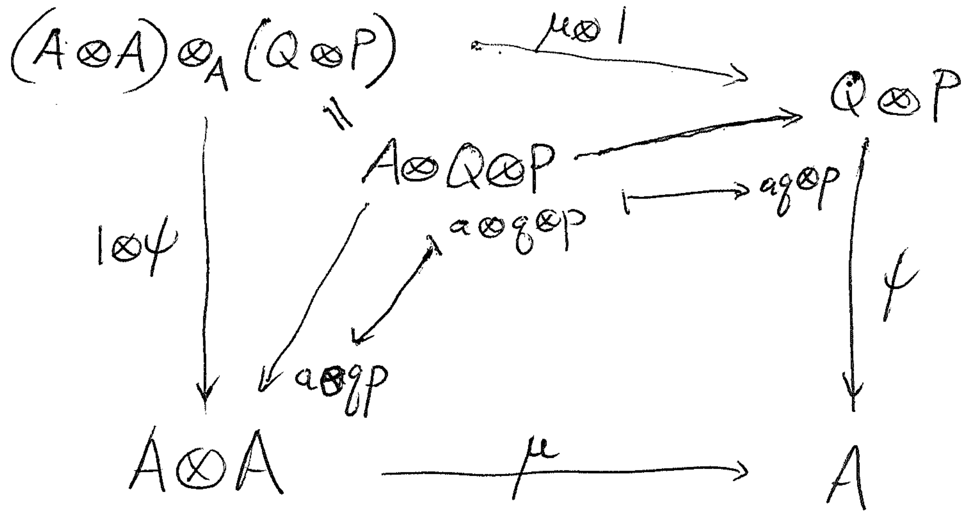
<u>October 24, 1995</u>

The bimodule approach to HC suggests trying to link two Meq rings A, B using tensor products rather than $\oplus$. To be specific, given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$, then this Mcontext results by taking the direct sum of (A, A, μ) and $(Q, P, Q\otimes P \xrightarrow{\psi} A)$. But suppose instead [illegible] we use the A-bimodule $(A\otimes A)\otimes_A (Q\otimes P)$ to link $A\otimes A$ and $Q\otimes P$:



Thus we have
$$\begin{array}{ccc} A\otimes Q\otimes P & \longrightarrow & A \\ a\otimes q\otimes p & \longmapsto & aqp \end{array}$$
, whose cyclic module maps ~~both~~ to $(A\otimes]^{(*)}$ and $[B\otimes]^{(*)}$. The thing that doesn't work is that there is no obvious ring structure on $(A\otimes Q\otimes P)\otimes_A$. We can write this bimodule either as $(A\otimes Q)\otimes P$ or $A\otimes (Q\otimes P)$. These lead respectively to the Morita contexts

$$\begin{pmatrix} A & A\otimes Q \\ P & P\otimes_A (A\otimes Q) \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} A & A \\ Q\otimes P & (Q\otimes P)\otimes_A A \end{pmatrix}$$

where $P\otimes_A(A\otimes Q) = P\otimes Q$ and $(Q\otimes P)\otimes_A A = Q\otimes P$.

One calculates that the ring structures induced on $P \otimes Q$ and $Q \otimes P$ are resp.

$$(p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 q_1 p_2 \otimes q_2$$

$$(q_1 \otimes p_1)(q_2 \otimes p_2) = q_1 \otimes p_1 q_2 p_2$$

Another thing I can do is to consider the bimodule $(Q \otimes P) \otimes_A (A \otimes A) = Q \otimes P \otimes A$ which can be split either as $Q \otimes (P \otimes A)$ or $(Q \otimes P) \otimes A$ leading to the Morita contexts

$$\begin{pmatrix} A & Q \\ P \otimes A & (P \otimes A) \otimes_A Q \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & Q \otimes P \\ A & A \otimes_A (Q \otimes P) \end{pmatrix}$$

$$(P \otimes A) \otimes_A Q = P \otimes Q \qquad A \otimes_A (Q \otimes P) = Q \otimes P$$

The induced ring structures on $P \otimes Q$ and $Q \otimes P$ are resp.

$$(p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \otimes q_1 p_2 q_2$$

$$(q_1 \otimes p_1)(q_2 \otimes p_2) = q_1 p_1 q_2 \otimes p_2$$

These four products arise from the two associative products on the two dialgebras given by $Q \otimes P \twoheadrightarrow A$ and $P \otimes Q \twoheadrightarrow B$.

[redacted] Let $X_{nk}(P,Q)$ be the subset of $M_{nk}(P)\times M_{kn}(Q)$ consisting of $(p;q)$ such that $1-pq$ is invertible. Fix [redacted] n and form the category with object set $\coprod_{k\geq 0} X_{nk}(P,Q)$, where a map $(p;q)\longleftarrow(p';q')$ is given by a matrix a over A such that $pa=p'$ and $q=aq'$. Notice that if there is a map $(p';q')\xrightarrow{a}(p;q)$, then $p'q'=paq'=pq$. Thus π_0 of this category maps to GL_nB by $(p;q)\mapsto 1-pq$. Denote this category by $\mathcal{X}_n$. I claim that $\pi_0\mathcal{X}_n\xrightarrow{\sim}GL_nB$, i.e. $1-p_1q_1=1-p_2q_2 \iff$ there is a path in $\mathcal{X}_n$ joining $(p_1;q_1)$ to $(p_2;q_2)$.

Proof. First we show $(p_1;q_1)\sim\left((p_1\ p_2);\begin{pmatrix}q_1\\0\end{pmatrix}\right)$ where $\sim$ means [redacted] in the same component of $\mathcal{X}_n$. We can write $q_1=aq'$ since $Q=AQ$. Then

$$\left((p_1\ p_2);\begin{pmatrix}q_1\\0\end{pmatrix}\right)=\left((p_1\ p_2);\begin{pmatrix}a\\0\end{pmatrix}q'\right)$$

$$\uparrow$$

$$(p_1a;q')\longrightarrow(p_1;aq')=(p_1;q_1)$$

Next given $(p_1;q_1)$ and $(p_2;q_2)$ such that $p_1q_1=p_2q_2$, we know $\exists\ a_1,a_2,a_3,p_3,q'$ such that (uses $P\otimes_A Q\xrightarrow{\sim}$ [redacted] B)

$$(p_1\ p_2\ p_3)\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=0 \qquad \begin{pmatrix}q_1\\-q_2\\0\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}q'$$

Then $$(p_2;q_2)\sim\left((p_2\ p_3);\begin{pmatrix}q_2\\0\end{pmatrix}\right)=\left((p_2\ p_3);\begin{pmatrix}-a_2\\-a_3\end{pmatrix}q'\right)$$

$$\uparrow$$

$$(p_1;\underbrace{a_1q'}_{q_1})\longleftarrow(p_1a_1;q')=(-p_2a_2-p_3a_3;q')$$

Next note that given $(\underset{p'}{\underbrace{pa}} ; q') \to (p ; \underset{q}{\underbrace{aq'}})$

then the invertible matrices over A

$$1 - q'p' = 1 - (q'p)a \quad , \quad 1 - qp = 1 - a(q'p)$$

represent the same element of $K_1 A$. Thus we have a well defined map

$$GL_n B \longrightarrow K_1 A \;, \qquad 1 - pq \mapsto [1 - qp]$$

Next we show this is a group homomorphism.

$$(1 - p_1 q_1)(1 - p_2 q_2) = 1 - p_1 q_1 - p_2 q_2 + p_1 q_1 p_2 q_2$$

$$= 1 - (p_1 \;\; p_2)\begin{pmatrix} q_1 - q_1 p_2 q_2 \\ q_2 \end{pmatrix}$$

$$= 1 - (p_1 \;\; p_2)\begin{pmatrix} 1 & -q_1 p_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

This product goes to the element of $K_1 A$ represented by

$$* = 1 - \begin{pmatrix} 1 & -q_1 p_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}(p_1 \;\; p_2)$$

$$= 1 - \begin{pmatrix} q_1(1 - p_2 q_2) \\ q_2 \end{pmatrix}(p_1 \;\; p_2)$$

$$= 1 - \begin{pmatrix} q_1(1 - p_2 q_2) p_1 & q_1(1 - p_2 q_2) p_2 \\ q_2 p_1 & q_2 p_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - q_1(1 - p_2 q_2) p_1 & -q_1 p_2 (1 - q_2 p_2) \\ -q_2 p_1 & 1 - q_2 p_2 \end{pmatrix}$$

Recall $\begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - bd^{-1}c & 0 \\ c & d \end{pmatrix} \sim \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix}$

Thus $*$ is conjugate mod elementaries to $\begin{pmatrix} \Xi & \\ & 1-q_2p_2 \end{pmatrix}$ where

$$\Xi = a - bd^{-1}c$$

$$= 1 - q_1(1-p_2q_2)p_1 - \cancel{q_1p_2(1-q_2p_2)(1-q_2p_2)^{-1}}q_2p_1$$

$$= 1 - q_1p_1.$$

Oct 26, 1995: The above calculation and the one on p37 use the following identity

$$1 - \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}(p_1 \ p_2) = \begin{pmatrix} 1-q_1p_1 & -q_1p_2 \\ -q_2p_1 & 1-q_2p_2 \end{pmatrix} \sim \begin{pmatrix} 1-q_1p_1 & 0 \\ 0 & * \end{pmatrix}$$

$$* = d - ca^{-1}b = 1 - q_2p_2 - q_2p_1(1-q_1p_1)^{-1}q_1p_2$$
$$= 1 - q_2\left[1 + p_1(1-q_1p_1)^{-1}q_1\right]p_2 = 1 - q_2(1-p_1q_1)^{-1}p_2$$

In the composition situation above p_2 is changed to $(1-p_1q_1)p_2$: $(1-p_1q_1)(1-p_2q_2) = 1 - p_1q_1 - (p_2 - p_1q_1p_2)q_2$ so $* = 1 - q_2p_2$.

In the situation on p37: $p_1q_1 = p_2q_2$, q_2 is changed to $-q_2$, and $* = 1 + q_2(1-p_2q_2)^{-1}p_2 = (1-q_2p_2)^{-1}$.

October 29, 1995

Another calculation with adjoint functors; compare p. 24. To show unicity of an adjoint.

$$\mathrm{Hom}(X, GY) = \mathrm{Hom}(FX, Y) = \mathrm{Hom}(X, G'Y)$$

$$\begin{array}{ccccc} \mathrm{Hom}(X,GY) & & \mathrm{Hom}(FX,Y) & & \mathrm{Hom}(X,G'Y) \\ \searrow & \nearrow (\alpha_Y)_* & & \searrow & \uparrow (\beta'_X)^* \\ & \mathrm{Hom}(FX, FGY) & & \mathrm{Hom}(G'FX, G'Y) & \end{array}$$

Taking $X = GY$, then 1_{GY} goes to $GY \xrightarrow{\beta'_{GY}} G'FGY \xrightarrow{G'(\alpha_Y)} G'Y$.
Similary $1_{G'Y}$ goes back to $G'Y \xrightarrow{\beta_{G'Y}} GFG'Y \xrightarrow{G(\alpha'_Y)} GY$.
Thus to show the composition is the identity.

$$\begin{array}{ccccc} GY & \xrightarrow{\beta'_{GY}} & G'FGY & \xrightarrow{G'(\alpha_Y)} & G'Y \\ \downarrow \beta_{GY} & & \downarrow \beta_{G'FGY} & & \downarrow \beta_{G'Y} \\ GFGY & \xrightarrow{GF(\beta'_Y)} & GFG'FGY & \xrightarrow{GFG'(\alpha_Y)} & GFG'Y \\ & \diagdown\!\!\diagdown & \downarrow G(\alpha'_{FGY}) & & \downarrow G(\alpha'_Y) \\ & & GFGY & \xrightarrow{G(\alpha_Y)} & GY \end{array}$$

triangle commutes by applying G to

$$\begin{array}{ccc} FX & \xrightarrow{F(\beta'_X)} & FG'FX \\ & \diagdown\!\!\diagdown & \downarrow \alpha'_{FX} \\ & & FX \end{array}$$

when $X = GY$.

Thus the composition equals

$$G(\alpha_Y)\,\beta_{GY} = 1_{GY}.$$

November 9, 1995

Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ completely firm, $w: A \to B$ a ring homomorphism, and $u': B \otimes_A A \xrightarrow{\sim} P$ a (B,A)-bimodule isom. To construct $u: A \to P$, $v: A \to Q$ such that

$$\begin{cases} u(a_1 a_2) = u(a_1)a_2 = w(a_1)u(a_2) \\ v(a_1 a_2) = a_1 v(a_2) = v(a_1) w(a_2) \\ v(a_1)u(a_2) = a_1 a_2 \\ u(a_1)v(a_2) = w(a_1 a_2) \end{cases}$$

<u>First proof via adjunction</u>: u' gives $B\otimes_A - \simeq P \otimes_A -$, so

$$\begin{aligned} \mathrm{Hom}_A(M, Q\otimes_B N) &= \mathrm{Hom}_B(P\otimes_A M, N) \\ &\xrightarrow{\sim} \mathrm{Hom}_B(B\otimes_A M, N) \\ &= \mathrm{Hom}_A(M, A\otimes_A N) \end{aligned}$$

whence $Q\otimes_B - \simeq A\otimes_A -$, an isom of the right adjoints. This gives an (A,B)-bimodule isom $v': A\otimes_A B \xrightarrow{\sim} Q$ such that u', v' respect the adjunction maps, i.e.

① $u'\otimes v'$: $B\otimes_A A \otimes_A A\otimes_A B \xrightarrow{\sim} P\otimes_A Q = B$

is $\alpha(b_1\otimes a_1 \otimes a_2 \otimes b_2) = b_1 w(a_1 a_2) b_2$, and

② $v'\otimes u'$: $A\otimes_A B \otimes_B B\otimes_A A \xrightarrow{\sim} Q\otimes_B P = A$

is the inverse of $\beta(a_1 a_2 a_3 a_4) = a_1 \otimes w(a_2)\otimes w(a_3) \otimes a_4$.

Now define u, v to be the compositions

[illegible]

$$A = A\otimes_A A \xrightarrow{w\otimes 1} B\otimes_A A \xrightarrow{\sim} P$$

$$A = A\otimes_A A \xrightarrow{1\otimes w} A\otimes_A B \xrightarrow{\sim} Q$$

i.e. $u(a_1 a_2) = u'(w(a_1)\otimes a_2)$, $v(a_1 a_2) = v'(a_1 \otimes w(a_2))$.

Clearly u, v are A-bimodule maps, while ① $\Longrightarrow$

$$u(a_1a_2)\,v(a_3a_4) = (u'\otimes v')\big(w(a_1)\otimes a_2\otimes a_3\otimes w(a_4)\big)$$
$$= \alpha(\text{———})$$
$$= w(a_1)\,w(a_2a_3)\,w(a_4)$$
$$= w(a_1a_2a_3a_4)$$

and ② $\Longrightarrow$

$$v(a_1a_2)\,w(a_3a_4) = (v'\otimes u')\big(a_1\otimes w(a_2)\otimes w(a_3)\otimes a_4\big)$$
$$= \beta^{-1}(\text{———})$$
$$= a_1a_2a_3a_4\,.$$

Second proof. Define $u: A \to P$ as above i.e.

$$\boxed{u(a_1a_2) = u'(w(a_1)\otimes a_2)}$$

Clearly $\boxed{u(a_1a_2) = a(a_1)a_2 = w(a_1)u(a_2)}$

Also $u'(b\otimes a_1a_2) = u'(bw(a_1)\otimes a_2) = b\,u'(w(a_1)\otimes a_2) = b\,u(a_1a_2)$, hence

$$\boxed{u'(b\otimes a) = b\,u(a)}$$

Next from $u': B\otimes_A A \xrightarrow{\sim} P$ we get an isomorphism

$$Q\otimes_A A = Q\otimes_B B\otimes_A A \xrightarrow[\sim]{1\otimes u'} Q\otimes_B P = A$$
$$gb\otimes a \longmapsto g\otimes b\otimes a \longmapsto g\otimes bu(a) \mapsto gbu(a)$$

so we have
$$\boxed{\begin{array}{c} Q\otimes_A A \xrightarrow{\sim} A \\ g\otimes a \longmapsto gu(a) \end{array}}$$

Define $v: A \to Q$ to be the composition

$$A \xleftarrow{\sim} Q\otimes_A A \longrightarrow Q$$
$$qu(a) \longmapsto q\otimes a \longmapsto qw(a)$$

Thus $\boxed{v(qu(a)) = qw(a)}$. It's clear that v is an A-bimodule map where A acts on the right of Q via w:

$$\boxed{v(a_1a_2) = a_1v(a_2) = v(a_1)\,w(a_2)}$$

Better to write

$$\begin{array}{ccccc} qb\otimes a & & q\otimes b\otimes a & & q\otimes bu(a) \quad qbu(a) \\ Q\otimes_A A & = & Q\otimes_B B\otimes_A A & \xrightarrow{\sim} & Q\otimes_B P = A \end{array}$$

$$q\otimes a \longleftrightarrow qu(a)$$

$$q\otimes aa_2 \qquad\qquad qu(a)a_2$$

$$\| $$

$$qw(a)\otimes a_2$$

$$v(qu(a))$$

Thus we have

$$\boxed{\begin{array}{ccc} Q\otimes_A A & \sim & A \\ q\otimes a & \longmapsto & qu(a) \\ v(a_1)\otimes a_2 & \longleftarrow & a_1a_2 \end{array}}$$

and tensoring with P we have

$$\boxed{\begin{array}{ccc} B\otimes_A A & \sim & P \\ b\otimes a & \longmapsto & bu(a) \\ pv(a_1)\otimes a_2 & \longleftarrow & pa_1a_2 \end{array}}$$

Since $a_1a_2 \longmapsto v(a_1)\otimes a_2 \longmapsto v(a_1)u(a_2)$ is the identity we get

$$\boxed{v(a_1)u(a_2) = a_1a_2}$$

Next $\quad q \otimes a_1 a_2 a_3 \longmapsto q\,u(a_1 a_2 a_3)$

$= q\,u(a_1)\,a_2 a_3 \longmapsto v(q\,u(a_1) a_2) \otimes a_3$

$= q\,u(a_1)\,v(a_2) \otimes a_3$ is the identity, so

we have $\quad q\,u(a_1)\,v(a_2) \otimes a_3 = q \otimes a_1 a_2 a_3$,

hence $\quad q\,u(a_1)\underbrace{v(a_2)\,w(a_3)}_{v(a_2 a_3)} = q\,w(a_1 a_2 a_3)$

and so $\quad \boxed{q\,u(a_1)\,v(a_2) = q\,w(a_1 a_2)}$

hence $\quad \boxed{b\,u(a_1)\,v(a_2) = b\,w(a_1, a_2)}$

Thus $\quad w(a_3 a_4)\,w(a_1 a_2) = w(a_3 a_4)\,u(a_1)\,v(a_2)$

$= u(a_3 a_4 a_1)\,v(a_2) = u(a_3 a_4)\,a_1\,v(a_2)$

$w(a_3 a_4 a_1 a_2) \quad = u(a_3 a_4)\,v(a_1 a_2)$

so $\quad \boxed{w(a_1 a_2) = u(a_1)\,v(a_2)}$.

Given $\quad u : A \longrightarrow P, \quad v : A \longrightarrow Q \quad$ usual props.

$$\begin{pmatrix} A \\ A \end{pmatrix} \otimes_A (A \quad A) \longrightarrow \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q)$$

$$\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : M_2(A) \longrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

get homomorphism of Morita contexts. Note 8 conditions

$a_1 a_2 = a_1 a_2$	$a_1 a_2 = v(a_1)\,u(a_2)$
$u(pa) = u(p)\,a$	$u(bp) = w(b)\,u(p)$
$v(aq) = a\,v(q)$	$v(qa) = v(q)\,w(a)$
$w(a_1 a_2) = w(a_1)\,w(a_2)$	$w(a_1 a_2) = u(a_1)\,v(a_2)$

November 26, 1995

We work over a commutative unital ground ring k. Let $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ be a completely firm Morita context (over k).

Prop. Assume A is h-unital and k-flat. Then B is h-unital iff $P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \longrightarrow B$ is a quis.

(Previously I proved this ignoring the problem of flat bimodules being left and right flat, so the proof works over a field, probably also if we assume A, P, Q, B are all k-flat. I now want to check the above version carefully.)

[redacted] Recall the previous argument:

$$\begin{array}{ccc} B \overset{L}{\otimes}_B P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q & \longrightarrow & B \overset{L}{\otimes}_B B \\ \downarrow & & \downarrow \\ P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q & \longrightarrow & B \end{array}$$

We know $B \overset{L}{\otimes}_B P \longrightarrow P$ is an A^{op}-nil-quis. Hence A h-unital $\Rightarrow$ $B \overset{L}{\otimes}_B P \overset{L}{\otimes}_A A \xrightarrow{\sim} P \overset{L}{\otimes}_A A$ $\Rightarrow$ left vertical arrow is a quis.

Then if the bottom arrow is quis, so is the top arrow and we conclude that B is h-unital. This proves the direction ($\Leftarrow$).

We know $P \overset{L}{\otimes}_A A \longrightarrow P$ and $P \overset{L}{\otimes}_A Q \longrightarrow B$ are B-nil-quis. If B is h-unital, then we have quis $B \overset{L}{\otimes}_A P \overset{L}{\otimes}_A A \xrightarrow{\sim} B \overset{L}{\otimes}_A P$ and $B \overset{L}{\otimes}_B P \overset{L}{\otimes}_A Q \xrightarrow{\sim} B \overset{L}{\otimes}_B B$. So the top arrow is a quis, as well as the left + right vertical arrows, and we conclude the bottom arrow is a quis, proving ($\Rightarrow$).

[redacted] To [redacted] carry out this argument we need to make sense of the derived tensor products. Let $E \to \tilde{A}$ be a flat A-bimodule resolution, let $\hat{B} \to B$ and $\hat{Q} \to Q$ be flat B^{op}-module and flat A-module resolutions respectively. Consider the square

$$
(*) \qquad
\begin{array}{ccc}
\hat{B} \otimes_B P \otimes_A E \otimes_A A \otimes_A \hat{Q} & \longrightarrow & \hat{B} \otimes_B B \\
\downarrow & & \downarrow \\
P \otimes_A E \otimes_A A \otimes_A \hat{Q} & \longrightarrow & B
\end{array}
$$

Now $E \otimes_A A$ is a flat A-module resolution of A. In effect, E is a flat A^{op}-module complex (as $\tilde{A} \otimes \tilde{A}$ is A^{op}-flat, $\tilde{A}$ being k-flat) so $E \otimes_A A = E \overset{L}{\otimes}_A A \xrightarrow{\sim} \tilde{A} \overset{L}{\otimes}_A A = A$. Thus $E \otimes_A A$ is a resolution of A. Also $E \otimes_A A$ is A-flat since $(\tilde{A} \otimes \tilde{A}) \otimes_A A = \tilde{A} \otimes A$ and A is k flat.

So $(\hat{B} \otimes_B P) \otimes_A E \otimes_A A = (\hat{B} \otimes_B P) \overset{L}{\otimes}_A A$ and similarly for P in place of $\hat{B} \otimes_B P$. Thus we see [redacted] that $\hat{B} \otimes_B P \otimes_A E \otimes_A A \xrightarrow{\sim} P \otimes_A E \otimes_A A$, because we know $\hat{B} \otimes_B P \to P$ is an A^{op}-mil-quis and A is h-unital. Then applying $- \otimes_A \hat{Q}$ yields a quis since $\hat{Q}$ is A-flat. Thus the left vertical arrow is a quis.

[redacted]

Something I should have mentioned earlier is that the condition $P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \xrightarrow{\sim} B$ can be interpreted as saying that $\hat{P} \otimes_A A \otimes_A \hat{Q} \xrightarrow{\sim} P \otimes_A A \otimes_A Q = B$ is a quis, where $\hat{P} \to P$ is a flat A^{op}-module resolution.

We have quis

$$\hat{P}\otimes_A A\otimes_A \hat{Q} \longleftarrow \hat{P}\otimes_A E\otimes_A A\otimes_A \hat{Q} \longrightarrow P\otimes_A E\otimes_A A\otimes_A \hat{Q}$$

the first because $\hat{P}, \hat{Q}$ are flat and $E\otimes_A A \to A$ is a quis, the second because $\hat{P}\to P$ is a quis and $E\otimes_A A$ and Q are A-flat.

At this point we can identify the bottom arrow in * with the map $P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A Q \longrightarrow B$. If this map is a quis, then so is the top arrow in the square *, whence $\hat{B}\otimes_B B \to B$ is a quis, and B is h-unital. This proves ($\Leftarrow$).

Now $P\otimes_A E\otimes_A A \longrightarrow P$ is $P\overset{L}{\otimes}_A A \to P$ which we know is a B-mod quis. Assuming B is h-unital we get a quis $\hat{B}\otimes_B P\otimes_A E\otimes_A A \longrightarrow \hat{B}\otimes_B P$, hence a quis $\hat{B}\otimes_B P\otimes_A E\otimes_A A\otimes_A \hat{Q} \longrightarrow \hat{B}\otimes_B P\otimes_A \hat{Q}$. Also $P\otimes_A \hat{Q} = P\overset{L}{\otimes}_A Q \longrightarrow B$ is a B-mod-quis, so we have a quis $\hat{B}\otimes_B P\otimes_A \hat{Q} \longrightarrow \hat{B}\otimes_B B$. Combining we see the top arrow in * is a quis, as well as the vertical arrows, so the bottom arrow is a quis, proving ($\Rightarrow$).

December 1, 1995

Multipliers again. Recall:

A *left multiplier* on B is an operator $b \mapsto xb$ [redacted] satisfying $x(b_1 b_2) = (xb_1)b_2$. These form a ring [redacted] $M_\ell(B) = \mathrm{Hom}_{B^{op}}(B,B)$ with product given by $(x_1 x_2)b = x_1(x_2 b)$.

A *right multiplier* on B is $b \mapsto bx$ s.t. $(b_1 b_2)x = b_1(b_2 x)$; we get ring $M_r(B) = \mathrm{Hom}_B(B,B)^{op}$ with product $b(x_1 x_2) = (bx_1)x_2$.

The ring of *multipliers* on B is the subring

$$M(B) = \left\{ (x^\ell, x^r) \in \mathrm{Hom}_{B^{op}}(B,B) \times \mathrm{Hom}_B(B,B)^{op} \,\middle|\, b_1(x^\ell b_2) = (b_1 x^r) b_2 \right\}$$

Suppose now $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ is a completely firm Morita context.

We have

$$M_\ell(B) = \mathrm{Hom}_{B^{op}}(B,B) \simeq \mathrm{Hom}_{A^{op}}(P,P) \quad [redacted] \qquad \begin{matrix} -\otimes_B P \\ \rightleftarrows \\ -\otimes_A Q \end{matrix}$$

[redacted]

$$x^\ell \longmapsto \{ y^\ell : bp \mapsto (x^\ell b)p \}$$

$$\{ x^\ell : pq \mapsto (y^\ell p)q \} \longleftarrow y^\ell$$

$$M_r(B) = \mathrm{Hom}_B(B,B)^{op} \simeq \mathrm{Hom}_A(Q,Q)^{op} \qquad \begin{matrix} Q\otimes_B - \\ \rightleftarrows \\ P\otimes_A - \end{matrix}$$

$$x^r \longmapsto \{ y^r : qb \mapsto q(bx^r) \}$$

$$\{ x^r : pq \mapsto p(qy^r) \} \longleftarrow y^r$$

Now check that if $x^r \leftrightarrow y^r$, $x^\ell \leftrightarrow y^\ell$ in this way, then the condition $(b_1 x^r)b_2 = b_1(x^\ell b_2)$ is equivalent to the condition $(q_1 y^r)p_1 = q_1(y^\ell p_1)$. Assume the first.

$$((qb_1)y^r)b_2p = (q(b_1x^r))b_2p = q(b_1x^r)b_2p$$
$$qb_1(y^\ell(b_2p)) = qb_1((x^\ell b_2)p) = qb_1(x^\ell b_2)p$$

Conversely assume $(qy^r)p = q(y^\ell p)$. Then

$$((p_1q_1)x^r)p_2q_2 = (p_1(q_1y^r))p_2q_2 = p_1(q_1y^r)p_2q_2$$
$$p_1q_1(x^\ell(p_2q_2)) = p_1q_1((y^\ell p_2)q_2) = p_1q_1(y^\ell p_2)q_2$$

Conclude then that

$$\boxed{M(B) \simeq \{(y^\ell, y^r) \in \mathrm{Hom}_A(P,P) \times \mathrm{Hom}_A(Q,Q)^{op} \mid (qy^r)p = q(y^\ell p)\}}$$

Recall that $B = B^2 \Longrightarrow$ any left multiplier commutes with any right multiplier:

$$(x^\ell(b_1b_2))y^r = ((x^\ell b_1)b_2)y^r = (x^\ell b_1)(b_2y^r)$$
$$x^\ell((b_1b_2)y^r) = x^\ell(b_1(b_2y^r)) = (x^\ell b_1)(b_2y^r).$$

Put another way, we have $B \otimes_B B \longrightarrow B$ with $x^\ell \otimes 1$ (resp. $1 \otimes y^r$) inducing x^ℓ (resp y^r) on B; hence $x^\ell \otimes 1$, $1 \otimes y^r$ commute $\Longrightarrow$ x^ℓ, y^r commute.

It follows that B is naturally a bimodule over $M_\ell(B) \times M_r(B)$ and that multiplication is a bimodule map from $B \otimes B$ to B. The multiplier condition $(b_1x^r)b_2 = b_1(x^\ell b_2)$ means this multiplication map descends to $B \otimes_{M(B)} B$. Thus B becomes an algebra over $M(B)$ in some noncommutative sense.

Center of $M(B)$.

First recall we have ring homomorphisms

$$B \longrightarrow \mathrm{Hom}_{B^{op}}(B,B) = M_\ell(B) \qquad b \longmapsto (b^\ell : b' \mapsto bb')$$

$$B \longrightarrow \mathrm{Hom}_{B}(B,B) = M_r(B) \qquad b \longmapsto (b^r : b' \mapsto b'b)$$

$$B \longrightarrow M(B) \qquad b \longmapsto (b^\ell, b^r)$$

Let $x = (x^\ell, x^r) \in M(B)$ centralize the image of B, i.e. $b^\ell x^\ell = x^\ell b^\ell$, $b^r x^r = x^r b^r$ $\forall b \in B$. Thus

$$b_1(x b_2) = x(b_1 b_2) \qquad (b_1 b_2)x = (b_1 x) b_2$$

Then $x(b_1 b_2) = b_1 x(b_2) = (b_1 x) b_2 = (b_1 b_2) x$, so that if $B = B^2$, then $xb = bx$, $\forall b \in B$. In other words x^ℓ and x^r are the same map from B to B. Thus x is a bimodule map $B \longrightarrow B$.

Let's examine the unital ring $\mathrm{Hom}_{(B,B)}(B,B)$ of B-bimodule maps $z: B \longrightarrow B$. Then $(z,z) \in M(B)$, since $z(b_1 b_2) = b_1 z(b_2) = b_1(z b_2)$ and $z(b_1 b_2) = z(b_1) b_2 \Longrightarrow (b_1 z) b_2 = b_1(z b_2)$; here $z(b) = zb = bz$ in the left + right multiplier notation. If $x = (x^\ell, x^r)$ is any multiplier, then since any left + any right multiplier commute, we have $x^\ell z = z x^\ell$, $x^r z = z x^r$ so (z,z) is in the center of $M(B)$. In particular $\mathrm{Hom}_{(B,B)}(B,B)$ is a <u>commutative unital ring.</u>

Better version of preceding: Introduce the canonical homomorphism

$$B \xrightarrow{\mu} M(B) \qquad \begin{aligned} \mu(b)b' &= bb' \\ b'\mu(b) &= b'b \end{aligned}$$

Then for any $x \in M(B)$ we have

(*) $\boxed{\mu(b)x = \mu(bx)}$ $\boxed{x\mu(b) = \mu(xb)}$

Check: $(\mu(b)x)b' = \mu(b)(xb') = b(xb') = (bx)b' = \mu(bx)b'$

$b'(\mu(b)x) = (b'\mu(b))x = (b'b)x = b'(bx) = b'\mu(bx)$.

and similarly for the other.

(*) shows that μ is a bimodule map over $M(B)$, in particular the image of μ is an ideal in $M(B)$.

Prop. The following unital rings are the same.

1) the center of $M(B)$
2) the centralizer in $M(B)$ of $\mu(B)$
3) $\mathrm{Hom}_{(B,B)}(B,B)$, the ring of bimodule maps $B \xrightarrow{z} B$.

Proof. Notice that a B-bimodule map $z: B \to B$, i.e. $z(b_1 b_2) = z(b_1)b_2 = b_1 z(b_2)$ is the same thing as a multiplier x such that $xb = bx$, i.e. such that $x^l = x^r : B \to B$. (Assuming $B = B^2$) such a multiplier commutes with any other $y = (y^l, y^r)$, since left and right multipliers commute:

$$xy = (x^l, x^r)(y^l, y^r) = (x^l y^l, y^r x^r) = (x^r y^l, y^r x^l)$$
$$= (y^l x^r, x^l y^r) = (y^l, y^r)(x^r, x^l) = yx$$

Thus $3) \subset 1) \subset (2)$.

Now let x commute with all $\mu(b)$. Then

$$x(bb') = (x\mu(b))b' = (\mu(b)x)b' = b(xb')$$
$$= b\mu(xb') = b(x\mu(b')) = b(\mu(b')x) = (bb')x$$

showing $xb = bx$ for all b since $B = B^2$.
Thus 3) ⊂ 2).

Recall what we know about left multipliers. The canonical map

$$A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A) \qquad a \mapsto (a' \mapsto aa')$$

is an A^{op}-nil-isom, whence (assuming A firm)

$$A \xrightarrow{\sim} \mathrm{Hom}_{A^{op}}(A,A) \otimes_A A$$

The other tensor product gives

$$A \longrightarrow A \otimes_A \mathrm{Hom}_{A^{op}}(A,A) = Q_{univ}$$

where Q_{univ} is universal wrt firm A-modules Q equipped with a bimodule map $Q \otimes A \longrightarrow A$. This leads to a Morita context studied by Steffan

$$\begin{pmatrix} A \to Q_{univ} \\ \| \qquad \| \\ A \to B \end{pmatrix}$$

Similarly we have canonical maps

$$A \longrightarrow \mathrm{Hom}_A(A,A)^{op} \qquad a \mapsto (a' \mapsto a'a)$$

which is an A-nil-iso, whence

$$A \xrightarrow{\sim} A \otimes_A \mathrm{Hom}_A(A,A)$$

but

$$A \longrightarrow \mathrm{Hom}_A(A,A) \otimes_A A = P_{univ}$$

is universal wrt firm A^{op}-modules P equipped with $A \otimes P \longrightarrow A$.

The Morita contexts where

$$\begin{pmatrix} A & Q \\ \parallel & \parallel \\ P & B \end{pmatrix} \quad \left(\text{also} \begin{pmatrix} A = Q \\ P = B \end{pmatrix} \right)$$

form an interesting ~~class~~* containing inclusions $A \subset B$ such that $BA = A$, $AB = B$, i.e. left ideal generating the ring. Also you get the situation where $A \twoheadrightarrow B$ is a B-module map, and then $B = A/I$ with $IA = 0$; $(a_1 a_2 = f(a_1)a_2)$.
(*rings A, B with $A \otimes_A B \xrightarrow{\sim} B$, $B \otimes_B A \xrightarrow{\sim} A$)

Question: Is there any interest in considering pairs of maps: $u: P \longrightarrow P'$ of A^{op}-modules and $u^*: Q' \longrightarrow Q$ of A-modules, which are adjoint:

$$q' u(p) = u^*(q') p \qquad \forall q' \in Q', p \in P?$$

Such a pair (u, u^*) should be analogous to a map of C^*-modules.

Roos' theorem link: Let Q be a generator of $\mathcal{M}(A)$, $R = \mathrm{End}_A(Q)^{op}$, $\tilde{B} = \mathrm{Im}\{\mathrm{Hom}_A(Q, A) \otimes_A Q \longrightarrow R\}$. We have a Morita context $\begin{pmatrix} A & Q \\ \mathrm{Hom}_A(Q,A) & R \end{pmatrix}$ yielding an equivalence of categories $\mathcal{M}(A) \simeq \mathrm{mod}(R)/\mathrm{nil}(R, \tilde{B}) = \mathcal{M}(R, \tilde{B})$. The firm A^{op}-module corresponding to Q is $P = \mathrm{Hom}_A(Q, A) \otimes_A A$.
When $Q = A$, $P = B$ (the firm ring $\tilde{B}^{(2)}$) is Steffan's ring.

~~[illegible]~~

Consider the canonical maps

$$B \xrightarrow{\mu^l} \mathrm{Hom}_{B^{op}}(B,B)$$

$$B \xrightarrow{\mu^r} \mathrm{Hom}_B(B,B)^{op}$$

$$B \xrightarrow{\mu} M(B)$$

Then μ^l is a B^{op}-mil isom, so $B \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(B,B) \otimes_B B$

μ^r —— B-mil isom, so $B \xrightarrow{\sim} B\otimes_B \mathrm{Hom}_B(B,B)$

μ is both so $B \xrightarrow{\sim} M(B) \otimes_B B$, $B \xrightarrow{\sim} B \otimes_B M(B)$

(To remember which side occurs you look at the contravariant variable of the Hom(,).)

December 7, 1995

I now want to record some observations I found when working on ring homomorphisms which induce Meq's.

One idea was to consider a homomorphism of Mcontexts

$$\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : \begin{pmatrix} A & A \\ A & A \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

i.e.

$$v(a_1)u(a_2) = a_1 a_2$$
$$u(a_1 a_2) = u(a_1)a_2 = w(a_1)\,u(a_2)$$
$$v(a_1 a_2) = a_1 v(a_2) = v(a_1)\,w(a_2)$$
$$w(a_1 a_2) = w(a_1)w(a_2) = u(a_1)v(a_2)$$

(Recall there are 8 products to consider in a Mcontext.)

Now assume $QP = A$, $PQ = B$ (also A, B idempotent). Then

$$A \xrightarrow{u} P \quad \text{is an } A\text{-nil-isom}$$

because $u(a) = 0 \Rightarrow a_1 a = v(a_1)u(a) = 0 \quad \forall a_1$

and $w(a_1 a_2)p = u(a_1)v(a_2)p = u(a_1 v(a_2)p) \Longrightarrow$

$w(a)$ kills $\mathrm{Coker}(u)$, $\forall a$.

Then

$$A^{(2)} \xrightarrow{\sim} A^{(2)} \otimes_A P$$
$$a_0 a_1 \otimes a_2 \longmapsto a_0 \otimes a_1 u(a_2)$$

so for M firm over A we have

$$M \xrightarrow{\sim} A^{(2)} \otimes_A P \otimes_A M$$
$$a_0 a_1 a_2 m \longmapsto a_0 \otimes a_1 \otimes u(a_2) \otimes m$$

Taking $M = Q \otimes_B N$, N firm over B we get

$$Q \otimes_B N \xrightarrow{\sim} A^{(2)} \otimes_A N$$

$$a_0 a_1 a_2 g \otimes n \longmapsto a_0 \otimes a_1 \otimes u(a_2) g n$$

Now consider the map in the opposite direction given by $a_0 v(a_1) \otimes n \longleftarrow a_0 \otimes a_1 \otimes n$. We have

$$a_0 v(a_1) \otimes_B u(a_2) g n \longleftarrow a_0 \otimes a_1 \otimes u(a_2) g n$$

$$\| $$

$$a_0 v(a_1) u(a_2) g \otimes n = a_0 a_1 a_2 g \otimes n$$

so this map is a one-sided, hence two-sided inverse. But note that $a_0 v(a_1) \otimes n = v(a_0 a_1) \otimes n$ so that this inverse map factors through $A \otimes_A N$. Thus it seems that $A^{(2)} \otimes_A N = A \otimes_A N$ for N firm over B. This can be checked much more simply as follows.

Observe that the maps

$$Q \otimes_B N \longleftrightarrow A \otimes_A N$$

$$a_1 a_2 g \otimes n \longmapsto a_1 \otimes u(a_2) g n$$

$$v(a) \otimes n \longleftarrow a \otimes n$$

are well-defined (use that $Q \otimes_B N$ is A-firm), and are inverse to each other. For example

$$v(a_0 a_1 a_2) \otimes n \longleftarrow a_0 a_1 a_2 \otimes n$$

$$\|$$

$$a_0 a_1 v(a_2) \otimes n \longmapsto a_0 \otimes \underbrace{u(a_1) v(a_2)}_{w(a_1 a_2)} n = a_0 a_1 a_2 \otimes n$$

Similarly $v : A \longrightarrow Q$ is an A^{op}-nil-isom so for M firm over A we have

$$M = A \otimes_A M \xrightarrow{\sim} Q \otimes_A M$$

$$a \otimes m \longmapsto v(a) \otimes m$$

whence $\quad P\otimes_A M \xrightarrow{\sim} P\otimes_A Q\otimes_A M \xrightarrow{\sim} B\otimes_A M$

$\qquad\qquad p\otimes am \longmapsto pv(a)\otimes m$

Here are the formulas

$A\otimes_A N \xrightarrow{\sim} Q\otimes_B N$	$B\otimes_A M \xrightarrow{\sim} P\otimes_A M$
$a\otimes n \longmapsto v(a)\otimes n$	$b\otimes am \longmapsto bu(a)\otimes m$
$a_1\otimes u(a_2)q_b n \longleftarrow a_1a_2q_b\otimes n$	$pv(a)\otimes m \longleftarrow p\otimes am$

Here M is firm$/A$ and N is firm$/B$. We note that then the right sides $Q\otimes_B N$, $P\otimes_A M$ are firm over A, B resp. Thus $A\otimes_A N \xleftarrow{\sim} A^{(2)}\otimes_A N$, $B\otimes_A M \xleftarrow{\sim} B^{(2)}\otimes_A M$. (The former seems special, but the latter is true in general as $B^{(2)} \twoheadrightarrow B$ is a B^{op}-nil-isom, in particular an A^{op}-nil iso.)

Now we can check compatibility of these isomorphisms with the adjunction maps in the case of $B^{(2)}\otimes_A - = B\otimes_A -$ and $A^{(2)}\otimes_A - = A\otimes_A -$ and the canonical isos $(P\otimes_A -)\circ(Q\otimes_B -) = 1$, $(Q\otimes_B -)\circ(P\otimes_A -) = 1$.

$$\begin{array}{ccc}
a\otimes b\otimes a'm & \longmapsto & v(a)\otimes bu(a')\otimes m \\
A\otimes_A B\otimes_A M & \xrightarrow{\sim} & Q\otimes_B P\otimes_A M \\
\uparrow & & \downarrow\wr \\
M & = & M
\end{array}$$

$$\begin{array}{ccc}
a_1\otimes w(a_2)\otimes a_3 m & \longmapsto & v(a_1)\otimes w(a_2)u(a_3)\otimes m \\
\uparrow & & \downarrow \\
a_1a_2a_3 m & & \underbrace{v(a_1)w(a_2)u(a_3)}_{v(a_1a_2)u(a_3) = a_1a_2a_3} m
\end{array}$$

$$b \otimes a_1 a_2 \otimes n \longmapsto b u(a_1) \otimes v(a_2) \otimes n$$
$$B \otimes_A A \otimes_A N \xrightarrow{\sim} P \otimes_A Q \otimes_B N$$
$$b \otimes a \otimes n \longmapsto b w(a) n$$
$$B \otimes_A A \otimes_A N \xrightarrow{\sim} [redacted] N$$

$$\begin{array}{ccc} b \otimes a_1 a_2 \otimes n & \longmapsto & b u(a_1) \otimes v(a_2) \otimes n \\ B \otimes_A A \otimes_A N & \xrightarrow{\sim} & P \otimes_A Q \otimes_B N \\ \downarrow & & \downarrow \\ N & = & N \\ b\, w(a_1 a_2)\, n & & b\, u(a_1) v(a_2)\, n \end{array}$$

The preceding [redacted] constructs a canonical isom between the Morita equivalence given by the functors $P \otimes_A -$, $Q \otimes_B -$ and the pair $B^{(2)} \otimes_A -$, $A^{(2)} \otimes_A -$.

Better to say we have constructed an isomorphism between the adjoint pair $(B^{(2)} \otimes_A - , A^{(2)} \otimes_A -)$ and the adjoint pair $(P \otimes_A -, Q \otimes_B -)$. As latter is an equivalence, so is the former.

Recall that when $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ is strictly idempotent: $A = A^2 = QP$, $P = PA = BP$, $Q = AQ = QB$, $B = B^2 = PQ$ then we have $P \otimes_A A = B \otimes_B P \otimes_A A = B \otimes_B P$ is a firm B,A bimodule, sim for $A \otimes_A Q = Q \otimes_B B$, and we have $Q \otimes_B P = A^{(2)}$, $P \otimes_A Q = B^{(2)}$. For the last point, note that $P \otimes_A Q = P \otimes_A A \otimes_A Q$ as $PA = P$, $AQ = Q$

and $P \otimes_A A$ is A-firm.

Consider next a homom. $A \longrightarrow B$ which induces a Meq, so that it factors $A \longrightarrow A/I = \bar{A} \subset B$ where $AIA = 0$ and $\bar{A}B\bar{A} = \bar{A}$, $B\bar{A}B = B$. This Meq is the composition of ones belonging to the Mcontexts

$$\begin{pmatrix} A & A/AI \\ A/IA & A/I \end{pmatrix} \qquad \begin{pmatrix} \bar{A} & \bar{A}B \\ B\bar{A} & B \end{pmatrix}$$

The functors are $M \longmapsto (B\bar{A} \otimes_{\bar{A}} A/IA) \otimes_A M$.

$N \longmapsto (A/AI \otimes_{\bar{A}}$ $\bar{A}B$ [redacted]$) \otimes_B N$

Note that $B\bar{A} \otimes_{\bar{A}} A/IA = \underbrace{B\bar{A} \otimes_{\bar{A}} \bar{A}}_{B\text{-firm}} \underbrace{\otimes_{\bar{A}} A/IA}_{A^{op}\text{-firm}}$ is a firm B, A bimodule.

Check:

$$B\bar{A} \otimes_{\bar{A}} A/IA \simeq B \otimes_{\bar{A}} \bar{A} \otimes_{\bar{A}} A/IA \qquad B\bar{A} \twoheadleftarrow B \otimes_{\bar{A}} \bar{A}$$

$$= B \otimes_A A \otimes_A A \,/\, \mathrm{Im}\{B \otimes_A I \otimes_A A + B \otimes_A A \otimes_A IA\}$$

$$b \otimes x \otimes a_1 a_2 = bx \otimes a_1 \otimes a_2 = 0$$

$$b \otimes a_1 \otimes x a_2 = b a_1 \otimes x \otimes a_2 = 0$$

$$= B \otimes_A A^{(2)} \simeq B^{(2)} \otimes_A A^{(2)}$$

Sim

$$A/AI \otimes_{\bar{A}} \bar{A}B = A/AI \otimes_{\bar{A}} \bar{A} \otimes_{\bar{A}} B$$

$$= A \otimes_A A \otimes_A B \,/\, \mathrm{Im}\{AI \otimes_A A \otimes_A B + A \otimes_A I \otimes_A B\}$$

$$= A^{(2)} \otimes_A B = A^{(2)} \otimes_A B^{(2)}$$

<u>December 8, 1995</u>

Here's a new idea for handling whether a ring hom $w: A \to B$ induces a Meguir. The condition that

$$\beta: A^{(2)} = A^{(6)} \longrightarrow A^{(2)} \otimes_A B^{(2)} \otimes_A A^{(2)}$$

is an isomorphism is equivalent to $A^{(2)} \to B^{(2)}$ being an $A \otimes A^{op}$-nil-isom. This is equivalent to $A \to B$ being an $A \otimes A^{op}$-nil-isom., because the kernels of $A^{(2)} \twoheadrightarrow A$, $B^{(2)} \twoheadrightarrow B$ are killed by A (resp B hence A) on both sides. Thus

1) β is an isomorphism
2) $w_! : \mathcal{M}(A) \to \mathcal{M}(B)$ is fully faithful
3) $A \xrightarrow{w} B$ is an $A \otimes A^{op}$-nil-isom
3') $AIA = 0$ and $\bar{A} B \bar{A} \subset \bar{A}$ where $I = \mathrm{Ker}(w)$ and $\bar{A} = \mathrm{Im}(w)$

are equivalent.

Now suppose this holds. Since $F = w_!$, $G = w^*$ are adjoint functors with F fully faithful, equivalently $\beta: 1 \xrightarrow{\sim} GF$, we can invert β and rewrite the adjunction conditions as

$$FGF \underset{F.\beta^{-1}}{\overset{\alpha.F}{\rightrightarrows}} F \quad (\text{equal}) \qquad GFG \underset{\beta^{-1}.G}{\overset{G.\alpha}{\rightrightarrows}} G \quad (\text{equal})$$

This should tell us that $\begin{pmatrix} 1 & G \\ F & 1 \end{pmatrix}$ is a Morita context, more precisely that

$$\begin{pmatrix} A^{(2)} & A^{(2)} \otimes_A B^{(2)} \\ B^{(2)} \otimes_A A^{(2)} & B^{(2)} \end{pmatrix}$$

is a Morita context. All we need to obtain a Meq is the surjectivity of

$$\left(B^{(2)} \otimes_A A^{(2)}\right) \otimes \left(A^{(2)} \otimes_A B^{(2)}\right) \longrightarrow B,$$

which amounts to the remaining condition

$B\bar{A}B = B$.

I think I should have first done the preceding when A, B are firm rings.

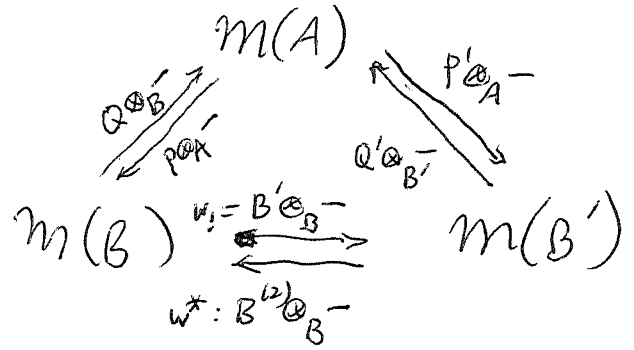
December 10, 1995

Consider a map of Morita contexts

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & v \\ u & w \end{pmatrix}} \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

$A = A^2 = QP$
$B = B^2 = PQ$
similarly with 's

To prove that w is a Morita equivalence homomorphism and that the triangle of equivalences



commutes up to canonical isomorphism. One reason for your difficulties with this is the number of canonical isomorphisms you can write down. Going between two different categories yields 6 isomorphisms:

1) $P' \otimes_A Q \otimes_B N \simeq B' \otimes_B N$

2) $B' \otimes_B P \otimes_A M \simeq P' \otimes_A M$

3) $Q' \otimes_{B'} B' \otimes_B N \simeq Q \otimes_B N$

4) $P \otimes_A Q' \otimes_{B'} N' \simeq B^{(2)} \otimes_B N'$

5) $B^{(2)} \otimes_B P' \otimes_A M \simeq P \otimes_A M$

6) $Q \otimes_B B^{(2)} \otimes_B N' \simeq Q' \otimes_{B'} N'$

On the other hand going from a category to itself either clockwise or counterclockwise leads to 6 isomorphisms.

Another reason for difficulties is the fact that you've relied on the naive transformations involving the Morita equivalences with P, Q and P', Q'. Thus 1)–3) concerning $w_! = B' \otimes_B -$ are equivalent naively, and similarly for 4)–6) concerning w^*. But a priori $w_!$ and w^* are only adjoint, not inverse, so you must bring in adjunction considerations to get between these groups, e.g. 1) is equivalent to 4) by

$$\mathrm{Hom}_{B'}(B' \otimes_B N, N') = \mathrm{Hom}_B(N, B^{(2)} \otimes_B N')$$

$$\mathrm{Hom}_{B'}(P' \otimes_A Q \otimes_B N, N') = \mathrm{Hom}_B(N, P \otimes_A Q' \otimes_{B'} N')$$

You might first prove 1) then deduce 4) by adjunction, but if you give a formula for 1), then it's probably simpler to also give a formula for 4) and check compatibility with adjunction maps than to compute the isomorphism 4) corresponding to 1).

Start this.

Claim $v : Q \longrightarrow Q'$ is a B^{op}-nil-isom:

$$v(q) = 0 \implies q p_1 q_1 = v(q) u(p_1) q_1 = 0$$

$$q' w(p_1 q_1) = (q' u(p_1)) v(q_1) = v((q' u(p_1)) q_1) \in v(Q)$$

If N is B-firm, then

$$Q \otimes_B N \xrightarrow{v} Q' \otimes_B N$$

$$q \otimes n \longmapsto v(q) \otimes n$$

$$(q' u(p) q) \otimes n \longleftarrow q' \otimes p q n$$

Thus

$$\boxed{\begin{aligned} (P'\otimes_A Q)\otimes_B N &\xrightarrow{\sim} B'\otimes_B N \\ p'\otimes q\otimes n &\longmapsto p'v(q)\otimes n \\ b'u(p)\otimes q\otimes n &\longleftarrow b'\otimes pq n \end{aligned}}$$

maps well-defd ✓ $\circlearrowright$: $b'u(p)v(q)\otimes n = b\, w(pq)\otimes n = b\otimes pqn$.

$\circlearrowleft$: $p'\otimes q\otimes p_1q_1 n \longmapsto p'v(q)\otimes p_1 q_1 n \longmapsto p'v(q)u(p_1)\otimes q_1\otimes n$

$= p'qp_1\otimes_A q_1\otimes n = p'\otimes qp_1q_1\otimes n = p'\otimes q\otimes p_1q_1 n$

Next $u: P\to P'$ is B-nil-isom so

$$B^{(2)}\otimes_B P \xrightarrow{\sim} B^{(2)}\otimes_B P' \quad \Big|\quad \begin{aligned} b_1\otimes b_2\otimes p &\mapsto b_1\otimes b_2\otimes u(p) \\ b_1\otimes b_2\otimes p_1 v(q_1)p' &\leftarrow b_1\otimes b_2 p_1 q_1\otimes p' \end{aligned}$$

If N' is B'-firm, then

$$B^{(2)}\otimes_B \underbrace{P\otimes_A Q'\otimes_{B'} N'}_{B\text{-firm}} \xrightarrow{\sim} B^{(2)}\otimes_B \underbrace{P'\otimes_A Q'\otimes_{B'} N'}_{N'}$$

$$\boxed{\begin{aligned} (P\otimes_A Q')\otimes_{B'} N' &\xrightarrow{\sim} B^{(2)}\otimes_B N' \\ b_1b_2\otimes p\otimes q'\otimes n' &\longmapsto b_1\otimes b_2\otimes u(p)q'n' \\ b_1b_2\otimes p_1\otimes v(q_1)\otimes n' &\longleftarrow b_1\otimes b_2 p_1q_1\otimes n' \end{aligned}}$$ ← can drop b_2

maps well-defined: Use $B^{(2)}\otimes_B P\otimes_A Q \xrightarrow{\sim} B^{(2)}\otimes_B \tilde{B} = B^{(2)}$. ✓

$\circlearrowright$: $b_1b_2\otimes u(p_1)v(q_1)n' = b_1b_2\otimes w(p_1q_1)n' = b_1\otimes b_2p_1q_1\otimes n'$

$\circlearrowleft$: $b_1b_2 p_1q_1 p\otimes q'\otimes n' \longmapsto b_1\otimes b_2p_1q_1\otimes u(p)q'n' \longmapsto$

$b_1b_2\otimes p_1\otimes v(q_1)\otimes_{B'} u(p)q'n' = b_1b_2\otimes p_1\otimes \underbrace{v(q_1)u(p)}_{q_1p}q'\otimes n' =$

$b_1b_2\otimes p_1q_1p\otimes q'\otimes n'$ ~~[struck out]~~.

Next check compatibility with the adjunction maps.

$$N = (P \otimes_A Q') \otimes_{B'} (P' \otimes_A Q) \otimes_B N \qquad b_1 b_2 p q' p' q n = b_1 b_2 p \otimes q' \otimes p' \otimes q \otimes n$$

$$\Big| \wr$$

$$P \otimes_A Q' \otimes B' \otimes_B N \qquad b_1 b_2 p \otimes q' \otimes p' v(q) \otimes n$$

$$\Big| \wr$$

$$B^{(2)} \otimes_B B' \otimes_B N \qquad b_1 \otimes b_2 \otimes \underbrace{u(p) q' p' v(q)}_{w(p q' p' q)} \otimes n$$

$$\uparrow \beta \qquad\qquad \uparrow$$

$$N \qquad b_1 b_2 \, p q' p' q \, n$$

$$N' \qquad p' q b_1 b_2 p q' n'$$

$$\| \qquad\qquad \|$$

$$(P' \otimes_A Q) \otimes_B (P \otimes_A Q') \otimes_{B'} N' \qquad p' \otimes q \otimes b_1 b_2 p \otimes q' \otimes n'$$

$$\downarrow \wr \qquad\qquad \wr$$

$$P' \otimes_A Q \otimes_B B^{(2)} \otimes_B N' \qquad p' \otimes q \otimes b_1 \otimes b_2 \otimes u(p) q' n'$$

$$\downarrow \wr \qquad\qquad \|$$

$$B' \otimes_B B^{(2)} \otimes_B N' \qquad p' v(q) \otimes b_1 \otimes b_2 \otimes u(p) q' n'$$

$$\downarrow \qquad\qquad \downarrow$$

$$N' \qquad p \, v(q) \, w(b_1) \, w(b_2) \, u(p) \, q' n' = p q b_1 b_2 p q' n'$$

Record some additional formulas

$$B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M$$

$$b' \otimes p \otimes m \longmapsto b' u(p) \otimes m$$

$$p' v(q) \otimes p \otimes m \longleftarrow p' \otimes qpm$$

$$Q \otimes_B B^{(2)} \otimes_B N' \xrightarrow{\sim} Q' \otimes_{B'} N'$$

$$q \otimes b_1 \otimes b_2 \otimes n' \longmapsto v(qb_1b_2) \otimes n'$$

$$q \otimes b_1 \otimes b_2 \otimes u(p) q' n' \longleftarrow q b_1 b_2 p q' \otimes n'$$

correspond under adjunction (i.e. are transpose)

December 17, 1995

Let's discuss the problem of defining iterated derived tensor products of bimodules. Let A, B, C be (unital) rings let ${}_A U_B$, ${}_B V_C$ be (unitary) bimodule complexes. We wish to define $U \overset{L}{\otimes}_B V$ as a complex of (A,C)-bimodules determined up to quis. It should also be functorial as U, V range over the appropriate derived categories.

The obvious thing to do is to set $U \overset{L}{\otimes}_B V = \hat{U} \otimes_B \hat{V}$ where $\hat{U}$ (resp. $\hat{V}$) is a projective (A,B)-bimodule (resp (A,C)-bimodule) resolution of U (resp V). These resolutions are defined and functorial up to homotopy, so you get a bifunctor on the derived categories. The same construction should be possible for iterated derived tensor products, even circular ones.

The problem with this construction is that the homology of this $U \overset{L}{\otimes}_B V$ might be different if the $A + C$ structures were ignored. You would want to know that $\hat{U}$ is flat over B^{op} or $\hat{V}$ is flat over B. You want there to be enough (A,B)-bimodules which are flat over B^{op}, which forces $A \otimes_k B$ to be flat over B^{op}. Sufficient for this is for A to be flat over the ground ring k. Similarly if C is flat over k, then $B \otimes_k C$ is B-flat, so $\hat{V}$ is B-flat.

Another thing you would like is associativity (which should be related to the composite functor condition). Suppose given ${}_C W_D$ and ask whether

$$U \overset{L}{\otimes}_B V \overset{L}{\otimes}_C W = U \overset{L}{\otimes}_B (V \overset{L}{\otimes}_C W) = (U \overset{L}{\otimes}_B V) \overset{L}{\otimes}_C W \; ?$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\hat{U} \otimes_B \hat{V} \otimes_C \hat{W} \qquad\qquad \hat{U} \otimes_B \left(\widehat{\hat{V} \otimes_C \hat{W}}\right)$$

You want $\hat{U}\otimes_B -$ to respect qeis, or a better sufficient condition would be for $\hat{V}\otimes_C \hat{W}$ to 'good' for $\hat{U}\otimes_B -$. We know $\hat{V}\otimes_C \hat{W}$ is made up of $(B\otimes_k C)\otimes_C (C\otimes_k D)$ $= B\otimes_k C\otimes_k D$. Thus if C is k-flat, then $\hat{V}\otimes_C \hat{W}$ is a flat (B,D)-bimodule complex.

<u>Summary</u>: 1) Given bimodule complexes ${}_A U_B$, ${}_B V_C$ let $\hat{U}\longrightarrow U$, $\hat{V}\longrightarrow V$ be flat bimodule resolutions. Then

$U\overset{L}{\otimes}_B V \overset{def}{=} \hat{U}\otimes_B \hat{V}$ is a flat (A,C)-bimodule complex if $A\otimes_k B\otimes_k C$ is flat over $A\otimes C^{op}$, e.g. B is k-flat

$\sim \hat{U}\otimes_B V$ if $A\otimes_k B$ is B^{op} flat, e.g. A is k-flat

$\sim U\otimes_B \hat{V}$ if $B\otimes_k C$ is B-flat, e.g. C is k-flat

2) Given M_A, ${}_A U_B$, ${}_B N$ complexes, then

$$M\overset{L}{\otimes}_A U\overset{L}{\otimes}_B N \simeq M\otimes_A \hat{U}\otimes_B N$$

provided $\hat{M}\otimes_k \hat{N} \xrightarrow{\sim} M\otimes_k N$, e.g. either A or B k-flat* and $M\overset{L}{\otimes}_k N \simeq M\otimes_k N$.

* can be weakened to $A\overset{L}{\otimes}_k B \simeq A\otimes_k B$

3) $M\overset{L}{\otimes}_A \overset{def}{=} \hat{M}\otimes_A = \hat{M}\otimes_A A\otimes_A \simeq \hat{M}\otimes_A \hat{A}\otimes_A \simeq M\otimes_A \hat{A}\otimes_A$ is true without flatness assumptions.

December 22, 1995

More adjoint functor stuff. Let

$$(F, G, \alpha, \beta) \qquad (F', G', \alpha', \beta')$$

be two pairs of adjoint functors between the same two categories. Consider $\theta : F \to F'$. [redacted]

$$1) \qquad \begin{array}{ccc} \mathrm{Hom}(F'X, Y) & \xrightarrow{\theta^*} & \mathrm{Hom}(FX, Y) \\ \| & & \| \\ \mathrm{Hom}(X, G'Y) & \overset{\theta^t_*}{\dashrightarrow} & \mathrm{Hom}(X, GY) \end{array}$$

This diagram shows, thanks to Yoneda's lemma, that there is a induced map $\theta^t : G' \to G$ which is called the transpose of θ. Clearly we have

$$\mathrm{Hom}(F, F') = \mathrm{Hom}(G', G)$$

$$\theta \longmapsto \theta^t$$

$$\theta \text{ isom.} \iff \theta^t \text{ isom.} \quad \text{and } (\theta^{-1})^t = (\theta^t)^{-1}.$$

One has the following commutative squares

$$\begin{array}{ccc} FG'Y & \xrightarrow{F.\theta^t} & FGY \\ \theta.G' \downarrow & & \downarrow \alpha \\ F'G'Y & \xrightarrow[\alpha']{} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\beta} & GFX \\ \beta' \downarrow & & \downarrow G.\theta \\ G'F'X & \xrightarrow[\theta^t.F']{} & GF'X \end{array}$$

obtained from 1). [redacted]

Consequently the isomorphisms $(\theta, (\theta^t)^{-1}) : (F, G) \to (F', G')$ (assuming θ is an isom.) are compatible with the adjunction maps, i.e.

$$\begin{array}{ccc} FGY & \xrightarrow{\theta.(\theta^t)^{-1}} & F'G'Y \\ & \alpha \searrow \quad \swarrow \alpha' & \\ & Y & \end{array} \qquad \begin{array}{ccc} & X & \\ & \beta \swarrow \quad \searrow \beta' & \\ GFX & \xrightarrow[(\theta^t)^{-1}.\theta]{} & G'F'X \end{array}$$

commute

<u>January 3, 1996</u>

Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be an equivalence of categories, i.e. a fully faithful and essentially surjective functor. Then there is a quasi-inverse (G, ε, η) for F, which is unique up to canonical isomorphism and obtained as follows.

F essentially surjective means we can choose for each Y a GY in $\mathcal{X}$ together with an isomorphism $\varepsilon_Y : FG(Y) \xrightarrow{\sim} Y$. Because F is fully faithful we can define G uniquely on morphisms in $\mathcal{Y}$ such that G becomes a functor and $\boxed{\varepsilon : FG \xrightarrow{\sim} 1}$ is an isomorphism. Then we have $\varepsilon . F : FGF \xrightarrow{\sim} F$, so again as F is fully faithful, there is a unique isomorphism $\boxed{\eta : GF \xrightarrow{\sim} 1}$ such that $\boxed{F.\eta = \varepsilon . F}$. I claim that also $\boxed{\eta . G = G . \varepsilon}$ holds.

Proof. By definition given $v : Y \to Y'$, then $G(v) : G(Y) \to G(Y')$ is the unique map such that

$$\begin{array}{ccc} FG(Y) & \xrightarrow{F(G(v))} & FG(Y') \\ \varepsilon_Y \downarrow \wr & & \wr \downarrow \varepsilon_{Y'} \\ Y & \xrightarrow{\quad v \quad} & Y' \end{array}$$

commutes. Thus $G(\varepsilon_Y)$ is unique such that

$$\begin{array}{ccc} FGFGY & \xrightarrow{F(G(\varepsilon_Y))} & FGY \\ \varepsilon_{FGY} \downarrow & & \downarrow \varepsilon_Y \\ FGY & \xrightarrow{\quad \varepsilon_Y \quad} & Y \end{array}$$

commutes and as ε_Y is an isomorphism, this means $G(\varepsilon_Y)$ is unique such that $FG(\varepsilon_Y) = \varepsilon_{FGY}$. But $\eta_X : GFX \to X$ by definition is unique such that $F(\eta_X) = \varepsilon_{FX}$. Taking $X = GY$, we find $\eta_{GY} = G(\varepsilon_Y)$ i.e. $\eta . G = G . \varepsilon$.

Notation: Given $\xi : F \to F'$, $\zeta : G \to G'$ of functors which can be composed we write $\xi \circ \zeta$ for the induced map on compositions:

$$\begin{array}{ccc} FG & \xrightarrow{F \circ \zeta} & FG' \\ {\scriptstyle \xi \circ G}\downarrow & \searrow^{\xi \circ \zeta} & \downarrow{\scriptstyle \xi \circ G'} \\ F'G & \xrightarrow[F' \circ \zeta]{} & F'G' \end{array}$$

Also we ~~write~~ $F \circ \zeta$ instead of $1_F \circ \zeta$. (Maybe $*$ is a traditional notation).

Uniqueness of quasi-inverses. Let (G, ε, η), $(G', \varepsilon', \eta')$ be two quasi-inverses for F. Then we have

$F \circ \eta = \varepsilon \circ F : FGF \to F$ $\qquad$ $F \circ \eta' = \varepsilon' \circ F : FG'F \to F$

$G \circ \varepsilon = \eta \circ G : GFG \to G$ $\qquad$ $G' \circ \varepsilon' = \eta' \circ G' : G'FG' \to G'$

Now define $\xi : G \simeq G'$ by either

1) $\quad G \xleftarrow{G \circ \varepsilon'} GFG' \xrightarrow{\eta \circ G'} G'$

1') $\quad G \xleftarrow{\eta' \circ G} G'FG \xrightarrow{G' \circ \varepsilon} G'$

Let's check 1) is compatible with ε-maps:

$$\begin{array}{ccccc} FG & \xleftarrow{FG \circ \varepsilon'} & FGFG' & \xrightarrow[\varepsilon \circ F]{F \circ \eta \circ G'} & FG' \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon \circ \varepsilon'} & & \downarrow{\scriptstyle \varepsilon'} \\ 1 & = & 1 & = & 1 \end{array}$$

and with η-maps:

$$\begin{array}{ccccc} GF & \xleftarrow{G\varepsilon' \circ F} & GFG'F & \xrightarrow{\eta \circ G'F} & G'F \\ {\scriptstyle \eta}\uparrow & & \uparrow{\scriptstyle \eta \circ \eta'} & & \uparrow{\scriptstyle \eta'} \\ 1 & = & 1 & = & 1 \end{array}$$

$GF \circ \eta'$

Next show 1) = 1') by applying F to 1').

$$\begin{array}{ccccc} & & \varepsilon'\cdot FG & & \\ & & \| & & \\ FG & \xleftarrow{F\cdot\eta'\cdot G} & FG'FG & \xrightarrow{FG'\cdot\varepsilon} & FG' \\ \varepsilon\downarrow & & \varepsilon'\cdot\varepsilon \downarrow & & \downarrow\varepsilon' \\ 1 & = & 1 & = & 1 \end{array}$$

Thus F applied to 1) and 1') yield the same map, namely $(\varepsilon')^{-1}\varepsilon$, so 1) = 1') as F is fully faithful.

Jan. 4, 1996 [illegible]

Let (G, ε, η) and $(G', \varepsilon', \eta')$ be quasi-inverses for F:

$$\begin{array}{ll} \varepsilon : FG \xrightarrow{\sim} 1 & \varepsilon\cdot F = F\cdot\eta : FGF \to F \\ \eta : GF \xrightarrow{\sim} 1 & G\cdot\varepsilon = \eta\cdot G : GFG \to G \\ \varepsilon' : FG' \xrightarrow{\sim} 1 & \varepsilon'\cdot F = F\cdot\eta' : FG'F \to F \\ \eta' : G'F \xrightarrow{\sim} 1 & G'\cdot\varepsilon' = \eta'\cdot G' : G'FG' \to G' \end{array}$$

Because F is fully faithful $\exists!\ \xi : G \to G'$ such that

$$\begin{array}{ccc} FG & \xrightarrow{F\cdot\xi} & FG' \\ \varepsilon\downarrow & & \downarrow\varepsilon' \\ 1 & = & 1 \end{array}$$

commutes.

Then

$$\begin{array}{ccc} FGF & \xrightarrow{F\cdot\xi\cdot F} & FG'F \\ \downarrow{\scriptstyle \varepsilon\cdot F = F\cdot\eta} & & \downarrow{\scriptstyle \varepsilon'\cdot F = F\cdot\eta'} \\ F & = & F \end{array} \text{ comm. } \overset{F \text{ f.f.}}{\Longrightarrow} \begin{array}{ccc} GF & \xrightarrow{\xi\cdot F} & G'F \\ \downarrow\eta & & \downarrow\eta' \\ 1 & = & 1 \end{array} \text{ comm.}$$

Hence we have $\xi : (G, \varepsilon, \eta) \xrightarrow{\sim} (G', \varepsilon', \eta')$. To get a formula for ξ apply G

$$\begin{array}{ccccccc} G & \xleftarrow{G\cdot\varepsilon} & GFG & \xrightarrow{GF\cdot\xi} & GFG' & \xrightarrow{G\cdot\varepsilon'} & G \\ \| & & \downarrow\eta\cdot G & & \downarrow\eta\cdot G' & & \\ & & G & \xrightarrow{\xi} & G' & & \end{array}$$

hence

$$\begin{array}{ccc} & GFG' & \\ \swarrow{\scriptstyle G\cdot\varepsilon'} & & \searrow{\scriptstyle \eta\cdot G'} \\ G & \xrightarrow{\xi} & G' \end{array}$$

which is 1) on p85. Similarly

$$\begin{array}{ccccccc}
 & & & \overset{1}{\frown\frown\frown\frown\frown} & & & \\
G & \xleftarrow{\eta\cdot G} & GFG & \xrightarrow{\xi\cdot FG} & G'FG & \xrightarrow{\eta'\cdot G} & G \\
 & \diagdown\!\!\diagdown & \downarrow{\scriptstyle G\cdot\varepsilon} & & \downarrow{\scriptstyle G'\cdot\varepsilon} & & \\
 & & G & \xrightarrow{\xi} & G' & &
\end{array}
\quad \text{yields} \quad
\begin{array}{ccc}
 & G'FG & \\
{\scriptstyle \eta'\cdot G}\swarrow & & \searrow{\scriptstyle G'\cdot\varepsilon} \\
G & \xrightarrow{\xi} & G'
\end{array}$$

which is 1') on p85.

Next I want to show using the essential uniqueness of a quasi-inverse that given $(F, G, \varepsilon, \eta)$, $(F', G', \varepsilon', \eta')$ and $\theta : F \xrightarrow{\sim} F'$ there is a corresponding $\xi : G \xrightarrow{\sim} G'$ such that (θ, ξ) is an isomorphism $(F, G, \varepsilon, \eta) \xrightarrow{\sim} (F', G', \varepsilon', \eta')$

i.e.

$$\begin{array}{ccc}
FG & \xrightarrow{\theta\cdot\xi} & F'G' \\
{\scriptstyle\varepsilon}\searrow & & \swarrow{\scriptstyle\varepsilon'} \\
 & 1 &
\end{array}
\qquad
\begin{array}{ccc}
GF & \xrightarrow{\xi\cdot\theta} & G'F' \\
{\scriptstyle\eta}\searrow & & \swarrow{\scriptstyle\eta'} \\
 & 1 &
\end{array}
\qquad \text{commute}$$

in other words we have $\varepsilon = \varepsilon'(\theta\cdot\xi)$, $\eta = \eta'(\xi\cdot\theta)$.

The idea is that $\theta : F \xrightarrow{\sim} F'$ makes G' into a quasi-inverse for F. More precisely we have an isom.

$$(\theta, 1) : (F, G', \varepsilon'(\theta\cdot G'), \eta'(G'\theta)) \xrightarrow{\sim} (F', G', \varepsilon', \eta')$$

Then we know there is a $\xi : G \to G'$ such that

$$(1, \xi) : (F, G, \varepsilon, \eta) \xrightarrow{\sim} (F, G', \varepsilon'(\theta\cdot G'), \eta'(G'\cdot\theta))$$

i.e.

$$\begin{aligned}
\varepsilon &= \varepsilon'(\theta\cdot G')(F\cdot\xi) & \text{and} \quad \eta &= \eta'(G'\cdot\theta)(\xi\cdot F) \\
&= \varepsilon'(\theta\cdot\xi) & &= \eta'(\xi\cdot\theta).
\end{aligned}$$

The first equation says:

$$\begin{array}{ccccc}
FG & \xrightarrow{F\cdot\xi} & FG' & \xrightarrow{\theta\cdot G'} & F'G' \\
\downarrow{\scriptstyle\varepsilon} & & & & \downarrow{\scriptstyle\varepsilon'} \\
1 & & = & & 1
\end{array}$$

This determines ξ since the other maps are isomorphisms and F is fully-faithful

March 12, 1996

I want to record formulas involved in the equivalences:

$$\begin{array}{ccc} \mathrm{mod}(\mathbb{R}) & \underset{\text{res}}{\overset{\text{extn}}{\rightleftarrows}} & \mathrm{mod}(\mathbb{H}) \\ \wr\| & & \| \\ \mathrm{mod}(\mathbb{C}[\sigma]_+) & \underset{H\otimes -}{\overset{H\otimes -}{\rightleftarrows}} & \mathrm{mod}(\mathbb{C}[\sigma]_-) \end{array}$$

First consider the adjoint functor relations

(1)
$$\mathrm{Hom}(H\otimes V, W) = \mathrm{Hom}(V, H^*\otimes W)$$
$$\mathrm{Hom}(V, H\otimes W) = \mathrm{Hom}(H^*\otimes V, W)$$

where H is a finite-diml vector space. The adjunction maps in the former arise from the canonical maps

$$\alpha: H\otimes H^* \longrightarrow \mathbb{C} \qquad h\otimes h^* \longmapsto (h|h^*) = (h^*|h)$$
$$\beta: \mathbb{C} \longrightarrow H^*\otimes H \qquad 1 \longmapsto \sum e_i^*\otimes e_i$$

The adjunction maps in the latter arise from the canonical maps

$$\alpha': H^*\otimes H \longrightarrow \mathbb{C} \qquad h^*\otimes h \longmapsto (h^*|h)$$
$$\beta': \mathbb{C} \longrightarrow H\otimes H^* \qquad 1 \longmapsto \sum e_i\otimes e_i^*$$

obtained from the preceding via the flips. Notice that $\alpha'\beta = \alpha\beta'$ is $\mathrm{tr}(1) = \dim H$.

In the $\mathbb{R}, \mathbb{H}$ situation, $H \approx \mathbb{C}^2$ comes equipped with a volume $\Lambda^2 H = \mathbb{C}$ which we use to identify H^* and H. We have a single adjoint functor relation

$$\mathrm{Hom}(H\otimes V, W) = \mathrm{Hom}(V, H\otimes W)$$

arising from the canonical maps

$$\alpha : H \otimes H \longrightarrow \mathbb{C} \qquad h_1 \otimes h_2 \longmapsto h_1 \wedge h_2$$

$$\beta : \mathbb{C} \longrightarrow H \otimes H \qquad 1 \longmapsto e_2 \otimes e_1 - e_1 \otimes e_2$$

(here $e_1 \wedge e_2 = 1$)

Check: Let $h = z_1 e_1 + z_2 e_2$ so that $z_2 = e_1 \wedge h$, $z_1 = -e_2 \wedge h$

$$H \xrightarrow{\beta \otimes 1} H \otimes H \otimes H \xrightarrow{1 \otimes \alpha} H$$

$$h \longmapsto (e_2 \otimes e_1 - e_1 \otimes e_2) \otimes h \longmapsto e_2 (e_1 \wedge h) - e_1 (e_2 \wedge h) = h$$

$$H \xrightarrow{1 \otimes \beta} H \otimes H \otimes H \xrightarrow{\alpha \otimes 1} H$$

$$h \longmapsto h \otimes (e_2 \otimes e_1 - e_1 \otimes e_2) \longmapsto (h \wedge e_2) e_1 - (h \wedge e_1) e_2 = h$$

Notice that $\mathbb{C} \xrightarrow{\beta} H \otimes H \xrightarrow{\alpha} \mathbb{C}$ is

$1 \mapsto e_2 \wedge e_1 - e_1 \wedge e_2 = -2(e_1 \wedge e_2) = -2$. (I don't really understand this sign. Somehow it arises from the fact that the volume $\Lambda^2 H = \mathbb{C}$ determines two isos. of H with H^*, i.e. $\Lambda^2 H \subset H \otimes H$ and you can contract either factor of H with H^*; these two isos. have opposite sign. Perhaps also this sign is related to what happens with the Fourier transform.)

Next recall $\mathbb{C}[\sigma]_{\pm} = \mathbb{C} + \mathbb{C}\sigma$ where $\sigma z = \bar{z}\sigma$ and $\sigma^2 = \pm 1$. $\mathbb{C}[\sigma]_- = \mathbb{H}$ where $\sigma = j$, so $\text{mod}(\mathbb{H}) = \text{mod}(\mathbb{C}[\sigma]_-)$ trivially. $\mathbb{C}[\sigma]_+ \cong M_2 \mathbb{R}$ so $\text{mod}(\mathbb{R}) = \text{mod}(\mathbb{C}[\sigma]_+)$ is a Morita equivalence, the functors being $V_r \mapsto \mathbb{C} \otimes_{\mathbb{R}} V_r$, $V \longmapsto V^{\sigma}$.

Now take $H = \mathbb{H} = \mathbb{C} + \mathbb{C}j$ with $\mathbb{C}$ acting by left multiplication and $\Lambda^2 H = \mathbb{C}$ given by $1 \wedge j = 1$. σ on H is is left mult by j. (Reason for notation H is to avoid confusion arising from $\mathbb{H} \otimes_{\mathbb{C}} V$ when $\mathbb{C}$ is left acting on $\mathbb{H}$.)

If $V \in \text{mod}(\mathbb{C}[\sigma]_+)$, then $H \otimes V$ equipped with $\sigma \otimes \sigma$ is in $\text{mod}(\mathbb{C}[\sigma_-])$. Conversely $W \in \text{mod}(\mathbb{C}[\sigma]_-)$ $\Rightarrow H \otimes W \in \text{mod}(\mathbb{C}[\sigma]_+)$.

Recall that restriction of scalars has both left and right adjoints. In the case of $\mathbb{R}\subset\mathbb{C}$ these two adjoints are isomorphic:

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, V^{\sigma}) = \underbrace{\mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{R})}_{\mathbb{H}^*}\otimes_{\mathbb{R}} V^{\sigma} \xleftarrow{\sim} \mathbb{H}\otimes_{\mathbb{R}} V^{\sigma}$$

where any nonzero element of $\mathbb{H}^*$ yields an isom. In practice one takes a trace $\tau: \mathbb{H}\to\mathbb{R}$ which is unique up to a scalar, since $\mathbb{H} = \mathbb{R}\oplus[\mathbb{H},\mathbb{H}]$.

We use the following isomorphism

$$\mathbb{H}\otimes_{\mathbb{R}} V^{\sigma} \simeq H\otimes V$$

$$(z_1 1 + z_2 j)\otimes v \longmapsto 1\otimes z_1 v + j\otimes z_2 v = \begin{pmatrix} z_1 v \\ z_2 v \end{pmatrix}$$

to link the left adjoint (extension of scalars) from $\mathrm{mod}(\mathbb{R})$ to $\mathrm{mod}(\mathbb{H})$ with $V\mapsto H\otimes V$.

Then we have isos. (this uses that σ compatible with α,β)

$$\mathrm{Hom}(H\otimes V, W)^{\sigma} = \mathrm{Hom}(V, H\otimes W)^{\sigma}$$

$$\| \qquad\qquad\qquad \|$$

$$\mathrm{Hom}_{\mathbb{H}}(H\otimes V, W) \qquad \mathrm{Hom}_{\mathbb{R}}(V^{\sigma}, (H\otimes W)^{\sigma})$$

$$\|$$

$$\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}\otimes_{\mathbb{R}} V^{\sigma}, W)$$

$$\|$$

$$\mathrm{Hom}_{\mathbb{R}}(V^{\sigma}, W)$$

By Yoneda this yields a canonical isomorphism

$$\rho W \simeq (H\otimes W)^{\sigma}$$

$$w \longmapsto j\otimes w - 1\otimes jw$$

where ρ stands for $\mathrm{res}^{\mathbb{H}}_{\mathbb{R}}$. This identifies ρ with $H\otimes -$ from $\mathrm{mod}(\mathbb{C}[\sigma]_-)$ to $\mathrm{mod}(\mathbb{C}[\sigma]_+)$.

Next we have isos

$$
\begin{array}{ccc}
\mathrm{Hom}(H\otimes W, V)^\sigma & = & \mathrm{Hom}(W, H\otimes V)^\sigma \\
\| & & \| \\
\mathrm{Hom}_{\mathbb{R}}((H\otimes W)^\sigma, V^\sigma) & & \mathrm{Hom}_{\mathbb{H}}(W, H\otimes V) \\
\| & & \| \\
\mathrm{Hom}_{\mathbb{R}}(\rho W, V^\sigma) & & \mathrm{Hom}_{\mathbb{H}}(W, \mathbb{H}\otimes_{\mathbb{R}} V^\sigma) \\
\| & & \\
\mathrm{Hom}_{\mathbb{H}}(W, \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, V^\sigma)) & &
\end{array}
$$

Thus we get a canon. isom. $\boxed{\mathbb{H}\otimes_{\mathbb{R}} V^\sigma \simeq \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, V^\sigma)}$

which amounts to an element τ of $\mathbb{H}^*$, (take $V^\sigma = \mathbb{R}$).

Calculation gives $\tau(1) = -2$, $\tau(i) = \tau(j) = \tau(k) = 0$.

April 4, 1996

Observation

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \ell'_z\otimes V & \overset{b'a'}{\dashrightarrow} & & & \\ & & & & \downarrow a' & & & & \\ 0 & \longrightarrow & W & \xrightarrow{a} & H\otimes V & \xrightarrow{b'} & Q & \longrightarrow & 0 \\ & & & \searrow^{ba} & \downarrow b & & & & \\ & & & & \ell_z\otimes V & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

The six term exact sequence of kernels and cokernels becomes

$$0 \longrightarrow \mathrm{Ker}(ba) \longrightarrow \underset{\substack{\parallel \\ \ell'_z\otimes Q}}{\mathrm{Ker}(b)} \xrightarrow{\;b'a'\;} \underset{\substack{\parallel \\ Q}}{\mathrm{Coker}(a)} \longrightarrow \mathrm{Coker}(ba) \longrightarrow 0$$

so that it looks as if the complexes $W \xrightarrow{ba} \ell_z\otimes V$ and $\ell'_z\otimes V \xrightarrow{b'a'} Q$ are quasi-isomorphic.

When you have more time examine this carefully. Recall something similar appeared in connection with Vaserstein's lemma, more specifically, when you proved Morita invariance for K^1.

May 24, 1996

Canonical resolutions over P^1. If F is a regular sheaf over P^1 then it has a resolution of the form

$$0 \to \mathcal{O}(-1)\otimes W \to \mathcal{O}\otimes V \to F \to 0$$

Tensor this short exact sequence with

$$0 \to \Lambda^2 H \otimes \mathcal{O}(-1) \to H\otimes\mathcal{O} \to \mathcal{O}(1) \to 0$$

to get

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda^2 H\otimes\mathcal{O}(-2)\otimes W & \to & \Lambda^2 H\otimes\mathcal{O}(-1)\otimes V & \to & \Lambda^2 H\otimes F(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H\otimes\mathcal{O}(-1)\otimes W & \to & H\otimes\mathcal{O}\otimes V & \to & H\otimes F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}\otimes W & \to & \mathcal{O}(1)\otimes V & \to & F(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whence

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \Lambda^2 H\otimes H^0(F(-1)) & & \\
 & & & & & & \downarrow & & \\
 & & & & H\otimes V & \xrightarrow{\sim} & H\otimes H^0(F) & & \\
 & & & & \| & & \downarrow & & \\
0 & \to & W & \to & H\otimes V & \to & H^0(F(1)) & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

which identifies $W \to H\otimes V$ with the map $\Lambda^2 H\otimes H^0(F(-1)) \to H\otimes H^0(F)$ induced by $\Lambda^2 H\otimes F(-1) \to H\otimes F$.

Next suppose G is a negative vector bundle. Then it has a dual canonical resolution of the form

$$0 \to G \to \mathcal{O}(-1)\otimes W \to \mathcal{O}\otimes V \to 0$$

Again we get by tensoring

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda^2 H\otimes G(-1) & \to & \Lambda^2 H\otimes \mathcal{O}(-2)\otimes W & \to & \Lambda^2 H\otimes \mathcal{O}(-1)\otimes V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H\otimes G & \to & H\otimes \mathcal{O}(-1)\otimes W & \to & H\otimes \mathcal{O}\otimes V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G(1) & \to & \mathcal{O}\otimes W & \to & \mathcal{O}(1)\otimes V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whence

$$\begin{array}{ccccc}
 & & H^0(G(1)) & & \\
 & & \downarrow & & \\
 & & \Lambda^2 H\otimes H^1(G(-1)) & \xrightarrow{\sim} & \underbrace{H^1(\Lambda^2 H\otimes \mathcal{O}(-2)\otimes W)}_{W} \\
 & & \downarrow & & \\
H\otimes V & \xrightarrow{\sim} & H\otimes H^1(G) & & \\
 & & \downarrow & & \\
 & & H^1(G(1)) & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

The problem is now to identify the map $W \to H\otimes V$ arising from this diagram with the map on H^0 induced by $(\mathcal{O}(-1)\otimes W \to \mathcal{O}\otimes V)$ tensored with $\mathcal{O}(1)$.

We will construct various maps of complexes linked by $R\Gamma$-isos. First map

$$\begin{array}{ccc} \Lambda^2 H \otimes G(-1)[1] & \dashrightarrow & \Lambda^2 H \otimes \mathcal{O}(-2) \otimes W [1] \\ \downarrow & & \downarrow \\ H \otimes G[1] & & \left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O}(1) \otimes V \right) \end{array}$$

where $\dashrightarrow$ is essentially obtained from the first two rows of the 3x3 diagram above. Second map

$$\begin{array}{ccc} \Lambda^2 H \otimes \mathcal{O}(-2) \otimes W [1] & \dashrightarrow & \left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O} \otimes W \right) \\ \downarrow & & \downarrow \\ \left(H \otimes \mathcal{O}(-1) \otimes W \to \mathcal{O}(1) \otimes V \right) & & \left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O}(1) \otimes V \right) \end{array}$$

Third map is inclusion

$$\begin{array}{ccc} \left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O} \otimes W \right) & \dashleftarrow & \mathcal{O} \otimes W \\ \downarrow & & \downarrow \\ \left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O}(1) \otimes V \right) & & \mathcal{O}(1) \otimes V \end{array}$$

One can check that the dotted arrows induce isos on $R\Gamma$ for both source & target of the vertical arrow. So applying $R^0\Gamma$ we get a commutative square

$$\begin{array}{ccc} \Lambda^2 H \otimes H^1(G(-1)) & \simeq & W \\ \downarrow & & \downarrow \\ H \otimes H^1(G) & \simeq & H \otimes V \end{array}$$

as desired.

August 1, 1996

Consider the problem of Morita invariance of K-theory for h-unital rings, but restrict one of the rings to be unital. Suppose then A is unital and (P,Q) is a firm dual pair over A. $B = P\otimes_A Q$ is h-unital iff $P \overset{L}{\otimes}_A Q = P\otimes_A Q$, e.g. if either P or Q is flat over A. If Q is flat, then Q is an inductive limit of fg free modules, and similarly for P.

Note that surjectivity of $Q\otimes P \longrightarrow A$ means $\exists\, p_i, q_i$ with $\sum_{i=1}^n q_i p_i = 1$. In this case, replacing (P,Q) by $(P,Q)^n$ and B by $M_n B$ we reduce to the case where $\exists\, p\in P,\ q\in Q$ with $qp = 1$. Then $(P,Q) = (A,A)\oplus(X,Y)$, $X = \{x\in P \mid xq = 0\}$, $Y = \{y\in Q \mid yp = 0\}$, so $B = \begin{pmatrix} A & Y \\ X & X\otimes_A Y\end{pmatrix}$ and the pairing $Y\otimes X \to A$ can be arbitrary. Also $e = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}$ is an idempotent in B such that $A = eBe$, $P = Be$, $Q = eB$, so we have the familiar Morita context

$$\begin{pmatrix} A = eBe & eB \\ Be & BeB = B \end{pmatrix}$$

we examined in connection with Drozdov's thm.

Let's review this result. Start with R, $e = e^2\in R$, $A = eRe$, $P = Re$, $Q = eR$, $B = ReR$. Hypotheses are: $Re\otimes_A eR \xrightarrow{\sim} B$ (i.e. B firm) and $eR \in \mathcal{P}(A)$. We have functors

$$\begin{array}{ccccccc} \mathrm{mod}(R) & \longrightarrow & \mathrm{mod}(A) & \xrightarrow{\sim} & \mathcal{M}(B) & \subset & \mathrm{mod}(R) \\ L & \longmapsto & eL & & N & \mapsto & N \\ & & M & \longmapsto & Re\otimes_A M & & \end{array}$$

where $\mathcal{M}(B) = \mathcal{M}(R,B)$.

which induce

$$\mathcal{P}(R) \longrightarrow \mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(R)$$

Here $\mathcal{P}(B) \simeq \mathcal{P}(R,B)$ is the full subcategory of small projectives in $\mathcal{M}(R,B)$, i.e. $L \in \mathcal{P}(R)$ such that $L = BL$.

We have some obvious maps

$$\begin{array}{ccccc} K_*(\mathcal{P}(B)) & \hookrightarrow & K_*(\mathcal{P}(R)) & \longrightarrow & K_*(\mathcal{P}(R/B)) \\ & \searrow{\scriptstyle\sim} & \downarrow & & \\ & & K_*(\mathcal{P}(A)) & & \end{array}$$

Now $B = Re \otimes_A eR$, $eR \in \mathcal{P}(A) \implies B \in \mathcal{P}(B)$, so we have a resolution by f.g projective modules

$$0 \longrightarrow B \longrightarrow R \longrightarrow R/B \longrightarrow 0.$$

This should imply that any object V in $\mathcal{P}(B)$ has a $0 \to P_1 \to P_0 \to V \to 0$ with $P_i \in \mathcal{P}(R)$. Hence by resolution we get a map $K_*(\mathcal{P}(R/B)) \to K_*(\mathcal{P}(R))$.

Claim $K_*(\mathcal{P}(R/B)) \longrightarrow K_*(\mathcal{P}(R)) \longrightarrow K_*(\mathcal{P}(R/B))$ is the identity. Given $0 \to P_1 \to P_0 \to V \to 0$ f proj R-res. of $V \in \mathcal{P}(R/B)$ one has

$$0 \longrightarrow \mathrm{Tor}_1^R(R/B, V) \longrightarrow P_1/BP_1 \longrightarrow P_0/BP_0 \longrightarrow V \longrightarrow 0$$

This Tor $= 0$ since V is a summand of $(R/B)^n$ and $\mathrm{Tor}_1^R(R/B, R/B) = B/B^2 = 0$.

At this point we know $K_*(\mathcal{P}(B))$ and $K_*(\mathcal{P}(R/B))$ are direct summands of $K_*(\mathcal{P}(R))$. Consider the exact sequence of exact functors

$$0 \longrightarrow B \otimes_B L \longrightarrow L \longrightarrow L/BL \longrightarrow 0$$

from $\mathcal{P}(R)$ to $\mathcal{P}'(R)$ (= modules admitting length ≤ 1 resolutions from $\mathcal{P}(R)$). [redacted] By additivity and $K_*(\mathcal{P}(R)) \xrightarrow{\sim} K_*(\mathcal{P}'(R))$, we get that

$$K_*(R) \longrightarrow K_*(\mathcal{P}(B)) \oplus K_*(R/B) \longrightarrow K_*(R)$$

is the identity. It follows that

$$\begin{matrix} K_*(\mathcal{P}(B)) \oplus K_*(R/B) \xrightarrow{\sim} K_*(R) \\ \| \\ K_*(A) \end{matrix}$$

I think this is correct. When $R = \tilde{B}$ we then get $K_*(A) = K_*(\mathcal{P}(B)) \xrightarrow{\sim} K_*(B)$ $\left(\overset{\text{def}}{=} K_*(\tilde{B})/K_*(\mathbb{Z})\right)$, which is the Morita invariance [redacted] result I am after.

Let's now return to the original setting $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with A unital, [redacted] $Q \otimes P \twoheadrightarrow A$, $P \otimes_A Q = B$, and suppose $Q \in \mathcal{P}(A)$. Now Q is a generator for $\mathrm{mod}(A)$ since we have $Q \otimes P \twoheadrightarrow A$, so without affecting the Morita invariance question we should be able to replace A by the nonunital ring $A' = \mathrm{Hom}_A(Q,Q)^{op}$. We have to compose the maps given by

$$\begin{pmatrix} A' & Q^* \\ Q & A \end{pmatrix} \qquad \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

where $Q^* = \mathrm{Hom}_A(Q, A)$.

$$\begin{pmatrix} A' & Q^* & Q^* \otimes_A Q \\ Q & A & Q \\ P \otimes_A Q & P & B \end{pmatrix} \qquad \therefore \begin{pmatrix} A' & Q^* \otimes_A Q = A' \\ P \otimes_A Q = B & B \end{pmatrix}$$

This transformation reduces us to a Morita context (put A for A') of the form

$$\begin{pmatrix} A & Q=A \\ P=B & B \end{pmatrix}$$

where P can be any A^{op}-module. [redacted] The pairing

$A\otimes P \longrightarrow A$ which must be surjective is given by an A^{op}-module map $f: P\longrightarrow A$, namely $a\otimes p \longmapsto af(p)$. Surjectivity means the right ideal $f(P)$ in A generates A in the sense that $Af(P)=A$. Suslin's excision theory should take care of the surjection $P\otimes_A Q \twoheadrightarrow f(P)\otimes_A Q$, so the important case to consider is when $P=B$ is a right ideal in A such that $AB=A$.

Let's try to understand the case where B is a right ideal in A unital and $\exists\, y\in A,\ x\in B$ such that $yx=1$.

August 2, 1996

Recall setup: $\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ A unital, B right ideal in A satisfying $AB = A$.

We know the following.

- This Morita context is sfirm, being associated to the firm dual pair (B, A) over A where the pairing is $A \otimes B \longrightarrow A$, $a \otimes b \mapsto ab$. Hence $A \otimes_B B \xrightarrow{\sim} A$.

- Since A is unital we know $B \in \mathcal{P}(B)$, $A \in \mathcal{P}(B^{op})$ are dual to each other and $A \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(A, A)$, $A \xrightarrow{\sim} \mathrm{Hom}_B(B^\bullet, B)^{op}$.

- functors on modules

$\downarrow$

$$\mathcal{M}(B) \subset \mathrm{mod}(\tilde{B}) \longrightarrow \mathrm{mod}(A) \xrightarrow{\sim} \mathcal{M}(B) \subset \mathrm{mod}(\tilde{B})$$

$$\mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(\tilde{B})$$

$$L \longmapsto A \otimes_B L \mapsto B \otimes_A A \otimes_B L = B \otimes_B L$$

So just from $\quad \mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A)$

$$V \mapsto B \otimes_A V \longmapsto A \otimes_B B \otimes_A V = V$$

we find $\boxed{K_*(A) \xrightarrow{i} K_*(\tilde{B}) \xrightarrow{j} K_*(A) \quad \text{is the identity.}}$

j is induced by $L \mapsto A \otimes_B L$, i.e. extension of scalars wrt $\tilde{B} \to A$, so j is induced by this homomorphism. Now i is induced by $V \longmapsto B \otimes_A V$, where B is regarded as a representation of A in $\mathcal{P}(B)$; in fact we have $A = \mathrm{Hom}_B(B, B)^{op}$. If we choose an embedding of B as a direct summand of $\tilde{B}^n$, then we get a homomorphism $A \longrightarrow M_n(\tilde{B})$. This homom. induces i.

For example suppose $\exists\, y \in A$, $x \in B$ satisfying $yx = 1$. Then we have

$$\tilde{B} = By \oplus \tilde{B}(1-xy)$$

Why? $1-xy$ is idempotent and $\cdot(1-xy)$ kills By.
$\tilde{b} = \underbrace{\tilde{b}x}_{\in B}y + \tilde{b}(1-xy) \in By + \tilde{B}(1-xy)$.
Also we have $B \xrightarrow{\cdot y} By \xrightarrow{\cdot x} B$ is the identity so $\cdot y : B \xrightarrow{\sim} By$.

It's better to give the pair of maps of B-modules $B \xrightarrow{\cdot y} \tilde{B} \xrightarrow{\cdot x} B$ with composition 1. The corresponding homomorphism $A \longrightarrow \tilde{B}$ is then $a \longmapsto xay$. Check: $(xa_1y)(xa_2y) = xa_1a_2y$.

Now our problem becomes showing that $K_*(\tilde{B}) \xrightarrow{f} K_*(A) \xrightarrow{i} K_*(\tilde{B})$ is projection onto $K_*(B)$. Look at this from the viewpoint of $H_*(GL(-))$. Use Suslin's result that because B is h-unital $H_*(GL(B))$ is the homology of the fibre of $BGL(\tilde{B})^+ \longrightarrow BGL(\mathbb{Z})^+$. Then it seems we want to know that the homomorphism

$$B \hookrightarrow A \longrightarrow B$$
$$a \longmapsto xay$$

induces the identity on $H_*(GL(B))$.

Another way to say this might be the obvious representations of $GL_n(B)$ on B^n and $\tilde{B}^n$ in $P(\tilde{B})$ have the same stable characteristic classes. Somehow you want to deduce this from the exact sequence

$$0 \longrightarrow B^n \longrightarrow \tilde{B}^n \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

Consider the chain of homoms.

$$\hookrightarrow A \longrightarrow B \hookrightarrow A \longrightarrow B \hookrightarrow A \longrightarrow$$
$$a \longmapsto xay$$

Notice that $A \longrightarrow A$, $a \mapsto xay$ is a nonunital ring homomorphism between unital rings. Is it a morphomorphism? We have M. equiv. given by

$$\begin{pmatrix} A & Ay \\ xA & xAy \end{pmatrix}$$

Setting: $B < A$ unital, $BA = B$, $\exists y \in A$, $x \in B$ s.t. $yx = 1$.

We have homomorphisms: $A \to B$, $a \mapsto xay$ and the inclusion $B < A$. These induce maps $BGL(A)^+ \to BGL(B)^+$ and $BGL(B)^+ \longrightarrow BGL(A)^+$. The question is whether they are inverse up to homotopy. Looks at the compositions.

Consider $A \xrightarrow{\phi} A$, $a \longmapsto xay$. This is a non unity preserving homomorphism, but it still induces group homomorphisms $GL_n(A) \longrightarrow GL_n(A)$ for all n. How? If $e = \phi(1) = xy$, then one has a homom. of unital rings $A \longrightarrow eAe$ followed by the inclusion $eAe < A$. The idea is that $Ae \in \mathcal{P}(A)$ has $Hom_A(Ae, Ae) = eAe$, so $\mathcal{P}(eAe)$ is equivalent to the full Karoubian subcat of $\mathcal{P}(A)$ which is generated by Ae. [redacted] We get the functor

$$\mathcal{P}(A) \longrightarrow \mathcal{P}(eAe) \subset \mathcal{P}(A)$$

$$V \mapsto Ae_\phi \otimes_A V \longmapsto Ae_\phi \otimes_A V$$

Here Ae_ϕ means Ae with A acting on the right via ϕ.

Let's calculate this for $\phi(a) = xay$. Note that

$Ae = Axy \subseteq Ay$ and $Ay \subset Ayxy \subset Axy$. $\therefore Ae = Ay$

Take $V = A^n$. Then $Ay_\phi \otimes_A A^n \xrightarrow{\sim} Ay_\phi^n$. Now you choose a split embedding of Ay into a free A^m in order to get a representation of $G = \mathrm{Aut}(V)$ by matrices. In this case

$$Ay \oplus A(1-xy) \simeq A$$
$$(axy \quad a(1-xy)) \longleftarrow a$$
$$(a_1 y \quad a_2(1-xy)) \longmapsto a_1 y + a_2(1-xy)$$

We have isom.
$$Ay_\phi \otimes_A M \simeq M$$
$$a'y \otimes m \longmapsto a'yxm = a'm$$
$$y \otimes m \longleftarrow m$$

Check: $a'y\otimes m \mapsto a'm \mapsto y \otimes a'm = yxa'y \otimes m$,
$m \mapsto y\otimes m \mapsto m$,

Take $M = A$ get isom

$$Ay_\phi \simeq A$$
$$a'y \longmapsto a'yx = a'$$
$$a'y \longleftarrow a'$$

Better $A \to Ay$, $a' \mapsto a'y$
note that $a'y\,\phi(a) = a'yxay = a'ay$, so right mult by a on A corresp. to right mult by xay on Ay.

If $g \in \mathrm{Aut}_A(A^n)$, you get induced autom on $Ay \otimes_A A^n \simeq (Ay)^n \underset{\cdot y}{\overset{\cdot x}{\rightleftarrows}} A^n$: $g = 1+\alpha$ on A^n becomes $1 \otimes (1+\alpha)$ on $Ay \otimes_A A^n$, i.e. $\phi(1+\alpha) = x(1+\alpha)y$ on Ay^n, to which you add $1-xy$ on $A(1-xy)$. Thus we get $1-xy+x(1+\alpha)y = 1+x\alpha y$.

This calculation identifies the effect of the homom. $a \mapsto xay$ on $GL_n(A)$ with what one gets from the functor

$$P(A) \longrightarrow P(eAe) \subset P(A)$$
$$V \longmapsto Ae_\phi \otimes_A V$$

August 4, 1996

Assume $B = B^2$ such that $B \in \mathcal{P}(B)$, i.e. B is a fg proj $\tilde{B}$-module which is firm (since $B = BB$). We have functors

$$\mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(B)$$
$$L \longmapsto B \otimes_{\tilde{B}} L = BL$$

whose composition is $\simeq 1$. On the other hand B is a generator from $\mathcal{P}(B)$, so one has an equivalence $\mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B)$, $V \mapsto B \otimes_A V$ where $A = \operatorname{Hom}_B(B,B)^{op}$. Consequently $K_*(\mathcal{P}(B)) = K_* A$. The above functors give maps $K_* A \xrightarrow{i} K_* \tilde{B} \xrightarrow{j} K_* A$ with composition the identity. Consider next the other composition $K_* \tilde{B} \xrightarrow{j} K_* A \xrightarrow{i} K_* \tilde{B}$ induced by $L \mapsto B \otimes_{\tilde{B}} L = BL$. One has functorial exact sequences from $\mathcal{P}(\tilde{B})$ to $\operatorname{mod}(\tilde{B})$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B \otimes_{\mathbb{Z}} \bar{L} & = & B \otimes_{\mathbb{Z}} \bar{L} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & BL & \longrightarrow & F(L) & \longrightarrow & \tilde{B} \otimes_{\mathbb{Z}} \bar{L} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & BL & \longrightarrow & L & \longrightarrow & \bar{L} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $F(L) = L \times_{\bar{L}} (\tilde{B} \otimes_{\mathbb{Z}} \bar{L})$. In $K_0(\tilde{B})$ we have from the two exact sequences involving $F(L)$.

$$[F(L)] = [B \otimes_{\tilde{B}} L] + [\tilde{B}]\, r(L) = [B]\, r(L) + [L]$$

where $r(L) = \text{rank}_{\mathbb{Z}}(\bar{L})$. Write this

$$[L] = [B \otimes_B L] + ([\tilde{B}] - [B])\, r(L)$$

This yields a direct sum decomposition

K_* [redacted] $A \underset{j}{\overset{\iota}{\rightleftarrows}} K_* \tilde{B} \underset{(\tilde{B}/B \otimes_B -)}{\overset{(\tilde{B}\otimes_{\mathbb{Z}} -) - (B \otimes_{\mathbb{Z}} -)}{\leftrightarrows}} K_* \mathbb{Z}$

in degree 0 at least. But it should hold for all degrees, since functorial exact sequences are additive. $\therefore\ K_* \mathcal{P}(B) = K_* B \overset{\text{def}}{=} K_*(\tilde{B})/K_*(\mathbb{Z})$.

We want to understand the above arguments better. We have $F(L) = F \otimes_{\tilde{B}} L$, where F is the B bimodule $F = \tilde{B} \times_{\mathbb{Z}} \tilde{B}$, $b(x,y) = (bx, by)$, $(x,y)b = (xb, 0)$.

We have B-bimodule exact sequences

$$0 \longrightarrow B_\varepsilon \xrightarrow{b \mapsto (0,b)} F \xrightarrow{pr_1} \tilde{B} \longrightarrow 0$$

$$0 \longrightarrow B \xrightarrow{b \mapsto (b,0)} F \xrightarrow{pr_2} \tilde{B}_\varepsilon \longrightarrow 0$$

where $\tilde{B}_\varepsilon, B_\varepsilon$ mean the right [redacted] action of B is [redacted] via the augmentation $\varepsilon : \tilde{B} \to \mathbb{Z}$. We can split these exact sequences compatibly with left B-action using $\Delta : \tilde{B} \to F$. Thus

$$F = (B,0) \oplus \Delta\tilde{B} = (0,B) \oplus \Delta \tilde{B}$$

giving two isomorphisms of F with $B \oplus$ [redacted] $\tilde{B}$ in $\mathcal{P}(\tilde{B})$.

Take the former.

$$F \xrightarrow{\sim} B \oplus \tilde{B}$$

$$(u+v, v) \longleftarrow (u\ \ v)$$

$$(x,y) \longmapsto (x-y\ \ y)$$

Then right mult by b is

$$(u\ \ v) \longmapsto (u+v, v)b = (ub+vb, 0) \longmapsto (ub+vb\ \ 0) = (u\ \ v)\begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}$$

Take the latter isom.

$$F \simeq B \oplus \tilde{B}$$

$$(v', u'+v') \longleftarrow (u' \quad v')$$

$$(x, y) \longmapsto (y-x \quad x)$$

and right mult by b is

$$(u' \; v') \mapsto (v', u'+v') \xrightarrow{b\cdot} (v'b, 0) \mapsto (-v'b \quad v'b) = (u' \; v')\begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}$$

Observe $(u', v') \mapsto (v', u'+v') \longmapsto (-u' \quad u'+v') = (u', v')\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}.$$

What does this mean? The first homomorphism $b \mapsto \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}$ from B to $\mathrm{Aut}_B(B \oplus \tilde{B}) = \begin{pmatrix} A & A \\ B & \tilde{B} \end{pmatrix}$ arises from the exact sequence $0 \to B \to F \to \tilde{B}_\varepsilon \to 0$. It extends to $\begin{pmatrix} B & 0 \\ B & 0 \end{pmatrix}$ which by Suslin should be K-equivalent to $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$. The second homomorphism $b \mapsto \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}$ extends to $\begin{pmatrix} 0 & 0 \\ B & B \end{pmatrix}$ which should be K-equiv. to $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$. Since these are conjugate this should mean that the representations $B \to A = \mathrm{Aut}_B(B)$ and $B \to \tilde{B} = \mathrm{Aut}_B(\tilde{B})$ are somehow equivalent

August 13, 1996

Let A be a left ideal in R unital. Recall that a) R/A is projective $\iff$ A has a right identity: $a = ae$ $\forall a$.

b) R/A is flat $\iff$ A has local right identities: $\forall a_1, \dots, a_n$ $\exists a$ $a_j(1-a) = 0$ (resp. this holds for $n=1$.)

Suppose A is an ideal in R such that R/A is right flat, so that $\forall$ modules M

$$A \otimes_R M \xrightarrow{\sim} AM$$

Then taking $M = R/A$ we get $A \otimes_R R/A = A/A^2 = 0$. Also we have $M = AM \implies M$ is firm.

Conversely assume these two conditions, and let M be any module. Since $A = A^2$, $AM = A(AM)$ so AM is firm. Also $A \otimes_R (M/AM) = 0$. So we have a diagram with exact rows

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \| & & \\ & & A \otimes_R AM & \longrightarrow & A \otimes_R M & \longrightarrow & A \otimes_R (M/AM) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & AM & \longrightarrow & M & \longrightarrow & M/AM & \longrightarrow & 0 \end{array}$$

showing that $A \otimes_R M \xrightarrow{\sim} AM$ for all M. $\therefore$ R/A is right flat. $\therefore$

<u>Prop.</u> R/A is right flat for an ideal A iff $A = A^2$ and $M = AM \implies M$ is firm.

It would be better to formulate this independently of R as follows

<u>Prop.</u> A has local left identities $\iff$ $A = A^2$ and $M = AM \implies M$ is firm for all modules M.

[redacted] Assume A is such a ring. Then

$$M = AM \implies \mathrm{Hom}_R(R/A, M) = 0.$$

In effect if $K = \mathrm{Hom}_R(R/A, M)$, then $AK = 0$ and

$$\begin{array}{ccccccc} & & \overset{0}{A \otimes_A K} & \longrightarrow & A \otimes_A M & \xrightarrow{\sim} & A \otimes_A (M/K) \longrightarrow 0 \\ & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K \longrightarrow 0 \end{array}$$

using $M = AM$ and $M/K = A(M/K)$. $\therefore K = 0$.

Alternate proof using local left identities: Let $Am = 0$, write $m = \sum a_i m_i$ and choose $a \in A$ such that $(1-a)a_i = 0$. Then [redacted] $m = am = 0$.

Prop. Let A be a ring satisfying $A = A^2$. TFAE

1) A has local left identities
2) $AM = M \implies M$ firm for all modules M
3) ${}_A M = \{m \mid Am = 0\}$ is zero for all firm modules

It remains to check $3) \implies 2)$. [redacted] Take [redacted] a module M s.t. $AM = M$. Then $A \otimes_A M$ is firm and the kernel of $A \otimes_A M \twoheadrightarrow AM$ is killed by A. By 3) the kernel is zero, and so M is firm.

Question: Is any idempotent ring Morita equivalent to a ring with local left identities?

Suppose $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ strictly firm such that B has local left identities. Then P as a B-module satisfying $P = BP$ has local identities in the sense that $\forall p_1, \ldots, p_n$ $\exists b \in B$ such that $(1-b)p_i = 0$. Conversely if this condition holds then as $B = PQ$, the ring B has local left identities. In this situation we also know that B is B^{op} flat, hence P is A^{op}-flat. Now, starting with A idempotent we have a sequential way to construct firm flat right modules P. Can this be modified to yield local left identities or is there an obstruction?

I think we can arrange Q to be essentially free in the following sense. We want, starting from a finite set of p'_μ, to construct $b = \sum_i p_i q_i$ satisfying $p'_\mu = \sum_i p_i (q_i p'_\mu)$ for all μ. Here p_i, q_i can be added to what we already have. The function of q_i is to provide an A^{op}-linear map $P \longrightarrow A$ (or maybe $\tilde{A}$).

Imagine constructing P, Q inductively adding at each stage the necessary p_i, q_i. Then P is a flat firm module over A and the q_i give linear functionals on P. So we can replace the Q we might have with AF, where F is a free $\tilde{A}$-module whose basis elements map to the q_i. In other words we have $F \otimes_{\mathbb{Z}} P \twoheadrightarrow A$ hence $AF \otimes P \twoheadrightarrow A$.

Consider A = maximal ideal in a valuation ring R such that the principal ideals are Rz^{ε}, $\varepsilon \in \bigcup \frac{1}{2^n}\mathbb{Z}$. A firm flat A^{op}-module P is a torsion free R^{op}-module such that for any $p \in P$ $\exists\, \varepsilon > 0, p_1 \in P$ such that $p = p_1 z^{\varepsilon}$. Suppose we have an sfirm Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where B has local left identities. Then we know P is A^{op} flat firm and for every finite set p'_j $\exists\, b = \sum p_i q_i$ such that

$$p'_j = \sum_i p_i q_i p'_j \qquad \forall j$$

Take a single p'. We have $p' \in \sum p_i R$ which is a torsion free finitely generated R-module.

Replacing the p_i by suitable linear combinations over R, we can assume they form an R-basis, and also that $p' \in p_1 R$. Then $p' = \sum p_i q_i p' \Rightarrow q_i p' = 0$ for $i \neq 1$. If $p' = p_1 z^{\varepsilon} u$, then $p' = p_1 q_1 p'$, so $p_1 z^{\varepsilon} u = p_1 q_1 p_1 z^{\varepsilon} u$, so $p_1 = p_1 q_1 p_1$, and so $q_1 p_1 = 1 \in R$. This contradicts the facts that $q_1 p_1 \in A$.

Try for a less computational proof as follows. The condition $p'_j = \sum_i p_i q_i p'_j$ says the B-module $W = \sum \tilde{B} p'_j$ satisfies $W \subset BW$. So W is finitely generated and $W = BW$, so there should exist a simple object in $\mathcal{M}(B)$. Strictly speaking theres a non nil simple B-module. But $\mathcal{M}(B) \simeq \mathcal{M}(A)$ and A is a radical ring so $\mathcal{M}(A)$ has no simple objects.

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Continue with the nondegeneracy question – whether any idempotent A is Morita equiv. to a B which injects into its multiplier ring. I consider a special A where factoring: $a = \sum a_i' a_i''$ can be done explicitly and simply.

Let R be a valuation ring with value group $\bigcup_{n\geq 0} \mathbb{Z}\frac{1}{2^n}$, say there are powers z^ε for $\varepsilon \in \bigcup \mathbb{Z}\frac{1}{2^n}$ so the principal ideals are $\{Rz^\varepsilon\}$. Let $m = \bigcup_{\varepsilon>0} Rz^\varepsilon$ be the maximal ideal of R. Take $A = m/mz$ and let $\bar{A} = m/Rz$. Actually we start with $\bar{A} \subset \bar{R} = R/Rz$, and note that $A = m/mz = \bar{R} \otimes_R m$ is flat over $\bar{R}$ so that A is firm flat over $\bar{A}$ and satisfies $\bar{A}A = A$. It should be clear from $0 \to k \xrightarrow{z} A \to \bar{A} \to 0$ that $A = \bar{A}^{(2)}$. So we have a firm flat commutative ring with a nonzero element z killed by A. When I consider a s-firm Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ this element z kills everything, creating difficulties with making things nondegenerate.

Claim there are firm flat A-modules M such that ${}_A M = 0$. Let $F = \bigcup_{\varepsilon > t} Rz^\varepsilon$ where t is a real no. so F is a flat R-module such that $mF = F$ $M = F/Fz$ is a firm flat A-module. Let $x \in F$ satisfy $mx \in Fz$. Up to units I can suppose $x = z^\varepsilon$ with $\varepsilon > t$. Then $z^{2^{-k}} z^\varepsilon \in Fz \overset{(\forall k)}{\Longrightarrow} 2^{-k} + \varepsilon > t+1$ $(\forall k)$ $\Rightarrow \varepsilon \geq t+1$. If $t \notin \bigcup 2^{-n}\mathbb{Z}$ then $\varepsilon > t+1$, so $x \in Fz$. Thus in this case $M = F/Fz$ has no nonzero element killed by A.

Let's take $Q = F_t / F_t z$, $F_t = \bigcup_{\varepsilon > t} R z^{\varepsilon}$.

I'd like to find an appropriate P. The obvious candidate which pairs nicely with Q is $P = F_{-t} / F_{-t} z$. The pairing $Q \otimes P \longrightarrow A$ is surjective and in fact it looks like $Q \otimes_A P \xrightarrow{\sim} A$, whence $B = P \otimes_A Q$ is also A. So although I've managed to make ${}_A Q$, $P_{(A)}$ zero, B is still degenerate.