November 13, 1997

Let $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ be an arbitrary M. context such that $PQ = B$. To proves that $K_0(A) \xrightarrow{\sim} K_0(C)$. Note that if in addition $QP = A$, then we have

$$K_0(A) \xrightarrow{\sim} K_0(C) \xleftarrow{\sim} K_0(B).$$

Let $C' = \begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix}$ be the M. context assoc. to the dual pair (P,Q) over A. Since $PQ = B$ we have a ring extension

$$\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \hookrightarrow C' \twoheadrightarrow C$$

where $K = \mathrm{Ker}\{P\otimes_A Q \longrightarrow B\}$. From the identity $(p\otimes q)p_1 = p\langle q, p_1\rangle = (pq)p_1$ we have $KP = 0$, and similarly $QK = 0$, whence $K(P\otimes_A Q) = 0 = (P\otimes_A Q)K$. Thus $\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$ is killed by C' on both sides, so C' is a nilpotent extension of C, whence $K_0(C') \xrightarrow{\sim} K_0(C)$.

Thus we can suppose $C = \begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix}$ if we want. By a colimit argument we can suppose P, Q are finitely presented right & left A-modules.

Consider a surjection $(P', Q) \xrightarrow{(f,1)} (P,Q)$ of dual pairs over A and let $N = \mathrm{Ker}(f)$. Then we have

$$N\otimes_A Q \longrightarrow P'\otimes_A Q \longrightarrow P\otimes_A Q \longrightarrow 0$$

where $(P'\otimes_A Q)(N\otimes_A Q) = P'\langle Q, N\rangle \otimes Q = 0$, since $\langle q, p'\rangle = \langle q, f(p')\rangle$ so $\langle Q, N\rangle = \langle Q, f(N)\rangle = 0$.

Thus $P'\otimes_A Q \twoheadrightarrow P\otimes_A Q$ is a ring surjection whose kernel I is killed by $P'\otimes_A Q$ on the left. In particular $I^2=0$ so $K_0(P'\otimes_A Q) \xrightarrow{\sim} K_0(P\otimes_A Q)$.

It follows more generally, that

$$0 \to \begin{pmatrix} 0 & 0 \\ N & I \end{pmatrix} \to \begin{pmatrix} A & Q \\ P' & P\otimes_A Q \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix} \to 0$$

is a square zero extension, in fact $\begin{pmatrix} A & Q \\ P' & P'\otimes_A Q \end{pmatrix}\begin{pmatrix} 0 & 0 \\ N & I \end{pmatrix} = 0$.

Thus we can suppose P' is a free f.gen. A^{op} module. Similarly can assume $Q = \tilde{A}^{\oplus m}$.

Next, to show we can replace A by $\tilde{A}$.

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r] \arrow[d] & K_0(\tilde{A}) \arrow[r] \arrow[d] & K_0(\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & K_0\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[r] & K_0\begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix} \arrow[r] & K_0\begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix} \arrow[r] & 0
\end{tikzcd}$$

The rows are split exact by the short exact sequence for a ring extension C/I

$$K_1(C) \to K_1(C/I) \to K_0(I) \to K_0(C) \to K_0(C/I)$$

So we can assume A unital, $P = A^{\oplus n}$, $Q = A^{\oplus m}$ with arbitrary pairing $Q\otimes_{\mathbb{Z}} P \to A$. Then we have a homomorphism

$$\begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix} \longrightarrow \begin{pmatrix} A & P^{\vee} \\ P & P\otimes_A \check{P} \end{pmatrix} = M_{n+1}(A)$$

hence $K_0(A) \xrightarrow{\sim} K_0(M_{n+1}(A))$, we have now reached the following situation: A unital ring $D = M_{n+1}(A)$, a D-module map $f: C \to D$, which we can be viewed as a ring homom. where $c_1 c_2 = f(c_1)c_2$.

Assuming $f(C)D = D$ we wish to prove $K_0(C) \xrightarrow{\sim} K_0(D)$. Check:

$$\begin{pmatrix} A & \bar{Q} \\ P & \overline{P \otimes_A Q} \end{pmatrix} \begin{pmatrix} A & \check{P} \\ P & P \otimes_A \check{P} \end{pmatrix} = \begin{pmatrix} A & \check{P} \\ P & P \otimes_A \check{P} \end{pmatrix}$$

But in this case I know by the Davydov analysis that $K_i(C) \xrightarrow{\sim} K_i(D)$ for all i.

November 24, 1997

Reading Pimsner's treatment of the Toeplitz algebra, I realized that interesting maps $K_*(B) \longrightarrow K_*(I)$ arise from a quasi-homom. $B \rightrightarrows R \supset I$ assuming I satisfies excision. In effect one has two maps $\mathbb{K}_B \rightrightarrows \mathbb{K}_R$ which become equal in $\mathbb{K}_{R/I}$, so the difference is a map from $\mathbb{K}_B$ to the fibre of $\mathbb{K}_R \to \mathbb{K}_{R/I}$, which is $\mathbb{K}_I$ when I satisfies excision. Here $\mathbb{K}_A$ denotes the space (spectrum?) yielding the K-theory of A.

I propose to understand well this construction in the case of K_0, which I believe satisfies excision. The Toeplitz algebra example suggests that I want to follow $K_*(B) \longrightarrow K_*(I)$ by a Morita equivalence $K_*(I) = K_*(A)$. So let's consider a dual pair (Y, X) over A, let R be its multiplier ring:

$$\begin{tikzcd}
R = \mathrm{Mult}(Y,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Hom}_A(X,X)^{op} \arrow[d] \\
\mathrm{Hom}_{A^{op}}(Y,Y) \arrow[r] & \mathrm{Hom}_{A,A^{op}}(X \otimes Y, A)
\end{tikzcd}$$

I can take B to be Milnor's ring $R \times_{R/I} R$. We have a canonical map $K_0(B) \to K_0(A)$ defined as follows. B is canonically $\Delta R \ltimes I$ – NO there are two obvious embeddings of I in $R \times_{R/I} R$. But there should be a canonical map $K_0(B) \longrightarrow K_0(I)$ given by $(pr_1)_* - (pr_2)_*$, at least up to signs. Here's how this arises.

We have $\Delta R \subset B = R \times_{R/I} R$ and two complementary ideals $I \times 0$, $0 \times I$. One can see this from the pull-back situation

$$\begin{tikzcd}
 & I \arrow[r, equal] \arrow[d] & I \arrow[d] \\
I \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & R \arrow[d] \\
I \arrow[r] & R \arrow[r] & R/I
\end{tikzcd}$$

So $K_0(B) = K_0(\Delta R) \oplus K_0(I)$ where there are possibly two choices for the second summand corresponding to the two embeddings $I \rightrightarrows B$. But we have $I \times I \subset B$ and $K_0(I \times I) = K_0(I) \oplus K_0(I)$. So the two maps $K_0(I) \to K_0(B)$ mentioned above arise from:

$$K_0(I) \underset{(0,1)}{\overset{(1,0)}{\rightrightarrows}} K_0(I) \oplus K_0(I) = K_0(I \times I) \to K_0(B)$$

On the other hand we know the diagonal map $I \xrightarrow{\Delta} I \times I \subset B$ will map $K_0(I)$ to $K_0(B)$ by the sum of the $(1,0)$ and $(0,1)$ embeddings. So it seems that these embeddings induce isomorphisms $K_0(I) \xrightarrow{\sim} K_0(B)/\Delta_* K_0(R)$ which are opposite in sign.

Better approach. Use Milnor's description of objects of $\mathcal{P}(B)$ as triples $(P, \bar{P}, \theta : P/IP \simeq \bar{P}/I\bar{P})$. By adding a diagonal object $(Q, Q, 1)$ one can suppose either P or $\bar{P}$ is free. Thus we get

$$(\underbrace{P \oplus \bar{P}^{\perp}}_{\text{reduces to an object of } \mathcal{P}(\tilde{I}) \text{ using } \theta \oplus 1}, \underbrace{\bar{P} \oplus \bar{P}^{\perp}}_{\text{free}}, \theta \oplus 1) \quad \text{or} \quad (\underbrace{P \oplus P^{\perp}}_{\text{free}}, \underbrace{\bar{P} \oplus P^{\perp}}_{\text{reduces to an object of } \mathcal{P}(\tilde{I}) \text{ using } \theta \oplus 1}, \theta \oplus 1)$$

This needs more work!

Given a triple $(P, \bar{P}, \theta)$ representing an object of $\mathcal{P}(B^{?})$. Lift $\theta : P/PI \xrightarrow{\sim} \bar{P}/\bar{P}I$ and θ^{-1} to maps $P \underset{p}{\overset{q}{\rightleftarrows}} \bar{P}$. Let $U = P \oplus \bar{P}$. We can think of U as a complex with diff'l p and homotopy operator q, or conversely. For the moment try to treat them symmetrically. We have an exact sequence (actually extension) of rings

$$0 \longrightarrow U \underset{R}{\otimes} I \underset{R}{\otimes} U^\vee \longrightarrow U \underset{R}{\otimes} U^\vee \longrightarrow U/UI \underset{R/I}{\otimes} U^\vee/IU^\vee \longrightarrow 0$$

$$1_U = \begin{pmatrix} 1_P & 0 \\ 0 & 1_{\bar{P}} \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}}_{\alpha} \longmapsto \varepsilon = \begin{pmatrix} 0 & \theta^{-1} \\ \theta & 0 \end{pmatrix}$$

Since $\alpha = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \longmapsto \varepsilon$ and $\varepsilon^2 = 1$, we have

$$1 - \alpha^2 = \begin{pmatrix} 1 - qp & 0 \\ 0 & 1 - pq \end{pmatrix} \equiv 0 \quad \text{modulo the ideal } U \underset{R}{\otimes} I \underset{R}{\otimes} U^\vee$$

of operators on U which are zero mod I.

I want to understand how to refine an α such that $1 - \alpha^2 \equiv 0 \mod I$ to an α' such that $1 - \alpha'^2 \equiv 0 \mod I^n$. Use the formula

$$\alpha' = \alpha (\alpha^2)^{-1/2} = \alpha \left(1 - (1 - \alpha^2)\right)^{-1/2}$$

$$= \alpha \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2^n \, n!} (1 - \alpha^2)^n$$

where the coefficients involve only powers of 2 in the denominator:

$$\frac{1 \cdot 3 \cdots (2n-1)}{2^n \, n!} = \frac{2n!}{2^n n! \, 2^n n!} = \underbrace{\binom{2n!}{n!}}_{\in \mathbb{Z}} \frac{1}{4^n}$$

Formally
$$1 - \alpha'^2 = 1 - \left(\alpha \left(1 - (1 - \alpha^2)\right)^{-1/2}\right)^2$$
$$= 1 - \alpha^2 \left(1 - (1 - \alpha^2)\right)^{-1} = 0$$
and this

should hold modulo I^n for all n.

Look at an iteration method which replaces α satisfying $1-\alpha^2 \in I$ by $\alpha' \equiv \alpha \bmod I$ such that $1-\alpha'^2 \in I^2$. Set

$$\alpha' = \alpha\left(1 + \tfrac{1}{2}(1-\alpha^2)\right)$$

Then
$$\begin{aligned} 1-\alpha'^2 &= 1 - \alpha^2\left(1 + 1-\alpha^2 + \left(\frac{1-\alpha^2}{2}\right)^2\right) \\ &= 1 - 2\alpha^2 + \alpha^4 + \alpha^2\left(\frac{1-\alpha^2}{2}\right)^2 \\ &= \left(1-\alpha^2\right)^2 + \alpha^2\left(\frac{1-\alpha^2}{2}\right)^2 \in I^2 \end{aligned}$$

Thus if $\alpha = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$, then

$$\alpha' = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}\begin{pmatrix} 1+\frac{1}{2}(1-qp) & 0 \\ 0 & 1+\frac{1}{2}(1-pq) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & q + \frac{q-qpq}{2} \\ p + \frac{p-pqp}{2} & 0 \end{pmatrix}$$

is the first refinement.

November 30, 1997

Some ideas about an algebraic Kasparov theory. Let (U, V) be a super dual pair over A, and form the basic ~~map~~ homomorphism

$$U \otimes_A V \longrightarrow \mathrm{Mult}(U,V) = \mathrm{Hom}_{A^{op}}(U,U) \times \mathrm{Hom}_A(V,V)^{op}$$

$$\mathrm{Hom}_{A,A^{op}}(V \otimes U, A)$$

One should think of this map as $\underset{\text{unital}}{}$ unital DGA of length one, i.e. having the form $M \xrightarrow{d} R$, where M is a unitary bimodule over the unital ring R, where d is an R-bimodule map satisfying $m_1 dm_2 = (dm_1)m_2$. By Kasparov module we mean such a dual power equipped with an odd element $x \in R$ and even element $y \in M$ such that $d(y) = 1 - x^2$. Thus

$$x = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} : U_1 \rightleftarrows U_0, \quad y = \begin{pmatrix} y_1 & 0 \\ 0 & y_0 \end{pmatrix} \quad \text{and} \quad \begin{matrix} y_1 \mapsto 1 - qp \\ y_0 \mapsto 1 - pq \end{matrix}$$

Let's examine the operators one can generate from x and y in a universal way. We form the free DGA $k\langle x, y\rangle$ with $|x| = 0$, $|y| = 1$, $dy = 1 - x^2$. which is

$$\longrightarrow N \otimes_R N \longrightarrow N \longrightarrow \overset{R}{\overset{\parallel}{k[x]}}$$

where $N = RyR \simeq R \otimes R$. Then truncate by dividing by the differential ideal $\to N \otimes_R N \to d(N \otimes_R N) \to 0 \to 0$.

This gives $$M = R \otimes R \Big/ \Big\{ d(f_0 \otimes f_1 \otimes f_2) = f_0(1-x^2) f_1 \otimes f_2 - f_0 \otimes \overbrace{f_1 (1-x^2) f_2}^{=0} \Big\}$$

i.e. $$M = R \otimes_{(1-x^2)R} R.$$ Thus M is the $R = k[x]$-bimodule generated by the element y subject to $f(x)y = yf(x)$ when $f(x) \in (1-x^2)R$. Use the basis $1, x, \{(1-x^2)x^n, \ n \geq 0\}$ for R and you see M is spanned by $y, xy, (1-x^2)x^n y^{\,n\geq 0}$, yx, xyx. Using

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$d : M \twoheadrightarrow (1-x^2)R$, you see $\mathrm{Ker}(d)$ is spanned by the two elements $z = xy - yx$ and $xz = x^2y - xyx$. We know $\mathrm{Ker}(d) = H_1$ of our DGA is a bimodule over $H_0 = R/(1-x^2)R$. So $x^2 z = z$. Also

$$zx = (xy-yx)x = xyx - yx^2 = xyx - y(x^2-1) - y = xyx - x^2y$$

$\therefore$ $\boxed{zx = -xz}$. It follows that z, xz are a basis for $\mathrm{Ker}(d)$, in fact $H_1 \simeq \Omega^1_{k[x]/(1-x^2)}$ (and I recall conjecturing that $H_*(T_R(N), d) = \Omega_{k[x]/(1-x^2)}$).

Note also that $x(xy+yx) = x^2y + xyx = yx^2 + xyx = (xy+yx)x$. Consider $\mathrm{ad}(x) : M \to M$. It maps M into $\mathrm{Ker}(d)$. Let M^x be the kernel of $\mathrm{ad}(x)$. Then M^x contains the elements $(x^2-1)x^n y$, $n \geq 0$, and $xy + yx$.

Actually one should first note that $\mathrm{ad}(x) z = xz - zx = 2xz$, also $\mathrm{ad}(x)(xz) = 2x^2 z = 2z$, so that $\frac{1}{4}\mathrm{ad}(x)^2$ is a projector with kernel $= M^x$ and image $= \mathrm{Ker}(d)$. Conclude (when 2 invertible) that $d : M^x \xrightarrow{\sim} (1-x^2)k[x]$. A basis for M^x is given by $(x^2-1)x^n y$ $n \geq 0$ and $y - \frac{1}{4}(\mathrm{ad}(x))^2 y = y - \frac{1}{2}xz$ and $xy - \frac{1}{4}\mathrm{ad}(x)^2(xy) = xy - \frac{1}{4}\mathrm{ad}(x)(xz) = xy - \frac{1}{2}z = \frac{xy+yx}{2}$

So in doing calculations one can assume an element of M commutes with x without changing its image under d.

Remark: The DGA $M \xrightarrow{d} R$ above arises from the dual pair $(R, R, \langle\ ,\ \rangle)$, $\langle r_1, r_2 \rangle = r_1(1-x^2)r_2$ over $(1-x^2)R$. This amounts to $R \xrightarrow{\sim} \mathrm{Hom}_{(1-x^2)R}(R,R)$ and also for the left multiplier ring. This follows from

$$\mathrm{Hom}_{(1-x^2)R}(R,R) = R \oplus \mathrm{Hom}_{(1-x^2)R}(R/(1-x^2)R, R).$$

I have been trying to determine the K-theory of $\mathcal{T}_E$ following the topological cases. [redacted] The basic idea is to use the Kasparov $\mathcal{T}_E, k$ bimodule furnished by

$$U: \quad T(E)\otimes E \rightleftarrows T(E)$$

$$V: \quad E^*\otimes T(E^*) \overset{?}{\rightleftarrows} T(E^*)$$

This somehow gives a map from $K_*(\mathcal{T}_E)$ to $K_*(k)$ which should be inverse to the one the other way given by the homom. $k \longrightarrow \mathcal{T}_E$. The hard point (even on the K_0 level) is to show that the "Kasparov product" of $P \in \mathcal{P}(\mathcal{T}_E^{op})$ and the above Kasparov bimodule, then tensored with R is equivalent to P.

Picture:

$$P \otimes \left(T(E)\otimes E \rightleftarrows T(E) \right) \otimes_k R$$

One might want to replace $T(E)$ by the resolution

$$0 \longrightarrow \mathcal{T}_E \otimes E^* \longrightarrow \mathcal{T}_E \longrightarrow T(E) \longrightarrow 0$$

and this might lead to a $R = \mathcal{T}_E$ - bimodule resolution of the form

$$0 \to R\otimes E\otimes E^*\otimes R \longrightarrow R\otimes (E\oplus E^*)\otimes R \longrightarrow R\otimes R \longrightarrow R \longrightarrow 0$$

I think it can be shown that such a canonical R-bimodule resolution exists. Write $R = T/J$, $T = T(E\oplus E^*)$ and use

$$0 \to J/J^2 \longrightarrow R\otimes_T \Omega^1(T)\otimes_T R \longrightarrow \Omega^1(R) \longrightarrow 0.$$

Use obvious basis of T given by monomials in s_i, s_i^*, and also the obvious increasing filtration on T. In $gr(T)$, J spanned by monomials containing a $s_i^* s_j$ transition, J^2 by monomials with 2 such transitions, so J/J^2 has the basis $s_\alpha s_\beta^* s_i s_j^* s_\gamma s_\delta^*$ leading to

$J/J^2 \simeq R \otimes E \otimes E^* \otimes R$.

There is already of problem with this approach in the case of the tensor algebra $R = T(E)$. Here one has two homoms. $k \to R \to k$ and the problem is to see that $R \to k \to R$ induces 1 on K_*. You have the bimodule resolution

$$0 \longrightarrow R \otimes E \otimes R \longrightarrow R \otimes R \longrightarrow R \longrightarrow 0.$$

The problem will be to take $P \in \mathcal{P}(R^{op})$, then ~~[illegible]~~ make

$$P \otimes E \otimes R \longrightarrow P \otimes R$$

in a Kasparov gadget which is equivalent to P. We know there are Nil phenomena arising in the case of general poly extension $A[X]$.

December 3, 1997

Let $U_1 \xrightarrow{d} U_0$ be a complex of unitary R-modules such that the identity map can be deformed to a nuclear map. Here's a simple proof that U is heq to a finite projective complex of length 1. By assumption $\exists\ h: U_1 \to U_0$ such that $1-dh: U_0 \to U_0$ and $1-hd: U_1 \to U_1$ are nuclear, i.e. factor through a finite projective complex. Let $1-dh = j_0 i_0 : U_0 \to T_0 \to U_0$ with $T_0 \in P(R)$, and define T_1 to be the fibre product of d and j_0:

$$\begin{tikzcd} T_1 \arrow[r, "d"] \arrow[d, "j_1"'] & T_0 \arrow[d, "j_0"] \\ U_1 \arrow[r, "d"'] & U_0 \end{tikzcd}$$

The total complex of this double complex is the cone on $j: T \to U$. One knows j is a heq $\Leftrightarrow$ the cone is contractible (resp j is a qis $\Leftrightarrow$ $\mathrm{Cone}(j)$ is acyclic). Consider

$$\begin{tikzcd} T_1 \arrow[r, shift right, "\begin{pmatrix} -j_1 \\ d \end{pmatrix}"'] & U_1 \oplus T_0 \arrow[l, shift right, "(-i_1\ \ h)"'] \arrow[r, shift right, "(d\ \ j_0)"'] & U_0 \arrow[l, shift right, "\begin{pmatrix} h \\ i_0 \end{pmatrix}"'] \end{tikzcd}$$

where the arrows $\to$ $\to$ are the differentials in $\mathrm{Cone}(j)$ and hence yield a short exact sequence; note that $(d\ \ j_0)\begin{pmatrix} h \\ i_0 \end{pmatrix} = 1$ on U_0, so this sequence has a splitting - this defines $i_1: U_1 \to T_1$ and $H: T_0 \to T_1$. This splitting implies $U \xrightarrow{i} T \xrightarrow{j} U$ are homotopy inverse. The only remaining point is to see that $T_1 \in P(R)$ equivalently 1 on T_1 is nuclear. But the identity map of T_1 factors

$$T_1 \xrightarrow{\binom{-j_1}{d}} \begin{matrix} U_1 \\ \oplus \\ T_0 \end{matrix} \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}-\binom{h}{i_0}(d\ \ j_0)} \begin{matrix} U_1 \\ \oplus \\ T_0 \end{matrix} \xrightarrow{(-i_1,\ h)} T_1$$

and the map $\begin{pmatrix} 1-hd & -hj_0 \\ -i_0 d & 1-i_0 j_0 \end{pmatrix}$ is nuclear

because $1-hd$ is nuclear by assumption and the other three components have source or target in $\mathbb{P}(R)$.

December 7, 1997

Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ such that $QP = A$, $PQ = B$ we shall prove $K_0(A) \cong K_0(B)$, starting from $K_0'A \simeq K_0'B$ and $K_0'(A) \twoheadrightarrow K_0(A)$ and similarly for B. <u>First argument</u>: Take $\xi \in K_0(A)$, represent ξ by a triple $(U_1, U_0, \alpha : U_1/AU \xrightarrow{\sim} U_0/AU_0)$ with $U_i \in \mathcal{P}(\tilde{A})$. Lift α to d and α^{-1} to h. Then the complexes $U_1 \xrightarrow{d} U_0$ and $U_0 \xrightarrow{h} U_1$ are finite projective over A and acyclic modulo A, which determine two classes in $K_0'A$ mapping to ξ. We know that applying $P\otimes_A -$ carries these complexes to ones over B which are homotopy equivalent to finite projective complexes acyclic modulo B. Specifically, put $U_i' = P\otimes_A U_i$ and write d, h for $1\otimes d, 1\otimes h$. The maps $1-hd$ on U_0' and $1-dh$ on U_1' are B-nuclear so we obtain a split exact sequence

$$T_1 \underset{\begin{pmatrix} j_1 \\ -d \end{pmatrix}}{\overset{(\iota_1,\ -h)}{\leftrightarrows}} \begin{matrix} U_1' \\ \oplus \\ T_0 \end{matrix} \underset{(d\ \ j_0)}{\overset{\begin{pmatrix} h \\ \iota_0 \end{pmatrix}}{\leftrightarrows}} U_0'$$

where $T_i \in \mathcal{P}(\tilde{B})$. This implies that $U_1' \xrightarrow{d} U_0'$ is homotopy equivalent to $T_1 \xrightarrow{d} T_0$ and that $U_0' \xrightarrow{h} U_1'$ is homotopy equivalent to $T_0 \xrightarrow{h} T_1$. Thus the two liftings of $\xi \in K_0A$ to $K_0'A$ we have go into the elements of $K_0'B$ represented by $T_1 \xrightarrow{d} T_0$ and $T_0 \xrightarrow{h} T_1$. Now d, h on the U_i' are inverse modulo B, since $1-[d,h]$ is B nuclear. This implies d, h on T are inverse modulo B, because the squares:

$$\begin{tikzcd} T_1 \arrow[r, "d"] \arrow[d] & T_0 \arrow[d] \\ U_1' \arrow[r, "d"] & U_0' \end{tikzcd} \qquad \begin{tikzcd} T_0 \arrow[r, "h"] \arrow[d] & T_1 \arrow[d] \\ U_0' \arrow[r, "h"] & U_1' \end{tikzcd}$$

are cartesian mod B.

Thus $[T_1 \xrightarrow{d} T_0]$ and $[T_0 \xrightarrow{h} T_1]$ in $K_0'B$ yield the same element of K_0B. It follows that the complexes $[P\otimes_A U_1 \xrightarrow{1\otimes d} P\otimes_A U_0]$ and $[P\otimes_A U_0 \xrightarrow{1\otimes h} P\otimes_A U_1]$ yield the same element of K_0B. (Here I'm appealing to the fact that these complexes are htpy equivalent to finite proj complex over B acyclic mod B, and that the choice of such a htpy equiv. yields the same K-class.) The latter complex is independent of the choice of d, so we have a well-defined map from K_0A to K_0B.

<u>Second argument.</u> Consider (U_1, U_0, α) as before and two liftings $d, d' : U_1 \to U_0$ for α. We want to show that the classes in K_0B represented by $P\otimes_A U_1 \xrightarrow{1\otimes d} P\otimes_A U_0$ and $P\otimes_A U_1 \xrightarrow{1\otimes d'} P\otimes_A U_0$ coincide. Choosing $h : U_0 \to U_1$ lifting α^{-1}, whence $1-dh, 1-d'h : U_0 \to U_0$ are A-nuclear, and then $1-dh, 1-d'h : P\otimes_A U_0 \to P\otimes_A U_0$ are nuclear. (Here simplify $1\otimes d$ etc to d). Choose factorizations with $T_0, T_0' \in P(\tilde{B})$
$1-dh = \underbrace{j_0 i_0}_{: P\otimes_A U_0 \xrightarrow{i_0} T_0 \to P\otimes_A U_0}$, $1-d'h = j_0' i_0' : P\otimes_A U_0 \to T_0' \to P\otimes_A U_0$. Replacing T_0 by $T_0 \oplus T_0'$ and i_0 by $(i_0, i_0') : P\otimes_A U_0 \to T_0 \oplus T_0'$ we can suppose $T_0 = T_0'$, $i_0 = i_0'$, but we have different $j_0, j_0' : T_0 \to P\otimes_A U_0$ to yield $1-dh$ and $1-d'h$. Now let T_1 be defined so that the following square is cocartesian:

$$\begin{tikzcd}
T_1 & T_0 \arrow[l, "h"'] \\
P\otimes_A U_1 \arrow[u, "i_1"] & P\otimes_A U_0 \arrow[l, "h"] \arrow[u, "i_0"']
\end{tikzcd}$$

This square is bicartesian and has two splittings given by (d, j_0), (d', j_0'), whence we get d, d' and j_1, j_1' defined on T_1. We know $T_1 \in P(\tilde{B})$. Our problem is to show that $d, d' : T_1 \to T_0$ are congruent modulo B. On the $P \otimes_A U$ level d and d' coincide mod B because they are inverses to h. ?

Observation: An invertible 2×2 matrix is the same thing as 2 splittings of the same module. Given an isomorphism

$$\begin{tikzcd}[column sep=huge] P \oplus Q \arrow[r, shift right, "\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}"'] & X \oplus Y \arrow[l, shift right, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"'] \end{tikzcd}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then we have 8 maps:

$$\begin{tikzcd} Q \arrow[r, shift right, "\beta"'] \arrow[d, shift left, "\delta"] & X \arrow[l, shift right, "c"'] \arrow[d, shift left, "a"] \\ Y \arrow[u, shift left, "d"] \arrow[r, shift right, "b"'] & P \arrow[l, shift right, "\gamma"'] \arrow[u, shift left, "\alpha"] \end{tikzcd}$$

such that ① the four squares you get by choosing the directions of the horizontal and vertical arrows anticommute
② at each corner the two paths add up to the identity.

In this situation: if b, γ are inverse then so are β, c.
Because we have cartesian square

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "a"] & Q \arrow[r, "\beta"] \arrow[d, "\delta"] & X \arrow[d, "a"] \\ P \arrow[r, "\gamma"] & Y \arrow[r, "b"] & P \end{tikzcd}$$

so the composite square is cartesian, so $b\gamma = 1 \Rightarrow \beta c = 1$, etc.

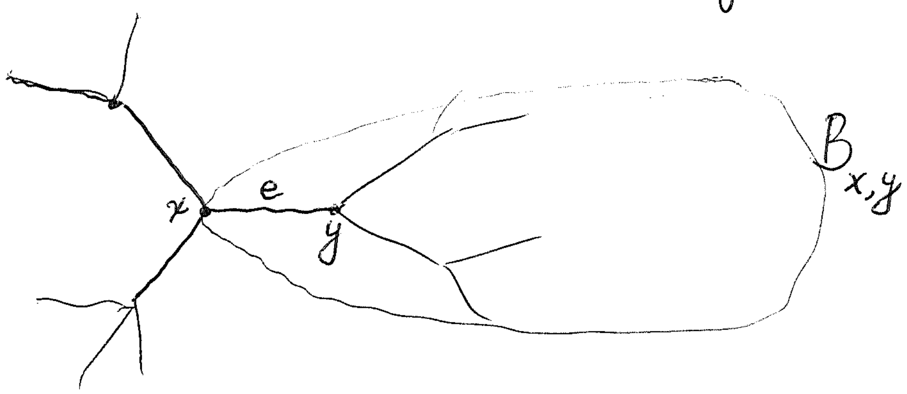
<u>December 14, 1997</u>

Let X be a tree, let M be a cosheaf over X (contravariant functor on the poset of simplices). Assume that $C_\cdot(X,M)$:

$$\begin{array}{ccc} C_1(X,M) & \xrightarrow{\ d\ } & C_0(X,M) \\ \| & & \| \\ \bigoplus\limits_{\text{edges}} M_{\{x_0,x_1\}} & & \bigoplus\limits_{\substack{x \\ \text{vertices}}} M_x \end{array}$$

is acyclic. (Note d on $M_{\{x_0,x_1\}}$ is defined by changing the sign of one of the maps $M_{\{x_0,x_1\}} \begin{array}{l} \nearrow M_{x_0} \\ \searrow M_{x_1} \end{array}$.)

Let x_0 be a vertex. Because X is a tree any vertex (or even edge) is joined to x by a unique path without cancellations. X is the union of branches:



corresponding to the edges issuing from x, where the branch $B_{x,y}$ corresponding to the edge $\{x,y\}$ is the subcomplex with vertices x and all vertices which can be joined to y without passing through x. Note that the branch $B_{x,y}$ depends on the oriented simplex (x,y). Two branches issuing from x meet only at x.

Because d is an isomorphism for every $\xi \in M_x$ there is a unique 1 chain $d^{-1}(\xi)$, whose boundary is ξ. Thus

$$M_x \xleftarrow[d]{\sim} \underbrace{Z_1(X,x;M)}_{\substack{\text{one chains which} \\ \text{are cycles away from } x}} = \bigoplus_y \underbrace{Z_1(B_{x,y},x;M)}_{\substack{\text{one chains on } B_{xy} \\ \text{which are cycles away from } x.}}$$

This gives a canonical splitting
$$M_x = \bigoplus_y M_x^y$$
indexed by the edges leaving x. An element ξ of M_x lies in M_x^y iff ξ is the boundary of a 1-chain η in $B_{x,y}$ which is a cycle away from x; note $\eta = d^{-1}\xi$ is unique.

Consider a simplex $\sigma = \{x,y\}$. d yields an isom.
$$Z_1(X, \{x,y\}; M) \xrightarrow[\sim]{d} M_x \oplus M_y$$
Now
$$Z_1(X, \{x,y\}; M) = \bigoplus_w Z_1(B_{x,w}, x; M) \oplus M_{\{x,y\}} \oplus \bigoplus_z Z_1(B_{y,z}, y; M)$$
where $\{x,w\}$ ranges over edges leaving x & $\neq \{x,y\}$, and where $\{y,z\}$ —————— y & $\neq \{y,x\}$. Also
$$M_x = M_x^y \oplus \bigoplus_{w \neq y} M_x^w, \quad M_y = M_y^x \oplus \bigoplus_{z \neq x} M_y^z.$$
Now d maps $Z_1(B_{x,w}, x; M)$ isom. onto M_x^w
and —————— $Z_1(B_{y,z}, y; M)$ —————— M_y^z,

so we are left with an isomorphism
$$M_{\{x,y\}} \subset Z_1(X, \{x,y\}; M) \xrightarrow{\sim} M_x \oplus M_y \twoheadrightarrow M_x^y \oplus M_y^x$$
which yields an intrinsic splitting of $M_{\{x,y\}}$ which we denote $M_{\{x,y\}} = M_{\{x,y\}}^x \oplus M_{\{x,y\}}^y$. An element $\eta \in M_{\{x,y\}}$ lies in $M_{\{x,y\}}^x$ when $(d\eta)_y \in \bigoplus_{z \neq x} M_y^z$, and η lies in $M_{\{x,y\}}^y$ when $(d\eta)_x \in \bigoplus_{w \neq x} M_x^w$.

Here's a way to look at this splitting, or rather the isom. $M_{\{x,y\}} \xrightarrow{\sim} M_x^y \oplus M_y^x$.

Given $\xi \in M_x^y$ we know $d^{-1}\xi$ is a chain in $B_{x,y}$ which is a cycle away from x. Write $d^{-1}\xi = \eta \oplus \eta'$ where $\eta \in M_{\{x,y\}}$ and η' has support "to the right" of y, i.e. in $\bigcup_{z \neq x} B_{y,z}$. Then $(d\eta)_y + (d\eta')_y = 0$, where $\eta' \in \bigoplus_{z \neq x} Z_1(B_{y,z}, y; M)$. Thus $\eta \in M^x_{\{x,y\}}$.

It should be clear now that the inverse to $M_{\{x,y\}} \xrightarrow{\sim} M_x^y \oplus M_y^x$ sends $(\xi_1, \xi_2) \in M_x^y \oplus M_y^x$ to $(d^{-1}\xi_1 + d^{-1}\xi_2)_{\{x,y\}} \in M_{\{x,y\}}$.

At this point we have decomposed $C_0(X, M)$ into $\bigoplus_x \bigoplus_y M_x^y$ (here y runs over neighbors of x) and decomposed $C_1(X,M)$ into $\bigoplus_{\{x,y\}} M^x_{\{x,y\}} \oplus M^y_{\{x,y\}}$. Both of these are decompositions indexed by oriented edges. What does d do? d maps $M^x_{\{x,y\}}$ to $M_x^y \oplus \bigoplus_{z \neq x} M_y^z$. So we have

$$\bigoplus_{(x,y)} M^x_{\{x,y\}} \xrightarrow[\sim]{\theta + \nu} \bigoplus_{(x,y)} M_x^y$$

where θ is a graded isomorphism i.e. $\theta : M^x_{\{x,y\}} \xrightarrow{\simeq} M_x^y$ and where ν moves 1 step away from x. You can think of $\nu\theta^{-1}$ as an operator acting on chains on $B_{x,y}$ with coefficients in the space "away from" x.

It should be clear that $1 + \nu\theta^{-1} + (\nu\theta^{-1})^2 + \dots$ is the operator which essentially lifts $\xi \in M_x^y$ into the chain $d^{-1}(\xi)$ supported on $B_{x,y}$.

Consider Waldhausen's free product situation. Suppose $G = G_A *_{G_C} G_B$ where $G_C \rightrightarrows \begin{smallmatrix} G_A \\ G_B \end{smallmatrix}$ injective. Then there is a tree with G acting and an edge fundamental domain $\underset{G_A}{\bullet}\overset{G_C}{\text{———}}\underset{G_B}{\bullet}$ with the indicated stabilizer. Let M be a G-equivariant cosheaf on X. This is equivalent to giving a G_A-module M_A, G_B-module M_B, and G_C-module M_C and maps $M_C \rightrightarrows \begin{smallmatrix} M_A \\ M_B \end{smallmatrix}$ consistent with the homos. $G_C \rightrightarrows \begin{smallmatrix} G_A \\ G_B \end{smallmatrix}$.
Assume $C_\cdot(X, M)$ acyclic. The canonical splittings discussed above must be preserved by the G-action, and so can be described by data over the fundamental domain. First we have the splitting of the vertex groups

$$M_A \simeq \mathbb{Z}[G_A] \otimes_{\mathbb{Z}[G_C]} M_A^B$$

$$M_B \simeq \mathbb{Z}[G_B] \otimes_{\mathbb{Z}[G_C]} M_B^A$$

since G_A acts transitively on the edges leaving A and G_C is the stabilizer of the edge AB. We also have the splitting of the edge group

$$M_C = M_C^A \oplus M_C^B$$

The boundary $(d_0, d_1): M_C \to M_A \times M_B$ relative to these isos can be described as follows. d_0, d_1 when restricted to M_C^+ are respectively an isomorphism $M_C^+ \xrightarrow{\sim} M_A^B$ and a map $M_C^+ \longrightarrow \mathbb{Z}[G_B - G_C] \otimes_{\mathbb{Z}[G_C]} M_B^A$, while d_0, d_1 restricted to M_C^- are resp. a map $M_C^- \longrightarrow \mathbb{Z}[G_A - G_C] \otimes_{\mathbb{Z}[G_C]} M_A^B$ and isomorphism $M_C^- \xrightarrow{\sim} M_B^A$.

Much more work is needed to understand.

Consider the <u>example</u> $G = \mathbb{Z}/2 * \mathbb{Z}/2$. Modules for this group are the same as abelian groups equipped with two splittings (at least if 2 is inverted). The tree in this case is $\mathbb{R}$ with $\mathbb{Z}$ as vertices and G acting as reflections at the integer points and translation by even integers. First note that because the tree is so simple there is canonical splitting of the cosheaf into right and left moving pieces.

$$\to M_x^y \to M_y^z \to$$
$$\overset{\cdot}{x} \qquad \overset{\cdot}{y} \qquad \overset{\cdot}{z}$$

Now the reflections interchange right + left movers, so the right moving cosheaf is equivariant only under the even translation subgroup. You have then

$$\begin{tikzcd}
M_A^B & & M_B^{T(A)} & & M_{T(A)}^{T(A)} \\
& M_{AB}^A \arrow[ul, "s"] \arrow[ur, "y"'] & & M_{BT(A)}^B \arrow[ul, "s"] \arrow[ur, "v"']
\end{tikzcd}$$

So it appears that the system is equivalent to a $\mathbb{Z}/2$-graded group $M_A^B \oplus M_B^{T(A)}$ equipped with an an odd operator which is nilpotent.

<u>December 22, 1997</u>

Carey-Evans result that $U(n,1)$ acts naturally on the Cuntz C^* algebra $\mathcal{O}_n$.

Background on indefinite unitary groups and pseudo-Hilbert spaces (a.k.a. Krein spaces).

Let $H = H_+ \oplus H_-$ be a $\mathbb{Z}_2$-graded Hilbert space, let ω be the pseudo-scalar product $\omega(\xi,\eta) = (\xi_+,\eta_+) - (\xi_-,\eta_-)$, let $U(H,\omega)$ be the group of bounded invertible operators on H which preserve ω. Then $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $U(H,\omega)$ iff

$$g^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e. $$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}$$

Let's describe <u>closed subspaces W of H such that $\omega \geq 0$ on H.</u> Consider the projections $w \mapsto w_+$, $w \mapsto w_-$ from W to $H_\pm$ resp. We have

$$0 \leq \omega(w,w) = \|w_+\|^2 - \|w_-\|^2, \quad \|w\|^2 = \|w_+\|^2 + \|w_-\|^2, \text{ so}$$

$$\|w_+\|^2 \leq \|w\|^2 \leq 2\|w_+\|^2$$

Since W is closed it follows that $w \mapsto w_+$ is a bounded isomorphism between W and a closed subspace W' of H_+. Then W is the graph of a contraction $\alpha : W' \to H_-$. If $W' < H_+$, then by choosing $0 \neq \xi \in H_+ \ominus W'$ and setting $\alpha(\xi) = 0$, we can extend W to a larger closed $\omega \geq 0$ subspace. We conclude that maximal closed $\omega \geq 0$ subspaces are the graphs $\begin{pmatrix} 1 \\ \alpha \end{pmatrix} H^+$ of contractions $\alpha : H^+ \to H^-$. (contraction means $\|\alpha\xi\| \leq \|\xi\|$ for all ξ.)

Consider next <u>closed isotropic subspaces</u> W, i.e. such that $\omega = 0$ on W. These are described bijectively as the graphs of partial isometries between H^+ and H^-, (i.e. $H^+ \supset W' \xrightarrow[\text{unitary}]{\sim} W'' \subset H^-$). Such a W is maximal iff $W' = H^+$ or $W'' = H^-$. There are thus three cases. a) $W = \binom{1}{u} H^+$ where $u : H^+ \xrightarrow{\sim} H^-$ is a unitary isom. b) $W = \binom{1}{u} H^+$, where $u : H^+ \to H^-$ is a non-unitary isometry $u^*u = 1$, $uu^* \neq 1$. c) $W = \binom{u}{1} H^-$ where $u : H^- \to H^+$ is such that $u^*u = 1$, $uu^* \neq 1$. Because W isotropic we have $W \subset W^\circ \overset{\text{def}}{=} \{ \eta \in H \mid \omega(H, \eta) = 0 \}$. If $W = \binom{1}{u} H^+$, then $W^\circ = \left\{ \eta \;\middle|\; \omega\left(\binom{\xi}{u(\xi)}, \eta \right) = (\xi, \eta_+) - (\xi, u^*(\eta_-)) = 0 \right.$ for all $\left. \xi \in H^+ \right\} = \binom{u^*}{1} H^-$. If u is unitary then $W = W^\circ$. If u is a non-unitary isometry, then $\operatorname{Ker} u^* \neq 0$ and $W^\circ = W \oplus \operatorname{Ker}(u^*)$ so that ω is < 0 on $\operatorname{Ker}(u^*) - 0$. In case (c) $W^\circ = \binom{1}{u^*} H^+ = W \oplus \operatorname{Ker} u^*$, where $\operatorname{Ker}(u^*) \subset H^+$ and $\omega > 0$ on $W^\circ - W$. So cases a), b), c) are distinguished by $W = W^\circ$, $W^\circ > W$ and $\omega < 0$ on $W^\circ - W$, $W^\circ > W$ and $\omega > 0$ on $W^\circ - W$ respectively.

By a <u>polarization</u> of H we mean a splitting $H = W^+ \oplus W^-$, where W^+, W^- are closed subspaces such that $\omega > 0$ on W^+, $\omega < 0$ on W^-, and $\omega(W^+, W^-) = 0$. We know $W^+ = \binom{1}{\alpha} H^+$, $W^- = \binom{\beta}{1} H^-$ where $\alpha : H^+ \to H^-$, $\beta : H^- \to H^+$ are contractions. Now $\omega(W^+, W^-) = 0$ means $\beta = \alpha^*$. Thus we have a continuous bijection

$$\begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix} \xrightarrow{\begin{pmatrix} 1 & \alpha^* \\ \alpha & 1 \end{pmatrix}} \begin{matrix} W^+ \\ \oplus \\ W^- \end{matrix} = \begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix}$$

which by the closed graph thm. implies $\begin{pmatrix} 1 & \alpha^* \\ \alpha & 1 \end{pmatrix}$ is invertible. Put $X = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$. Then $1+X$ and $\varepsilon(1+X)\varepsilon = 1-X$ are invertible - here $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ - so $1-X^2 = \begin{pmatrix} 1-\alpha^*\alpha & 0 \\ 0 & 1-\alpha\alpha^* \end{pmatrix}$ is invertible. This implies that $\|\alpha^*\alpha\| = \|\alpha\|^2 < 1$, i.e. α is a strict contraction. Thus polarizations of H are described bijectively by strict contractions $\alpha : H^+ \to H^-$.

Put $\boxed{g = \dfrac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix} (1-\alpha^*\alpha)^{-1/2} & \alpha^*(1-\alpha\alpha^*)^{-1/2} \\ \alpha(1-\alpha^*\alpha)^{-1/2} & (1-\alpha\alpha^*)^{-1/2} \end{pmatrix}}$.

Then $g \in U(H,\omega)$ because

$$g^*\varepsilon g = \frac{1+X}{\sqrt{1-X^2}}\,\varepsilon\,\frac{1+X}{\sqrt{1-X^2}} = \frac{(1+X)(1-X)}{1-X^2}\,\varepsilon = \varepsilon$$

Consequences: $U(H,\omega)$ acts transitively on the set $\mathcal{P}$ of polarizations $\simeq$ set of strict contractions $\alpha : H^+ \to H^-$. Moreover there's a lift from polarizations to $U(H,\omega)$, i.e. a cross section of $U(H,\omega) \longrightarrow U(H,\omega)/\underbrace{U(H^+)\times U(H^-)}_{\text{stabilizer of } \varepsilon} = \mathcal{P}$,

and this leads to the polar decomposition $U(H,\omega) = \mathcal{P}(U(H^+)\times U(H^-))$.

If you want familiar formulas to the Sl_2 action on $\mathbb{P}^1$ you should emphasize the negative spaces $W^- = g(H^-)$, because the z in $\frac{az+b}{cz+d}$ is the line $\binom{z}{1}\mathbb{C}$ in $\mathbb{P}^1$. Thus $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(H,\omega)$ carries $H^- = \binom{0}{1}H^-$ to $\binom{b}{d}H^- = \binom{bd^{-1}}{1}H^-$.

So the action on the space of contractions is

$$\alpha^* \longleftrightarrow \binom{\alpha^*}{1}H^- \xrightarrow{\;g\;} \begin{pmatrix} a\alpha^*+b \\ c\alpha^*+d \end{pmatrix}H^- = \begin{pmatrix} (a\alpha^*+b)(c\alpha^*+d)^{-1} \\ 1 \end{pmatrix}H^-$$

$\longleftrightarrow (a\alpha^*+b)(c\alpha^*+d)^{-1}$. Acting on $\alpha^* = 0$ gives $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto bd^{-1} = \alpha^*$, and the cross section is the above formula for $g = \frac{1+X}{\sqrt{1-X^2}}$

Consider next $U(n,1)$ acting on $V = \mathbb{C}^n \oplus \mathbb{C} = V^+ \oplus V^-$. Let E be any Hilbert space, form $V \otimes E = \underbrace{E^n}_{H^+} \oplus \underbrace{E}_{H^-}$, so we get a group homom. from $U(n,1)$ to $U(H,\omega)$. Recall that a full isotropic subspace $(W = W^0)$ is the same as a unitary isomorphism between E^n and E, and this is the same as an $*$-action of $\mathcal{O}_n$ on H. Thus $U(n,1)$, and in fact $U(H,\omega)$, acts on $*$ homoms. $\mathcal{O}_n \longrightarrow \mathcal{L}(H)$. We need to understand why we get an action of the group $U(n,1)$ by automorphisms of $\mathcal{O}_n$.

Given $W = W^0$, we have

$W = \begin{pmatrix} 1 \\ s \end{pmatrix} H^+ = \begin{pmatrix} s^* \\ 1 \end{pmatrix} H^-$ where $s : E^n \longrightarrow E$, $s\xi = \sum_{i=1}^n s_i \xi_i$ and $s^* : E \longrightarrow E^n$, $s^*\xi_0 = \begin{pmatrix} s_1^*\xi_0 \\ \vdots \\ s_n^*\xi_0 \end{pmatrix}$ are inverse: $s_i^* s_j = \delta_{ij}$, $\sum s_i s_i^* = 1$. Note the s_i and s_i^* are operators on E, s is the row vector $(s_1, \dots, s_n)$ and s^* is the column vector $\begin{pmatrix} s_1^* \\ \vdots \\ s_n^* \end{pmatrix}$.
The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} s^* \\ 1 \end{pmatrix} H^- \longmapsto \begin{pmatrix} as^*+b \\ cs^*+d \end{pmatrix} H^-$
$= \begin{pmatrix} (as^*+b)(cs^*+d)^{-1} \\ 1 \end{pmatrix} H^-$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(s^*) = (as^*+b)(cs^*+d)^{-1}$$

Notice that $(as^*+b)_i = \sum_j a_{ij} s_j^* + b_i$, $cs^*+d = \sum_j c_j s_j^* + d$ are in $\mathcal{O}_n$, and so, assuming cs^*+d is invertible, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(s^*)$ is a well-defined column vector in $\mathcal{O}_n$.

To get an action on $\mathcal{O}_n$ ~~yours~~ replace E by the Hilbert C^*-module $\mathcal{O}_n$ over itself, so you effectively work inside $M_{n+1}(\mathcal{O}_n) = \mathrm{End}_{\mathcal{O}_n^{op}}(\mathcal{O}_n^{n+1})$, and use functional calculus inside this C^*-algebra.

Argument omitted above. $U(n,1)$ action on maximal isotropic subspaces preserves the three types. So it preserves $\binom{1}{s}H^+$ types where $s: H^+ \hookrightarrow H^-$ is an isometry: $s^*s = 1$, as well as $\binom{s^*}{1}H^-$ types where $s^*: H^- \hookrightarrow H^+$ is an isometry: $ss^* = 1$. The former should mean that $U(n,1)$ acts on the Toeplitz algebra $\mathcal{T}_n$. (You wasted a lot of time trying to use $\frac{as+b}{cs+d}$ instead of $(c+ds)(a+bs)^{-1}$. s is a row vector so $as+b$ has no meaning apparently.)

In the above ~~we~~ chose a particular polarization as basepoint. There's an intrinsic notion of Krein space: A top. v.s. H, whose topology underlies some Hilbert space structure, together with a hermitian form ω which is nondegenerate in the sense that the associated antilinear map $H \longrightarrow H'$ is bijective. Choosing a Hilbert space structure, ω becomes an invertible self-adjoint operator A on H, and then $\frac{A}{|A|}$ gives a polarization.