begin{tikzcd} B = P''\otimes_A \check{P}' \arrow[r] \arrow[d, hook] & P'\otimes_A \check{P}' \\ P''\otimes_A \check{P}'' & \end{tikzcd}$$

~~these~~ How do I distinguish these?

In the former ~~you have~~ P is a subrepresentation of $P''=P\oplus N$ and B acts trivially on P''/P so we have $\left(\begin{array}{c|c} * & * \\ \hline 0 & 0 \end{array}\right)$

$P^{\#} = N \oplus P'$

In the latter P' is a quotient of $P^{\#}$ and B acts trivially on the kernel, so we have

$$\left(\begin{array}{c|c} 0 & * \\ \hline 0 & * \end{array}\right)$$

or if we write $P=P'\oplus N$ then $\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$. Indeed you get the two affine groups.

So now how am I going to get anywhere???

I really have to understand Suslin. His fundamental idea is that excision results from $GL(\mathbb{Z})$ acting trivially on $H_*(BGL(A)^+)$. So maybe I can find another approach

 I think I need to go back to stability arguments – to derive Suslin's K^M theorem.

To understand $G_n \subset G_{n+1}$. How to prove stability? What about symmetric groups Σ_n? The basic idea was to make G_n act on an acyclic space, or nearly acyclic space, and somehow to arrange inductive information. Buildings and the Q category! Suslin's idea of the first interesting group.

For the moment look at the Morita invariance question.

So you think you can prove M inv of K_* for A right flat. Why $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ A, B right flat $\Rightarrow$ so are P, Q so we have maps $K_*(B) \longrightarrow K_*(A)$ and $K_*(A) \longrightarrow K_*(B)$ induced by the bimodules P and Q resp. The composition is induced by the A bimod A, B bimod B and I think this must be the identity. This seems very easy.

Now take $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ form with A rt flat.

Then B h-unital $\Longleftrightarrow$ $P \overset{L}{\otimes}_{\tilde{A}} Q = P \otimes_A Q = B$.

Proof. A A^{op} flat $\Longrightarrow$ $Q = A \otimes_A Q$ is B^{op} flat which implies we have $K_*(A) \longrightarrow K_*(B)$ canonical. We have a representation of A on Q a B^{op}-flat module.

$$\begin{array}{ccc} B \overset{L}{\otimes}_{\tilde{B}} P \overset{L}{\otimes}_{\tilde{A}} Q & \longrightarrow & B \overset{L}{\otimes}_{\tilde{B}} B \\ \downarrow{\scriptstyle \text{quis}} & & \downarrow \\ P \overset{L}{\otimes}_{\tilde{A}} Q & \longrightarrow & B \end{array}$$

suppose A is left flat $\Longrightarrow$ $P = P \otimes_A A$ is B-flat.

$$\begin{array}{ccc} \hat{B} \otimes_{\tilde{B}} P \otimes_{\tilde{A}} \hat{Q} & \longrightarrow & \hat{B} \otimes_{\tilde{B}} B \\ \downarrow & & \downarrow \\ P \otimes_{\tilde{A}} \hat{Q} & \longrightarrow & B \end{array}$$

~~So we need to understand~~

This means we understand when a ring B mod a flat ring A is h-unital. We need to understand when B is K-equiv. to A. ~~Now we can choose~~ Now given B we can choose a B-module surjection

Idea. Given B pick $P \twoheadrightarrow B$ in $\mathcal{M}(B)$ with P flat

Then get $\begin{pmatrix} A & B \\ \| & \| \\ P & B \end{pmatrix}$ $\qquad a_1 a_2 = f(a_1) a_2$

A is B-flat mapping onto B

and $A/I = B$ where $IA = 0$ $\qquad \mathcal{M}(A) = \mathcal{M}(B)$

$$B = B \otimes_B B \leftarrow B$$
$$A = B \otimes_B A \qquad A.$$

So now we need to understand

$$1 \to \underset{\text{abelian}}{GL(I)} \longrightarrow GL(A) \longrightarrow GL(B) \to 1$$

We get $GL(B)$ acts on $GL(I) = M(I)$ by left mult since $IA = 0$. We need to understand why, or when, $H_*\left(GL(B), \tilde{H}_*(M(I))\right) = 0$. A first step might be to understand something about the functor $I \mapsto H_*(GL(B), V(I))$. Here $V(I)$ is column vectors. Now one can assume a lot is known about $H_*(M(I))$. Rationally this is an exterior algebra on $M(I)$. Suslin knows [redacted] the answer. Basically there is the multilinearity.

Look at h-unital question. $B = A/I$ is h-unital iff $\exists$ a resolution of B by firm flat B-modules. But these are the same as firm flat A-modules. So B is h-unital $\Leftrightarrow$ B is an h-unitary B-module $\Leftrightarrow$ B is an h-unitary A-module $\Leftrightarrow$ i.e. $\mathbb{Z} \overset{L}{\otimes}_{\tilde{A}} B = 0$ and as A is flat firm $\Leftrightarrow$ $\mathbb{Z} \overset{L}{\otimes}_{\tilde{A}} I = 0$ i.e. I is h-unitary over A. So the h-unital question is quite [redacted] easy.

$$\begin{array}{ccccccccc} 0 & \to & I & \to & A\oplus I & \to & A & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & I & \to & A & \to & B & \to & 0 \end{array}$$

crazy functor theory. So what is to be done?
$GL(A)$

Let's go over the linear algebra. $0 \to I \to A \to B \to 0$
Examine ~~flat~~ $A \twoheadrightarrow B$ with A B-flat. Any such A gives a group extn $1 \to M(I) \to GL(A) \to GL(B) \to 1$. So what do I know about the possible A.

03/01/97 ~~At~~ 9/0 over for K_1, maybe K_2. What about K_0?
You do get M inv for K_0 in general for idempotent rings at least. Pro object. So what is it all about?

Before building castles in the sky you need to check details.

Given a unitary dual pair $(P, Q, Q\otimes P \to \tilde{A})$ over $\tilde{A}$ with P right flat one has a canonical map $K_*(P\otimes_A Q) \to K_*(\tilde{A})$. Construction - suffices to define a natural map for P in the cat $\mathcal{P}(\tilde{A}^{op}) / \mathrm{Hom}_A(Q, \tilde{A})$. Basic construction: If $P \in \mathcal{P}(\tilde{A}^{op})$ have canonical ~~map~~ homom. $P\otimes_A Q \to P \otimes_A \check{P} = \mathrm{End}_{A^{op}}(P)$ whence $K_*(P\otimes_A Q) \to K_*(\mathrm{End}_{A^{op}}(P)) \to K_*(\tilde{A})$. What about a map $P_1 \to P_2 \to \mathrm{Hom}_A(Q,\tilde{A})$. Can factor $P_1 \to P_1 \oplus P_2 \hookrightarrow P_2$ two cases etc.

There's an idea here - namely working in the category $\mathcal{P}(\tilde{A}^{op})/P$. My guess is that the homology of the functor $F \rightsquigarrow F\otimes_A Q$ taken over this category is $\mathrm{Tor}_*^{\tilde{A}}(P,Q)$. Another idea is Goodwillie's calculus of functors. You want to consider the behavior of group homology on extensions.

Idea. ~~Consider the cat~~ Given a right module U over A consider the cat of f. free $\tilde{A}^{op}$ modules equipped with a map to U. For each $n \geq 0$ get $U^n = \mathrm{Hom}_{A^{op}}(\tilde{A}^n, U)$ acted on by $\mathrm{Hom}_{A^{op}}(\tilde{A}^n, \tilde{A}^{n'})$

121 Why consider this? Because one might try to probe $K_*(U \otimes_A Q)$ as a limit over this category. Not clear enough, although there may ∃ a germ of truth. ~~But it~~

Question: We know that if U is a flat $\tilde{A}^{op}$ module, then the cat of pair $\tilde{A}^n \to U$ is filtering. Is it possible to get a link between h-unitary and the acyclicity of a category.

Look at one-sided meq's Begin with $f: A \to B$ a B-module surjection. Assume A, B are B firm, A flat over B. Meant is $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\quad B \otimes_B A \cong A$, $\quad A \otimes_A B = B$

A is A-flat $\overset{A=}{\Longleftrightarrow}$ $A \otimes_A A$ is B-flat. ~~Now A is h-unital~~ Now A is left flat, we know B is h-unital iff it has a firm flat res. over B

iff ——————————————————— A

iff $\mathbb{Z} \overset{L}{\otimes}_{\tilde{A}} B = 0$ iff $\mathbb{Z} \overset{L}{\otimes}_{\tilde{A}} I = 0$, where $B = A/I$.

Now the angle I would like to develop is what? Somehow you want to understand ~~subsets~~.

Fix A and examine possible B one sided equivalent to A and then try to analyze the GL appropriately. Take A unital. Need unitary A-module B with surj. pairing $B \otimes A \to A$ equivalently $B \to \mathrm{Hom}_A(A, A) = A$ such that image generates A as left ideal. Any surjective map e.g. Note A is leftflat ~~so this wa~~ so

$B = \underbrace{A}_{\in P(A^{op})} \otimes_A \underset{Q}{B}$ so $B \in P(B^{op})$. So we can't find an ~~important~~ example of this sort with B not flat.

122 But I still should look at B which are meq a unital ring $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ P, Q, $\langle\,\rangle$ surj. unitary /A

What ~~situation~~ to examine? I want A left flat but not right flat, and meq to a unital ring. Suppose A contain e idemp such that $ae = a$ $\forall a$. If true $A \in \mathcal{P}(A_{\text{unital}})$ In fact $\mathbb{Z} \in \mathcal{P}(\tilde{A})$. then $A = \begin{matrix} eA = A' \\ \oplus \\ e^{\perp}A \end{matrix}$ $\therefore$ Right A' module.

Try again. B given firm ring.

Let P be a flat firm B-module equipped w a B-mod. surj $P \xrightarrow{f} B$. get firm dual pair over B

$\left(B, P, \quad P \otimes B \xrightarrow{\langle\,\rangle} B\right)$

$\langle p, b\rangle = f(p)b$

Put $A = B \otimes_B P \simeq P$

$(b_1 \otimes p_1)(b_2 \otimes p_2) = b_1 \otimes f(p_1) b_2 p_2 = f(b_1 p_1) b_2 \otimes p_2$

$a_1 \cdot a_2 = f(a_1)\, a_2$.

get $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\quad B \otimes_B A = A$, $\quad A \otimes_A B = B$.

Now A B-flat $\Rightarrow$ $\exists$ canon map $K_*(A) \to K_*(B)$ which must be the map induced by the homom. f.

Idea: Fix B, You want to construct a flat firm ring A which is one-sided morita equivalent to B. This is equiv. to a firm dual pair/B: $(B, A, \langle\,\rangle : A \otimes B \twoheadrightarrow B)$ such that A is B-flat. Equiv. $A \to \mathrm{Hom}_{B^{op}}(B, B)$ suff. big image So the simplest way to proceed is to construct a filtered ind. limit of free B^{op}-mods $P_i \to \mathrm{Hom}_{B^{op}}(B,B)$ such that $P_i \oplus BP_{i+1}$

123 Suppose I start with $P^\circ = \tilde{B}$. This doesn't work since the identity of B does not usually come from $B \otimes_B \mathrm{Hom}_{B^{op}}(B,B)$. Only thing seems to be to apply your divisibility ~~g...~~ arg.

03/02/97. Consider B a firm ring, ~~th~~ we can construct a firm left flat ring A which is left meq B, i.e. ~~...~~ the Mcont has the form $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$, equiv. is given by a B-module map $A \longrightarrow B \otimes_B \mathrm{Hom}_{B^{op}}(B,B)$ suff. non deg. The problem is then to relate $K_*(B)$ to $K_*(A)$. You know that $K_*(A)$ is ind of the choice of A, moreover there's a canonical map $K_*(A) \longrightarrow K_*(B)$. ~~...~~ We have some sort of derived functor situation.

~~...~~ What does one know about this? You should understand K_0, K_1, K_2.

For every B-module P equipped with ~~...~~ pairing $P \otimes B \longrightarrow B$, i.e. $P \longrightarrow \mathrm{Hom}_{B^{op}}(B,B)$ you get a ring $B \otimes_B P$ and M.cont. $\begin{pmatrix} B\otimes_B P & B \\ P & B \end{pmatrix}$

~~For the best modules~~ I think you want to consider surjections $P \twoheadrightarrow B$. First examine K_0 and K_1. Choose $A \overset{f}{\twoheadrightarrow} B$ ~~...A...~~ a map in $M(B)$ with A B-flat; let I be the kernel. Then A is a firm ring, I ideal in $A \ni IA = 0$, in particular $I^2 = 0$, so we know $K_0 A \xrightarrow{\sim} K_0 B$. Also have $1 \longrightarrow M(I) \longrightarrow GL(\tilde{A}) \longrightarrow GL(\tilde{B}) \longrightarrow 1$ so that

$$H_2(GL(\tilde{A})) \longrightarrow H_2(GL(\tilde{B})) \longrightarrow H_0(GL(\tilde{B}), H_1(M(I))) \longrightarrow H_1(GL(\tilde{A})) \longrightarrow H_1(GL(\tilde{B})) \to 0$$

$$H_0(GL(\tilde{B}), M(I)) \longrightarrow K_1\tilde{A} \longrightarrow K_1\tilde{B} \to 0$$

So what? You have $E(A)$. Maybe you are faced with the necessity of working out the Suslin theory.

Consider $P = \tilde{B}^n \xrightarrow{f} B$ form the ring $B \otimes_B P = B^n$ equipped with $(b_i)(b'_i) = f((b_i))\, b'_i = \left(\sum_i b_i \beta_i\right) b'_i$

These are the kind of rings to understand. ~~So~~ the ideal is image of

$$\begin{aligned} P \otimes B &\longrightarrow B \\ (b_i) \otimes b &\longmapsto \left(\sum b_i \beta_i\right) b \end{aligned}$$

The image is apparently $\sum B b_i B$. As far as I can see I know nothing about these rings.

Let's look more carefully at the category theory. I want to ~~handle~~ understand firm B-modules P equipped with a surjection $P \twoheadrightarrow B$. We have functor $P \to K_*(P)$ on this category.

---

anyway ~~we have consider~~ let B be a firm

B firm ring, consider the cat of surjections $f : A \to B$ in $\mathcal{M}(B)$. Such an A is a ring with $a_1 a_2 = f(a_1) a_2$, so with have a functor from this category to firm rings. Given $A \twoheadrightarrow B$, let $B = A/I$.

Group extension

$$1 \longrightarrow M(I) \longrightarrow GL(A) \longrightarrow GL(B) \longrightarrow 1$$

What structure? The point is that I have a category - the objects are ~~firm modules~~ B-modules P mapping onto B. ~~the~~ One can replace P by $P \otimes_B B$?

Take $P \twoheadrightarrow B$ $\quad P \in \mathrm{Mod}(\tilde{B})$, form the firm ring $P \otimes_B B$

$$\begin{pmatrix} P \otimes_B B & B \\ P & B \end{pmatrix}$$

Let F be a free $\tilde{B}$-module mapping onto?

125 First suppose B flat firm. Choose $F \twoheadrightarrow B$, F free $\tilde{B}$-module, whence $B \otimes_B F \twoheadrightarrow B$. Now choose $P \twoheadrightarrow B \otimes_B F$ with P flat. Then you have

$$P \to BF \to B$$

ring homos (surjective) and we know that $K_*(P) \to K_*(BF) \to K_*(B)$ is an isomorphism

So in fact there is some surprising point here. Given B firm ring, then the good $K_*^L(B)$ is actually a summand of ?

So what

$\cancel{B \otimes_B F}$

Suppose B left flat. Can you see that the affine rings $B \oplus M$, have same K-theory as B? Here M = the bimodule given by the left B-module B with right mult, or with l + r reversed.

$$\begin{pmatrix} B & B \\ B & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B & 0 \\ B & 0 \end{pmatrix}$$

i.e. assoc. to dual pairs B, $B \oplus B$, $(B \oplus B) \otimes B \xrightarrow{\langle\,\rangle} B$
$(b_1, b_2) \otimes b \mapsto b_1 b$

or $B \oplus B$, B, $B \otimes (B \oplus B) \to B$
$b \otimes (b_1, b_2) \mapsto b b_1$

~~We have that~~

The first case is $Q \otimes_B P =$
You probably have them wrong.

$$\begin{pmatrix} B & 0 \\ B & B \end{pmatrix} \simeq \overset{Q}{(B \ \ B)} \otimes_B \overset{P}{\begin{pmatrix} 0 \\ B \end{pmatrix}}$$

$$\begin{pmatrix} 0 \\ B \end{pmatrix} \otimes (B \ \ B)$$
$$b \otimes (b_1, b_2) \mapsto b b_2$$

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$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

B left flat $\Rightarrow$ $P = \begin{pmatrix} 0 \\ B \end{pmatrix}$ is B flat $\Rightarrow$ $Q \otimes_B P = A$ is A flat.

B rt flat $\Rightarrow$ $Q = (B \;\; B)$ is B^{op}-flat

$\Rightarrow$ $Q \otimes_B P = A$ is A^{op}-flat.

As a check take 2nd ring.

$$\begin{pmatrix} 0 & B \\ 0 & B \end{pmatrix} = \overset{Q}{(0 \;\; B)} \otimes_B \overset{P}{\begin{pmatrix} B \\ B \end{pmatrix}}$$

$$\begin{pmatrix} B \\ B \end{pmatrix} \otimes (0 \;\; B) \longrightarrow B$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \otimes b \longmapsto b_2 b$$

B left flat $\Rightarrow$ $P = \begin{pmatrix} B \\ B \end{pmatrix}$ is B-flat

$\Rightarrow$ $Q \otimes_B P = A$ is A-flat.

B rt flat $\Rightarrow$ $Q = (0 \;\; B)$ is B^{op} flat $\Rightarrow$ $Q \otimes_B P = A$ is rt flat.

03/03/97 Review material for lectures

firm dual pair over A $\quad (P, Q, \langle , \rangle : Q \otimes P \to A)$

firm M contexts $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

firm ring B together with ~~an~~ Morita equiv. $\Phi : M(A) \simeq M(B)$

these are the objects, maps are

$(P, Q, \langle , \rangle) \to (P', Q', \langle \rangle)$ ~~is a pair (u, v~~

consists of a B^{op}-mod map $u : P \to P'$, and a B-mod map $v : Q \to Q'$

such that $\langle q, p \rangle = \langle u(p), v(q) \rangle$.

A map $\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \to \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$ is a hom. of M contexts extending the identity map on A.

A map $(B, \Phi) \to (B', \Phi')$ ~~is a~~ consists of a morphism $w : B \to B'$ and an ism $\Phi_w \Phi \simeq \Phi'$.

127 Functors

$(P, Q, \langle\ \rangle) \longmapsto \begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix}$ equivs

$(P, Q, \langle q,p\rangle = qp) \longleftarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ forgetful fun.

~~$\longmapsto (B, \Phi)$ where~~

$\Phi = (F, G, \varepsilon : FG \xrightarrow{\sim} 1, \eta : GF \xrightarrow{\sim} 1)$

$$\begin{tikzcd}
\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[r, mapsto] \arrow[d, "\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}"] & (B, \Phi) \arrow[d, "(w, \theta, \xi)"] \\
\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix} \arrow[r, mapsto] & (B', \Phi')
\end{tikzcd}$$

where $F(M) = P\otimes_A M$
$G(N) = Q\otimes_B N$
$\varepsilon : P\otimes_A Q \otimes_B N \longrightarrow N$
$p\otimes q\otimes n \longmapsto (pq)n$

$\theta : w_! F \xrightarrow{\sim} F'$ defd by
$\xi : G w^* \xrightarrow{\sim} G'$

$\theta : B'\otimes_B P\otimes_A M \longrightarrow P'\otimes_A M$
$b'\otimes p\otimes m \longmapsto b'u(p)\otimes m$

$\xi : Q\otimes_B B\otimes_B N' \longrightarrow Q'\otimes_{B'} N'$
$q\otimes \otimes n' \longmapsto$ ~~$v(q)\otimes w(b)n' = v(qb)\otimes n'$~~ $v(q)\otimes n'$

~~nos. $v: Q \to Q'$ is a B^{op} mil isom. $Q\otimes_B N \xrightarrow{\sim} Q'\otimes_B N$~~

$u: P \longrightarrow P'$ is a B-mil iso.

$P \xrightarrow{\sim} B\otimes_B P'$

$B'\otimes_B P \xrightarrow{\sim} B'\otimes_B B\otimes_B P' \xrightarrow{\sim} P'$

$b'\otimes bp \longrightarrow b'\otimes b\otimes p \longmapsto b'\, w(b)u(p) = b'\, w(bp)$

$v: Q \longrightarrow Q'$ is a B^{op}-mil isom

$Q \xrightarrow{\sim} Q'\otimes_B B \qquad Q\otimes_B B' \xrightarrow{\sim} Q'\otimes_B B\otimes_B B' \xrightarrow{\sim} Q'$

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$ given $\rightsquigarrow$ $\left(\begin{matrix} B'\otimes_B P \xrightarrow{\sim} P' \\ Q\otimes_B B' \xrightarrow{\sim} Q' \end{matrix} \right.$

get isom.
$$\begin{pmatrix} A & Q\otimes_B B' \\ B'\otimes_B P & B' \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

Claim $\nearrow$ is the Mcont descr'g $\Phi_w \Phi$.

~~[illegible]~~ A map of Mcont $\rightsquigarrow$ w, $\left(\begin{matrix} \Phi_w \Phi \simeq \Phi' \\ \Phi_w \simeq \Phi' \Phi^{-1} \end{matrix} \right.$

But what's the best way to recover the homom. of Mconts? Suppose given w and $\theta : \Phi_w \Phi \simeq \Phi'$. How do you find u, v? [redacted]

Suppose given (B, Φ), (B', Φ'). Consider all pairs (w, Θ) where w a ring hom and $\theta : \Phi_w \Phi \xrightarrow{\sim} \Phi'$ equiv. $\Theta : \Phi_w \xrightarrow{\sim} \Phi' \Phi^{-1}$:

$$\begin{pmatrix} B & B \\ B & B \end{pmatrix} \xrightarrow{\text{canon}} \begin{pmatrix} B & B\otimes_B B' \\ B'\otimes_B B & B' \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} B & P\otimes_A Q' \\ P'\otimes_A Q & B' \end{pmatrix}$$

So you get $\quad B \longrightarrow P'\otimes_A Q \qquad B \longrightarrow P\otimes_A Q'$

<u>You</u> need to find the simplest form for all this.

Fully faithful: Given $(P, Q, \langle\,\rangle)$, $(P', Q', \langle\,\rangle)$ get $B = P\otimes_A Q$, ~~[illegible]~~ $B' = P'\otimes_A Q'$, and $\Phi = (P\otimes_A -, Q\otimes_B -, \varepsilon, \eta)$ sim. with primes. To show

$$\{(u,v) \mid \cdots\} \longrightarrow (w, \theta, \xi)$$

is bijective. Here $(u,v) \in \operatorname{Hom}_{A^{op}}(P, P') \times \operatorname{Hom}_A(Q, Q')$ s.t. $\langle q, p\rangle = \langle v(q), u(p)\rangle$. $\emptyset$

129 $(\theta, \xi) : \Phi_w \Phi \xrightarrow{\sim} \Phi'$

~~Thus~~. If we take $\Phi = (P\otimes_A -, Q\otimes_B -, \text{etc.})$.

Then $\theta : B'\otimes_B P \otimes_A M \xrightarrow{\sim} P'\otimes_A M$ isom of functors of M!

corresp. firm B', A bimodule map

$$\theta : B'\otimes_B P \xrightarrow{\sim} P'$$

and $\xi : GW^* \xrightarrow{\sim} G'$ $\quad \xi : Q\otimes_B B\otimes_B N' \xrightarrow{\sim} Q'\otimes_{B'} N'$

corresp firm A, B' bimod map is and isom:

$$\xi : Q\otimes_B B' \xrightarrow{\sim} Q'$$

Now ~~this~~ compatible with ε, η.

$$w_! F G w^* \xrightarrow{w_! \varepsilon w^*} w_! w^* \xrightarrow{\alpha} 1$$

$$\begin{array}{ccc}
(B'\otimes_B P) \otimes_A (Q \otimes_B B') & \longrightarrow \; B'\otimes_B B \otimes_B B' \xrightarrow{\alpha} & B' \\
b_1'\otimes p \otimes q \otimes b_2' & \longmapsto & b_1' w(pq) b_2' \\
\downarrow \theta\otimes\xi & & \\
P'\otimes_A Q' & &
\end{array}$$

$$\theta(b_1'\otimes p)\, \xi(q\otimes b_2')$$

Got stuck in today's lecture – how?

Prob: $\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$ given, get

$$B'\otimes_B P \xrightarrow{\sim} P' \qquad Q\otimes_B B' \xrightarrow{\sim} Q'$$

missing the idea ~~that if~~ of extension of scalars for dual pairs?

Given $(A, Q, P, \langle\ \rangle)$ and a homom. $P \otimes_A Q \to B'$, then can extend scalars. Alternative – given ~~...~~ $(Q, P, P \otimes Q \to B)$ and $B \to B'$, then get dual pair $\left(Q \otimes_B B',\ B' \otimes_B P,\ (B' \otimes_B P) \otimes (Q \otimes_B B') \xrightarrow{} B' \otimes_B B \otimes_B B' \to B'\right)$.

How does this work in the present case?

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad \begin{pmatrix} \cancel{A} & Q \otimes_B B' \\ B' \otimes_B P & B' \end{pmatrix} \qquad \text{dual pair}$$

$$\underline{(Q \otimes_B B') \otimes_{B'} (B' \otimes_B P) = A} \qquad \text{if } B \to B' \qquad B \otimes B^{\text{op}}\text{-nil iso.}$$

So what ~~...~~ to understand.

First direction: Given $\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}$, get $\quad w_! F \simeq F'$, $\ G w^* \simeq G'$

i.e. $B' \otimes_B P \xrightarrow[\tilde{u}]{\sim} P'$, $Q \otimes_B B' \xrightarrow[\tilde{v}]{\sim} Q'$

v $\quad B^{\text{op}}$ nil iso $\Rightarrow$ $Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N$

$\Rightarrow$

$$\begin{array}{ccc} P' \otimes_A Q \otimes_B P \otimes_A M & \xrightarrow{\sim} & P' \otimes_A Q' \otimes_B P \otimes_A M \\ \wr\vert & & \wr\downarrow \\ P' \otimes_A M & \text{——} & B' \otimes_B P \otimes_A M \end{array}$$

u $\quad B$-nil isom $\Rightarrow$ $B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$

$\Rightarrow$

$$\begin{array}{ccc} B' \otimes_B B^{(2)} \otimes_B P \otimes_A M & \xrightarrow{\sim} & B' \otimes_B B^{(2)} \otimes_B P' \otimes_A M \\ \Vert & & \wr\downarrow \\ B' \otimes_B P \otimes_A M & & P' \otimes_A M \end{array}$$

[3] So what's at stake? You have so much to do before it becomes clear. What sort of things? Idea: Given $\begin{pmatrix} & v \\ u & w \end{pmatrix}$ get $\begin{pmatrix} w_! F \xrightarrow{\sim} F' \\ G w^* \xrightarrow{\sim} G' \end{pmatrix}$.
how? $\begin{pmatrix} u : P \to P' & \text{is a } B\text{-nil isom}, \\ v : Q \to Q' & \text{— } B^{op}\text{-nil isom}, \end{pmatrix}$

you get $P \xrightarrow{\sim} B \otimes_B P'$, $Q \xrightarrow{\sim} Q' \otimes_B B$
but you don't ~~k~~ yet know that $B \to B'$ is a meq homom. However $F \xrightarrow{\sim} w^* F'$ $\Rightarrow$ $FG' \xrightarrow{\sim} w^*$
$1 \doteq FG \xrightarrow{\sim} w^* F' G$. $F \xrightarrow{\sim} w^* F' \Rightarrow G \simeq G' w_!$
$\Rightarrow F' = F' G F \xrightarrow{\sim} F' G' w_! F = w_! F$. Or $FG' \xrightarrow{\sim} w^* \Rightarrow$
$w_! \xrightarrow{\sim} F' G \Rightarrow w_! F \xrightarrow{\sim} F'$

$Q^\# \to Q'$ B^{op}-nil $\to$ $Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N$

$$\begin{array}{ccc} \Rightarrow \; P' \otimes_A Q \otimes_B P \otimes_A M & \xrightarrow{\sim} & P' \otimes_A Q' \otimes_B P \otimes_A M \\ \Big| & & \Big\downarrow \cong \\ P' \otimes & & B' \otimes_B P \otimes_A M \end{array}$$

$Q \to Q'$ B^{op}-nil isom $\Rightarrow$

$Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N$ this doesn't mean much functor wise.

except when translated to bimodules it says that

$Q \xrightarrow{\sim} Q' \otimes_B B$ so $P' \otimes_A Q \xrightarrow{\sim} P' \otimes_A Q' \otimes_B B = B' \otimes_B B$

so $P' \otimes_A Q \otimes_B P \xrightarrow{\sim} B' \otimes_B B \otimes_B P$

$$\overset{\shortparallel}{P'} \xrightarrow{\sim} B' \otimes_B P$$

There seems to be a difficulty translating v as B^{op}-nil isom into the functors.

 ~~1st attempt to use~~

Try the other side. Take $u : P \to P'$ is a B-nil isom.

$$B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$$\underset{\substack{\parallel \\ P \otimes_A M}}{B^{(2)} \otimes_B P \otimes_A M} \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A M$$

$$\therefore \quad P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} \underset{\substack{\parallel \\ B^{(2)} \otimes_B N'}}{B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N'}$$

$$P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} B^{(2)} \otimes_B N'$$

$$\therefore \quad FG' \xrightarrow{\sim} w^*$$

$$P \otimes_A Q' \xrightarrow{\sim} B^{(2)} \otimes_B B'$$

Better: $u : P \to P'$ B-nil isom

$$\Longrightarrow \quad B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$$\Longrightarrow \quad \underset{\substack{\parallel \\ P \otimes_A M}}{B^{(2)} \otimes_B P \otimes_A M} \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A M$$

$$\Longrightarrow \quad F \xrightarrow{\sim} w^* F'$$

$v : Q \to Q' \leftarrow Q' \otimes_{B'} B'$ B^{op}-nil isoms

$$Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N = (Q' \otimes_{B'} B') \otimes_B N$$

$$G \xrightarrow{\sim} G' w_!$$

$\therefore$ get $\quad F'G \xrightarrow{\sim} w_! \qquad FG' \xrightarrow{\sim} w^*$

$$P' \otimes_A Q \otimes_B N \xrightarrow{\sim} B' \otimes_B N \qquad P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} B^{(2)} \otimes_B N'$$

$$b_1 b_2\, p \otimes q' \otimes n' \longmapsto b_1 \otimes b_2 \otimes u(p) q' n'$$

133 Let's take a break. Go back to the idea of ~~__~~ fixing a left flat B and trying to understand those firm B-modules A equipped with $A \longrightarrow B \otimes \mathrm{Hom}_{B^{op}}(B, B)$ such that $K_*(A) = K_*(B)$.

For example restrict A to be a firm B-module ~~__~~ equipped with a surjection $A \twoheadrightarrow B$

Morita context
$$\begin{matrix} A & B \\ A & B \end{matrix}$$

B left flat $\Rightarrow$ $Q \otimes_B B = B \otimes_B B = B$ is A-flat

If B is right flat the $B \otimes_B P = B \otimes_B A \xrightarrow{\sim} A$ would be A^{op}-flat, and this should gives a map $K_*(B) \to K_*(A)$.
YES.

How hard is it to find a B which is left flat but not right flat. Suppose B arises from firm $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ then B left flat $\Longleftrightarrow Q = Q \otimes_B B$ is A flat

B is rt flat $\Longleftrightarrow P = B \otimes_B P$ is A^{op} flat.

In other words the type of flatness B might have is mirrored in the flatness of the dual pair over A.

Restrict to $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ then B right flat iff $P = A$ is right flat, and B left flover $B \Leftrightarrow B$ is A-flat. I want to find in fact really understand possible A's.

One idea: instead of fixing B and looking at A arising from firm B-modules equipped w. pairing $(B, A, A \otimes B \longrightarrow B)$, let's select another coord. system say $\begin{pmatrix} \Lambda & Y \\ X & B \end{pmatrix}$. Then it might be true that only X is varying "over" $\mathrm{Hom}_\Lambda(Y, \Lambda)$

$Y \otimes X \longrightarrow A$

Check this. If $X \otimes_\Lambda Y = B$ and if $X' \to X$ over Λ then we have $X' \otimes_\Lambda Y \longrightarrow X \otimes_\Lambda Y$

$$\begin{matrix} \Lambda & Y & Y_0 \\ X & B & A \\ & B & A \end{matrix}$$

If $X \otimes_\Lambda Y_0 = A$

Start with $\begin{pmatrix} \Lambda & Y_0 \\ X & B \end{pmatrix}$ and consider $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

The composite m.eq. is
$$\begin{pmatrix} \Lambda & Y_0 \otimes_B A \\ \overset{X}{\overset{\shortparallel}{B \otimes_B X}} & \underset{\underset{B \otimes_B A = A}{\shortparallel}}{X \otimes_\Lambda Y_0 \otimes_B A} \end{pmatrix}$$

So as $\overbrace{A}^{\text{the } B\text{-module}}$ varies, as the dual pair $(B, A, A \otimes B \to B)$ varies, B staying fixed, the corresponding dual pair over Λ: $\left(X, Y_0 \otimes_B A, (Y_0 \otimes_B A) \otimes X \underset{p}{\to} Y_0 \otimes_\Lambda X \to \Lambda\right)$

$A \to \mathrm{Hom}_{B^{op}}(B, B)$
so A acts on $X = B \otimes_B X$.

So it seems that referred to Λ I have the right Λ-module X fixed and the left module Y varying. This agrees when I put B in Λ's place: $\begin{pmatrix} B & A \\ B & A \end{pmatrix}$. So what viewpoint do I have at this point??? YES

What's a good picture? Fix $\begin{pmatrix} \Lambda \\ X \end{pmatrix}$ say Λ unital, X unitary, and consider all possible left unital Λ-modules Y over $\mathrm{Hom}_{\Lambda^{op}}(X, \Lambda)$.

135 Now I want to understand when ~~[illegible]~~ This maybe too hard. Instead, go back to $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ with B fixed and A varying. What do I know?

B is B-flat $\iff$ B is A-flat

B is B^{op}-flat $\iff$ A is A^{op}-flat

$B \overset{Q}{\otimes}_B \widehat{B}$

In either of these cases I know $K_*(A) = K_*(B)$

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

P_A flat $\iff$ $P \otimes_A Q = B_B$ flat

${}_B P$ flat $\iff$ $Q \otimes_B P = {}_A A$ flat

$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

A_A flat $\iff$ $A \otimes_A B = B_B$ is flat

${}_B A$ flat $\iff$ $B \otimes_B A = {}_A A$ flat.

${}_A B$ flat $\iff$ ${}_B B$ flat

same

B_B flat $\iff$ A_A flat

So let's start with B firm, and consider firm dual pairs of the form $(B, A, \langle\ \rangle : A \otimes B \twoheadrightarrow B)$. Assume B is left flat, but not right flat. Then B is h-unital and we know that $A = B \otimes_B A$ is h-unital iff $B \overset{L}{\otimes}_{\widehat{B}} A \xrightarrow{\sim} A$, e.g. if A is a flat B^{op} module. ~~We can produce~~ Since B is not right flat we ought to be able to arrange this condition to fail. Then ~~there is nothing~~ look at $K_*(A)$ versus $K_*(B)$. Example: Suppose we take $A = B \oplus M$ where $\langle M, B \rangle = 0$ and $\langle b_1, b_2 \rangle = b_1 b_2$. Then $B \overset{L}{\otimes}_B (B \oplus M) = B \oplus B \overset{L}{\otimes}_B M$.

 More generally if A

$$0 \to M \to A \to B \to 0$$

is an extension of B, then we have a Δ

$$\begin{array}{ccccc} B \overset{L}{\underset{B}{\otimes}} M & \longrightarrow & B \overset{L}{\underset{B}{\otimes}} A & \longrightarrow & B \overset{L}{\underset{B}{\otimes}} B \\ \downarrow & & \downarrow & & \| \\ M & \longrightarrow & A & \longrightarrow & B \end{array}$$

so A is h-unitary iff M is. Then you want to see problems arising with homology. If B is left flat, then $A = B \oplus M$ should give problems with Morita invariance if M is not h-unitary.

Now let's look at K_1. I shouldn't see anything here for firm rings. But in any case you can look at $K_1(B \oplus M)$ as a functor of M

03/04/97 0632

Lecture A firm

cat of firm dual pairs over A $(P, Q, \langle\, \rangle : Q \otimes_{\mathbb{Z}} P \to\to A)$

cat of firm M. cont. with upper left corner A $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

cat of firm rings equipped with a meq to A $(B, F : m(A) \to m(B))$

given

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

what's the simplest way to show

$$B' \otimes_B P \xrightarrow{\sim} P', \quad Q \otimes_B B' \xrightarrow{\sim} Q'$$

Note these imply

$$B' \otimes_B B \otimes_B B' \xleftarrow{\sim} B' \otimes_B P \otimes_A Q \otimes_B B' \xrightarrow{\sim} P' \otimes_A Q' = B'$$

$$B \otimes_B B' \otimes_B B \xleftarrow{\sim} Q \qquad ?$$

$u : P \to P'$ B-mil isom

$v : Q \to Q'$ B^{op}-mil isom.

$\Rightarrow$ $P = B\otimes_B P \xrightarrow{\sim} B\otimes_B P'$ $\quad bp \mapsto b\otimes u(p)$

$Q = Q\otimes_B B \xrightarrow{\sim} Q'\otimes_B B$ $\quad qb \mapsto v(q)\otimes b$

$\Rightarrow$ $A = Q\otimes_B P = (Q'\otimes_B B)\otimes_B (B\otimes_B P') = Q'\otimes_B B\otimes_B P'$

$B' = P'\otimes_A A\otimes_A Q' = P'\otimes_A Q'\otimes_B B\otimes_B P'\otimes_A Q' = B'\otimes_B B\otimes_B B'$

earlier should have put

$B = P\otimes_A Q = B\otimes_B P'\otimes_A Q'\otimes_B B = B\otimes_B B'\otimes_B B$

$b_1 p q b_2 \mapsto$ $\quad b_1\otimes w(pq)\otimes b_2$

$v : Q \to Q'$ B^{op}-mil isom

$Q\otimes_B N \xrightarrow{\sim} Q'\otimes_B N$

$P'\otimes_A Q\otimes_B P\otimes_A M \xrightarrow{\sim} P'\otimes_A Q'\otimes_B P\otimes_A M$

$P'\otimes_A M \xrightarrow{\sim} B'\otimes_B P\otimes_A M$

$p'\otimes qpm \mapsto p'v(q)\otimes p\otimes m$

$b'u(p)\otimes m \leftarrow b'\otimes p\otimes m$

$v : Q \to Q'$ is a B^{op}-mil isom

$\therefore Q = Q\otimes_B B \xrightarrow{\sim} Q'\otimes_B B$

$P' = P'\otimes_A Q\otimes_B P \xrightarrow{\sim} P'\otimes_A Q'\otimes_B P = B'\otimes_B P$

$p'qp$ $\quad p'\otimes q'\otimes p \mapsto p'\otimes v(q)\otimes p \mapsto p'v(q)\otimes p$

$b'u(p) \leftarrow b'\otimes p$

Similary $u : P \to P'$ B-mil isom.

$P = B\otimes_B P \xrightarrow{\sim} B\otimes_B P'$

$Q' = Q\otimes_B P'\otimes_A Q' \xrightarrow{\sim} Q\otimes_B B\otimes_B P'\otimes_A Q' = Q\otimes_B B'$

qpq' $\quad q\otimes u(p)q'$

 u B-nil isom. $\Longrightarrow$ $P = B\otimes_B P \xrightarrow{\sim} B\otimes_B P'$

This says $F \simeq w^*F'$ so w^* is an equivalence (with inverse $\simeq FG'$) so w is a meg hom. ~~Also get~~

~~$w_!F \simeq w_!w^*F' \xrightarrow{\sim} F'$ WHY?~~

Also the isom $F \xrightarrow{\sim} w^*F'$ on passing to induced adjoints becomes $G'w_! \xrightarrow{\sim} G$ $\qquad Q'\otimes_B \cancel{B'}\otimes_B B \xrightarrow{\sim} Q$

$$P \xrightarrow{\sim} B\otimes_B P'$$

$$\begin{array}{lccc}
 & & q \quad b \quad u(p) \quad q' & \\
q\otimes p\otimes q' & Q\otimes_B P\otimes_A Q' \xrightarrow{\sim} & Q\otimes_B B\otimes_B P'\otimes_A Q' & q \\
 & \| & \| & \\
qpq' & Q' \xrightarrow{\sim} & Q\otimes_B B' & qb\otimes u(p)\otimes q' \\
qpq' & qpq' \longmapsto & q\otimes u(p)q' &
\end{array}$$

$$Q = Q\otimes_B B \xrightarrow{\sim} Q'\otimes_B B$$

$$\begin{array}{ccc}
P'\otimes_A Q\otimes_B P & \xrightarrow{\sim} & P'\otimes_A Q'\otimes_B P \\
\| & & \| \\
P' & \xrightarrow{\sim} & B'\otimes_B P
\end{array}$$

Spend time on left Morita equivalences. ~~Th~~ These are Meg's of the form $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ and they are given by a dual pair $(A, B, B\otimes_{\mathbb{Z}} A \to A)$ over A, or by $(B, A, A\otimes_{\mathbb{Z}} B \to B)$. More generally if we start with A and form right mod P, then keeping these fixed by allowing Q to vary, here $Q \to A\otimes \mathrm{Hom}_{A^{op}}(P, A)$, better Q is equipped with $Q\otimes_{\mathbb{Z}} P \to A$

Then the resulting rings $P\otimes_A Q$ as Q varies are all left Morita equivalent, namely the actual ring modules are the same

$$\begin{array}{ccc} A & Q & Q' \\ P & B & B' \\ P & B & B' \end{array}$$

Now at some point we have to ~~find~~ handle K-theory.

So let's fix B and consider dual pairs $(B, A, A\otimes B \to B)$. I'm interested in the functor $K_*(A)$, e.g. $K_1(A)$, $K_1 = H_1(GL(A))$. The only thing I can see worthwhile doing is to assume $A \to B$ surjective whence we have a group ext

$1 \to GL(I) \longrightarrow GL(A) \longrightarrow GL(B) \to 1$

Actually, it might help to take B idempotent, then consider $A \twoheadrightarrow B$ A a firm B-module. Other things I can do - take $A = B \oplus M$, M a B-module. You need to get some feeling for the cats.

The cat of B-modules A equipped with pairing $A\otimes B \longrightarrow B$, has a final object, namely $\mathrm{Hom}_{B^{op}}(B,B)$ if A any B-module and $B\otimes_B \mathrm{Hom}_{B^{op}}(B,B)$ if A is firm. What about variations?

Let's write up things cleanly.

Assume B flat. Show the rings $\left(\begin{array}{c|c} B & B \\ \hline 0 & 0 \end{array}\right)$ and $\left(\begin{array}{c|c} B & 0 \\ \hline B & 0 \end{array}\right)$ have the same K-theory as B. Shift notation to A.

$$\left(\begin{array}{c|c} A & A \\ \hline 0 & 0 \end{array}\right) = \begin{pmatrix} A \\ 0 \end{pmatrix} \otimes_A \begin{pmatrix} A & A \end{pmatrix}$$

$$B = \left(\begin{array}{c|c} A & A \\ \hline 0 & 0 \end{array}\right) = \underbrace{\begin{pmatrix} A \\ 0 \end{pmatrix}}_{P=A} \otimes_A \underbrace{\begin{pmatrix} A & A \end{pmatrix}}_{Q}$$

A left flat $\Longrightarrow$ Q is A flat $\Longrightarrow$ $\overset{P\otimes_A Q}{\overset{\|}{B}}$ is left flat
So both A, B are left flat ~~so we h~~ and meq so
$K_*A = K_*B$.

A rt flat $\Longrightarrow$ $P = A$ is A^{op} flat
$\Longrightarrow$ $P \otimes_A Q = B$ is B^{op} flat.

So in both cases the point is ~~that the assumption that A is flat implies the con~~

P left B ft $\Longleftrightarrow$ A left flat $\Longleftrightarrow$ Q is left flat $\Longleftrightarrow$ B is left flat
Q rt flat $\Longleftrightarrow$ A rt flat $\Longleftrightarrow$ P is rt flat $\Longleftrightarrow$ B is rt flat

general — nature of $(P, Q, \langle\ \rangle)$ over A — general

but is this enough? $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

Let's see what I can do with flatness. ~~Consider a~~ ~~study~~ The goal would be to prove somehow that if $A \to B$ is a surjective B-module map such that A is flat and B is h-unital, then $K_*(A) \cong K_*(B)$.

$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ A is A-flat and B acts on A so there is a homom. $K_*(B) \longrightarrow K_*(A)$ NO This is wrong because B acts on the left on A, so you [redacted] need A right flat.
All you know is that A A-flat $\Rightarrow$ $P \otimes_A A = P$ is B-flat so A acting on the right of A yields $K_*A \to K_*B$.