8] 02/02/97 1713

~~[illegible]~~ To study Mor. for K_* for rings Morita equivalent to unital rings. B Morita equivalent to a unital ring $\Leftrightarrow$ $\mathcal{P}(B)$ contains a generator for $\mathcal{M}(B)$. Suppose $P \in \mathcal{P}(B)$ generates $\mathcal{M}(B)$. Wait. Suppose B Morita equivalent to a unital ring A. Let $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ give the Morita equivalence. Then A unital $\Rightarrow A \in \mathcal{P}(A)$ so $P \otimes_A A \in \mathcal{P}(B)$ and generates $\mathcal{M}(B)$. ~~The rest follows easily I think. [illegible]~~

$$Q \xrightarrow{\sim} \mathrm{Hom}_B(P, B)$$

$$Q \otimes_B P = \mathrm{Hom}_B(P, B) \otimes_B P = \mathrm{Hom}_B(P,P)$$

All this stuff is trivial and uninteresting. I need ~~however a coherent thing~~ to do the h.unital stuff.

Significance of $B \in \mathcal{P}(B) \Leftrightarrow Q \otimes_B B = Q \in \mathcal{P}(A)$.

I think I need to determine ~~[illegible]~~.

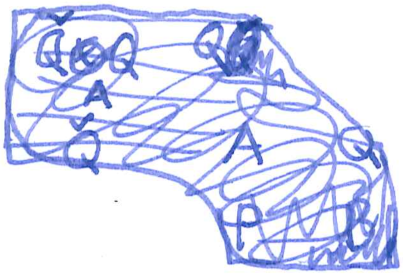
Philosophy as before. ~~You start with~~ ~~$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. This~~

The first case is where B is left flat:

B is B-flat $\Leftrightarrow$ $Q \otimes_B B$ is A-flat. So we have a triple $(Q, P, Q \otimes P \twoheadrightarrow A)$ with Q flat. Then can write $Q = \varinjlim F_\alpha = \varinjlim A F_\alpha$. Because A unital, have $F_\alpha \otimes P \twoheadrightarrow A$ for large α, so $B = P \otimes_A Q = \varinjlim \overbrace{P \otimes_A F_\alpha}^{B_\alpha}$ where now $B_\alpha \in \mathcal{P}(B_\alpha)$ $\begin{pmatrix} A & F_\alpha \\ P & \end{pmatrix}$?

So suppose $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where $Q \in \mathcal{P}(A)$ $Q = A^n$. Want $B \longrightarrow \mathrm{Hom}_B(B, B) = \mathrm{Hom}_A(Q, Q)$. Is there something I can find? How to construct zilch? ~~Still~~

So is there any hope of proving things. It seems c]



$$\begin{pmatrix} \Lambda = \check{Q}\otimes_A Q & \check{Q} & \Lambda \\ Q & A & Q \\ B & P & B \end{pmatrix}$$

$$\mathcal{M}(B) \qquad \mathcal{M}(A) \qquad \mathcal{M}(\Lambda)$$

$$N \longmapsto Q\otimes_B N \longmapsto \check{Q}\otimes_A Q\otimes_B N = \Lambda\otimes_B N$$

$$M \longmapsto \check{Q}\otimes_A M$$

$$P\otimes_A Q\otimes_\Lambda W \longleftarrow\!\shortmid\; Q\otimes_\Lambda W \longleftarrow\!\shortmid\; W$$

$$\| \qquad\qquad$$

$$B\otimes_\Lambda W$$

right mult

$$\mathrm{Hom}_B(B,B)^{op}$$

$$\|$$

typical situation $B \longrightarrow \mathrm{Hom}_A(Q,Q)^{op}$

B sort of a right ideal in Λ.

At this point you know how to prove the Morita invariance essentially using the functors

$$B\otimes_\Lambda - \; : \; \mathcal{P}(\Lambda) \longrightarrow \mathcal{P}(B)$$

equivalence.

$$\Lambda\otimes_B - \; : \; \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(\Lambda)$$

?

Just because of $\begin{pmatrix} \Lambda & \Lambda \\ B & B \end{pmatrix}$ we have an equiv. of $\mathcal{P}(\Lambda)$ and $\mathcal{P}(B)$. The point is that because $B \in \mathcal{P}(B)$ we can relate $K_*(B) = K_*(\tilde{B})/K_*\mathbb{Z}$ with $K_*(\mathcal{P}(B))$.

K] 02/03/97 0540

~~Is~~ Decide whether you can prove M inv. for K theory of h-unital rings mag unital rings. You need to use Suslin + Davydov.

Start with B h-unital. The first step is to reduce to B ~~right flat~~ left flat. Choose a firm flat B-module P mapping onto B. Then have dual pair $B \otimes B \twoheadrightarrow B$ $\quad p \otimes b \mapsto f(p)b$

so $\begin{pmatrix} A & B \\ P & B \end{pmatrix}$ $\qquad A = B \otimes_B P = P$

$$(b_1 p_1)(b_2 p_2) = \underbrace{b_1 f(p_1)}_{f(b_1 p_1)} b_2 p$$

$$p_1 p_2 = f(p_1) p_2$$

P $\quad B$-flat ~~$\Rightarrow A = B \otimes_B$~~ $\Rightarrow A = Q \otimes_B P$ ~~P~~ is A flat

So we have this Morita equivalence homom.

$f: P \twoheadrightarrow B$ $\quad$ where P is left flat. Also $\operatorname{Ker}(f) P = 0$. $\qquad 0 \to I \to P \to B \to 0$

~~Pick as Assumption that B is~~ Now you have

$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\qquad A \overset{L}{\otimes}_A B$

Basic result I remember is that if A is biflat, then B is h-unital $\Leftrightarrow$ $P \overset{L}{\otimes}_A Q \simeq B$. Proof?

$$\begin{array}{ccc} B \overset{L}{\otimes}_B P \overset{L}{\otimes}_A Q & \longrightarrow & B \overset{L}{\otimes}_B B \\ \downarrow s & & \downarrow \\ P \overset{L}{\otimes}_A Q & \longrightarrow & B \end{array}$$

2]

$$0 \to I \to A \to B \to 0 \qquad IA = 0$$

$$0 \to M(I) \to GL(A) \to GL(B) \to 0$$

$$E^2 \simeq H(GL(B), H(M(I))) \Rightarrow H(GL(A))$$

So when is B h-unital?

$$0 \to I \to A \to B \to 0 \qquad IA = 0$$

$\therefore {}_B A_A$

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

$A \overset{L}{\otimes}_A B \simeq B$ if A right flat this means $A \otimes_A B \xrightarrow{\sim} B$ which should be OK if B is firm.

$$\begin{tikzcd}
 & A\otimes_A I \arrow[r] \arrow[d] & A\otimes_A A \arrow[r] \arrow[d, equal] & A\otimes_A B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & AI \arrow[r] & A \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

so you need $AI = I$, more generally $A \overset{L}{\otimes}_A I \xrightarrow{\sim} I$. Certainly OK if A is right flat. In our case A is left flat, so we have Δ

$$\begin{tikzcd}
A\overset{L}{\otimes}_A I \arrow[r] \arrow[d] & A\overset{L}{\otimes}_A A \arrow[r] \arrow[d, equal] & A\overset{L}{\otimes}_A B \arrow[r] \arrow[d] & {} \\
I \arrow[r] & A \arrow[r] & B \arrow[r] & {}
\end{tikzcd}$$

So you get the condition $A \overset{L}{\otimes}_A I \xrightarrow{\sim} I$

μ] You have a lot of reviewing to do.

Today's lecture.

$$\mathcal{M}(A^{op}) = \text{rtcontfun}(\mathcal{M}(A), ab).$$

independence of R.

A ideal in R, [$A=A^2$] have canon. hom. of unital ring $\tilde{A} \to R$, restriction of scalars. Claim $\mathcal{F}(R, A) \longrightarrow \mathcal{F}(\tilde{A}, A)$ isom. of cat.

$$R\otimes_{\tilde{A}} M \longleftarrow M$$

$$N \longmapsto N$$

$$A\otimes_{\tilde{A}} N \longrightarrow A\otimes_R N \xrightarrow{\sim} N$$

Claim $\cong$ Suffices to show $(a, n) \longmapsto a\otimes_{\tilde{A}} n$ is R-bil.

$ar\otimes_{\tilde{A}} n \overset{?}{=} a\otimes_{\tilde{A}} rn$. can ass. $n = a'n'$

$ar \otimes a'n' = ara' \otimes n' = a\otimes ra'n'$

Conversely let $A\otimes_{\tilde{A}} M \xrightarrow{\sim} M$. Note R acts on left side by $r(a\otimes m) = ra\otimes m$. ~~[illegible]~~ $\therefore \exists!$ R-action on $M \ni r(am) = (ra)m$. This is the unique extension of the A-module structure to an R-mod. str.

$$\begin{tikzcd} A \arrow[r, phantom, "\subset"] \arrow[dr] & R \arrow[d, dotted] \\ & \mathrm{Hom}_{\mathbb{Z}}(M, M) \end{tikzcd}$$

$r(am) = (ra)m$

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$$\mathrm{Mod}(R^{op}) = \mathrm{rtcontfun}(\mathrm{Mod}(R), Ab)$$

$$N \longmapsto (M \longmapsto V \otimes_R M).$$

Claim if $A = A^2$, then

$$\mathcal{F}(R^{op}, A^{op}) \xrightarrow{\sim} \mathrm{rtcontfun}(\mathcal{F}(R,A), Ab)$$

$$V \qquad\qquad (\,$$

Lecture YES. Get 2nd paper done. The problem is how best to handle the details. ~~So what~~ You really have to finish Ch II.

Key point is description of firm rings meq to A
Key point is how to get back into the main stream.

$$m(A) = \mathcal{F}(\tilde{A}, A) \qquad A\text{-modules} \ni A \otimes_A M \xrightarrow{\sim} M$$

$$\begin{array}{ccc} N & & M \\ \mathrm{Mod}(\tilde{A}) & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & m(A) \end{array}$$

$$\mathrm{Hom}_{\mathrm{Mod}(\tilde{A})}(j_! M, N) = \mathrm{Hom}_{m(A)}(M, j^* N)$$

$$\qquad\qquad \| \qquad\qquad\qquad\qquad \|$$

M firm $\quad \mathrm{Hom}_A(M, N) \qquad \mathrm{Hom}_A(M, A^{(2)} \otimes_A N)$

$$A \in m(A) \iff A^{(2)} \simeq A. \iff A \in m(A^{op}).$$

$$\mathrm{Hom}_B(P \otimes_A M, N) = \mathrm{Hom}_A(M, \mathrm{Hom}_B(P, N))$$

$$= \mathrm{Hom}_A(M, A^{(2)} \otimes_A \mathrm{Hom}_B(P, N))$$

ξ] How about

$$m(R^{op}, A^{op}) \xrightarrow{\sim} \mathrm{rtcontfun}(m(R,A), ab).$$

$$V \longmapsto \left(V \otimes_R - : m(R,A) \subset Mod(R) \to ab\right)$$

$$F(A^{(2)}) \longleftarrow\!\shortmid \; F$$

$$\begin{tikzcd}
 & V \arrow[r, mapsto] & V \otimes_R - \arrow[dl, mapsto] & F \arrow[d, mapsto] & \\
V \otimes_R A^{(2)} \arrow[ur, "s"] & & & F(A^{(2)}) \arrow[r, mapsto] & F(A^{(2)}) \otimes_R -
\end{tikzcd}$$

~~Define~~ $\cancel{F(A^{(2)}) \otimes_R M \longrightarrow F(M)}$

$$\cancel{\xi \otimes m \longmapsto F(A^{(2)} \xrightarrow{\cdot m} M)(\xi)}$$

Given $F : m(R,A) \to ab$ right cont, compose with $\underset{j^* = A^{(2)} \otimes_R -}{Mod(R) \to m(R,A)}$ to get rt cont $Mod(R) \to ab$ so we have canon isom $F(A^{(2)} \otimes_R N) \xleftarrow{\sim} F(A^{(2)}) \otimes_R N$. LHS inverts nil isos $\Rightarrow$ same for RHS $\Rightarrow$ $F(A^{(2)})$ is firm, therefore $F \simeq F(A^{(2)}) \otimes_R -$.

$$m(A^{op}) \xrightarrow{\sim} \mathrm{rtcontfun}(m(A), ab).$$

$$V \longmapsto V \otimes_A -$$

$$F(A^{(2)}) \longleftarrow\!\shortmid \; F$$

same arguments should work.

0]
02/05/97 firm rings. I am ready for Part II.
0630 I have to go over what I've done and print out a version. Yes.

1) ~~firm~~ firm bimodules and rtcontfuns

~~(A)~~ A, B rintal

$$\mathrm{Mod}(A\otimes B^{op}) \longrightarrow \mathrm{rtcontfun}(\mathrm{Mod}(B), \mathrm{Mod}(A))$$
$$Q \longmapsto (N \mapsto Q\otimes_B N)$$
~~What~~
$$F(B) \longleftarrow F$$

(*) $$m(B\otimes A^{op}) \xrightarrow{\sim} \mathrm{rtcontfun}(m(A), m(B))$$
$$P \longmapsto (M \mapsto P\otimes_A M)$$
$$F(A^{(2)}) \longleftarrow F$$

Define firm B,A-bimodule to be one $\exists$ firm on both sides. Same as $\tilde{B}\otimes\tilde{A}^{op}$ unitary module firm wrt $B\otimes A^{op}$. Why?

Ass: $B\otimes_B P\otimes_A A \xrightarrow{\sim} P$

Then $$B^{(2)}\otimes_B P\otimes_A A^{(2)} \xrightarrow{\sim} B\otimes_B P\otimes_A A \xrightarrow{\sim} P$$

so P is B-firm and A^{op}-firm.

General case also true because M firm wrt an ideal A in $R \Longleftrightarrow -\otimes_R M$ inverts A^{op}-nil iso etc.

Proof of (*). ~~This is easy enough to~~

$$\mathrm{Mod}(\tilde{A}) \xrightarrow{A^{(2)}\otimes_A -} m(A) \xrightarrow{F} m(B) < \mathrm{Mod}(B)$$

$$M \longmapsto F(A^{(2)}\otimes_A M) \quad \text{rtcont so}$$

~~one has~~ $$F(A^{(2)})\otimes_A M \xrightarrow{\sim} F(A^{(2)}\otimes_A M)$$

II] next step is ~~functors~~. homomorphisms

$$w : A \longrightarrow B \qquad \mathcal{M}(A) \underset{w^*}{\overset{w_!}{\rightleftarrows}} \mathcal{M}(B)$$

$$M \longmapsto B^{(2)} \otimes_B \tilde{B} \otimes_A M = B \otimes_A N$$

$$A^{(2)} \otimes_A N \longleftarrow\!\!| \quad N$$

$B^{(2)} \longrightarrow B \longrightarrow \tilde{B}$ are B^{op}-nil isos.

$\therefore$ A^{op}-nil isos. $\therefore$ $B^{(2)} \otimes_A N \xrightarrow{\sim} B \otimes_A N \xrightarrow{\sim} \tilde{B} \otimes_A N$

$$\mathrm{Hom}_A(M, A^{(2)} \otimes_A N) \xrightarrow{\sim} \mathrm{Hom}_A(M, N)$$

$$\xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_B(B, N))$$

$$= \mathrm{Hom}_B(B \otimes_A M, N)$$

adj. maps.

$$\alpha : B \otimes_A A^{(2)} \otimes_A M \longrightarrow N \qquad b \otimes a_1 \otimes a_2 \otimes m \longmapsto b\, w(a_1 a_2) m$$

$$\beta : M \xrightarrow{\sim} A^{(3)} \otimes_A M \longrightarrow A^{(2)} \otimes_A B \otimes_A M$$

$$a_1 a_2 a_3 m \longmapsto a_1 \otimes a_2 \otimes w(a_3) \otimes m$$

---

adj for $P \otimes_A -$.

$$\mathrm{Hom}_B(P \otimes_A M, N) = \mathrm{Hom}_A(M, \mathrm{Hom}_B(P, N))$$

$$\xleftarrow{\sim} \mathrm{Hom}_A(M, A^{(2)} \otimes_A \mathrm{Hom}(P, N))$$

$$\cancel{w_!(M) = B \otimes_A M} \qquad \cancel{P = B \otimes_A A^{(2)}}$$

$$\cancel{w_*(N) = A^{(2)} \otimes_A \mathrm{Hom}_B(B \otimes_A A^{(2)}, N)}$$

$\mathrm{Hom}_A(N, M) = \mathrm{Hom}_A(B \otimes_B N, M) = \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, M))$

initial

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$$\mathrm{Hom}_B(B \otimes_A M, N) = \mathrm{Hom}_A(M, \mathrm{Hom}_B(B,N))$$ 1996+

$$\begin{tikzcd} & \mathrm{Hom}_A(M, \mathrm{Hom}_B(B,N)) \\ \mathrm{Hom}_A(M, A^{(2)} \otimes_A N) \arrow[r, "\sim"] & \mathrm{Hom}_A(M, N) \arrow[u, "\wr"'] \end{tikzcd}$$

$\mathrm{Hom}_N(N,$

$w : A \longrightarrow B$

$$\mathcal{M}(A) \underset{w^*}{\overset{w_! = B\otimes_A -}{\rightleftarrows}} \mathcal{M}(B)$$

$A^{(2)} \otimes_A N \qquad N$

$$\mathrm{Hom}_A(A^{(2)} \otimes_A N, M) = \mathrm{Hom}_A(B \otimes_B N, \mathrm{Hom}_A(A^{(2)}, M))$$

$$= \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, \mathrm{Hom}_A(A^{(2)}, M)))$$

$$= \mathrm{Hom}_B(N, \mathrm{Hom}_A(A^{(2)} \otimes_A B, M))$$

$$= \mathrm{Hom}_B(N, \boxed{B^{(2)} \otimes_B \mathrm{Hom}_A(A^{(2)} \otimes_A B, M)})$$

$$w_*(M) = B^{(2)} \otimes_B \mathrm{Hom}_A(A^{(2)} \otimes_A B, M)$$

$$w_!(M) = B \otimes_A M$$

Looking at $w^*(N) = A^{(2)} \otimes_A N \qquad Q = A^{(2)} \otimes_A B$

adj should be $\boxed{B^{(2)} \otimes_B \mathrm{Hom}_A(A^{(2)} \otimes_A B, -)}$

So now things are very clear.

σ] Summarize what you've gone over.

firm bimodules represent rt cont fcns

firm ~~A~~ B,A bimod = firm $B\otimes A^{op}$ mod

adj of $P\otimes_A -$ is $A^{(2)}\otimes_A \mathrm{Hom}_B(P,-)$

adjoint fcns assoc. to $w: A\to B$

$w^*N = A^{(2)}\otimes_A N$ $\qquad w_!M = B^{(2)}\otimes_A M = B\otimes_A M = \tilde{B}\otimes_A M$

$w_*M = B^{(2)}\otimes_B \mathrm{Hom}_A(A^{(2)}\otimes_A B, M)$

~~Apply preceding stuff~~

Call $w: A\to B$ a meq hom when w^* is an equivalence of categories. Adj maps.

$\alpha: B\otimes_A A^{(2)}\otimes_A B \longrightarrow B^{(2)}$ $\qquad \alpha: w_!w^*\to 1$

$b\otimes a_1\otimes a_2\otimes b_2 \longmapsto b_1 \overset{w}{(a_1 a_2)} b_2$

~~$\beta: A^{(2)}\otimes_A B\otimes_A A^{(2)} \to A^{(2)}$~~

$\beta: A^{(2)} = A^{(2)} \longrightarrow A^{(2)}\otimes_A B \otimes_A A^{(2)}$

β is an isom $\Leftrightarrow$ $A\xrightarrow{w} B$ is an $A\otimes A^{op}$ mil iso.

i.e. $A\,\mathrm{Ker}(w)A = 0$ and ~~$Bw(A)B = B$~~. $w(A)Bw(A) = w(A)$

β an equivalence $\Leftrightarrow$ in addition $Bw(A)B = B$.

Does it help to assume A, B firm. Maybe a little.

Analyze basic proof.

τ] $A \xrightarrow{f} w(A) \overset{g}{\subset} B$

Conditions $\quad A\,\mathrm{Ker}(w)\,A = 0 \quad$ $w(A)\,B\,w(A) = w(A)$

$B\,w(A)\,B = B.$

$\Rightarrow$ true for inclusion $w(A) \subset B$.

red. to w inj and w surjective. $\quad A \to A/I = B$, $\; AIA = 0$.

$A \subset B \quad \ni \quad ABA = A, \quad BAB = B.$

$A \xrightarrow{} A/IA \xrightarrow{} A/I = B$

$A \cdot IA = 0 \qquad (I/IA)(A/IA) = 0$

two cases $\quad A \cdot I = 0 \quad$ and $\quad I \cdot A = 0$

Take case $A \subset B$ right ideal in B gen. B. $\quad AB = A$, $\; BA = B$

$$\begin{pmatrix} A & A \\ B & B \end{pmatrix}$$

$\mathcal{M}(A) \qquad \mathcal{M}(B)$

$M \longmapsto B \otimes_A M$

$A \otimes_A N \longleftarrow N$

$M \mapsto A \otimes_A M = 0$

$B \otimes_B N \leftarrow N$

$\mathcal{M}(A^{op}) \xrightarrow{\sim} \mathcal{M}(B^{op})$

$U \longmapsto U \otimes_A A = U$

$V \otimes_B B \longleftarrow V$

Reverse left and right so that $BA = A$, $AB = B$

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

Suppose $A \otimes_A M \xrightarrow{\sim} M$. Then $\exists!$ B-module structure on M ~~$\ni\; b(am) = (ba)m$~~ extending the A-mod str. it is given by $b(am) = (ba)m$. Check firm

$A \otimes_A N \longrightarrow B \otimes_B N \longrightarrow N$

U] Play more carefully. Start with N. $B \otimes_B N = N$.

Then $B = AB \Rightarrow$

$$\cancel{A \otimes_A N \longrightarrow B \otimes_B N \longrightarrow B \otimes_B N \xrightarrow{\sim} N}$$

$$A \otimes_A N \longrightarrow B \otimes_B N \xrightarrow{\sim} N$$

Recall proof. $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

$$Q \otimes_B P \longrightarrow A$$

$$B \otimes_B A \twoheadrightarrow A$$

$$A \otimes_A B \twoheadrightarrow B$$

~~surj $(ab)(a_i \otimes b_i) = a \otimes b a_i b_i$~~

Start first with $P \otimes_A Q \to B$

surj $(a_i \otimes b_i) ab = a_i b_i a \otimes b$

$(b_i \otimes a_i) a = b_i a_i \otimes a$

$A \subset B$ $\quad BA = A \quad AB = B \quad \begin{pmatrix} A & B \\ A & B \end{pmatrix}$

$$0 \longrightarrow K \longrightarrow B \otimes_B A \longrightarrow A \longrightarrow 0$$

$$\Big(\sum_i b_i \otimes a_i\Big) a = \Big(\sum b_i a_i\Big) \otimes a \Rightarrow KA = 0$$

$$A \otimes_A M \xrightarrow{\sim} M \Rightarrow B \otimes_B M \xrightarrow{\sim} M$$

$$0 \longrightarrow K \longrightarrow A \otimes_A B \longrightarrow B \longrightarrow 0$$

$$\Big(\sum a_i \otimes_A b_i\Big) ab = \sum a_i b_i a \otimes b \Rightarrow KAB = KB = 0.$$

$$B \otimes_B N \xrightarrow{\sim} N \Longrightarrow A \otimes_A N \xrightarrow{\sim} N.$$

On right modules. $\mathcal{U} \in \mathcal{M}(A^{op})$ $\quad \cancel{U \otimes_A N}$

[redacted] $\mathcal{M}(B) \longrightarrow \mathcal{M}(A) \longrightarrow \mathcal{Ab}$

$$N \longmapsto N \longmapsto \mathcal{U} \otimes_A N = \mathcal{U} \otimes_A B^{(2)} \otimes_B N$$

$$V \otimes_B M \longleftarrow M \longleftarrow M$$

$$\| \; V \otimes_B A^{(2)} \otimes_A M$$

$$\mathcal{U} \mapsto \mathcal{U} \otimes_A B^{(2)}$$

$$V \mapsto V \otimes_B A^{(2)}$$

$\phi]$ So what does it amount to.

$V \mapsto V \otimes_B A^{(2)}$ bimodule is $B \otimes_B A^{(2)}$

Q. Is $A^{(2)} \to A$ a B mil iso.

$ab(a_i \otimes a_i') = a \otimes b a_i a_i'$ $B = AB$

OK.

So I guess the functors are $U \mapsto U \otimes_A B$

$V \mapsto V \otimes_B A$. Check $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ OKAY because $BA = A$, $AB = B$.

$A \subset B$ $BA = A$ $AB = B$ $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$

$M \mapsto A \otimes_A M = M$

$N = B \otimes_B N \longleftarrow N$

$\therefore$ for inclusion of left ideal $\mathcal{M}(A) = \mathcal{M}(B)$

---

for $A \twoheadrightarrow A/I = B$ $IA = 0$, $\mathcal{M}(A) = \mathcal{M}(B)$

$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ Note in both cases you have $A \to B$ A acting on left, B on the right

This is the general setup where $P = A$, $B = P \otimes_A Q = A \otimes_A Q = Q$, but $Q \otimes P \longrightarrow A$ is ?

$Q \longrightarrow \mathrm{Hom}_{A^{op}}(A, A)$? You're missing the map

$A \to B$ i.e. $(A, A, \mu) \longrightarrow (Q, P, \phi)$. Keep on trying.

Claim: Given $P = A$, $Q \longrightarrow \mathrm{Hom}_{A^{op}}(A, A)$ with Q big enough, i.e. you get $B = A \otimes_A Q$ and you get $\mathcal{M}(A) = \mathcal{M}(B)$. Stefan's observation that the left mult alg acts on any firm A-module.

X] A ring, $B \longrightarrow \mathrm{Hom}_{A^{op}}(A,A)$ $\ni$ $BA = A$

$B = Q$ is an A-module, have A-mod map

~~Generalization: Let us see some examples.~~

I seem to have something ~~slightly more than~~ weaker than a homom. You have ${}_AQ$, ${}_AQ\otimes A \twoheadrightarrow A$ and then $B = A\otimes_A Q$. Does firmness cause problems? Restrict to Q firm A-module w $Q\otimes A \twoheadrightarrow A$. Then claim $\mathcal{M}(A) = \mathcal{M}(Q)$.

$$\begin{pmatrix} A & Q \\ A & Q \end{pmatrix} = \begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

What's the mechanism? .

$A\otimes_A M \xrightarrow{\sim} M$ has $Q = B$ action.

$B\otimes_B N \xrightarrow{\sim} N$ has A action.

two firm rings A, B A left acts on B_B $AB = B$

B left acts on A_A $BA = A$

Go over it again. ${}_AB_B$ ${}_BA_A$ $B\otimes_B A \twoheadrightarrow A$, $A\otimes_A B \twoheadrightarrow B$

Then can conclude $\mathcal{M}(A) = \mathcal{M}(B)$ in the sense that $\exists$ equivalence between firm A, and firm B module structures on any abelian group.

Observe the obvious symmetry ${}_AB_B$ yields $\mathcal{M}(B) \to \mathcal{M}(A)$ while ${}_BA_A$ yields $\mathcal{M}(A) \to \mathcal{M}(B)$. Is it possible to? Working within <u>isomorphisms</u> of categories is it possible to find a canonical ring?

Somehow you have managed a stranger kind of equivalence on firm rings than Morita equivalence. If I fix A then I am looking at all firm Q in $\mathcal{M}(A)$ equipped with a map $Q \longrightarrow \mathrm{Hom}_{A^{op}}(A,A)$, such that $QA = A$. I get a category out of these where maps

4] ~~correap~~ to morphisms, Final object is $A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$

~~Thus~~ Let's go back to the viewpoint. ~~[illegible]~~.

Analyzing $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\quad {}_B A_A \quad {}_A B_B$ $\quad A \otimes_A B \xrightarrow{\sim} B$

$B \otimes_B A \xrightarrow{\sim} A$

$$B \longrightarrow \mathrm{Hom}_{A^{op}}(A,A) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B,B)$$

In general $\mathrm{Hom}_{A^{op}}(P,P) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B,B)$ $\quad P \otimes_A Q$

Start again. I'm trying to understand more about this special kind of Morita equivalence, ~~[illegible]~~ where $\mathcal{M}(A) = \mathcal{M}(B)$ in the sense that firm module structures for A, B are identical. So part of it involves the identity of left mult. algebras.

Assume ~~[illegible]~~

~~[illegible]~~

$$\begin{tikzcd}
a \otimes b' & A \otimes_A B \arrow[r, "\sim"] \arrow[d] & B & ab' \arrow[d, mapsto] \\
ba \otimes b' \arrow[u, mapsfrom] \arrow[rrr, bend right, mapsto] & A \otimes_A B \arrow[r, "\sim"] & B & \\
& & & bab'
\end{tikzcd}$$

$$b(p \otimes q) = bp \otimes q \rightsquigarrow bpq$$

There is some idea here that I am missing!

$$\begin{tikzcd}
(p_2 \mapsto p_1\langle q_1, p_2\rangle) \arrow[r, mapsto] & (p_2 q_2 \mapsto p_1\langle q_1, p_2\rangle q_2) \\
\mathrm{Hom}_{A^{op}}(P,P) \arrow[r] & \mathrm{Hom}_{B^{op}}(B,B) \\
P \otimes_A Q \arrow[u] \arrow[r] & B \arrow[u] \\
p_1 \otimes q_1 \arrow[uuu, mapsto] &
\end{tikzcd}$$

YES!!

$p_1 q_1 \cdot p_2 q_2$ So in this case you find

$$\begin{tikzcd}
\mathrm{Hom}_{A^{op}}(A,A) \arrow[rr, "\sim"] & & \mathrm{Hom}_{B^{op}}(B,B) \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

c0] $B \longrightarrow \mathrm{Hom}_{B^{op}}(B, B)$ is a B^{op}-mil isom.

this map is like, similar, to the inclusion of a left ideal.

Apparently what happens is that B, and ~~defines a coh~~ yield the same ideal in the common multiplier alg Λ. This point is not very deep, but perhaps worth focusing upon. We might start the other way round, namely, with a Roos cat $\mathcal{M}(A^{op})$ and generator P, let $\Lambda = \mathrm{Hom}_{A^{op}}(P, P)$. Looks like wrong side again. But I guess the idea is that we have this unital Λ and ideal in it which we try to generate by a Λ module map $B \xrightarrow{f} \Lambda$

02/06/97 0536 Yesterday I started looking at special mg's where one has equality $\mathcal{M}(A) = \mathcal{M}(B)$. in the sense that firm module structures on any abelian group are in 1-1 corresp for A and B.

~~the same sides~~ Ex. $A \subset B$ s.t. $BA = A$, $AB = B$ (A left ideal in B which generates B). $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ is the Morita context. Other ex. $A \twoheadrightarrow A/I = B$ where $IA = 0$.

e.g. $A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A)$, ?

$M \longmapsto A \otimes_A M = M$

Note that under $\mathcal{M}(A) = \mathcal{M}(B)$, $A \longmapsto A \otimes_A A = A$

So $\mathrm{Hom}_A(A, A) \xrightarrow{\sim} \mathrm{Hom}_B(\overset{A}{A}, \overset{A}{A})$

$\mathrm{Hom}_{A^{op}}(A, A) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(\overbrace{A \otimes_A B}^{B}, \overbrace{A \otimes_A B}^{B})$

In general $\mathrm{Hom}_{A^{op}}(M, M') \simeq \mathrm{Hom}_{B^{op}}(M \otimes_A Q, M' \otimes_A Q)$

[α ~~So what do we do next? The point is~~ ~~that yields~~ So A, B have the same left mult. alg. ~~What Now the generators~~

The point is that the forgetful functors from $\mathcal{M}(A)$ and $\mathcal{M}(B)$ to $\mathcal{Ab}$ coincide

How to say this? In general given a Roos category $\mathcal{M}$ one picks a Q, P and gets a repn as $\mathcal{M}(B)$. $\quad M \mapsto P \otimes_A M \qquad \mathcal{M} \longrightarrow \mathrm{Mod}(\tilde{B})$

$U \mapsto U \otimes_A Q \qquad \mathcal{M}^{op} \longrightarrow \mathrm{Mod}(\tilde{B}^{op})$.

In my situation I have

$$\begin{tikzcd} \mathcal{M}(A) \arrow[r, phantom, "="] \arrow[d, "A\otimes_A -"'] & \mathcal{M}(B) \arrow[d, "B\otimes_B -"] \\ \mathcal{Ab} \arrow[r, phantom, "="] & \mathcal{Ab} \end{tikzcd}$$

commutes

general case
$$\begin{tikzcd} M \arrow[r, mapsto] & P\otimes_A M \\[-2ex] \mathcal{M}(A) \arrow[r, "\sim"] & \mathcal{M}(B) \arrow[d, hook] \\ & \mathcal{Ab} \end{tikzcd}$$

You have to play ~~things against~~ rt exact fun to $\mathcal{Ab}$ against Hom functors.

Roos then describes $\mathcal{M}(A)$ as $\mathrm{Mod}\left(\mathrm{Hom}_A(A,A)^{op}\right) / \text{nil modules}$

nil mods for ideal generated by image of ~~ρA~~

$\rho : A \longrightarrow \mathrm{Hom}_A(A,A)^{op}$. This image should be a ~~left~~ right ~~right~~ ideal left mult alg. ? $\quad f \in \mathrm{Hom}_A(A,A)^{op}$ is a right mult. $\quad (a\,a')f = a(a'f)$, the image ~~of~~ ρA should be a right ideal. $\quad a' \mapsto (a'a)f = a'(af)$

[β The ideal gen. by gA is $\mathrm{Hom}_A(A,A)\cdot A$
image of $\mathrm{Hom}_A(A,A)\otimes_A A$.

$$\begin{pmatrix} A & A \\ \mathrm{Hom}_A(A,A) & \mathrm{Hom}_A(A,A)^{op} \end{pmatrix}$$ Yes.

So things are clearer, but where are we. I seem to be homing in onto a fixed unital ring and idempotent ideal. The problem then becomes to understand essentially the different generating left ideals.

It would be nice to summarize what's been learned. Basically ~~once I fix~~ I start with A say firm and I want to consider all firm dual pairs of the form $(A, Q, Q\otimes A \xrightarrow{\phi} A)$. ϕ is equiv. to a map $Q \longrightarrow A\otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ in $\mathcal{M}(A)$. The condition that ϕ be onto probably amounts to the fact that the image of Q in $\mathrm{Hom}_{A^{op}}(A,A)$ generates the ideal $A\,\mathrm{Hom}_{A^{op}}(A,A)$. Let's try to prove this. Careful analysis.

$\mathrm{Hom}_{A^{op}}(A,A)$ is left mult ring $\lambda(aa') = (\lambda a)a'$.

$$(\lambda\circ(a\cdot))(a') = \lambda(aa') = (\lambda a)a' = ((\lambda a)\cdot)(a')$$

$\therefore$ $\lambda\bar{a}$ in mult alg is $\overline{\lambda a}$.

So $\bar{A} = A/\{a \mid aA = 0\}$ ($a\cdot = 0$) is a left ideal in $\mathrm{Hom}_{A^{op}}(A,A)$.
so $\bar{A}\,\mathrm{Hom}_{A^{op}}(A,A)$ ~~should be~~ is an ideal.

[γ ~~[illegible]~~. What to do? $Q \longrightarrow \mathrm{Hom}_{A^{op}}(A,A) = R$ such that

$$Q \otimes A \longrightarrow \mathrm{Hom}_{A^{op}}(A,A) \otimes A \xrightarrow{ev} A$$ is surjective.

A firm, $R = \mathrm{Hom}_{A^{op}}(A,A)$ left mult. ring

A is a left R module. ${}_R A_A$

and have $A \xrightarrow{f} R$ $\quad f(a) = a\cdot \quad f(a_1 a_2) = f(a_1)f(a_2)$

$$f(ra) = (a' \longmapsto (ra)a' = r(aa'))$$
$$rf(a) = (a' \longmapsto r(f(a)a') = r(aa'))$$

Image of f is a left ideal in R. $\blacksquare$ Keep at it.

Now suppose given Q a firm A-module, and an A-module map $f : Q \longrightarrow \mathrm{Hom}_{A^{op}}(A,A)$ i.e. $f(aq) = (a' \longmapsto a f(q) a')$. Same as A-bimod map $Q \otimes A \longrightarrow A$, $q \otimes a \mapsto \langle q, a\rangle \ni$ $\langle a'q, a\rangle = a'\langle q, a\rangle$, $\langle q, aa'\rangle = \langle q, a\rangle a'$. The image of f is a left ideal in R, so $f(Q)R$ is an ideal. Wait: you need $f(rq) = rf(q)$, where the r action is defined by $r(aq) = (ra)q$. So $rf(q)a'$

$$f(r(aq)) = f((ra)q) = (a' \longmapsto \langle (ra)q, a'\rangle) \qquad r\langle aq, a'\rangle$$
$$= (ra)\langle q, a'\rangle = r(a\langle q, a'\rangle)$$

So what comes next?

You have $Q \xrightarrow{f} R$

$$q \xmapsto{f} (a' \mapsto \langle q, a'\rangle)$$

To understand the ideal QR

Assume $Q \otimes A \twoheadrightarrow A$

First point is that $Q \xrightarrow{f} R$ factors $\left(\text{because } A \otimes_A Q = Q\right)$

$$Q \twoheadrightarrow A \otimes_A R = A \otimes_A \operatorname{Hom}_{A^{op}}(A,A)$$

Thus the image of $f : Q \longrightarrow R$ is contained in AR. Next suppose $Q \otimes A \longrightarrow A$ is surjective. I want

$$\bar{Q} \operatorname{Hom}_{A^{op}}(A,A) = \bar{A} \operatorname{Hom}_{A^{op}}(A,A)$$

It's probably obvious because $QA = A$

Because $AQ = Q$ should have $\bar{Q} \subset \bar{A}\bar{Q} \subset \bar{A}R$.

Any a has form $\sum \langle g_i, q_i \rangle$ so $\bar{A} \subset \bar{Q}R$?

Start again with A firm, let $R = \operatorname{Hom}_{A^{op}}(A,A)$ be the left mult. ring. Consider $Q \otimes A \twoheadrightarrow A$ $\langle g, a \rangle$ with $Q \in \mathcal{M}(A)$. Let $B = A \otimes_A Q = Q$. Then have

$$\begin{pmatrix} A & Q \\ A & \overset{\shortparallel}{B} \end{pmatrix}$$

$$\operatorname{Hom}_{A^{op}}(A,A) \xrightarrow{\sim} \operatorname{Hom}_{B^{op}}\left(A \otimes_A Q , A \otimes_A Q\right) = \operatorname{Hom}_{B^{op}}(B,B)$$

Thus any of these rings have the same left multiplier ring. Also although the rings $A = A \otimes_A A$, $B = A \otimes_A Q$ are different, they generate the same ideal in $\operatorname{Hom}_{A^{op}}(A,A) = R$. Now how ~~do~~ I get central? At some point I should ~~take central of~~ look at K_1.

So where do I begin? Basically you are stuck with

$$A \longrightarrow \operatorname{Hom}_{A^{op}}(A,A) = R$$

$$B = A \otimes_A \operatorname{Hom}_{A^{op}}(A,A)$$

[ε

~~So Parker~~ Is B idempotent. $ARAR = A^2R = AR$.

OK. Next - what?

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

A flat over A^{op}

$\Rightarrow B$

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

Recall $A \in \mathcal{P}(\dot{A}^{op})$ set $R = B = \mathrm{Hom}_{A^{op}}(A, A)$

and then ~~$R \otimes B$~~ $A \in \mathcal{P}(\dot{A}^{op}) \Rightarrow A \otimes_A B = B \in \mathcal{P}(\dot{B}^{op})$

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13 26 A firm, $P = A$, $Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A) = B$

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

${}_B A_A$ think $BA = A$, $AB = B$, A is roughly a left ideal gen B as ideal.

$$\mathrm{Hom}_{A^{op}}(A, A) = \mathrm{Hom}_{B^{op}}(\overset{B}{\cancel{A \otimes_A B}}, \overset{B}{\cancel{A \otimes_A B}})$$

So the first situation to understand perhaps is when $A = B$, i.e. where $A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A)$ is an A-mil isom.

This seems to be a good question

A firm eg. A unital.

I did this before I think.

Given A take $P = A$, $Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$

Then
$$\mathrm{Hom}_{A^{op}}(P, P) \times' \mathrm{Han}_A(Q, Q)^{op}$$

~~So I think~~ To compute the left multiplier ring of B I just use $\mathrm{Hom}_{A^{op}}(P, P)$ turns out to be $R = \mathrm{Hom}_{A^{op}}(A, A)$ ~~So if were to start with ideals.~~ The fact is that for any $B = P \otimes_A Q$ the left mult. ring is

$$\mathrm{Han}_{A^{op}}(P, P) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B, B).$$

[3 Thus replacing A by $A\otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ does what?

~~[illegible]~~

Start with A firm, set $B = A\otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ – evident Mcont $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\quad BA = A$, $AB = B$. Then

$$\mathrm{Hom}_{A^{op}}(A,A) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(\underset{\underset{B}{\parallel}}{A\otimes_A B}, \underset{\underset{B}{\parallel}}{A\otimes_A B})$$

what I am trying to show is that

$$B \xrightarrow{\sim} B\otimes_B \mathrm{Hom}_{B^{op}}(B,B)$$

enough to have $B \longrightarrow \mathrm{Hom}_{B^{op}}(B,B)$ a B-nil iso.

why the difficulty? Somehow you should be able to argue that any ~~[illegible]~~ f'dual pair $Q\otimes A \twoheadrightarrow A$ maps uniquely to $B\otimes A \longrightarrow A$.

So try this. namely A, B, C

$$B = A\otimes_A \mathrm{Hom}_{A^{op}}(A,A) \qquad C = B\otimes_B \mathrm{Hom}_{B^{op}}(B,B)$$

$$\parallel$$

$$B\otimes_B \mathrm{Hom}_{A^{op}}(A,A)$$

$$\begin{matrix} A & B & C \\ A & B & C \\ A & B & C \end{matrix} \qquad \begin{pmatrix} A & B \\ A & B \end{pmatrix}$$

what is the mult. ring for B?

$$\mathrm{Hom}_{B^{op}}(B,B) \times' \mathrm{Hom}_B(B,B)$$

$$\parallel$$

$$\mathrm{Hom}_{A^{op}}(A,A) \times' \mathrm{Hom}_A(B,B)$$

~~[illegible]~~ and these are the same.

[7 So what I'm failing to see is why

$$B = B \otimes_B \operatorname{Hom}_{B^{op}}(B,B)$$

normally the image of A, call it $\bar{A}$ in $\operatorname{Hom}_{A^{op}}(A,A)$ is a left ideal but not a right ideal. Assume it is a two-sided ideal, i.e. a right ideal. Then

$$\bar{A} \hookrightarrow \operatorname{Hom}_{A^{op}}(A,A)$$

is a ~~right~~ left A-nil ism. In general $B = A \otimes_A \operatorname{Hom}_{A^{op}}(A,A)$ should be such that the map

$$B \longrightarrow \operatorname{Hom}_{B^{op}}(B,B)$$

is both a left and right nil ism. This amounts to some property of the multiplier ring.

Need control.

Go back to A firm $\quad B = A \otimes_A \operatorname{Hom}_{A^{op}}(A,A)$

basic meq $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$. Then

$$\operatorname{Hom}_{A^{op}}(A,A) \xrightarrow{\sim} \operatorname{Hom}_{B^{op}}(B,B)$$

$$\operatorname{Hom}_A(B,B) \xrightarrow{\sim} \operatorname{Hom}_B(B,B)$$

$B \otimes_B A = A$

$A \otimes_A B = B$

$B = Q \otimes_B B = B.$

$$\begin{array}{ccc} B & & \\ \| & & \\ A \otimes_A \operatorname{Hom}_{A^{op}}(A,A) & \xrightarrow{\sim} & \operatorname{Hom}_{B^{op}}(B,B) \\ \uparrow & & \uparrow \\ A \otimes_A A & & B \end{array}$$

There seems to be a homom. $A \to B$ which I have been ignoring.

$A \quad Q \qquad\qquad Q \otimes P \longrightarrow A$

$P \quad P \otimes_A Q \qquad\qquad Q \longrightarrow \mathrm{Hom}_{A^{op}}(P, A)$

suppose you have $(\phi, \psi) \in \mathrm{Hom}_{A^{op}}(P,P) \times \mathrm{Hom}_A(Q,Q)^{op}$ such that $\langle q, \phi p\rangle = \langle q\psi, p\rangle$. This should amount to

$$\begin{tikzcd}
Q \arrow[r] & \mathrm{Hom}_{A^{op}}(P, A) \\
Q \arrow[r] \arrow[u, "\psi"] & \mathrm{Hom}_{A^{op}}(P, A) \arrow[u, "\phi^t"']
\end{tikzcd}$$

commuting.

$$\begin{tikzcd}
q\psi \arrow[r, mapsto] & (p \mapsto \langle q\psi, p\rangle) & p \mapsto \phi p \mapsto \langle q, \phi p\rangle. \\
q \arrow[r, mapsto] \arrow[u, mapsto] & (p' \mapsto \langle q, p'\rangle) \arrow[ur, bend right] &
\end{tikzcd}$$

Thus $Q \xrightarrow{\sim} A \otimes_A \mathrm{Hom}_{A^{op}}(P,A) \Rightarrow$ the multiplier ring of B is actually $\mathrm{Hom}_{A^{op}}(P,P) = \mathrm{Hom}_{B^{op}}(B,B)$, the left multiplier ring of B. In other words the projection $\mathrm{Mult}(B) \longrightarrow \mathrm{Hom}_{B^{op}}(B,B)$ is an isom.

Does this help me any???

~~there seems to be an~~

Take $P = A \qquad Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) = B$ etc.

OKAY, so much to do - . First example to understand carefully is $\boxed{A \in \mathcal{P}(A)}$. Then B should be unital.

Start again. I seem to have understood why $B = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ is such that $\mathrm{Mult}(B) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B,B)$, so that ~~the~~ $B \to \mathrm{Hom}_{B^{op}}(B,B)$ is both a B and B^{op} mod iso.

[6 Start with A, put $B = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$.

Let $R = \mathrm{Hom}_{A^{op}}(A, A)$. R is the left mult ring ~~algebra~~ of A. Replace A by its image in R $\bar{A} = A/\{a \mid aA = 0\}$. Then $\bar{A}$ is reduced as $\bar{A}^{op}$-module, so $R = \mathrm{Hom}_{\bar{A}^{op}}(\bar{A}, \bar{A})$. Then $\bar{A} \subset R$ is a left ideal and we can consider the ideal it generates, namely $\bar{A}R$. $B = A \otimes_A R \twoheadrightarrow \bar{A}R \subset R$. ~~so by M-inv~~

Start again with firm picture. A firm, $R = \mathrm{Hom}_{A^{op}}(A, A)$ $B = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$ is firm. By Meg we have $R = \mathrm{Hom}_{A^{op}}(A, A) \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B, B)$. It seems clear that $B = A \otimes_A R$ has image $\bar{A}R$ in R. Thus the essential point seems to be A is a slight generalization of a left ideal in R generating $\bar{B}$. One can ask when the image of A in the left mult alg is an ideal? This seems to have a simple answer, namely when $A \to \mathrm{Hom}_{A^{op}}(A, A)$ is an A-nil isom. 2nd slightly stronger. Need kernel $I = \{a \mid aA = 0\}$ to be killed by A. Let $I = \{a \in A \mid aA = 0\}$. This is the largest nil submod for A as right module, so $IA = 0$

$R = \mathrm{Hom}_{\bar{A}^{op}}(\bar{A}, \bar{A})$ $\quad \bar{A} = A/I$ so

need $AI = 0$.

Next suppose $A \in \mathcal{P}(A^{op})$. Then $1 \in R$ lies in B i.e. $B = R$, so we are concerned with gen. left ideals^A in R $\quad A \xrightarrow{f} R$ R-hom. $\ni$ $(fA)R = R$

$A \quad R$

$$A \in \mathcal{P}(A^{op}) \Longrightarrow \underset{R}{A \otimes_A R} \in \mathcal{P}(R^{op}) \quad \text{obv}$$

$A \quad R$

$$R \in \mathcal{P}(R^{op}) \Rightarrow R \otimes_R P = P \in \mathcal{P}(A^{op}).$$

LK Nothing is very clear. I think I have to get started with K_1 and K_2.

Other general stuff $\quad A \quad A^{op}$ flat $\iff$ ~~[illegible]~~ $A \otimes_A B = B$ is B^{op}-flat.

Will find out that $B \in \mathcal{P}(R^{op})$

The important point is the flatness I think.

So if A is A^{op}-flat, then B is B^{op}-flat so ~~[illegible]~~ B should be R^{op}-flat. But now you may have a problem because $B \to R$ ~~[illegible]~~ may not be injective. This might not be important

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~~[illegible]~~ 02/07/97 0830 I want to write up something

A firm, say left flat. Example start with a firm ~~[illegible]~~ ring C and choose a C-module surjection $W \overset{f}{\twoheadrightarrow} C$ with W firm + flat. ~~[illegible]~~ Then get

$\langle w, c\rangle = f(w)c$

$W \otimes C \twoheadrightarrow C$, hence a ring $A = C \otimes_C W \simeq W$

with $a_1 a_2 = f(a_1) a_2$ $\qquad c_1 w_1 (c_2 w_2) = \underbrace{c_1 f(w_1)}_{f(c_1 w_1)} (c_2 w_2)$

~~[illegible]~~ It seems we have passed from

$C \otimes C \overset{\mu}{\twoheadrightarrow} C$ to $W \otimes C \twoheadrightarrow C$. $\begin{pmatrix} C & W = A \\ C & A \end{pmatrix}$

(Digress: Write the context oppositely $\begin{pmatrix} A & C \\ \overset{\shortparallel}{W} & C \end{pmatrix}$

I'm trying to say that the processes $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$ $\begin{pmatrix} B & A \\ B & A \end{pmatrix}$ have the same form. ~~the bimodule which has the~~

Starting from a ring say A, the bimodule having the ring acting on the right is just the ring and the other ring B acts on the left. So there's no change to the form

modules $M \mapsto A \otimes_A M = M$ & sim $N \mapsto B \otimes_B N = N$.)
So what is the next point? Other point is
$\mathrm{Hom}_{A^{op}}(A, A) \simeq \mathrm{Hom}_{B^{op}}(B, B)$ since $A \mapsto A \otimes_A^Q B = B$.

examine flatness. In the situation I began with
$\begin{pmatrix} C & A \\ C & A \end{pmatrix}$ you have $A = W \xrightarrow{f} C$

$A \otimes C \longrightarrow C$ Thus have surj $A \otimes C \twoheadrightarrow C$
$a \otimes c \mapsto f(a)c$

$$\begin{array}{ccc} A \otimes C & \twoheadrightarrow & C \\ \downarrow{\scriptstyle f \otimes 1} & & \| \\ C \otimes C & \xrightarrow{\mu} & C \end{array}$$

$C \otimes_C A = A$

$A \otimes_A C = C$.

$$\begin{tikzcd} \mathrm{Hom}_{C^{op}}(C, C) \arrow[r, "\simeq"] & \mathrm{Hom}_{A^{op}}(A, A) \\ C \arrow[u] & A \arrow[l, two heads, "f"'] \arrow[u] \end{tikzcd}$$

commutes.

$a_1 a_2 = f(a_1) a_2$ So what am I trying to see? A is a C-module eg with C-surj $f : A \to C$
$a_1 a_2 = f(a_1) a_2$.

Assume A is C-flat. Then $C \otimes_C A = A$ is A-flat and conversely.

Check: Let $a \in A$, consider $a \cdot$ on $A \in \mathcal{M}(A^{op}) = \mathcal{M}(C^{op})$

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes_A C & \xrightarrow{\sim} & C \\ a' & & a' \otimes c & & \\ a\cdot \downarrow & & \downarrow \quad \downarrow & & \\ aa' & & aa' \otimes c \quad A \otimes_A C & = & C \\ \| & & & & \\ f(a)a' & & & & \end{array}$$

$a'c = f(a')c$
$\downarrow$
$aa'c \quad f(a)f(a')c$.

Compare flatness to

6μ Let's go back to ~~[illegible]~~ C nog to a unital ring. Then $\mathcal{M}(C)$ has a small proj gen. U and we have a Mcat. $\begin{pmatrix} C & U \\ U^* & U^*\otimes_C U \end{pmatrix}$

$U^*\otimes_C U = \mathrm{Hom}_C(U,U)^{op}$. Change notation

$\begin{pmatrix} C & U \\ U^* & A \end{pmatrix}$ from another viewpt you have A unital and unitary module U_A ${}_AU^*$ surj pairing $U^*\otimes U \twoheadrightarrow A$.

I'm especially interested in the case where ${}_AU^* \in \mathcal{P}(A)$, since C left flat reduces to this case. Then

$C = U\otimes_A U^* \in \mathcal{P}(C)$.

~~I have gone over the reduction of~~.

The critical case to understand is ~~[illegible]~~ an idempotent ring C such that $C\in\mathcal{P}(C)$ (or $\mathcal{P}(C^{op})$).

~~Theor~~ Suppose $C\in\mathcal{P}(C^{op})$, consider $B = \mathrm{Hom}_C$

$A\in\mathcal{P}(A^{op})$ assume. $R = \mathrm{Hom}_{A^{op}}(A,A) = A\otimes_A \mathrm{Hom}_{A^{op}}(A,A)$

$\begin{pmatrix} A & \mathrm{Hom}_{A^{op}}(A,A)=R \\ A & R \end{pmatrix}$ $\begin{pmatrix} A & R \\ A & R \end{pmatrix}$ $A\otimes_A R \xrightarrow{\sim} R$, $R\otimes_R A \xrightarrow{\sim} A$.

It seems ~~[illegible]~~ we have an R-map $A\xrightarrow{t} R$ whose image generates R as ideal. $\mathcal{M}(A) = \mathcal{M}(R)$ have A like a left ideal in R generating R: $AR=R$

$R\in\mathcal{P}(R^{op}) \implies R\otimes_R A = A \in \mathcal{P}(A^{op})$

$R\in\mathcal{P}(R) \implies R\otimes_R R = R\in\mathcal{P}(A)$.

[V] How can I make this more clear? ~~...~~
Davydov situation. Logic I used.

$$A \in \mathcal{P}(\tilde{A}^{op}) \text{ left acted on by } R$$

defines $K_*(R) \longrightarrow K_*(\tilde{A})$
while the homom. $A \to R$ induces $K_*(\tilde{A}) \to K_*(R)$.
What?

1317 $A \in \mathcal{P}(A^{op})$ A idempotent

$R = \mathrm{Hom}_{A^{op}}(A,A)$ $\begin{pmatrix} A & R \\ A & R \end{pmatrix}$ $R \otimes_R A \xrightarrow{\sim} A$

$A \otimes_A R \xrightarrow{\sim} R$.

have homom. $A \xrightarrow{f} R$ $f(a) = a\cdot$

Also we have $f(ra) = \{a' \mapsto (ra)a' = r(aa')\} = rf(a)$

~~The facts~~ $AR = R \iff A \in \mathcal{P}(A^{op})$

$U \in \mathcal{P}(A^{op}) \iff 1 \in \text{Image of } A \otimes \mathrm{Hom}_{A^{op}}(A,A) \to R$

$$1 = \sum a_i r_i .$$

What should be true? Should $K_* A \xrightarrow{\sim} K_* R$ for any such A? ~~This amounts to~~ Is any such A h-unital. Yes A is right A-flat. Thus clear!

Perhaps I should try to work out a complete proof for K_1 and K_2. So you want to

Understanding K_1 and K_2 are important.

First do K_1 carefully. To compare $GL(A)$ with $GL(R)$. There's a homomorphism $A \to R$ which you want to show induces $H_*(GL(A)) \xrightarrow{\sim} H_*(GL(R))$. Take the various viewpoints — besides K_* you have HH and HC. Let us understand the main ingredients.

[redacted] GL(A) subgroup of GL(R)

main idea is that we have ~~[illegible]~~

$$(1+r)(1+a) = 1 + r + a + ra$$

$RA \subseteq A$ $\quad \underline{AR \doteq R}$. If you have an invertible

$r \in R^\times$ $\quad r = 1 - \underbrace{(1-r)}_{ar}$ and you find that

$1 - ar \sim 1 - ra$ by Vaserstein identity

$$\begin{pmatrix} 1 & 0 \\ (1-yx)^{-1}y & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}\begin{pmatrix} 1 & -x(1-yx) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-xy & 0 \\ 0 & (1-yx)^{-1} \end{pmatrix}$$

$$(1-yx)^{-1}y = y(1-xy)^{-1}$$

So I need to understand everything.

$$\begin{pmatrix} 1-xy & x \\ 0 & (1-yx)^{-1} \end{pmatrix}$$

$1 + y(1-xy)^{-1}x$. Here's what I have to play with.

You have the homom. $A \longrightarrow R$ and $R \longrightarrow \mathrm{Hom}_{A^{op}}(A, A)$ so that $GL_n(R)$. ~~[illegible]~~

The ring homos in this situation I basically understand. You have $A \longrightarrow R$ and $R \longrightarrow \mathrm{Hom}_{A^{op}}(A,A)$ and $A \in \mathcal{P}(A^{op})$ whence a split embedding $A \rightleftarrows \tilde{A}^m$ of A^{op} modules, hence a [redacted] hom. $R \to \mathrm{Hom}_{A^{op}}(A,A) \hookrightarrow \mathrm{Hom}_{A^{op}}(\tilde{A}^m, \tilde{A}^m) = M_m(\tilde{A})$.

In fact it seems that maybe R maps to $M_m(A)$.

How to prove. Suppose to simplify that we have $yx = 1$ with $y \in A$, $x \in R$.

Then $A \xrightarrow{x\cdot} A \subset \tilde{A} \xrightarrow{y\cdot} A$

so as A^{op}-module we have

$$A \underset{y\cdot}{\overset{x\cdot}{\rightleftarrows}} eA \qquad e = xy$$

$$a \longmapsto xa$$

$$ya' \longleftarrow\!\!| \; xya'$$

This identifies A as A^{op}-module with the $eA = e\tilde{A}$

So now try you idiot to prove $H_*(GL(A)) \simeq H_*(GL(R))$.

You have $GL(A) \to GL(R)$ induced by the inclusion

and $R \longrightarrow Hom_{A^{op}}(A,A) \simeq Hom_{A^{op}}(eA, eA) = e\tilde{A}e$

which gives $GL(R) \longrightarrow GL(A)$ maybe. But now you need to look at the composition. R acts on A which is a direct summand $e\tilde{A}$. So we have

$R \longrightarrow e\tilde{A}e \subset A \longrightarrow R$. Why am I so stupid.

Viewpoint: have $A \longrightarrow R$ homom. and have homom.

$R \longrightarrow Hom_{A^{op}}(A,A) \simeq Hom_{A^{op}}(e\tilde{A}, e\tilde{A}) = e\tilde{A}e \subset A$. You want to see these induce inverse maps on K-theory.

How to start. It should be easy to show $R \longrightarrow A \longrightarrow R$ yields 1 on K_* as a consequence [π

of stabilization. ~~[illegible]~~ I think you have
$R \longrightarrow e\tilde{A}e \longrightarrow eRe \subset R$. $\quad r \mapsto xry$

~~[illegible]~~ To simplify suppose A left ideal in R such that $AR = R$ ~~[illegible]~~ and that $yx = 1$ where $y \in A$ $x \in R$. Then $e = xy$ is idempotent in A. We have

$$\begin{pmatrix} eAe & eA \\ Ae & A \end{pmatrix} \subset \begin{pmatrix} eRe & eR \\ Re & R \end{pmatrix}$$

$$Re \subset A \Rightarrow Re \subset Ae \subset Re \quad \therefore Re = Ae$$

so also $eAe = eRe$. $\quad ReR = R \Longrightarrow AeA = A$.

$$R \Longrightarrow \mathrm{Hom}_{A^{op}}(A, A) \simeq \mathrm{Hom}_{A^{op}}(e\tilde{A}, e\tilde{A}) = eAe$$

$$A \underset{y\cdot}{\overset{x\cdot}{\rightleftarrows}} eA = xyA \subset A$$

so $r \longmapsto xry$ is a homom., isom of $R \simeq$ ~~[illegible]~~ $xRy = eRe = eAe$. So I am asking whether $r \longmapsto xry$ induces iso on K_*. This I think is basic stabilization: no matter how $P \in \mathcal{P}(R^{op})$ is embedded in a free module you get the same homom. $\mathrm{Aut}(P) \longrightarrow GL(R)$ up to conjugacy.

Next comes the hard part. Review: have homom. $R \longrightarrow eRe = eAe \subset A \subset R$
$$r \longmapsto xry$$

We also know that $eRe \subset R$ induces ~~[illegible]~~ isom. on K_*, again by stab. args. Now we have this.

[8] ~~anyway~~ Thus we have

$$\begin{array}{ccccccccc} x^2Ry^2 \subset & xAy & \subset & xRy & \subset & A & \subset & R \\ & \uparrow{\scriptstyle S} & & \uparrow{\scriptstyle S} & & & & \\ & A & \subset & R & & & & \end{array}$$

~~etc~~.

So now we want to know that $A \xrightarrow{a \mapsto xay} A$ induces the identity on K_*. I can try the same argument, namely,

$$0 \longrightarrow A \xrightarrow{\hat{x}\cdot} A \longrightarrow A/xA \longrightarrow 0 \qquad \text{What?}$$

A as right A-module splits $A = eA \oplus e^\perp A$ where $eA \underset{y\cdot}{\overset{x\cdot}{\rightleftarrows}} A$. So it seems that you have managed to embed A_A as a summand of itself. But now comes the problem with ~~K~~ K_* or $\underline{H_*Gl}$.

$$\begin{pmatrix} eAe & eA \\ Ae & A \end{pmatrix} \subset \begin{pmatrix} eRe & eR \\ Re & R \end{pmatrix}$$

Re and eR are dual f proj modules over $eRe = eAe$

so eA any $\overset{eAe}{\text{submodule}}$ of eR such that $TAe = eAe$

02/08/97, 0500

So what happens next.

So it is probably time to summarize; go over what I did yesterday. Clean it up, try to ~~...~~.

Let us start again

[σ 02/08/97 0507

$A = A^2 \qquad A \in \mathcal{P}(A^{op}) \qquad R = \mathrm{Hom}_{A^{op}}(A, A)$

$$\begin{pmatrix} A & R \\ A & R \end{pmatrix} \qquad R \otimes_R A \xrightarrow{\sim} A \qquad A \otimes_R R \xrightarrow{\sim} R$$

Since R unital, hence $R \in \mathcal{P}(R^{op})$ and $\mathcal{P}(R)$, we know $A = R \otimes_R A \in \mathcal{P}(A^{op})$, $R = A \otimes_R R \in \mathcal{P}(A)$. In fact these are dual over A: $R = \mathrm{Hom}_{A^{op}}(A, A)$ and $A = \mathrm{Hom}_A(R, A)$.

We have basic representations

$$A \longrightarrow R = \mathrm{Hom}_{R^{op}}(R, R) \quad \text{inducing} \quad K_* A \longrightarrow K_* R$$

$$R = \mathrm{Hom}_{A^{op}}(A, A) \quad \text{inducing} \quad K_* R \longrightarrow K_* A.$$

The latter requires maybe some care because what does one mean by $K_* A$. fibre of $BGL(\tilde{A})^+ \to BGL(\mathbb{Z})^+$ or something constructed from $GL(A)$.

Yesterday I looked at the case where A left ideal $\subset R$ $\exists x \in R, y \in A$ st $yx = 1$ in $AR = R$. Then we have an explicit direct embedding of A_A as the summand $eA = e\tilde{A}$ of $\tilde{A}$. $e = xy$

$$A \underset{y\cdot}{\overset{x\cdot}{\rightleftarrows}} \tilde{A}$$

and this gives the homom. $r \mapsto xry$ from R into A. Compose

$$R \longrightarrow A \hookrightarrow R$$
$$r \longmapsto xry \longmapsto xry$$

This composition should give the identity on K for more or less obvious reason. The point maybe is that the effect on GL is ~~[illegible]~~?

[I ~~That seems~~ You have homs. $R \to A \subset \tilde{A} \to R$
meaning. ~~$P \in \mathcal{P}(R^{op})$~~ Given G acting on $P \in \mathcal{P}(R^{op})$
then get on $P \otimes_R A \in \mathcal{P}(A^{op})$ a rep of G

Given a rep of G on $V \in \mathcal{P}(R^{op})$ you get a rep of G on $V \otimes_R A \in \mathcal{P}(A^{op})$, then you get a rep. of G on $V \otimes_R A \otimes_A R \simeq V$. Is this true? This should be OKAY although its might be tricky on the level of GL's.

The other direction $A \subset \tilde{A} \to R \to A$

$a \longmapsto a \mapsto xay \in RA = A.$

is more subtle. Given $U \in \mathcal{P}(\tilde{A}^{op})$ a rep of G, you get $U \otimes_A R$ then $U \otimes_A R \otimes_R A \simeq UA$. So you don't get back to the point of departure. ~~This simplifies~~.

~~So I did get somewhere~~

So take the module viewpoint. meaning?

Have two reps. one from the homo $\tilde{A} \to R$ yielding $\mathcal{P}(\tilde{A}^{op}) \to \mathcal{P}(R^{op})$ $U \mapsto U \otimes_A R$, the other from $R \to \mathrm{Hom}_{A^{op}}(A, A)$ yielding $\mathcal{P}(R^{op}) \to \mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op})$, $V \mapsto V \otimes_R A$. These induce $K_*\tilde{A} \to K_* R \to K_* \tilde{A}$. ~~So life go~~ Compositions on the level of $\mathcal{P}(\tilde{A}^{op})$ and $\mathcal{P}(R^{op})$ are identity. So the problem does not really involve R so much. It mainly is a question of understanding the relation between $\mathcal{P}(\tilde{A}^{op})$ and $\mathcal{P}(A^{op})$. You have

$$\mathcal{P}(A^{op}) \subset \mathcal{P}(\tilde{A}^{op}) \longrightarrow \mathcal{P}(A^{op})$$
$$U \longmapsto U \otimes_A A = UA.$$

[4] ~~otherwise. What next. Yes. Anyway what next.~~

confusion reigns. Ex. seq

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0$$

What methods? Main method you have is the exact sequence of exact functors

$$0 \longrightarrow AU \longrightarrow U \longrightarrow U/AU \longrightarrow 0$$

from $P(\tilde{A}^{op})$ to $P_1(\tilde{A}^{op})$. You would like to understand the implications of this for the ~~.....~~ GL.

Question: Is Suslin's result enough here to deduce Davydov's result? Suslin's result is an excision thm. and roughly says that $BGL(A)^+$ is an h-space. But does it say much about ~~the~~ link between $BGL(A)^+$ and $P(A)$?

1325 ~~.....~~ Let's review: $A \in P(A^{op})$. ~~.....~~ For example a ring (idempotent) $\ni$ $y \in A$ and $x \in \mathrm{Hom}_{A^{op}}(A, A)$ such that $yx = 1$. This means $yx(a) = a$ $\forall a$.

~~.....~~ Does this condition imply A idempotent? Why $A \in P(\tilde{A}^{op})$. You have the identity in the image of $A \otimes_A \mathrm{Hom}_{A^{op}}(A, \tilde{A}) \longrightarrow \mathrm{Hom}_{A^{op}}(A, A)$.

Simpler $\quad A \xrightarrow{x\cdot} A \subset \tilde{A} \xrightarrow{y\cdot} A \quad$ A^{op}-module maps comp. $= 1$.

It implies A idempotent, in fact $A = yA \subset A^2$.

Same argument holds for $\sum y_i x_i(a) = a$. $\forall a$.

Have a Morita equiv. A with unital ring $\mathrm{Hom}_{A^{op}}(A, A)$.

[ϕ So now where are we?   So what?

Back to $A \in \mathcal{P}(A^{op})$ $\exists\, y_i \in A$, $x_i \in Hom_{A^{op}}(A, A) = R$ such that $\sum y_i x_i(a) = a$ $\forall a \in A$. Then $A = \sum y_i A \subset A^2$.

$$A \xrightarrow{(x_i)} A^n \subset \tilde{A}^n \xrightarrow{(y_i \cdot)} A$$

What point needs understanding? $H_*(GL(A))$ Look at this carefully. Group homology. So how to proceed? The problem will be to relate invertible matrices over A to something else. Basically you have [redacted] the category $\mathcal{P}(A^{op}) \simeq \mathcal{P}(R^{op})$, which is not very close to $GL(A)$. You can consider cats, rings, groups – basic analogies. Wodzicki

A idempotent $\longleftrightarrow$ G perfect

A firm $\longleftrightarrow$ G superperfect.

analogies which become almost functors $A \mapsto GL(A)$ in some [redacted] complicated way. Anyway, what info do we have. We have to deal with invertible matrices over A. These are [redacted] linked to finite free modules over $\tilde{A}$.

You want to relate $K_* A$ say $H_* GL(A)$ to $K_* R$. The difficulty [redacted] arises from $GL(A)$ being defined using f free $\tilde{A}$ modules. You don't have as yet an "intrinsic" definition in the sense [redacted] of being related to autos of A-objects. So begin with zilch

I somehow don't have the proper approach.

[X Let's consider ~~.....~~ the possible homos.

$A \subset \tilde{A} \longrightarrow R$. Also have $R \longrightarrow A$, $r \mapsto xry$, defined because $y \in A$, $x, r \in \mathrm{Hom}_{A^{op}}(A,A)$, [redacted] ~~YES~~.

$$A \xrightarrow{x} \dot{\tilde{A}} \subset \tilde{A} \xrightarrow{y\cdot} A \qquad yx(a) = a.$$

$$\tilde{A} \xrightarrow{y\cdot} A \xrightarrow{x} \tilde{A} \quad \text{is left mult by } e = x(y).$$

$e^2 = x(y)x(y) = x(yx(y)) = x(y) = e$. The important point ~~then~~ I think is that we have $R \longrightarrow \dot{\tilde{A}}$, $r \mapsto xry$ the basic nonunital homom. of R to A which induces $GL(R) \longrightarrow GL(A)$. And this is to be combined w. $A \subset \tilde{A} \longrightarrow R$.

Let's try to do a little bit on K_1 and K_2. How much to decide? First handle K_1. ~~YES~~. No first do your analysis. What was the point? You have $P(\tilde{A}^{op}) \longrightarrow P(A^{op}) \subset P(\tilde{A}^{op})$

$$u \longmapsto u \otimes_A A = uA$$

~~.....~~ point is the exact seq of ex funs

$$0 \to uA \longrightarrow u \longrightarrow u/uA \longrightarrow 0$$

$$u/uA = u \otimes_A \mathbb{Z} = \bar{u}$$

together with your res. thm. ~~So we have something~~

But now look for an elementary argument. What do we know? We have functors $- \otimes_A u : P(A^{op}) \to P(A^{op})$. We have also $\mathbb{Z} \to \tilde{A}$. Because A augmented we have another res. $0 \longrightarrow \bar{u} \otimes_{\mathbb{Z}} A \longrightarrow \bar{u} \otimes_{\mathbb{Z}} \tilde{A} \longrightarrow \bar{u} \longrightarrow 0$

[4 So now use fib prod.

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & \bar{u}\otimes_Z \Lambda \arrow[r, equal] \arrow[d] & \bar{u}\otimes_Z \Lambda \arrow[d] & \\ 0 \arrow[r] & uA \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & \bar{u}\otimes_Z \tilde{A} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & uA \arrow[r] & u \arrow[r] \arrow[d] & \bar{u} \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

Then ordinary additivity for exact sequences of representations works. This settles the Milnor question in an elementary ~~[illegible]~~ way ~~[illegible]~~ reducing to ~~vanishing~~ additiv. for Δ matrices, but it doesn't ~~[illegible]~~ prove Davydov's thm I ~~[illegible]~~ feel.

So where am I? I think I can prove that if A is right flat and meg a unital ring B, then K_* agrees for A and that unital ring. ~~[illegible]~~ $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

A A^{op} flat $\Rightarrow$ $A\otimes_A Q = Q$ is B^{op} flat. Replace Q by $\varinjlim B^{n_\alpha}$. Also $P\otimes Q \twoheadrightarrow B$ says OKAY for Q replaced by B^{n_α}. But then this reduces A A^{op} flat to $A\in P(A^{op})$ which I think I can handle.

Let's change notation: $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ A unital, B right flat

$\Rightarrow$ $B\otimes_B P = P$ is right A-flat. So can take P f. free.

Can keep $Q\otimes P \twoheadrightarrow A$

[ω 02/09/97 Consider Davydov's situation $\begin{pmatrix} \cancel{eRe} & \cancel{eR} \\ \cancel{Re} & \cancel{ReR} \end{pmatrix}$

$$\begin{pmatrix} ReR & Re \\ eR & eRe \end{pmatrix}$$

assume

$$Re \otimes_{eRe} eR \xrightarrow{\sim} ReR$$

$$Re \in \mathcal{P}(eRe^{op})$$

Viewpoint $\quad A = Q \otimes_B P \quad \begin{matrix} & Q \\ P & B \end{matrix}$

where B is unitary, ${}_BP$, Q_B unitary. Assume we have $B, B \xrightarrow{\sim} Q, P$ compatible with pairings so $(Q,P) = (B,B) \oplus (Q', P')$. What's important here is the result $K_*(R) \cong K_*(R/ReR) \oplus K_*(eRe)$. One has R-bimod exact sequence

$$0 \longrightarrow A \longrightarrow R \longrightarrow R/A \longrightarrow 0 \qquad A = ReR$$

where $A \in \mathcal{P}(R^{op})$. So get exact seq. of exact functors

$$0 \longrightarrow U \otimes_R A \longrightarrow U \longrightarrow U \otimes_R R/A \longrightarrow 0$$

from $\mathcal{P}(R^{op})$ to $\mathcal{P}_1(R^{op})$. $\quad \mathcal{P}(R^{op}) \longrightarrow \mathcal{P}(R/A^{op})$

$$\mathcal{P}((R/A)^{op}) \longrightarrow \mathcal{P}_1(R^{op})$$

Here use resolution thm. $V \in \mathcal{P}(R/A)$

$$0 \to U_1 \to U_0 \to V \to 0$$

Work in exact cat of such resolutions?

$0 \to \mathrm{Tor}_1^R(V, R/A) \to U_1 \otimes_R R/A \to$ (0 as $A = A^2$)

YES

[α] I want to know whether anything I have been doing with $\mathrm{Hom}_{A^{op}}(A, A)$ is relevant.

Check: $A = Re \otimes_{eRe} eR \in \mathcal{P}(A^{op})$ as $Re \in \mathcal{P}^{r}(eRe)$

~~so~~ Let $\Lambda = \mathrm{Hom}_{A^{op}}(A, A) = \mathrm{Hom}_{\underbrace{eRe}_{B}{}^{op}}(\underset{Q}{Re}, \underset{Q}{Re})$

$$\| $$

$$\mathrm{Hom}_{B^{op}}(A \otimes_A Q, A \otimes_A Q)$$

Basically we know little about [redacted] Λ. We do know that $R \to \mathrm{Mult}(A)$ [redacted] $\subset \mathrm{Hom}_{B^{op}}(Q, Q) \times \mathrm{Hom}_B(P, P)^{op}$.

~~If we understand this!! OKAY~~

It's going to be very difficult to say anything about R/A.

1146 Davydov situation

$$\begin{array}{cc} A & Q \\ \left(\begin{array}{cc} ReR & eR \\ Re & eRe \end{array}\right) & \\ P & B \end{array}$$

Bimodal

$A = Q \otimes_B P$

$B = P \otimes_A Q$ ~~$= \mathrm{Hom}_A(Q,Q)$~~

$B = P \otimes_A Q = \mathrm{Hom}_{A^{op}}(P, P) = \mathrm{Hom}_A(Q, Q)^{op}$

We know ~~[illegible]~~ $P \in \mathcal{P}(A^{op})$, $Q \in \mathcal{P}(A)$ are dual.

$P = \underset{\mathcal{P}(B^{op})}{\overset{\cap}{B}} \otimes_B P \Rightarrow P \in \mathcal{P}(A^{op})$

$Q = Q \otimes_B B \in \mathcal{P}(B^{op}) \Rightarrow Q \in \mathcal{P}(A)$

Assume $A \in \mathcal{P}(A^{op})$ equiv. $A \otimes_A Q = Q \in \mathcal{P}(B^{op})$.

$\Lambda = \mathrm{Hom}_{A^{op}}(A, A) = \mathrm{Hom}_{B^{op}}(Q, Q)$

$R \to \mathrm{Mult}(A) \subset \mathrm{Hom}_{B^{op}}(Q, Q) \times \mathrm{Hom}_B(P, P)^{op}$

[B] A natural question is whether $\mathrm{Mult}(A) \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(A,A)$. This is unlikely. So how to handle? We can start with ${}_BP$, Q_B unit over B and $P \otimes Q \twoheadrightarrow B$ arb. surjective. Condition for $(\phi,\psi) \in \mathrm{Hom}_{B^{op}}(Q,Q) \times \mathrm{Hom}_B(P,P)^{op}$

$$\langle p\psi | q\rangle = \langle p | \phi q\rangle$$

$$\begin{tikzcd}
p\psi & P \arrow[r] & \mathrm{Hom}_{B^{op}}(Q,B) & \{q \mapsto \langle p\psi | q\rangle\} \\
 & & & \{q \mapsto \langle p | \phi q\rangle\} \\
 & P \arrow[uu, "\psi"] \arrow[r] & \mathrm{Hom}_{B^{op}}(Q,B) \arrow[uu, "\phi^t"'] & \\
 & p & & \{q' \mapsto \langle p | q'\rangle\} \arrow[uu, mapsto]
\end{tikzcd}$$

So there is no reason in general why
$\mathrm{Mult}(A) \to \mathrm{Hom}_{A^{op}}(A,A) = \mathrm{Hom}_{B^{op}}(Q,Q) = \Lambda$
has special properties.

Question: Assume A ideal in R unital such that $A \in \mathcal{P}(A^{op})$, equiv. $A = A^2$, $A \in \mathcal{P}(R^{op})$. It seems that Davydov's args say that
$K_*R = K_*A \oplus K_*(R/A)$. Possible pf.
A h-unital $\Rightarrow$ by Suslin excision that we has ~~exact~~ A

$$\begin{tikzcd}
K_*A \arrow[r] \arrow[dr, "1"] & K_*R \arrow[r] \arrow[d, "A \otimes_R -"] & K_*(R/A) \arrow[r] & {} \\
 & K_*A & &
\end{tikzcd}$$

But we know $\exists$ so done.

[8] Also have $\begin{pmatrix} A & \Lambda \\ A & \Lambda \end{pmatrix}$ $\Lambda = \mathrm{Hom}_{A^{op}}(A, A)$

$\Lambda \otimes_{\Lambda} A = A \qquad A \otimes_{A} \Lambda \xrightarrow{\sim} \Lambda$

What am I trying to understand

I keep on trying to understand A such that $A \in \mathcal{P}(A^{op})$ and I find such an A is equivalent to a unital ring Λ, ~~and a b~~ a unitary Λ module M, and a Λ-module map $f : M \twoheadrightarrow \Lambda$ such that $f(M)\Lambda = \Lambda$. Alt. a unital ring Λ and a firm dual pair over Λ of the form $M \otimes \Lambda \twoheadrightarrow \Lambda$.

In this case one has $A = M$ with $m_1 m_2 = f(m_1) m_2$, and I have a simple proof that $K_* A \simeq K_* \Lambda$ using only K thms for unital Δular matrix rings.

In the Davydov situation $A = Re \otimes_{eRe} eR$ $\quad Q \otimes_B P$ where $Q \in \mathcal{P}(B^{op})$ B unital. Here $\Lambda = \mathrm{Hom}_{B^{op}}(Q, Q)$.

Also R itself is close to

$$\mathrm{Mult}(A) \cong \left\{ \overset{(\lambda,\xi)}{\Lambda \times \mathrm{Hom}_B(P,P)^{op}} \;\middle|\; \begin{array}{l} P \to \mathrm{Hom}_{B^{op}}(Q, B) \\ \xi \text{ compat with } \lambda^t \end{array} \right\}$$

Can you do Davydov simply?

$$1 \to GL(A) \to GL(R) \to GL(R/A) \to 1$$

know for elem. reasons that $K_* R = KA + KR/A$ in dim 0,1.

It's possible that, using the fact that $K_* A \to K_* R \to K_* A$ is the identity, we can see that $GL(R/A)$ doesn't act on $H_*(GL(A))$, then use comparison thm.

[8] A flat firm as A^{op}-module ~~and~~. meg to B unital

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad A \otimes_A Q = Q \quad \text{is flat firm } B^{op}\text{-mod}$$

Then ~~can replace~~ $Q = \varinjlim F_\alpha$ F_α fin. free B^{op}-mod.

and one has $PF_\alpha = B$ for α large, so $\therefore Q \in P(B^{op})$

can assume Q finite free over B^{op}, in which

can $A = Q \otimes_B P \in P(A^{op})$, then we know

that

$$\boxed{\begin{matrix} A & \longrightarrow & \mathrm{Hom}_{B^{op}}(Q,Q) \\ \| & & \| \\ Q \otimes_B P & \longrightarrow & Q \otimes_B \mathrm{Hom}_{B^{op}}(Q,B) \end{matrix}}$$

induces an isom. on K_*.

General case. Assume A meg a unital ring i.e. $\mathcal{M}(A)$ has a generator $Q \in P(A)$. Let $P = \mathrm{Hom}_A(Q,A)$ $B = P \otimes_A Q = \mathrm{Hom}_A(Q,Q)^{op} = \mathrm{Hom}_{A^{op}}(P,P)$. First step is to replace A by a ring which is right flat, use Suslin. So anyway.

Go back to $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

What's the point!

$$K_*(\tilde{A}) = K_*(\mathbb{Z}) \oplus \mathrm{Ker}(\varepsilon)$$

$$0 \longrightarrow \mathrm{Ker}(\varepsilon) \longrightarrow K_*(\tilde{A}) \longrightarrow K_*(\mathbb{Z}) \longrightarrow 0$$

[Σ] 02/10/97 M20. ~~[struck out]~~ The M-inv: question

$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ Ass: A both left and right flat

Claim B h-unital $\Longleftrightarrow$ $P \overset{\mathbb{L}}{\underset{A}{\otimes}} Q = P \otimes_A Q$

idea.
$$\begin{tikzcd} B \overset{\mathbb{L}}{\underset{B}{\otimes}} P \overset{\mathbb{L}}{\underset{A}{\otimes}} Q \arrow[r] \arrow[d, "\alpha"'] & B \overset{\mathbb{L}}{\underset{B}{\otimes}} B \arrow[d] \\ P \overset{\mathbb{L}}{\underset{A}{\otimes}} Q \arrow[r] & B \end{tikzcd}$$

$\wr\!\wr$ //

α always a quis, why? A left A-flat $\Longrightarrow$ $P = P \otimes_A A$ is B flat so $B \overset{\mathbb{L}}{\underset{B}{\otimes}} P = P$. $P \overset{\mathbb{L}}{\underset{A}{\otimes}} Q \to B$ always a B-nil quis.

Consider left mult by $b = pq$ on P; it factors through $\tilde{A}$:
$$\begin{tikzcd} P \arrow[r, "q\cdot"] \arrow[rrr, bend right, "pq\cdot"'] & A \subset \tilde{A} \arrow[rr, "p\cdot"] & & P \end{tikzcd}$$

$\therefore$ $pq\cdot$ on $\mathrm{Tor}_n^{\tilde{A}}(P, Q)$ factors through $\mathrm{Tor}_n^{\tilde{A}}(\tilde{A}, Q) = 0$ for $n \geq 1$

So far have used only A is A-flat, but suppose we try to make sense of the argument by interpreting $\overset{\mathbb{L}}{\otimes}$

Choose $\hat{Q}$ an A-flat res. of Q

and $\hat{B}$ an B^{op}-flat res. of B.

Now consider $\hat{B} \otimes_B P \otimes_A \hat{Q}$ double complex.

$$\begin{tikzcd} \hat{B} \otimes_B P \otimes_A \hat{Q} \arrow[r] \arrow[d, "\alpha"'] & \hat{B} \otimes_B P \otimes_A Q = \hat{B} \otimes_B B \arrow[d] \\ P \otimes_A \hat{Q} \arrow[r] & B \otimes_A Q = B \end{tikzcd}$$

α is a quis as P is left B-flat and $\hat{Q}$ is A-flat.

If $P \otimes_A \hat{Q} \to P \otimes_A Q$ is a quis, then both horizontal arrows are quis. YES

[5] 02/11/97 0630

Prop: $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ s.firm, assume A left or right flat

Then B is h-unital iff $P \otimes_A^{\mathbb{L}} Q \xrightarrow{\sim} B$ (i.e. $\mathrm{Tor}_n^A(P,Q) = 0$ $n \geq 1$.)

<u>Proof.</u> ~~Choose~~ ~~Assume~~ Suppose A left flat.

Choose $E \to Q$ a flat A-mod res. of Q

$F \to B$ —— B^{op}-mod res. of B.

Consider double complex and augmentations

$$\begin{tikzcd} F \otimes_B P \otimes_A E \arrow[r, "(2)"] \arrow[d, "(1)"] & F \otimes_B P \otimes_A Q \arrow[d, "(3)"] \\ B \otimes_B P \otimes_A E \arrow[r, "(4)"] & B \otimes_B P \otimes_A Q \end{tikzcd}$$

i.e.

$$\begin{tikzcd} F \otimes_B P \otimes_A E \arrow[r, "(2)"] \arrow[d, "(1)"] & F \otimes_B B \arrow[d, "(3)"] \\ P \otimes_A E \arrow[r, "(4)"] & B \end{tikzcd}$$

(1) is a quis because ${}_B P$ ${}_A E$ are flat, so $- \otimes_B P \otimes_A E$ is exact, and $F \to B$ is a quis.

Assume $\mathrm{Tor}_n^A(P,Q) = 0$ for $n \geq 1$. Then (4) is a quis and (2) also as F_B is flat. $\therefore$ (3) is a quis and B is h-unital.

Assume B h-unital. In general we know (4) is a B-nil quis i.e. $\mathrm{Tor}_{>0}^A(P,Q)$ killed by B.

pf: $P \xrightarrow{\quad} A \subset \tilde{A} \xrightarrow{p} P$. As B h-unital $\Rightarrow B \otimes_B^{\mathbb{L}} -$ kills complexes with B nil homology, so (2) is a quis also. (3) is as B is h-unital. $\therefore$ (4) quis.

[7] So let's look at lecture.

firm bimodules.

Def. P an B, A^{op} - bimodule (unitary $\tilde{B} \otimes_{\mathbb{Z}} \tilde{A}$-module) is called firm when firm as both B-module and A^{op}-module.

Prop. a firm B, A - bimodule same as a firm $B \otimes_{\mathbb{Z}} A^{op}$-module.

$$B \otimes A^{op} \quad \text{ideal in} \quad \tilde{B} \otimes \tilde{A} = \mathbb{Z} \oplus A \oplus B \oplus B \otimes A^{op}$$

P firm B, A bimod $\Longleftrightarrow$ P in $\mathcal{F}(\tilde{B} \otimes \tilde{A}^{op}, B \otimes A^{op})$.

$$\Downarrow \qquad \qquad \mathcal{F}(\widetilde{B \otimes A^{op}}, B \otimes A^{op})$$

$$B \otimes_B P \otimes_A A \simeq P \otimes_A A \xrightarrow{\sim} P \qquad B \otimes_B P \simeq B$$

$$B \otimes_B P \otimes_A \tilde{A}$$

same as firm wrt. ideal $B \otimes \tilde{A}$

Prop.

$$m(B \otimes A^{op}) \simeq \text{rtcontfun}(m(A), m(B))$$

$$P \longmapsto P \otimes_A -$$

$A^{(2)} = A \otimes_A A$

$$F(A^{(2)}) \longleftarrow F$$

Proof:

$$\text{Mod}(\tilde{A}) \longrightarrow \text{Mod}(\tilde{B})$$

$$M \longmapsto F(A^{(2)} \otimes_A M)$$

knows from initial thm.

$$\underbrace{F(A^{(2)})}_{P} \otimes_A M \xrightarrow{\sim} F(A^{(2)} \otimes_A M)$$

and B, A - bimodule

[θ] 2nd thm. $\mathcal{M}(A) \simeq \mathcal{M}(B)$

You need to discuss homos.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

Claim here is that you have

$$\theta : B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M \qquad b' \otimes p \otimes n \mapsto b' u(p) \otimes n$$

$$\xi : Q \otimes_B B' \otimes_B N' \qquad Q' \otimes_{B'} N'$$

$u : P \to P'$

$$u(pa) = u(p)a$$
$$u(bp) = w(b)u(p).$$

$$p' \otimes q \otimes p \otimes m \longmapsto p' v(q) \otimes p \otimes m$$

$$P' \otimes_A M = P' \otimes_A Q \otimes_B P \otimes_A M \longrightarrow B' \otimes_B P \otimes_A M$$

$Q \xrightarrow{v} Q'$ $\qquad v(bq) = w(b)v(q)$

B'-mil iso. since

$\cancel{q' w(pq) = q' u(p) v(q)}$

$$\begin{array}{ccc} Q & \xrightarrow{v} & Q' \\ \downarrow{\cdot pq} & \swarrow{(q'u(p))q \leftarrow q'} & \downarrow{\cdot w(pq) = u(p)v(q)} \\ Q & \xrightarrow{v} & Q' \end{array}$$

$q_1 \longmapsto v(q_1)$

$q_1(pq) \qquad v(q_1)u(p)q \qquad q'$

$$q'u(p)q \longmapsto q'u(p)v(q) = q' w(pq)$$

[c]

$$B'\otimes_B P\otimes_A M \longrightarrow P'\otimes_A M$$

Apply $Q'\otimes_{B'}-$ get

$$Q'\otimes_B P\otimes_A M \longrightarrow A\otimes_A M$$

so you should be able to prove

$$Q'\otimes_B P \longrightarrow A \qquad q'\otimes p \mapsto q'u(p)$$

is an A^{op}-mil isom. Certainly onto, so again let $\sum q_i'\otimes p_i$ be in kernel: $\sum q_i' u(p_i)=0$. Then

$$\left(\sum q_i'\otimes p_i\right)qp = \sum q_i' w(p_i q)\otimes p$$
$$= \sum q_i' u(p_i)v(q)\otimes p = 0.$$

Then $Q'\otimes_B P\otimes_A M \xrightarrow{\sim} M \qquad q'\otimes p\otimes m \mapsto q'u(p)m$

so $P'\otimes_{B'} Q'\otimes_B P\otimes_A M \xrightarrow{\sim} P'\otimes_A M \qquad p'\otimes q'\otimes p\otimes m \mapsto$ ~~$p'\otimes q'u(p)\otimes m$~~

$$\begin{array}{ccc} P'\otimes_{B'} Q'\otimes_B P\otimes_A M & \xrightarrow{\sim} & P'\otimes_A M \\ \| & & \| \\ B'\otimes_B P\otimes_A M & \xrightarrow{\sim} & P'\otimes_A M \end{array}$$

$p'\otimes_A q'u(p)m$

$= p'q'u(p)\otimes m$

$b'\otimes p\otimes m \mapsto b'u(p)\otimes m.$

~~Other functor $Q'\otimes_{B'} N' = Q\otimes_B B'\otimes_{B'} N'$~~

So we have $w_!(P\otimes_A -) = (P'\otimes_A -)$ whence $w_!$ is an equivalence

$$w_! = (P'\otimes_A -)(P\otimes_A -)^{-1}$$
$$= (P'\otimes_A -)(Q\otimes_B -) = (P'\otimes_A Q)\otimes_B -$$

so you want. $w^* = (P\otimes_A -)(P'\otimes_A -)^{-1} = P\otimes_A Q'\otimes_{B'} -$

[K] So you would like ~~ad~~ an isom.

$$B^{(2)}\otimes_B N' \simeq P\otimes_A Q'\otimes_{B'} N'$$

Maybe take the viewpoint that ~~[illegible]~~ given $\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}$, ~~[illegible]~~ then w is a [illegible] and

$$\begin{cases} w_! = (P'\otimes_A -)(P\otimes_A -)^{-1} = P'\otimes_A Q\otimes_B - \\ w^* = (P\otimes_A -)(P'\otimes_A -)^{-1} = P\otimes_A Q'\otimes_{B'} - \end{cases}$$

Proof of first: ~~[illegible]~~ $\qquad Q'w(B) \subset vQ$

want $B'\otimes_B N \simeq P'\otimes_A Q\otimes_B N$. $\qquad v$

$$Q' \xleftarrow{v} Q \qquad \Uparrow$$

~~[illegible]~~

is a B-mil isom. $\qquad$ ~~[illegible]~~ $q' w(pq) = q' u(p) v(q)$ ~~[illegible]~~ ~~[illegible]~~

$q_1 P q = v(q_1) u(p) q$

$= 0$ if $v(q_1) = 0$. $\qquad = v(q' u(p))$.

$\Rightarrow Q'\otimes_B N \longleftarrow Q\otimes_B N$

$\Rightarrow B'\otimes_B N \xleftarrow{\sim} (P'\otimes_A Q)\otimes_B N$

Proof of 2nd. want $B^{(2)}\otimes_B N' = P\otimes_A Q'\otimes_{B'} N'$

$Q\otimes_B B^{(2)}\otimes_B N' = Q'\otimes_{B'} N'$?

$Q \xrightarrow{v} Q'$ $\qquad$ ~~[illegible]~~ $\qquad q' w(pq) = q' u(p) v(q)$

$\qquad = v(q' u(p) q)$

$v(q) = 0 \qquad q_1 P q = v(q_1) u(p) q = 0.$

[7] $\therefore$ ~~B^{op} nil-iom whence $Q \otimes_B B^{(2)} \xrightarrow{\sim} Q' \otimes_B B^{(2)}$~~

$$B^{(2)} \otimes_B B' \simeq P \otimes_A Q' \quad ?$$

$$B^{(2)} \otimes_B P' \simeq P$$

so we use $u : P \to B'$ is B-nil iom.

$$(pq)p' = u(p)(\underset{\in A}{v(q)p'}) = u(p\, v(q) p')$$

$$u(p_1) = 0 \implies (pq)p_1 = p\, v(q) u(p_1) = 0.$$

so we find
$$B^{(2)} \otimes_B P' \xleftarrow{\sim} B^{(2)} \otimes_B P$$

so
$$B^{(2)} \otimes_B \boxed{P' \otimes_A Q'} \otimes_{B'} N' \xleftarrow{\sim} B^{(2)} \otimes_B P \otimes_A Q' \otimes_{B'} N'$$

$$\therefore \quad B^{(2)} \otimes_B N' \xleftarrow{\sim} P \otimes_A Q' \otimes_{B'} N'$$

$$\longleftarrow\!\shortmid \quad p \otimes q' \otimes n'$$

induced by u

~~I guess the point of all this~~

Let's try the reduction to the fin case.

$$\begin{array}{ccc} A^{(2)} & \xrightarrow{\tilde{w}} & B^{(2)} \\ \downarrow & & \downarrow \\ A & \xrightarrow{w} & B \end{array}$$

~~$N = B^{(2)} \otimes_{A^{(2)}}$~~

$$\begin{array}{ccc} A^{(2)} \otimes_{A^{(2)}} N & \longleftarrow\!\shortmid & N \\ m(A^{(2)}) & \longleftarrow & m(B^{(2)}) \\ \| & & \| \\ m(A) & \longleftarrow & m(B) \\ A^{(2)} \otimes_A N & \longleftarrow\!\shortmid & N \end{array}$$

$$M \longmapsto B^{(2)} \otimes_{A^{(2)}} M$$

$$B^{(2)} \otimes_A M \xrightarrow{\sim} B \otimes_A M \xrightarrow{\sim} \tilde{B} \otimes_A M$$

[μ] $w_!$ fully faithful iff $w : A \to B$ is an $A \otimes A^{op}$ mod isom.

$$\begin{tikzcd}
K' \arrow[d] & \subset & K \arrow[d] \\
B^{(2)} w(A^{(2)}) B^{(2)} \arrow[d] & \subset & B^{(2)} \arrow[d] \\
B\, w(A)\, B \arrow[d] & \supset & B \arrow[d] \\
0 & & 0
\end{tikzcd}$$

~~$B\,w(A)\,B \rightleftarrows B$~~

$$B^{(2)} = \underbrace{B^{(2)} w(A^{(2)}) B^{(2)}} + K$$

$$\therefore \quad B^{(2)} = B^{(2)}(\cdots\cdots\cdots) B^{(2)} + \underset{\overset{\shortparallel}{0}}{B^{(2)} K B^{(2)}}$$

Suppose you have $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. Then you get ~~can construct~~

$$\begin{pmatrix} A^{(2)} & Q' \\ P' & B^{(2)} \end{pmatrix}$$

$$P' = P \otimes_A A^{(2)} \xleftarrow{\sim} B^{(2)} \otimes_B P \otimes_A A^{(2)}$$
$$\downarrow$$
$$B^{(2)} \otimes_B P$$

$$Q' = Q \otimes_B B^{(2)} = A^{(2)} \otimes_A Q \otimes_B B^{(2)} = A^{(2)} \otimes_A Q.$$

Alt. ~~$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q \end{pmatrix} =$~~

notice that if $P = PA$ and $Q = \underline{AQ}$, then

$$B^{(2)} = P \otimes_A Q = P \otimes_A A \otimes_A Q = P \otimes_A A^{(2)} \otimes_A Q$$

$$0 \longrightarrow \underset{KA=0}{K} \longrightarrow P \otimes_A A \longrightarrow P \longrightarrow 0 \quad \text{is } B\text{-finite one}$$

[v] Anyway suppose given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with $QP = A = A^2$ and $PQ = B = B^2$. Then get M context.

$$\begin{pmatrix} A^{(2)} & Q' \\ P' & B^{(2)} \end{pmatrix}$$ describing the M-equiv. $m(A) \xrightarrow{\sim} m(B)$

so $P' = P \otimes_A A^{(2)} \xleftarrow{\sim} B^{(2)} \otimes_B P \otimes_A A^{(2)} \xrightarrow{\sim} B^{(2)} \otimes_B P$

$Q' = Q \otimes_B B^{(2)} \xleftarrow{\sim} \overbrace{A^{(2)} \otimes_A Q \otimes_B B^{(2)}} \xrightarrow{\sim} A^{(2)} \otimes_A Q.$

$$Q' \otimes_{B^{(2)}} P' = A^{(2)}$$
$$\|$$
$$Q' \otimes_B P' = Q \otimes_B B^{(2)} \otimes_B P \xrightarrow{\sim} Q \otimes_B P$$ if $\left| \begin{array}{l} QB = Q \\ BP = P. \end{array} \right.$

$A = A^2 = QP \qquad AQ = QPQ = QB$

$PA = PQP = BP \qquad B = B^2 = PQ$

$AQ \cdot PA = A^3 = A.$

Still I have much problems.

What am I after.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

$$\begin{tikzcd} & m(A) \arrow[dl, "P \otimes_A -"'] \arrow[dr, "P' \otimes_A -"] & \\ m(B) \arrow[rr, dashed, shift left, "w_!"] & & m(B') \arrow[ll, shift left, "w^*"] \end{tikzcd}$$

Replace by f.i.m.

$$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q \end{pmatrix} \qquad \begin{pmatrix} A \\ P' \end{pmatrix} \otimes_A \begin{pmatrix} A & Q' \end{pmatrix}$$

so you have map of f.i.m. dual pairs $(Q \otimes P \to A)$
$\to (Q' \otimes P' \to A)$

You want to prove that w is a meg hom. and

$w_! = P' \otimes_A Q \otimes_B -$	$w^* = P \otimes_A Q' \otimes_{B'} -$

[ξ] Proof

$$\begin{array}{ccc} B' \otimes_B P & \xrightarrow{\sim} & P' \quad ? \\ \| & & \| \\ P' \otimes_A Q' \otimes_B P & & P' \otimes_A Q \otimes_B P \end{array}$$

Point is that $v : Q \longrightarrow Q'$ is B^{op}-nil isom.
~~A nil iso?~~ NO

$\cancel{v(q_1) = 0 \implies v(q_1)u(p) = q_1 p = 0 \quad \forall p}$

$\cancel{\implies q_1 p q = 0}$

$v(q_1) = 0 \implies q_1(pq) = v(q_1)u(p)q = 0$

$q' w(pq) = \underbrace{q' u(p)}_{\in A} v(q) = v\big((q'u(p))q\big)$

$\therefore \; Q' w(B) \subset v(Q)$.

$\therefore \; Q' \otimes_B P \xleftarrow{\sim} Q \otimes_B P = A$

$B' \otimes_B P = P' \otimes_A Q' \otimes_B P \xleftarrow{\sim} P' \otimes_A Q \otimes_B P = P' \otimes_A A = P'$.

$\underbrace{p'v(q)}_{b'} \otimes p = p' \otimes v(q) \otimes p \longleftarrow p' q p \in P'$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad p' v(q) u(p)$

$b' \otimes p \longmapsto b' u(p)$.

$B' \otimes_B P \xrightarrow{\sim} P' \implies B \otimes_B B' \otimes_B P \longrightarrow B \otimes_B P'$

$\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad\qquad P$

$P \xrightarrow{u} P'$ is a B-nil iso? $\implies P = B \otimes_B P \xrightarrow{\sim} B \otimes_B P'$

$u(p_1) = 0 \qquad (pq)p_1 = p\,v(q)u(p_1) = 0$

$w(pq)p' = u(p)\underbrace{v(q)p'} = u(p \cdot v(q)p')$

[0] So $B'\otimes_B P = B'\otimes_B B\otimes_B P \longrightarrow B'\otimes_B B\otimes_B P'$ no good.

How to do this? No way. I will have problems organizing.

Main step: $v: Q \longrightarrow Q'$ is a B^{op}-nil isom.

$$\Longrightarrow \quad Q\otimes_B N \xrightarrow{\sim} Q'\otimes_B N \quad \text{for } N\in \mathcal{M}(B)$$

$$\Longrightarrow \quad P'\otimes_A Q\otimes_B N \xrightarrow{\sim} P'\otimes_A Q'\otimes_B N \xrightarrow{\sim} B'\otimes_B N$$

use $P'\otimes_A Q' \to B'$ is a B'^{op}-nil isom hence B^{op}-nil isom.

$$\boxed{w_!(N) = P'\otimes_A Q\otimes_B N}$$

hence w is a neg hom.

Now take $N = P\otimes_A M$ get

$$P'\otimes_A M = P'\otimes_A Q\otimes_B P\otimes_A M \xrightarrow{\sim} P'\otimes_A Q'\otimes_B P\otimes_A M$$
$$= B'\otimes_B P\otimes_A M$$

$$p'\otimes qpm \longmapsto p'v(q)\otimes p\otimes m$$

$$\underbrace{p'v(q)u(p)}\otimes m$$
$$p'qp\otimes m = p'\otimes qpm$$

But there's an obvious ~~map~~ partial inverse namely $b'\otimes p\otimes m \mapsto b'u(p)\otimes m$

get $\boxed{B'\otimes_B P\otimes_A M \xrightarrow{\sim} P'\otimes_A M}$

$$w_!(P\otimes_A -) \simeq (P'\otimes_A -)$$

~~Dually we should get~~ $\cancel{Q'\otimes_B B' \longrightarrow Q'}$ ~~YES!!!~~

$$\cancel{Q\otimes_B (B^{(2)}\otimes_B N') \xrightarrow{\sim} Q'\otimes}$$

quasi-inv. fns.

$$Q'\otimes_{B'} N' \xrightarrow{\sim} Q\otimes_B B^{(2)}\otimes_B N'$$

$$P\otimes_A Q'\otimes_{B'} N' \xrightarrow{\sim} B^{(2)}\otimes_B N'$$

[π] I vaguely remember having a maps

$$\theta : B' \otimes_B P \otimes_A M \longrightarrow P' \otimes_A M$$

and computing its transpose. But actually you should try to do better.

Perhaps the central point is the equivalence going from ~~triples~~ firm dual pairs /A to firm rings eq with map to A.

Suppose given $(v,u) : (Q, P, \langle\rangle) \to (Q', P', \langle\rangle)$

Greenlees Strickland

you construct

$$\begin{tikzcd} & \mathcal{M}(A) \arrow[dl, "P\otimes_A -"'] \arrow[dr, "P'\otimes_A -"] & \\ \mathcal{M}(B) \arrow[rr, "w_!"'] & \arrow[ur, phantom, "\theta\Nearrow"] & \mathcal{M}(B') \end{tikzcd}$$

Suppose you have

$$[\text{redacted}]\overbrace{(P \otimes_A Q) \otimes_B N}^{F} \simeq \overbrace{B' \otimes_B N}^{F'}$$

$$p' \otimes q \otimes n \longmapsto p' v(q) \otimes n$$

$$b' u(p) \otimes q \otimes n \longleftarrow\!\!| \; [\text{redacted}]\; b' \otimes pqn$$

$$\overbrace{P \otimes_A Q' \otimes_{B'} N'}^{G} \simeq \overbrace{B \otimes_B N'}^{G'}$$

Compute the corresp. isom. of adjoints

$$\begin{array}{ccc} \mathrm{Hom}(FX, Y) & = & \mathrm{Hom}(X, GY) \\ \uparrow \theta^* & & \uparrow \theta^t_* \\ \mathrm{Hom}(F'X, Y) & = & \mathrm{Hom}(X, G'Y) \end{array}$$

$$G'Y \longrightarrow GFG'Y \longrightarrow GF'G'Y \longrightarrow GY$$

[p]

$$
\begin{array}{lll}
G' & B^{(2)} \otimes_B N' & p_1 q_1 \otimes p_2 q_2 \otimes b_1 b_2 \otimes n' \\
& \downarrow & \\
GFG' & P \otimes_A Q' \otimes_{B'} P' \otimes_A Q \otimes_B B^{(2)} \otimes_B N' & p_1 \otimes v(q_1) \otimes u(p_2) \otimes q_2 \otimes b_1 \otimes b_2 \otimes n' \\
& \downarrow & \\
GF'G' & P \otimes_A Q' \otimes_{B'} B' \otimes_B B^{(2)} \otimes_B N' & p_1 \otimes v(q_1) \otimes u(p_2) v(q_2) \otimes b_1 \otimes b_2 \otimes n' \\
& \downarrow & \\
G & P \otimes_A Q' \otimes_{B'} N' & p_1 \otimes v(q_1) \otimes w(p_2 q_2 b_1 b_2) \otimes n'
\end{array}
$$

should have started with ~~$p_1 \otimes q_1 \otimes p_2 \otimes q_2$~~ $p_1 q_1 p_2 q_2 b_1 \otimes b_2 \otimes n'$

$p_1 \otimes v(q_1) w(p_2 q_2 b_1 b_2) \otimes n'$

$p_1 \otimes v(q_1 p_2 q_2 b_1 b_2) \otimes n'$

$p_1 q_1 \otimes b \otimes n'$ ⟵ should be $p_1 q_1 b_1 \otimes b_2 \otimes n'$

$\downarrow$

$p_1 \otimes v(q_1 b) \otimes n'$

since you need $B^{(2)} \otimes_B N'$ form $/B$ to have $P \otimes_A Q \otimes_B B^{(2)} \otimes_B N' = B^{(2)} \otimes_B N'$

You want to avoid this junk

But how do I organize things.

[G] 02/11/97 0615

Use the black pen for a change

Problem: What to say about

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

$$\begin{tikzcd} & \mathcal{M}(A) \arrow[dl, "P\otimes_A -"', "\wr"] \arrow[dr, "P'\otimes_A -", "\wr"'] & \\ \mathcal{M}(B) \arrow[rr, "w_! = B'\otimes_B -"] & & \mathcal{M}(B') \end{tikzcd}$$

Result is that ~~there is~~ this commute up to a canon. isom:

$$\theta : B'\otimes_B P \otimes_A M \xrightarrow{\sim} P'\otimes_A M$$

$$b'\otimes p \otimes m \longmapsto b'u(p)\otimes m$$

It follows that w is a meghom.

Proof. Cl. $v : Q \to Q'$ is a B^{op}-nil iso.

$q_1 \longmapsto v(q_1)$

$$\begin{tikzcd} Q \arrow[r, "v"] \arrow[d, ".pq"'] & Q' \arrow[d, ".w(pq)"] \arrow[dl] \\ Q \arrow[r] & Q' \end{tikzcd}$$

$\underline{v(q_1)\,u(p)q}$

$q_1 p q$

$$q' \swarrow$$

$(q'u(p))q \longmapsto q'u(p)v(q) = q'w(pq)$

So ~~$Q\otimes_B P \otimes_A M$~~

~~$Q\otimes_B B^{(2)} \xrightarrow{\sim} Q'\otimes_B B^{(2)}$~~

Since $u : P \to P'$ is a B-nil isom

$$\Rightarrow \quad B^{(2)}\otimes_B P\otimes_A M \xrightarrow{\sim} B^{(2)}\otimes_B P'\otimes_A M$$

$$\wr|$$

$$P\otimes_A M$$

[I]

$1 \otimes u: \quad B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P' \quad \Rightarrow \quad P \otimes_A M \xrightarrow{\sim} \underset{\overset{\shortparallel}{w^*(P' \otimes_A M)}}{B^{(2)} \otimes_B P' \otimes_A M}$

$v \otimes 1: \quad Q \otimes_B B^{(2)} \longrightarrow Q' \otimes_B B^{(2)}$

$$M \xleftarrow{\sim} \underbrace{Q \otimes_B P \otimes_A M} \xrightarrow{\sim} Q' \otimes_B P \otimes_A M$$

---

v a B^{op}-nil isom.

$$P' \otimes_A Q \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A Q' \otimes_B P \otimes_A M \simeq B' \otimes_B P \otimes_A M$$

Yes. $\quad Q \longrightarrow Q'$ is a B^{op}-nil isom.

$\Rightarrow \quad P' \otimes_A Q \longrightarrow P' \otimes_A Q' \to B'$ is a B^{op}-nil isom.

$\Rightarrow \quad P' \otimes_A Q \otimes_B N \xrightarrow{\sim} B' \otimes_B N = w_! N$

$$P' \otimes_A M \xleftarrow{\sim} P' \otimes_A Q \otimes_B P \otimes_A M \xrightarrow{\sim} B' \otimes_B P \otimes_A M$$

for M firm.

$$\begin{array}{ccc} p' \otimes \underset{\shortparallel}{q} p m & \longmapsto & p' v(q) \otimes p \otimes m \\ & & \downarrow \\ p' v(q) u(p) \otimes m & \longleftarrow & \end{array}$$

Simplest proof. $\quad P \xrightarrow{u} P' \quad$ B-nil isom.

$\Rightarrow \quad B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$

$\Rightarrow \quad \underset{\overset{\shortparallel}{P \otimes_A M}}{B^{(2)} \otimes_B P \otimes_A M} \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A M$

other proof. $\quad Q \xrightarrow{v} Q' \quad B^{op}$-nil isom.

$\Rightarrow \quad Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N$

$\Rightarrow \quad (P' \otimes_A Q) \otimes_B N \xrightarrow{\sim} B' \otimes_B N$

$P' \otimes_A Q' \to B'$ is B'-nil hence B-nil isom.

[u]

$$\Rightarrow \quad P'\otimes_A M \xleftarrow{\simeq} P'\otimes_A Q \otimes_B P \otimes_A M \xrightarrow{\sim} B'\otimes_B P \otimes_A M$$

$$p'\otimes q \otimes p \otimes m \longmapsto p'v(q)\otimes p\otimes m$$

$$p'\otimes_{\mu} qpm \qquad p'v(q)u(p) \qquad \cancel{b'\otimes p\otimes m}$$

$$p'qp\otimes m$$

These details are not very important. However, let's review the pairings.

$$P \xrightarrow{u} P' \qquad B\text{-bil isom}$$

$$B^{(2)}\otimes_B P \xrightarrow{\sim} B^{(2)}\otimes_B P'$$

$$B^{(2)}\otimes_B P\otimes_A M \xrightarrow{\sim} B^{(2)}\otimes_B P'\otimes_A M$$

$$\downarrow\wr$$

$$P\otimes_A M$$

anyway

$$\therefore (P\otimes_A -) \simeq \omega^*(P'\otimes_A -)$$

$$(P\otimes_A Q'\otimes_{B'} -) \simeq \omega^* = B^{(2)}\otimes_B -$$

$$B'\otimes_B P\otimes_A M \xleftarrow{\sim} B'\otimes_B B^{(2)}\otimes_B P\otimes_A M \xrightarrow{\sim} B'\otimes_B B^{(2)}\otimes_B P'\otimes_A M$$

Argument amounts to $\omega^* \simeq P\otimes_A Q'\otimes_{B'} - : \mathcal{M}(B') \to \mathcal{M}(B)$.

so ω^* has a quasi-inverse namely $P'\otimes_A Q\otimes_B -$

Since $\omega_! = B'\otimes_B - : \mathcal{M}(B) \to \mathcal{M}'(B)$ is left adjoint to ω^*

] one gets an isom $B'\otimes_B - \simeq P\otimes_A Q'\otimes_{B'} -$

specifically ~~to go from given~~

$$P\otimes_A Q'\otimes_{B'} N' \xleftarrow{\sim} B^{(2)}\otimes_B P\otimes_A Q'\otimes_{B'} N' \xrightarrow{\sim} B^{(2)}\otimes_B P'\otimes_A Q'\otimes_{B'} N'$$

$$b_1b_2p\otimes q'\otimes n' \longmapsto b_1\otimes b_2\otimes u(p)q'n' \qquad B^{(2)}\otimes_B N'$$

[ϕ] What would be simplest? Yes. What would be simplest? What you would like

~~What you need~~ is

What you need to do is to find assertions and get out this ~~[illegible]~~ theory of Morita equivalence.

I need to organize the ideas

need to get your act together.

Organize this paper and finish it. This means going over many things, especially the cat theory.

adjoint functors $\mathrm{Hom}_{\mathcal{D}}(FX, Y) = \mathrm{Hom}_{\mathcal{C}}(X, GY)$

$$\alpha = \alpha_Y : FGY \longrightarrow Y \qquad g : X \to GY$$

$$\beta = \beta_X : X \longrightarrow GFX \qquad F.g : FX \to FGY \xrightarrow{\alpha} Y$$

$f : FX \to Y$ goes to $X \xrightarrow{\beta} GFX \xrightarrow{G.f} GY$

$$f \longmapsto (G.f)\beta \qquad g \longmapsto \alpha(F.g)$$

$$\begin{tikzcd}
FX \arrow[r, "F.\beta"] & FGFX \arrow[r, "FG.f"] \arrow[d, "\alpha.F"'] & FGY \arrow[r, "\alpha"] \arrow[d, "\alpha"] & Y \\
& FX \arrow[r, "f"] & Y &
\end{tikzcd}$$

$$\alpha.F = \alpha_{FX} \qquad G.\alpha = G(\alpha)$$

need $FX \xrightarrow[F.\beta]{} FGFX \xrightarrow[\alpha.F]{} FX$ is 1.

Sim. need $GY \xrightarrow{\beta.G} GFGY \xrightarrow{G.\alpha} GY$ is 1.

$$\mathrm{Hom}(X, X') \longrightarrow \mathrm{Hom}(FX, FX') \xrightarrow{\sim} \mathrm{Hom}(X, GFX')$$

F fully faithful iff $\beta : X' \xrightarrow{\sim} GFX'$ $\forall X'$.

[X] So what else do we do?

$w: A \to B$ homom.

$$\mathrm{Hom}_A(M, A^{(2)} \otimes_A N) \xrightarrow{\sim} \mathrm{Hom}_A(M, N)$$
$$\xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_B(B, N))$$
$$= \mathrm{Hom}_B(B \otimes_A M, N).$$

note that $B^{(2)} \to B \to \vec{B}$ are all B^{op}-nil isos.

hence A^{op}-nil isos. $\Rightarrow$ for $M \in \mathcal{M}(A)$ that

$$B^{(2)} \otimes_A M \xrightarrow{\sim} B \otimes_A M \xrightarrow{\sim} \vec{B} \otimes_A M.$$

adj arrows

$$\alpha: B \otimes_A A^{(2)} \otimes_A N \to N \qquad b \otimes a_1 \otimes a_2 \otimes n \mapsto b\,w(a_1 a_2) n$$

$$\beta: M \simeq A^{(3)} \otimes_A M \longrightarrow A^{(2)} \otimes_A B \otimes_A M$$
$$a_1 a_2 a_3 m \longmapsto a_1 \otimes a_2 \otimes w(a_3) \otimes m.$$

So w^* is fully faithfully $\Leftrightarrow$ β isom. for all M

$\Leftrightarrow$ β an isom for $M = A^{(2)}$ $\qquad A^{(6)} \to A^{(2)} \otimes_A B \otimes_A A^{(2)}$ $\quad 1 \otimes 1 \otimes w \otimes 1 \otimes 1$

$\Leftrightarrow$ w is an $A \otimes A^{op}$-nil isom.

$\Leftrightarrow$ $A\,\mathrm{Ker}(w)\,A = 0$ and $w(A)\,B\,w(A) \subset w(A)$

w^* equivalence of cats $\Leftrightarrow$ α, β are isos.

$\Rightarrow$ β isom for $N = B^{(2)}$. $\quad BA^2B^2 = B \Rightarrow BAB = B$

[4] What's the best way to proceed? You have
You know what the firm bimodules are

$$w^*(N) = A^{(2)} \otimes_A N \quad \leftrightarrow \quad A^{(2)} \otimes_A B = Q$$

$$w_!(M) = B \otimes_A M \quad \leftrightarrow \quad B \otimes_A A^{(2)} = P.$$

These are firm ~~A,B~~ bimodules with pairing.

$$Q \otimes_B P = A^{(2)} \otimes_A B^{(2)} \otimes_A A^{(2)} \simeq A^{(2)}$$

We get a ring with firm ring

$$P \otimes_A Q = B \otimes_A A^{(2)} \otimes_A B$$

But have surjection $B \otimes_A A^{(2)} \otimes_A B \twoheadrightarrow B$

The point is that we have a M cont.

$$\begin{pmatrix} A & A^{(2)} \otimes_A B \\ B \otimes_A A^{(2)} & B \end{pmatrix}$$ Check $(pq)p' = p(qp')$ etc.

Useful generalization $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ Say QP, PQ idemp. ideals

Suppose A, B firm. Then you have

$$\begin{pmatrix} A \\ B \otimes_A A \end{pmatrix} \otimes_A \begin{pmatrix} A & A \otimes_A B \end{pmatrix} = \begin{pmatrix} A & A \otimes_A B \\ B \otimes_A A & B \otimes_A A \otimes_A B \end{pmatrix}$$

Look: have B bimodule map $B \otimes_A A \otimes_A B \to B$

$$(b_1 \otimes a_2 \otimes b_3)(b_4 \otimes a_5 \otimes b_6) = (b_1 \otimes a_2 \otimes b_3) b_4 a_5 b_6$$

[ω]

$$(b_1 w(a_2) b_3)(b_4 \otimes a_5 \otimes b_6)$$

$$(b_1 w(a_2' a_2'') b_3)(b_4 \otimes a_5' a_5'' \otimes b_6)$$

$$= b_1^{w(a_2')} w(a_2'') b_3 b_4 w(a_5') \otimes a_5'' \otimes b_6$$

$$= b_1 \otimes w(a_1') \otimes a_3 a_5'' \otimes b_6$$

$$= b_1 \otimes w(a_2') w(a_2'') b_3 b_4 w(a_5') w(a_5'') \otimes b_6$$

$$= b_1 \otimes w(a_2') \otimes \underline{w(a_2'') b_3 b_4 w(a_5')}$$

$$(b_1 w(a_2) b_3)(b_4 \otimes a_5 \otimes b_6$$

$$b_1^{w(a_2')} \underbrace{w(a_2'') b_3 b_4 w(a_5')}_{w(a)} \otimes a_5'' \otimes b_6$$

$$b_1 \otimes a_2' a a_5'' \otimes b_6$$

$$b_1 \otimes a_2' \otimes w(a a_5'') b_6$$

$$w(a_2'') b_3 b_4 w(a_5') w(a_5'') b_6$$

$$(b_1 \otimes a_2 \otimes b_3) b_4 w(a_5) b_6$$

"correct" proof. Take $w: A \to B$ and you consider separately $A \twoheadrightarrow w(A) \hookrightarrow B$.

Check these two maps satisfy condition. Reduce to $A \to A/I$ $\quad AIA = 0$ $\quad \begin{pmatrix} A & A/AI \\ A/IA & A/I \end{pmatrix}$ $\quad Q \otimes P$ $\quad A/AI \otimes A/IA \to A/I$

$A \hookrightarrow B$ where $ABA = A$ $\quad BAB = B$.

d) So what goes on? $\begin{pmatrix} A & AB \\ BA & B \end{pmatrix}$

It seems that the Gr way to establish the result about morphisms is ~~first to prove~~

- $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\quad A = A^2 = QP$, $B = B^2 = PQ$ $\rightsquigarrow$ $\mathcal{M}(A) \simeq \mathcal{M}(B)$

$$M \longmapsto P\otimes_A M$$
$$Q\otimes_B M \longleftarrow Q\otimes_B$$

- $w: A \to B$ $\rightsquigarrow$ adj functors

$$\mathcal{M}(A) \underset{w_*}{\overset{w_!}{\rightleftarrows}} \mathcal{M}(B) \qquad (w^* : \mathcal{M}(B)\to\mathcal{M}(A))$$

$w_!(M) = B\otimes_A M$

$w^*(N) = A^{(2)}\otimes_A N$

$w_*(M) = B^{(2)}\otimes_B \mathrm{Hom}_A(A^{(2)}\otimes_A B, M)$

~~Next we find~~

Next want

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

a hom. of Mcont w first comp. A. Assume $A = A^2 = QP$, $B = B^2 = PQ$ as before, sim with primes.

~~Then~~ (Question is there a composite Mcont.

$\begin{pmatrix} B & P\otimes_A Q' \\ P'\otimes_A Q & B' \end{pmatrix}$? $\quad (P\otimes_A Q')\otimes_B (P'\otimes_A Q) \xrightarrow{?} B$ Yes

$(P'\otimes_A Q)\otimes_B (P\otimes_A Q') \longrightarrow B'$ Yes.

So there is one, does it receive a map from $\begin{pmatrix} B & B \\ B & B \end{pmatrix}$?

$B \to P'\otimes_A Q \qquad B \longleftarrow P\otimes_A Q \xrightarrow{u\otimes 1} P'\otimes_A Q$

β}
$$\begin{pmatrix} B & P\otimes_A Q \\ P\otimes_A Q & B \end{pmatrix} \qquad \begin{aligned}&(P\otimes_A Q)\otimes_B (P\otimes_A Q)\\ &\to P\otimes_A A\otimes_A Q \to B\end{aligned}$$

$$\cancel{p_1\otimes q_2}$$

$$\underbrace{(p_1\otimes q_1')\big(\overbrace{(p_2'\otimes q_2)}}\,(p_3\otimes q_3')\big) = (p_1\otimes q_1')(p_2\otimes \langle q_3,p_3\rangle q_3')$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$(p_1\otimes\langle q_1,p_2'\rangle q_2)(p_3\otimes q_3') \qquad\qquad p_1\otimes\langle q_1',p_2\rangle\langle q_2,p_3\rangle q_3'$$

$$\|$$

$$p_1\otimes\langle\langle q_1,p_2'\rangle q_2,p_3\rangle q_3'$$

Thus have

$$\begin{pmatrix} B & P\otimes_A Q \\ P\otimes_A Q & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 1\otimes v \\ u\otimes 1 & w\end{pmatrix}} \begin{pmatrix} B & P\otimes_A Q' \\ P'\otimes_A Q & B' \end{pmatrix}$$

But how does this help? It might help if you knew you had $B \to P'\otimes_A Q$. Maybe replace B by $P\otimes_A Q$.

Perhaps at the outset you should restrict to s. idemp. $\begin{pmatrix} A & Q \\ P & B\end{pmatrix}$. Then you know that

$\cancel{A\otimes_A P = A\otimes_A P}$ $\qquad P\otimes_A A \xleftarrow{\sim} B\otimes_B P\otimes_A A \xrightarrow{\sim} B\otimes_B P$

etc. Also $\quad Q\otimes_B P \xrightarrow{\sim} A^{(2)}$

g} ~~What next??? Nothing!!~~

Conclude that you need

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

$$\mapsto \quad w_! \circ (P \otimes_A -) \xrightarrow{\sim} (P' \otimes_A -)$$

Might try for $\begin{cases} w_! \simeq P' \otimes_A Q \otimes_B - \\ w^* \simeq P \otimes_A Q' \otimes_{B'} - \end{cases}$

see if can do this:

$$Q \xrightarrow{v} Q' \qquad B^{op}\text{-mod isom}$$

$$\therefore \quad Q \otimes_B B^{(2)} \xrightarrow{\sim} Q' \otimes_B B^{(2)}$$

$$P' \otimes_A Q \otimes_B \cancel{B^{(2)}}\, N \xrightarrow{\sim} P' \otimes_A Q' \otimes_B \cancel{B^{(2)}}\, N \qquad N \in \mathcal{M}(B)$$

$$P' \otimes_A Q \otimes_B N \xrightarrow{\sim} B' \otimes_B N$$

$$p' \otimes q \otimes n \longmapsto p' v(q) \otimes n$$

inverse map

$$\begin{array}{ccc} & & p'v(q) \otimes n \\ & & \uparrow S \\ P' \otimes_A Q \otimes_B N & \longleftarrow & B' \otimes_B P \otimes_A Q \otimes_B N \\ b'u(p) \otimes q \otimes n & \longleftarrow & b' \otimes p \otimes q \otimes n \end{array}$$

This isom identifies $(P' \otimes_A Q \otimes_B -) \simeq w_!$

It follows then that w is a mag homom.

and there's a corres isom $(P \otimes_A Q' \otimes_{B'} -) \simeq w^*$

δ} Set this up

$P \xrightarrow{u} P'$ B-nil isom.

$$B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$b_1 \otimes b_2 \otimes u(p) \otimes q' \otimes n'$

$b_1 \otimes b_2 \otimes p \otimes q \otimes n'$

$$\begin{tikzcd}
B^{(2)} \otimes_B P \otimes_A Q' \otimes_B N' \arrow[r, "\sim"] \arrow[d, "s"] & B^{(2)} \otimes_B P' \otimes_A Q' \otimes_B N' \arrow[d, "s"] \\
P \otimes_A Q' \otimes_B N' \arrow[ur, dotted] \arrow[r, phantom, "="] & B^{(2)} \otimes_B N' \\
& P \otimes_A Q \otimes_B B^{(2)} \otimes_B N' \arrow[u, "s"] \arrow[ul]
\end{tikzcd}$$

$b_1 \otimes b_2 \otimes u(p) q' \otimes n'$

$b_1 b_2 p \otimes q' \otimes n'$

Why is $P \otimes_A Q \otimes_B B^{(2)} \xrightarrow{\sim} B^{(2)}$?

$P \otimes_A Q \otimes_B P \otimes_A Q$? work

$b_3^{\varphi} b_4^{\varphi} p \otimes q \otimes b_1^{\varphi} \otimes b_2 \otimes n'$

$$b_3 b_4 p \otimes q \otimes b_1 \otimes b_2 \otimes n'$$
$$\downarrow$$
$$b_3 b_4 p \otimes v(q) w(b_1 b_2) \otimes n'$$
$$\downarrow$$
$$b_3 \otimes b_4 \otimes u(p) \otimes v(q) w(b_1 b_2) \otimes n \mapsto b_3 \otimes b_4 \otimes w(pq) w(b_1 b_2) \otimes n$$

$$b_3 \otimes b_4 p q \otimes b_1 \otimes b_2 \otimes n'$$
$$\downarrow$$
$$b_3 \otimes b_4 \otimes w(pq) w(b_1 b_2) \otimes n$$

$\varepsilon\}$ Repeat: $\quad Q \xrightarrow{v} Q' \quad B^{op}$-nil iso.

$$Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N \qquad N \in \mathcal{M}(B)$$

$$(P' \otimes_A Q) \otimes_B N \xrightarrow{\sim} P' \otimes_A Q' \otimes_B N$$

$$p' \otimes q \otimes n \longmapsto p' v(q) \otimes n \qquad \qquad \wr\downarrow$$

$$B' \otimes_B N$$

because $P' \otimes_A Q' \to B'$ is B'-op nil hence B-op nil iso.

$$F \longrightarrow F'$$

Thus $(P' \otimes_A Q \otimes_B -) \xrightarrow{\sim} w_!$

$$G \qquad G'$$

Thus w is a megham, and $\exists\ (P \otimes_A Q' \otimes_{B'} -) \xleftarrow{\sim} w^*$ compatible with preceding. How

$$G' \longrightarrow G F G' \longrightarrow G F' G' \longrightarrow G$$

$$B^{(2)} \otimes_B N' \xrightarrow{\sim} P \otimes_A Q' \otimes_B P' \otimes_A Q \otimes_B B^{(2)} \otimes_B N'$$

$$p_1 \otimes q_1' \otimes p_1' \otimes q_2 \otimes b_1 \otimes b_2 \otimes n'$$

$$\downarrow$$

$$P \otimes_A Q' \otimes_B B' \otimes_B B^{(2)} \otimes_B N'$$

$$p_1 \otimes q_1' \otimes p_1' v(q_2) \otimes b_1 \otimes b_2 \otimes n'$$

$$\downarrow$$

$$P \otimes_A Q' \otimes_{B'} N'$$

~~$p_1 \otimes q_1' \otimes p$~~

$$p_1 q_1' p_1' q_2 b_1 \otimes b_2 \otimes n' \qquad p_1 \otimes q_1' p_1' v(q_2) w(b_1 b_2) \otimes n'$$

$$\underline{p_1 q_1' p_1' q_2} \otimes p_3 q_3 \otimes p_4 q_4 \otimes n' \longmapsto p_1 \otimes q_1' p_1' v(q_2) u(p_3) v(q_3) \otimes b_2 \otimes n'$$

$$p_1 q_1' p_1' q_2 p_3 \otimes v(q_3) w(b_2) \otimes n'$$

]}

$$B^{(2)} \otimes_B N'$$

$$\shortparallel$$

$$B^{(2)} \otimes_B P \otimes_A Q \otimes_B N'$$

$\cancel{p_1 q_1' p_1' q_2 \otimes b_1 \otimes b_2 \otimes n'}$

$$p_1 q_1' p_1' q_2 \otimes b_1 \otimes b_2 \otimes n' \longleftarrow p_1 \otimes q_1' \otimes p_1' \otimes q_2 \otimes b_1 \otimes b_2 \otimes n'$$

$$\downarrow$$

$\cancel{p_1 q_1 p_1' v(q_2)}$

$$p_1 \otimes q_1' \otimes p_1' v(q_2) \otimes b_1 \otimes b_2 \otimes n'$$

$$\downarrow$$

$$p_1 \otimes q_1' \otimes p_1' v(q_2) w(b_1) w(b_2) \otimes n'$$

$$B^{(2)} \otimes_B N' \qquad\qquad P \otimes_A Q' \otimes_{B'} N'$$

$$\underset{\underset{pq}{\shortparallel}}{p_1 q_1' p_1' q_2 \otimes b \otimes n'} \longmapsto p_1 \otimes q_1' \otimes p_1' \underline{v(q_2 b)} n'$$

$$p_1 \otimes q_1' \otimes p_1' v(q_2)\, u(p)\, v(q)\, n'$$

$$p_1 \otimes q_1' p_1' v(q_2) \otimes \overline{u(p) v(q)}\, n'$$

$$p_1 q_1' p_1' \otimes v(q_2) \otimes w(pq) n'$$

$$pq \otimes b \otimes n' \longmapsto p \otimes v(qb) \otimes n'$$

$$\shortparallel$$

$$p \otimes v(q) \otimes w(b) n'$$

$$pq\, b_1 \otimes b_2 \otimes n' \longmapsto p \otimes v(q) \otimes w(b_1 b_2) n'$$

73 I think ultimately I decided to give the isom.

$P' \otimes_A Q \otimes_B P \otimes_A M$
$\shortparallel$

$$B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M$$
$$b' \otimes p \otimes m \longmapsto b' u(p) \otimes m$$

$Q \otimes_B P \otimes_A Q \otimes_B P \otimes_A Q' \otimes_{B'} P \otimes_A Q' \otimes_{B'} N'$

$$Q \otimes_B B^{(2)} \otimes_B N' \xrightarrow{\sim} Q' \otimes_{B'} N'$$
$$q \otimes b_1 \otimes b_2 \otimes n' \longmapsto v(q b_1 b_2) \otimes n'$$

02/12/97

$$\begin{array}{ccc} B' \otimes_B P \otimes_A M & \rightleftarrows & P' \otimes_A M \\ \shortparallel & & \shortparallel \\ \cdots & \longleftarrow & P' \otimes_A Q \otimes_B P \otimes_A M \\ p' v(q) \otimes p \otimes m & \longleftarrow\!\shortmid & p' \otimes q \otimes p \otimes m \end{array}$$

$b' \otimes p \otimes m \mapsto b' u(p) \otimes m$
$p' v(q) \otimes p \otimes m \mapsfrom p' \otimes q p m$

$$b' \otimes p \otimes q_1 p_1 m \longmapsto b' u(p) \otimes q_1 p_1 m$$
$$\overbrace{b' u(p) v(q_1)}^{w(pq_1)} \otimes p_1 \otimes m \longleftarrow\!\shortmid$$
$$\shortparallel$$
$$b' \otimes p q_1 p_1 \otimes m$$

θ} The maps are

$$B' \otimes_B P \otimes_A M \longrightarrow P' \otimes_A M \qquad b' \otimes p \otimes m \longmapsto b' u(p) \otimes m$$

$$Q \otimes_B B^{(2)} \otimes_B N' \longrightarrow Q' \otimes_{B'} N' \qquad q \otimes b_1 \otimes b_2 \otimes n' \longmapsto v(q) \otimes w(b_1 b_2) n'$$

$$\| $$

$$v(q b_1, b_2) \otimes n'$$

check comp. with pairings.

$$\begin{array}{ccc} B' \otimes_B P \otimes_A Q \otimes_B B^{(2)} \otimes_B N' & \longrightarrow & P' \otimes_A Q' \otimes_{B'} N' \\ \downarrow & b' \otimes p \otimes q \otimes b_1 \otimes b_2 \otimes n' & b' u(p) \otimes v(q) \otimes w(b_1 b_2) n' \\ B' \otimes_B B^{(2)} \otimes_B N' & \downarrow & \downarrow \\ \downarrow & b' \otimes pq b_1 \otimes b_2 \otimes n' & b' \underbrace{u(p) v(q) w(b_1 b_2)}_{w(pq b_1 b_2).} n' \\ N' & \downarrow & \\ & b' w(pq b_1 b_2) n' & \end{array}$$

$$\begin{array}{ccc} Q \otimes_B B^{(2)} \otimes_B B' \otimes_B P \otimes_A M & q \otimes b_1 \otimes b_2 \otimes w(b_3) \otimes p \otimes m & Q' \otimes_{B'} P' \otimes_A M \\ \| & \uparrow & v(q) \otimes w(b_1 b_2) w(b_3) u(p) \otimes m \\ Q \otimes_B P \otimes_A M & q \otimes b_1 b_2 b_3 p \otimes m & \\ \downarrow & \downarrow & \\ M & (q b_1 b_2 b_3 p) m & \end{array}$$

c} So the basic result is that given

$$\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

you get a canonical isom ~~between the pairs of q.inv.fun~~

~~$(P' \otimes_A -, Q \otimes_{B'} -)$~~ ~~$(P \otimes_A$~~

$$B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M$$

i.e. $\quad w_! (P \otimes_A -) \xrightarrow{\sim} (P' \otimes_A -)$

It follows that $w_! \simeq (P' \otimes_A -)(Q \otimes_B -) = (P' \otimes_A Q \otimes_B -)$ is an equiv of cats. $\therefore$ w is a mor hamom. One can check that the corresp isom of q. inv. functors is

$$Q \otimes_B B^{(2)} \otimes_B N' \xrightarrow{\sim} Q' \otimes_{B'} N'$$

$$q \otimes b_1 \otimes b_2 \otimes n' \longmapsto v(q b_1 b_2) \otimes n'$$

It might be better to give the isom

$$B' \otimes_B N \simeq P' \otimes_A Q \otimes_B N$$

$$p' v(q) \otimes n \qquad p' \otimes q \otimes n$$

shows immed $w_! = (P' \otimes_A Q) \otimes_B$

e.g. $\quad Q \xrightarrow{v} Q' \quad B^{op}$-mil iso so

$$Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N \qquad \text{if } N \text{ is } B\text{-firm}$$

$$P' \otimes_A Q \otimes_B N \xrightarrow{\sim} B' \otimes_B N$$

$$p' \otimes q \otimes n \longmapsto p' v(q) \otimes n$$

observe $\quad P' \otimes_A Q \otimes_B P \otimes_A M \xrightarrow{\sim} B' \otimes_B P \otimes_A M$

K} ~~First result~~.

$$P \xrightarrow{u} P' \quad \text{a} \quad B\text{-nil isom.}$$

$$B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$$B^{(2)} \otimes_B P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N'$$

$$\|$$

$$(P \otimes_A Q') \otimes_{B'} N' \xrightarrow{\sim} B^{(2)} \otimes_B N'$$

$$b_1 b_2 p \otimes q' \otimes n' \longmapsto b_1 \otimes b_2 \otimes u(p) q' n'$$

$$B^{(2)} \otimes_B N' \qquad p q b_1 \otimes b_2 \otimes n' \longmapsto p \otimes v(q b_1 b_2) \otimes n'$$

$$s \uparrow$$

$$P \otimes_A Q \otimes_B B^{(2)} \otimes_B N' \longrightarrow P \otimes_A Q' \otimes_{B'} N'$$

$$p \otimes q \otimes b_1 \otimes b_2 \otimes n' \longmapsto p \otimes v(q b_1 b_2) \otimes n'$$

~~[illegible]~~ simplest maybe is:

(a) $v: Q \longrightarrow Q'$ B^{op} nil-iso ...

$$\Rightarrow \quad Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N$$

$$\Rightarrow \quad \boxed{\begin{array}{l} P' \otimes_A Q \otimes_B N \xrightarrow{\sim} P' \otimes_A Q' \otimes_B N \xrightarrow{\sim} B' \otimes_B N \\ p' \otimes q \otimes n \longmapsto \qquad\qquad\qquad p' v(q) \otimes n \end{array}}$$

use $P' \otimes_A Q' \to B'$ is a B'^{op} nil isom hence B^{op}-nil iso.

(b) $u: P \longrightarrow P'$ is B-nil iso

$$\Rightarrow \quad B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$$\Rightarrow \quad B^{(2)} \otimes_B P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N'$$

$$\wr\wr$$

$$\boxed{\begin{array}{l} (P \otimes_A Q') \otimes_{B'} N' \xrightarrow{\sim} B^{(2)} \otimes_B N' \\ b_1 b_2 p \otimes q' \otimes n' \longmapsto b_1 \otimes b_2 \otimes u(p) q' n' \end{array}}$$

$\lambda\}$ compatible with pairings.

$$\begin{array}{ccc} P\otimes_A Q'\otimes_{B'} P'\otimes_A Q\otimes_B N & \longrightarrow & B^{(2)}\otimes_B B'\otimes_B N \\ \downarrow & & \wr\!\downarrow \\ N & = & N \end{array}$$

$$b_1b_2p\otimes q'\otimes p'\otimes q\otimes n \mapsto b_1\otimes b_2\otimes u(p)q'\otimes p'v(q)\otimes n$$

$b_1b_2p\,q'p'q\,n$ $\qquad$ $b_1b_2\,\underbrace{u(p)q'p'v(q)}_{u}\,n$, $\quad u(pq'p')v(q)$, $\quad w(pq'p'q)$

$$b_1\otimes b_2\otimes \underbrace{u(p)q'p'v(q)}_{w(pq'p'q)}\otimes n \qquad\qquad B^{(2)}\otimes_B B'\otimes_B N$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow S$$

$$b_1\otimes b_2\otimes pq'p'q\otimes n \qquad\qquad B^{(3)}\otimes_B N = N$$

$$P'\otimes_A Q\otimes_B P\otimes_A Q'\otimes_{B'} N' \longrightarrow B'\otimes_B B^{(2)}\otimes_B N'$$

$$p'\otimes q\otimes b_1b_2p\otimes q'\otimes n' \qquad\qquad p'v(q)\otimes b_1\otimes b_2\otimes u(p)q'n'$$

$$\downarrow$$

$$p'qb_1b_2pq'n' \qquad\qquad p'\underbrace{v(q)w(b_1b_2)u(p)}_{qb_1b_2p}q'n'$$

Summary: This result identifies $w_!, w^*$ with $(P'\otimes_A Q\otimes_B -,\ P\otimes_A Q'\otimes_B -)$.

---

next points: Go over the equivalence between two cats. ~~Fix~~ A fixed firm ring.

First cat has objects (B, F) $\quad$ B firm ring, $F: \mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$ equiv. ~~objects~~ maps $(B,F) \longrightarrow (B', F')$ consist of

$\mu\}$ a $w: B \to B'$ and $\theta: w_! F \xrightarrow{\sim} F'$.

composition is clear ~~$(w'_! w_!)_! \cong w'_! w_! \xrightarrow{w'_!\theta} w'_! F'$~~

$$(w'w)_! F \cong w'_! w_! F \xrightarrow[\sim]{w'_!(\theta)} w'_! F' \xrightarrow[\sim]{\theta'} F''.$$

Different Objects types.

firm dual pair $Q \otimes P \longrightarrow A$ $\left(\begin{array}{l} A \otimes_A Q \xrightarrow{\sim} Q \\ P \otimes_A A \xrightarrow{\sim} P \\ \langle Q, P \rangle = A. \end{array} \right.$

s firm M context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

Lemma: $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ a M cont $\ni P \otimes_A Q \xrightarrow{\sim} B$ determined by $\underbrace{\text{ring } A,\ P_A,\ {}_AQ,\ \langle\,,\rangle : Q \otimes P \longrightarrow A}$

point: Given the data you get $B = P \otimes_A Q$ etc.

$$P' = \tilde{A} \oplus P \qquad Q' = \tilde{A} \oplus Q$$

pairing $Q' \otimes P' \longrightarrow \tilde{A}$ obvious

$\Longrightarrow$ $P' \otimes_{\tilde{A}} Q'$ ring etc.

$$\overset{\shortparallel}{\begin{pmatrix} \tilde{A} & Q \\ P & B \end{pmatrix}} \supset \begin{pmatrix} A & Q \\ P & B \end{pmatrix}.$$

I'm going over the equivalence between firm dual pairs and firm rings meq A. There's awkwardness, but basically an equivalence of cats.

Given B, $F: \mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$ can ~~assoc~~ complete to $F, G, \ \varepsilon: FG \simeq 1,\ \eta: GF \xrightarrow{\sim} 1 \ \ni \ \varepsilon.F = F.\eta \quad \eta.G = G.\varepsilon$

Given $(F, G, \varepsilon, \eta)$ get $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $P = F(A)$ $Q = G(B)$

$F(M) = F(A) \otimes_A M \qquad G(N) = G(B) \otimes_B N$

ν} [redacted]

functorality. ~~$(F,G,\varepsilon,\eta) \xrightarrow{(\theta,\xi)} (F',G',\varepsilon',\eta')$~~ η

A map $(B,F) \to (B',F')$ is $w: B \to B'$, $\theta: w_! F \xrightarrow{\sim} F'$.
This can be enhanced uniquely by $\xi: G w^* \xrightarrow{\sim} G'$

$$
\begin{tikzcd}
 & m(A) \arrow[dl, shift left, "G"] \arrow[dr, shift left, "G'"] & \\
m(B) \arrow[ur, shift left, "F"] \arrow[rr, shift left, "w_!"] & & m(B') \arrow[ul, shift left, "F'"] \arrow[ll, shift left, "w^*"]
\end{tikzcd}
$$

Trying to say $\theta: w_! F \xrightarrow{\sim} F'$ can be enhanced to isom.

$(w_!, w^*, \alpha, \beta^{-1})(F, G, \varepsilon, \eta) \simeq (F', G', \varepsilon', \eta')$

$$w_! F G w^* \xrightarrow{w_! \cdot \varepsilon \cdot w^*} w_! w^* \xrightarrow{\alpha} 1$$

$$G w^* w_! F \xrightarrow{G \cdot \beta^{-1} \cdot F} GF \xrightarrow{\eta} 1$$

What's the point? Answer: A quasi-inverse to F is G, ε, η $\quad \varepsilon: FG \xrightarrow{\sim} 1$, $\eta: GF \xrightarrow{\sim} 1$ + $\begin{array}{l} \varepsilon.F = F.\eta \\ G.\varepsilon = \eta.G \end{array}$

It's unique up to canonical isom. A quasi-inv. can also be desc. as left adjoint (adj arrows are $GF \xrightarrow{\eta} 1$, $1 \xrightarrow{\varepsilon^{-1}} FG$)
or as right ——————— $FG \xrightarrow{\varepsilon} 1$, $GF \xrightarrow{\eta^{-1}} 1$).

Steps: Given B, F choose q-inv. G, $\varepsilon: FG \xrightarrow{\sim} 1$, $\eta: GF \to 1$.
Then get M cont. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\quad P = F(A)$, $Q = G(B)$

$F(m) \simeq F(A) \otimes_A M \qquad G(N) \xleftarrow{\sim} G(B) \otimes_B N$

$\varepsilon: FG(N) \to N$, with $FG(N) = P \otimes_A Q \otimes_B N \to N$

$FG(A) = P \otimes_A Q \to A$

Yes.

Review B, F yields $B, F, G, \varepsilon, \eta$ yields $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$.
yields $Q \otimes P \to A$. Next
$(B,F) \to (B',F')$ yields $(B, F, G, \varepsilon, \eta) \to (B', F', G', \varepsilon', \eta')$?

ξ} Let's try to say what needs saying.

Given
$$\begin{tikzcd} & \mathcal{M}(A) \arrow[dl] \arrow[dr] & \\ \mathcal{M}(B) \arrow[rr] \arrow[ur, phantom, "\Nearrow"] & & \mathcal{M}(B') \end{tikzcd}$$

What you have to do now is get the theorems proved and written out.

<u>Roos theorem.</u> Start with abelian cat $\mathcal{A}$ AB5. a generator U, let $R = \mathrm{End}(U)^{op}$. Then get functor
$$\mathrm{Mod}(R) \longrightarrow \mathcal{A}$$
$$M \longmapsto U \otimes_R M$$

Is this functor exact. It should be left adjoint to $N \longmapsto \mathrm{Hom}_{\mathcal{A}}(U, N)$ which ~~that~~ should be faithful.
$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r, shift left, "M \mapsto U \otimes_R M"] & \mathcal{A} \arrow[l, shift left] \end{tikzcd}$$
$$\mathrm{Hom}_{\mathcal{A}}(U, N) \qquad\qquad N$$

—

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{A}}(U,N)) = \mathrm{Hom}_{\mathcal{A}}(U \otimes_R M, N)$$

I think there's no problem, here $U \otimes_R N$ for N free is obvious so the rest is pretty clear. Now I think the GP thm. says this functor is exact because U is a generator. If so then the modules killed by $F = U \otimes_R -$ is a torsion theory in $\mathrm{Mod}(R)$.

Take now $\mathcal{A} = \mathcal{M}(B)$ $\quad B$ ~~...~~ idemp.

U is ~~an~~ a B, R-bimodule and generates $\mathcal{M}(B)$

this means $\exists$ ~~enough~~ $\bigoplus_{(I)} U \longrightarrow B$

of}

$$\mathrm{Mod}(R) \longrightarrow \mathcal{M}(B)$$
$$M \longmapsto \mathcal{U} \otimes_R M$$

we know $R = \mathrm{Hom}_B(\mathcal{U}, \mathcal{U})^{op}$. Because $\mathcal{U}$ is a generator, $\exists$ V and $\langle\ \rangle : \mathcal{U} \otimes V \twoheadrightarrow B$

example $V = \mathrm{Hom}_B(\mathcal{U}, B) \otimes_B B^{(2)}$.

$$\begin{pmatrix} R = \mathrm{Hom}_B(\mathcal{U},\mathcal{U})^{op} & \mathrm{Hom}_B(\mathcal{U},B)\otimes_B B^{(2)} \\ \mathcal{U} & B \end{pmatrix}$$

You seem to end up with some ideal in R, essentially $\mathrm{Im}\{\mathrm{Hom}_B(\mathcal{U}, B) \otimes_B \mathcal{U} \longrightarrow \mathrm{Hom}_B(\mathcal{U},\mathcal{U})\}$.

Here's a basic question – can you see why $M \longmapsto \mathcal{U} \otimes_R M$ should be exact? You ~~have~~ need to go from $0 \to M' \hookrightarrow M$ to $\mathcal{U} \otimes_R M' \hookrightarrow \mathcal{U} \otimes_R M$

One key idea is that ~~this def~~ is a derived functor, i.e. to show that $\mathrm{Tor}_1^R(\mathcal{U}, -) = 0$. Look at class of R-modules for which this is true. Enough to take $R/\sum R x_i$. You need to worry about $M' = \sum R x_i \subset R$.

So

$$\longrightarrow R^n \xrightarrow{(x_i)} R \longrightarrow R/\sum R x_i \longrightarrow 0$$

$$0 \longrightarrow K \longrightarrow \mathcal{U}^n \xrightarrow{(\cdot x_i)} \mathcal{U} \longrightarrow \mathcal{U} \otimes_R R/\sum R x_i \to 0$$

$\mathcal{U}$ generates so that K is a quotient $\bigoplus_I \mathcal{U} \to K$.
and this will translate into the fact that

π} At this point I have recovered the GP type argument, which ultimately ~~should be very~~ might be important. But perhaps I can see things better now in the idempotent ring context. This time you have P generating $\mathcal{M}(B)$, $R = \mathrm{Hom}_B(Q,Q)^{op}$, take $Q = \mathrm{Hom}_B(P, B)$, should get Mcontext

$$\begin{pmatrix} R & Q = \mathrm{Hom}_B(P, B) \\ P & B \end{pmatrix}$$

hence an ideal $A = QP$ in R. We know $PQ = B$ by the assumption that P generator in $\mathcal{M}(B)$. Also $QPQP = QBP = QP$ as P is firm. Now why is $M \longmapsto P \otimes_R M$ exact? Look at $\mathrm{Tor}_1^R(P, N)$. It isn't exact necessarily from $\mathrm{Mod}(R)$ to $\mathrm{Mod}(B)$, but it should be exact from $\mathrm{Mod}(R)$ to $\mathrm{Mod}(B)/\mathrm{Mod}(\mathbb{Z})$. So why is $\mathrm{Tor}_1^R(P, N)$ a left B-nil module. Answer $b = pq$ then mult by b on P factors thru R: $P \xrightarrow{q\cdot} QP \subset R \xrightarrow{p\cdot} P$

So at this point I have learned that there's a connection between my "factoring through a free" arguments and the proof of GP and Roos' thms.

Now to see if this stuff is good for something?

Q! ~~Let us consider~~

Go back to problem of $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. A flat in one side, ~~flat~~ and either P or Q is. What's going on?

83 ~~The idea Go back to the~~ suppose A mog to a unital ring, i.e. $M(A)$ has a gen ~~e~~ lying in $P(A)$ - get $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with B unital. Now ~~you want me~~ for m ino of K_* you need to bring in A h-unital eg. $Q \overset{L}{\otimes}_B P = Q \otimes_B P = A$. The first step here is to assume Q_B or ${}_BP$ is flat which is equivalent to A being either left or right flat. Then use inductive limit argument to reduce to either $Q \in P(B^{op})$ or $P \in P(B)$, whence $A \in P(A^{op})$ or $A \in P(A)$.

In the general case what might you do? Critical case is when both A, B ~~say~~ are say left flat.

Roughly you start with A both left and right flat say. Wait. Up to now I have looked at the condition $A \in P(A^{op})$ and tried to generalize this to A is right flat. So what do you propose? ~~You need~~ By Suslin you can always replace ~~an A~~ up to Morita equiv. any A by one which is both left and right flat. So what does one do? So if A is left + right flat, then ${}_BP$ Q_B are flat. Basic hypothesis is that $P \overset{L}{\otimes}_A Q \xrightarrow{\sim} B$ ~~etc.~~ A first step would be the understand the case where A is l+r flat and B is one sided flat. This is like A unital + B one sided-flat which we know how to handle.

Let's try to understand when ~~A~~ A one-sided flat and B two-sided flat. Generalizing $A \in P(A^{op})$ and B unital.

83

So is there anything we can do? How about $\mathrm{Tor}_n^R(P,-)$. So what is the important part?

Let's ~~...~~ recall that the exactness of $\mathrm{Mod}(R) \to m(B)$ $M \mapsto P \otimes_R M$ depends on $\mathrm{Tor}_1^R(P,M)$ being B-nil.

You got $B = PQ = P\,\mathrm{Hom}_B(P,B)$

$$\begin{pmatrix} R & Q = \mathrm{Hom}_B(P,B) \\ P & B \end{pmatrix} \qquad A = QP = \mathrm{Hom}_B(P,B)P$$

Important are ~~...~~

Do you have any feeling about $\mathrm{Tor}_n^R(P,-)$?

I considered the case B unital?

$$A \qquad Q = A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$$

$$P = A \qquad B = \mathrm{Hom}_{A^{op}}(A,A)$$

~~So~~ Let's stop this and try to outline things a bit.

Things are unclear.

Let's go over the main steps again.

rt cont fun.

begin with $m(A)$ = full subcat of $\mathrm{Mod}(\tilde{A})$ cons. of M such that $A \otimes_A M \xrightarrow{\sim} M$. When $A = A^2$ $m(A) \subset \mathrm{Mod}(\tilde{A})$ has a right adjoint $N \mapsto A^{(2)} \otimes_A N$, i.e.

~~...~~

M firm $\Rightarrow$ $\mathrm{Hom}_A(M, A^{(2)} \otimes_A N) \xrightarrow{\sim} \mathrm{Hom}_A(M,N)$.

Actually things might go smoother if you go over results and their proofs.

$\tau 3$ $A \xrightarrow{w} B$ homo.

$$\mathcal{M}(A) \underset{w^*}{\overset{w_!}{\rightleftarrows}} \mathcal{M}(B) \qquad \begin{array}{l} M \mapsto B^{(2)} \otimes_A M = B \otimes_A M = B \otimes_A M \\ N \mapsto A^{(2)} \otimes_A N \end{array}$$

$$\mathrm{Hom}_A(M, A^{(2)} \otimes_A N) \xrightarrow{\sim} \mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_B(B, N))$$

$$= \mathrm{Hom}_B(B \otimes_A M, N).$$

adjunction maps.

$$\alpha : B \otimes_A A^{(2)} \otimes_A N \longrightarrow N \qquad b \otimes a_1 \otimes a_2 \otimes n \mapsto b\,w(a_1 a_2) n$$

$$\beta : M = A^{(3)} \otimes_A M \longrightarrow A^{(2)} \otimes_A B \otimes_A M$$

$$a_1 a_2 a_3 m \longmapsto a_1 \otimes a_2 \otimes w(a_3) \otimes m.$$

~~Def w a map [illegible]~~

$$w_*(M) = B^{(2)} \otimes_B \mathrm{Hom}_A(A^{(2)} \otimes_A B, M)$$

$$w_!(A^{(2)}) = B \otimes_A A^{(2)} \qquad w^*(B^{(2)}) = A^{(2)} \otimes_A B^{(2)} = A^{(2)} \otimes_A B$$

$w_!$ is fully faithful iff $\beta : 1 \xrightarrow{\sim} w_* w^!$.

i.e. $A^{(2)} = A^{(3)} \longrightarrow A^{(2)} \otimes_A B \otimes_A A^{(2)}$

is an isom, i.e. iff $A \xrightarrow{w} B$ is an $A \otimes_A A^{op}$ - ~~module~~ nil-quis, i.e. $A\,\mathrm{Ker}(w) A = 0$ and $w(A) B w(A) \subseteq w(A)$.

$w_!$ equiv iff in addition $\alpha : B \otimes_A A^{(2)} \otimes_A B^{(2)} \xrightarrow{\sim} B^{(2)}$

This implies $B w(A) B = B$. Conversely assume all this

Then $B \otimes_A A^{(2)} \otimes_A B \longrightarrow B$. Need only show this is a B^{op}-rel isom. identity

$$(b_1 \otimes a_1 \otimes a_2 \otimes b_2)(b_3 w(a_3 a_4) b_4) = (b_1 w(a_1 a_2) b_2)(b_3 \otimes a_3 \otimes a_4 \otimes b_4)$$

$a\}$ $$(b_1 \otimes a_1 \otimes a_2' a_2'' \otimes b_2)(b_3 \otimes w(a_3' a_3'')\, w(a_4)\, b_4)$$

$$b_1 \otimes a_1 \otimes a_2' \otimes \underbrace{w(a_2'')\, b_2 b_3\, w(a_3')}_{w(a)}\, w(a_3'')\, w(a_4)\, b_4$$

$$= b_1 \otimes a_1 \otimes a_2' \, a\, a_3'' a_4 \otimes b_4$$

$$= b_1 w(a_1) w(a_2')\, w(a) \otimes a_3'' \otimes a_4 \otimes b_4$$

$$= b_1 w(a_1) w(a_2') w(a_2'') b_2 b_3\, w(a_3') \otimes a_3'' \otimes a_4 \otimes b_4$$

$$= b_1\; w(a_1 a_2)\, b_2 (b_3 \otimes a_3 \otimes a_4 \otimes b_4).$$

alternative would be to first prove ~~...~~ ~~suppose~~

Thm. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ $\quad A = QP \quad B = PQ \implies \mathcal{M}(A) \simeq \mathcal{M}(B)$

$$M \longmapsto P \otimes_A M$$
$$Q \otimes_B N \longleftarrow N$$

Pf. $B \otimes_B P \to P$ is an A^{op}-mil iso.

$$\sum (b_i \otimes p_i)\, qp = \left(\sum b_i p_i q\right) \otimes p \implies \text{kernel} \cdot A = 0$$

$$PA = PQP = BP \implies (P/BP) \cdot A = 0$$

$$\therefore\ B \otimes_B (P \otimes_A M) \xrightarrow{\sim} P \otimes_A M \qquad \text{so fun. defined.}$$

Next. $Q \otimes_B P \to A$ is A^{op}-mil iso. onto.

$$(q_i \otimes p_i)\, qp = q_i p_i q \otimes p$$

$$\left(\sum q_i \otimes p_i\right) qp = \sum q_i p_i q \otimes p$$

§3 Thm. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$

$\Rightarrow$ isom $\theta : B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M$, $\theta(b' \otimes p \otimes m) = b' u(p) \otimes m$

between $w_!(P\otimes_A -)$ and $(P' \otimes_A -)$ from $\mathcal{M}(A)$ to $\mathcal{M}(B')$.

There are two isos.

first $\quad v : Q \to Q'$ is B^{op}-nil iso.

$$\Rightarrow \quad Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N \qquad N \in \mathcal{M}(B)$$

$$\Rightarrow \quad \boxed{P' \otimes_A Q \otimes_B N \xrightarrow{\sim} P' \otimes_A Q' \otimes_B N \xrightarrow{\sim} B' \otimes_B N}$$

as $P' \otimes_A Q' \to B'$ is B' ~~op~~ nil isom
$\therefore$ B^{op} " " .

$$\therefore \quad w_! \simeq P' \otimes_A Q \otimes_B - \; : \; \mathcal{M}(B) \to \mathcal{M}(A) \to \mathcal{M}(B')$$

It follows that we have a corresp. isom of quasi-inv.

$$w^* \simeq P \otimes_A Q' \otimes_{B'} - \; : \; \mathcal{M}(B') \to \mathcal{M}(A) \to \mathcal{M}(B).$$

Obtain as follows. $\quad u : P \to P' \quad$ B-nil isom.

$$B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$$\begin{array}{ccc} B^{(2)} \otimes_B P \otimes_A Q' \otimes_{B'} N' & \xrightarrow{\sim} & B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N' \\ \| & & \| \\ P \otimes_A Q' \otimes_{B'} N' & \xrightarrow{\sim} & B^{(2)} \otimes_B N' \\ b_1 b_2 p \otimes q' \otimes n' & \longmapsto & b_1 \otimes b_2 \otimes u(p) q' n' \\ p \otimes v(q b_2) \otimes n' & \longleftarrow & p q \otimes b_2 \otimes n' \end{array}$$

As an exercise let's calculate the adjoint to above

$$F = P' \otimes_A Q \otimes_B - \xrightarrow{\theta} F' = B' \otimes_B N$$

$$G' \longrightarrow GFG' \longrightarrow GF'G' \longrightarrow G$$

$x\}$

$$\cancel{B^{(2)} \otimes_B N' \longrightarrow B^{(2)} \otimes_B P' \otimes_A Q}$$

$$B^{(2)} \otimes_B N' \rightleftarrows P \otimes_A Q' \otimes_{B'} P' \otimes_A Q \otimes_B B^{(2)} \otimes_B N'$$

~~etc.~~ etc.

$$\downarrow$$

$$P \otimes_A Q' \otimes_{B'} B' \otimes_B B^{(2)} \otimes_B N'$$

$$\downarrow$$

$$P \otimes_A Q' \otimes_{B'} N'$$

$$pqp'qb_1 \otimes b_2 \otimes n' \longleftrightarrow p \otimes q' \otimes p' \otimes q \otimes b_1 \otimes b_2 \otimes n'$$

$$\downarrow$$

$$p \otimes q' \otimes p'v(q) \otimes b_1 \otimes b_2 \otimes n'$$

$$\downarrow$$

$$p \otimes q' \otimes p'v(qb_1b_2)n'$$

$$p_1q_1p_2q_2b_1 \otimes b_2 \otimes n' \longmapsto p_1 \otimes v(q_1) \otimes u(p_2) \otimes q_2 \otimes b_1 \otimes b_2 \otimes n'$$

$$\downarrow$$

1035

$$p_1 \otimes v(q_1) \otimes w(p_2q_2b_1b_2)n'$$

$$\shortparallel$$

$$p_1 \otimes v(q_1p_2q_2b_1b_2)n'$$

$$p_1q_1b \otimes b_2 \otimes n' \longmapsto p_1 \otimes v(q_1 b b_2) \otimes n'$$

4] go over again $\begin{pmatrix}1 & v\\ u & w\end{pmatrix} : (\quad) \longrightarrow (\quad)$

$v: Q^{*} \longrightarrow Q'$ $\quad B^{op}$-mod iso $\quad v($

$$\begin{array}{ccc} q_1 & \longmapsto & v(q_1) \\ & \swarrow & \\ q\,u(p)\,v(q_1) & & \end{array}$$

$$\begin{tikzcd} Q \arrow[r, "v"] \arrow[d, "\cdot pq"'] & Q' \arrow[d, "\cdot w(pq)"] \arrow[dl, "\phi"'] \\ Q \arrow[r, "v"] & Q' \end{tikzcd}$$

~~$\phi(q') = q\,u(p)\,q'$~~

$q' \longmapsto q\,u(p)\,q'$

$$v(qb) = v(q)\,w(b).$$

$$\phi(q') = q'u(p)\,q$$

$$\begin{array}{ccc} q_1 & \longmapsto & v(q_1) \\ & \swarrow & \\ v(q_1)\,u(p)\,q & & \\ \| & & \\ q_1 p q & & \end{array}$$

$$\begin{tikzcd} Q \arrow[r, "v"] \arrow[d, "\cdot pq"'] & Q' \arrow[d, "\cdot w(pq)"] \arrow[dl, "q' \mapsto q'u(p)q"'] \\ Q \arrow[r, "v"] & Q' \end{tikzcd}$$

$$\begin{array}{c} q' \\ \swarrow \\ q'u(p)q \longmapsto v(q'u(p)q) \\ q'u(p)v(q) = q'w(pq) \end{array}$$

$$\therefore\quad Q\otimes_B N \xrightarrow{\sim} Q'\otimes_B N$$

$$P'\otimes_A Q\otimes_B N \xrightarrow{\sim} P'\otimes_A Q'\otimes_B N \xrightarrow{\sim} B'\otimes_B N.$$

$$p'\otimes q\otimes n \longmapsto p'v(q)\otimes n$$

$$b'u(p)\otimes q\otimes n \longmapsfrom b'\otimes pqn$$

identifies $w_!$ with $P'\otimes_A Q\otimes_B -$. $\therefore$ w is a meghan. Converse. Suppose $w: A\to B \ni A\,\mathrm{Ker}(w)A = 0$, $w(A)Bw(A) \subset B$ and $Bw(A)B = B$. ~~Red. to case w surj.~~

$A \twoheadrightarrow A/K = \bar A \subset B.$ $\quad AKA = 0$ $\quad \bar A B \bar A \subset \bar A + B\bar A B = B.$

$$\begin{pmatrix} A & A/AK \\ A/KA & A/K \end{pmatrix} \longleftarrow \begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

identifies $w_!(M) = \overset{M/KM}{\overset{\|}{A/K\otimes_A M}}$ ~~with $A/KA\otimes_A M$.~~

wf Next $\cancel{\begin{pmatrix} \bar{A} & \bar{A}B \\ B\bar{A} & B \end{pmatrix}}$ $\begin{pmatrix} B & BA \\ AB & A \end{pmatrix} \subset \begin{pmatrix} B & B \\ B & B \end{pmatrix}$, $\xrightarrow{v}$, $\xrightarrow{w}$

$$\begin{pmatrix} A & \dot{A} \\ A & A \end{pmatrix} \subset \begin{pmatrix} A & AB \\ BA & B \end{pmatrix} \quad \begin{pmatrix} B & B \\ B & B \end{pmatrix}$$

$$\mathcal{M}(A) \xrightarrow{w_!} \mathcal{M}(B)$$

from first you get incl. $\quad w_!(M) \simeq P' \otimes_A Q \otimes_B M = BA \otimes_A A \otimes_A M$

from 2nd you get $\quad \cancel{w_!(M) \simeq P' \otimes_A Q \otimes_B M = AB \otimes_B BA \otimes_A M}$

$$w_!(M) = B \otimes_B BA \otimes_A M$$

and so your notation is terrible

It's now 1500 on 02/14/97 Valentine's Day.
I want to make some progress on M invariance for K/X.
I believe ~~an important~~ the crucial case to understand is when both A, B are biflat. The point is that given A firm, say, we can choose a firm flat A^{op}-module P mapping onto A. Then get firm dual pair $P \otimes A \to A$, $p \otimes a \mapsto f(p)a$, ~~$\langle p, a \rangle = f(p)a$~~ whence a firm M cont. $\begin{pmatrix} A & A \\ P & B \end{pmatrix}$

$P = P \otimes_A A = B$

$(p_1 a_1)(p_2 a_2) = p_1 a_1 f(p_2 a_2) a_2$

$\quad \therefore p_1 p_2 = p_1 f(p_2)$.

Now P A^{op} flat $\Longrightarrow$
$P \otimes_A A = P$ is B^{op} flat so B is right flat. Change notation to $A \twoheadrightarrow A/I = B$ where A is right flat, I ideal in A such that $AI = 0$.