<u>September 5, 1995</u>

I want to reduce Morita invariance to the simplest steps. Consider $\mathcal{M}(A)$, $A=A^2$ a Roos category, and let Q be a generator which is firm. Put $S = \mathrm{Hom}_A(Q,Q)^{op}$. Then we have a functor

$$\begin{aligned} \mathrm{mod}(S) &\longrightarrow \mathcal{M}(A) \\ N &\longmapsto Q\otimes_S N \end{aligned}$$

Roos' theorem should tell us that there an idempotent ideal J in S such that this functor induces an equivalence

$$\mathcal{M}(S,J) \xrightarrow{\sim} \mathcal{M}(A)$$

One has the Morita context

$$\begin{pmatrix} A & Q \\ \mathrm{Hom}_A(Q,A) & S \end{pmatrix}$$

so it should be clear that

$$J = \mathrm{Im}\left\{ \mathrm{Hom}_A(Q,A)\otimes_A Q \longrightarrow \overset{S}{\overset{\|}{\mathrm{Hom}_A(Q,Q)}} \right\}$$

In fact I might as well dispense with S and consider the triple $(Q, \mathrm{Hom}_A(Q,A)\otimes_A A, \varphi)$ where φ:

$$Q\otimes_{\mathbb{Z}} \mathrm{Hom}_A(Q,A)\otimes_A A \longrightarrow A\otimes_A A \twoheadrightarrow A$$

This triple is the "maximum" one containing the Q given at the outset. More precisely, given $(Q, P, Q\otimes_{\mathbb{Z}} P \xrightarrow{\varphi} A)$ one has

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Hom}_A(Q,A) \\ \wr\uparrow & & \uparrow \\ P\otimes_A A & \longrightarrow & \mathrm{Hom}_A(Q,A)\otimes_A A \end{array} \qquad p \mapsto (q \mapsto \varphi(q,p))$$

whence a map $P \to \mathrm{Hom}_A(Q,A)\otimes_A A$ inducing φ from ψ.

[redacted] Let's fix $(Q, P, Q\otimes_{\mathbb{Z}} P \overset{\psi}{\twoheadrightarrow} A)$ and put $B = P\otimes_A Q$, $C = \mathrm{Hom}_A(Q,A)\otimes_A A \otimes_A Q = \mathrm{Hom}_A(Q,A)\otimes_A Q$. We have Morita equivalences

$$
\begin{array}{ccccc}
\mathcal{M}(B) & = & \mathcal{M}(A) & = & \mathcal{M}(C) \\
P\otimes_A M & \longleftarrow\!\shortmid & M & \longmapsto & \mathrm{Hom}_A(Q,A)\otimes_A M \\
N & \longmapsto & Q\otimes_B N & \longmapsto & \mathrm{Hom}_A(Q,A)\otimes_A Q\otimes_B N
\end{array}
$$

(M firm)

The last functor is base extension wrt $B \to C$.
The relevant Morita contexts here are contained in

$$
\begin{pmatrix} A & Q & Q \\ P & B & B \\ \mathrm{Hom}_A(Q,A)\otimes_A A & C & C \end{pmatrix}
$$

I want to understand $\begin{pmatrix} B & B \\ C & C \end{pmatrix}$ better.
We have a canonical map

$$
\rho : \underset{B}{\underbrace{P\otimes_A Q}} \longrightarrow \underset{C}{\underbrace{\mathrm{Hom}_A(Q,A)\otimes_A Q}}
$$

which is a ring homom. on one hand, and a right C-module map on the other.

Let's consider a ring C and a right C-module map $\rho : B \to C$. [redacted] Define a product on B by

$$b_1 \circ b_2 = b_1\rho(b_2)$$

Then $(b_1\circ b_2)\circ b_3 = (b_1\circ b_2)\rho(b_3) = (b_1\rho(b_2))\rho(b_3) = b_1(\rho(b_2)\rho(b_3))$

and $b_1\circ(b_2\circ b_3) = b_1\rho(b_2\rho(b_3)) = b_1(\rho(b_2)\rho(b_3))$. Thus B is a ring. Also $\rho(b_1\circ b_2) = \rho(b_1\rho(b_2)) = \rho(b_1)\rho(b_2)$ so

ρ is a ring homom. $\blacksquare$

An example of such a $\rho : B \to C$ is the inclusion of a right ideal in C.

<u>September 6, 1995</u>

Start with A firm and a generator Q for $\mathcal{M}(A)$. Take $P = \mathrm{Hom}_A(Q,A) \otimes_A A$, let $B = P \otimes_A Q$. Then the triple $(Q, P, Q\otimes P \twoheadrightarrow A)$ has the property that P is the 'dual' of Q. Under the Morita equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$ associated to this triple one has $Q \mapsto P\otimes_A Q$, $P \mapsto P \otimes_A Q$, hence the triple goes into $(B, B, B\otimes B \xrightarrow{\mu} B)$. It should follow then that B as right module should be dual to B as left module, i.e. $B \simeq \mathrm{Hom}_B(B,B)\otimes_B B$.

Let's check this. Consider more generally an arbitrary completely firm Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. I claim there is a canonical isom.

$$\mathrm{Hom}_A(Q,A)\otimes_A Q = \mathrm{Hom}_B(B,B)\otimes_B B$$

In other words the dual of ${}_A Q$ under the Morita equivalence is the dual of ${}_B B$. Pf. One has a comm. diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(Q,Q)\otimes_B P \otimes_A Q \otimes_B P \otimes_A Q \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_A(Q,Q)\otimes_B P\otimes_A Q \arrow[d] \\
\mathrm{Hom}_A(Q,A)\otimes_A Q \otimes_B P \otimes_A Q \arrow[r, "\sim"] \arrow[ur] & \mathrm{Hom}_A(Q,A)\otimes_A Q
\end{tikzcd}$$

$$\begin{aligned}
&\lambda\otimes p\otimes q\otimes p_0\otimes q_0 \longmapsto \lambda\otimes p\otimes qp_0q_0 = \\
&\qquad (\lambda . pq)\otimes p_0\otimes q_0 = \lambda\otimes pqp_0\otimes q_0 \\
&(\lambda . p)\otimes q\otimes p_0\otimes q_0 \\
&\qquad (\lambda' . q)\otimes p_0\otimes q_0 \\
&\qquad\qquad (\lambda' . qp_0)\otimes q_0 \\
&\lambda'\otimes q\otimes p_0\otimes q_0 \longmapsto \lambda'\otimes qp_0q_0 =
\end{aligned}$$

On the other hand we have

$$\mathrm{Hom}_A(Q,Q) = \mathrm{Hom}_B(B,B)$$

by Morita equivalence, i.e. induced by the functors $P\otimes_A -$, $Q\otimes_B -$. Thus

$$\mathrm{Hom}_A(Q,A)\otimes_A Q \xleftarrow{\sim} \mathrm{Hom}_A(Q,Q)\otimes_B P\otimes_A Q = \mathrm{Hom}_B(B,B)\otimes_B B$$

<u>September 8, 1995</u>

Fix a Roos category $\mathcal{M}(A)$, A firm. For each "coordinatization" (Q,P,φ) we have a ring $P\otimes_A Q$, hence an abelian group $((P\otimes_A Q)^\times)_{ab}$. Can we take an appropriate inductive limit of these abelian groups?

To fix the ideas consider $\mathcal{M}(k) = \mathrm{mod}(k)$ where k is a unital ring. Among all coordinatizations are those $(V, U, V\otimes_{\mathbb{Z}} U \twoheadrightarrow k)$, where $V\in\mathcal{P}(k)$, $U = \mathrm{Hom}_k(V,k) \in \mathcal{P}(k^{op})$ and the pairing is the evident pairing. For such a triple $U\otimes_k V = \mathrm{End}_k(V)$ and $(U\otimes_k V)^\times = \mathrm{Aut}_k(V)$. (I should have pointed out above that $(P\otimes_A Q)^\times$ is $GL(P\otimes_A Q)$ = grp of invertible elts in $1 + P\otimes_A Q$.)

Given triples (V,U,φ), (V_1,U_1,φ_1) there is an obvious way a homomorphism $U\otimes_k V \to U_1\otimes_k V_1$ arises, namely from a pair of maps $V\to V_1$, $U\to U_1$ such that φ is the restriction of φ_1.

Assume now that these triples are both f. proj. reflexive, i.e. $V\in\mathcal{P}(k)$, U = dual of V, φ = canonical pairing. Then a map $(V,U,\varphi) \longrightarrow (V_1,U_1,\varphi_1)$ arises when

$$(V_1,U_1,\varphi_1) = (V,U,\varphi)\oplus(V',U',\varphi')$$

i.e. when we are given a retract situation $V \underset{i}{\overset{r}{\rightleftarrows}} V_1$.

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The converse seems likely, namely a [redacted] map [redacted] $(V, V^*, \langle\ \rangle) \longrightarrow (V_1^*, V_1, \langle\ \rangle)$ is equivalent to a retract situation $V \rightleftarrows V_1$, assuming the triples are f proj reflexive. Proof. Let $a : V \longrightarrow V_1$, $b : V^* \longrightarrow V_1^*$ be compatible with the pairings: $\langle v, \lambda \rangle = \langle a(v), b(\lambda) \rangle$ for all $v \in V$, $\lambda \in V^*$. Then $\langle v, \lambda \rangle = \langle b^t a(v), \lambda \rangle \Longrightarrow b^t a = 1_V$, so V is a retract of V_1.

<u>September 14, 1995</u>

Let B be an idempotent ring, let $f: P \twoheadrightarrow B$ be a surjection of left B-modules, where P is firm. Then we get a coordinate system on $\mathcal{M}(B)$ given by the triple

$$\Big(P,\ B^{(2)},\ \underset{p \otimes b_1 \otimes b_2 \ \longmapsto\ f(p) b_1 b_2}{P \otimes_{\mathbb{Z}} B^{(2)} \twoheadrightarrow B} \Big)$$

Let $A = B^{(2)} \otimes_B P$ be the corresponding $\overset{\text{firm}}{\text{ring}}$. Since P is a firm B module we have $A \xrightarrow{\sim} P$. To keep things simple, suppose B firm. Let's calculate the product in $A \simeq P$. By def. then if $a_1 = b_1 \otimes p_1$, $a_2 = b_2 \otimes p_2$ in $A = B \otimes_B P$,

$$a_1 a_2 = b_1 \otimes f(p_1) b_2 p_2 \longmapsto \underbrace{b_1 f(p_1)}_{f(b_1 p_1)} b_2 p_2$$

Thus if we use $A \xrightarrow{\sim} P$ to identify A and P we have the product in A:

$$a_1 a_2 = f(a_1) a_2$$

and $f(a_1 a_2) = f(f(a_1) a_2) = f(a_1) f(a_2)$. So $f: A \to B$ is a surjective homom. Let $K = \mathrm{Ker}(f)$. Then K is an ideal in A such that $KA = 0$ and A is a $B = A/K$ module.

This time start with a ring B, a B-module A and a B-module map $f: A \to B$. Define

$$a_1 \cdot a_2 = f(a_1) a_2$$

This is an associate product:

$$(a_1 \cdot a_2) \cdot a_3 = f(f(a_1) a_2) a_3$$

$$a_1 \cdot (a_2 \cdot a_3) = f(a_1) f(a_2) a_3$$

and $\quad f(a_1 \cdot a_2) = f(f(a_1)a_2) = f(a_1)f(a_2)$

so $f : A \to B$ is a homomorphism. I have encountered this ~~situation~~ before; it generalizes the inclusion of a left ideal.

When f is surjective we have $A/K \simeq B$ where $K = \mathrm{Ker}(f)$ is an ideal in A such that $KA = 0$.

Let's start now with A a firm ring, K an ideal such that $KA = 0$, and put $B = A/K$.

$$A/K \otimes_{A/K} A/K = A/K \otimes_A A/K \xrightarrow{\sim} A/\cancel{KA} + AK$$

Thus B firm $\iff$ $AK = K$.

Suppose A h-unital. One has the M context

$\begin{pmatrix} A & B \\ A & B \end{pmatrix} = \begin{pmatrix} A & A/AK \\ A/KA & A/K \end{pmatrix}$ linking A and B. Recall that B is h-unital $\iff$ $\underset{\overset{\|}{A}}{P} \overset{L}{\otimes}_A A \overset{L}{\otimes}_A \underset{\overset{\wr}{B}}{Q} \xrightarrow{\text{qis}} B$ actually here I use the M context $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$.

Thus B is h-unital $\iff$ $A \overset{L}{\otimes}_A A \overset{L}{\otimes}_A B \xrightarrow{\text{qis}} B$

$\iff$ $A \overset{L}{\otimes}_A A/K \xrightarrow{\text{qis}} A/K$

But one has a map of Δ's

$$\begin{tikzcd}
A \overset{L}{\otimes}_A K \arrow[r] \arrow[d] & A \overset{L}{\otimes}_A A \arrow[r] \arrow[d, "\text{qis}"] & A \overset{L}{\otimes}_A (A/K) \arrow[r] \arrow[d] & {} \\
K \arrow[r] & A \arrow[r] & A/K \arrow[r] & {}
\end{tikzcd}$$

so we obtain

<u>Claim:</u> A h-unital, $K \subset A$ an ideal s.t. $KA = 0$. Then $B = A/K$ is h-unital $\iff$ $A \overset{L}{\otimes}_A K \xrightarrow{\text{qis}} K$ (i.e. K is an h-unitary A-module.)

Consider the map on K_* induced by $A \to A/K = B$. Note that $K^2 \subset KA = 0$, so A is a square zero extension of B by the B-bimodule K where the right multiplication is zero. We have then a group extension

$$1 \longrightarrow M(K) \longrightarrow GL(A) \longrightarrow GL(B) \longrightarrow 1$$

with abelian kernel. If $1+\beta \in GL(B)$ (β a matrix over B) and $\kappa \in M(K)$, then

$$(1+\beta)(1+\kappa)(1+\beta)^{-1} = (1+\beta)(1+\beta)^{-1} + (1+\beta)\kappa(1+\beta)^{-1}$$

$$= 1 + (1+\beta)\kappa$$

Thus the action of $GL(B)$ on $M(K)$ [redacted] defined by this group extension is given by left multiplication $(1+\beta), \kappa \longmapsto (1+\beta)\kappa$.

I would like to understand when

$$K_1 A = GL(A)_{ab} \longrightarrow K_1 B = GL(B)_{ab}$$

is an isomorphism. Conjecturally this happens [redacted] if A and B are firm. Recall that assuming A is firm, then $B = A/K$ is firm $\iff AK = K$.

Now we have an exact sequence

$$M(K)/[GL(A), M(K)] \to GL(A)_{ab} \to GL(B)_{ab} \to 0$$

so it would be nice to show that $AK = K \implies M(K) = [GL(A), M(K)] = \{ \alpha\kappa \mid 1+\alpha \in GL(A), \kappa \in M(K) \}$. Thus I think we can take $1+\alpha$ in $E(A)$.

$$[redacted]\,(a e_{ij})(k e_{jh}) = ak\, e_{ih}$$

here $j \neq i, h$. Thus the subgroup $[E(A), M(K)]$ contains $AK\, e_{ih}$ for all i, h, so contains $M(AK)$; in fact $[E(A), M(K)] = M(AK)$.

It seems I have proved

Claim: A idempotent, $K \subset A$ ideal st. $KA = 0$, $B = A/K$. If $AK = K$, then $K_1(A) \xrightarrow{\sim} K_1(B)$.

Recall that ~~starting~~ with B idempotent and choosing a surjection $P \twoheadrightarrow B$ of B-modules with P firm flat, we obtain a ring $A \simeq P$ which is left flat such that $A/K \xrightarrow{\sim} B$. From the preceding we know B firm $\iff AK = K \implies K_1(A) \xrightarrow{\sim} K_1(B)$. On the other hand I think I've show that for two left flat Morita equivalent rings A, A' ~~are~~ has a canonical iso $K_1 A \simeq K_1 A'$. So it might be true that $K_1 B \xrightarrow{\sim} K_1 B'$ when B, B' are firm M.eq. rings.

Let $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q) = \begin{pmatrix} Q \\ B \end{pmatrix} \otimes_B (P \quad B)$

be a completely-firm M context. Then

A is A-flat $\iff P \otimes_A A = P$ is B-flat $\iff \begin{pmatrix} A \\ P \end{pmatrix}$ is C-flat

B is B-flat $\iff Q \otimes_B B = Q$ is A-flat $\iff \begin{pmatrix} Q \\ B \end{pmatrix}$ is C-flat

$(A \quad Q)$ is A-flat $\iff (P \quad B)$ is B-flat $\iff C$ is C-flat

(the third is obtained by combining the first & second). ∎

C is C-flat $\iff A$ is A-flat and B is B-flat.

September 15, 1995

Recall equivalence between the data:

1) ring B, and B-module surjection $A \twoheadrightarrow B$.

2) ring A and ideal $K \subset A$ such that $KA = 0$.

Claim: 3) If A is a firm ring, then B is firm ring $\Leftrightarrow$ $AK = K$.

4) If B is firm ring, then A is firm ring $\Leftrightarrow$ A is firm B-module.

Pf. 3):
$$B \otimes_B B = A/K \otimes_A A/K = A \otimes_A A / \mathrm{Im}(K \otimes_A A + A \otimes_A K)$$
$$\xrightarrow{\sim} A / KA + AK = A/AK \quad \text{is iso } B \Leftrightarrow AK = K$$

4) One has exact sequence
$$\begin{array}{ccccccc} K \otimes_A A & \longrightarrow & A \otimes_A A & \longrightarrow & B \otimes_A A & \longrightarrow & 0 \\ \| \ \text{as } KA=0 & & & & \| & & \\ 0 \ \text{and } A=A^2 & & & & B \otimes_B A & & \end{array}$$

Thus $A \otimes_A A \xrightarrow{\sim} B \otimes_B A$ proving 4).

When A, B both firm, then we have the completely firm $\mathcal{M}$ context.
$$\begin{pmatrix} A & B \\ A & B \end{pmatrix} = \begin{pmatrix} A \\ A \end{pmatrix} \otimes_A (A \quad B) = \begin{pmatrix} B \\ B \end{pmatrix} \otimes_B (A \quad B)$$

In this case this amounts to four isos:
$$A \otimes_A A \xrightarrow{\sim} A \qquad A \otimes_A B \xrightarrow{\sim} B$$
$$B \otimes_B A \xrightarrow{\sim} B \qquad B \otimes_B B \xrightarrow{\sim} B$$

Now let's look at h-unitality. One has
$$P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q = A \overset{L}{\otimes}_A A \overset{L}{\otimes}_A B$$
$$Q \overset{L}{\otimes}_B B \overset{L}{\otimes}_B P = B \overset{L}{\otimes}_B B \overset{L}{\otimes}_B A$$

Thus we get from our h-unital criterion:

5) If A is h-unital, then B is h-unital $\Leftrightarrow$ $A \overset{L}{\otimes}_A B \to B$ quis

6) If B is h-unital, then A is h-unital $\Leftrightarrow$ $B \overset{L}{\otimes}_B A \to A$ quis

From the maps of Δ's.

$$\begin{tikzcd} A \overset{L}{\otimes}_A K \arrow[r] \arrow[d] & A \overset{L}{\otimes}_A A \arrow[r] \arrow[d] & A \overset{L}{\otimes}_A B \arrow[d] \\ K \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

$$\begin{tikzcd} B \overset{L}{\otimes}_B K \arrow[r] \arrow[d] & B \overset{L}{\otimes}_B A \arrow[r] \arrow[d] & B \overset{L}{\otimes}_B B \arrow[d] \\ K \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

we get:

5') If A is h-unital, then B is h-unital $\Leftrightarrow$ $A \overset{L}{\otimes}_A K \to K$ quis.

6') If B ———————— A ———— $\Leftrightarrow$ $B \overset{L}{\otimes}_B K \to K$ quis.

[redacted]

In the situation $A/K = B$, $KA = 0$, both A, B h-unital I want to show that $K_* A \xrightarrow{\sim} K_* B$.

There is a group extn

$$1 \to M(K) \to GL(A) \to GL(B) \to 1$$

and Hochschild-Serre spec sequence

$$E^2_{pq} = H_p(GL(B), H_q(M(K))) \Longrightarrow H_*(GL(A))$$

so it's enough to have

$$H_*(GL(B), H_q(M(K))) = 0 \qquad q > 0.$$

I think Suslin proves this with $B^{(\infty)}$ is place of $M(K)$ when he shows that
$$H_*(GL(B) \ltimes B^{(\infty)}) \xrightarrow{\sim} H_*(GL(B))$$
for B h-unital. In any case from $H_*(GL(B), B^{(\infty)}) = 0$ one can deduce that $H_*(GL(B), K^{(\infty)}) = 0$ for any B-module K such that $B \otimes^L_B K \to K$ is a quis, by using a pseudo-free resolution of K.

Another point that gives some confidence in these ideas is the fact that the semi-direct product $C = B \ltimes K$, where right multiplication by B on K is trivial, is h-unital iff B is h-unital and K is h-unitary over B. In effect $\mathbb{Z} \otimes^L_B B$ is a retract of $\mathbb{Z} \otimes^L_C C$, so C h-unital $\Rightarrow$ B h-unital. The rest is clear from 6') above.

At some point I have to learn Suslin's methods, probably based on Volodin's model, and also the new ideas involving stable K-theory & THH.

Formulas:

$$F \longrightarrow BGL(A) \longrightarrow BGL(A^+), \qquad P \text{ A-bimod, } A \text{ unital}$$

$$K^S_*(A;P) = H_*(F, M(P))$$ 1st order variation of $K^S_*(A)$. maybe?

$$\begin{array}{ccc} K_*R & \xrightarrow{\text{W's Dennis trace}} & HH_*(R) \\ \downarrow & & \uparrow \\ K^S_*R & \xrightarrow{\ \gamma\ } & H^{ML}_*(R) \end{array}$$

Pirashvili factorization

Maclane $H^{ML}_*(R,M) = HH_*(Q_*R, M)$

$H_*(Q_*\mathcal{B}) = H_{*+\infty}(K(\mathcal{B},\infty))$ $\mathcal{B}$ abelian

Thm. (D.. & McCarthy Annals 94) $K^S(R) = THH(R)$

Thm. (Pirash. Wald JPAA 92) $THH(R) = H^{ML}_*(R)$

<u>September 16, 1995</u>

Suppose $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ such that A is unital,

P_A, ${}_AQ$ are finite ~~proj~~ A-modules, $B = P \otimes_A Q$.

A pairing $Q \otimes_{\mathbb{Z}} P \xrightarrow{\psi} A$ is equivalent to a map $P \xrightarrow{f} Q^* = \mathrm{Hom}_A(Q, A)$ of fproj A^{op} modules. This can be factored $P \hookrightarrow P_1 \twoheadrightarrow Q^*$ where the injection is a direct injection of fproj right modules. For example one can take $P_1 = P \oplus Q^*$, $f = pr_2 \circ \Gamma_f$. Put $Q_1 = P_1^* = \mathrm{Hom}_{A^{op}}(P_1, A)$. Then we have direct injections $P \hookrightarrow P_1$, $Q \hookrightarrow Q_1$ such that ψ is the restriction of the canonical perfect pairing $Q \otimes_{\mathbb{Z}} P_1 \to A$.

In this way we embed $B = P \otimes_A Q$ into the "matrix" ring $P_1 \otimes_A Q_1 = \mathrm{Hom}_A(Q_1, Q_1)$. (Picture:

$$\begin{tikzcd}
 & & Q^* \otimes_A Q \\
 & P \otimes_A Q \arrow[r, hook] \arrow[d, hook] & P_1 \otimes_A Q \arrow[u, two heads] \arrow[d, hook] \\
 & P \otimes_A Q_1 \arrow[r, hook] \arrow[dl, two heads] & P_1 \otimes_A Q_1 \\
P \otimes_A P^* & &
\end{tikzcd}$$

The square is part of a 9 lemma diagram.)

Consider A a field. $P_1 \otimes_A Q_1$ is a $\overset{\text{square}}{}$ matrix algebra, $P \otimes_A Q_1$ is a right ideal which can be roughly viewed as made of "rows" $p \otimes Q_1$, while $P_1 \otimes_A Q$ is a left ideal made of "columns". Their intersection $P \otimes_A Q$ is a subring which has zero multiplication when $\langle Q, P \rangle = 0$. This situation is ruled out in the case of a M equivalence.

(In fact the situation $A = P\otimes_A Q \subset P_1 \otimes_{A'} Q_1 = B$ satisfies $B = B^2 = BAB$ (assuming $P, Q \neq 0$) and $ABA = A$, but not $A = A^2$ where $\langle Q, P\rangle = 0$.)

Next I would like to extend the field situation in a geometric direction, i.e. take P, Q to [redacted] correspond to vector bundles over X and $A = C(X)$. Then [redacted] $\psi: Q\otimes_{\mathbb{C}} P \longrightarrow A$ is onto iff $f: P \longrightarrow Q^*$ is nonzero at each $x \in X$.

(Actually since A is commutative, the fact that ψ is an A-bimodule map implies that ψ descends to a pairing $Q\otimes_A P \longrightarrow A$: $\psi(q a p) = \psi(a_1 q, p) = a_1 \psi(q, p) = \psi(q, p) a_1 = \psi(q, p a_1) = \psi(q, a_1 p)$.)

Suppose $P = A^k$, $Q = A^\ell$, [redacted] let p_i, q_j be bases for P, Q. ([redacted] column, row vectors, resp.)

Then $\psi: Q\otimes_{\mathbb{Z}} P \longrightarrow A$ is given by a $\ell\times k$ matrix over A: $\beta_{ji} = \psi(q_j, p_i)$. We can identify elts of $B = P\otimes_A Q$ with [redacted] $k\times \ell$ matrices $b = \sum p_i \otimes a_{ij} q_j$. Then

$$\left(\sum p_i \otimes a^1_{ij} q_j\right)\left(\sum p_{i'} \otimes a^2_{i'j'} q_{j'}\right)$$

$$= \sum p_i \otimes a^1_{ij}\beta_{ji'} a^2_{i'j'} q_{j'}$$

Thus $B = M_{k\ell}(A)$ with product

$$\alpha^1 * \alpha^2 = \alpha^1 \beta \alpha^2 .$$

<u>September 22, 1995</u>

I want to prove Morita invariance for cyclic homology of h-unital rings. The key idea is to make use of ring homomorphisms which are Morita equivalences.

Let A be a nonunital ring. $HC_*(A)$ is defined to be the homology of the Connes-Tsygan bicomplex of A, equivalently the homology of the pre-cyclic module $[n] \longmapsto A^{\otimes n+1}$. The mixed complex behind $HC_*(A)$ is the cone on $1-\lambda : (A^{\otimes *+1}, b') \longrightarrow (A^{\otimes *}, b)$:

$$
\begin{array}{ccc}
b\downarrow & & -b'\downarrow \\
A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} \\
b\downarrow & & -b'\downarrow \\
A & \xleftarrow{1-\lambda} & A
\end{array}
$$

The Hochschild homology corresponding to this cyclic homology is the homology of this mixed complex, which can also be described as $(\bar{\Omega}\tilde{A}, b, B)$. Thus $HH_*(A)$ is what I called the reduced Hochschild homology $\overline{HH}_*(\tilde{A})$.

Now $A \overset{L}{\otimes}_A$ can be calculated as follows. Start with the standard A-bimodule resolution of $\tilde{A}$:

$$\cdots \xrightarrow{b'} \tilde{A}\otimes A^{\otimes 2}\otimes\tilde{A} \xrightarrow{b'} \tilde{A}\otimes A\otimes\tilde{A} \xrightarrow{b'} \tilde{A}\otimes\tilde{A}$$

I will assume from now on that A is flat over the groundring (default groundring is $\mathbb{Z}$). Then the above complex is a flat A-bimodule res. of $\tilde{A}$. Now

$$M \overset{L}{\otimes}_A \;=\; M \overset{L}{\otimes}_{\tilde{A}\otimes\tilde{A}^{op}} \tilde{A} \qquad \begin{array}{c} M \text{ an} \\ A\text{-bimodule} \end{array}$$

so that $M \overset{L}{\otimes}_A$ is given by the complex

$$M \otimes_{\tilde{A}\otimes\tilde{A}^{op}} (\tilde{A}\otimes A^{\otimes *}\otimes\tilde{A}, b') = (M\otimes A^{\otimes *}, b)$$

In particular $A \overset{L}{\otimes}_A = (A^{\otimes *+1}, b)$, which does not give the reduced Hochschild homology $\overline{HH}_*(\tilde{A})$ in general.

However for h-unital rings the b' complex is acyclic, so the Hochschild homology $HH_*(A)$ belonging to $HC_*(A)$ is the homology of $A \overset{L}{\otimes}_A$. In fact there's a Δ:

$$\begin{array}{c} (A^{\otimes *+1}, b) \longrightarrow (\tilde{\Omega}\tilde{A}, b) \longrightarrow (A^{\otimes *}, b')[1] \longrightarrow \\ \wr \\ A \overset{L}{\otimes}_A \end{array}$$

showing that $H_*(A \overset{L}{\otimes}_A) \xrightarrow{\sim} \tilde{H}H_*(\tilde{A}) \iff A$ is h-unital

---

If $u : A \longrightarrow B$ is a homomorphism rings, then u induces a map of mixed complexes $\tilde{\Omega}\tilde{A} \longrightarrow \tilde{\Omega}\tilde{B}$, and formally one has that $HH_*(A) \xrightarrow{\sim} HH_*(B) \iff HC_*(A) \xrightarrow{\sim} HC_*(B)$. For h-unital rings we have $\dot{H}H_*(A) = H_*(A \overset{L}{\otimes}_A)$ and so we have $A \overset{L}{\otimes}_A \to B \overset{L}{\otimes}_B$ is a quis $\iff HC_*(A) \xrightarrow{\sim} HC_*(B)$. This is what I want to use to prove Morita invariance for HC_* of h-unital rings.

Let $u : A \to B$ be a homom. between h-unital (hence firm) rings which is a Morita equivalence. The corresponding Morita context is

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A & A \otimes_A B \\ B \otimes_A A & B \end{pmatrix}$$

Notice that there are homom.

$$M_2 A \longrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow M_2 B$$

The idea now is to use say the latter map of Morita contexts to produce

$$\begin{array}{ccc} A \overset{L}{\otimes}_A & \sim & B \overset{L}{\otimes}_B \\ \downarrow{\scriptstyle u_*} & & \| \\ B \overset{L}{\otimes}_B & = & B \overset{L}{\otimes}_B \end{array}$$

from our proof of M inv. of HH.

More precisely we have

$$\begin{tikzcd}
A\overset{L}{\otimes}_A \arrow[d, "u_*"'] & Q\overset{L}{\otimes}_B B\overset{L}{\otimes}_B P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A \arrow[l, "\sim"'] \arrow[d] & = & P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A Q\overset{L}{\otimes}_B B\overset{L}{\otimes}_B \arrow[r, "\sim"] \arrow[d] & B\overset{L}{\otimes}_B \arrow[d, "1"] \\
B\overset{L}{\otimes}_B & B\otimes B\otimes B\otimes B\otimes \arrow[l, "\sim"'] & = & B\overset{L}{\otimes}_B B\overset{L}{\otimes}_B B\overset{L}{\otimes}_B B\overset{L}{\otimes}_B \arrow[r] & B\overset{L}{\otimes}_B
\end{tikzcd}$$

and so we can conclude that $u_* : HH_*(A) \xrightarrow{\sim} HH_*(B)$, hence $HC_*(A) \longrightarrow HC_*(B)$. This proves

<u>Prop.</u> If $u : A \to B$ is a homom. of h-unital rings which is also a Morita equivalence, then u_* induces isos on HH_* and HC_*.

The next step will be to try to handle a Morita equivalence with context $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ by means of the evident homos $A \hookrightarrow C \hookleftarrow B$. The problem is that C need not be h-unital even if A, B are.

Recall that if A is h-unital, then

1) B h-unital $\iff P\overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \xrightarrow{\sim} B$

2) C h-unital $\iff \begin{pmatrix} A \\ P \end{pmatrix} \overset{L}{\otimes}_A A \overset{L}{\otimes}_A (A \;\; Q) \xrightarrow{\sim} C$

$$\iff \begin{pmatrix} A & A\overset{L}{\otimes}_A Q \\ P\overset{L}{\otimes}_A A & P\overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

Here are cases where C is h-unital

1) A h-unital, B left + right flat (equiv ${}_AP, {}_AQ$ flat)

1') A left + right flat (hence h-unital) and B h-unital

Note 1) and 1') are symmetric.

2) A, Q A-flat (This $\Rightarrow$ P, Q B-flat and C is C-flat i.e. we have a completely left-flat situation)

The next discussion is perhaps of no real interest. I am interested in situations where $P\overset{L}{\otimes}_A Q \xrightarrow{\sim} B$, $Q\overset{L}{\otimes}_B P \xrightarrow{\sim} A$ so that the following simpler proof works:

$$A\overset{L}{\otimes}_A \simeq Q\overset{L}{\otimes}_B P\overset{L}{\otimes}_A = P\overset{L}{\otimes}_A Q\overset{L}{\otimes}_B \xrightarrow{\sim} B\overset{L}{\otimes}_B$$

The question is whether this situation occurs for (A,C) and (C,B) in the cases 1), 1'), 2) above.

Consider 2) the completely left flat situation. Then $C = \begin{pmatrix}A\\P\end{pmatrix}\otimes_A(A\ \ Q)$ is left C-flat, so $\begin{pmatrix}A\\P\end{pmatrix}, \begin{pmatrix}Q\\B\end{pmatrix}$ are C-flat.

Then $C = \begin{pmatrix}A\\P\end{pmatrix}\overset{L}{\otimes}(A\ \ Q)$ and from $C = C\overset{L}{\otimes}_C C$ we get $A = (A\ \ Q)\overset{L}{\otimes}_C\begin{pmatrix}A\\P\end{pmatrix}$. Thus the short proof (probably the one used by Block + Getzler) works for $A\subset C$, and also for $B\subset C$ by symmetry.

Next take 1) A h-unital, ${}_AQ$, P_A flat ($\Leftrightarrow$ B left + right flat)

Then $C = C\overset{L}{\otimes}_C C \implies A = (A\ \ Q)\overset{L}{\otimes}_C\begin{pmatrix}A\\P\end{pmatrix}$, $B = (P\ \ B)\overset{L}{\otimes}_C\begin{pmatrix}Q\\B\end{pmatrix}$.

But also

$$\begin{pmatrix}A\\P\end{pmatrix}\overset{L}{\otimes}_A(A\ \ Q) = \begin{pmatrix}\overset{A\ \text{h-unital}}{\overset{\shortparallel}{A\overset{L}{\otimes}_A A}} & \overset{Q\ \text{flatness}}{\overset{\shortparallel}{A\overset{L}{\otimes}_A Q}} \\ \underset{P}{\underset{\shortparallel}{P\overset{L}{\otimes}_A A}} & \underset{P\otimes_A Q = B}{\underset{\shortparallel}{P\overset{L}{\otimes}_A Q}}\end{pmatrix} = \begin{pmatrix}A & Q\\ P & B\end{pmatrix}$$

$$\begin{pmatrix}Q\\B\end{pmatrix}\overset{L}{\otimes}_B(P\ \ B) = \begin{pmatrix}Q\overset{L}{\otimes}_B P & Q\overset{L}{\otimes}_B B\\ B\overset{L}{\otimes}_B P & B\overset{L}{\otimes}_B B\end{pmatrix} = \begin{pmatrix}Q\overset{L}{\otimes}_B P & Q\\ P & B\end{pmatrix}$$

as B left + rt flat

finally

$$Q\overset{L}{\otimes}_B P \underset{\substack{\uparrow\\ B\text{ left}\\ \text{flat}}}{=} Q\overset{L}{\otimes}_B B\overset{L}{\otimes}_B P \underset{\substack{\uparrow\\ A,B\ \text{h-unital}}}{\xrightarrow{\sim}} B$$

So in case 1) the short proof works for both $A\subset C$ and $B\subset C$. Same for 1') by symmetry.

At this point I understand somewhat when C is h-unital, and I should be able to prove

Morita invariance of HC for h-unital rings.

Let's proceed by considering a Roos category $\mathcal{M}$ and ~~[illegible]~~ all its coordinate systems. Suppose to fix the ideas that $\mathcal{M} = \mathcal{M}(D)$ with D some idempotent ring. Then we can choose a new coordinate system $({}_D V, W_D, V \otimes W \twoheadrightarrow D)$ where V, W are flat firm over D. Then $A = W \otimes_D V$ is both left and right flat.

Now let B_1, B_2 be rings M. eq. to D, hence to A. Then we get an isom $HC(B_1) \simeq HC(B_2)$ from the homos.

$$B_1 \subset \begin{pmatrix} A & * \\ * & B_1 \end{pmatrix} \supset A \subset \begin{pmatrix} A & * \\ * & B_2 \end{pmatrix} \supset B_2$$

This is OK because A both left + right flat, B h-unital $\Rightarrow \begin{pmatrix} A & * \\ * & B \end{pmatrix}$ is h-unital.

Next check the independence of the choice of A. Picture of the different coordinate systems

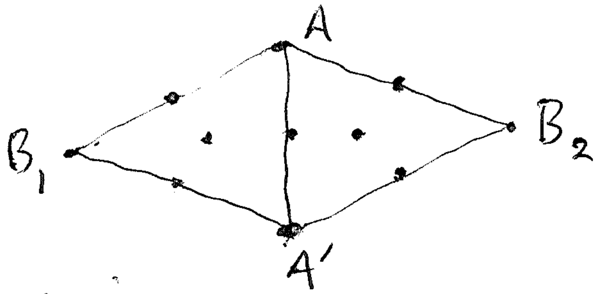


Diagram of inclusions:

$$\begin{tikzcd}
 & \begin{pmatrix} A & \cdot \\ \cdot & B_1 \end{pmatrix} \arrow[d] & A \arrow[l] \arrow[r] \arrow[dl] \arrow[d] \arrow[dr] & \begin{pmatrix} A & \cdot \\ \cdot & B_2 \end{pmatrix} \arrow[d] & \\
B_1 \arrow[ur] \arrow[r] \arrow[dr] & \begin{pmatrix} A & \cdot & \cdot \\ \cdot & A' & \cdot \\ \cdot & \cdot & B_1 \end{pmatrix} & \begin{pmatrix} A & \cdot \\ \cdot & A' \end{pmatrix} \arrow[l] \arrow[r] & \begin{pmatrix} A & \cdot & \cdot \\ \cdot & A' & \cdot \\ \cdot & \cdot & B_2 \end{pmatrix} & B_2 \arrow[ul] \arrow[l] \arrow[dl] \\
 & \begin{pmatrix} A' & \cdot \\ \cdot & B_1 \end{pmatrix} \arrow[u] & A' \arrow[l] \arrow[r] \arrow[ul] \arrow[u] \arrow[ur] & \begin{pmatrix} A' & \cdot \\ \cdot & B_2 \end{pmatrix} \arrow[u] &
\end{tikzcd}$$

This shows the isom $HC(B_1) \simeq HC(B_2)$ is independent of the choice of A.

Here's another way to see that B_1, B_2 h-unital M. eq $\Rightarrow \begin{pmatrix} B_1 & * \\ * & B_2 \end{pmatrix}$ h-unital. Let A be left + right flat, e.g. A unital. Then we know given a M. eq $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ that

$$B \text{ h-unital} \iff P \overset{L}{\otimes}_A Q \underset{\substack{\uparrow \\ A \text{ flat}}}{=} P \overset{L}{\otimes}_A A \overset{L}{\otimes} Q \xrightarrow{\sim} B.$$

~~$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ h-unital $\iff \begin{pmatrix} A \\ P \end{pmatrix} \overset{L}{\otimes}_A (A, Q) \xrightarrow{\sim} C$~~

~~$\begin{pmatrix} A \\ P \end{pmatrix} \overset{L}{\otimes}_A (A, Q) = \begin{pmatrix} A & Q \\ P & P \overset{L}{\otimes}_A Q \end{pmatrix}$~~

~~$\iff B$ unital~~

Consider

$$\begin{pmatrix} A \\ P_1 \\ P_2 \end{pmatrix} \otimes_A \begin{pmatrix} A & Q_1 & Q_2 \end{pmatrix} = \begin{pmatrix} A & Q_1 & Q_2 \\ P_1 & B_1 & P_1 \otimes_A Q_2 \\ P_2 & P_2 \otimes_A Q_1 & B_2 \end{pmatrix}$$

Then

$$B_1 \text{ h-unital} \iff P_1 \overset{L}{\otimes}_A Q_1 \xrightarrow{\sim} P_1 \otimes_A Q_1$$

$$B_2 \text{ h-unital} \iff P_2 \overset{L}{\otimes}_A Q_2 \xrightarrow{\sim} P_2 \otimes_A Q_2$$

$$\begin{pmatrix} B_1 & P_1 \otimes_A Q_2 \\ P_2 \otimes_A Q_1 & B_2 \end{pmatrix} \text{ h-unital} \iff P_i \overset{L}{\otimes}_A Q_j = P_i \otimes_A Q_j \quad \forall i, j$$

So you get a non-h-unital example (as before) by arranging $\operatorname{Tor}^A_{>0}(P_i, Q_j) = 0$ for $i = j$ and $\neq 0$ for some $i \neq j$.

It seems there is a natural category structure on coordinate systems given by homomorphisms of the corresponding rings which induce the M. equivalence.

September 24, 1995

If A is a nonunital ring its multiplier ring $Mult(A)$ consists of pairs $x = (x^r, x^\ell)$ of operators on A which we write $a \mapsto ax^r$ and $a \longmapsto x^\ell a$ satisfying

$$(a_1 a_2)x^r = a_1(a_2 x^r)$$
$$(a_1 x^r) a_2 = a_1(x^\ell a_2)$$
$$x^\ell(a_1 a_2) = (x^\ell a_1)a_2$$

The first condition says $x^r \in End({}_A A)$, the third says that $x^\ell \in End(A_A)$, and the second says that x^r and x^ℓ are adjoint with respect to the pairing $\mu : A \otimes A \longrightarrow A$. This obviously generalizes to a triple (Q, P, φ), namely let $Mult(Q,P,\varphi)$ be the set of pairs $(x^r, x^\ell) \in End_A(Q)^{op} \times End_{A^{op}}(P)$ such that $\langle x^r q, p\rangle = \langle q, x^\ell p\rangle$, i.e.

$$Q \otimes P \xrightarrow{x^r \otimes 1 - 1 \otimes x^\ell} Q \otimes P \xrightarrow{\langle\ \rangle} A$$

has composition zero, equivalently

$$\begin{array}{ccc} P & \longrightarrow & Hom_A(Q,A) \\ {\scriptstyle x^\ell}\downarrow & & \downarrow{\scriptstyle (x^r)^t} \\ P & \longrightarrow & Hom_A(Q,A) \end{array}$$

commutes. $M = Mult(Q,P,\varphi)$ is clearly a subring of $End_A(Q)^{op} \times End_A(P)$.

Assume the triple (Q,P,φ) such that Q, P are firm and φ is surjective. Then I claim that

$$M(Q,P,\psi) \simeq \mathrm{Mult}(P \overset{B}{\overbrace{\otimes_A}} Q)$$

because $\mathrm{End}_A(Q) = \mathrm{End}_B(B)$, $\mathrm{End}_{A^{op}}(P) = \mathrm{End}_{B^{op}}(B)$ and adjointness condition is preserved under the Morita equivalence. Here I use that

$$Q\otimes P \xrightarrow{x^r\otimes 1 - 1\otimes x^\ell} Q\otimes P \xrightarrow{\langle\ ,\ \rangle} A$$

$$B\otimes B \xrightarrow{x^r\otimes 1 - 1\otimes x^\ell} B\otimes B \xrightarrow{\mu} B$$

go into each other via $P\otimes_A - \otimes_A Q$ and $Q\otimes_B - \otimes_B P$. (Strictly speaking $Q\otimes_B B\otimes_B P = A^{(2)}$, but then one can follow with $A^{(2)} \twoheadrightarrow A$. Note also that because Q, P are firm any A-bimodule map $Q\otimes P \longrightarrow A$ lifts uniquely to $A^{(2)}$.)

We can use this to compute multiplier rings. For example let A be a field and let Q, P be finite dimensional. If the pairing $Q\otimes P \xrightarrow{\psi} A$ is non-degenerate then $B = \mathrm{Mult}(B)$ because $P \xrightarrow{\sim} \mathrm{Hom}_A(Q,A)$ so $x^\ell = (x^r)^t$.

In general one has

$$0 \longrightarrow P_0 \longrightarrow P \longrightarrow P/P_0 \longrightarrow 0$$

$$0 \longrightarrow Q_0 \longrightarrow Q \longrightarrow Q/Q_0 \longrightarrow 0$$

where $P_0 = Q^\perp$, $Q_0 = P^\perp$ for the pairing

$$\begin{array}{ccc} P & \longrightarrow & Q^* \\ \downarrow & & \uparrow \\ P/P_0 & \xrightarrow{\sim} & (Q/Q_0)^* \end{array}$$

so the multiplier ring is a fibre product consisting of pairs [redacted] (x^r, x^l) with x^r an endo of Q [redacted] respecting Q_0, and x^l an endo of P respecting P_0 such that on $P/P_0 = (Q/Q_0)^*$ one has $x^l = (x^r)^t$.

If Q, P are infinite dimensional and the pairing is non degenerate so that $P \hookrightarrow Q^*$ and $Q \hookrightarrow P^*$, then the multiplier ring is the ring of endos of P having transposes defined on Q. This might be equivalent to some sort of continuity for the weak topology on P coming from Q.

[redacted] Another comment: If A is a ring such that $A = A^2$ then any left multiplier $x^l \in \mathrm{Hom}_{A^{op}}(A, A)$ commutes with any right multiplier $y^r \in \mathrm{Hom}_A(A, A)$:

$$\big(x^l(a_1 a_2)\big)y^r = \big((x^l a_1)a_2\big)y^r = (x^l a_1)(a_2 y^r) = x^l\big(a_1(a_2 y^r)\big) = x^l\big((a_1 a_2)y^r\big)$$

In diagrams:

$$\begin{tikzcd} A \otimes A \arrow[r] \arrow[d, "x^l \otimes 1"', "1 \otimes y^r"] & A \arrow[d, "x^l"', "{[redacted]\ y^r}"] \\ A \otimes A \arrow[r] & A \end{tikzcd}$$

Thus A is a bimodule over $M = \mathrm{Mult}(A)$ when $A = A^2$. (Also it seems when A has trivial left-~~annihilators~~-and [redacted]-right annihilator in general.)

Also you have $A \xrightarrow{\mu} M$ satisfying $x\mu(a) = \mu(xa)$, $\mu(a)y = \mu(ay)$ so that $\mu(A)$ is an ideal in M. Thus $\mu(x(ay)) = \mu((xa)y)$, so if μ is injective (same as trivial left + right annihilator) we see A is a bimodule over M. It's this step that you need either $A = A^2$ or μ injective.

October 1, 1995

Here's an exercise I found difficult. Let F, G be adjoint functors, $\alpha : FG \to 1$, $\beta : 1 \to GF$ the adjunction maps. One knows that F is fully faithful $\iff$ β is an isom. In effect there is a commutative triangle

$$\begin{tikzcd} \mathrm{Hom}(F(X), F(X')) \arrow[rr, equal] & & \mathrm{Hom}(X, GF(X')) \\ & \mathrm{Hom}(X, X') \arrow[ul, "F"] \arrow[ur, "(\beta_{X'})_*"'] & \end{tikzcd}$$

etc.

. The exercise is to give a proof at least of $\Rightarrow$ using properties of α, β and avoiding Yoneda's lemma. (I wanted this for bimodule arguments in connection with Morita equivalence.)

Assuming F fully faithful we know that $\alpha.F : FGF \to F$ has the form $F.\gamma$ where $\gamma : GF \to 1$ is unique. Since

$$F \xrightarrow{F.\beta} FGF \xrightarrow{\alpha.F} F$$

is the identity, we have $(F.\gamma)(F.\beta) = 1_F$, so $F.(\gamma\beta) = 1_F$ and then $\gamma\beta = 1$. Next by naturality of β

$$\begin{tikzcd} GF \arrow[r, "\gamma"] \arrow[d, "\beta.GF"'] & 1 \arrow[d, "\beta"] \\ GFGF \arrow[r, "GF.\gamma"] & GF \end{tikzcd}$$

commutes. On the other hand $GF.\gamma = G.\alpha.F$ and we know that $G \xrightarrow{\beta.G} GFG \xrightarrow{G.\alpha} G$ is $1.G$, so

$\beta\gamma = (GF.\gamma)(\beta.GF) = 1.GF$, showing that $\beta\gamma = 1_{GF}$.

above square

October 7, 1995

Higgins thesis on Leibniz algebras.

It seems one first wants to understand dialgebras, just like before doing Lie theory one ought to understand associative algebras.

A dialgebra D is a k-module with two assoc. operations $d_1 \cdot d_2$ and $d_1 * d_2$ satisfying

$$d_1 \cdot (d_2 \cdot d_3 - d_2 * d_3) = 0$$

$$(d_1 \cdot d_2 - d_1 * d_2) * d_3 = 0$$

$$d_1 * (d_2 \cdot d_3) = (d_1 * d_2) \cdot d_3$$

Example. Let A be an assoc. algebra, M an A-bimodule, and $f: M \longrightarrow A$ a bimodule map. Then $m_1 \cdot m_2 = m_1 f(m_2)$, $m_1 * m_2 = f(m_1) m_2$ are associative operations on M (recall the former makes sense when $f: M \to A$ is only a right A-module map, and the latter requires only that f be a left A-module map). Then

$$m_1 \cdot (m_2 \cdot m_3 - m_2 * m_3) = m_1 \underbrace{f(m_2 f(m_3) - f(m_2) m_3)}_{f(m_2) f(m_3) - f(m_2) f(m_3) = 0}$$

$$(m_1 \cdot m_2 - m_1 * m_2) * m_3 = f(m_1 f(m_2) - f(m_1) m_2) m_3 = 0$$

$$m_1 * (m_2 \cdot m_3) - (m_1 * m_2) \cdot m_3 = f(m_1)(m_2 f(m_3)) - (f(m_1) m_2) m_3 \overset{\shortparallel}{0}$$

so M is a dialgebra.

Let $(D, \cdot, *)$ be a dialgebra, let

$$N = \mathrm{Im}\{ D \otimes D \xrightarrow{\cdot - *} D \}, \qquad d_1 \otimes d_2 \mapsto d_1 \cdot d_2 - d_1 * d_2 .$$

Then $D \cdot N = N * D = 0$. In particular $N \cdot N \subset D \cdot N = 0$ and $N * N \subset N * D = 0$.

Also $d_1 \cdot d_2 \equiv d_1 * d_2$ modulo the subspace N. Thus

$$D * N + N = D \cdot N + N = N$$

$$N \cdot D + N = N * D + N = N$$

so N is an ideal for both $\cdot$ and $*$ of square zero. On D/N, $\cdot = *$ so D/N is an assoc. algebra regarded as a dialgebra in a trivial way. N is a D/N-bimodule with left action given by $*$ and right action by $\cdot$; these commute by the third axiom.

So D is a square zero dialgebra extension

$$(*) \quad 0 \longrightarrow N \longrightarrow D \longrightarrow D/N \longrightarrow 0$$

of the associative algebra D/N by the D/N-bimod N. Presumably there is some sort of analogue of Hochschild cohomology connected with these extensions.

D is a unital dialgebra where $\exists\, 1 \in D$ such that $d \cdot 1 = 1 * d = d$, $\forall d$. In this case D/N is unital and N is a unitary D/N bimodule.

Let's split $*$ linearly (possible if D/N proj. as k-module). Then $\cdot, *$ are given by appropriate 2-cocycles on $A = D/N$ for the two A-bimodule structures on N having zero on one side. It looks like the Hochschild cohomology $H^*(\tilde{A}, N)$ vanishes in degrees >0 and is ${}_A N$ in degree 0 when the right multiplication of A on N is zero. In effect

$$H^*(\tilde{A}, N) = H^*\left\{ \mathrm{Hom}_{\tilde{A} \otimes \tilde{A}^{op}} \left(\tilde{A} \otimes A^{\otimes *} \otimes \tilde{A}, N \right) \right\}$$

$$= H^*\{ \mathrm{Hom}_{\tilde{A}}(\tilde{A} \otimes A^{\otimes *}, N)\} = 0$$

because the right mult. of $\tilde{A}$ on N factors through $\tilde{A}/A = k$, and because

$$\to \tilde{A} \otimes A \longrightarrow \tilde{A} \otimes A \longrightarrow \tilde{A}$$

should be a proj. resolution of k.

If so then we can assume $D = {}_A N_0 \rtimes D/N$ for the $*$ product, arbitrariness is a derivation $D/N \longrightarrow {}_A N_0$, which should be inner. Then the $\circ$ product should be given by a 2-cocycle $A^{\otimes 2} \longrightarrow {}_0 N_A$, which should be a coboundary. ?

A Leibniz algebra L is a k-module equipped with bilinear operation $l \cdot l'$ satisfying

$$(l \cdot m) \cdot n = (l \cdot n) \cdot m + l \cdot (m \cdot n).$$

In other words right mult by any $n \in L$ is a derivation of $(L, \cdot)$. Thus we have a map

$$L \xrightarrow{R} \mathrm{Der}(L, \cdot) \qquad n \longmapsto - \cdot n$$

such that $R(m \cdot n) = R_n R_m - R_n R_m$, so R is a homomorphism of Leibniz algebras (maybe $-R$?).

Note that $l \cdot (m \cdot m) = 0$; alt: $R(m \circ m) = -[R_m, R_m] = 0$.

Let $K = \mathrm{Im}\left\{ \begin{matrix} L \otimes L \longrightarrow L \otimes L \\ l \otimes l' \longmapsto l \cdot l' + l' \cdot l \end{matrix} \right\} = \mathrm{span}\{ m \circ m \mid m \in L \}$

(char $k \neq 2$). Then $L \cdot K = 0$ and $K \cdot L \subset L$ because right mult. $-R(l)$ is a derivation of L hence preserves K. Thus we have an extension

$$0 \to K \to L \to L/K \to 0$$

of Leibniz algebras, where $K.K = 0$ and L/K is a Lie algebra (since modulo K we have $l.l = 0$, which is the anti-symmetry condition) Right multiplication makes K a Lie module over L/K.

A functor from dialgebras to Leibniz algebras.

Given $(D, \cdot, *)$ a dialgebra, then $(D, d\cdot d' - d' * d)$ is a Leibniz algebra.

If $M \xrightarrow{f} A$, $m \cdot m' = m f(m')$, $m * m' = f(m) m'$, then the assoc. Liebniz alg is M with operation $m \otimes m' \longmapsto m f(m') - f(m') m$.

Important example: $M = A \oplus A$, $f(a, b) = b$.

Then we get the Leibniz algebra

$$(a,b)\cdot(a',b') = (a,b)b' - b'(a,b) = ([a,b'],[b,b'])$$

denoted $\mathcal{L}^{\oplus}A$ by Higgins. His universal enveloping algebra $U_{\mathcal{L}^{\oplus}}(L)$ is left-adjoint to this. Apparently there is a more interesting universal enveloping dialgebra of a Leibniz algebra.

October 8, 1995

Let $(D, \cdot, *)$ be a dialgebra: $\cdot, *$ associative +

$$d_1 \cdot (d_2 \cdot d_3 - d_2 * d_3) = 0$$

$$(d_1 \cdot d_2 - d_1 * d_2) * d_3 = 0$$

$$d_1 * (d_2 \cdot d_3) = (d_1 * d_2) \cdot d_3$$

Example: Let A be an associative alg, M an A-bimodule, $f: M \to A$ an A-bimodule map. Then $m_1 \cdot m_2 = m_1 f(m_2)$, $m_1 * m_2 = f(m_1) m_2$ makes M into a dialgebra.

Conversely, given a dialgebra D, let N be the image of $D \otimes D \to D$, $d \otimes d' \mapsto d \cdot d' - d * d'$. Then $A = D/N$ is an associative algebra with product induced by both $\cdot$ and $*$, D is an A-bimodule with left A-mult (resp. right A-mult) induced by $*$ (resp. $\cdot$), and the canonical surjection $f: D \to A$ is an A-bimodule map such that $d_1 \cdot d_2 = d_1 f(d_2)$, $d_1 * d_2 = f(d_1) d_2$.

Check this: The first and second identity above give $D \cdot N = N * D = 0$. Now

$$d * n = d * n - d \cdot n \in N \implies D * N \subset N$$

$$n \cdot d = n \cdot d - n * d \in N \implies N \cdot D \subset N$$

so both $*, \cdot$ on D descend to $A = D/N$ making A an assoc. algebra. Next $D \cdot N = 0 \implies \cdot$ on D descends to a right mult. $D \otimes A \to D$ making D a ~~right module~~ over A, since $\cdot$ is associative. Similarly $\boxed{N * D = 0 \implies}$ $*$ descends to $A \otimes D \to D$ making D a left A-module. The third identity above implies D is a bimodule over A. Finally I should have said that the left A-module structure is defined by $f(d_1) d_2 = d_1 * d_2$ and the right module structure by $d_1 f(d_2) = d_1 \cdot d_2$.

Suppose now that $M \xrightarrow{f} A$ is given as above, with f surjective, and assume that A is unital and that M is [redacted] a unitary A-bimodule. Choose $\xi \in M$ such that $f(\xi) = 1 \in A$.

[redacted] Let N = span of $m_1 * m_2 - m_1 \cdot m_2 = f(m_1)m_2 - m_1 f(m_2)$. Then $N \subset \mathrm{Ker}(f)$ and conversely given $n \in \mathrm{Ker}(f)$ we have

$$\xi * n - \xi \cdot n = f(\xi)n - \xi f(n) = 1 n = n.$$

This shows that the dialgebra M determines A.

So start with a unital dialgebra D. This means we are given 1_D such that $1_D * d = d \cdot 1_D = d$, $\forall d$. [redacted] We've seen that if N = span of $d_1 * d_2 - d_1 \cdot d_2$, then $A = D/N$ is an assoc. algebra, [redacted] D is an A-bimodule, $f: D \to A$ is an A-bimod. map such that $d_1 * d_2 = f(d_1)d_2$, $d_1 \cdot d_2 = d_1 f(d_2)$. Now $f(d) = f(1_D * d) = f(1_D)f(d)$, $f(d) = f(d \cdot 1_B) = f(d)f(1_B)$, which implies that A is unital with $f(1_D) = 1_A$. Clearly D is a unital bimodule over A. In fact it's clear that 1_D is not uniquely determined; it can be any $\xi \in D$ such that $f(\xi) = 1_A$. [redacted]

[redacted]

The conclusion of the above discussion is that a unital dialgebra D is equivalent to [redacted] a quadruple (A, M, f, ξ), where A is a unital assoc. alg, [redacted] $f: M \to A$ an A-bimodule map, and $\xi \in M$ satisfies $f(m) = 1$.

Let's now work out the Lie analogue, rather Leibniz analog. Start with a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module M, and a $\mathfrak{g}$-module map $f: M \to \mathfrak{g}$. Then define $m \cdot m' = f(m)m' - f(m')m$. NO This gives an opposite Leibniz algebra, i.e. where L_m is a derivation of $(M,\cdot)$ for each $m \in M$. Check

$$m_1 \cdot m_2 = f(m_1)m_2 - f(m_2)m_1$$

$$f(m_1 \cdot m_2) = [f(m_1), f(m_2)]$$

$$m\cdot(m_1 \cdot m_2) = f(m)\big(f(m_1)m_2 - f(m_2)m_1\big) - [f(m_1), f(m_2)]m$$

$$(m\cdot m_1)\cdot m_2 = [f(m), f(m_1)]m_2 - f(m_2)\big(f(m)m_1 - f(m_1)m\big)$$

$$m_1\cdot(m\cdot m_2) = f(m_1)\big(f(m)m_2 - f(m_2)m\big) - [f(m), f(m_2)]m_1 \quad \text{OK}$$

Conversely given a Leib° alg J:

$$l\cdot(m\cdot n) = (l\cdot m)\cdot n + m\cdot(l\cdot n)$$

we have a map $J \longrightarrow Der(J,\cdot)$, $l \mapsto L_l = l\cdot$. which is a homomorphism $L_{l\cdot m} = [L_l, L_m]$. Let $N =$ span of $\{m\cdot m \mid m \in J\}$. Then $L_{m\cdot m} = [L_m, L_m] = 0$

$\Rightarrow$ $J \longrightarrow Der(J,\cdot)$ descends to $J/N \longrightarrow Der(J,\cdot)$, making J a Lie module over J/N. I should have noted earlier that $N\cdot J = 0$, $J\cdot N \subset N$, so that J/N is a quotient algebra of J, and that J/N is a Lie algebra $\mathfrak{g}$. Moreover $f: J \longrightarrow J/N = \mathfrak{g}$ is a $\mathfrak{g}$-module map. But $j\cdot k = f(j)k$ not $f(j)k - f(k)j$?

Start again with a Lie module map $M \xrightarrow{f} \mathfrak{g}$ and define $m\cdot m' = f(m)m'$. Then

$$m_0\cdot(m_1\cdot m_2) = f(m_0)\big(f(m_1)m_2\big)$$

$$(m_0\cdot m_1)\cdot m_2 = f\big(f(m_0)m_1\big)m_2 = \overbrace{\big(f(m_0)f(m_1)\big)}^{[f(m_0), f(m_1)]} m_2$$

$$m_1\cdot(m_0\cdot m_2) = f(m_1)\big(f(m_0)m_2\big)$$

$$\therefore \boxed{m_0\cdot(m_1\cdot m_2) = (m_0\cdot m_1)\cdot m_2 + m_1\cdot(m_0\cdot m_2)} \quad \text{Leibniz° alg.}$$

Let J be a Leib° alg:

$$j\cdot(k\cdot l) = (j\cdot k)\cdot l + k\cdot(j\cdot l)$$

i.e. $J \xrightarrow{L} \mathrm{Der}(J)$

$$j \longmapsto L_j = j\cdot$$

and also $L_{j\cdot k} = [L_j, L_k]$. Let N = span of $j\cdot j$. Then $J\cdot N \subset N$, $N\cdot J = 0$ so $L\otimes L \to L$, $j\otimes k \mapsto j\cdot k$ descends to $L/N\otimes L \to L$.. $\mathfrak{g} = L/N$ is a Lie alg, L is a $\mathfrak{g}$-module and if $f: L \twoheadrightarrow L/N$ is the canonical surjection we have $j\cdot k = f(j)k$.

So we learn that any Leib° alg J arises from a triple $(\mathfrak{g}, M, f: M \to \mathfrak{g})$ where f is surjective. When can I conclude that $\mathfrak{g}$ is determined by $(M, \cdot)$?

Look at assoc. analogue. Given $M \xrightarrow{f} A$ an A-bimodule map, we want to know when

$$M\otimes_A M \xrightarrow{f\cdot 1 - 1\cdot f} M \xrightarrow{f} A \longrightarrow 0$$

is exact. Better: Start with $M \xrightarrow{f} A$, then divide M by span of $\{f(m)m' - m f(m')\}$ to get an associative algebra extension $0 \to K \to B \xrightarrow{f} A \to 0$ such that $B\cdot K = K\cdot B = 0$. The natural condition for this sort of extension not to exist is for A to be firm: $A\otimes_A A \xrightarrow{\sim} A$, i.e. $HB_1(A) = HB_2(A) = 0$.

The analogue of this in the Lie case would be $H_1(\mathfrak{g}) = H_2(\mathfrak{g}) = 0$. Consider $M \xrightarrow{f} \mathfrak{g}$ and divide M by the span of $m\cdot m = f(m)m$. Then we should have a Lie algebra extension

$$0 \to \mathfrak{k} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

such that k is a trivial $\mathfrak{g}$-module, i.e. h is a central extension of $\mathfrak{g}$. So we want $H_2(\mathfrak{g}) = 0$, possibly also $H_1(\mathfrak{g}) = 0$, to recover $\mathfrak{g}$ from the Leibniz algebra M.

Next consider homology. If $(D, \cdot, *)$ is a dialgebra then the analogue of HH_0 for D is
$$D / \operatorname{span}\{d \cdot d' - d' * d\}.$$

As $m \cdot m' - m' * m = m f(m') - f(m') m$, it's clear that, when D arises from $M \overset{f}{\twoheadrightarrow} A$, one has $D/[D,D]_{\cdot,*} = M/[A,M]$.

The following assertions seem correct.

1) A dialgebra D is equivalent to a triple (A, M, f), where A is an (nonunital) assoc. alg, M an A-bimodule, and $f: M \to A$ is an A-bimodule map, such that f is surjective and the kernel of f is spanned by $m f(m') - f(m) m'$, $\forall m, m' \in M$.

1') A unital dialgebra (without a specific choice of 1) D is equivalent to a triple (A, M, f) where A is unital, M is a unitary bimodule over A, and f is surjective.

2) A Leib° algebra is equivalent to a triple $(\mathfrak{g}, J, f)$, where $\mathfrak{g}$ is a Lie alg, J a $\mathfrak{g}$-module, $f: J \to \mathfrak{g}$ a $\mathfrak{g}$-module map (adjoint repn on $\mathfrak{g}$) such that f is surjective and the kernel of f is spanned by $f(j)j$, $\forall j \in J$.

The above should be equivalences of categories.

Let's assume this and try to calculate free objects. Let V be a k-module and look for the free nonunital dialgebra generated by V.

$$\begin{tikzcd} & M \arrow[r, two heads, "f"] & A \\ V \arrow[ur] \arrow[r] & T(V)\otimes V \otimes T(V) \arrow[u] \arrow[r, two heads] & \bar{T}(V) \arrow[u] \end{tikzcd}$$

I don't understand the unital case.

Next the free Leib° algebra.

free $L(V)$ module generated by V

$$\begin{tikzcd} J \arrow[r] & \mathfrak{g} \\ \underbrace{U(L(V))\otimes V}_{\bar{T}(V)} \arrow[u] \arrow[r] & L(V) \arrow[u] \end{tikzcd}$$

$\therefore$ $\bar{T}(V)$ is the free Leib° algebra generated by V

Next there is a functor Dialg $\longrightarrow$ Leib° which takes $M \xrightarrow{f} A$ to $\mathfrak{g} = (A, [\ ,\ ])$ and $J = M$ considered as $\mathfrak{g}$-module via $a \cdot m = [a, m]$. This functor has left adjoint - the universal dialg generated by a Leib° algebra. It takes $J \xrightarrow{f} \mathfrak{g}$ to $A = U(\mathfrak{g})$ and $M = (U(\mathfrak{g}) \otimes J \otimes U(\mathfrak{g}))_{\mathfrak{g}}$.

Here $\mathfrak{g}$ acts on $U(\mathfrak{g}) \otimes J \otimes U(\mathfrak{g})$ via

$$\rho(X)(\alpha \otimes j \otimes \beta) = -\alpha X \otimes j \otimes \beta + \alpha \otimes Xj \otimes \beta + \alpha \otimes j \otimes X\beta.$$

Put another way $\mathfrak{g}$ acts internally on the $U(\mathfrak{g})$ bimodule $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and we couple this to the action on J. M is the universal $U(\mathfrak{g})$-bimodule generated by the $\mathfrak{g}$-module J. As a vector space it should be true that $M \simeq U(\mathfrak{g}) \otimes J$.

Higgins' $U_{L\oplus}$: Leib $\longrightarrow$ Aalg.

Given B assoc alg, consider the dialgebra $B \oplus B \xrightarrow{pr_1} B$. This gives a functor Aalg $\longrightarrow$ Dialg, and we look for the left adjoint

$$\begin{tikzcd} B \oplus B \arrow[r, "pr_1"] & B \\ M \arrow[u, "{(v, uf)}"] \arrow[r, "f"] & A \arrow[u, "u"'] \end{tikzcd}$$

$u : A \to B$ is a hom.
$v : M \to B$ is an A-bimodule map.

Thus the universal B seems to be $T_A(M) = A \oplus M \oplus M \otimes_A M \oplus$

Now take $M \longrightarrow A$ to be

$(U(\mathfrak{g}) \otimes J \otimes U(\mathfrak{g}))_{\mathfrak{g}} \longrightarrow U(\mathfrak{g})$ and we

get $T_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes J)$ which is approx $U(\mathfrak{g}) \otimes T(J)$.

This agrees with Higgins' PBW thm. saying that $\mathrm{gr}\, U_{\mathfrak{g}\oplus} J = S(\mathfrak{g}) \otimes T(J)$.

October 15, 1995

Morita invariance of K_1 for firm rings, a direct approach. Consider A, P_A, ${}_AQ$, $\varphi: Q \otimes P \to A$ arbitrary A-bimodule map, and $B = P \otimes_A Q$ the associated ring. I propose to define a map $GL(B) \to K_1 A$. Suppose given ~~[illegible]~~ $1-b \in GL_s(B)$. We can choose $p_i = (p_{ji}) \in P^s$, $q_i = (q_{ij}) \in Q^s$, $\begin{array}{l} 1 \leq i \leq n \\ 1 \leq j \leq s \end{array}$

such that $b = p_i q_i = (p_{ji} q_{ik})$ using summation convention. Let $p = (p_1 \cdots p_n) \in M_{s,n}(P)$, $q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in M_{n,s}(Q)$:

$$\left(\begin{array}{c|c} M_n(A) & M_{ns}(Q) \\ \hline M_{sn}(P) & M_s(B) \end{array}\right)$$

Because $1-b = 1-pq$ is invertible over B we know that $1-qp \in GL_n(A)$. Our task is to show that the class $[1-qp] \in K_1 A$ is independent of the choice of p, q.

By replacing B by $M_s B$, P by $P^s = M_{s1}(P)$, Q by $Q^s = M_{1s}(Q)$ we should be able to reduce to $s = 1$. So $b = p_i q_i = (p_1 \cdots p_n)\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$.

Suppose now that we have two choices:

$$b = p_i' q_i' = p_j'' q_j'' \qquad \begin{array}{l} 1 \leq i \leq n' \\ 1 \leq j \leq n'' \end{array}$$

Consider $p = (p' \;\; p'')$, $q = \begin{pmatrix} q' \\ -q'' \end{pmatrix}$. Then $pq = 0$.

$$1 - qp = \begin{pmatrix} 1-q'p' & -q'p'' \\ q''p' & 1+q''p'' \end{pmatrix}$$

Claim this is congruent mod $E(A)$ to

$$\begin{pmatrix} 1-q'p' & 0 \\ 0 & (1-q''p'')^{-1} \end{pmatrix}$$

In effect

$$\begin{pmatrix} 1 & 0 \\ -q''p'(1-q'p')^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1-q'p' & -q'p'' \\ q''p' & 1+q''p'' \end{pmatrix} = \begin{pmatrix} 1-q'p' & -q'p'' \\ 0 & * \end{pmatrix}$$

$$* = 1+q''p'' + q''p'(1-q'p')^{-1}q'p'' = 1+q''\left(1+(1-p'q')^{-1}p'q'\right)p''$$

$$= 1+q''(1-p'q')^{-1}p'' = 1+q''(1-p''q'')^{-1}p'' \qquad \text{since } p'q'=p''q''$$

$$= (1-q''p'')^{-1}$$

Now we are reduced to showing that $pq=0$ $\Longrightarrow$ $1-qp \in E(A)$, because then it follows that $\left[\begin{pmatrix} 1-q'p' & 0 \\ 0 & (1-q''p'')^{-1} \end{pmatrix}\right] = [1-q'p'] - [1-q''p'']$ is zero in K_1A.

So we have to understand the ~~[illegible]~~ consequences of the ~~[illegible]~~ condition $\sum_{i=1}^{n} p_i \otimes q_i = 0$ in a tensor product $P \otimes_A Q$. (Here ~~[illegible]~~ is where we will use the fact that $B = P\otimes_A Q$ and not just PQ.) One way this condition arises is when $q=(q_i)$ can be factored $q = aq'$ $(q_i = a_{ij} q'_j)$ such that $pa = 0$.

For then $\quad p\otimes q = p\otimes aq' = pa\otimes q' = 0$.

(With indices $\quad p_i \otimes q_i = p_i \otimes a_{ij} q'_j = p_i a_{ij} \otimes q'_j = 0$.)

It seems that the converse, i.e. $p\otimes q = 0 \Rightarrow \exists\ q = aq'$ such that $pa = 0$, is not true. But we have

Lemma: If $p\otimes q = 0$ in $P\otimes_A Q$, then $\exists\, p', q', a, a'$ such that

$$(p\ p')\underline{\begin{pmatrix} a \\ a' \end{pmatrix}} = 0 \ , \quad \begin{pmatrix} q \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a' \end{pmatrix} q' .$$

In other words, if we enlarge p to $(p\ p')$ and q to $\begin{pmatrix} q \\ 0 \end{pmatrix}$, then we get the desired factorization. In the above a, a' are matrices over $\tilde{A}$, but when $Q = AQ$ we can further factor $q' = a''q''$ with a'' a matrix over A and so assume a, a' are over A.

Proof. We can suppose P is finitely generated. Let p' be a finite set of generators. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow T \oplus T' \xrightarrow{f} P \longrightarrow 0$$

where T, T' are f.free $\tilde{A}$-modules with bases x, x', ~~[illegible]~~ and $f(x) = p$, $f(x') = p'$. We have an exact sequence

$$\begin{array}{ccccccc} K\otimes_A Q & \longrightarrow & T\otimes_A Q \oplus T'\otimes_A Q & \longrightarrow & P\otimes_A Q & \longrightarrow & 0 \\ & & \cup\!\!\!\!\!\!\;\shortmid & & & & \\ & & x\otimes q & \longmapsto & p\otimes q = 0 & & \end{array}$$

so $\exists\ k_j \in K$, $q'_j \in Q$ such that

$$k \otimes q' \longmapsto x \otimes q$$

We have $k = xa + x'a'$ for unique a, a' since $(x\ x')$ is a basis for $T \oplus T'$. Then

$$x \otimes q = (xa + x'a') \otimes q'$$
$$= x \otimes aq' + x' \otimes a'q'$$

whence $q = aq'$, $a'q' = 0$. Also $k = xa + x'a' \longmapsto 0$ in P implies that $pa + p'a' = 0$. $\therefore (p\ p')\begin{pmatrix} a \\ a' \end{pmatrix} = 0$

and $\begin{pmatrix} q \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a' \end{pmatrix} q'$.

Here's the problem you run into when you don't allow the extra elements p'. Suppose $p \in P$, $q \in Q$ such that $p \otimes q = 0$ in $P \otimes_A Q$. Consider the exact sequence

$$0 \longrightarrow \tilde{A}/\mathfrak{a} \longrightarrow P \longrightarrow P/p\tilde{A} \longrightarrow 0 \qquad 1 \longmapsto p$$

where $\mathfrak{a} = \{a \in \tilde{A} \mid pa = 0\}$. Then

$$\operatorname{Tor}_1^A(P,Q) \longrightarrow \operatorname{Tor}_1^A(P/p\tilde{A}, Q) \longrightarrow \tilde{A}/\mathfrak{a} \otimes_A Q \longrightarrow P \otimes_A Q$$

$$\tilde{A}/\mathfrak{a} \otimes_A Q \cong Q/\mathfrak{a}Q \xrightarrow{p\otimes -} P \otimes_A Q, \qquad Q/\mathfrak{a}Q \ni q \longmapsto 0$$

I want $q \in \mathfrak{a}Q$, for then $q = aq'$ with $pa = 0$. So if I take P projective, $P/p\tilde{A}$ not right flat, then I can find Q such that $\operatorname{Tor}_1^A(P/p\tilde{A}, Q) = Q/\mathfrak{a}Q \neq 0$, and I get a counterexample.

Go back now to $p \otimes q = 0$ and the problem of showing that $1-qp \in E(A)$. Use the lemma to get $(p \;\; p')\begin{pmatrix} a \\ a' \end{pmatrix} = 0$, $\begin{pmatrix} q \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a' \end{pmatrix} q'$. Consider

$$1 - \begin{pmatrix} q \\ 0 \end{pmatrix}(p \;\; p') = \begin{pmatrix} 1-qp & -qp' \\ 0 & 1 \end{pmatrix}$$

This is equivalent to $1-qp$ modulo $E(A)$. Next

$$1 - \begin{pmatrix} q \\ 0 \end{pmatrix}(p \;\; p') = 1 - \begin{pmatrix} a \\ a' \end{pmatrix} q'(p \;\; p') = 1 - \begin{pmatrix} a \\ a' \end{pmatrix}(q'p \;\; q'p')$$

$$= 1 - \alpha\alpha' \qquad \text{where} \qquad \alpha'\alpha = 0.$$

But Vaserstein's identity tells us that $\begin{pmatrix} 1-\alpha\alpha' & 0 \\ 0 & (1-\alpha'\alpha)^{-1} \end{pmatrix}$ is in $E(A)$ in general, so we conclude that $1-\alpha\alpha'$ and $1-qp$ are in $E(A)$ as desired.

I forgot to give the simpler example, namely if $pa = 0$, $q = aq'$, then

$$1 - qp = 1 - a\underbrace{(q'p)}_{a'} \qquad \text{when} \qquad a'a = 0$$

so $\begin{pmatrix} 1-qp & 0 \\ 0 & 1 \end{pmatrix}$ is a product of elementaries.

At this point I should know that given $1-b = 1-pq \in GL(P \otimes_A Q)$, that $[1-qp] \in K_1 A$ depends only on $1-b$ and not on the choice of p, q.

Basic forms of the Vaserstein identity.

$$\begin{pmatrix}1 & -q\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & q\\ p & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -p & 1\end{pmatrix} = \begin{pmatrix}1-qp & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ -p & 1\end{pmatrix}\begin{pmatrix}1 & q\\ p & 1\end{pmatrix}\begin{pmatrix}1 & -q\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1-pq\end{pmatrix}$$

Thus

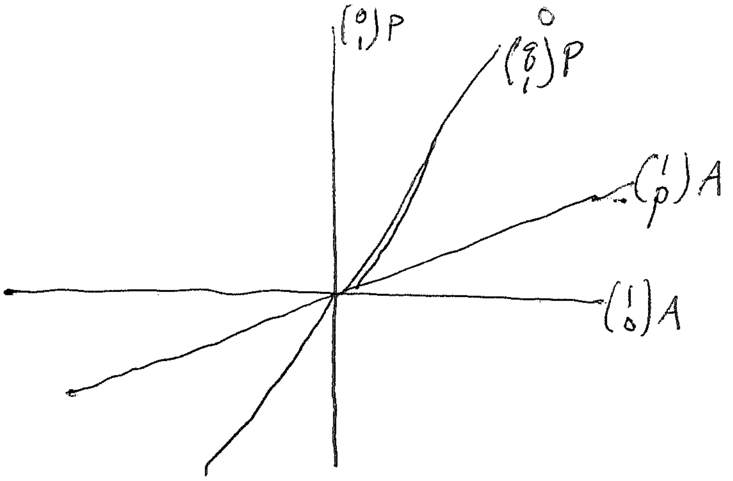
$$\begin{pmatrix}1 & q\\ p & 1\end{pmatrix} = \begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix}\begin{pmatrix}1-qp & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ p & 1\end{pmatrix} = \begin{pmatrix}1-qp & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & (1-qp)^{-1}q\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ p & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ p & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1-pq\end{pmatrix}\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1-pq\end{pmatrix}\begin{pmatrix}1 & 0\\ (1-pq)^{-1}p & 1\end{pmatrix}\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix}$$

$$\boxed{\begin{pmatrix}1-qp & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & (1-pq)^{-1}\end{pmatrix} = \begin{pmatrix}1 & 0\\ (1-pq)^{-1}p & 1\end{pmatrix}\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -p & 1\end{pmatrix}\begin{pmatrix}1 & -(1-qp)^{-1}q\\ 0 & 1\end{pmatrix}}$$

Pictures

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & P \arrow[d, "(q\ 1)"] \arrow[dr, bend left, "1-pq"] & & \\
0 \arrow[r] & A \arrow[r, "\binom{1}{p}"] \arrow[dr, "1-qp"'] & A\oplus P \arrow[r, "(1\ -p)"] \arrow[d, "(1\ -q)"] & P \arrow[r] & 0 \\
 & & A \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$



$\binom{0}{1}P$, $\binom{q}{1}P$ are both complementary to both $\binom{1}{0}A$, $\binom{1}{p}A$

<u>October 19, 1995</u>

Attempt to define HH intrinsically for a Roos category $\mathcal{M}$.

The first idea is to pick $P \in \mathcal{M}'$, $Q \in \mathcal{M}$ and a surjection $\psi : Q \otimes P \to \mathbb{1}$, where P and Q are flat. Suppose I choose a coordinate system: $\mathcal{M} \simeq \mathcal{M}(A)$, A [redacted] firm. Then P becomes a firm flat A^{o}-~~mod~~ Q becomes a firm flat A-module, and $\psi : Q \otimes P \twoheadrightarrow A$ is an A-bimodule surjection. In general, if $M \xrightarrow{f} A$ is an A-bimodule map one gets a precyclic abelian group $[M \otimes_A]^{(*)}$. [redacted] In fact before this one has a presimplicial A-bimodule with augmentation to A:

$$\cdots M \otimes_A M \otimes_A M \rightrightarrows M \otimes_A M \rightrightarrows M \longrightarrow A$$

(this is $T_A(M[1]) = R_A(M \to A)$ which is a DG algebra after making it a complex.)

In the case of $Q \otimes P \twoheadrightarrow A$ we get the augmented complex

$$\cdots Q \otimes B \otimes B \otimes P \longrightarrow Q \otimes B \otimes P \longrightarrow Q \otimes P \longrightarrow A$$

where $B = P \otimes_A Q$, and the precyclic object

$$\left[(Q \otimes_{[redacted]} P) \otimes_A\right]^{(*)} = \left[P \otimes_A Q \otimes\right]^{(*)} = [B \otimes]^{(*)}$$

[redacted]

which gives the Hochshild homology

$$H_*(\tilde{B}, B) = H_*(B \overset{L}{\otimes}_B)$$

of $\tilde{B}$ with coefficients equal to the bimodule B. Recall that $HH_*(B) = HH_*(\tilde{B})/\mathbb{Z}$ where

$$HH_*(\tilde{B}) = H_*(\tilde{B} \overset{L}{\otimes}_B)$$

and the exact sequence $0 \to B \to \tilde{B} \to \mathbb{Z} \to 0$

of B-bimodules yield a Δ

$$B \overset{L}{\otimes}_B \longrightarrow \tilde{B} \overset{L}{\otimes}_B \longrightarrow \mathbb{Z} \overset{L}{\otimes}_B \longrightarrow$$

hence a long exact sequence

$$H_*(\tilde{B}, B) \longrightarrow HH_*(\tilde{B}) \longrightarrow \underset{\substack{\| \\ \text{bare homology } HB_*(B) \\ \text{except for } \mathbb{Z} \text{ in degree } 0.}}{H_*(\mathbb{Z} \overset{L}{\otimes}_B)} \longrightarrow$$

Thus one has

$$H_*(\tilde{B}, B) \longrightarrow HH_*(B) \longrightarrow HB_*(B) \longrightarrow$$

which [redacted] one can also see using the s.e.s. of complexes

$$0 \longrightarrow (B^{\otimes(*+1)}, b) \longrightarrow \mathrm{Cone}(1-\lambda) \longrightarrow \Sigma(B^{\otimes(*+1)}, b') \longrightarrow 0.$$

The point is that there is a canonical map

$$H_*(\tilde{B}, B) \longrightarrow HH_*(B)$$

which is an isom. $\iff$ $HB_*(B) = 0$ i.e. B is h-unital.

When P_A, ${}_AQ$ are flat, B is left and right flat, and conversely. In particular B is h-unital so we see that $[(Q \otimes P) \otimes_A]^{(*)}$ gives the Hochschild homology.

It seems that we have proved

<u>Prop.</u> [redacted] Given $({}_AQ, P_A, \psi : Q \otimes P \twoheadrightarrow A)$ firm as usual. Then $[(Q \otimes P) \otimes_A]^{(*)}$ gives the Hochschild homology iff $P \otimes_A Q$ is h-unital.

October 20, 1995

Intrinsic construction of HH and HC for a Roos category $\mathcal{M}$.

Let's choose a coordinate system $\mathcal{M} = \mathcal{M}(A)$ to do our calculations. [redacted] Let $M \twoheadrightarrow A$ be an A-bimodule surjection [redacted] where M is a firm flat [redacted] A-bimodule. Then we have a cyclic module $[M\otimes_A]^{(*)}$ which we claim computes the Hochschild and cyclic homology of $\mathcal{M}$. (The latter may be defined [redacted] as $HH_*(B)$ and $HC_*(B)$ for any coordinate system $\mathcal{M} \simeq \mathcal{M}(B)$ such that B is h-unital.) [redacted]

The first case to consider is when $M = Q\otimes P$ where ${}_AQ$, P_A are firm flat over A. Then

$$[M\otimes_A]^{(*)} = [(Q\otimes P)\otimes_A]^{(*)} = [P\otimes_A Q\otimes]^{(*)} = B^{\otimes *}$$

is the standard cyclic module associated to the ring B. Now B is left and right flat in this case, in particular h-unital. Thus we know [redacted] $B^{\otimes *}$ yields $HH_*(B)$ and $HC_*(B)$, which are the HH_* and HC_* associated to $\mathcal{M}$ as B is h-unital. (I recall [redacted] it is not a complete triviality that $H_*(B^{\otimes *}, b) = HH_*(B)$, but that this uses the [redacted] h-unitality of B.)

~~Suppose we are given $M_1 \to A$, $M_2 \to A$~~
~~with M_1, M_2 firm flat A-bimodules~~

Next case: A h-unital, M firm flat A-bimodule. I propose to identify $H_*([M\otimes_A]^{(*)}, b)$ with $H_*(A\overset{L}{\otimes}_A)$. [redacted] Consider the DG alg.

$$\cdots \longrightarrow M\otimes_A M \longrightarrow M \longrightarrow A$$

where the diff'l d is the unique degree -1 derivation on $T_A(M)$ which is zero on A and $f: M \twoheadrightarrow A$ on M.

Since M is a flat A-bimodule so is $T_A^n M$ for all $n \geq 1$. Indeed, M is a filtered inductive limit of fg free bimodules, $(\tilde{A} \otimes \tilde{A})^n$, so we ~~restrict~~ to seeing that $(\tilde{A} \otimes \tilde{A}) \otimes_A (\tilde{A} \otimes \tilde{A}) = \tilde{A} \otimes \tilde{A} \otimes \tilde{A}$ is a flat A-bimodule, which results from $\tilde{A}$ being flat over the groundring $\mathbb{Z}$.

Also, since M is a firm A-bimodule, so is $T_A^n(M)$, $\forall n \geq 1$. This is clear since firm for a bimodule means firm on both sides.

Thus

$$\longrightarrow M \otimes_A M \longrightarrow M \longrightarrow A \longrightarrow 0$$

is a complex of firm flat A-bimodules with augmentation to A.

Next we show this is a resolution modulo nil left A-modules. Look at the homology of this DG ring: $H_*(T_A M, d)$. Left and right multiplication by A on this homology factor through left + right mult by $H_0(T_A M, d) = 0$. Thus $H_*(T_A M, d)$ is killed by $\tilde{A} \otimes A^{op} + A \otimes \tilde{A}^{op} \subset \tilde{A} \otimes \tilde{A}^{op}$. (More concretely given $a \in A$ choose $\xi \in M$ such that $d(\xi) = a$, then $h = \xi \cdot$ satisfies $[d, h]\alpha = d(\xi\alpha) + \xi\, d\alpha = a\alpha$, showing that $A \cdot H_*(T_A M) = 0$.

Now A h-unital $\iff$ $A \overset{L}{\otimes}_A -$ kills complexes with nil-homology. Apply this functor to the complex

$$T_{\tilde{A}} M: \quad \longrightarrow M \otimes_A M \longrightarrow M \longrightarrow \tilde{A}$$

Because all modules are flat over A we know that

$$A \overset{L}{\otimes}_{A} T_{\tilde A} M \cong A \otimes_{A} T_{\tilde A} M = T_A M$$

On the other hand since $T_{\tilde A} M$ has nil-homology, and A is h-unital we know this complex is acyclic.

At this point we have a flat bimodule resolution of A

$$\longrightarrow M \otimes_A M \longrightarrow M \longrightarrow A$$

so applying $-\otimes_A$ gives

$$\left([M\otimes_A]^{(*)}, b\right) \simeq A \overset{L}{\otimes}_A$$

Thus for A h-unital, M firm flat bimod $\twoheadrightarrow A$, we have $H_*\left([M\otimes_A]^{(*)}, b\right) \simeq HH_* A$, a canon isom.

A remaining point is that given two such M's, say M_1 and M_2 we can form either $M_1 \otimes_A M_2$ (resp. $M_1 \oplus M_2$) and then get

$$\begin{tikzcd}
 & \left[(M_1\otimes_A M_2)\otimes_A\right]^{(*)} \arrow[dl, "\text{quis}"'] \arrow[dr, "\text{quis}"] & \\
\left[M_1 \otimes_A\right]^{(*)} \arrow[dr, dashed] & & \left[M_2\otimes_A\right]^{(*)} \arrow[dl, dashed] \\
 & \left[(M_1 \oplus M_2)\otimes_A\right]^{(*)} &
\end{tikzcd}$$

resp. $\}$ $\}$ resp.

and thus get not only a canonical isom. of HH_*, but also HC_*, for the cyclic modules $[M_1\otimes_A]^{(*)}$ and $[M_2\otimes_A]^{(*)}$. Then taking $M_2 = Q\otimes P$, we identify these with $HH_*(B)$, $HC_*(B)$.

Finally we want to get beyond assuming A is h-unital. The point will be that if we have $Q\otimes P \twoheadrightarrow A$ with Q, P firm flat$/A$, then the transport of M to the bimodule $P\otimes_A M \otimes_A Q$ over $P\otimes_A Q = B$ is a firm flat bimodule over B.

Consider the rings $A \otimes A^{op}$, $B \otimes B^{op}$. Then $P \otimes Q$ is a left-$B \otimes B^{op}$, right-$A \otimes A^{op}$ bimod. and $Q \otimes P$ is a left-$A \otimes A^{op}$, right-$B \otimes B^{op}$ bimodule. Moreover these rings are firm and the bimodules are firm on both sides. Also

$$(P \otimes Q) \otimes_{A \otimes A^{op}} (Q \otimes P) = (P \otimes_A Q) \otimes (Q \otimes_{A^{op}} P) = B \otimes B$$

$$(Q \otimes P) \otimes_{B \otimes B^{op}} (P \otimes Q) = (Q \otimes_B P) \otimes (P \otimes_{B^{op}} Q) = A \otimes A$$

Thus we should have a completely firm Morita context:

$$\begin{pmatrix} A \otimes A^{op} & Q \otimes P \\ P \otimes Q & B \otimes B^{op} \end{pmatrix}$$

We can then conclude that

$$M \longmapsto (P \otimes Q) \otimes_{A \otimes A^{op}} M = P \otimes_A M \otimes_A Q$$

carries firm flat A-bimodules to firm flat B-bimods.
(I use here that firm A-bimodules, i.e. A-bimods M which are firm on either side, are the same as unital $\tilde{A} \otimes \tilde{A}^{op}$-modules which are firm wrt the ideal $A \otimes A^{op}$, equivalently firm modules for the ring $A \otimes A^{op}$. Check:

$$(A \otimes A) \otimes_{\tilde{A} \otimes \tilde{A}^{op}} M = A \otimes_A M \otimes_A A$$

and $\underbrace{A \otimes_A M \otimes_A A \xrightarrow{\sim} M}_{\text{implies}} \implies M = AMA \subset AM,\ MA \subset M$

Also

$$\begin{tikzcd} A \otimes_A A \otimes_A M \otimes_A A \arrow[r, "\sim"] \arrow[d] & A \otimes_A M \arrow[d] \\ A \otimes_A M \otimes_A A \arrow[r, "\sim"] & M \end{tikzcd}$$

iso since $AM = M \implies A \otimes_A M$ firm.

$\therefore A \otimes_A M \xrightarrow{\sim} M$
$M \otimes_A A \xrightarrow{\sim} M$
so M is left and right firm.)

Lem: Let A be an ideal in B and M a B-module. Then $A\otimes_B M \xrightarrow{\sim} M \implies \tilde{B}\otimes_B M \xrightarrow{\sim} M$.

Proof. The hypothesis implies $-\otimes_B M$ inverts A^{op}-nil-isos, in particular $A \subset B \subset \tilde{B}$. □

Application: If M is a bimodule over A, ~~[illegible]~~ then $A\otimes_A M\otimes_A A \xrightarrow{\sim} M \iff A\otimes_A M \xrightarrow{\sim} M$ and $M\otimes_A A \xrightarrow{\sim} M$.

Pf. ($\Leftarrow$) clear. ($\Rightarrow$): The hypothesis says that the $\tilde{A}\otimes\tilde{A}^{op}$ module M is firm wrt $A\otimes A^{op}$. The lemma says M is also firm wrt the ideal $A\otimes\tilde{A}^{op}$, i.e. $A\otimes_A M\otimes_A \tilde{A} \xrightarrow{\sim} M$. But $M\otimes_A\tilde{A} \xleftarrow{\sim} M$, so $A\otimes_A M \xrightarrow{\sim} M$. □.

added Nov 10

Direct proof ~~[illegible]~~ for a (B,A)-bimodule P that $B\otimes_B P\otimes_A A \xrightarrow{\sim} P \iff B\otimes_B P \xrightarrow{\sim} P$ and $P\otimes_A A \xrightarrow{\sim} P$.

$\Leftarrow$ clear. $\Rightarrow$: $B^{(2)}\otimes_B P\otimes_A A^{(2)} \xrightarrow{\sim} B\otimes_B P\otimes_A A \xrightarrow{\sim} P$ and clear $B^{(2)}\otimes_B P\otimes_A A^{(2)}$ is firm on both sides.