1 02/14/97 1513

~~Discuss~~ Review M invariance of K_* for firm rings.

Key reductions. In view of Suslin's theorem the critical case I think is meq rings A, B which are both left and right flat. Go over this.

Start with B idempotent. We can choose a surjective homom. $f: A \to B$ of ~~[illegible]~~ B^{op}-modules with A a firm flat B^{op}-module. ~~Then have f dual Make~~ Define $a_1 a_2 = a_1 f(a_2)$, find f is a homom. so $A/I \xrightarrow{\sim} B$, $I = \ker(f)$ satisfies $AI = 0$. Get M cont. $\begin{pmatrix} A & A \\ A/I & A/I \end{pmatrix}$

so A and B are meq. In fact, since $AI = 0$ $UI = (U \otimes_A A)I = 0$ for any firm A^{op}-module, and one can see $M(A^{op}) = M(A/I^{op})$. Check $U \mapsto U \otimes_A A = U$. Also $A = Q$ is B^{op}-flat $\Rightarrow Q \otimes_B P = A \otimes_B B = A$ is A^{op}-flat.

~~Note that B is firm iff [illegible]~~

~~Thus~~ Recall that all A is one-sided flat we know that B is h-unital $\Leftrightarrow$ $P \overset{L}{\otimes}_A Q = A/I \overset{L}{\otimes}_A A = A/I$ i.e. $I \overset{L}{\otimes}_A A = I$. ~~So~~ Should have A/I ~~[illegible]~~ firm iff $IA = I$. Point is that $A/I \in M(A^{op})$ iff it $\in M(A/I^{op})$. Condition is then that $A/I \otimes_A A \xrightarrow{\sim} A/I$ $\left(= A/IA\right)$

clear. Now Suslin should handle the case of the extension $A \to A/I$. ~~[illegible]~~ <u>Observe</u>: You don't apply Suslin's excision thm. to this extension because $I^2 \subset AI = 0$. I definitely isn't h-unital.

~~Keep on going. Next. Next.~~

2 Summarize. The key point in this example is that we need to handle the one-sided case $A \to A/I \simeq B$ where A is right flat, $AI = 0$ and $I \overset{L}{\otimes}_A A = I$. This is handled by Suslin's work, I think, because you have LHS spec. seq. Do I understand this at all?

$$E^2_{pq} = H_p(GL(A/I), H_q(GL(I))) \Rightarrow H_{p+q}(GL(A)).$$

$$0 \to I \to A \to A/I \to 0 \qquad AI = 0$$

so I is an A-module regarded as a bimodule with 0 right multiplication. Since $I^2 = 0$, we have

$$1 \to GL(I) \to GL(A) \to GL(A/I) \to 1$$

$GL(I) \cong M(I)$ the additive abelian group and the conjugation action of $GL(A/I)$ is just the natural left mult. I is an A/I-module so matrices over A/I act on matrices over I. You restrict to invertible matrices. ~~Now you have to understand the reduction of the functor H_*~~ You need to prove that

$$E^2_{pq} = 0 \quad \text{for } q \geq 1.$$

First step is $q=1$, where $H_1(GL(I))$ is $M(I)$. So now we get down to $H_*(GL(A/I), M(I))$ being zero. Then comes higher coh. If you work over $\mathbb{Q}$ you find $H_*(M(I), \mathbb{Q}) = \Lambda^*_{\mathbb{Q}}(M(I) \otimes_{\mathbb{Z}} \mathbb{Q})$. Suslin must understand leading terms. YES.

But it's likely that

Question: If $A \overset{L}{\otimes}_A I \simeq I$, this means that I is an h-unitary module over A, the question is whether you expect this to imply that

$$H_p\big(GL(A), H_q(M(I), \mathbb{Z})\big) = 0 \qquad q \geq 1.$$

or do you expect to need some h-unital condition on A in addition. In the situation I am looking at A is right flat, but I don't know if this is a relevant point.

How am I to fix this???? NO WAY.

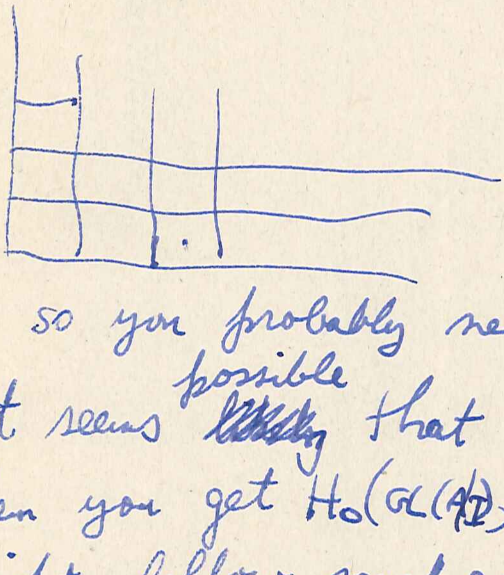
Maybe look at low degrees.

For $K_1 = GL(A)_{ab}$

$$\to H_0\big(GL(A/I), H_1(M(I))\big) \to H_1(GL(A)) \to H_1(GL(A/I)) \to 0$$

$H_1(M(I)) = M(I)$

$$H_2(GL(A)) \to H_2(GL(A/I)) \to$$



What do you need for K_2?

You want $H_2(GL(A)) \twoheadrightarrow H_2(GL(A/I))$

so you probably need $H_1\big(GL(A/I), H_1(M(I))\big) = 0$.

It seems possible that once you have $H_0(GL(A), M(I)) = 0$ then you get $H_0\big(GL(A/I), H_q(M(I))\big) = 0$ for $q \geq 1$. This step might follow somehow by stabilizing. The rough idea is that ~~M(I) enters~~ you want to use the fact that the columns are subrepresentations. Thus let $V_n(I)$ denote column vectors of length n over I. $H_q(M_n(I))$ is some messy nonlinear functor. Goodwillie's calculus

4 idea? Let's try to push this through dim 2.

B a ring, I a B-module, propose to link h-unitality condition $B\overset{L}{\otimes}_B I = I$ to $H_*(GL(B), V(I))$ somehow.

Is there some way to link (↑) with $A\overset{L}{\otimes}_A I = I$?

$B = A/I$. Question. Go back to $\begin{pmatrix} A & A \\ A/I & A/I \end{pmatrix}$ ~~$\cdot$~~ $=0$ ~~$AI=0$~~

Get left + right straight. $A \xrightarrow{f} B$ $\quad a_1 a_2 = a_1 f(a_2)$ $\therefore$ ~~$AI=0$~~

so $M(I)$ is trivial on the left, nontrivial on right

Go to other case then $A \to A/I = B$ where $AI \neq 0$ ~~$IA=0$~~

$\begin{pmatrix} A & A/I \\ A & A/I \end{pmatrix}$. When is B h-unital? Suff. condition is A one sided flat and $Tor^{\tilde{A}}_{>0}(A, A/I) = 0$.

$\therefore A \overset{L}{\otimes}_{\tilde{A}} A/I = A/I$ Why

$$\begin{array}{ccc} B\overset{L}{\otimes}_B P \overset{L}{\otimes}_A Q & \longrightarrow & B\overset{L}{\otimes}_A B \\ \downarrow & & \downarrow \\ P\overset{L}{\otimes}_A Q & \longrightarrow & B \end{array}$$

If A left flat then $P\otimes_A A \otimes P$ is left B-flat qis

$$\begin{array}{ccc} B\overset{L}{\otimes}_B A \overset{L}{\otimes}_A B & & B\overset{L}{\otimes}_B B \\ & & \downarrow \\ A\overset{L}{\otimes}_A B & \longrightarrow & B \end{array}$$

What can you say about $B\overset{L}{\otimes}_B A$?

$B\overset{L}{\otimes}_B I \longrightarrow B\overset{L}{\otimes}_B A \longrightarrow B\overset{L}{\otimes}_B B \longrightarrow$

If B is h-unital, then $B\overset{L}{\otimes}_B I = 0$ as I is ·

If A is B flat and B is h-unital, then $A\overset{L}{\otimes}_A B = B$

then A is A flat

$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $B = A/I$ where $\boxed{IA = 0}$

A is A-flat $\iff$ $A \otimes_A A = A$ is B-flat. Then

$$\begin{array}{ccccccc}
B \overset{L}{\otimes}_B I & \to & B \overset{L}{\otimes}_B A & \to & B \overset{L}{\otimes}_B B & \to \\
\downarrow & & \downarrow & & \downarrow & \\
I & \to & A & \to & B & \to
\end{array}$$

so we do get $B \overset{L}{\otimes}_B I = I$

Because A is A-flat, know B h-unital if $A \overset{L}{\otimes}_A B = B$ i.e. $A \overset{L}{\otimes}_A I = 0$. I'm still not getting very far.

~~A→B~~ $A \to A/I = B$ $IA = 0$

~~Relate~~ Relate $A \overset{L}{\otimes}_A I = I$ to $B \overset{L}{\otimes}_B I = I$.

$\iff$ I has a resolution by firm flat A-modules and the firm flat module cats for A and B are ~~equiv~~ the same.

Actually this is good idea. Look at $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ again. Suppose A left flat so that P is B flat firm. Take a firm flat B^{op}-resolution F mod nil modules of Q. ~~Then~~ Consider $F \otimes_B P$. π

B is h-unital iff $\exists$ firm flat resolution of B, say a complex F of ~~ff~~ flat firm B-modules + quis $F \to B$. Then $Q \otimes_B F$ ~~...~~ $\mathrm{Tor}_n^{\tilde{B}}(Q, B)$

Suppose $A \twoheadrightarrow A/I = B$ $IA = 0$ so that $M(A) = M(B)$ on the nose. Now A is h-unital iff h-unitary A-module, means ∃ resolution by flat firm A-modules, and then A is h-unitary over B. Thus it seems that B is h-unital iff B is h-unitary over A (or B).

Observe that if B is h-unitary then we can can organize the possible A's which are flat firm having surj $A \to B$ whose kernel $I \ni IA = 0$. Is it possible to enlarge B somewhat? I guess we look at the left mult. alg, which is $\mathrm{Hom}_{A^{op}}(A,A)$

$$\begin{pmatrix} A & A/I \\ A & A/I \end{pmatrix} \quad A \twoheadrightarrow B \to \mathrm{Hom}_{A^{op}}(A,A)$$

Look at Mcontexts with P fixed,
$$\begin{pmatrix} A & A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) \\ P & P \otimes_A \mathrm{Hom}_{A^{op}}(P,A) \end{pmatrix}$$
(with $\mathrm{Hom}_{A^{op}}(P,P)$ crossed out)

So the maximum B is $A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$

Here you should think of A as a left ideal in $\mathrm{Hom}_{A^{op}}(A,A)$, and then $A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ is roughly the ideal it generates. Basically ind of A, depends only on the gen. P. If you start with P you get the ring $\underbrace{P \otimes_A \mathrm{Hom}_{A^{op}}(P,A)}_{\text{ideal of finite rank ops.}}$ "$\subset$" $\mathrm{Hom}_{A^{op}}(P,P)$.

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Consider $A \twoheadrightarrow A/I = B$ where $IA = 0$ ~~and~~ recall that $M(A) = M(B)$ strongly. I want eventually to prove that $BGL(A)^+ \xrightarrow{\sim} BGL(B)^+$ using the HLS spec. seq for $1 \to M(I) \to GL(A) \to GL(B) \to 1$. to prove $H_p(GL(B), H_q(M(I))) = 0$ for $q \geq 1$. This should require B to be h-unital.

First example – $H_0(GL(B), M(I))$

Let's place ourselves in the rational homology situation. Note that $M(I) = V(I) \otimes_{\mathbb{Z}} \mathbb{Z}^{(S)}$ where $V(I) = M_{\infty,1}(I)$ column vectors over I. So rationally

$$H_*(M(I)) = \Lambda^*_{\mathbb{Q}}(M(I) \otimes_{\mathbb{Z}} \mathbb{Q})$$

$$= \Lambda^*_{\mathbb{Q}}(V(I) \otimes_{\mathbb{Z}} \mathbb{Q}^{(S)})$$

so as a representation of $GL(B)$ it ~~involves~~ splits into tensor powers of $\Lambda^j V(I) \otimes \mathbb{Q}$. ~~So the~~ Using that ~~exterior~~ $\Lambda^q V$ is a summand of $V^{\otimes q}$ the vanishing result we need is that $H_*(GL(B), V(I)_{\mathbb{Q}}^{\otimes q}) = 0$ for $q \geq 1$. Get act together.

~~[illegible]~~ It seems that you need some machine for analyzing such homology, stable homology. ~~[illegible]~~ One question to ask concerns extensions. Suppose B given and I a B-module. ~~And an~~ Consider any extn.

$$0 \to I \to A \to B \to 0$$

Then these are all B mods.

1107 make serious effort

~~Main~~ focus on what's needed. ~~Suppose~~ Assume B h-unital, and let's consider A meq B in a special way i.e. $\mathcal{M}(A) = \mathcal{M}(B)$ strictly. We know how to describe these A, namely as firm B-modules equipped with $A \otimes B \xrightarrow{\langle , \rangle} B$ surjective

$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ B-bimodule map, equivalently with B-module map $A \xrightarrow{\phi} \mathrm{Hom}_{B^{op}}(B, B)$ which is sufficiently nonzero, which I guess means that A generates the ideal $B\,\mathrm{Hom}_{B^{op}}(B, B)$. Let's check this carefully before proceeding. I need the pairing $\langle , \rangle : A \otimes B \longrightarrow B$ to be surjective, i.e. in the Mcontext $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ I need $AB = B$. If this is true then ~~...~~ ?

$$B\,\mathrm{Hom}_{B^{op}}(B,B) = AB\,\mathrm{Hom}_{B^{op}}(B,B) \subset A\,\mathrm{Hom}_{B^{op}}(B,B)$$
$$\subset BA\,\mathrm{Hom}_{B^{op}}(B,B) \subset B\,\mathrm{Hom}_{B^{op}}(B,B)$$

This seems correct, but it requires clarification

So we are in effect considering B-modules maps $A \xrightarrow{\phi} B \otimes_B \mathrm{Hom}_{B^{op}}(B,B)$ ^ A-B pairing such that the ϕA generates the ring ↑ as ideal. ~~A first step~~ An ~~...~~ important step will be to show that for ~~A~~ any A flat of this sort $K_* A$ is the same.

I'm trying to get a clear picture of all the implications in this ~~situation~~. It seems you know something special about the ring $B \otimes_B \mathrm{Hom}_{B^{op}}(B,B)$. ~~More~~

For the moment let's worry about B-module surjections $A \twoheadrightarrow B$. Assuming B is h-unital I want to know when A is h-unital. Use $\mathcal{M}(A) = \mathcal{M}(B)$, then A is h-unital iff A is h-unitary as A-module, which is equiv to A having a flat firm A-res., equiv. a flat firm B resolution i.e. A being h-unitary as B-module.

~~So not find~~ Assuming B h-unital, then B is h-unitary as B-mod, so A is h-unitary $\Leftrightarrow$ $I = \mathrm{Ker}(A \to B)$ is h-unitary over B. ~~...~~ Our aim is to show that $H_*(GL(A)) \xrightarrow{\sim} H_*(GL(B))$. This follows from $H_*(GL(B), M(I)^{\otimes 8}) = 0$ at least rationally. So as a consistency check it would be nice to know that

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1345 viewpoint - taking a derivative, ~~this~~ linearizing a functor.

Let's begin with one-sided Morita equivalences. Try to understand Morita invariance for one-sided Morita equivalences. To define: Ex. inclusion of a left ideal $A \subset B$ generating B as ideal.

$$A \subset B \qquad BA \subset A \qquad AB = B$$

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

Also surjection $f: A \twoheadrightarrow A/I = B$ where $IA = 0$

~~...~~ Can combine these two to a B-mod map $f: A \longrightarrow B$ such that $f(A)B = B$, where $a_1 a_2 = f(a_1) a_2$

What's important is that we have bimodules $P = {}_B A_A$ and $Q = {}_A B_B$ ~~together with ... $B \otimes A$ ... such that~~ ${}_B\!\longrightarrow$

such that $B\otimes A \twoheadrightarrow A$, $A\otimes B \twoheadrightarrow B$

anything else? $(pq)p' = p(qp')$ $(qp)q' = q(pq')$?

A B A $(ab)a' =$

A ring, B ring, [redacted] $A \to Hom_{B^{op}}(B,B)$

$B \to Hom_{A^{op}}(A,A)$

and $(ab)a' = a(ba')$ too hard.

1510 concept of one-sided meq. Fix B firm. [redacted]

[redacted] Consider a firm B-module A together with B-bimod map $A\otimes B \twoheadrightarrow B$, equivalently a firm triple $A\otimes B \twoheadrightarrow B$ over B, ~~of a ring~~ ~~$\begin{pmatrix}A & P\\ A & B\end{pmatrix}$~~ equivalently a [redacted] B-module map $A \to B\otimes_B Hom_{B^{op}}(B,B)$ generating the latter as left ideal. too confusing.

Fix B firm. Consider all firm triples $A\otimes B \xrightarrow{\langle\rangle} B$ over B, equiv. Mcont $\begin{pmatrix}A & B\\ A & B\end{pmatrix}$. Have $\mathcal{M}(A) = \mathcal{M}(B)$ strongly. ~~[illegible]~~ Such an $A, \langle\rangle$ should amount to a B-module map $A \to Hom_{B^{op}}(B,B)$ whose image generates the ideal $B\,Hom_{B^{op}}(B,B)$. These pairs $(A, \langle\rangle)$ form a category with a final object namely ~~[illegible]~~ $B\otimes_B Hom_{B^{op}}(B,B)$.

Next ask whether this

Special feature of $I = B\otimes_B Hom_{B^{op}}(B,B)$ is that

[redacted] $I \xrightarrow{\sim} I\,Hom_{I^{op}}(I,I)$. ~~For some reason B~~

[redacted] The question is how intrinsic? Start with $\mathcal{M}$ and a choice of $P: \mathcal{M} \to Ab$ right continuous. can then look at the possible $Q \in \mathcal{M}$ such that $Q\otimes P \twoheadrightarrow 1$. [redacted] To such a Q you get $P\otimes_A Q = B$ firm ring. My

Change notation but be intrinsic. Consider a Rory cat $\mathcal{M}$ and a generator of the dual $\mathcal{M}^\vee$.
I need names. Suppose $\mathcal{M} = \mathcal{M}(A)$, $P \in \mathcal{M}(A^{op})$.
Then I look at Q in $\mathcal{M}$ such that $Q \otimes P \twoheadrightarrow id$

Let's change from coords (A, A, μ) to $(Q, P, \langle , \rangle)$, whence
$\mathcal{M} = \mathcal{M}(B)$ and

$$\begin{array}{ccc} \mathcal{M}(A^{op}) & \xrightarrow{\sim} & \mathcal{M}(B^{op}) \\ P & \longmapsto & P \otimes_A Q = B \end{array}$$

whereas

$$\begin{array}{ccc} \mathcal{M}(A) & \xrightarrow{\sim} & \mathcal{M}(B) \\ Q & \longmapsto & P \otimes_A Q = B. \end{array}$$

So we can assume that $\mathcal{M} = \mathcal{M}(B)$
$P = B \in \mathcal{M}(B^{op})$ and now we can look at the possible firm dual pairs $P' \otimes B$ [redacted] $\twoheadrightarrow B$

Can assume $\mathcal{M} = \mathcal{M}(B)$ and $B \in \mathcal{M}(B^{op})$ is fixed
Look at possible $Q \otimes B \twoheadrightarrow B$.

Again:

So the idea is that I have. Let's get the argument straight about the final object. If $P \in \mathcal{M}(A^{op})$ is a generator, then what? $Q \otimes P \to A$ yields
$Q \longrightarrow Hom_{A^{op}}(P, A)$ whence $Q \longrightarrow A \otimes_A Hom_{A^{op}}(P, A)$. Look at

$$P \otimes_A Q \longrightarrow P \otimes_A Hom_{A^{op}}(P, A) \longrightarrow Hom_{A^{op}}(P, P)$$

Spend next hour trying to straighten the rest out.

I have to concentrate on this. What I know in general is

$$Hom_{A^{op}}(P, P) \times Hom_A(Q, Q)^{op} \ni (\phi', \phi'') \qquad \langle q\phi'' | p \rangle = \langle q | \phi' p \rangle$$

$$\|$$

$$Hom_{B^{op}}(B, B) \times Hom_B(B, B)^{op} \qquad (p_1 q_1 \phi'') p q_1 = p_1 q (\phi' p q_1)$$

So therefore you get the multiplier ring.

Consider $B \longrightarrow \mathrm{Hom}_{B^{op}}(B, B) = R$

$$r(b\cdot) = \{b' \longmapsto r(bb')\} = r(b)\cdot$$
$$r(bb') = r(b)\,b'$$

so the image is a left ideal, and

Argument: $Q \otimes P \longrightarrow A$

$$Q \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$

OKAY

If this map is ~~[illegible] then~~ an isom, then

$$\mathrm{Mult}(Q \otimes P \to A) = \mathrm{Hom}_{A^{op}}(B, P)$$

So if $B \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P, A) = P \otimes_A \mathrm{Hom}_{B^{op}}(B, Q)$?

$$Q \xrightarrow{\sim} A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$
$$\| $$
$$\mathrm{Hom}_{B^{op}}(B, Q)$$

Basic idea is that $\mathrm{Mult}(Q \otimes P \to A)$ should be Morita inv. $\therefore$ if $Q \xrightarrow{\sim} A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$ OKAY

~~then~~

Consider property of $Q \otimes P \to A$ that ~~it is final obje~~ $Q \xrightarrow{\sim} A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$, this means it is the final object of the cat of triples with P fixed. But cat triples over A $\simeq$ cat of triples over B. So

$$\{Q' \otimes P' \longrightarrow A\} \rightsquigarrow \{(P \otimes_A Q') \otimes (P' \otimes_A Q) \longrightarrow P \otimes_A Q = B\}.$$

$$\{Q \otimes P \longrightarrow A\} \rightsquigarrow \{B \otimes B \longrightarrow B\}$$

so $\{B \otimes B \longrightarrow B\}$ should be final in trips over B with 2nd comp B i.e. $B \xrightarrow{\sim} B \otimes_B \mathrm{Hom}_{B^{op}}(B, B)$.

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$$Q \xrightarrow{\sim} A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$

$$P \otimes_A Q \xrightarrow{\sim} P \otimes_A \mathrm{Hom}_{B^{op}}(B, Q)$$

$$\|$$

$$B \otimes_B P \otimes_A \mathrm{Hom}_{B^{op}}(B, Q)$$

$$\downarrow$$

$$B \otimes_B \mathrm{Hom}_{B^{op}}(B, B)$$

$$\|$$

$$B \otimes_B \mathrm{Hom}_{B^{op}}(P, P)$$

$$p \otimes (b \mapsto f(b)) \mapsto (b \mapsto pf(b))$$

There's a lot here I don't understand. I really must work on this.

i) Fix $P_0 \in \mathcal{M}(A^{op})$ consider all fdp with $P = P_0$. These form a cat. equiv. to form $Q \longrightarrow \mathrm{Hom}_{A^{op}}(P_0, A)$ such that $Q \otimes P_0 \twoheadrightarrow A$ surj. ∎

Start with A form P_0 gen. for $\mathcal{M}(A^{op})$. Consider cat of fdp $Q \otimes P_0 \twoheadrightarrow A$, equiv $Q \xrightarrow{A\otimes_A} \mathrm{Hom}_{A^{op}}(P_0, A)$ such that corresp map $Q \otimes P_0 \twoheadrightarrow A$. This cat $\neq \phi$ since P_0 gen.

Ask what $Q \otimes P \twoheadrightarrow A$ means.

$$P \otimes_A Q \longrightarrow \underbrace{P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)}_{J} \xrightarrow{\alpha} \mathrm{Hom}_{A^{op}}(P, P)$$

$\mathrm{Im}(\alpha)$ is an ideal J in $\mathrm{Hom}_{A^{op}}(P, P) = R$.

my idea is that $PQR = J$. First point is that

Assume $QP = A$. $\bar{B}$ = span of $p\theta : P \to P$ in R.

sitting inside $\bar{B}$ in $\overline{PQ}$ " " $pq : P \to P$

We assume $QP = A$. So ask about $QR < \mathrm{Hom}_{A^{op}}(P, A)$

Given $P \xrightarrow{\theta} A$

14 02/16/97 So I still would like to understand why $B = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$ has the property that $B \longrightarrow \mathrm{Hom}_{B^{op}}(B, B)$ is a B-mod isom. Let $R = \mathrm{Hom}_{A^{op}}(A, A)$ be the left multiplier alg. Let B be the ideal in R spanned by products $\lambda_a r$. ~~Note $(r\lambda_a)(b') = r(bb') = r(b)b' = \lambda_{rb}(b')$. Thus λ_B.~~ $\lambda(A)$ is a left ideal in R $(r\lambda_a)(a') = r(aa') = r(a)a' = \lambda_{r(a)}(a')$. The first point is that we have the M-context $\begin{pmatrix} A & R \\ A & B \end{pmatrix}$

Let's set up again. ~~Let A be idempotent~~ Start with A firm, set $R = \mathrm{Hom}_{A^{op}}(A, A)$, let $\bar{A} = A/\{a \mid aA = 0\} = \mathrm{Im}\{\lambda : A \to R\}$. $\bar{A}$ is the reduced version of the right A-module A, so I know that $R = \mathrm{Hom}_{A^{op}}(\bar{A}, \bar{A}) = \mathrm{Hom}_{\bar{A}^{op}}(\bar{A}, \bar{A})$. At this point I want to simplify notation. No.

~~To begin with A~~ $\bar{A} = \lambda(A)$. Let $B = \bar{A}R \subset R$.
~~Look at Hom~~ We have B faithfully rep on $\bar{A}$. If $bB = 0$, then $bB\bar{A} = 0$, but $bB\bar{A} = b\bar{A}^2 = b\bar{A}$, $\therefore$ ~~$b\bar{A}$~~ $\cdot\ b = 0$. So B is reduced. But ~~now~~ you ~~know~~ have the Mcont. $\begin{pmatrix} \bar{A} & B \\ \bar{A} & B \end{pmatrix}$ ~~and Im $(\bar{A} \otimes_{\bar{A}} B \to \mathrm{Hom}_{\bar{A}}(\bar{A}, B))$~~

$$\mathrm{Im}\{M \otimes_{\bar{A}} Q \longrightarrow \mathrm{Hom}_{\bar{A}^{op}}(P, M)\}$$
(with $M = \bar{A}$, $Q = B$)

$$= \mathrm{Im}\{\bar{A} \otimes_{\bar{A}} B \longrightarrow \mathrm{Hom}_{\bar{A}^{op}}(\bar{A}, B)\}$$

$$\mathrm{Hom}_{\bar{A}^{op}}(\bar{A}, \bar{A}) = R$$

$\therefore$ under the map $\bar{A} \in \mathcal{M}_{red}(A^{op})$ corresp to B.

$\therefore \mathrm{Hom}_{B^{op}}(B, B) = \mathrm{Hom}_{\bar{A}^{op}}(\bar{A}, \bar{A})$

In any case I can try to prove directly

~~A firm $R = \mathrm{Hom}_{A^{op}}(A,A)$~~

Start with $A = A^2$, $\{a \in A \mid aA = 0\} = 0$ i.e.

$\lambda : A \hookrightarrow \mathrm{Hom}_{A^{op}}(A,A) = R$. Then set $B = AR$

To prove ~~$\mathrm{Hom}_{B^{op}}(A,A)$~~ $R \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B,B)$

Take $f : B \to B$ B^{op} linear and restrict to A

f must carry ~~AB into AB~~ $A = A^2$ into $BA = A$, so

~~[illegible]~~ $\exists\, r$ with $f(a) = ra$ whence $f(ab) = r(ab)$.

So then B is already an ideal in R.

Back to one-sided meg's. Start with A firm and consider B having the same firm modules as A, means you want $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where $P = A$. $\therefore Q = B$ Thus you are looking at f.d.p. $B \otimes A \twoheadrightarrow A$, equivalently a ~~[illegible]~~ firm A-mod. B equipped with a map

$B \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ such that ?

$Q \otimes A \twoheadrightarrow A$ $\qquad B = A \otimes_A Q = Q$ $\qquad a_1 q_1 a_2 q_2 = a_1 \langle q_1, a_2 \rangle q_2$

$Q \xrightarrow{A\otimes} {}_A\mathrm{Hom}_{A^{op}}(A,A) \longmapsto \mathrm{Hom}_{A^{op}}(A,A)$

$aq \longmapsto a \otimes (a' \mapsto \langle q, a' \rangle) \longmapsto (a' \mapsto \langle aq, a' \rangle)$

$Q \xrightarrow{f} A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ such that

$Q \otimes A \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) \otimes A \xrightarrow{1 \otimes ev} A \otimes_A A \longrightarrow A$ onto

$Q \to \mathrm{Hom}_{A^{op}}(A,A) = R$

I think you would like to say that f is an A-module map such that $f(Q)R = \bar{A}R$, i.e. $f(Q)$ and $\bar{A}$ generate the same ideal in R. Note R acts on Q so that $f(Q)$ is a left ideal.

you waste too much time. Try again

A firm $R = \mathrm{Hom}_{A^{op}}(A, A)$ $r(aa') = (ra)a'$; $r\lambda_a(a') = \lambda_{ra}(a')$.

$A \xrightarrow{\lambda} R$ $\lambda(ra) = r\lambda(a)$

let $\bar{A} = \lambda A \subset R$. $\bar{A}$ is a left ideal in R.

Put $\bar{B} = \bar{A}R = \mathrm{Im}\{A \otimes R \longrightarrow R\}$

Morita cont $\begin{pmatrix} A & R \\ A & \end{pmatrix}$ $\begin{pmatrix} A & R \\ A & R \end{pmatrix}$

I'm actually looking at a dual pair $R \otimes A \to A$ so I should get a Morita $\begin{pmatrix} A & R \\ A & A \otimes_A R \end{pmatrix}$ and a quotient

Mcont. $\begin{pmatrix} A & R \\ A & R \end{pmatrix}$. Then get $m(A) = m(\bar{A}R)$, $\bar{A}R = \bar{B}$.

$$P \otimes_A Q \longrightarrow P \otimes_A \check{P} \qquad \begin{pmatrix} P\otimes_A Q & P \otimes_A \check{P} \\ P \otimes_A Q & P \otimes_A \check{P} \end{pmatrix}$$

note $(P \otimes_A Q)(P \otimes_A \check{P}) = P \otimes_A \check{P}$ when $QP = A$.

can I get the converse? I can get $\boxed{PQP = P}$

by acting on P $(P \otimes_A Q)\underbrace{(P \otimes_A \check{P})}_{}P = (P \otimes_A Q)P = P(QP)$

$P \underbrace{\check{P}P}_{A} = P$ $\therefore$ $\underbrace{\underbrace{\check{P}P}_{A}(QP)}_{QP} = \check{P}P = A$ if Q firm.

17 A firm $R = \mathrm{Hom}_{A^{op}}(A, A)$

have $\lambda : A \longrightarrow R$ $\qquad \lambda_{ra}(a') = (ra)a' = r(aa') = (r\lambda_a)(a')$.

then $\bar{A} = \lambda A \subset R$ is a left ideal in R.

set $\bar{B} = \bar{A}R$. ~~Morita~~

have dual pair $R \otimes A \twoheadrightarrow A$, hence M cont.

$$\begin{pmatrix} A & R \\ A & A \otimes_A R \end{pmatrix}$$

$r \otimes a \longmapsto r(a)$

$a_1 r \otimes a \longmapsto (a_1 r)(a) = a_1 r(a)$

$r \otimes aa_1 \longmapsto r(aa_1) = r(a)a_1$

What's relevant?

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good viewpoint: Begin with A firm and $P \in \mathcal{M}(A^{op})$ a generator. Then consider all pairs $Q, \langle\, , \rangle$ where $Q \in \mathcal{M}(A)$ and $\langle\ \rangle : Q \otimes P \longrightarrow A$ surj A-bimod map.

Equiv. description of a bimod map $Q \otimes P \to A$ is a map $Q \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$. Analyze surj condition $\langle Q, P \rangle = A$. surj of $\langle\ \rangle$ means $\exists (q_i)$ such that the corresp family $\langle q_i, - \rangle \in \mathrm{Hom}_{A^{op}}(P, A)$ yields surjection.

$$\bigoplus_I P \twoheadrightarrow A \qquad \sum_i q_i P = A$$

Now suppose given $\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$. ~~A, P became~~

Then any f.d.p $(Q \otimes P \twoheadrightarrow A)$ over A corresponds to a f.d.p. $\big((P' \otimes_A Q) \otimes (P \otimes_A Q') \twoheadrightarrow P' \otimes_A Q' = B'\big)$ and conversely.

Take $P' = P$ and $Q' = A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$, and you get the ring B' the right B'-module $P' \otimes_A Q' = B'$

$$B' = P' \otimes_A Q' = P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$

So we are dealing with, have replaced A, P by B, B where $B = P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$ like ring of compact operators on P.

The point is that $A \otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ (assoc. to A,P) is the final object of the category of the Q's considered before, so it should follow that this B is the final object of the corresp category assoc. to (B,B). Thus

$$B \xrightarrow{\sim} B \otimes_B \mathrm{Hom}_{B^{op}}(B,B) \text{ should be true.}$$

$$\| $$

$$B \underset{P\otimes_A}{\otimes} \mathrm{Hom}_{A^{op}}(P,P)$$

$$A \otimes_A \mathrm{Hom}_{A^{op}}(P,A) \underset{P \otimes_A \mathrm{Hom}_{A^{op}}(P,A)}{\otimes} \mathrm{Hom}_{A^{op}}(P,P)$$

Simplest might be to set $B = P \otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ and to calculate $B \otimes_B \mathrm{Hom}_{B^{op}}(B,B)$. Take $R = \mathrm{Hom}_{A^{op}}(P,P) = \mathrm{Hom}_{B^{op}}(B,B)$. Consider $B = P \otimes_A \mathrm{Hom}_{A^{op}}(P,A) \longrightarrow R$. B is an R-bimodule and you have $B \otimes_R B \to B$. B firm.

A firm $P \in \mathcal{M}(A^{op})$ generator

$B = P \otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ "compact ops on P".

Morita context. $\begin{pmatrix} A & \overbrace{\mathrm{Hom}_{A^{op}}(P,A)}^{Q_0} \\ P & P\otimes_A Q_0 \end{pmatrix}$. so $P \in \mathcal{M}(A^{op}) \Longrightarrow P \otimes_A Q_0 = B \in \mathcal{M}(B^{op})$.

Thus B is firm. To prove that the canon. homom.

$B \xrightarrow{\lambda} \mathrm{Hom}_{B^{op}}(B,B)$ is a B nil isom. It's automatically a B^{op}-nil isom.

$$V \xrightarrow{\mu} \mathrm{Hom}_{B^{op}}(B,V) \qquad \text{Note } \mu(v) = v\cdot : B \to V$$
$$v \longmapsto (b \mapsto vb) \qquad \text{which is what } \lambda \text{ does.}$$

Let's try direct approach, namely.

$$P \otimes_A Q_0 \longrightarrow Hom_{A^{op}}(P,P)$$

$$p \otimes q \longmapsto (p' \mapsto p\langle q, p' \rangle)$$

$$B = P \otimes_A Hom_{A^{op}}(P, A) \longrightarrow Hom_{A^{op}}(P,P)$$

so I seem to be trying to prove that this homom. ~~is a map~~ is both a B-mil and a B^{op}-mil isom. Take $p_0 \otimes q_0 \in P \otimes_A Q_0$.

$p_1 \otimes q_1$ $\qquad$ $p_1 \langle q_1|$

$$P \otimes_A Q_0 \longrightarrow Hom_{A^{op}}(P,P)$$

T

$T(p_1)\langle q_1|$

$T(p_1) \otimes q_1$

$$P \otimes_A Q_0 \longrightarrow$$

$$p_1 \otimes q_1 \longmapsto |p_1\rangle\langle q_1|$$

$$P \otimes_A Q_0 \longrightarrow Hom_{A^{op}}(P,P)$$

T

$T(p_0) \otimes q_0$

$$P \otimes_A Q_0 \longrightarrow Hom_{A^{op}}(P,P)$$

$$p_1 \otimes q_1 \longmapsto p_1\langle q_1|-\rangle \qquad\qquad T$$

rt mult by $p_0 \otimes q_0$

$$p_1\langle q_1|p_0\rangle \otimes q_0 \qquad\qquad T(p_0)\otimes q_0 \longmapsto T(p_0)\langle q_0|-\rangle$$

right mult by $p_0 \otimes q_0$

20 Other side $T \longmapsto p_0\langle g_0| \rangle \circ T = p_0\langle g_0|T-\rangle$

Now because $Q = \mathrm{Hom}_{A^{op}}(P,A)$ this should be defined

It does seem to work. The point is that

$P\otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ form a bimodule for $\mathrm{Hom}_{A^{op}}(P,P)$
"finite rank ops" (from $P\otimes_A \mathrm{Hom}_{A^{op}}(P,A)$); all operators (for $\mathrm{Hom}_{A^{op}}(P,P)$).

Another way to see this maybe is to observe that $\mathrm{Hom}_{A^{op}}(P,A)$ inherits a unique R^{op}-module structure from the R-mod. structure on P.

Given $Q \longrightarrow \mathrm{Hom}_{A^{op}}(P,A) = Q_0$, when will $Q\otimes P \longrightarrow A$ be surjective? Q_0 is an R^{op}-module. Look at $g\in Q$ and $r\in R$. Then $P \xrightarrow{r} P \xrightarrow{g} A$

$\langle\ \rangle$ surj means $\exists \bigoplus_I P \xrightarrow{(g_i\cdot)} A$

For example suppose $\exists\ f: P \twoheadrightarrow A$ given $g: P \longrightarrow A$

$$\begin{array}{ccc} P & & P \\ \downarrow & & \downarrow f \\ A & = & A \end{array} \qquad\qquad \begin{array}{ccc} P\times_A P & \longrightarrow & P \\ \downarrow & & \downarrow f \\ P & \xrightarrow{g} & A \end{array}$$

This is not working. Instead look at

$$P\otimes_A Q \longrightarrow P\otimes_A \mathrm{Hom}_{A^{op}}(P,A) = B$$

and now you can ask whether the image of this map, which should be a left ideal, generates B as ideal.

Let's simplify: assume i.e. look at $Q \longrightarrow \mathrm{Hom}_{A^{op}}(A,A)$? This time have $Q' \longrightarrow B$ left mod map. Suppose $P\otimes_A \mathrm{Hom}_{A^{op}}(P,A) = B$. We know $B \longrightarrow \mathrm{Hom}_{B^{op}}(B,B)$ is both B and B^{op} mod iso. Suppose we have $Q \longrightarrow \mathrm{Hom}_{A^{op}}(P,A)$ $\ni$ $Q\otimes P \twoheadrightarrow A$, does it follow that $P\otimes_A Q \longrightarrow B$ generates B as ideal. Should be yes.

$$\begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix} \longrightarrow \begin{pmatrix} A & \mathrm{Hom}_{A^{op}}(P,A) \\ P & P\otimes_A \mathrm{Hom}_{A^{op}}(P,A) \end{pmatrix}$$

Thus $P\otimes_A Q \xrightarrow{w} P\otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ is an meq hom.

~~$(P\otimes_A Q)w$~~

$$(P\otimes_A \check{P})(P\otimes_A Q)(P\otimes_A \check{P}) \doteq P\otimes_A \check{P}$$

In fact we know that

$$(P\otimes_A \check{P})(P\otimes_A Q) = P\otimes_A Q$$
$$(P\otimes_A Q)(P\otimes_A P^\vee) = P\otimes_A \check{P}$$ seems to check

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~~Return to the beginning~~

Go back to problem of M-inv. My original idea was? Somehow I was trying to ~~define things~~ understand what happens to $H_*(\mathfrak{GL})$. First if you begin with B firm, then look at? Say B firm

The idea was to focus on one-sided meq's and discuss minv of K_*. This led to looking at ~~the~~ all rings 1-sided mor. eq. to a given firm ring. What do we get?. Start with A firm, take $P = A$, then you consider the category a firm A-modules Q equipped with $Q \to \mathrm{Hom}_{A^{op}}(A,A)$. Take your time! In the end your cat becomes a cat of mods.

Start with A firm, P gen. of $\mathcal{M}(A^{op})$. Then your category consists of A maps $Q \longrightarrow \check{P} = \mathrm{Hom}_{A^{op}}(P,A)$ with $Q \in \mathcal{M}(A)$ such that $QP = A$, i.e. $Q\otimes P \longrightarrow \check{P}\otimes P \xrightarrow{ev} A$ is surj.

Digression Given A firm, P gen of $\mathcal{M}(A^{op})$, and $Q \longrightarrow P^\vee = \mathrm{Hom}_{A^{op}}(P,A)$ (firm) have M cat.

$$\begin{pmatrix} P\otimes_A Q & P\otimes_A \check{P} \\ P\otimes_A Q & P\otimes_A \check{P} \end{pmatrix}, \text{ i.e. } \begin{aligned} (P\otimes_A \check{P})(P\otimes_A Q) &= PA\otimes_A Q = P\otimes_A Q \\ (P\otimes_A Q)(P\otimes_A \check{P}) &= PQP\otimes_A \check{P} \overset{\uparrow}{=} P\otimes_A \check{P} \end{aligned}$$

(provided $QP = A$.)

Conversely if $PQP\otimes_A \check{P} = P\otimes_A \check{P}$, apply this to P to get $P(QP) = P$ and now pair with $\check{P}$ to get $A = \check{P}P = \check{P}P(QP) = AQP = QP$ as Q is firm. Thus it seems that $QP = A \iff$ the image of $P\otimes_A Q$ is $P\otimes_A \check{P}$ generates $P\otimes_A \check{P}$ as right ideal.

22 Return to situation as follows. Start w A firm P gen of $\mathcal{M}(A^{op})$, consider $Q \longrightarrow \mathrm{Hom}_{A^{op}}(P,A) = \check{P}$ ∋ $QP = A$. Such Q's describe a ^firm^ class of one-sided meq's. All the rings $P \otimes_A Q$, as Q varies, are 1-sided meq. This cat has a final object, namely $Q = A \otimes_A \check{P}$ and the corresp ring is $P \otimes_A \check{P}$, call this B. Then we know that $B \xrightarrow{\sim} B \otimes_B \mathrm{Hom}_{B^{op}}(B,B)$ is a special type of ring.

I think it's best to choose some Q and change A, P via $\begin{pmatrix} A & Q \\ P & A \otimes_A Q \end{pmatrix}$ to $P \otimes_A Q$ and $P \otimes_A Q$. And we can assume A is left flat. Thus if we start with A, $P = A$ we can choose Q to be A-flat whence $P \otimes_A Q$ is left flat.

Much more interesting ~~might~~ be to have $P \otimes_A \check{P}$ flat, so at least by ~~starting~~ with $P \in \mathcal{M}(A^{op})$ right flat we can assume $P \otimes_A \check{P}$ to be right flat. might be useful

If A right flat + firm, then $B = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ should be right flat.

Go back to matrices. ~~[illegible]~~

I need to understand GL

Take $A \in \mathcal{P}(A^{op})$ set $B = \mathrm{Hom}_{A^{op}}(A,A)$
$= \mathrm{Hom}_{A^{op}}(A, \tilde{A}) \otimes_A A$ $\qquad \begin{pmatrix} A & B \\ A & B \end{pmatrix}$

So what am I doing? ~~[illegible]~~

So what is the business to understand

The point was h-unital ~~[illegible]~~ All the rings being considered – these are essentially left ideals A in $B = B \otimes_B \mathrm{Hom}_{B^{op}}(B,B)$ generating B. The category $\mathcal{M}(A)$ is ind. of A.
$$\mathcal{M}(A) = \mathcal{M}(B)$$
$$M \longmapsto A \otimes_A M = M.$$
and the firm flat modules are ~~the same~~ ind. of A. Recall we have left B-maps $A \longrightarrow B$ such that $\overline{AB = B}$.

A h-unital $\iff$ A h-unitary over A $\iff$ A has a res. by firm flat A-modules $\iff$ A has a res by firm flat B-modules.

$\therefore$ A h-unital $\iff B \overset{L}{\underset{B}{\otimes}} A = 0$.

02/17/97 What I learned yesterday:

A firm P gen of $\mathcal{M}(A^{op})$ then $B = P \otimes_A \underbrace{Hom_{A^{op}}(P, A)}$ is such that $B \xrightarrow{\lambda} Hom_{B^{op}}(B, B)$ is a B-mil iso. $P^\vee$ and B^{op}-mil iso.

NO. Somehow the point is that $B = P \otimes_A P^\vee$ is a bimodule over $Hom_{B^{op}}(B, B) = Hom_{A^{op}}(P, P)$ and λ is a bimodule map.

You know that any $Q \longrightarrow \check{P}$ such that $QP = A$ yields a ring $P \otimes_A Q$ with

$$\mathcal{M}(P \otimes_A Q) = \mathcal{M}(\underbrace{P \otimes_A \check{P}}_{B}) \quad \text{one-sided m.eq.}$$

From an invariant viewpoint you have $P \in \mathcal{M}$ generator, $R = Hom(P, P)$, $B = P \otimes \check{P}$, where $\check{P}$ in $\mathcal{M}$ is universal for obj Q together with a "pairing" $Q \otimes P \longrightarrow 1$

Each Q such that $Q \otimes P \twoheadrightarrow 1$ gives rise to a ring $A = P \otimes Q$ m.eq. to $B = P \otimes \check{P}$.

To analyze start with B such that $B \xrightarrow{\sim} B \otimes_B \check{B}$ (with $\check{B} = Hom_{B^{op}}(B, B)$), then consider all $A \longrightarrow \check{B}$ in $\mathcal{M}(B)$ such that $BA = B$. So we have $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ A varying, B fixed.

By starting with P flat get B right flat, whence all $A \simeq B \otimes_B A$ are right flat. Thus we get

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Recap. You have $\mathcal{M}(A)$ and select a generator $P \in \mathcal{M}(A^{op})$ Then you get immediately $R = Hom_{A^{op}}(P, P)$ and the distinguished idempotent ideal $P \otimes \check{P} \subset R$ such that

$$P\check{P} \otimes_R P\check{P} = P \underset{A}{\otimes} \check{P}$$

 Intrinsically start with M and the generator P of $M^\vee$ = ~~rtcontfun~~(M, Ab).
There is then $\check{P} \in M^*$ universal for Q equipped with pairing $Q \otimes_{\mathbb{Z}} P \longrightarrow \mathbb{Z}$. Let $B = P \otimes \check{P}$, then can assume $M = M(B)$ $P = B$, $\check{P} = B = B \otimes_B \mathrm{Hom}_{B^{op}}(B, B)$

Things are confused. I guess the interesting point is the fact that if $P \in M(A^{op})$ is a generator, and $\check{P} = \mathrm{Hom}_{A^{op}}(P, A)$, then $B = P \otimes_A \check{P}$ is not only a left $\mathrm{Hom}_{A^{op}}(P, P) = R$ module but also a right one.

How can I get started? Basically you have a special class of firm rings B such that

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How to get started? ~~Roos thm.~~

Let's look at the problem of Mor. of K_*, one-sided Mor. this means Morita of the form $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$, I know these leads to $M(A) = M(B)$ as $M \mapsto P \otimes_A M = A \otimes_A M = M$. Moreover $\mathrm{Hom}_{A^{op}}(A, A) = \mathrm{Hom}_{B^{op}}(A \otimes_A Q, A \otimes_A Q) = \mathrm{Hom}_{B^{op}}(B, B)$, so the ~~base~~ left multiplier ring, the ring that acts naturally on any firm module, the ring of endos of the forgetful functor from $M(A)$ to Ab stays the same. Now what ~~do~~ I do next is to consider h-unitality.

Given A idempotent we can always choose ~~the~~ ~~...~~ a firm flat A-module B with surj $B \twoheadrightarrow A$. Then B is flat firm A-mod $\Longrightarrow P \otimes_A B = A \otimes_A B = B$ is a flat firm B-module. Thus there are left flat rings B in our category. If B is right flat, then $B \otimes_B A = A$ is right flat. Thus if we start with A right-flat, any B one-sided mor to it is right flat. Another way to see this is that $A \in M_{fl}(A^{op}) \iff M(A) \longrightarrow \mathrm{Mod}(R)$ is exact

So now we have an interesting case, namely take A right flat (e.g. $A \in P(A^{op})$) and then any B we are considering is also right flat. Then both rings are h-unital, so it should be possible to see $K_*(A) = K_*(B)$. For example if B unital we know what to do

Special case: $A \in \mathcal{M}_{pe}(A^{op})$, then it should be possible to show $K_*(A) = K_*(B)$ for any $B \in \mathcal{M}(A)$ equipped with $B \otimes A \twoheadrightarrow A$. So we are working on a certain category of suff. big $B \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$

By Suslin the K_* result should be true for surjections, so essentially we reduce to the poset of left ideals in $A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ which generate this ring.

What did we do when $A \in \mathcal{P}(A^{op})$?

Change notation $B = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) \longrightarrow \mathrm{Hom}_{A^{op}}(A,A) = R$

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new notation. A right flat idemp. ring, $B = A \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(A,A)}_{R}$

Then $A \otimes_A B = B$ is right flat.

$$\begin{pmatrix} A & \mathrm{Hom}_{A^{op}}(A,A) \\ A & B \end{pmatrix} \qquad A = \mathrm{Hom}_{A^{op}}(A,A) \otimes_A A \quad \text{we know.}$$

First case to understand is when $A \in \mathcal{P}(A^{op})$, then $B = R$. The problem is to get a map $K_*(B) \to K_*(A)$. We have B operating on A_A, something like compact operators on A. A technique you tried was to write $P = A$ as a $\varinjlim$ of fg free A-modules. We know some facts about A since it's right flat. On the level of K_1 I have a proof.

One idea is the injectivity $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

if P right flat, then $P = \varinjlim F_\alpha = \varinjlim F_\alpha A$ where $F_\alpha = \tilde{A}^n$ right free A-module

$$Q \longrightarrow \mathrm{Hom}_{A^{op}}(P, A) \longrightarrow \mathrm{Hom}_{A^{op}}(F_\alpha, A) = AF_\alpha$$

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$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \varinjlim \begin{pmatrix} A & Q \\ F_\alpha & F_\alpha \otimes_A Q \end{pmatrix}$$

so there's a problem with C_α not being idempotent

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix} \qquad \begin{pmatrix} B & A \\ B & A \end{pmatrix}$$

So A A^{op}-flat
$\Rightarrow B = A \otimes_A B$ is B^{op}-flat

so certainly ~~the~~ the injectivity argument has a chance of working.

Idea. $\begin{pmatrix} A \\ F_\alpha A \end{pmatrix} \otimes_A (A \quad Q)$ ~~[illegible]~~ is idempotent. in fact ~~[illegible]~~ right flat.

I need to check this. Consider $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s firm

P is A^{op}-flat $\Longleftrightarrow$ $P \otimes_A Q = B$ is B^{op}-flat
A is A^{op}-flat $\Longleftrightarrow$ $A \otimes_A P$ is B^{op} flat.

So it appears then that when A, B are both right flat rings that we have $K_*(A) \hookrightarrow K_*\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \hookleftarrow K_*(B)$

So we have $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

So how to handle it all

Suppose B such that $B \xrightarrow{\sim} B \otimes_B \operatorname{Hom}_{B^{op}}(B, B)$.
then the same is true for matrices over B. ~~cut proof.~~

~~Take~~ Consider $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ where $B =$ ~~[illegible]~~ $A \otimes_A \operatorname{Hom}_{A^{op}}(A, A)$

If A is right flat as ring, then $A \in \mathcal{M}_{fl}(A^{op}) \Longleftrightarrow$
$A \otimes_A B = B \in \mathcal{M}_{fl}(B^{op})$. ~~Note that~~ ~~iff~~.

Note that if we have $A \xrightarrow{\ ?\ } A$, then ?

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longleftarrow \begin{pmatrix} A & Q \\ A^n & A^n \otimes Q \end{pmatrix} \longrightarrow \begin{pmatrix} A \\ A^n \end{pmatrix}$$

 ~~Try again~~

Assume $A \simeq A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$

~~then~~

what does ~~requiring~~ saying that $A \longrightarrow \mathrm{Hom}_{A^{op}}(A,A)$ is an A-nil isom. mean. ~~Then~~ It means that given $\phi: A \to A$ A^{op}-linear and $a \in$

Wait. It means first that if $aA = 0$ then $Aa = 0$. ~~And that given~~ $r \in R = \mathrm{Hom}_{A^{op}}(A,A)$ and $a \in A$, $\exists\, a_1$ such that ~~$ar = a_1$~~

$$\boxed{a\, r(a') = a_1 a' \quad \forall a',}$$

Review previous mistake: Given $\begin{pmatrix} A & Q \\ A & Q \end{pmatrix}$ i.e.

given $Q \xrightarrow{f} \mathrm{Hom}_{A^{op}}(A,A)$ w big enough image. $QA = A$ then f must factor $\searrow A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) \nearrow$. Mistake was in believing $\nearrow$ is A. But this applies when Q is the final object. So we get $\begin{pmatrix} A & Q \\ A & Q \end{pmatrix} \longrightarrow \begin{pmatrix} A & A \\ A & A \end{pmatrix}$

Suppose now

02/18/97 ~~Rel multip~~

$A \in \mathcal{M}_{fe}(A^{op})$ $\qquad B = A \otimes_A \overbrace{\mathrm{Hom}_{A^{op}}(A,A)}^{R}$

$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ ~~satisfies~~ $\mathcal{M}(A) \rightleftarrows \mathcal{M}(B)$
$M \longmapsto A \otimes_A M = M$

To show $B = B \otimes_B \mathrm{Hom}_{B^{op}}(B,B)$

First $\mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}_{B^{op}}(B,B)$ as $A \otimes_A B = B$.

~~Also~~ R is a B-mod. hence an A mod.

$A \otimes_B \tilde{B}$

$$\begin{pmatrix} A & \mathrm{Hom}_{A^{op}}(P,A)=\check{P} \\ P & B = P\otimes_A \check{P} \end{pmatrix} \leftarrow \begin{pmatrix} A & Q \\ P & P\otimes_A Q = C \end{pmatrix}$$

$$\mathrm{Hom}_{A^{op}}(P,P) = \mathrm{Hom}_{B^{op}}(B,B)$$

$$\mathrm{Hom}_{A^{op}}(P,A) = \mathrm{Hom}_{B^{op}}(B, A\otimes_A \check{P})$$

Easiest way to see that $B = B\otimes_B \mathrm{Hom}_{B^{op}}(B,B)$ is to show that $B \longrightarrow \mathrm{Hom}_{B^{op}}(B,B)$ is a B-nilciso.

$$\begin{array}{ccc} P\otimes_A \check{P} & \longrightarrow & \mathrm{Hom}_{A^{op}}(P,P) \\ \| & \nearrow & \\ P\otimes_A \mathrm{Hom}_{A^{op}}(P,A) & & \end{array}$$

$$\begin{array}{ccc} P\otimes_A \check{P} & \longrightarrow & \mathrm{Hom}_{A^{op}}(P,P) \\ {\scriptstyle p_0 q_0 \cdot}\downarrow & \overset{F}{\swarrow} & \downarrow{\scriptstyle p_0 q_0 \cdot} \\ P\otimes_A \check{P} & \longrightarrow & \mathrm{Hom}_{A^{op}}(P,P) \end{array}$$

given $p_0, q_0 \in \check{P}$ and define $F(T) = p_0 \otimes q_0 T$

$$\begin{array}{ccc} p\otimes q & \longmapsto & (p' \longmapsto p(qp')) \\ & \overset{F}{\swarrow} & T \end{array} \qquad T(p') = pqp' \qquad q_0 T p' = q_0 p q p'$$

$$p_0 q_0 p \otimes q = p_0 \otimes q_0 p q$$

$$p_0 \otimes q_0 T \longmapsto (p' \longmapsto p_0 q_0 T p')$$

Anyway this seems OKAY but

Take $P=A$. $\quad A\otimes_A \check{A} \longrightarrow \check{A} \qquad \check{A} = \mathrm{Hom}_{A^{op}}(A,A)$

fits the pattern of a R-bimodule map $M \to R$ together with $M\otimes_R M \longrightarrow M$ R-bim maps, assoc.

It follows that M is a ring and one has $R \to \mathrm{Mult}(M)$

29. $Mult(A) = \{ \in Hom_{A^{op}}(A,A) \times Hom_A(A,A)^{op} \mid (a_1\xi)a_2 = a_1(\xi a_2)\}$

So let $M \xrightarrow{f} R$ be an R-bimod map such that the two dialg structures coincide $f(m_1)m_2 = m_1 f(m_2)$

Then M is a ring, and we have

$$R \longrightarrow Hom_{M^{op}}(M,M) \times Hom_M(M,M)^{op}$$
$$r \qquad (r\,\cdot) \qquad (\cdot\, r)$$

You need [illegible] what about $(m_1 \cdot r)m_2 \overset{?}{=} m_1(r \cdot m_2)$

This means that $M \otimes M \longrightarrow M$ descends to $M \otimes_R M$.

and this should be obvious here.

$$(m_1 r)m_2 = m_1 r f(m_2) = m_1 f(rm_2) = m_1(rm_2).$$

Converse. Start with A put $R = Mult(A)$ $\subset Hom_{A^{op}}(A,A) \times Hom_A(A,A)^{op}$. Then R is a unital ring and A is a left and a right A-module. You check that R is a subring of this product and A is both a left & right module over it. When $A = A^2$ left and right mults. commute. λ, ρ

$$(\lambda(a_1 a_2))\rho = ((\lambda a_1)a_2)\rho = (\lambda a_1)(a_2 \rho)$$

Look at $B = A \otimes_A \check{A} \xrightarrow{f} \check{A}$ ✓ $\qquad a \otimes \phi \longmapsto \lambda_a \phi$

check B is an $\check{A}$-bimodule ✓, f an $\check{A}$-bimod map ✓

$$a \otimes \phi \longmapsto \lambda_a \phi$$

$$\phi'(\lambda_a \phi) = (\phi' \lambda_a)\phi = \lambda_{\phi' a}\phi$$

So we should have $Mult(B) \longleftarrow \check{A}$

$Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N \qquad v \otimes 1$

$$P' \otimes_A Q \otimes_B N \xrightarrow{\sim} P' \otimes_A Q' \otimes_B N \xrightarrow{\sim} B' \otimes_B N$$

$B^{(2)} \otimes_B P \otimes_{AQ'} \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A Q' = B^{(2)} \otimes_A B'$ $\qquad \cong$ because $P' \otimes_A Q' \to B'$ is a B'^{op}-mil isom $\therefore$ also a B^{op}-mil

$\otimes_{B'} N'$

$B^2 \quad b_1 b_2 p \otimes q' \otimes n' \longmapsto b_1 \otimes b_2 \otimes u(p) q' n'$

 02/18/97 1130. So where are we?

Consider a right flat idemp ring A, set $B = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ so B acts as some sort of compact operators on A.

It seems to be easier to write $B = P \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(P,A)}_{\check{P}}$

and to consider $B = P \otimes_A \check{P} \longrightarrow \mathrm{Hom}_{A^{op}}(P,P)$.

Something is going on ~~$P \otimes_A \check{P}$~~ which I don't understand.

Suppose $A \in \mathcal{P}(A^{op})$, let ~~$B = \mathrm{Hom}_{A^{op}}(A,A) = A \otimes_A \check{A}$~~

$B = A \otimes_A \check{A} = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}_{A^{op}}(A,A)$. Then $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ with B unital. Points are that $\exists$ a homom. $A \to B$ and $B \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(A,A)$, so we get maps on K_*. I would like to generalize.

First suppose $\exists$ $P \in \mathcal{P}(A^{op})$ gen $\mathcal{M}(A^{op})$, put

$$B = \mathrm{Hom}_{A^{op}}(P,P) = P \otimes_A \check{P} \qquad \check{P} = \mathrm{Hom}_{A^{op}}(P,A)$$

$\begin{matrix} A & \check{P} \\ P & B \end{matrix}$ where B unital. Have what sort of K-maps.

$B \longrightarrow \mathrm{Hom}_{A^{op}}(P,P)$ $\quad P \in \mathcal{P}(A^{op})$ yields $K_*B \longrightarrow K_*A$

need to pick up.

~~Because you do~~

~~Suppose P gen for $\mathcal{M}(A^{op})$, P right flat.~~

Suppose A right flat, write P for $A \in \mathcal{M}(A^{op})$

In fact maybe I should take $P = A^n$.

$P = \varinjlim F_\alpha$ $\qquad Q \longrightarrow \mathrm{Hom}_{A^{op}}(F_\alpha, A)$ $\qquad \begin{pmatrix} A & Q \\ F_\alpha A & \end{pmatrix}$

$P = \varinjlim F_\alpha$ $\qquad Q \longrightarrow \mathrm{Hom}_{A^{op}}(F_\alpha, A) = A\check{F}_\alpha$

$= \varinjlim F_\alpha A$

Start with $\mathcal{M} = \mathcal{M}(A)$ and choose P a gen. of $\check{\mathcal{M}} = \mathcal{M}(A^{op})$ s.t. P is right flat. Then there is $Hom_{A^{op}}(P,P)$ and $\check{P} = Hom_{A^{op}}(P,A)$ and $P \otimes_A \check{P} \longrightarrow Hom_{A^{op}}(P,P)$ intrinsically defined

~~start~~

$$\begin{pmatrix} A & A \otimes_A \check{P} \\ P & B \otimes_A \check{P} = B \end{pmatrix}.$$

Change coords from A to B ?

Start with $\mathcal{M}(A)$ and $P \in \mathcal{M}_{fl}(A^{op})$ gen.

Start with a Roos cat $\mathcal{M}$ and a flat P in $\check{\mathcal{M}}$ which generates $\check{\mathcal{M}}$. Consider all way of extending P to a firm dual pair $Q \otimes P \to 1$. To each one we get a ring $P \otimes_A Q$. To prove these all have the same K_*. There is a "largest" of those rings corresp to $\check{P}$ (study $A \otimes_A Hom_{A^{op}}(P,A)$)

Canonical coordinate system given by the f.d.p. $\check{P} \otimes P \longrightarrow 1$. This gives us ~~an idemp. sum~~ a firm B such that $P = B$ and $B \xrightarrow{\sim} B \otimes_B Hom_{B^{op}}(B,B)$. So the situation we reach is that of a right flat ring B satisf.

The other way to put this is to say that

$$B \xrightarrow{\lambda} Hom_{B^{op}}(B,B) \qquad \lambda(b) = (b' \mapsto bb')$$

has the 2 properties: $\lambda(b) = 0$ (i.e. $bB = 0$) $\Longrightarrow Bb = 0$. and also that given $T: B \to B$ commuting with ~~left mult~~ rt mult, and $b \in B$, then $(bT)(b' \mapsto bT(b'))$ ~~has the form~~ is in the image of λ, i.e. [redacted] has form $b' \mapsto b_0 b$ for some b_0.

Curious operation on ~~idempotent~~ firm rings.

$$B \mapsto B \otimes_B Hom_{B^{op}}(B,B)$$

 I'll have to go over this several times.

P gen. for $\mathcal{M}(A^{op})$, set $B = P \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(P, A)}_{\check{P}}$

$P \otimes_A \check{P} \longrightarrow \mathrm{Hom}_{A^{op}}(P, P)$ $\qquad p \otimes \phi \longmapsto (p' \mapsto p\phi(p'))$ $\quad |p\rangle\langle\phi|$

$p_0 \otimes \phi_0 T \longleftarrow T$ $\quad \delta$

$P \otimes_A \check{P} \longrightarrow$ $\qquad T$

$p_0 \otimes \phi_0(p)\phi$

$p_0 \otimes (\phi_0 T) \longmapsto (p_0 \phi_0) \circ T$

$\delta(T)$ should be like $(p_0 \otimes \phi_0) T = p_0 \otimes \phi_0 T$.

$$(p_0 \otimes \phi_0)(p \otimes \phi) = p_0 \otimes \phi_0(p)\phi$$

$$(p_1 \otimes \lambda_1)(p_2 \otimes \lambda_2) \overset{\text{defn}}{=} p_1 \otimes \underbrace{\lambda_1(p_2)}_{\langle\lambda|p_2\rangle} \lambda_2$$

simpler notation $\quad p \otimes \lambda \longmapsto p\lambda = (p' \mapsto p\lambda(p'))$

product in $P \otimes_A \check{P}$ is $\quad (p_1 \otimes \lambda_1)(p_2 \otimes \lambda_2) = p_1 \otimes \lambda_1 p_2 \lambda_2$. YES!

So lets try to see if we can do it by equivalent conditions. ~~Start with the firm ring A, let~~

Let B be a firm ring, use $\bar{B} = B/I \quad I = \{b \mid bB = 0\}$. So $\bar{B}$ is reduced as right B-module, and we should have $\mathrm{Hom}_{B^{op}}(B, B) = \mathrm{Hom}_{\bar{B}^{op}}(\bar{B}, \bar{B})$. Have canon ~~map~~ homom.

$B \longrightarrow \mathrm{Hom}_{B^{op}}(B, B) \qquad b \longmapsto (b\,\cdot). \qquad$ Image is $B/\{b \mid bB = 0\} = I$.

Here's a question: It is possible for there to be an element b of I such that $Bb \neq 0$. Somehow this is too difficult.

33. Start again. $\mathcal{M}$ Roos cat say $\mathcal{M}(A)$, P flat gen. by $\mathcal{M}^\vee = \mathcal{M}(A^{op})$. Define $\check{P} \in \mathcal{M}$ to be universal equipped with $\check{P} \otimes_{\mathbb{Z}} P \to 1$

$R = \mathrm{Hom}_{A^{op}}(P, P)$, $\quad B = P \otimes_A \check{P} \xrightarrow{\text{can}} \mathrm{Hom}_{A^{op}}(P, P)$

We know that the image $P\check{P} \subset \mathrm{Hom}_{A^{op}}(P, P)$ is an ideal. $\quad (pq)T = p(qT) \qquad T(pq) = (Tp)q$

Moreover the kernel of $P \otimes_A \check{P} \to P\check{P}$ is killed by B on either side. $\quad p_1 q_1 (p_i \otimes q_i) = p_1 \otimes q_1 p q$

So we can start with B firm such that the image of canon: $B \to \mathrm{Hom}_{B^{op}}(B, B)$ is an ideal, not just a left ideal.

$$(r(b\cdot))(b') = r(bb') = r(b)b' = (r(b)\cdot)b'.$$

r

~~Is it possible for B~~

So let's suppose given a firm ring B such that can: $B \to \mathrm{Hom}_{B^{op}}(B, B)$ is a B-nil isom, equiv.

$$bB = 0 \implies Bb = 0.$$

$$B \xrightarrow{\sim} B \otimes_B \mathrm{Hom}_{B^{op}}(B, B)$$

the image of B is an ideal.

so that $\forall\, b_1, r \quad \exists\, b_2$ such that $b_1 r(b') = b_2 b' \quad \forall b'$

$$(b_1 r - b_2)b' = 0 \quad \text{all}$$

$$(b_1 r - b_2)B = 0$$

I will suppose B right flat.

Can consider any $A \to B$ in $\mathcal{M}(B)$ such that $A \otimes B \twoheadrightarrow B$. Then get $\begin{pmatrix} B \otimes_B A & B \\ A & B \end{pmatrix}$.

Since B is right flat so is A.

 Now you have to see if there is any hope ~~at~~ all.
Yes!!!

Go back to old picture – namely $A,\ P \in \mathcal{M}_{fl}(A^{op})$ gen.

$$\begin{pmatrix} A & A\otimes_A \mathrm{Hom}_{A^{op}}(P,A) \\ P & P\otimes_A \underset{\substack{\parallel \\ B}}{\mathrm{Hom}_{A^{op}}(P,A)} \end{pmatrix}$$

I guess the first transition is to ~~go~~ choose $Q\otimes P \twoheadrightarrow A$ and then go via $\begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix}$ to the case

$$\mathcal{M}(A^{op}) = \mathcal{M}(P\otimes_A Q^{op}) \qquad \mathcal{M}(A) = \mathcal{M}(P\otimes_A Q)$$
$$P \longmapsto P\otimes_A Q \qquad\qquad Q \qquad P\otimes_A Q.$$

By this transition you end up with $P = A$, ~~this~~ so now ~~both~~ both A, P are right flat.

But look from the viewpoint of B ~~...~~ which is right flat and sats $\boxed{B \xrightarrow{\sim} B\otimes_B \mathrm{Hom}_{B^{op}}(B,B)}$. Anyway you are looking at the special case where

$$\begin{array}{ccccc}
\begin{pmatrix} A & A \\ A & A \end{pmatrix} & \longrightarrow & \begin{pmatrix} A & B \\ A & B \end{pmatrix} & \longrightarrow & \begin{pmatrix} B & B \\ B & B \end{pmatrix} \\
& & \Big\| \qquad \Big\uparrow & \nearrow & \\
& & \begin{pmatrix} B \\ B \end{pmatrix} \otimes_B (A \quad \overset{B}{B}) & &
\end{array}$$

 02/19/97

Go back to the idea that if $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ is a form with $\begin{cases} A \text{ rt flat (equiv } A\otimes_A Q = Q \text{ is rt flat)} \\ B \text{ rt flat (equiv } P \text{ is rt flat)} \end{cases}$

then we know the maps $K_*A \longrightarrow K_*\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longleftarrow K_*B$ are injective. Weaken this a bit to just requiring B rt flat (equiv. P rt flat) $\Rightarrow K_*A \to K_*\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ inj.

Recall argument: $P = \varinjlim F_\alpha$ filtered ind limit with $F_\alpha \in \mathcal{P}(\tilde{A}^{op})$.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \varinjlim \begin{pmatrix} A & Q \\ F_\alpha A & F_\alpha \otimes_A Q \end{pmatrix}$$

For each α you have $Q \to \mathrm{Hom}_{A^{op}}(F_\alpha, A) = A\check{F}_\alpha$ so for each α you have a diagram

$$A \hookrightarrow \begin{pmatrix} A & Q \\ F_\alpha A & F_\alpha \otimes_A Q \end{pmatrix} \longrightarrow \begin{pmatrix} A & A\check{F}_\alpha \\ F_\alpha A & F_\alpha A \otimes_A A\check{F}_\alpha \end{pmatrix}$$

(bottom row: essentially a matrix ring over A.)

So the question is whether I can arrange this argument so as to produce a map $K_*\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \to K_*(A)$ and I would actually like a retraction. Look at the case where [redacted] P is a generator of $\mathcal{P}(A^{op})$ first. You have to go over this case again for the next 2 hours.

Start with $A \in \mathcal{P}(A^{op})$, put $B = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}_{A^{op}}(A,A)$

$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ B is unital, so $A \in \mathcal{P}(A^{op})$, $B \in \mathcal{P}(A)$ is its dual.

We have B acting on $A \in \mathcal{P}(A^{op})$ whence $K_*B \to K_*A$ and we have a hom $A \to B$ — $K_*A \to K_*B$.

These corresp to functors $\mathcal{P}(\tilde{B}^{op}) \longrightarrow \mathcal{P}(\tilde{A}^{op}) \longrightarrow \mathcal{P}(B$

$V \longmapsto V \otimes_B A$

Actual functors $\mathcal{P}(B^{op}) \longrightarrow \mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op}) \longrightarrow \mathcal{P}(B^{op})$

$$V \longmapsto V \otimes_B A, \quad U \longmapsto U \otimes_A B$$

One direction the composition is the identity, namely

$$V \longmapsto V \otimes_B A \longmapsto V \otimes_B A \otimes_A B = V \qquad \text{since } A \otimes_A B = B.$$

The other direction gives the functor

$$U \longmapsto U \otimes_A B \longmapsto U \otimes_A B \otimes_B A = U \otimes_A A = UA.$$

from $\mathcal{P}(\tilde{A}^{op})$ to $\mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op})$. One has to understand this functor on the level of K_*. Since $A \in \mathcal{P}(\tilde{A}^{op}) \subset \mathcal{P}(\tilde{A}^{op})$ one has $U \longmapsto U \otimes_A A$ from $\mathcal{P}(\tilde{A}^{op})$ to $\mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op})$. This is an additive idempotent functor. In fact you have

$$K_* \tilde{A} \longrightarrow K_*(\mathcal{P}(A^{op})) \longrightarrow K_* \tilde{A} \longrightarrow K_*(\mathcal{P}(A^{op}))$$

$$\therefore \quad K_*(\tilde{A}) = (?) \oplus K_*(\mathcal{P}(A^{op})).$$

Rest comes from some exact sequences. You want to use

$$0 \longrightarrow U \otimes_A A \longrightarrow U \longrightarrow U/UA \longrightarrow 0$$

$$0 \longrightarrow U/UA \otimes_{\mathbb{Z}} A \longrightarrow U/UA \otimes_{\mathbb{Z}} \tilde{A} \longrightarrow U/UA \longrightarrow 0$$

Mhanel gives exact sequences of additive functors from $\mathcal{P}(\tilde{A}^{op})$ to itself.

$$0 \longrightarrow U \otimes_A A \longrightarrow U \times_{\bar{U}} \bar{U} \otimes_{\mathbb{Z}} \tilde{A} \longrightarrow \bar{U} \otimes_{\mathbb{Z}} \tilde{A} \longrightarrow 0$$

$$\|$$

$$0 \longrightarrow \bar{U} \otimes_{\mathbb{Z}} A \longrightarrow U \times_{\bar{U}} \bar{U} \otimes_{\mathbb{Z}} \tilde{A} \longrightarrow U \longrightarrow 0$$

$$[U] + \varepsilon(U)[A] = [UA] + \varepsilon(U)[\tilde{A}].$$

etc.

I really ought to examine this from the B-viewpoint. Namely, take B unital and let $A \subset B$ be any left ideal such that $AB = B$. For example if BeB with $e^2 = e$, then can take $A = Be$ which is $\begin{pmatrix} eBe & eB \\ Be & B \end{pmatrix}$

It ~~would~~ might be useful to really understand the exact sequences inside of $P(\tilde{\Lambda}^{op})$ that have been used.

Perhaps it would be useful to see the case of ~~elements~~ $A \subset B$, left ideal, $y \in A$, $x \in B$, $yx = 1$. So that $By \subset A \subset B$. Now $(xy)^2 = xy$, call this e. e is an idempotent in A, [redacted]

$$\left(\begin{array}{c|c} eAe & eA \\ \hline Ae & A \end{array}\right) \subset \left(\begin{array}{c|c} eBe & eB \\ \hline Be & B \end{array}\right)$$

(Ae and Be: same)

Now $e \in A \Rightarrow Be \subset A \Rightarrow Be \subset Ae \subset Be \Rightarrow Be = Ae$

So what have we done? ~~We have tried and trusted~~

Let $\Lambda = eAe = eBe$. eB is the dual $\mathrm{Hom}_{\Lambda^{op}}(Be, \Lambda)$

~~So eA is the any~~

OKAY

$$\left(\begin{array}{c|c} eBe & eAe^{\perp} \\ \hline e^{\perp}Be & e^{\perp}Ae^{\perp} \end{array}\right) \subset \left(\begin{array}{c|c} eBe & eBe^{\perp} \\ \hline e^{\perp}Be & e^{\perp}Be^{\perp} \end{array}\right)$$

So $eAe^{\perp}$ is any Λ-submodule of $eBe^{\perp}$ = dual of $e^{\perp}Be$

~~There seems to be a general principle which you should try to make clear, namely that once~~

~~So we have an coherent~~

Consider then $\mathcal{M}$ and a flat generator for $\check{\mathcal{M}}$ = rtcontfun $(\mathcal{M}, ab)$. Call this gen U. Choose $V \in \mathcal{M}$ with $V \otimes U \twoheadrightarrow 1$, then we get ring $A = U \otimes V$

Start again with $\mathcal{M}(A)$ and P gen. for $\mathcal{M}(A^{op})$

Get $\begin{pmatrix} A & Q \\ P & P\otimes_A Q = B \end{pmatrix}$ $\qquad Q = A \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$

$B = P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$.

If we use the meq $\mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$ $\qquad \mathcal{M}(A^{op}) \xrightarrow{\sim} \mathcal{M}(B^{op})$

$Q \longmapsto P \otimes_A Q = B \qquad P \longmapsto P \otimes_A Q = B$.

So we end up with $\mathcal{M} = \mathcal{M}(B)$, where

$$B = B \otimes_B \mathrm{Hom}_{B^{op}}(B, B). \quad ?$$

This B maybe irrelevant for the problem at hand but there's the analogy with $A \in \mathcal{P}(A^{op})$.

$$A \in \mathcal{M}_{fl}(A^{op}) \rightsquigarrow B = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A).$$

You want to proceed closely to the case $A \in \mathcal{P}(A^{op})$. We have a homom. $A \longrightarrow B$ which is an A^{op}-nil isom, in fact the comp. $A \to B \to \mathrm{Hom}_{A^{op}}(A, A)$ is a rt nil isom. Therefore we have a homom. $K_*(A) \longrightarrow K_*(B)$. Now use the rt flatness of A, and there might be some hope of constructing a map $K_*(B) \to K_*(A)$. The idea is that you

 You take A A^{op} flat and write it as $\varinjlim F_\alpha$, $F_\alpha \in \mathcal{P}(\tilde{A}^{op})$, whence $A = \varinjlim F_\alpha A$. Then can approx

$$\begin{pmatrix} A & A\otimes_A \check{P} \\ P & P\otimes_A \check{P} \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} A & A\otimes_A \check{P} \\ F_\alpha A & F_\alpha A \otimes_A \check{P} \end{pmatrix}.$$

But now you take.

So by looking carefully at the case $A \in \mathcal{P}(A^{op})$ I learn that B is irrelevant. ~~What you want to know~~ You ultimately the relation between $\mathcal{P}(A)$ and $\mathcal{P}(\tilde{A})$ and this you get from the Shanuel trick.

$$\begin{array}{ccccccc} & & & & \tilde{u}\otimes_{\mathbb{Z}} A & = & \tilde{u}\otimes_{\mathbb{Z}} A \\ & & & & \downarrow & & \downarrow \\ 0 & \to & uA & \longrightarrow & F & \to & \tilde{u}\otimes_{\mathbb{Z}} \tilde{A} \\ & & \| & & \downarrow & & \downarrow \\ 0 & \to & uA & \longrightarrow & u & \longrightarrow & u/uA & \longrightarrow 0 \end{array}$$

Now the issue is whether you can say anything in the case $A \in \mathcal{M}_{fl}(A^{op})$. Ring B may be useful, because it will play the role of $\mathcal{P}(A^{\#})$, which need not be large enough. The

Can still write $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\qquad A \to B = A\otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ gives $K_*A \to K_*B$

You however need a map $K_*B \to K_*A$. You have $K_*B \longrightarrow K_*\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ so it's enough to prove that K_*

You are going to play a funny game I think.

Let's see if any reasonable argument can be found. We assume A right flat, then $A\otimes_A B \cong B$ is right flat, so I know that ~~$K_*A \cong K_*B$ and~~

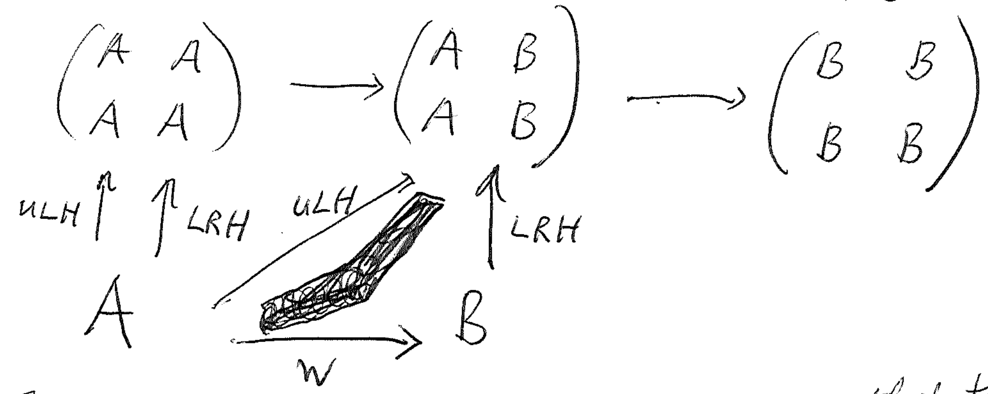
$$K_*A \hookrightarrow K_*\begin{pmatrix} A & B \\ A & B \end{pmatrix} \hookleftarrow K_*B$$

are injective. On the other hand we have the homomorphism $A \to B$ and perhaps we can show

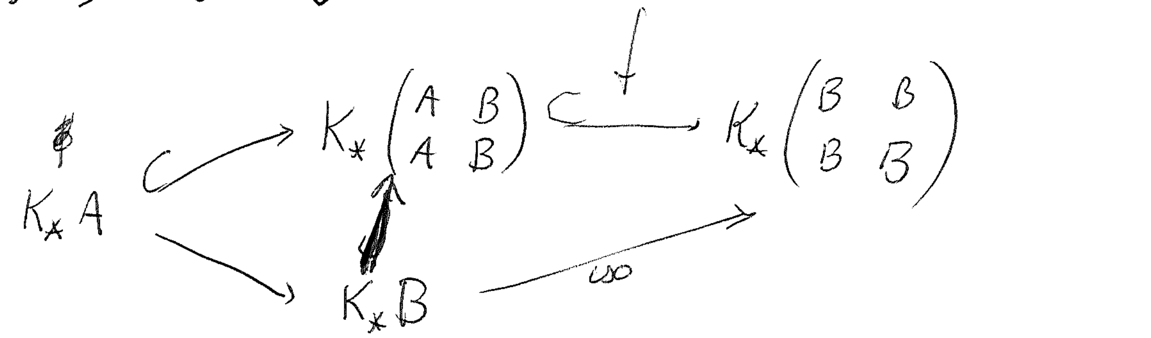
$$\begin{array}{ccc} K_*A & \longrightarrow & K_*\begin{pmatrix} A & B \\ A & B \end{pmatrix} \\ & \searrow & \uparrow \\ & & K_*B \end{array}$$

commutes.

 Thus should be clear because we have homos.



Thus it seems that



so we conclude $K_* A \hookrightarrow K_* B$ is injective

So you should be able to prove that $K_* B \xrightarrow{\sim} K_* \begin{pmatrix} A & B \\ A & B \end{pmatrix}$

But in any case you should know that a one-sided meq homomorphism is always injective on K_*. So then you have

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To see how much can be done with the injectivity arg.

Get theorem. Basic result is that given $\begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix} = C$ with P right A flat, $QP = A$, then $K_* A \to K_* C$ is injective.

P flat $\Rightarrow$ $P = \varinjlim F_\alpha$ $\quad F_\alpha \in \mathcal{P}(\tilde{A}^{op})$

$$P = P \otimes_A A = \varinjlim F_\alpha \otimes_A A \qquad C_\alpha = \begin{pmatrix} A & Q \\ F_\alpha \otimes_A A & F_\alpha \otimes_A Q \end{pmatrix}$$

~~But~~ Note the pairing $(A \quad Q) \begin{pmatrix} A \\ F_\alpha \otimes_A A \end{pmatrix} \longrightarrow A$ is surjective since $A \otimes A \to A$. ~~It should be true that~~

So C_α is the same sort of ring as C.

Situation: You have $B = P \otimes_A Q$ where P is a right A-module, Q is a left A-module and one is given a bimod map $Q \otimes P \to \tilde{A}$. Now assume P is $\tilde{A}^{op}$-flat, hence a filtered ind limit of modules $P_\alpha \in \mathcal{P}(\tilde{A}^{op})$, and $B = \varinjlim B_\alpha$, $B_\alpha = P_\alpha \otimes_A Q$. (Now everything said really makes sense for ~~mod~~ unitary modules over a unital ring R instead of $\tilde{A}$.)

Now $Q \otimes P_\alpha \to \tilde{A}$ is eq to $Q \to \mathrm{Hom}_{A^{op}}(P_\alpha, \tilde{A}) = P_\alpha^\vee$ and we get for each α a homom.

$$B_\alpha = P_\alpha \otimes_A Q \longrightarrow P_\alpha \otimes_A \check{P}_\alpha = \mathrm{Hom}_{A^{op}}(P_\alpha, P_\alpha).$$

So we really have for each α an induced map $K_*(B_\alpha) \to K_*(\tilde{A})$. The natural question is whether these are compatible for different α. Seems unlikely but one doesn't know. Notice that everything above ~~involves $\tilde{A}$ unital~~ generalizes from $\tilde{A}$ to any unital ring.

Let R be unital, $P \in Mod(R^{op})$, $Q \in Mod(R)$ given $Q \otimes P \to R$. ~~Now the question is~~ Form $B = P \otimes_R Q$ and ask whether there is a natural homom. $K_*(\tilde{B}) \to K_*(R)$.

So if P is flat

$$R \to \begin{pmatrix} R & Q \\ P & \tilde{B} \end{pmatrix} \leftarrow \tilde{B}$$

0945 So we have forgotten to treat the unital case thoroughly. I already ~~handled~~ treated the case $A \in \mathcal{P}(A^{op})$. Then I can prove that $K_*(A) = K_*(\tilde{A})/K_*(\mathbb{Z})$ is $K_*(\mathcal{P}(A^{op}))$. Thus if A, B are two meq rings satisfying $A \in \mathcal{P}(A^{op})$, $B \in \mathcal{P}(B^{op})$ then $K_*A = K_*B$.

For example if B is unital this holds.

I started today by considering the case of a ring $B = P \otimes_A Q$, P right, Q left A-module, with $Q \otimes P \xrightarrow{\langle , \rangle} A$. I want to show that if P is flat, then there is a natural map $K_*(B) \longrightarrow K_*(A)$. Argument: Can suppose $P \in \mathcal{P}(\tilde{A}^{op})$. All this firmness stuff becomes irrelevant here. Suppose $P \in \mathcal{P}(\tilde{A}^{op})$, $Q \in Mod(\tilde{A})$, $Q \otimes P \xrightarrow{\langle , \rangle} \tilde{A}$ any A-bimodule map. Then we have the ring $B = P \otimes_A Q$ non-unital, unital M context

$$\begin{pmatrix} \tilde{A} & Q \\ P & \tilde{B} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{A} & \check{P} \\ P & P \otimes_A \check{P} \end{pmatrix}$$

$$P \otimes_A \check{P} = Hom_{A^{op}}(P, P)$$

Thus we get a unital ring hom. $\tilde{B} \longrightarrow Hom_{A^{op}}(P,P)$, better P is a representation of $\tilde{B}$ in $\mathcal{P}(\tilde{A}^{op})$, so we get $K_*(\tilde{B}) \longrightarrow K_*(\tilde{A})$ posse

So fix A non-unital, and an A-module Q, suppose given $P \in \mathcal{P}(\tilde{A}^{op})$ and an A-map $P \longrightarrow Hom_A(Q, \tilde{A})$; latter same as a bimodule map $Q \otimes P \longrightarrow \tilde{A}$, and it gives rise to a ring structure on $P \otimes_A Q$. Canonical hom.

$$P \otimes_A Q \longrightarrow P \otimes_A Hom_{A^{op}}(P, \tilde{A}) = Hom_{A^{op}}(P, P)$$

hence a map $K_*(\widetilde{P \otimes_A Q}) \longrightarrow K_*(\tilde{A})$. Next suppose we have another $P' \in \mathcal{P}(\tilde{A}^{op})$ with $P' \longrightarrow Hom_A(Q, \tilde{A})$ and a map $P \longrightarrow P'$ compatible with pairings. Then get a homom. $P \otimes_A Q \longrightarrow P' \otimes_A Q$ and can ask whether

$$\begin{array}{ccc} K_*(\widetilde{P \otimes_A Q}) & \longrightarrow & K_*(\widetilde{P' \otimes_A Q}) \\ & \searrow & \downarrow \\ & & K_*(\tilde{A}) \end{array}$$

commutes.

situation to understand. Suppose first we ~~restrict~~ consider the case of [illegible] meq given by $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where A is unital and B is right flat. Then P is A^{op}-flat so we have $P = \text{filt.} \varinjlim P_\alpha$, $P_\alpha \in \mathcal{P}(\tilde{A}^{op})$. Then $B = \varinjlim_\alpha B_\alpha$ $\quad B_\alpha = P_\alpha \otimes_A Q$. Note that because A is unital one has $Q \otimes P_\alpha \to A$ surjective for $\alpha \geq$ some α_0. So ~~we have~~ B_α is meq A via

$$\begin{pmatrix} A & Q \\ P_\alpha & B_\alpha \end{pmatrix}$$

But $P_\alpha \in \mathcal{P}(A^{op}) \implies P_\alpha \otimes_A Q = B_\alpha \in \mathcal{P}(B_\alpha^{op})$ so we are in a situation we can handle. We know that $\mathcal{P}(B_\alpha^{op}) \hookrightarrow \mathcal{P}(\tilde{B}_\alpha^{op})$ induces $K_*(\mathcal{P}(B_\alpha^{op})) \xrightarrow{\sim} K_*(B_\alpha)$

~~Note~~ Perhaps it's worthwhile taking $\begin{pmatrix} A & Q \\ P & B = P \otimes_A Q \end{pmatrix}$ with $P \in \mathcal{P}(A^{op})$ A unital and trying to see directly why $K_*(B)$ is independent of Q. Here Q can be any A-mod equipped with $Q \to \overset{\vee}{P} = \operatorname{Hom}_{A^{op}}(P, A)$ such that QP is surjective. My method was to start with $V \in \mathcal{P}(\tilde{B}^{op})$. What do you have? ~~You have~~ ~~something ridiculously simple. Start~~

It seems the critical idea is that there are two maps

$$K_*(A) \to K_*(B) \subset K_*(\tilde{B}) \to K_*(A).$$

$$\underset{\mathcal{P}(A^{op})}{\mathcal{P}(A^{op}) \ni U} \longmapsto \underset{\mathcal{P}(B^{op}) \subset \mathcal{P}(\tilde{B}^{op})}{U \otimes_A Q} \longmapsto U \otimes_A Q \otimes_B P = U \otimes_A A = U$$

$$V \longmapsto V \otimes_B P \otimes_A Q = V \otimes_B B$$

Anyway the exact sequence show the image of the idempotent operator on $K_*(\tilde{B})$ ~~give~~ induced by $V \mapsto V \otimes_B B$ is exactly $K_*(B)$.

Next we need to see what can be done???? when A is not unital.

Let's begin with $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s.f.m., P rt flat

Write P as $\varinjlim P_i$ with $P_i \in \mathcal{P}(\tilde{A}^{op})$, ~~...~~

Then $B = \varinjlim B_i$ $B_i = P_i \otimes_A Q$. Assume that $QP_i = A$ so that A and B_i are meq. ~~...~~ In particular any transition $P_i \to P_j$ yields a meq hom $B_i \to B_j$. Note that B_i acts on P_i which is in $\mathcal{P}(\tilde{A}^{op})$. Thus there is a canon map $K_*(\tilde{B}_i) \to K_*(\tilde{A})$, hence $K_*(B_i) \to K_*(A)$. ~~...~~ To show these maps for different i are compatible. So what happens? ~~Yes Take pilch~~ Consider $P_1 \to P_2$ a map in $\mathcal{P}(\tilde{A}^{op})$. How to set this up?

Q is a fixed firm A-module. I know nothing about it except that ~~that~~ there are compatible pairings

$$\begin{array}{ccc} Q\otimes P_1 & \longrightarrow & Q \otimes P_2 \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

which I am assuming are surjective. The $P_i \in \mathcal{P}(\tilde{A}^{op})$, and the s.f. contexts are $\begin{pmatrix} A & Q \\ P_i A & P_i \otimes_A Q \end{pmatrix}$. ~~...~~

Focus on the repn of $B_i = P_i \otimes_A Q$ on $P_i \in \mathcal{P}(\tilde{A}^{op})$. You get $K_*(B_i) \to K_*(A)$. Relation between these representations?

You have

$$\begin{array}{ccc} B_1 \otimes P_1 & \longrightarrow & P_1 \\ w \otimes u \downarrow & & \downarrow u \\ B_2 \otimes P_2 & \longrightarrow & P_2 \end{array}$$

You want to see somehow that the repn P_2 of B_2, when restricted to B_1, is somehow "K-equivalent" to P_1 even though ~~we have~~ $P_1 \to P_2$ ~~is~~ pretty much arbitrary. We do know that u is a B-nil isom.

$$P \xrightarrow{u} P' \qquad p_0 \mapsto u(p_0) \qquad p'$$

$$p\,v(q)\,p' \leftarrow p'$$

$$P \qquad (pq)p_0 = p\,v(q)\,u(p_0) \qquad p\,v(q)\,p' \mapsto u(p\,v(q)\,p') = u(p)\,v(q)\,p' = w(pq)\,p'.$$

The point is that up to B-nil modules, you know that the rep. of B on P' is the same as the rep. of B' on P' restricted to B. So factor u as

$$P \xrightarrow{\Gamma_u} P \oplus P' \xrightarrow{pr_2} P'$$

So it seems you get ~~exact~~ exact sequences of repn of B in $\mathcal{P}(\tilde{A}^{op})$

Let's be careful. Start with A firm, ~~~~ of $\mathcal{M}(A)$

Start with A nonunital, let $P \xrightarrow{u} P'$ be a map in $\mathcal{P}(\tilde{A}^{op})$, let Q be an A-module, let $Q \otimes P' \to A$ be an A-bimod map. Put $B = P \otimes_A Q$, $B' = P' \otimes_A Q$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 1 \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q \\ P' & B' \end{pmatrix}$$

B acts on P in $\mathcal{P}(\tilde{A}^{op})$ so get $K_*(B) \to K_*(\tilde{A}_-)$

B' —— P' —— so get $K_*(B') \nearrow$

Are these maps comp. with the map $K_*(B) \to K_*(B')$ induced by w. We can factor P' into

$$P \xrightarrow[\binom{1}{u}]{\Gamma_u} P \oplus P' \xrightarrow[(0\ 1)]{pr_2} P'$$

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so can suppose u a direct injection or surj in $\mathcal{P}(\tilde{A}^{op})$. But other point is that [illegible] u is a B-mod isom. Thus B kills the kernel + cokernel of u. (Something involving the complex.)

Let [illegible] $p_0 q_0$ be a gen. of B. $u(p) = 0$. Then

$$w(p_0 q_0)p = p_0 v(q_0) u(p) = 0.$$

$$\begin{array}{ccc} P & \xrightarrow{u} & P' \\ & \swarrow \phi & \\ P & \longrightarrow & P' \end{array}$$

$$\phi(p') = p_0(v(q_0)p')$$

$$\phi(u(p)) = p_0(q_0 p) = (p_0 q_0) p$$

$$p' \overset{\phi}{\mapsto} p_0(v(q_0)p') \longmapsto u(p_0 v(q_0) p')$$

$$u(p_0) v(q_0) p' = w(p_0 q_0) p'$$

~~Thus we will have. I think this is due~~

Observe for $P \oplus P' \xrightarrow{pr_2} P'$ that B acts trivially on the factor P, since this factor ~~pairs~~ trivially with Q.

So it seems that for any $\begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$ with [illegible] [illegible] P $\tilde{A}^{op}$-flat.

Go over the argument. ~~Given~~ $\left(\begin{smallmatrix} A & Q \\ P & B \end{smallmatrix}\right)$ ~~[illegible]~~

Given A, $Q \otimes P \to A$, assume P A^{op}-flat, then write $P = \varinjlim P_\alpha$ with $P_\alpha \in \mathcal{P}(\tilde{A}^{op})$ and the natural rep of $B_\alpha = P_\alpha \otimes_A Q$ on P_α defines $K_*(B_\alpha) \longrightarrow K_*(\tilde{A})$ compatible with transition, whence we get $K_*(B) \to K_*(\tilde{A})$. ~~In fact~~ [illegible] Observe that because [illegible] the pairing has values in A, we get $Q \longrightarrow \mathrm{Hom}_{A^{op}}(P_\alpha, A)$ so

$$P_\alpha \otimes_A Q \longrightarrow P_\alpha \otimes_A \mathrm{Hom}_{A^{op}}(P_\alpha, A)$$

which means that the rep with by 1 mod A.

So what's going on? ~~End result is unclear.~~ It seems as if we should proceed with $\begin{pmatrix} \tilde{A} & Q \\ P & \tilde{B} \end{pmatrix}$

Where are we now? $\begin{pmatrix} \tilde{A} & Q \\ P & \tilde{B} \end{pmatrix}$

Assume P is $\tilde{A}^{op}$ flat. Then $P = \varinjlim P_\alpha$. Maybe simpler to have $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ everything unital. Assume P is R^{op}-flat, then get $\begin{pmatrix} R & Q \\ P_\alpha & P_\alpha Q \end{pmatrix}$. $P_\alpha Q$ not an ideal in S.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xleftarrow{\lim} \begin{pmatrix} A & Q \\ P_\alpha & P_\alpha Q \end{pmatrix} \qquad P_\alpha Q = B_\alpha$$

No you need $P_\alpha \otimes_A Q$ to have an action on P_α
∴ you need $B = P \otimes_A Q$

$$\begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix} = \varinjlim_\alpha \begin{pmatrix} A & Q \\ P_\alpha & P_\alpha \otimes_A Q \end{pmatrix} \qquad P\otimes_A Q = B, \quad P_\alpha \otimes_A Q = B_\alpha$$

Assume $P_\alpha \in \mathcal{P}(\tilde{A}^{op})$
Get a repn of B_α on P_α
hence $K_*(B_\alpha) \to K_*(\tilde{A})$

But $B_\alpha P_\alpha = P_\alpha Q P_\alpha \subset P_\alpha A$ so get $K_*(B_\alpha) \to K_*(A)$.
I've checked compat, so get $K_*(B) \longrightarrow K_*(A)$. In good cases (sfrim) P A^{op} flat $\Leftrightarrow$ B is B^{op}-flat. ~~So the next point is to take~~
If A, B both right flat, then this argument gives us homom. in both directions.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \supset \begin{pmatrix} A & Q \\ P_\alpha & B_\alpha \end{pmatrix}$$

Is it possible to choose both $P_\alpha \longrightarrow P$ over A^{op} and $Q_\beta \to Q$ over B^{op} and get approximations $Q_\beta \otimes_{P_\alpha Q_\beta} P_\alpha$ for A and $P_\alpha \otimes_{Q_\beta P_\alpha} Q_\beta$ for B?

Suppose we ~~have~~ are given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. Let $P' \to P$ be an A^{op}-module map, let $Q' \to Q$ be a B^{op}-mod map. Consider

$$B' = P' \otimes_{\mathbb{Z}} Q' \Big/ \; p_1'(q_1' p_2') \otimes q_2' = p_1' \otimes q_1' p_2' q_2'$$

Somebody has axiomatized these you take P, Q abelian gps and $P \otimes Q \otimes P \to P$, $Q \otimes P \otimes Q \to Q$ products. First identity $p_1 q_1 p_2$

Consider ~~tensor algebra~~ $S = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$ and the tensor alg over S of $\left(\begin{array}{c|c} 0 & Q \\ P & 0 \end{array}\right)$

$$\left(\begin{array}{c|c} \mathbb{Z} & \\ \hline & \mathbb{Z} \end{array}\right) \oplus \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \oplus \begin{pmatrix} Q \otimes P & 0 \\ 0 & P \otimes Q \end{pmatrix} \oplus \begin{pmatrix} 0 & Q \otimes P \otimes Q \\ P \otimes Q \otimes P & 0 \end{pmatrix} \oplus \cdots\cdots$$

We then want a ~~quotient~~ ring which is of the form

$$\begin{pmatrix} \tilde{A} & Q \\ P & \tilde{B} \end{pmatrix}$$

Maybe you need to specific $P \otimes Q \otimes P \to P$ and $Q \otimes P \otimes Q \to Q$

Some relations needed

$$(p_1 q_1 p_2) q_2 p_3 = p_1 q_1 (p_2 q_2 p_3)$$
$$\parallel \qquad\qquad \parallel$$
$$p_1 (q_1 p_2 q_2) p_3$$

similarly with p's and q's interchanged

Claim (if true) would be that

$$A = Q \otimes P \Big/ \; (q_1 p_1 q_2) \otimes p_2 = q_1 \otimes (p_1 q_2 p_2)$$
$$B = P \otimes Q \Big/ \; (p_1 q_1 p_2) \otimes q_2 = p_1 \otimes (q_1 p_2 q_2)$$

~~Def~~ define action $\begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \otimes \begin{pmatrix} A & \\ & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & \\ & B \end{pmatrix}$

$$\begin{pmatrix} & q_0 \\ p_0 & \end{pmatrix} \cdot \begin{pmatrix} q_1 \tilde{\otimes} p_1 & \\ & p_2 \tilde{\otimes} q_2 \end{pmatrix} = \begin{pmatrix} 0 & \langle q_0 p_2 q_2 \rangle \\ \langle p_0 q_1 p_1 \rangle & 0 \end{pmatrix}$$

What we are doing is to consider a ring gen by elements $x = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$ subject to relation $x_1 x_2 x_3 = \langle x_1 x_2 x_3 \rangle$ so immediately we know the ring is a ~~quotient~~ of $X \oplus X \otimes X$. Since this will take a while to get straight

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$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longleftarrow \begin{pmatrix} A & Q \\ P' & B' \end{pmatrix} \longleftarrow \begin{pmatrix} A' & Q' \\ P' & B' \end{pmatrix}$$

$B' = P' \otimes_A Q$ $\qquad$ $A' = Q' \otimes_{B'} P'$

here $P' \longrightarrow P$ is an A^{op}-map

$Q' \longrightarrow Q$ is a B^{op}-map

~~Things~~ The above ~~appears~~ is asymmetrical as $B' = P' \otimes_A Q$. Symmetric possibilities might use the triple products.

$$P' \otimes Q' \otimes P' \longrightarrow P' \otimes Q \otimes P \longrightarrow P' \otimes A \rightarrow P'$$

$$Q' \otimes P' \otimes Q' \longrightarrow Q' \otimes P \otimes Q \longrightarrow Q' \otimes B \rightarrow Q'$$

Let $p_i' \in P'$ $\quad q_i' \in Q'$ $\quad i = 1, 2, 3.$

$$p_1' \otimes q_1' \otimes p_2' \otimes q_2' \otimes p_3'$$

Before getting involved in this Γ theory you might see if its needed. Another poss. is to introduce $B \otimes P' \rightarrow P$ and $A \otimes Q' \rightarrow Q$. These reminds me of dialgs.

~~$B \otimes P' \otimes A \otimes Q'$~~

$$\begin{matrix} A & A \otimes Q' \\ B \otimes P' & B \otimes P' \otimes_A A \otimes Q' \end{matrix}$$

This is certainly very tricky.

Review what we know

Consider $\begin{pmatrix} A & Q \\ P & B = P\otimes_A Q \end{pmatrix}$ $\qquad P$ flat $\tilde{A}^{op}$-module

write $P = \varinjlim P_i$, $P_i \in \mathcal{P}(\tilde{A}^{op})$, put $B_i = P_i \otimes_A Q$.

Then $B = \varinjlim B_i$, $K_*(B) = \varinjlim K_*(B_i)$.

From $Q \to Hom_{A^{op}}(P, A) \to Hom_{A^{op}}(P_i, A) = A \otimes_A \check{P}_i \to \check{P}_i$

we get a homom. $B_i = P_i \otimes_A Q \longrightarrow P_i \otimes_A A \otimes_A \check{P}_i \to P_i \otimes_A \check{P}_i = Hom_{A^{op}}(P_i, P_i)$.

This is a representation of B_i on $P_i \in \mathcal{P}(\tilde{A}^{op})$ trivial on $P_i / P_i A$, so we get a homom. $K_*(B_i) \xrightarrow{c_i} K_*(A) = \mathrm{Ker}(K_*(\tilde{A}) \to K_*(\mathbb{Z}))$.

Claim that the c_i are compatible, so we get

$K_*(B) = \varinjlim K_*(B_i) \longrightarrow K_*(A)$.

Proof of compatibility. It suffices to show that given

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q \\ P' & B' \end{pmatrix}, \qquad B = P \otimes_A Q, \quad B' = P' \otimes_A Q$$

where $P, P' \in \mathcal{P}(\tilde{A}^{op})$, that the repn of B' on P' restricted to B is somehow equiv to the rep of B on P. Now we know $P \xrightarrow{u} P'$ is a B-mod isom. ~~Can suppose~~

Can factor $P \xrightarrow{\Gamma_u} P \oplus P' \xrightarrow{pr_2} P'$. Say u der. inj. w coker P''.

$$0 \to P \to P' \to P'' \to 0 \quad \text{exact in } \mathcal{P}(\tilde{A}^{op})$$

repns of B such that $BP'' = 0$.

$$\tilde{B} \rightrightarrows \begin{pmatrix} End_{A^{op}}(P) & Hom_{A^{op}}(P'', P) \\ 0 & \mathbb{Z} \end{pmatrix} \subset End_{A^{op}}(P')$$

$$\downarrow$$

$$End_{A^{op}}(P)$$

~~and then $K_*(\tilde{B}) \to$~~

CLEAR by the unital theorem.

$$K_* \quad \rightrightarrows \cdot \longrightarrow \cdot$$

$$\searrow \quad \downarrow \cong$$

$$\cdot$$

So the problem now becomes to consider comp.
Notice that in this argument no assumptions about $A = QP$ are made, although we do assume $B = P \otimes_A Q$. It's actually a central ring then.

Given R unital, $P \in Mod(R^{op})$, $Q \in Mod(R)$ and $Q \otimes P \longrightarrow R$ any R-biread map. Assume P flat, then there's a canon. hom. $K_*(P \otimes_R Q) \longrightarrow K_*(R)$. Probably

$K_*(P \otimes_R Q) \longrightarrow K_*(R \rightarrow R/QP)$

Suppose now given $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ P, Q right flat

can you calculate the composition??

You better take $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where P is $\tilde{A}$-flat
Q is $\tilde{B}$-flat
$P \otimes_A Q \xrightarrow{\sim} B$, $Q \otimes_B P \xrightarrow{\sim} A$

Then get $K_*(A) \rightleftarrows K_*(B)$
Maybe you want a transitivity result first.

$$\begin{pmatrix} R & & \\ P & S & \\ & & T \end{pmatrix}$$

So the real problem is what to do about A flat firm

$\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ if $A \in \mathcal{P}(A^{op})$

So I take the limit over $P \rightarrow A$ with $P \in \mathcal{P}(\tilde{A}^{op})$

$$\begin{matrix} A & A \\ P & P \otimes_A A = PA \end{matrix}$$

you get $P \otimes_A A \longrightarrow Hom_{A^{op}}(P, P)$

10:00 So the real problem is to show that for a firm flat ring A say right flat that the map $K_*A \to K_*A$ defined by $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ is the identity. Special case: $A \in \mathcal{P}(A^{op})$. Consider more generally $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with $P \in \mathcal{P}(A^{op})$. Then P itself is cofinal in the cat $\mathcal{P}(\tilde{A}^{op})/P$, so that the map $K_*B \to K_*A$ is induced by $B \to \mathrm{Hom}_{A^{op}}(P,P)$.

More precisely

$$\begin{array}{ccccccc} K_*B \subset K_*\tilde{B} & \longrightarrow & K_*(\mathrm{Hom}_{A^{op}}(P,P)) & \longrightarrow & K_*(\tilde{A}) \\ \| & & & & \| \\ K_*(\mathcal{P}(\tilde{B}^{op})) & \longrightarrow & K_*(\mathcal{P}(A^{op})) & \longrightarrow & K_*(\mathcal{P}(\tilde{A}^{op})) \end{array}$$

When $A \in \mathcal{P}(A^{op})$. $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ have

$$\begin{array}{ccc} \mathcal{P}(\tilde{A}^{op}) & \longrightarrow & \mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op}) \\ U & \longmapsto & U \otimes_A A \end{array}$$

get an idempotent operator on $\mathcal{P}(\tilde{A}^{op})$ with im $\mathcal{P}(A^{op})$.

Check first idempotent.

A firm + right flat

So what can one expect?

Case to ~~consider~~:

Perhaps it's easiest to prove <u>idempotence</u>.

~~How to prove~~. Probably part of trans.

$$\begin{pmatrix} A & Q & \\ P & B & Q' \\ & P' & C \end{pmatrix}$$

 ~~Suppose $B = P \otimes_A Q$~~

Think carefully about what is needed

Basic construction concerns R, P flat rgt, Q arb., $Q \otimes P \to R$ bimod map

Then can define

$$\begin{array}{ccc} K_*(P \otimes_R Q) & \longrightarrow & K_*(R) \\ \downarrow & \nearrow & \\ K_*(P \otimes_R \mathrm{Hom}_{R^{op}}(P, R)) & & \end{array}$$

~~construction obvious~~ $P = \varinjlim P_\mu$, $P_\mu \in \mathcal{P}(R^{op})$

$$K_*(P \otimes_R Q) = \varinjlim K_*(P_\mu \otimes_R Q)$$

for each μ get $P_\mu \otimes_R Q \longrightarrow \mathrm{Hom}_{R^{op}}(P_\mu, P_\mu)$

$$K_*(\quad) \longrightarrow K_*(\quad \cdot\cdot \quad) \to K_*(R).$$

and you can check consistency as I have done.

Lemma: Given R unitary, P flat in $\mathrm{Mod}(R^{op})$, $Q \in \mathrm{Mod}(R)$ and an R-bimod map $Q \otimes P \to R$ (equiv $Q \to \mathrm{Hom}_{R^{op}}(P, R)$) there's a well-defined map $K_*(P \otimes_R Q) \longrightarrow K_*(R)$.

Next point goes as follows.

What form should transitivity take

$$\begin{pmatrix} R & Q \\ P & P \otimes_R Q \end{pmatrix} \longrightarrow \begin{pmatrix} R & Q' \\ P' & P' \otimes_R Q' \end{pmatrix}$$

would say

$$\begin{array}{ccc} K_*(P \otimes_R Q) & \longrightarrow & K_*(P' \otimes_R Q') \\ & \searrow \quad \swarrow & \\ & K_*(R) & \end{array}$$

commutes. This should work easily, but I want something a bit more complicated

$$\begin{matrix} R & Q & Q' \\ P & P \otimes_R Q & P \otimes_R Q' \\ P' & P' \otimes_R Q & P' \otimes_R Q' \end{matrix} \qquad ?$$

What do I need? I have to get control of transitivity. It's not possible to work unitally. So let's suppose we have

$$\begin{pmatrix} A & Q & \\ P & B & U \\ & T & C \end{pmatrix}$$

We need the data to define $K_*(B) \to K_*(A)$.

This is $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with P A^{op}-flat and $P \otimes_A Q \xrightarrow{\sim} B$.

We need the same sort of data $\begin{pmatrix} B & U \\ T & C \end{pmatrix}$ T B^{op}-flat, $T \otimes_B U \xrightarrow{\sim} C$.

Then do we have composite data?

$$\begin{matrix} A & Q \otimes_B U \\ T \otimes_B P & T \otimes_B P \otimes_A Q \otimes_B U \\ & \| \\ & T \otimes_B B \otimes_B U \end{matrix}$$

$\therefore$ $T \otimes_B P$ is A^{op} flat.

P A^{op}-flat $\Rightarrow$ $M \mapsto P \otimes_A M$ exact $Mod(\tilde{A}) \to Mod(\tilde{B})$

T B^{op}-flat $\Rightarrow$ $N \mapsto T \otimes_B N$ exact $Mod(\tilde{B}) \to Mod(\tilde{C})$

It seems we need $T\otimes_B B\otimes_B \mathcal{U} = T\otimes_B \mathcal{U}$.

~~Which~~ How can I proceed?

Suppose we were to do everything from the viewpoint of A.

$$\begin{array}{ccc} A & Q & Q' \\ P & P\otimes_A Q \;(=B) & P\otimes_A Q' = U \\ P' & P'\otimes_A Q \;(=T) & P'\otimes_A Q' = C \end{array}$$

so certainly you have

$$T\otimes B\otimes U \rightrightarrows T\otimes U \longrightarrow T\otimes_B U \to 0$$

$$T\otimes B\otimes U = T\otimes P\otimes_A Q\otimes U$$

$$(P'\otimes_A Q)\otimes(P\otimes_A Q)\otimes(P\otimes_A Q') \rightrightarrows (P'\otimes_A Q)\otimes(P\otimes_A Q') \longrightarrow T\otimes_B U \longrightarrow 0$$

Look: $T\otimes_B U = P'\otimes_A Q\otimes_B P\otimes_A Q' = P'\otimes_A(Q\otimes_B P\otimes_A Q')$

$$\begin{array}{ccc} A & Q & Q' \\ P & P\otimes_A Q = B & \\ T\otimes_B P & T & \end{array}$$

$$\begin{array}{cccc} & A & Q & Q' \\ & & \| & \| \\ P = & \tilde{A} & B & U \\ & & & \| \\ P' = & \tilde{A} & \tilde{B} & C \end{array}$$

$\tilde{B}\otimes_B \tilde{A} = T\otimes_B P \qquad \tilde{B} \neq T$

model.

A firm ring

$$\begin{array}{ccc} A & Q & Q\otimes_B U \\ \tilde{A}^n & \tilde{A}^n\otimes_A Q = B & \text{--}\; U \end{array}$$

$$\tilde{A}^{kn} = \tilde{B}^k\otimes_B \tilde{A}^n \qquad \tilde{B}^k$$

to find what you need. At the beginning you have $P \in \mathcal{P}(\tilde{A}^{op})$ with B acting on P, maybe you even want $B = P \otimes_A Q$ where Q is an A-mod eq. to $Q \longrightarrow \mathrm{Hom}_{A^{op}}(P, \tilde{A})$. In any case you have $B \longrightarrow \mathrm{Hom}_{A^{op}}(P,P) = P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})$. The next thing you need is $T \in \mathcal{P}(\tilde{B}^{op})$ with C acting on T, which means $C \longrightarrow \mathrm{Hom}_{B^{op}}(T,T) = T \otimes_B \mathrm{Hom}_{B^{op}}(T, \tilde{B})$.

Let's see if I can describe the basic structure!

$$\left.\begin{array}{ll} {}_B P_A & P \in \mathcal{P}(\tilde{A}^{op}) \\ {}_C T_B & T \in \mathcal{P}(\tilde{B}^{op}). \end{array}\right) \Longrightarrow {}_C(T \otimes_B P)_A \overset{?}{\in} \mathcal{P}(\tilde{A}^{op})$$

ignore C, then T summand of $\tilde{B}$ and $\tilde{B} \otimes_B P = P$.

How big can B be? $B \longrightarrow \mathrm{Hom}_{A^{op}}(P,P)$, but you might it better to land in $P \otimes_A A \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})$.

But if you enlarge B to B', then you should be able to change T to $T \otimes_B \tilde{B}'$. Note that

$$T \otimes_B \tilde{B}' \otimes_{B'} P = T \otimes_B P$$

and that $T \in \mathcal{P}(\tilde{B}^{op}) \Longrightarrow T \otimes_B \tilde{B}' \in \mathcal{P}(\tilde{B}'^{op})$.

So what seems to emerge is that the choice of Q may not be so important at all.

Now look at the desired situation

You want $K_*(B) \to K_*(A)$ when there is a flat $\tilde{A}$ module P on which B acts by finite rank operators, e.g. $B = P \otimes_A Q$

$Q \longrightarrow Hom_{A^{op}}(P, \tilde{A})$, more gen. $B \longrightarrow P \otimes_A Hom_{A^{op}}(P, \tilde{A})$

Suppose given $A \quad P \quad B \quad T \quad C$ ~~such that~~ where P A^{op}-flat, T B^{op}-flat, $B \longrightarrow P \otimes_A Hom_{A^{op}}(P, \tilde{A})$, $C \longrightarrow T \otimes_B Hom_{B^{op}}(T, \tilde{B})$. Form $T \otimes_B P$ this is A^{op}-flat, and you have to check C acts by fin. rank (nuclear operators)

$$\left(T \otimes_B P\right) \otimes_A \underbrace{Hom_{A^{op}}\left(T \otimes_B P, \tilde{A}\right)}$$

$$\| $$

$$\left(T \otimes_B P\right) \otimes_A Hom_{B^{op}}\left(T, Hom_{A^{op}}(P, \tilde{A})\right)$$

$$\big\downarrow$$

$$C \longrightarrow T \otimes_B Hom_{B^{op}}\left(T, P \otimes_A Hom_{A^{op}}(P, \tilde{A})\right)$$

OKAY

There seems to be a problem here.

So let's go over this. You have P flat over A^{op}, $B \longrightarrow P \otimes_A Hom_{A^{op}}(P, \tilde{A})$, T flat over B^{op}, $C \to T \otimes_B Hom_{B^{op}}(T, \tilde{B})$. Then $T \otimes_B P$ flat over A^{op} and the question is whether $\exists$

$$C \overset{?}{\longrightarrow} (T \otimes_B P) \otimes_A Hom_{A^{op}}(T \otimes_B P, \tilde{A})$$

$$\|$$

$$T \otimes_B P \otimes_A Hom_{B^{op}}\left(T, Hom_{A^{op}}(P, \tilde{A})\right)$$

given

$$C \longrightarrow T \otimes_B Hom_{B^{op}}\left(T, P \otimes_A Hom_{A^{op}}(P, \tilde{A})\right)$$

 You have to work around this problem.

The point might be that ~~you want to lift~~ already you need to lift finite rank ops i.e. the image of $P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A}) \longrightarrow \mathrm{Hom}_{A^{op}}(P,P)$ back to this tensor product. Therefore it's not such a big deal to ask for a lifting

$$\begin{tikzcd}
 & (T \otimes_B P) \otimes_A \mathrm{Hom}_{B^{op}}(T, \mathrm{Hom}_{A^{op}}(P, \tilde{A})) \arrow[d] \\
C \arrow[ur, dashed] \arrow[r] & T \otimes_B \mathrm{Hom}_{B^{op}}(T, P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A}))
\end{tikzcd}$$

Start with A, P and $B \longrightarrow \overbrace{P \otimes_A \mathrm{Hom}_{A^{op}}(P, \tilde{A})}^{Q}$

and C with $C \longrightarrow (T \otimes_B P) \otimes_A \underbrace{\mathrm{Hom}_{A^{op}}(T \otimes_B P, \tilde{A})}_{U}$

$$\begin{array}{ccc}
A & Q & \\
P & B & U \\
T \otimes_B P & T &
\end{array}$$

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Review the basic idea. Given a ring A, a flat A^{op} module P, a ring B and homom.

$$B \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$

we get a map $K_*(B) \to K_*(A)$. How? Let $Q = \mathrm{Hom}_{A^{op}}(P, A)$. Enough to treat $B = P \otimes_A Q$. $P = \varinjlim P_i$ filtered colim $P_i \in \mathcal{P}(\tilde{A}^{op})$ and $K_*(B) = \varinjlim K_*(B_i)$ $B_i = P_i \otimes_A Q$

have $P_i \otimes_A Q \longrightarrow P_i \otimes_A \mathrm{Hom}_{A^{op}}(P_i, A) \longrightarrow \mathrm{Hom}_{A^{op}}(P_i, P_i)$

induces $K_*(B_i) \longrightarrow K_*(A) = \mathrm{Ker}(K_*(\tilde{A}) \to K_*(\mathbb{Z}))$

 ~~So what am I going to do?~~

Check compatible with a map $P_i \to P_j$ (with both mapping to P)

Now want to check trans.

A	P'	
P	B	T'
$T \otimes_B P$	T	C

P is A^{op}-flat

$B \to P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$ given

T is B^{op}-flat

$C \to T \otimes_B \mathrm{Hom}_{B^{op}}(T, B)$ given

Then we can form: $T \otimes_B P$ is A^{op}-flat, need

$$C \overset{?}{\longrightarrow} (T \otimes_B P) \otimes_A \mathrm{Hom}_{A^{op}}(T \otimes_B P, A)$$

$$\| $$

$$T \otimes_B P \otimes_A \mathrm{Hom}_{B^{op}}(T, \mathrm{Hom}_{A^{op}}(P, A))$$

$$\downarrow$$

$$C \downarrow \qquad T \otimes_{B^{op}} \mathrm{Hom}(T, B) \longrightarrow T \otimes_B \mathrm{Hom}_{B^{op}}(T, P \otimes_A \mathrm{Hom}_{A^{op}}(P, A))$$

Assume such a lifting is given, i.e. ~~an action of~~ ~~[illegible]~~ a ~~finite rank~~ nuclear action of C on $T \otimes_B P$ compatible with the ~~[illegible]~~ nuclear actions of B on P and C on T. Can we get transitivity?

~~[illegible]~~ Digress you would like to know what right flat modules T over $B = P \otimes_A P'$ look like. You would like

suppose $C \to T \otimes_B \mathrm{Hom}_{B^{op}}(T, B)$ given and also $B \to B'$

$$\|$$

$$T \otimes_B \tilde{B}' \otimes_{B'} \mathrm{Hom}_{B'^{op}}(T \otimes_B \tilde{B}', B')$$

$$\mathrm{Hom}_{B^{op}}(T, B)$$

 transitivity.

$$\begin{array}{ccc} A & & \\ P & B & \\ Q\otimes_B P & Q & C \end{array}$$

P A^{op} flat, $B \to P\otimes_A \mathrm{Hom}_{A^{op}}(P,A)$

Q B^{op} flat, $C \to Q\otimes_{B} \mathrm{Hom}_{B^{op}}(Q,B)$

$Q\otimes_B P$ is A^{op} flat

$$C \xrightarrow{\exists} (Q\otimes_B P)\otimes_A \mathrm{Hom}_{A^{op}}(Q\otimes_B P, A)$$

$$\| $$

$$Q\otimes_B P\otimes_A \mathrm{Hom}_{A^{op}}(Q, \mathrm{Hom}_{A^{op}}(P,A))$$

$$\downarrow$$

$$C \downarrow \qquad Q\otimes_B \mathrm{Hom}_{B^{op}}(Q,B) \longrightarrow Q\otimes_B \mathrm{Hom}_{B^{op}}(Q, P\otimes_A \mathrm{Hom}_{A^{op}}(P,A))$$

Let $B' = P\otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ $\qquad Q' = Q\otimes_B \tilde{B}'$

$$Q'\otimes_{B'} X = Q\otimes_B \tilde{B}'\otimes_{B'} X = Q\otimes_B X$$

Replacing B, Q by B', Q' doesn't affect the map.

bottom horizontal arrow is an isomorphism. Can suppose then that $B = P\otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ and $C = Q\otimes_B P\otimes_A \mathrm{Hom}_{A^{op}}(Q\otimes_B P, A)$

First step is to approximate Q by $Q_i \in \mathcal{P}(\tilde{B}^{op})$.

~~take a set of elements in~~

You want $C \longrightarrow Q\otimes_B \mathrm{Hom}_{B^{op}}(Q,B)$

$$\begin{array}{cccc} A & \mathrm{Hom}_{A^{op}}(P,A) & \mathrm{Hom}_{A^{op}}(Q\otimes_B P, A) & \\ P & B & \mathrm{Hom}_{B^{op}}(Q,B) & \\ Q\otimes_B P & Q & & C \end{array}$$

$$\mathrm{Hom}_{A^{op}}(P,A)\otimes_B \mathrm{Hom}_{B^{op}}(Q,B) \swarrow$$

$$\mathrm{Hom}_{A^{op}}(Q\otimes_B P, A) = \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P,A))$$

61 Find the hypotheses you need to understand things.

original viewpoint

A, P A^{op}-flat, $B \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$

Q, B^{op}-flat, $C \longrightarrow Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, B)$ but

you want

$$C \longrightarrow (Q \otimes_B P) \otimes_A \mathrm{Hom}_{A^{op}}(Q \otimes_B P, A)$$

$$\| $$

$$Q \otimes_B P \otimes_A \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P, P))$$

$$\downarrow \text{canon.}$$

$$C \downarrow \qquad Q \otimes \mathrm{Hom}_{B^{op}}(Q, B) \longrightarrow Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, P \otimes_A \mathrm{Hom}_{A^{op}}(P, A))$$

observe that given $B \to B'$, put $Q' = Q \otimes_B \tilde{B}'$, then

$$B \searrow, \ B' \swarrow \ P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$$

$$Q' \otimes_{B'} P = Q \otimes_B \tilde{B}' \otimes_{B'} P = Q \otimes_B P$$ more gen for $P \in \mathrm{Mod}(\tilde{B'})$

Also $\mathrm{Hom}_{B'^{op}}(Q', -) = \mathrm{Hom}_{B'^{op}}(Q \otimes_B \tilde{B}', -) = \mathrm{Hom}_{B^{op}}(Q, -)$

So can assume $B = P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$. Then we are faced with the map

$$(Q \otimes_B P) \otimes_A \mathrm{Hom}_{A^{op}}(Q \otimes_B P, A) = Q \otimes_B P \otimes_A \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P, A))$$

$$\downarrow$$

$$Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, B) \simeq Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, P \otimes \mathrm{Hom}_{A^{op}}(P, A))$$

not necessarily being an isom. ~~The important case for us~~

 An idea here would be to use

$$\mathrm{Hom}_{A^{op}}(Q \otimes_B P, A) = \mathrm{Hom}_{B^{op}}(Q, \mathrm{Hom}_{A^{op}}(P, A))$$
$$\uparrow$$
$$\mathrm{Hom}_{A^{op}}(P, A) \otimes_B \mathrm{Hom}_{B^{op}}(Q, B)$$

A sufficient condition I guess is then that our $C \longrightarrow Q \otimes_B \mathrm{Hom}_{B^{op}}(Q, P \otimes_A \mathrm{Hom}_{A^{op}}(P, A))$ lifts

$$\uparrow$$
$$Q \otimes_B P \otimes_A \mathrm{Hom}_{A^{op}}(P, A) \otimes_B \mathrm{Hom}_{B^{op}}(Q, B)$$

This basically means something like:

$$\begin{array}{ccc} A & \mathrm{Hom}_{A^{op}}(P, A) & \mathrm{Hom}_{A^{op}}(P, A) \otimes_B \mathrm{Hom}_{B^{op}}(Q, B) \\ P & B & \mathrm{Hom}_{B^{op}}(Q, B) \\ Q \otimes_B P & Q & C \end{array}$$

Basically I need $P \otimes_A \mathrm{Hom}_{B^{op}}(Q, \check{P}) = \mathrm{Hom}_{B^{op}}(Q, P \otimes_A \check{P})$ and the only reasonable situation where this might hold is when I replace $\mathrm{Hom}_{B^{op}}(Q, \check{P})$ with
$\check{P} \otimes_B \mathrm{Hom}_{B^{op}}(Q, B)$

So it looks reasonable to consider

$$\begin{array}{ccc} A & P' & P'\otimes_B Q' \\ P & B = P\otimes_A P' & Q' \\ Q\otimes_B P & Q & C = Q\otimes_B Q' \end{array} \qquad ?$$

Now we need to get the approximations straight

Roughly we have

$$\begin{array}{c} A \\ P \end{array}$$

~~Wait suppose we consider the strictly f~~

$$\begin{array}{cc} A & \\ P & B \\ & Q \end{array}$$

to simplify pick $P' \longrightarrow Hom_{A^{op}}(P, A)$ and let $B = P\otimes_A P'$. Next we need $Q \in \mathcal{P}(\tilde{B}^{op})$.

Alternative.

Idea Suppose $P \in \mathcal{P}(\tilde{A}^{op})$, $B \longrightarrow P\otimes_A Hom_{A^{op}}(P, A) < Hom_{A^{op}}(P, P)$

$Q \in \mathcal{P}(\tilde{B}^{op})$, $C \longrightarrow Hom_{B^{op}}(Q, Q)$. You have

$Q\otimes_B P \in \mathcal{P}(\tilde{A}^{op})$ $Hom_{A^{op}}(Q\otimes_B P, Q\otimes_B P) =$

$$\underbrace{Hom_{B^{op}}(Q, Hom_{A^{op}}(P, Q\otimes_B P))}_{Q\otimes_B P\otimes_A \check{P}\otimes_B \check{Q}} = Q\otimes_B P\otimes_A Hom_{A^{op}}(P, A)\otimes_B Hom_{B^{op}}(Q, B)$$

unital situation: suppose everything unitary.
$P \in \mathcal{P}(A^{op})$, $B \to \mathrm{Hom}_{A^{op}}(P, P)$, $Q \in \mathcal{P}(B^{op})$

Then $Q \otimes_B P$ summand of finitely many copies of $B \otimes_B P = P$ so $Q \otimes_B P \in \mathcal{P}(A^{op})$. Also

$$\mathrm{Hom}_{A^{op}}(Q \otimes_B P, A) = \mathrm{Hom}_{B^{op}}(Q, \check{P}) = \check{P} \otimes_B \check{Q}.$$

$$\mathrm{Hom}_{A^{op}}(Q \otimes_B P, Q \otimes_B P) = Q \otimes_B P \otimes_A \check{P} \otimes_B \check{Q}$$

How can I make use of this? Suppose $P = eA$. $\check{P} = Ae$

Then $Q \otimes_B eAe \otimes_B \check{Q}$

First you must understand the unital case. $P \in \mathcal{P}(A^{op})$, $B \to \mathrm{Hom}_{A^{op}}(P, P)$ give a functor

$$\mathcal{P}(B^{op}) \longrightarrow \mathcal{P}(A^{op})$$
$$Q \longmapsto Q \otimes_B P$$

and this induces $K_*(B) \to K_*(A)$.

Transitivity is now obvious since given, besides ${}_BP_A$ above, ${}_CQ_B$ with $Q \in \mathcal{P}(B^{op})$, the composite functor

$$\mathcal{P}(C^{op}) \longrightarrow \mathcal{P}(B^{op}) \longrightarrow \mathcal{P}(A^{op})$$
$$R \longmapsto R \otimes_C Q \longmapsto R \otimes_C Q \otimes_B P$$

is assoc. to ${}_C(Q \otimes_B P)_A$. Completely clear.

Now generalize. Suppose P is right flat over A, given $B \to P \otimes_A \mathrm{Hom}_{A^{op}}(P, A)$, B non unital here. Now write $P = \varinjlim P_i$ $P_i \in \mathcal{P}(\tilde{A}^{op})$

 ~~[illegible]~~ put

$$B_i = B \times_{P \otimes_A \mathrm{Hom}_{A^{op}}(P,A)} P_i \otimes_A \mathrm{Hom}_{A^{op}}(P,A)$$

~~[illegible]~~ have

$$P_i \otimes_A \mathrm{Hom}_{A^{op}}(P,A) \to P_i \otimes \mathrm{Hom}_A$$

Most generality: ~~Assume~~ given A unital, P unitary A^{op}-module flat ~~[illegible]~~, $B \longrightarrow P \otimes_A \mathrm{Hom}_{A^{op}}(P,A)$. Then write $P = \varinjlim P_i$, $P_i \in \mathcal{P}(A^{op})$, put

$$\begin{array}{ccccc} B_i & \longrightarrow & P_i \otimes_A \mathrm{Hom}_{A^{op}}(P,A) & \longrightarrow & P_i \otimes_A \mathrm{Hom}_{A^{op}}(P_i,A) = \mathrm{Hom}_{A^{op}}(P_i,P_i) \\ \downarrow & & \downarrow & & \\ B & \longrightarrow & P \otimes_A \mathrm{Hom}_{A^{op}}(P,A) & & \end{array}$$

so get

$$\begin{array}{ccc} K_*(B_i) & \longrightarrow & K_*(A) \\ \downarrow & & \\ K_*(B) & & \end{array}$$

Prove consistency of maps, and so get $\varinjlim K_*(B_i) \longrightarrow K_*(A)$

~~Note $P_i \to P$~~

Notice that B in this ~~exact~~ situation ~~[illegible]~~ is effectively $P \otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ which is nonunital. So we want A ~~before~~ to have form $\tilde{A}$, and $B \to P_i \otimes_A \mathrm{Hom}_{A^{op}}(P,A)$ so B acts trivial mod A.

Now how am I supposed to handle transdualy