<u>August 16, 1981</u>

Review of buildings, Lie gps, etc.

First consider the case of a compact connected Lie gp K acting on its Lie algebra $\mathfrak{k}$. Using Morse theory one shows that any orbit $K\cdot x$ has a cell decomposition with even dimensional cells, hence $K\cdot x = K/K_x$ is 1-connected and so K_x is connected. If x is chosen generic, then $\mathfrak{k}_x = \mathfrak{a}$ (= Lie algebra of maximal torus K_x) meets each K-orbit in a <u>Weyl group orbit</u>.

Calculate simple root systems for classical groups.

1) $K = SU_n$. a maximal torus is given by the diagonal matrices, so $\mathfrak{a}$ consists of $i\begin{pmatrix}\theta_1 & & \\ & \ddots & \\ & & \theta_n\end{pmatrix}$ with $\sum\theta_i = 0$. $W = \Sigma_n$. $\mathfrak{k}$ = skew-hermitian matrices. Root vectors are

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n}\end{pmatrix}\begin{pmatrix} & & \\ & z & \\ & & \end{pmatrix}\begin{pmatrix} e^{-i\theta_1} & & \\ & \ddots & \\ & & e^{-i\theta_n}\end{pmatrix} = \begin{pmatrix} & & \\ & e^{i(\theta_i-\theta_j)}z & \\ & & \end{pmatrix}$$

i,j th position

hence the roots are $\theta_i - \theta_j \qquad i \neq j$.

The obvious choice for fundamental Weyl chambre is

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$$

so the simple roots are

$$\theta_1-\theta_2,\ \theta_2-\theta_3,\ \ldots,\ \theta_{n-1}-\theta_n$$

Recall the convention that $\alpha(H_\alpha) = \langle H_\alpha, H_\alpha\rangle = 2$

e.g. in $\mathfrak{sl}_2$ one has

$$X = \begin{pmatrix}0&1\\0&0\end{pmatrix} \quad Y = \begin{pmatrix}0&0\\1&0\end{pmatrix} \quad H = [X,Y] = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

and $$[H,X] = 2X \qquad [H,Y] = -2Y.$$

Thus for the root $\theta_1 - \theta_2$ one has $H_\alpha = (1, -1, 0, \cdots, 0)$ in the natural inner product. The simple roots $\alpha_i(\theta) = \theta_i - \theta_{i+1}$ all have length $\sqrt{2}$ and the angle between consecutive ones has

$$\cos = \frac{-1}{2} \quad \therefore \quad \text{angle} = 120^\circ$$

The diagram then is

$$\begin{array}{ccccccccc} \alpha_1 & & & & & & & & \alpha_{n-1} \\ \circ & — & \circ & — & \circ & — & \circ & — & \circ \\ \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} \end{array}$$

2) $K = Sp_{2n}$ = autos of $\mathbb{H}^n$ preserving distance

$\mathbb{H}^n = \mathbb{C}^n \oplus \mathbb{C}^n j$ and a $\mathbb{H}$-linear endo of $\mathbb{H}^n$ is of the form

$$(u + vj) \longmapsto u' + v'j$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

This will be in $\mathfrak{k}$ when $\uparrow$ is skew-hermitian $\Longrightarrow$ $-\alpha = \alpha^*$, β is symmetric. Max torus is

$$\left(\begin{array}{ccc|ccc} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_n} & & & \\ \hline & & & e^{-i\theta_1} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_n} \end{array}\right)$$

the roots are

$$\theta_i - \theta_j \qquad i \neq j$$

$$\pm(\theta_i + \theta_j) \qquad i \geqslant j$$

Check: $\dim K = n^2 + 2\frac{n(n+1)}{2} = 2n^2 + n$

$$= \underbrace{n(n-1)}_{\theta_i - \theta_j} + \underbrace{2\frac{n(n+1)}{2}}_{\pm(\theta_i + \theta_j)} + \underbrace{n}_{\dim T} = 2n^2 + n.$$

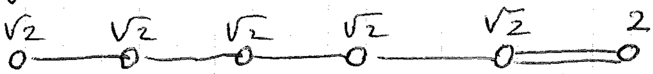
$$W = \Sigma_n \times (\mathbb{Z}_2)^n$$ so a fundamental chamber is

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \geq 0$$

Simple roots are

$$\theta_1 - \theta_2,\ \theta_2 - \theta_3,\ \ldots,\ \theta_{n-1} - \theta_n,\ 2\theta_n$$

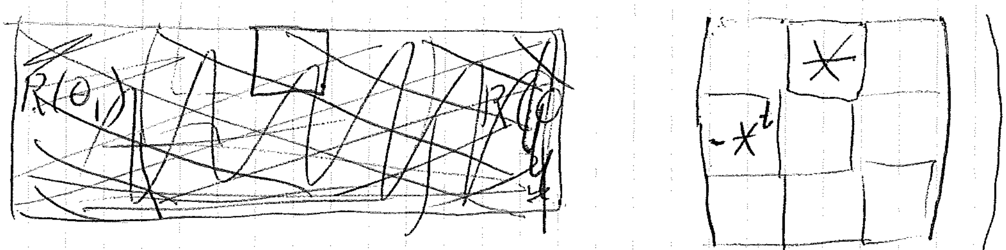
so the diagram is



angle has $\cos = \frac{-2}{\sqrt{2}\cdot 2} = -\frac{1}{\sqrt{2}}$

$\therefore$ angle $= 135^\circ$

3) $K = SO_{2n+1}$. Max. torus is $SO(2)^n$ and $W = \Sigma_n \times (\mathbb{Z}_2)^n$; note last coordinate is determined so that the det $= 1$. $\mathfrak{k}$ = skew-symm. matrices. For each $i > j$ one has a 2×2 block



which you should think of as $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}} + \overline{\mathrm{Hom}_{\mathbb{C}}}$.
Thus the roots are seen to be

$$\pm(\theta_i \pm \theta_j) \qquad i > j$$

$$\pm\ \theta_i$$

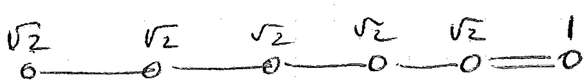
Check: $2\,\frac{n(n-1)}{2}\,2 + 2n + n = 2n^2 + n = \frac{2n(2n+1)}{2}$.

W acts by permuting + changing signs, so a fdl. domain is

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \geq 0$$

Simple roots $\quad \theta_1 - \theta_2,\ \theta_2 - \theta_3,\ \ldots,\ \theta_{n-1} - \theta_n,\ \theta_n$

Diagram



4) $K = SO_{2n}$. $\qquad T = SO(2)^n \qquad W = \Sigma_n \tilde{\times} \underbrace{(\mathbb{Z}/2)^{n-1}}_{\text{so } \det = 1}.$

This time the roots are

$$\pm(\theta_i \pm \theta_j) \qquad i > j$$

and W acts by permuting and changing an <u>even</u> no. of signs. Hence a fundamental domain is

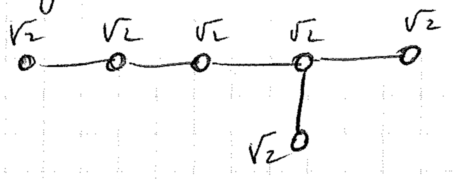
$$\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n \qquad \theta_{n-1} + \theta_n \geqslant 0$$

so the simple roots are

$$\theta_1 - \theta_2, \; \ldots, \; \theta_{n-2} - \theta_{n-1}, \; \theta_{n-1} - \theta_n, \; \theta_{n-1} + \theta_n$$

$$\perp$$

and so diagram is

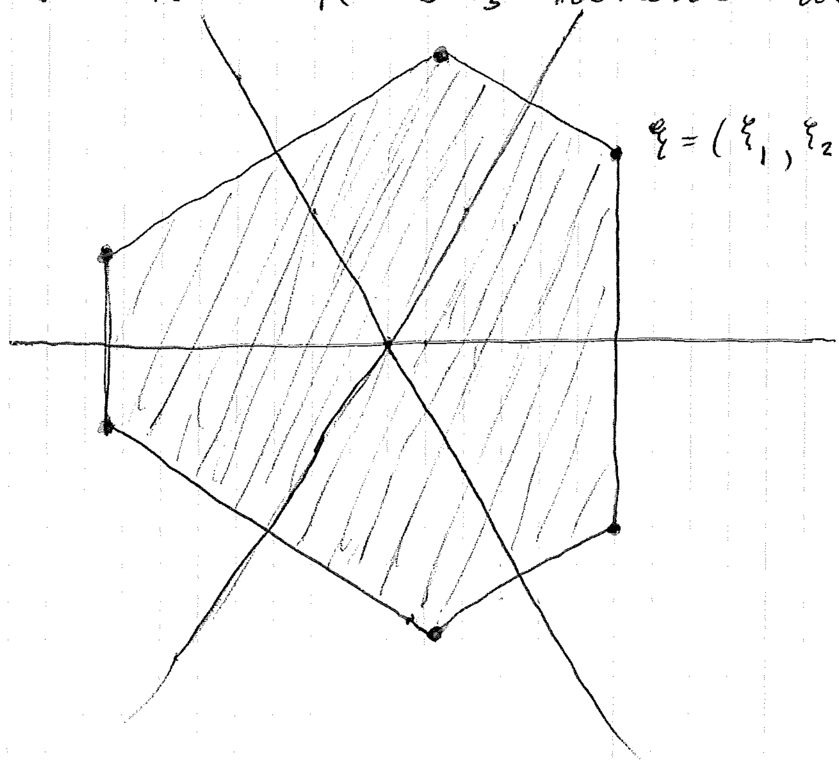


August 19, 1981

Atiyah gave Arbeitstagung talk on convexity + commuting Hamiltonians. He has a nice proof of the following theorem of Kostant:

Let K be a compact connected Lie gp and ~~$\mathfrak{k}$ its~~ Lie algebra, and $\mathfrak{t}$ a maximal abelian subspace of $\mathfrak{k}$. Then the orthogonal projection onto $\mathfrak{t}$ of any K-orbit $\mathcal{O}$ in $\mathfrak{k}$ is a convex set, in fact, it's the convex hull of the Weyl group orbit $\mathcal{O} \cap \mathfrak{t}$.

[redacted] In the following I prefer to work with [redacted] $\mathfrak{p} = i\mathfrak{k}$, $\mathfrak{a} = i\mathfrak{t}$ in the Lie algebra of the complexification G of K. So when $G = GL_n$, $K = U_n$ for instance, $\mathfrak{a}$ will be the diagonal hermitian matrices. Take $K = SU_3$ whence we have the picture:



$\xi = (\xi_1, \xi_2, \xi_3)$ in fundamental chambre $\xi_1 > \xi_2 > \xi_3$

I think it is actually very easy to see that the projection of the orbit $\mathcal{O}$ is contained in the convex hull of the points $\mathcal{O} \cap \mathfrak{a}$. Take a linear functional on $\mathfrak{a}$; it is of the form $\langle \eta, x \rangle$ $\quad x \in \mathfrak{a}$ [redacted] with $\eta \in \mathfrak{a}$. If we compose this with the projection of $\mathfrak{p}$ on $\mathfrak{a}$ we get the linear function

$$\langle \eta, x \rangle \qquad x \in \mathfrak{a}$$

[redacted] If this function is restricted to the orbit $\mathcal{O}$, then we [redacted] know the [redacted] critical points [redacted] are $\mathfrak{p}_\eta \cap \mathcal{O}$. Working in the centralizer of η one sees critical points are orbits wrt K_η of $\mathfrak{a} \cap \mathcal{O}$. In particular the max. and min. of the function $\langle \eta, x \rangle$, $x \in \mathcal{O}$ are taken on the set $\mathfrak{a} \cap \mathcal{O}$. Since η is arbitrary, it's clear that the projection of $\mathcal{O}$ in $\mathfrak{a}$ is contained in the convex hull of $\mathfrak{a} \cap \mathcal{O}$.

August 20, 1981

Still trying to prove Kostant's theorem.

$\mathcal{O}$ = orbit of K on $\mathfrak{p} = i\mathfrak{k}$

$p : \mathcal{O} \longrightarrow \mathfrak{a}$ orthogonal projection

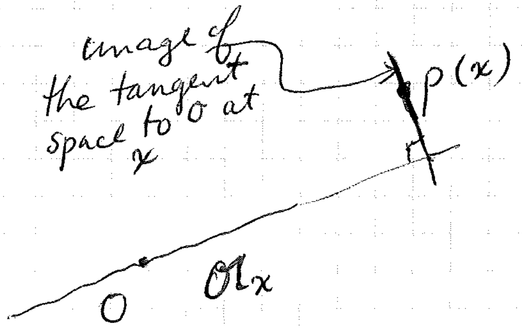
Let $x \in \mathcal{O}$. The tangent space to $\mathcal{O}$ at x is $[\mathfrak{k}, x]$ and its image $p[\mathfrak{k}, x]$ is $\perp$ to η when

$$\langle \eta, [\mathfrak{k}, x] \rangle = \langle [\eta, x], \mathfrak{k} \rangle = 0$$

i.e. $[\eta, x] = 0$.

i.e. $\eta \in \mathfrak{a}_x$ = elts of $\mathfrak{a}$ commuting with x.

Thus p maps the tangent space to $\mathcal{O}$ at x, to the space thru $p(x)$ parallel to $(\mathfrak{a}_x)^\perp$



Idea: Does this thm. have anything to do with J-matrices? Take $K = SU_n$. Then we have

$$J \subset \mathcal{O} \xrightarrow{p} \mathfrak{a}$$

where J is the set of J-matrices

$$\begin{pmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{pmatrix} \qquad \begin{array}{l} a_i > 0 \\ \sum_1^n b_i = 0 \end{array}$$

with the eigenvalues $\lambda_1, \dots, \lambda_n$ of the orbit $\mathcal{O}$ (thus these must be distinct for the problem to have a meaning) Then we know J can be described in terms of all probability measures $\sum_1^n r_i \delta_{\lambda_i}$ $\qquad \sum r_i = 1$, $r_i > 0$

supported on $\{\lambda_1, \dots, \lambda_n\}$. Thus

$$\dim \mathcal{J} = n-1 = \dim \mathfrak{a}.$$

and one can ask whether $\mathcal{J}$, $\mathcal{O}$ have the same image in $\mathfrak{a}$.

The answer seems to be NO because we know that the origin should be in $\rho(\mathcal{O})$. On the other hand a J-matrix with 0 entries on the diagonal belongs to a probability measure which is symmetric about the origin. So 0 will not be in $\rho(\mathcal{J})$ when $\{\lambda_1, \dots, \lambda_n\}$ is not symmetric.

<u>August 21, 1981</u>

Kostant's thm. in the case of SU_n. Take the orbit in $\mathfrak{p}$ = hermitian matrices of trace zero of the diagonal matrix Λ with entries $\lambda_1, \dots, \lambda_n$, say distinct. a typical element of the orbit is of the form

$$\underbrace{(v_1 \cdots v_n)^* \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{\Lambda} (v_1 \cdots v_n) = (v_i^* \Lambda v_j)$$

where v_i is an orthonormal basis. If $v_i = (v_{ji})_{1 \le j \le n}$ then the diagonal entries are

$$v_i^* \Lambda v_i = \sum_j \overline{v_{ji}}\, \lambda_j\, v_{ji}$$
$$= \sum_j \lambda_j |v_{ji}|^2$$

But recall that for any unitary matrix

$$(v_1 \cdots v_n) = (v_{ij})$$

that the matrix $(|v_{ij}|^2)$ is doubly-stochastic. $\blacksquare$ Also the Birkhoff-von Neumann thm. says that the set of doubly-stochastic matrices is the convex hull of the permutation matrices. Thus if we write $(|v_{ij}|^2)$ as a convex linear combination of permutation matrices we see that the vector $(v_i^* \Lambda v_i)$ is in the convex hull of the Weyl orbit of Λ.

Maybe it's true that the map

$$(*) \qquad \begin{array}{ccc} T \backslash SU_n / T & \longrightarrow & \text{doubly-stochastic matrices} \\ (v_{ij}) & \longmapsto & |v_{ij}|^2 \end{array}$$

is bijective. The conditions $\sum_j p_{ij} = 1$, $\sum_i p_{ij} = 1$ are really $2n-1$ conditions, so the dim of the d.s. mat. is

n^2-2n+1. SU_n/T has dim $(n^2-1)-(n-1) = n^2-n$ and so $T\backslash SU_n/T$ has dim $n^2-n-(n-1) = n^2-2n+1$.

Assuming ⊛ is sur[redacted]jective we get Kostant's thm. because if we take a convex linear combination of [redacted] points in the Weyl group orbit of Λ, this [redacted] lifts to a convex linear combination of permutation matrices, i.e. a doubly-stochastic matrix, which lifts (assuming ⊛ onto) to an element of SU_n.

Atiyah's version: Let M be a connected symplectic manifold, compact, and suppose one has a symplectic action $T^n \times M \longrightarrow M$, T^n an n-torus, such that each of the associated [redacted] vector fields comes from a function, e.g. M 1-connected. Thus [redacted] we have n Hamiltonian functions $f_1, \ldots, f_n$ such that X_{f_i} are commuting vector fields giving a periodic action. Then we have [redacted] the "moment" map

$$f = (f_1, \ldots, f_n) : M \longrightarrow \mathbb{R}^n$$

and he proves

(A) Image of the moment map is the convex hull of a finite set $\{c_i\}$ = Image of M^{T^n} under f

(B) $f^{-1}(c$[redacted]$)$ connected for all $c \in f(M)$.

Now [redacted] A[redacted] for T^n action follows from B for a suitable T^{n-1} action. Somehow you take a line in $\mathbb{R}^n$ and show the inverse image is connected. You can assume the line is "rational" so that the [redacted] linear combs. of the f_i constant on it have vector fields belonging to a subtorus $T^{n-1} \subset T^n$.

The interesting part is (B). First look at $n=1$ whence we have a circle action $S^1 \times M \longrightarrow M$ with vector field X_f. [redacted]
To show $f^{-1}(c)$ connected one uses [redacted] Morse theory — it's enough to show f has no critical submanifolds (assume these are non-degenerate) of index 1 or $n-1$. But the critical points of f are the fixpts for the circle action M^{S^1} and the normal bundle has a complex structure, the Hessian of f will be the real part of a hermitian form on the normal bundle, hence the index will be even.

August 23, 1981

Counterexample to the hope that

$$T\backslash \mathcal{U}_n / T \;=\; \text{doubly-stochastic } n\times n \text{ matrices.}$$

Take $n=3$. Consider a doubly-stochastic matrix

$$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}$$

where $a, b, c > 0$. Any unitary matrix giving rise to this can be premultiplied by something in T so that its first column is $\begin{pmatrix} \sqrt{a} \\ \sqrt{b} \\ \sqrt{c} \end{pmatrix}$. Then the second column will be of the form $\begin{pmatrix} \sqrt{a'}\, e^{i\theta_1} \\ \sqrt{b'}\, e^{i\theta_2} \\ \sqrt{c'}\, e^{i\theta_3} \end{pmatrix}$ where $\sqrt{aa'}\, e^{i\theta_1} + \sqrt{bb'}\, e^{i\theta_2} + \sqrt{cc'}\, e^{i\theta_3} = 0$

Thus if $c'=0$ we must have $\sqrt{aa'} = \sqrt{bb'}$. So if you take the matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

it is doubly-stochastic, yet it can't come from a unitary matrix, because there is no way to assign angles so that orthogonal unit vectors will give rise to the first two columns.

August 25, 1981

Idea: review $\Omega(K)$ and Laurent polynomials.

Here K is a compact connected Lie gp like U_n and the theorem is that $\Omega(K)$ has the minimal model given by the space of Laurent polynomial maps $S^1 \to K$ preserving basepoint.

Perhaps the good viewpoint is to think of ΩU_n as describing (n-dimensional) vector bundles over the Riemann sphere. One will get various models for (parameterizing) such vector bundles, all of which consist of a group acting on a space. Let's consist the Bott-Atiyah version. One fixes the rank and degree and considers a given C^∞ vector bundle E with this rank + degree and with hermitian structure. Then there is a 1-1 correspondence between holomorphic structures on E and connections preserving the hermitian structure. The space of connections is a contractible space $\mathfrak{X}$ which is acted on by the gauge group $\mathcal{H}$ of all automorphisms of E.

Next return to old idea of $\mathcal{K}$ = maps $S^1 \to K$. Then this is the group of autos. of the trivial K-bundle $S^1 \times K$ over S^1. Connections are essentially differential equations

$$\frac{dx}{dt} = A(t)x$$

where $A(t) \in \mathrm{Lie}(K)$ is periodic, hence a function of $z = e^{2\pi i t} \in S^1$. To the connection belongs the solution matrix $u(t) : \mathbb{R} \longrightarrow K$ defined by

$$\frac{d}{dt} u(t) = A(t)\, u(t)$$
$$u(0) = 1$$

Given $\varphi : S^1 \to K$, its action on $u(t)$ will be

$$\varphi(t)\, u(t)\, \varphi(0)^{-1}$$

To see this note $\quad u(t+1) = u(t)\,u(1)\quad$ and that $\tilde{u}(t) = \varphi(t)\,u(t)$ satisfies

$$\tilde{u}(t+1) = \varphi(t)\,u(t)\,u(1)$$
$$= \tilde{u}(t)\,u(1)$$

hence to normalize so that $\tilde{u} = 1$ at $t=0$ we put

$$\tilde{u}(t) = \varphi(t)\,u(t)\,\varphi(0)^{-1}$$

The corresponding action of φ on A is

$$\frac{d}{dt}\tilde{u}(t) = \varphi'\,u\,\varphi(0)^{-1} + \varphi\,\underbrace{u'}_{Au}\,\varphi(0)^{-1}$$
$$= \varphi'\varphi^{-1}\tilde{u} + \varphi A \varphi^{-1}\tilde{u}$$

Thus
$$\tilde{A} = \varphi'\varphi^{-1} + \varphi A \varphi^{-1}$$

which is the usual formula for the gauge group acting on connections.

So we recover the action of $\varphi \in \mathcal{K}$ on $\mathcal{X}$ by the formula

$$\varphi, h \longmapsto \varphi(t)\,h(t)\,\varphi(0)^{-1}$$

In the very good Laurent case, $\mathcal{K}$ = Laurent poly. maps $S^1 \to K$, and $\mathcal{X}$ consists of all transforms under $\mathcal{K}$ of the paths
$$h(t) = e^{tX} \qquad X \in \mathrm{Lie}(K).$$
This corresponds to constant connection forms A.

August 28, 1981

I want to understand why $\Omega(K)$ has a natural symplectic structure. More generally this should be true for any $\mathcal{K}$ orbit on $\mathfrak{X}$, i.e. paths joining 1 to a conjugacy class.

I am thinking of $\mathfrak{X}$ as a set of either paths $h : [0,1] \longrightarrow K$, $h(0) = 1$ or as connection forms $A : S^1 \to \mathfrak{k}$, these being related by

$$A = h' h^{-1}$$

The action of $\varphi \in \mathcal{K}$ upon $\mathfrak{X}$ is given by

$$\varphi * h = \varphi h \varphi(1)^{-1}$$

or

$$\varphi * A = (\varphi h \varphi(1))' (\varphi h \varphi(1)^{-1})^{-1}$$

$$= (\varphi' h + \varphi h') h^{-1} \varphi^{-1} = \varphi' \varphi^{-1} + \varphi A \varphi^{-1}.$$

Therefore if $\varphi = 1 + \varepsilon X \pmod{\varepsilon^2}$ is a tangent vector to the identity of $\mathcal{K}$, we have

$$\varphi * A = A + \left(X' + [X, A] \right)$$

Thus the tangent space to the $\mathcal{K}$ orbit thru A contains all

$$X' + [X, A]$$

with $X : S^1 \longrightarrow \mathfrak{k}$.

So the project now is to produce a ~~non~~-degenerate skew-symmetric form on this tangent space. Let's consider the case $K = S^1$. Then the tangent space consists of all $X' = B$, i.e. all $B : S^1 \longrightarrow i\mathbb{R} = \mathfrak{k}$ such that $\int_0^1 B\,dt = 0$. A convenient skew-symmetric form is

$$(B_1, B_2) = \int_0^1 B_1\, dB_2 = -\int_0^1 B_2\, dB_1$$

since $\int B_1 dB_2 + \int B_2 dB_1$
$= \int d(B_1 B_2) = 0$

This clearly vanishes if either B_1, B_2 is constant so it is a good form on the tangent space.

Compute the form on the complexification which has the basis $z^n \quad n \neq 0$.

$$z = e^{2\pi i t}$$
$$dz = e^{2\pi i t}\, 2\pi i\, dt$$

$$(z^m, z^n) = \int_0^1 z^m\, n z^{n-1}\, dz$$

$$= 2\pi i n \int_0^1 z^{m+n}\, dt = \begin{cases} 0 & m+n \neq 0 \\ 2\pi i n & m+n = 0. \end{cases}$$

So this is <u>clearly nice</u> and <u>non-degenerate</u>.

The next project is to extend this to a non-abelian group K. Now we have these minimal models where $\mathcal{K}$ and $\mathcal{G}$ are made up of Laurent polynomial matrices. Thus

$$\mathrm{Lie}(\mathcal{G}) = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$$

$\mathrm{Lie}(\mathcal{K})$ = fixed part for the involution of above which is $*$ on $\mathfrak{g}$ and $z^* = z^{-1}$!

Thus we have

$$\mathrm{Lie}(\mathcal{G}) = \mathfrak{g}[z^{-1}] \underset{\mathfrak{g}}{+} \mathfrak{g}[z]$$

$$= \underbrace{\mathfrak{g}[z^{-1}]z^{-1} \oplus \mathfrak{n}^*}\oplus \mathfrak{h} \oplus \underbrace{\mathfrak{n} \oplus z\mathfrak{g}[z]}$$

I think it should be true that these spaces are both isotropic for the symplectic structure, since they belong to unipotent part of an Iwahori subgroup.

So the first thing to understand is what happens to the symplectic structure in the case of $\mathfrak{g} = \mathfrak{n}^* \oplus \mathfrak{h} \oplus \mathfrak{n}$.

Let's therefore review the Kirilov business. Let G be a Lie group acting naturally on $\mathfrak{g}^*$, $\mathfrak{g} = \mathrm{Lie}(G)$. Take $\lambda \in \mathfrak{g}^*$ and consider the form $\lambda([x,y])$ on $\mathfrak{g} \times \mathfrak{g}$.

The stabilizer of λ has Lie alg. $\mathfrak{g}_\lambda$ consisting of all $x \in \mathfrak{g}$ such that [redacted] $x \cdot \lambda = 0$. But $x \cdot \lambda$ is the form [redacted] $y \mapsto -\lambda(\operatorname{ad} x(y)) = -\lambda([x,y])$. Thus $\lambda([x,y])$ is a skew-symmetric form on $\mathfrak{g}/\mathfrak{g}_\lambda$ clearly non-degenerate since $\lambda[x,y]=0$ for all $y \Rightarrow x \cdot \lambda = 0 \Rightarrow x \in \mathfrak{g}_\lambda$. Therefore the orbit $G\lambda$ [redacted] has a natural non-deg. 2 form [redacted] which is invariant for the G-action.

Why is this 2 form closed?

Conversely if M is a symplectic manifold on which G acts as autos., then we get a map of Lie algebras

$$\mathfrak{g} \longrightarrow \text{[redacted] Hamiltonian Vector fields}$$

But one has a [redacted] central extension of Lie algebras

$$\mathbb{R} \longrightarrow \text{[redacted] functions on } M \text{ under } \{\ ,\ \} \longrightarrow \text{Hamiltonian vect. flds.}$$

so that if $H^2(\mathfrak{g}, \mathbb{R}) = 0$, you get a lifting

$$\mathfrak{g} \longrightarrow \text{functions on } M$$

and hence a map $M \longrightarrow \mathfrak{g}^*$.

Finally for $\mathfrak{g}$ semi-simple one can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by the Killing form, so orbits in $\mathfrak{g}^*$ are the same as orbits in $\mathfrak{g}$. But we can also get a formula.

Let's consider then the group $K = SU_n$, $G = SL_n$. Take a generic $i\xi \in \operatorname{Lie}(T) = i\mathfrak{a}$, and consider the form

$$X, Y \longmapsto \langle i\xi, [X,Y] \rangle$$

on $\mathfrak{k}/i\mathfrak{a}$. Here $\langle x, y \rangle = -\operatorname{tr}(xy) = \operatorname{tr}(x^* y)$ is [redacted] a non-degenerate inner product. So

$$\langle i\xi, [X,Y] \rangle = \langle i[\xi, X], Y \rangle$$
$$= \sum_\alpha{}' \alpha(\xi) \langle iX_\alpha, Y_\alpha \rangle$$

Here iX_α has to be computed in terms of the complex structure on $\mathfrak{k}/i\mathfrak{a}$. Thus [redacted] the Euclidean inner product $\langle\ ,\ \rangle$ is the real part of the hermitian product

$$\langle iX_\alpha, Y_\alpha \rangle = \mathrm{Re}\,(iX_\alpha, Y_\alpha)$$
$$= \mathrm{Re}\; i(X_\alpha, Y_\alpha) = -\,\mathrm{Im}\,(X_\alpha, Y_\alpha)$$

and so the form is

$$X, Y \longmapsto -\sum_\alpha \alpha(\xi)\, \mathrm{Im}(X_\alpha, Y_\alpha)$$

So to be specific $\alpha = (i,j)$, $i<j$. [redacted]

Let's be more precise. A root of SL_n is a pair i,j with $1 \le i,j \le n$ and $i \ne j$. Thus one has root vectors X_{ij} = a single 1 in i-th row, jth column.

and
$$\alpha_{ij}(H) = [redacted]\ H_i - H_j$$

if H is a diagonal matrix. We are above all interested in the case where the fundamental chambre is [redacted] given by $\xi_1 > \xi_2 > \cdots > \xi_n$ whence positive roots are [redacted] α_{ij} with $i<j$.

Take
$$X = \begin{pmatrix} & x_{ij} \\ -\overline{x_{ij}} & \end{pmatrix} \qquad Y = \begin{pmatrix} & y_{ij} \\ -\overline{y_{ij}} & \end{pmatrix}$$

Then
$$X = \sum_{\alpha>0}{}' x_\alpha X_\alpha - \overline{x_\alpha} X_{-\alpha}$$
$$Y = \sum_{\alpha>0}{}' y_\alpha X_\alpha - \overline{y_\alpha} X_{-\alpha}$$

and
$$[i\xi, X] = \sum{}' i\alpha(\xi) x_\alpha X_\alpha + i\alpha(\xi)\overline{x_\alpha} X_{-\alpha}$$

so
$$\langle [i\xi, X], Y \rangle = \sum_{\alpha>0} \mathrm{Re}\left(i\alpha(\xi)\, x_\alpha(-\overline{y_\alpha})\right) = \sum_{\alpha>0}{}' \alpha(\xi)\, \mathrm{Im}(x_\alpha \overline{y_\alpha})$$

<u>August 29, 1981</u>

Kirillov setup. $\mathcal{O}$ = orbit in $\mathfrak{g}^*$. If $\lambda \in \mathcal{O}$, then

$$\mathfrak{g}/\mathfrak{g}_\lambda \longrightarrow \text{Tangent space to } \mathcal{O} \text{ at } \lambda$$

and $X, Y \longmapsto \langle \lambda, [X,Y] \rangle$ gives a skew-symmetric non-degenerate form on $T_\lambda(\mathcal{O})$. Thus we get a canonical 2-form on $\mathcal{O}$. Notice also that each $X \in \mathfrak{g}$ determines a function f_X on $\mathcal{O}$ by

$$f_X(\lambda) = \langle \lambda, X \rangle$$

Let's compute df_X. Thus we want to evaluate

$$i(Y)df_X = \theta(Y) f_X \quad \text{at} \quad \lambda \in \mathcal{O}$$

and we have

$$0 = \theta(Y)[f_X(\lambda)] = (\theta(Y)f_X)(\lambda) + \underbrace{f_X(\theta(Y)\lambda)}$$

$$\langle \lambda, [X,Y] \rangle = -\langle \lambda, [Y,X] \rangle = \langle \theta(Y)\lambda, X \rangle$$

$$\|$$

$$\Omega_\lambda(X \wedge Y) = i(Y)i(X)\Omega \quad \text{at } \lambda$$

Thus we get the formula

$$\boxed{df_X = i(X)\Omega}$$

which says that X is the vector field belonging to the function f_X. Also we have

$$i(X)d\Omega = \underbrace{\theta(X)\Omega}_{\substack{0 \text{ because} \\ \Omega \text{ is invariant}}} - d\underbrace{i(X)\Omega}_{df_X} = 0$$

for all X showing that $\boxed{d\Omega = 0}$.

<u>August 30, 1981</u>

$\mathcal{K} =$ maps $\varphi : S^1 \to K$ $\qquad\qquad$ $\mathfrak{k} = \mathrm{Lie}(\mathcal{K}) =$ maps $X: S^1 \to k$

$=$ autos of trivial principal bundle $S^1 \times K \to S^1$

$\mathcal{X} =$ connections on the bundle

$\cong$ maps $A : S^1 \to k$

Thus if $K =$ autos of a vector space (say $K = SU_n$) then the differential operator D measuring deviation from flatness is
$$D = \frac{d}{dt} - A .$$

The action of $\mathcal{K}$ on $\mathcal{X}$ is
$$\varphi * A = \varphi A \varphi^{-1} + \varphi' \varphi^{-1}$$
which results from
$$\varphi\left(\frac{d}{dt} - A\right)\varphi^{-1} = \frac{d}{dt} - \varphi\varphi^{-1}\varphi'\varphi^{-1} - \varphi A \varphi^{-1} .$$
Finally the induced action of $\mathfrak{k}$ on $\mathcal{X}$ is
$$X * A = [X, A] + X'$$
i.e.
$$\left[X, \frac{d}{dt} - A\right] = -[X, A] - X' .$$
Note that $X * A$ belongs to the tangent space to $\mathcal{X}$ at A, but this can be identified with $\mathfrak{k}$, because $\mathcal{X}$ is an affine space with group $\mathfrak{k}$.

For the symplectic structure, recall what happens in the Kirillov situation. Then $\mathcal{X} = k^*$ and on an orbit one define f_X by
$$f_X(\lambda) = \langle \lambda, X \rangle$$
where $\langle\ ,\ \rangle =$ pairing $k^* \times k \to \mathbb{R}$. Then
$$\begin{aligned}(i(Y)\,df_X)(\lambda) &= (\theta(Y) f_X)(\lambda) = -f_X(\theta(Y)\lambda) \\ &= -\langle \theta(Y)\lambda, X\rangle = \langle \lambda, \theta(Y) X\rangle \\ &= \langle \lambda, [Y, X]\rangle \overset{\text{definition of } \Omega}{=} (i(Y) i(X)\Omega)(\lambda)\end{aligned}$$

which yields the key formula

$$df_X = i(X)\,\Omega$$

saying that X is the Hamiltonian vector field on the orbit belonging to the function f_X.

So on $\mathcal{K}$ = maps $X: S^1 \to k$ we pick the inner product

$$\langle X, Y \rangle = \int (X, Y)\,dt$$

where $(\ ,\ )$ is an invariant inner product on k (e.g. $tr(X^*Y)$ in the case of U_n.) Now define f_X on $\mathcal{X}$ by the formula

$$f_X(A) = \langle A, X \rangle$$

where we have identified $\mathcal{K}$ and $\mathcal{X}$ by using $0 \in \mathcal{X}$ as origin (another origin changes f_X by a constant which doesn't affect df_X). Thus

$$\begin{aligned}
\big(i(Y)\,df_X\big)(A) &= -f_X(\theta(Y)A) \\
&= -\langle Y * A, X \rangle \\
&= -\langle [Y, A] + Y', X \rangle \\
&= \langle A, [Y, X] \rangle \underbrace{- \langle Y', X \rangle}_{-\int (Y', X)\,dt}
\end{aligned}$$

This is clearly skew-symmetric, and earlier calculations show that at least for $A = 0$, it is non-degenerate. So we should get a symplectic structure on at least the orbit of 0 which is $\mathcal{K}/K \cong \Omega(K)$, if not all orbits.

So let's now look at the moment map. One takes the functions f_X where X runs over $\mathfrak{t} = \mathrm{Lie}(T)$, viewed as constant loops. Then $\{f_X(A)$[redacted]$, X \in \mathfrak{t}\}$ is essentially the projection of $A \in \mathcal{K}$ onto $\mathfrak{t}$. Recall that

$$\mathcal{K} = \text{"real" subspace of } \underbrace{\mathfrak{g}[z, z^{-1}]}$$

[redacted]

$$z^{-1}\mathfrak{g}[z^{-1}] \oplus \overbrace{\mathfrak{g}}^{} \oplus z\,\mathfrak{g}[z]$$
$$\mathfrak{n}^* \oplus \mathfrak{h} \oplus \mathfrak{n}$$

so this orthogonal projection just takes the component in $\mathfrak{h}$. So the projection of [redacted] $\mathcal{K}$ onto $\mathfrak{t}$ takes [redacted] A and integrates it over S^1 to land in $\mathfrak{k}$, then you project onto $\mathfrak{t}$. Recall that

$$A = h' h^{-1}$$

so if h lies in T, better $h(t) \in T$ for all t, then our projection is just

$$\int_0^1 A\,dt = \int_0^1 d\log h = \log h(1).$$

calculated using $h(t)$.

Recall that a $\mathcal{K}$-orbit $\mathcal{O}$ on X can be thought of as all paths h joining 1 to a given conjugacy class. The fixpts $\mathcal{O}^T$ are paths lying in T ending in a Weyl orbit on T. Therefore the moment map[redacted] on $\mathcal{O}^T$ looks at the endpoint of the path lifted into $\mathfrak{t}$, and so you get an extended Weyl group orbit. The convex hull of this should be all of $\mathfrak{t}$.

September 4, 1981

New idea: There is an additional symmetry on the set $\mathcal{X}$ of connections which is furnished by translation on the group S^1. Recall that I am thinking of connections on the trivial principal bundle $S^1 \times K \longrightarrow K$. I like to think of $K = SU_n$ so that a connection is an operator on sections of the trivial vector bundle $S^1 \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$; the operator has the form

$$D = \partial - A \qquad \text{where } \partial = \frac{d}{dt} \text{ and } A : S^1 \longrightarrow \mathfrak{k}.$$

(Actually the whole group of diffeos of S^1 acts on $\mathcal{X}$).

What are the orbits on $\mathcal{X}$ for this larger group of symmetries $S^1 \tilde{\times} \mathcal{K}$? Recall that $\mathcal{K}$ orbits on $\mathcal{X}$ are described by conjugacy classes in K. One takes a connection and integrates it to get a path $h : [0,1] \longrightarrow K$ with $h(0) = 1$, $h'h^{-1} = A$. Then the $\mathcal{K}$-orbit of A is described by the conjugacy class of $h(1)$, because we know $\varphi * h$ is $\varphi(t)\, h(t)\, \varphi(1)^{-1}$, which has the endpoint $\varphi(1)\, h(1)\, \varphi(1)^{-1}$. Now we want to see if translation of A leads to the same or different conjugacy class. So consider $\tilde{A}(t) = A(t + t_0)$. Then

$$\tilde{h}(t) = h(t+t_0)\, h(t_0)^{-1}$$

since $\tilde{h}'\tilde{h}^{-1} = h'(t+t_0)\cancel{h(t_0)^{-1} h(t_0)}\, h(t+t_0)^{-1} = A(t+t_0) = \tilde{A}(t)$.

Thus the endpoint is

$$\tilde{h}(1) = h(1+t_0)\, h(t_0)^{-1}$$

But we know $h(1+t) = h(t)\, h(1)$ (take $t_0 = 1$ above then $\tilde{h}(t) = h(t) = h(t+1)\, h(1)^{-1}$). So

$$\tilde{h}(1) = h(t_0)\, h(1)\, h(t_0)^{-1}$$

and we conclude that the orbits of $S^1 \tilde{\times} \mathcal{K}$ on $\mathcal{X}$

coincide with the $\mathcal{K}$ orbits on $\mathcal{X}$.

Another proof is to take A and transform it under $\mathcal{K}$ to a constant A_0 which is then invariant under S^1-translation. The $S^1 \tilde{\times} \mathcal{K}$ orbits of A, A_0 coincide, and the latter coincides with the $\mathcal{K}$-orbit of A_0.

Recall how the symplectic structure on an orbit is defined. $\mathcal{X}$ is an affine space with linear space $Lie(\mathcal{K})$ so we can identify the two using $A=0$ as origin. Choose an invariant inner product [redacted] $(\, , \,)$ on $\mathbb{k}$ and define the functions on $\mathcal{X}$

$$f_X(A) = \int (X, A)\, dt \qquad f(A) = \tfrac{1}{2}\int (A, A)\, dt$$

Then

$$\begin{aligned}(Yf_X)(A) &= \int (X, Y * A)\, dt \\ &= \int (X, Y' + [Y, A])\, dt \\ &= \int (X, Y') + ([X, Y], A)\, dt\end{aligned}$$

and at least for [redacted] K abelian, this gives a[redacted] non-degenerate skew-symmetric product on any orbit. Here's the proof of non-degeneracy in general. Suppose $(Yf_X)(A) = 0$ for all X. Then from

$$\int (X, Y' + [Y, A])\, dt = 0$$

and the [redacted] fact that $(\, , \,)$ is an inner product we conclude that

$$Y' + [Y, A] = 0$$

i.e. that Y leaves A fixed.

Next

$$\begin{aligned}(Yf)(A) &= \int (A, Y * A)\, dt \\ &= \int (A, Y' + [A, Y])\, dt = -\int (A', Y)\, dt\end{aligned}$$

Thus $(Yf)(A) =$ symplectic form applied to the tangent vectors $Y*A$ and $A*A = A'$ at A.

September 5, 1981:

Here is a curious sign problem. Let a Lie group G act on a manifold M. Then the group acts on the functions on M by

$$(g*f)(m) = f(g^{-1}m)$$

hence the Lie algebra $\mathfrak{g}$ of G acts on functions by

$$(X*f)(m) = \lim_{\varepsilon\to 0} \frac{(e^{\varepsilon X}*f)(m) - f(m)}{\varepsilon}$$

But

$$\begin{aligned}(e^{\varepsilon X}*f)(m) &= f(e^{-\varepsilon X}m) \\ &= f(m + (-\varepsilon)Xm + \frac{(-\varepsilon X)^2}{2!}m + \cdots) \\ &= f(m) - \varepsilon\, df_m(X)\end{aligned}$$

and so

$$(X*f)(m) = -df_m(X)$$

or

$$X*f = -i(X)df = -Xf$$

This is not the expected formula, but I think it is forced if you want the formula

$$X*(Y*f) - Y*(X*f) = [X,Y]*f$$

Take the example where $G = Gl_n(\mathbb{R})$ acting on $\mathbb{R}^n$, and let f be a linear function on $\mathbb{R}^n$. The Lie algebra of G consists of all endos. X of $\mathbb{R}^n$, and the assoc. 1-par. gp is e^{tX}. Thus the vector field on $\mathbb{R}^n$ belonging to $X \in \mathfrak{gl}_n$ assigns to v the tangent vector Xv:

$$e^{tX}v = v + tXv + \frac{t^2X^2}{2!}v + \cdots$$

Next compute $Xf = i(X)df$ if f is linear:

$$(Xf)(v) = \frac{f(e^{tX}v) - f(v)}{t}\Big|_{t\to 0} = f(Xv)$$

and so

$$\begin{aligned}(XYf - YXf)(v) &= Yf(Xv) - Xf(Yv) \\ &= f((YX - XY)v) \\ &= -([X,Y]f)(v).\end{aligned}$$

This example shows that if you define Xf in the standard geometric way (rate of change of f in the direction given by X), then this is not an action of the Lie algebra of vector fields on functions, when the bracket of vector fields is defined by

$$[X,Y] = \frac{d}{dt}\, e^{tX}\, Y e^{-tX}\Big|_{t=0}$$

So for example take $X = \frac{\partial}{\partial x}$ $Y = x\frac{\partial}{\partial y}$. Then e^{tX} sends (x,y) to $(x+t, y)$. So $e^{tX}Ye^{-tX}$ at (x,y) should $(x-t)\frac{\partial}{\partial y}$, and so by the above formula the bracket should be

$$\frac{d}{dt}(x-t)\frac{\partial}{\partial y}\Big|_{t=0} = -\frac{\partial}{\partial y}.$$

But $$\left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right] = \frac{\partial}{\partial y} \quad \text{as operators.}$$

<u>September 6, 1981</u>

Review: $\mathcal{X}$ = space of connections on trivial bundle $S^1 \times K \longrightarrow S^1$, $\mathcal{K}$ = maps $S^1 \longrightarrow K$ = gauge group, $\mathfrak{k}$ = maps $X: S^1 \longrightarrow \mathbb{k}$ = gauge Lie algebra. We think of a connection D as differing from the flat connection $\partial = \frac{d}{dt}$ by an element A of $\mathfrak{k}$:

$$D = \partial - A.$$

Thus the action of $\varphi \in \mathcal{K}$ on $\mathcal{X}$ is

$$\begin{aligned} \varphi D \varphi^{-1} &= \partial + \varphi(\varphi^{-1})' - \varphi A \varphi^{-1} \\ &= \partial - \{\varphi'\varphi^{-1} + \varphi A \varphi^{-1}\} \end{aligned}$$

in other notation

$$\varphi * A = \varphi'\varphi^{-1} + \varphi A \varphi^{-1}.$$

If $X \in \mathfrak{k}$, this becomes

$$X * A = X' + [X, A].$$

Now given an [redacted] X one defines a function on $\mathcal{X}$ by

$$f_X(A) = \int (X, A)\, dt$$

where $(\ ,\ )$ is an invariant inner product on $\mathbb{k}$. Then if $Y \in \mathfrak{k}$, because f_X is linear

$$\begin{aligned} (Y f_X)(A) &= \int (X, Y * A)\, dt \\ &= \int \Big[(X, Y') + \underbrace{(X, [Y, A])}_{([X,Y], A)} \Big]\, dt \end{aligned}$$

and it's clear this is a skew-symmetric bilinear form on $\mathfrak{k}$, invariant under the action of $\mathcal{K}$. If Y, A are such that that this form vanishes for all X, then $Y * A = 0$, which means that the restriction of this two-form to $\mathcal{K}$-orbits on $\mathcal{X}$ is non-degenerate. Call this form Ω so that

$$Y f_X = i(Y) i(X) \Omega \qquad \text{or} \qquad d f_X = i(X) \Omega$$

As in the Kirillov situation we get an invariant symplectic structure on each orbit of $\mathcal{K}$ on $\mathcal{X}$ such that f_X generates the field X.

Furthermore the function

$$f(A) = \frac{1}{2}\int (A, A)\,dt$$

satisfies

$$(Yf)(A) = \int (A, Y * A)\,dt$$

$$= -\int (\underbrace{A * A}_{A'}, Y)\,dt\,.$$

This shows that f generates the vector field on the orbit associating to A the vector A'. This is the vector field resulting from the translation action of S^1 on $\mathcal{X}$.

Fix a max. torus T in K. Then inside the group $S^1 \tilde{\times} \mathcal{K}$ is the torus $S^1 \times T$, where we think of T as constant gauge transformations. The fixpts for the translation S^1 are the constant A, hence

$$\mathcal{X}^{S^1} = \mathfrak{k}$$

$$\mathcal{X}^{S^1 \times T} = \mathfrak{t}\,.$$

Now we recall that a $\mathcal{K}$ orbit in $\mathcal{X}$ can be identified with a conjugacy class in K, namely, given A you integrate

$$h(t) = T\{e^{\int_0^t A}\}$$

and then take the conjugacy class of $h(1)$. So if we have an orbit $\mathcal{O}$, then $\mathcal{O}^{S^1 \times T}$ will consist of $A \in \mathfrak{t}$ such that $e^A \in$ given conjugacy class corresp. to $\mathcal{O}$.

The moment map is determined by f and f_X where X ranges over a basis of $\mathfrak{t}$. Since

X is constant we have

$$f_X(A) = \int (X, A)\,dt = \left(X, \int A\,dt\right)$$

so that the part of the moment map belonging to the f_X, $X \in \mathfrak{t}$ can be identified with $A \mapsto \int A\,dt$

In fact we see that the moment map on all of $\mathcal{X}$ can be identified with

$$A \longmapsto \left(\frac{1}{2}\int_0^1 (A,A)\,dt,\ pr^{k}_{t} \int_0^1 A\,dt\right) \in \mathbb{R} \times \mathfrak{t}$$

Let's look at this in the case of $K = S^1$.

The orbit space $\mathcal{K}\backslash\mathcal{X}$ can be identified with S^1 via the map

$$A \longmapsto e^{\int_0^1 A\,dt}$$

so fixing an orbit amounts to looking at all $A(t)$ such that

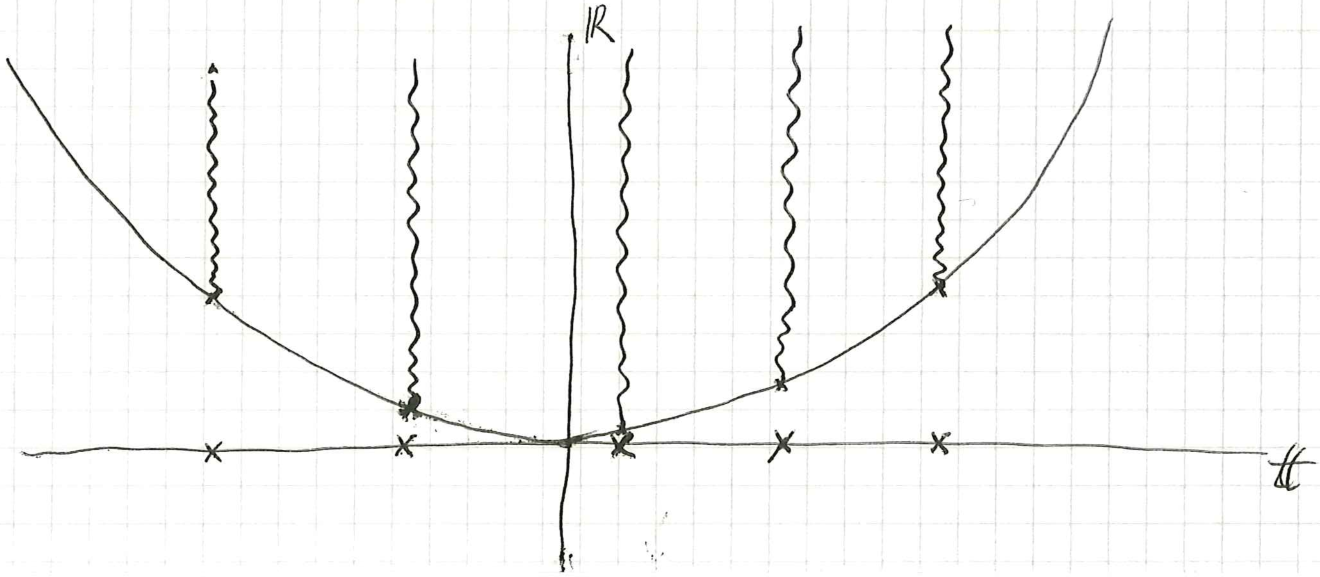
$$e^{\int_0^1 A\,dt} = \text{a fixed number } \zeta \in S^1.$$

Since $k = \mathfrak{t}$ in this example, the moment map is

$$\mathcal{X} \longrightarrow \mathbb{R} \times \mathfrak{t}$$

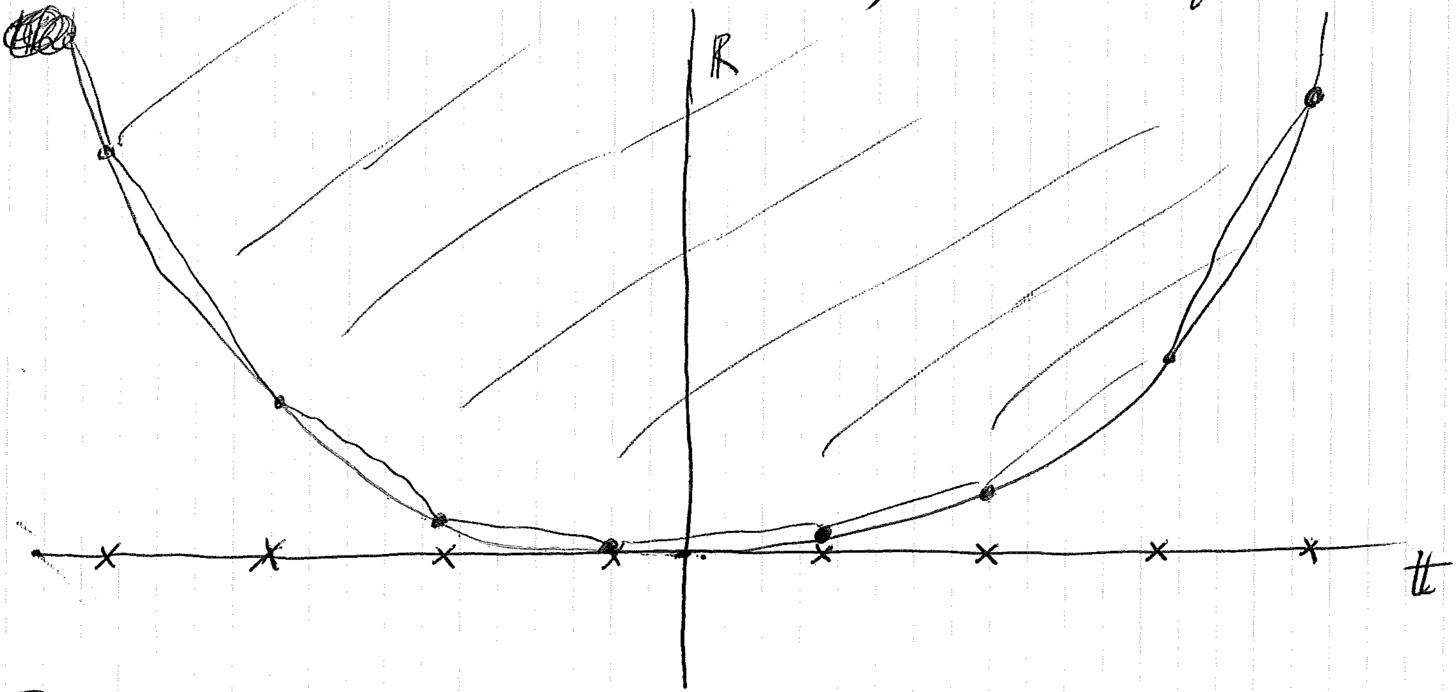
$$A \longmapsto \left(\frac{1}{2}\int |A|^2 dt,\ \int_0^1 A\,dt\right)$$

and the possible values for $\int_0^1 A\,dt$ on an orbit $\mathcal{O}$ are a coset of $2\pi i\mathbb{Z}$ in $i\mathbb{R} = \mathfrak{t}$. So the image of the moment map on $\mathcal{O}$ seems to look like



This image is clearly not convex, so it seems that the Atiyah conclusion breaks down in this example. The reason probably is that the orbits are not connected. In this case the orbits are $\cong K/S^1 = \Omega(S^1)$ which is disconnected.

The picture in the case of simply-connected K, where the orbits are connected, looks as follows.



The image of the moment map is the convex hull of the points on the paraboloid $y = \frac{1}{2}|x|^2$ lying over an extended (affine) Weyl group orbit.

September 6, 1981

The next project is to work out the representation theory of $\mathcal{K}$. The idea is that to a character of T one can associate a holomorphic bundle over $\mathcal{K}/T$ whose sections give interesting representations. Above all one wants a character formula giving the character of the representation restricted to T. In the case of $K/T = G/B$ this (Weyl) character formula can be derived from the holomorphic Lefschetz fixpoint formula, so let's review the relevant examples.

Example 1 let θ be an autom. of a complex vector space V with distinct eigenvalues $\alpha_1, \dots, \alpha_r$; $\dim V = r$. Then θ acts on $\mathbb{P}(V^\vee)$ and on the line bundle $\mathcal{O}(1)$. In effect we have the map

$$\mathbb{P}(V^\vee) \xrightarrow{\ \tau\ } \mathbb{P}(V^\vee)$$

$$\underset{\text{hyperplane}}{H} \longmapsto \theta^{-1}H$$

and a map of line bundles $\tau^*\mathcal{O}(1) \longrightarrow \mathcal{O}(1)$ which over H is

$$\begin{array}{ccc}
\underbrace{\tau^*\mathcal{O}(1)\,(H)} & \longrightarrow & \underbrace{\mathcal{O}(1)(H)} \\
\mathcal{O}(1)(\tau H) & & \| \\
\| & & \\
V/\theta^{-1}H & \xrightarrow{\ \theta\ } & V/H
\end{array}$$

The θ fixpoints are the hyperplanes

$$H_i = \bigoplus_{j \neq i} W_{\alpha_j}$$

where W_{α_i} is the eigenspace of θ belonging to α_i. What we want to do I think is to compute the trace of τ on the stalk $\mathcal{O}(n)$ at H_i formally, then the sum of these traces should be the trace on the cohomology $H^\cdot(\mathbb{P}(V^\vee), \mathcal{O}(n))$.

$$\mathcal{O}(1)(H_i) = V/H_i \cong W_{\alpha_i} \; ; \text{ here } \tau = \alpha_i$$

$$gr\{\mathcal{O}(n)_{H_i}\} = \mathrm{Sym}(m_{H_i}/m_{H_i}^2) \otimes \mathcal{O}(1)(H_i)^{\otimes n}$$

The tangent space to $\mathbb{P}(V^\vee)$ at H_i is

$$\mathrm{Hom}(\text{sub}, \text{quot.}) \cong \mathrm{Hom}(H_i, W_{\alpha_i})$$

so the cotangent space is

$$\mathrm{Hom}(W_{\alpha_i}, H_i) \; ; \text{ here } \tau \text{ has eigenvalues } \alpha_j \alpha_i^{-1} \quad j \neq i$$

Thus

$$\text{trace on } \mathcal{O}(n)_{H_i} \overset{\text{formally}}{=\!=} \frac{\alpha_i^n}{\prod_{j\neq i}(1-\alpha_j\alpha_i^{-1})}$$

and so the fixpoint formula should be

$$\text{trace on } H^\cdot(\mathbb{P}(V^\vee), \mathcal{O}(n)) = \sum_{i=1}^{r} \frac{\alpha_i^n}{\prod_{j\neq i}(1-\alpha_j\alpha_i^{-1})}$$

We know the cohomology is zero for $-r < n < 0$, so we get the formula

$$(*) \qquad \sum_{i=1}^{r} \frac{\alpha_i^m}{\prod_{j\neq i}(\alpha_i-\alpha_j)} = \begin{cases} 1 & m = r-1 \\ 0 & 0 \leq m < r-1 \end{cases}$$

Lagrange's formula is

$$\frac{X^m}{\prod_{i=1}^{r}(X-\alpha_i)} = \sum_i \frac{1}{X-\alpha_i} \frac{\alpha_i^m}{\prod_{j\neq i}(\alpha_i-\alpha_j)}$$

for $0 \leq m \leq r-1$, and this implies $(*)$ by letting $X \to \infty$.

Example 2: Consider the holomorphic line bundle $\mathcal{L}$ over $G/B = K/T$ associated to a character [redacted] of T, and take a generic element of T so that the fixpts on K/T are a Weyl group orbit. Let's compute the trace on the stalk of the line bundle over the identity coset. The tangent space is (as a repn. of T)

$$\mathfrak{k}/\mathfrak{t} = \sum_{\alpha>0} \mathfrak{g}^{\alpha}$$

so the cotangent space is $\sum_{\alpha>0} \mathfrak{g}^{-\alpha}$, hence the formal trace on the stalk is

$e^{i\lambda} : T \to \mathbb{C}^*$ orig. character

$$\frac{e^{i\lambda}}{\prod_{\alpha>0}(1-e^{-i\alpha})}$$

which we can write $\dfrac{e^{i(\lambda+\rho)}}{\prod_{\alpha>0}\left(e^{i\frac{\alpha}{2}}-e^{-i\frac{\alpha}{2}}\right)}$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$. Then to get the other points you act on this with the Weyl group. [redacted] The denominator is anti-symmetric, so one gets

$$\text{trace on } H^*(G/B, \mathcal{L}_\lambda) = \frac{\sum_{\sigma\in W}(-1)^{\sigma} e^{i\sigma(\lambda+\rho)}}{\prod_{\alpha>0}\left(e^{i\alpha/2}-e^{-i\alpha/2}\right)}.$$

If one [redacted] now uses vanishing results about the cohomology, e.g. for λ in the [redacted] fundamental chamber only H^0 is $\neq 0$, then one gets the Weyl character formula. Notice that taking $\lambda=0$ gives

$$\sum_{\sigma\in W}(-1)^{\sigma} e^{i\,[\text{redacted}]\,\sigma(\rho)} = \prod_{\alpha>0}\left(e^{i\alpha/2}-e^{-i\alpha/2}\right)$$

<u>September 9, 1981.</u> (illness the past few days + Carl starts school)

The project [redacted] now is to understand the repn. theory that goes with $\mathcal{K}$, and that would involve the Iwahori subgroup of [redacted] $\mathcal{G}$ analogously to the Borel B in G. The critical idea perhaps goes as follows. Take $K = S^1$. For $\mathcal{K}$ we take the "minimal" form of all Laurent polynomial loops. Thus

$$\mathcal{K} \cong \mathbb{Z} \times S^1 = \{ z^n \zeta \mid n \in \mathbb{Z}, \zeta \in S^1 \}$$

where I try the following notation. ζ is an arbitrary point of S^1, z is the function on S^1 sending ζ to ζ. Thus $z^n \zeta$ sends a point α to $\alpha^n \zeta$.

Now $\mathbb{Z} \times S^1$ is abelian and so its irreducible representations are just characters. But now bring in the translation S^1 acting on $\mathcal{K}$. In general let $\alpha \in S^1$ act on $f : S^1 \longrightarrow K$ by

$$(\alpha f)(\zeta) = f(\alpha \zeta).$$

Then the infinitesimal element $1 + i\varepsilon$ in S^1 acts as

$$\big((1+i\varepsilon) f\big)(\zeta) = f(\zeta + i\varepsilon\zeta) = f(\zeta) + i\zeta f'(\zeta)\varepsilon$$

and so the infinitesimal generator of S^1 acts as

$$i z \frac{d}{dz} = \frac{d}{d\theta} \qquad \begin{aligned} z &= e^{i\theta} \\ dz &= iz\, d\theta . \end{aligned}$$

on functions on the circle.

I believe that we want to look at representations of the group $S^1 \ltimes \mathcal{K} = S^1 \ltimes (\mathbb{Z} \times S^1)$. Let τ_α denote translation by α. Then

$$\tau_\alpha (z^n \zeta) \tau_\alpha^{-1} = (\alpha z)^n \zeta = \alpha^n z^n \zeta$$

and so the constant maps $\{\zeta\}$ are in the center. Thus we have a central extension

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^1 \ltimes (\mathbb{Z} \times S^1) & \longrightarrow & S^1 \times \mathbb{Z} \\ \{\zeta\} & & \{\tau_\alpha z^n \zeta\} & & \{\tau_\alpha z^n\} \end{array}$$

and it would be nice if [redacted] this were a Heisenberg group.

Recall that given an abelian group (locally compact) A one can form a canonical central extension of S^1 by $A \times \check{A}$ called the Heisenberg group which is faithfully represented on [redacted] $L^2(A)$ or $L^2(A^\vee)$. Hence we might be able to see a nice representation of $S^1 \ltimes (\mathbb{Z} \times S^1)$ on $L^2(\mathbb{Z})$. $L^2(\mathbb{Z})$ is square integrable Laurent series $\sum_{n \in \mathbb{Z}} a_n z^n$. Define

$$\tau_\alpha z^n \zeta \text{ acting on } \sum a_m z^m = \sum a_m (\alpha z)^{n+m} \zeta$$

in other words

$$\tau_\alpha * z^m = \alpha^m z^m$$
$$z^n * z^m = z^{n+m}$$

Then

$$\underbrace{\left(\tau_\alpha * \underbrace{\left(z^n * \underbrace{\left(\tau_\alpha^{-1} * z^m\right)}_{\alpha^{-m} z^m}\right)}_{\alpha^{-m} z^{m+n}}\right)}_{\alpha^{-m}\alpha^{m+n} z^{m+n}} = \alpha^n z^{m+n}$$

$$\underbrace{\left(\tau_\alpha z^n \tau_\alpha^{-1}\right)}_{\alpha^n z^n} * z^m = \alpha^n z^{m+n}$$

so it works.

Let's recall that for A finite of order n, the irreducible repns of the Heisenberg group of A are calculated as follows. First simplify by supposing A of exponent p, and let's work with the extension having ~~the~~ center μ_p. Then A has order $n = p^d$ say. One has characters of $A \times A^\vee$ and

then are irreducible repns for each embedding $\mu_p \hookrightarrow S^1$.
So
$$p^d p^d + (p-1)(p^d)^2 = p(p^d)^2 = p^{2d+1}$$
which is the order of the Heisenberg group.

Next consider A of exponent p and the H-group $\mu_q \tilde{\tilde{\times}} (A\times\check{A})$ where $p \mid q$. Then for each character $\mu_q \to S^1$ we look at its restriction to μ_p.

$$0 \to \mathrm{Hom}(\mu_q/\mu_p, S^1) \to \mathrm{Hom}(\mu_q, S^1) \longrightarrow \mathrm{Hom}(\mu_p, S^1) \to 0$$

$\therefore$ Each character $\mu_p \to S^1$ occurs q/p times, so [redacted] one counts a given [redacted] $\mu_p \tilde{\tilde{\times}} (A\times A^\vee)$ repn. q/p times. So one gets

$$q/p \cdot \underbrace{p^{2d}}_{\substack{\text{char}\\ \text{of } A\times\check{A}}} + q/p(p-1)(p^d)^2 = (q/p)\cdot p \cdot p^{2d} = \underbrace{q p^{2d}}_{\text{ord } \mu_q \tilde{\tilde{\times}}(A\times\check{A})}$$

[redacted]

In general given a semi-direct product $H \rtimes A$ with H, A abelian, irreducible reps are by Mackey given by A-orbits in $\check{H}$ and characters on the stabilizer. Thus if $\check{H} = \coprod \mathcal{O}_i$, $\mathcal{O}_i \cong A/B_i$ then for each character of B_i we can [redacted] induce from $H\times B_i$ to get an irred. repn of dim $= |A/B_i|$. So the sum of squares of dimensions is

$$\sum_i |A/B_i|^2 \cdot |B_i| = \sum_i |A||\mathcal{O}_i| = |A||H| = |H\rtimes A|.$$

So if we want to understand the irreducible repns. of $\underset{\check{A}}{S^1} \ltimes (\underset{\ddot{A}}{\mathbb{Z}} \times S^1) = $ [redacted] $\underset{A}{\mathbb{Z}} \ltimes (\underset{\check{A}}{S^1}\times S^1)$, we need the [redacted] action of A on $(\check{A}\times S^1)^\vee = A \times \mathbb{Z}$. In $A \ltimes (\check{A}\times S^1)$ we have elements $a\lambda\zeta$ with $a\lambda a^{-1} = \lambda\cdot\lambda(a)$, e.g. $\tau_\alpha z^n \tau_\alpha^{-1} = z^n \alpha^n$.
So if $\chi : A^\vee \times S^1 \to S^1$ is given by $\chi(\lambda\zeta) = \lambda(a_0)\zeta^p$ we

have $\quad \chi(a^{-1}\lambda\zeta a) = \chi(\lambda\,\lambda(a)^{-1}\zeta) = \lambda(a_0)\left(\lambda(a)^{-1}\right)^p\zeta^p$

Thus the effect of $a \in A$ on the character $\chi = (a_0, p)$ is the character $(a_0 - pa, p)$. Thus the action of A on $(A^\vee \times S^1)^\vee = A \times \mathbb{Z}$ is

$$a * \begin{pmatrix} a_0 \\ p \end{pmatrix} = \begin{pmatrix} a_0 - pa \\ p \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_0 \\ p \end{pmatrix}$$

In the case of $A = \mathbb{Z}$ we have $\mathbb{Z}$ acting on $\mathbb{Z}^2$ by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, so what are the orbits?

The action is generated by $\begin{pmatrix} a \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a-p \\ p \end{pmatrix}$

and so for each $p \neq 0$ there are $|p|$ different orbits, one for each coset mod p. There are infinitely many orbits if $p = 0$. In general for each p, the orbits are the cosets mod p.

September 10, 1981

Description of irreducible repns. of $S^1 \ltimes (\mathbb{Z} \times S^1)$
$= \{ \tau_\alpha z^n \xi \mid \alpha, \xi \in S^1, n \in \mathbb{Z} \}$ where

$$\tau_\alpha z^n \tau_\alpha^{-1} = z^n \alpha^n .$$

Use Mackey's description for semi-direct products: One begins with a character χ of $\mathbb{Z} \times S^1$. Such a character is of the form.

$$\chi_{\eta,p}(z^n \xi) = \eta^n \xi^p$$

Then

$$\begin{aligned} \chi_{\eta,p}(\tau_\alpha z^n \xi \tau_\alpha^{-1}) &= \chi_{\eta,p}(z^n \alpha^n \xi) \\ &= \eta^n (\alpha^n \xi)^p = (\eta \alpha^p)^n \xi^p \\ &= \chi_{\eta\alpha^p, p}(z^n \xi) \end{aligned}$$

Thus $S^1 = \{\tau_\alpha\}$ acts on $(\mathbb{Z} \times S^1)^\vee$ by

$$\alpha * \chi_{\eta p} = \chi_{\eta \alpha^p, p}$$

and so for $p \neq 0$, where $\alpha \mapsto \alpha^p$ from S^1 to S^1 is surjective we have a single orbit. The stabilizer of $\chi_{1,p}$ is μ_p so to this orbit belong p different irreducible repns. To obtain them one starts with a character on $\mu_p \ltimes \mathbb{Z} \times S^1$ of the form $\chi' \otimes \chi_{1,p}$ where $\chi' : \mu_p \to S^1$ and then induces up to $S^1 \ltimes (\mathbb{Z} \times S^1)$.

If $p = 1$ we then get the natural action of this group on $\mathbb{C}[z, z^{-1}]$ with

$$\begin{aligned} \tau_\alpha * z^m &= \alpha^m z^m \\ z^n * z^m &= z^{n+m} \\ \xi * z^m &= \xi z^m \end{aligned}$$

This is a repn. because

$$\begin{aligned} \tau_\alpha * (z^n * (\tau_\alpha^{-1} * z^m))) &= \tau_\alpha * (\alpha^{-m} z^{n+m}) \\ &= \alpha^{-m} \alpha^{n+m} z^{n+m} \\ (z^n \alpha^n) * z^m &= \alpha^n z^{n+m} \end{aligned}$$

This repn starts with the basic character on the subgroup of $\{\tau_\alpha \zeta\}$ as $S^1 \times S^1$ given by

$$(\tau_\alpha \zeta) * z^0 = \zeta z^0$$

and then this line is moved around by $Z = \{z^n\}$.

In general given $p \neq 0$ we have the characters

$$(\tau_\alpha \zeta) * e_m = \alpha^m \zeta^p e_m$$

where m lies in a coset $i + p\mathbb{Z}$ mod p. These are moved around by $\{z^n\}$ as follows:

$$z^n * e_m = e_{m+pn}$$

This is a repn. because

$$\begin{aligned} \tau_\alpha^*\left(z^n * \left(\tau_\alpha^{-1} * e_m\right)\right) &= \tau_\alpha^*\left(z^n * \left(\alpha^{-m} e_m\right)\right) \\ &= \tau_\alpha^* * \left(\alpha^{-m} e_{m+np}\right) \\ &= \alpha^{-m} \alpha^{m+np} e_{m+np} = \alpha^{np} e_{m+np} \\ z^n \alpha^n * e_m &= \alpha^{np} e_{m+np} \end{aligned}$$

According to the Mackey theory this should be a complete description of the irreducible repns, ~~because~~ for each p we do have (p) cosets. It doesn't work for $p=0$, because then there would be a single e_m and we should have a character for the action of $\mathbb{Z}$.

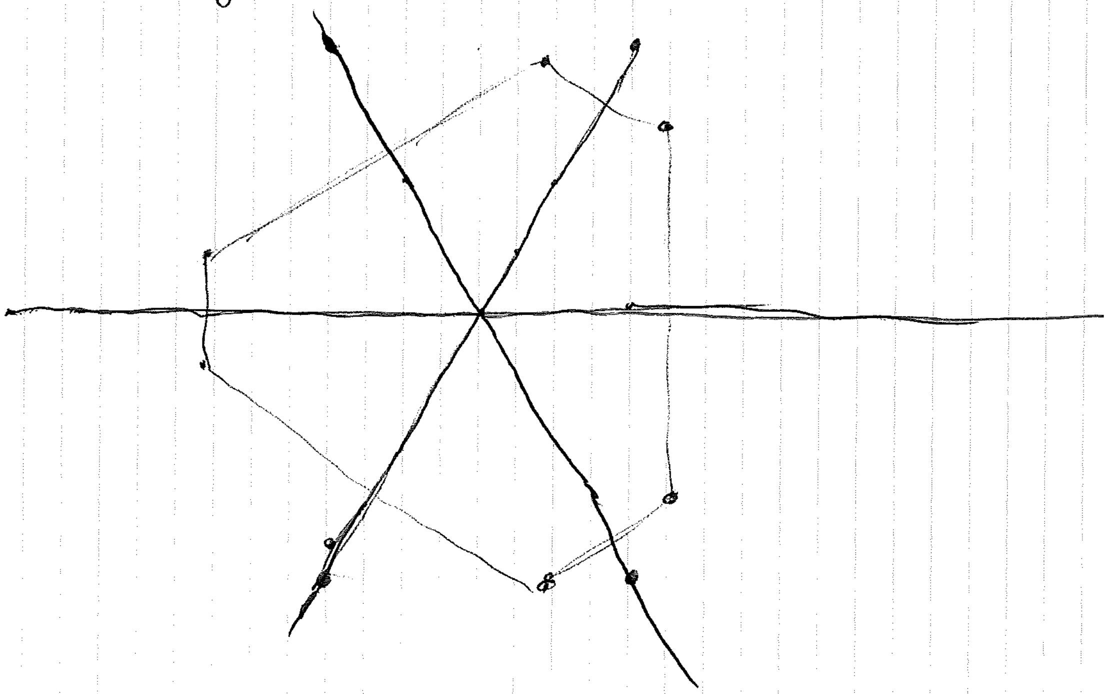
For later we will the situation for $S^1 \ltimes (\Gamma \times T)$ where T is a torus with $\pi_1(T) = \Gamma$, so that $T = \Gamma \otimes S^1$. Thus $\Gamma = \mathbb{Z}^k$, $T = (S^1)^k$ can be assumed. Then the above formulas

$$(\tau_\alpha \zeta) * e_m = \alpha^m \zeta^p e_m$$

$$z^n * e_m = e_{m+pn}$$

make sense provided $\zeta = (\zeta_1, \ldots, \zeta_k)$ $p = (p_1, \ldots, p_k)$ $z^n = (z_1^{n_1}, \ldots, z_k^{n_k})$. However we don't get all irreducibles maybe because the stabilizer of [redacted] e_m, i.e. those z^n for which [redacted] $pn =$ [redacted] 0 should act via a possibly non-trivial character. I'll worry about this later.

Here's the idea: [redacted] In the case of K/T if we take an irreducible repn V of K and look at its weights, that is, the characters of T that occur in V, then we get a convex piece of lattice points in $\mathfrak{t}$.



There should be some way ([redacted] at least heuristic) to deduce the Kostant thm about projection onto $\mathfrak{t}$ of a K orbit in $\mathfrak{k}$ being convex from this [redacted] representation fact. The repn. consists of a quantization of the orbit, so that the T characters which occur somehow should correspond in a sly way to projections of points of the orbit on $\mathfrak{t}$.

So the analogue for $\mathcal{K}$ is that if we look at the characters of the torus $S^1 \times T$ in a representation of $S^1 \ltimes \mathcal{K}$, then we should perhaps get integral points in an Atiyah type convex set.

Take $K = SU_n$ ($n=2$ eventually). Then we an obvious representation of $S^1 \times K$ on $L^2(S^1)^n$ which has the orthonormal basis $z^n e_i$, $i = 1, \dots, n$ $n \in \mathbb{Z}$. These are eigenvectors for $S^1 \times T$ with characters

$$\tau_\alpha \begin{pmatrix} \zeta_1 & & \\ & \ddots & \\ & & \zeta_n \end{pmatrix} \longmapsto \alpha^n \zeta_i$$

So for $n=2$ we get the characters

$$\tau_\alpha \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix} \longmapsto \alpha^n \zeta^{\pm 1}$$

Somehow this isn't the correct sort of answer so perhaps we have a <u>different type</u> of representation

This time let's take $\mathcal{K}$ to be all smooth maps from S^1 to S^1 and ask about representations of $\mathcal{K}$. First of all what are the characters of $\mathcal{K}$? There is obviously evaluation at any point and similarly to any divisor on S^1 is associated a character: If $D = \sum n_i \alpha_i$, then the character is

$$\varphi \longmapsto \prod \varphi(\alpha_i)^{n_i}$$

Since

$$\mathcal{K} \cong \Omega^1(S^1) \times S^1$$

and

$$\pi_0(\Omega^1 S^1) = \pi_1(S^1) = \mathbb{Z}$$

we also have a homomorphism

$$\mathcal{K} \longrightarrow \mathbb{Z}.$$

This is just the degree and is given by

$$\deg(\varphi) = \frac{1}{2\pi i} \int_{S^1} \frac{d\varphi}{\varphi}$$

Thus we get characters by following the degree by a map $\mathbb{Z} \longrightarrow S^1$. ~~The character given by a divisor is homotopy to the character $\varphi \longmapsto \varphi(1)$ which has degree~~.

This business of characters + divisors reminds me of the classical gas. Recall the configuration space of a gas on the space X is the free abelian monoid of positive divisors on X:

$$\coprod_{n\geq 0} SP_n(X) = \coprod_n \Sigma_n \backslash X^n$$

and that the grand canonical partition function is

$$\sum_n \frac{1}{n!}\int dx_1\cdots dx_n \prod_{i=1}^n z(x_i)\, e^{-\beta U_n(x_1,\dots,x_n)}$$

where $z(x)$ is a variable "activity". One way to write this is to think of $z: X \longrightarrow \mathbb{C}$ as being a homomorphism

$$\coprod SP_n(X) \longrightarrow \mathbb{C}$$
$$(x_1,\dots,x_n) \longmapsto \prod z(x_i)$$

and that $\sum \frac{1}{n!}\int dx_1\cdots dx_n\, e^{-\beta U_n(x_1,\dots,x_n)}$ (the Gibbs measure) is a measure on the monoid $SP(X)$. Thus the partition function becomes sort of a Fourier transform of a measure. But now one pushes forward this measure on the monoid to one on the abelian group generated by the space X.

So we have a measure on

$$\text{ab. group } \mathbb{Z}[X] = \text{divisors on } X$$

and we compute its F.T. using characters on $\mathbb{Z}[X]$ which come from maps $z: X \longrightarrow S^1$. But the example of the degree shows that there are very interesting examples of characters on $(S^1)^X$ which do not come from $\mathbb{Z}[X]$. Namely one has a homomorphism

$$(S^1)^X \longrightarrow [X, S^1] = H^1(X,\mathbb{Z})$$

and one can follow this by any character of $H^1(X,\mathbb{Z})$.

Hence it might be useful to keep in mind that classical gas configurations [redacted] may have to be **enlarged** so as to get the irreducible states of a classical gas.

Next project is to go over in the SU_2 case the cell decomposition of one of these orbits on the building, so as to compute the sections of a line [redacted] bundle.

Other viewpoint:

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$$\mathfrak{g} = \mathfrak{sl}_2 = (Y) \oplus (H) \oplus (X).$$

$$\tilde{\mathfrak{g}} = \mathfrak{g}[z, z^{-1}] = z^{-1}\mathfrak{g}[z^{-1}] \oplus (Y) \oplus \underbrace{(H) \oplus (X) \oplus z\mathfrak{g}[z]}_{\text{Iwahori subalg} \,=\, \mathfrak{h} \oplus \tilde{\mathfrak{n}}}$$

$\tilde{\mathfrak{n}} = (X) + z\mathfrak{g}(z)$ has the generators X, zY because

$$[X, zY] = zH \qquad [zH, X] = 2zX$$

so one gets all of $z\mathfrak{g}$, and then $[z\mathfrak{g}, z\mathfrak{g}] = z^2\mathfrak{g}$ etc.

Introduce the notation

$$X_1 = X$$
$$X_2 = zY$$
$$Y_1 = Y$$
$$Y_2 = z^{-1}X$$

Then $[X_i, Y_j] = 0 \quad i \neq j \qquad [X_1, Y_1] = H$

$$[X_2, Y_2] = -H$$

and $[H, X_1] = 2X_1 \qquad [H, Y_1] = -2Y_1$

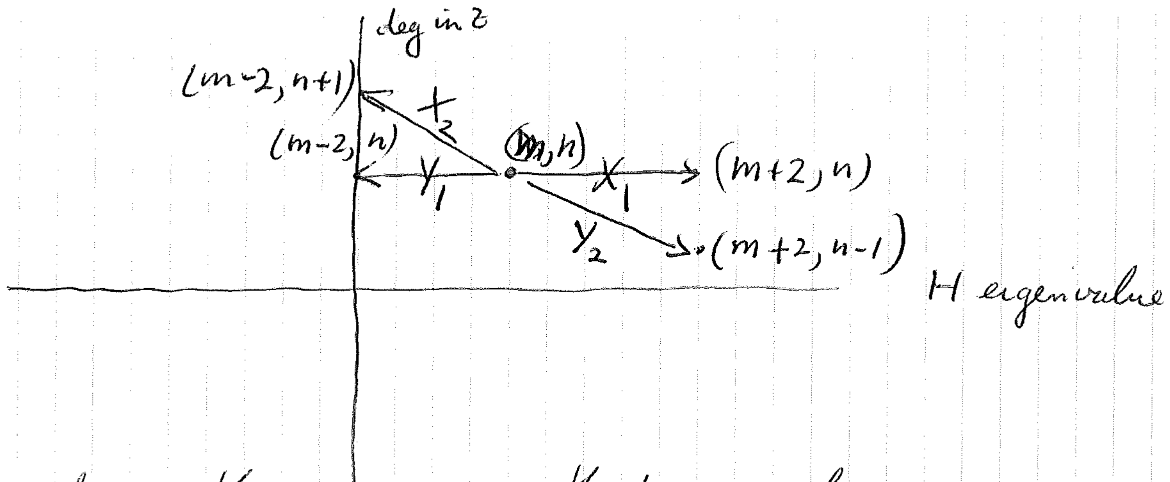
$$[H, X_2] = -2X_2 \qquad [H, Y_2] = 2Y_2$$

What I'm going to do is look at a repres. of $\tilde{\mathfrak{g}}$ with vector e_λ satisfying

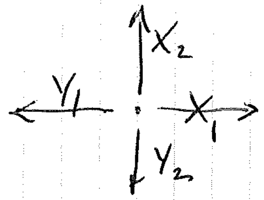
$$He_\lambda = \lambda(H)e_\lambda$$
$$X_1 e_\lambda = X_2 e_\lambda = 0$$

and then the repn. should be spanned by vectors of the form $\underbrace{\cdots Y_1 \cdots Y_2 \cdots}_{\text{word in } Y_1, Y_2} e_\lambda$

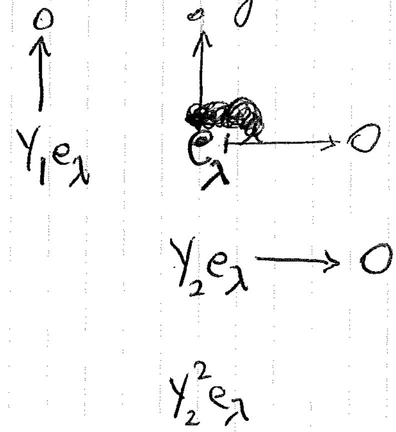
It seems useful to keep track of the degree in the variable z. Notice that each vector $\cdots Y_1 \cdots Y_2 \cdots e_\lambda$ has a z degree assuming e_λ does. Thus we can plot degrees and H-eigenvalues.



I will skew the axes so that we have



So now let us begin with $X_1 e_\lambda = X_2 e_\lambda = 0$.



So you see immediately that the weights are always in a lower left quadrant. If we come across a vector killed by X_1, X_2 which is different from e_λ it will generate a subrepresentation and hence the initial repn. is reducible.

Let's now take $\lambda(H) = 1$: $He_1 = e_1$. Then we know from repn. of SL_2 that $X_1 Y_1^2 e_1 = 0$ as well as $X_2 Y_1^2 e_1 = 0$. So by what I said about irreducibility we must have $Y_1^2 e_1 = 0$. In the other direction, we have to be careful about signs.

$$X_2 Y_2^2 e_1 = [X_2, Y_2^2] e_1 = (-H) Y_2 e_1 + Y_2 (-H) e_1$$
$$= -[H, Y_2] e_1 - Y_2 H e_1 - Y_2 e_1$$

$$= -(2Y_2)e_1 - 2Y_2 e_1 = -4Y_2 e_1$$

Check this: In the 2nd sl_2 one has the standard basis $X_2, Y_2, -H$ because of the relations

$$[X_2, \text{[redacted]}\ Y_2] = -H$$

$$[-H, X_2] = [-H, zY] = 2zY = 2X_2$$

$$[-H, Y_2] = [-H, z^{-1}X] = -2z^{-1}X = -2Y_2$$

so therefore in order to have finite dimensional reps of the first sl_2 with $X_1 e_\lambda = 0$, $He_\lambda = \lambda e_\lambda$ one needs $\lambda = 0, 1, 2, \cdots\cdots$, but for the second sl_2 to have f.d. repns [redacted] with $X_2 e_\lambda$, $(-H)e_\lambda = (-\lambda)e_\lambda$ one wants $\lambda = 0, -1, -2, \cdots\cdots$.

Thus let us take the example $\lambda = 1$.

$$\begin{array}{cccc} & Y_1 e_1^{(-1)} & e_1^{(1)} & \\ Y_1^3 Y_2 e_1^{(-3)} & Y_1^2 Y_2 e_1^{(-1)} & Y_1 Y_2 e_1^{(1)} & Y_2 e_1^{(3)} \end{array}$$

The first thing to ask now is whether $Y_1 Y_2 e_1 = Y_2 Y_1 e_1$? [redacted] These two vectors lie in the $\deg = -1$, $H = 1$ weight space which can't contain something killed by both X_1, X_2 and hence is at most 2 dimensional, and is detected by the effect of X_1, X_2. So

$$X_1(Y_1 Y_2 e_1) = \underbrace{[X_1, Y_1]}_{H} Y_2 e_1 + \cancel{Y_1 X_1 Y_2 e_1} = 3Y_2 e_1$$

$$X_1(Y_2 Y_1 e_1) = Y_2 X_1 Y_1 e_1 = Y_2 e_1$$

$$X_2(Y_1 Y_2 e_1) = Y_1 X_2 Y_2 e_1 = Y_1 \underbrace{[X_2, Y_2]}_{-H} e_1 = -Y_1 e_1$$

$$X_2(Y_2 Y_1 e_1) = \underbrace{[X_2, Y_2]}_{-H} Y_1 e_1 = Y_1 e_1$$

This shows $Y_1 Y_2 e_1$, $Y_2 Y_1 e_1$ [redacted] are linearly independent.

The next thing to do is to compute the terms of degree -2 such as $Y_2^2 e_1$ and to determine the lin. independent terms, but this gets progressively more complicated. Now the operators given by words in Y_1, Y_2 are not all independent, because this would mean that the Lie algebra generated by Y_1, Y_2 is free which I don't think is true, since it has polynomial growth. Thus there should be some relations.

Let's try to compute sections of the line bundle. The orbit $\mathcal{K}/T$ for $K = SU_2$ I can think of [redacted] in terms of pairs of lattices $\Lambda_0 > \Lambda_1$ with $\deg \Lambda_0 = 0$ and $\deg \Lambda_1 = -1$. Here the lattices are for $\mathbb{C}[[z]]$ in the vector space $\mathbb{C}[[z]][z^{-1}]^2$ over the field of Laurent series. On this orbit is a canonical exact sequence of vector bundles with fibres

$$0 \longrightarrow \Lambda_1/\pi\Lambda_0 \longrightarrow \Lambda_0/\pi\Lambda_0 \longrightarrow \Lambda_0/\Lambda_1 \longrightarrow 0$$

A simpler thing to look at is the following: Let's fix a lattice Λ_0 and consider all chains

$$\Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_r$$

of lattices each of codim 1 in the following. Call this space X_r. Then on X_r is a canonical 2 plane bundle E_r with fibre $\pi^{-1}\Lambda_r/\Lambda_r$ and $X_{r+1} = \mathbb{P}(E_r \text{ over } X_r)$, and the subline-bundle $\mathcal{O}(-1)$ has fibres Λ_{r+1}/Λ_r. Can one calculate the sections of the bundle $\mathcal{O}(n)$ over X_r?

Recall that for [redacted] $K = SU_2$ that $\mathcal{K}/K$ can be identified with the space of A-lattices Λ in F^2 of degree 0 relative to A^2. Here F = Laurent series (formal or convergent) and A = power series. [redacted] Over $\mathcal{K}/K$ is a holomorphic 2-plane bundle E with fibre $\Lambda/z\Lambda$ over Λ. If I take the projective bundle $\mathbb{P}(E)$ I get the orbit $\mathcal{K}/T$ and the quotient line bundle $\mathcal{O}(1)$ is the homogeneous line bundle of interest on this orbit. So if I denote $f : \mathbb{P}(E)$ [redacted] $= \mathcal{K}/T \longrightarrow \mathcal{K}/K$ the canonical map, the representation to be computed is [redacted] all sections of $\mathcal{O}(1)^{\otimes d}$ over $\mathbb{P}(E)$. But

$$H^0(\mathbb{P}(E), \mathcal{O}(1)^{\otimes d}) = H^0(\mathcal{K}/K, \underbrace{f_*\mathcal{O}(1)^{\otimes d}}_{\mathrm{Sym}^d(E)})$$

So [redacted] now we want [redacted] ways to associate to a lattice Λ an element of $\Lambda/z\Lambda$, say for $d=1$. For example suppose we restrict to lattices containing a fixed lattice Λ_1, then any element of $\Lambda_1/z\Lambda_1$ will gives us such a section.

September 13, 1981

Recall that we are thinking of $\mathcal{K}/K$ as the space of lattices of degree 0 in F^2, where F = Laurent series and we are trying to determine the holomorphic sections of the vector bundle E whose fibre at Λ is the space $\Lambda/z\Lambda$. I think that there are not very many of these sections.

So let us fix a lattice Λ_0 of degree $-r$ and look at the subspace of $\mathcal{K}/K$ consisting of lattices Λ of degree 0 such that $\Lambda_0 \subset \Lambda$. Thus in fact $\Lambda_0 \subset \Lambda \subset z^{-r}\Lambda_0$, so we have a closed subvariety of a suitable Grassmannian. We now want to determine the sections of E over this subvariety. First case is where $r = 1$, whence the space of Λ is $\mathbb{P}(z^{-1}\Lambda_0/\Lambda_0)$. The bundle E has fibre $\Lambda/z\Lambda$ at the point Λ, and one has an exact sequence

$$0 \longrightarrow \Lambda_0/z\Lambda \longrightarrow \Lambda/z\Lambda \longrightarrow \Lambda/\Lambda_0 \longrightarrow 0 .$$

$$\mathcal{O}(1)(\Lambda) \simeq z^{-1}\Lambda_0/\Lambda \qquad E(\Lambda) \qquad \mathcal{O}(-1)(\Lambda)$$

Since $\mathcal{O}(-1)$ has no holomorphic sections, we have

$$H^0(E) = H^0(\mathcal{O}(1)) \cong \Lambda_0/z\Lambda_0$$

So now let's generalize. In general let us consider all chains

$$(*) \qquad \Lambda_0 < \Lambda_1 < \cdots < \Lambda_r$$

of lattices, each of codim 1 in the following, with Λ_0 fixed. Denote the space of these X_r and by E_r the bundle with fibre $\Lambda_r/z\Lambda_r$ at the point $(*)$. Then $X_r = \mathbb{P}(E_{r-1} \text{ over } X_{r-1})$ and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_r(1)(\Lambda_\cdot) \to E_r(\Lambda_\cdot) \longrightarrow \mathcal{O}_r(-1)(\Lambda_\cdot) \to 0$$

$$0 \longrightarrow \Lambda_{r-1}/z\Lambda_r \to \Lambda_r/z\Lambda_r \longrightarrow \Lambda_r/\Lambda_{r-1} \to 0$$

So if $f: X_n \to X_{n-1}$ is the canonical map we have

$$f_*(E_n) = f_*(\mathcal{O}_n(1)) = E_{n-1}$$

and so therefore by induction

$$H^0(X_n, E_n) = E_0 = \Lambda_0 / z\Lambda_0 .$$

Since $R^1 f_*(\mathcal{O}(-1)) = 0$, this calculation holds for the whole cohomology.

Now consider the bundle $\mathrm{Sym}^2(E)$. Since E is an extension $0 \to \mathcal{O}(1) \to E \to \mathcal{O}(-1) \to 0$, we know $\mathrm{Sym}^2(E)$ has a filtration with quotients $\mathcal{O}(2), \mathcal{O}, \mathcal{O}(-2)$, so applying f_* gives an exact sequence

$$0 \to \underset{\substack{\parallel \\ \mathrm{Sym}^2(E_{n-1})}}{f_*(\mathcal{O}_n(2))} \longrightarrow f_*(\mathrm{Sym}^2(E_n)) \longrightarrow \mathcal{O}_{n-1} \to 0$$

It isn't clear what happens under iteration

Note: [redacted] In the algebra $\mathfrak{g}[z, z^{-1}]$, $\mathfrak{g} = \mathfrak{sl}_2$, the elements $X_1 = X$, $X_2 = zY$ are not free. In effect one has the relation $(\mathrm{ad}\, X)^3 = 0$ because this holds in $\mathfrak{g}$ and everything commutes with z.

Similarly we have $(\mathrm{ad}\, Y_1)^3 Y_2 = Y_1^3 Y_2 - 3Y_1^2 Y_2 Y_1 + 3Y_1 Y_2 Y_1^2 - Y_2 Y_1^3 = 0$. Thus we find that there will be non-trivial relations among elements of the form

$$Y_{i_n} \cdots\cdots Y_{i_1} e_\lambda$$

in our representation.