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K_0 for nonunital rings and Morita invariance

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The subject of this paper is a variant $K_0'A$ of K_0A for a nonunital ring A, which has the advantage of being invariant with respect to the kind of Morita equivalences considered in [Q].

To explain how $K_0'A$ arises, let R be a unital ring containing A as ideal, for example, the ring $\tilde{A}$ obtained by adjoining an identity to A. Let $\mathcal{L}(R,A)$ be the category of finite projective complexes U over R such that U/AU is acyclic, and let $K_0\mathcal{L}(R,A)$ be the associated Grothendieck group in which homotopy equivalent complexes are identified and short exact sequences provide the usual relations. Our main result says that this group is Morita invariant in the following sense.

Theorem 1) *If A, A' are ideals in R such that $A^n \subset A'$, $A'^n \subset A$ for some n, then $K_0\mathcal{L}(R,A) \simeq K_0\mathcal{L}(R,A')$.*
2) *Given a Morita context*

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$$

linking the unital rings R and S, then $K_0\mathcal{L}(R,QP) \simeq K_0\mathcal{L}(S,PQ)$.

As a corollary one obtains the excision result

$$K_0\mathcal{L}(\tilde{A},A) \xrightarrow{\sim} K_0\mathcal{L}(R,A)$$

which shows that $K_0\mathcal{L}(R,A)$ depends only on the nonunital ring A, and justifies using the notation $K_0'A$ for it.

To gain a better understanding of $K_0'A$ we prove

$$K_0\mathcal{L}^1(R,A) \xrightarrow{\sim} K_0\mathcal{L}(R,A)$$

where $\mathcal{L}^1(R,A)$ is the full subcategory of $\mathcal{L}(R,A)$ consisting of 1-dimensional chain complexes. An object of this subcategory is the same as a map $f : P \to Q$ in the category $\mathcal{P}(R)$ of finitely generated projective R-modules, such that the induced map $\bar{f} : P/AP \to Q/AQ$ is an isomorphism. In terms of these maps $K_0'A$ can be described as follows.

Theorem *$K_0'A \simeq K_0\mathcal{L}^1(R,A)$ is the abelian group generated by elements $[f]$, where f is any map as above, subject to the relations:*
1) $[f : P \to Q] + [f' : P' \to Q'] = [f \oplus f' : P \oplus P' \to Q \oplus Q']$
2) $[f : P' \to P] + [g : P \to Q] = [gf : P' \to Q]$
3) $[f] = 0$ *when f is an isomorphism.*

On the other hand, K_0A can be identified with the relative group $K_0(R \to R/A)$ consisting of classes of triples (P, Q, w) with P, Q in $\mathcal{P}(R)$ and $w : P/AP \xrightarrow{\sim} Q/AQ$. There is a canonical homomorphism

$$K_0'A \to K_0A$$

induced by sending $f : P \to Q$ to the triple $(P, Q, \bar{f})$, which is always surjective but not an isomorphism in general. When A is a C^*-algebra we prove this surjection is an isomorphism, and deduce the following application (compare [A], §2.6).

Theorem *Let Y be a closed subspace of a compact space X. Then the Grothendieck group of complexes of vector bundles on X which are acyclic over Y is isomorphic to the topological K-theory group $K^0(X, Y)$.*

In proving Morita invariance of K_0' we need to enlarge the category of finite projective complexes to include all complexes homotopy equivalent to some finite projective complex. In the first section we characterize such complexes as those whose identity map is homotopic to a 'nuclear' map, and the second section contains a corresponding A-nuclearity criterion for complexes homotopy equivalent to complexes in $\mathcal{L}(R, A)$. We use this criterion in the next two sections to establish Morita equivalence up to homotopy for the categories $\mathcal{L}(R, A)$ and $\mathcal{L}^1(R, A)$, and Morita invariance of their Grothendieck groups.

The fifth section is concerned with the reduction of $K_0\mathcal{L}(R, A)$ to $K_0\mathcal{L}^1(R, A)$. By Morita invariance it suffices to treat the case $R = \tilde{A}$, where we exploit the fact that a complex in $\mathcal{L}(\tilde{A}, A)$ can be easily compared to the contractible free complex obtained by reducing modulo A and then tensoring with $\tilde{A}$.

The next two sections discuss situations where the canonical map $K_0'A \to K_0A$ is, or fails to be, an isomorphism. In the eighth section we examine $K_0\mathcal{L}^1(R, A)$ in detail and derive the presentation of it we have mentioned. Finally, the last section is devoted to a Whitehead type formula which decomposes the class in $K_0'A$ of a complex in $\mathcal{L}(R, A)$ into classes of 1-dimensional complexes.

§1. We fix a unital ring R and work with unitary left modules over R and R-linear maps. Given modules M, N there is a canonical map of abelian groups

$$\mathrm{Hom}_R(M, R) \otimes_R N \to \mathrm{Hom}_R(M, N) \tag{1}$$

sending $\lambda \otimes n$ to the map $(\lambda.n)(m) = \lambda(m)n$.

A module map $f : M \to N$ will be called *nuclear* when it lies in the image of (1). This is equivalent to f factoring through a finitely generated free module. The identity map 1_M is nuclear iff M is a finitely generated projective module, and in this case (1) is an isomorphism for all N.

We next extend these ideas to complexes of R-modules: $U = \bigoplus_n U_n$, $n \in \mathbf{Z}$, with differential d of degree -1. All complexes will be assumed bounded unless stated otherwise.

Let U, V be complexes over R, and let $\mathrm{Hom}_R(U, V)$ be the mapping complex, where $\mathrm{Hom}_R(U, V)_n$ is the abelian group of graded module maps of degree n and the differential is $f \mapsto$ $[d, f] = d \cdot f - (-1)^{|f|} f \cdot d$. There is a canonical map of complexes of abelian groups

$$\mathrm{Hom}_R(U, R) \otimes_R V \to \mathrm{Hom}_R(U, V) \tag{2}$$

sending $\lambda \otimes v$ to $(\lambda.v)(u) = (-1)^{|u||v|} \lambda(u) v$. We call $f \in \mathrm{Hom}_R(U, V)$ nuclear when it lies in the image of (2), equivalently, when each component $f_{nm} : U_m \to V_n$ is a nuclear map of modules.

By a finite projective (resp. finite free) complex we mean a bounded complex of finitely generated projective (resp. free) modules. Clearly the identity map 1_U is nuclear iff U is a finite projective complex, and in this case (2) is an isomorphism for all V.

This nuclearity criterion will now be extended to describe complexes which are (chain) homotopy equivalent to finite projective complexes. We recall that U is said to be *dominated* by V when U is a homotopy retract of V, i.e. there exist maps $i : U \to V$, $j : V \to U$ of complexes, such that ji is homotopic to 1_U.

Proposition 1.1 *The following are equivalent:*
1) *U is homotopy equivalent to a finite projective complex.*
2) *For any V the map (2) is a homotopy equivalence.*
3) *1_U is homotopic to a nuclear map.*
4) *U is dominated by a finite free (resp. finite projective) complex.*

Proof. The implications 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 3) are easy.

3) $\Rightarrow$ 4). Assuming 3) there is a homotopy operator $h \in \mathrm{Hom}_R(U, U)_1$ such that $f = 1 - dh - hd$ is nuclear. In each degree f factors through a finite free module which we can take to be zero when f is zero in that degree. Thus there is a finite free graded module T and graded module maps $i : U \to T$, $j : T \to U$ such that $f = ji$. Since $idj \cdot idj = id(1 - dh - hd)dj = 0$, T becomes a complex with the differential idj. Also $i' = i(1 - dh) : U \to T$, $j' = (1 - hd)j : T \to U$ are maps of complexes, since $idj \cdot i(1 - dh) = id(1 - dh - hd)(1 - dh) = i(1 - dh) \cdot d$ and similarly for j'. Finally

$j'i' = (1-hd)(1-dh-hd)(1-dh) = 1-2dh-2hd+hdhd+dhdh = 1-dh'-h'd$ where $h' = 2h - hdh$. Thus U is dominated by the finite free complex T.

4) $\Rightarrow$ 1). See [R1], p. 106. □

§2. Let A be an ideal in R. We now study finite projective complexes over R which are acyclic modulo A, and also complexes which are homotopy equivalent to them.

Lemma 2.1 *Let U a complex over R whose identity map is homotopic to a map f such that $f(U) \subset AU$. Then for any ideal A' containing A^n for some n the inclusion $A'U \subset U$ is a homotopy equivalence.*

Proof. Let h be a homotopy operator such that $f = 1 - [d,h]$. Then $f(A^nU) \subset A^{n+1}U$, $h(A^nU) \subset A^nU$ for all $n \in \mathbf{N}$, and hence f, h provide a homotopy inverse for the inclusion $A^{n+1}U \subset A^nU$ for all n. Consequently $A^nU \subset U$ is a homotopy equivalence for all n. In particular 1_U is homotopic to a map with image contained in A^nU, and hence in $A'U$ when $A^n \subset A'$. By the same arguments with A' in place of A, we conclude that $A'U \subset U$ is a homotopy equivalence. □

Proposition 2.2 *Let U be a finite projective complex over R. The following are equivalent:*

1) *U/AU is acyclic.*
2) *U/AU is a contractible complex over R/A.*
3) *1_U is homotopic to a map with image contained in AU.*
4) *The map $A \otimes_R U \to U$, $a \otimes u \mapsto au$ is a homotopy equivalence.*

Proof. The hypothesis on U implies that U/AU is a finite projective complex over R/A. Thus 1) $\Rightarrow$ 2), because any right bounded acyclic complex of projective modules is contractible.

Assuming 2), let $\bar{h}$ be an R/A-linear homotopy operator on U/AU such that $[d, \bar{h}] = 1$. As U is projective in each degree, we can lift $\bar{h}$ to an R-linear homotopy operator h on U. Then $f = 1 - [d,h]$ induces zero on U/AU, so $f(U) \subset AU$, yielding 3).

Next note that the multiplication map $A \otimes_R U \to AU$ is an isomorphism when U is a complex of projective modules. Assuming 3), we know the inclusion $AU \subset U$ is a homotopy equivalence by the preceding lemma, so 4) holds.

Finally 4) $\Rightarrow$ 1) by the homology exact sequence arising from the short exact sequence $A \otimes_R U \to AU \to U/AU$. □

Define $\mathcal{L}(R,A)$ to be the category of finite projective complexes over R which are acyclic modulo A, and hence satisfy all the conditions of the preceding proposition.

Proposition 2.3 *Let A and A' be two ideals in R which define the same adic topology, i.e. $A' \supset A^n$ and $A \supset A'^n$ for some n. Then $\mathcal{L}(R,A) = \mathcal{L}(R,A')$.*

This follows from the preceding lemma and proposition.

Although we are mainly concerned with complexes in $\mathcal{L}(R, A)$, it will be necessary when we discuss Morita equivalence to consider more generally complexes homotopy equivalent to complexes in $\mathcal{L}(R, A)$. We now derive a useful nuclearity criterion for these complexes.

Let $\mathcal{U}(R, A)$ be the category of complexes over R which are homotopy equivalent to complexes in $\mathcal{L}(R, A)$. Clearly any complex in $\mathcal{U}(R,A)$ has the properties 1)-4) of 2.2.

Theorem 2.4 *A complex U over R is in $\mathcal{U}(R, A)$ iff 1_U is homotopic to an A-nuclear map f, by which we mean that f lies in the image of the canonical map*

$$\mathrm{Hom}_R(U, A) \otimes_R U \to \mathrm{Hom}_R(U, U)$$

sending $\lambda \otimes u$ to $(\lambda.u)(u') = (-1)^{|u||u'|}\lambda(u')u$.

Proof. Assume 1_U ~~be~~ is homotopic to the A-nuclear map f. Then f is nuclear and $f(U) \subset AU$, so U is homotopy equivalent to a finite projective complex by 1.1, and satisfies condition 3) of 2.2. Thus U is in $\mathcal{U}(R, A)$.

Conversely, assume U is in $\mathcal{U}(R, A)$, and consider the commutative square of canonical maps

$$\begin{array}{ccc}
\mathrm{Hom}_R(U, R) \otimes_R A \otimes_R U & \to & \mathrm{Hom}_R(U, R) \otimes_R U \\
\downarrow & & \downarrow \\
\mathrm{Hom}_R(U, A) \otimes_R U & \to & \mathrm{Hom}_R(U, U)
\end{array}$$

The top arrow is a homotopy equivalence by 2.2, 4), and the right arrow is a homotopy equivalence by 1.1, 2). Consequently 1_U is homotopic to a map f coming from the upper left corner, hence contained in the image of the bottom arrow. Thus f is A-nuclear. $\square$

§3. We consider a Morita context

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$$

that is, a unital ring equipped with a splitting into four abelian subgroups such that when the elements are written as 2×2 matrices the multiplication is consistent with matrix multiplication. It follows that R and S are unital rings, P is an (S, R) bimodule, and Q is a (R, S) bimodule. Also we have an R-bimodule map $Q \otimes_S P \to R$, $q \otimes p \mapsto qp$ and an S-bimodule map $P \otimes_R Q \to S$, $p \otimes q \mapsto pq$, satisfying the associativity conditions $(qp)q' = q(pq')$, $(pq)p' = p(qp')$. Conversely this data gives an equivalent description of a Morita context.

This Morita context determines a functor $P \otimes_R -$ from R-modules to S-modules, and a functor $Q \otimes_S -$ in the opposite direction, together with natural transformations $Q \otimes_S P \otimes_R M \to M$, $P \otimes_R Q \otimes_S N \to N$. These functors and natural transformations extend to complexes over R and S in an evident way.

Let A be the ideal QP in R, and let B be the ideal PQ in S. When $A = R$ and $B = S$ we have the classical Morita equivalence situation, where the categories of modules over R and S are equivalent.

Recall that $\mathcal{U}(R, A)$ is the category of complexes over R which are homotopy equivalent to finite projective complexes acyclic modulo A, and define $\mathcal{U}(S, B)$ similarly.

Theorem 3.1 *The functors $P\otimes_R-$, $Q\otimes_S-$ map $\mathcal{U}(R,A)$, $\mathcal{U}(S,B)$ into each other, and the canonical maps $Q\otimes_S P\otimes_R U \to U$, $P\otimes_R Q\otimes_S V \to V$ for U in $\mathcal{U}(R,A)$, V in $\mathcal{U}(S,B)$ are homotopy equivalences.*

Proof. We first show that $Q\otimes_S P\otimes_R U \to U$ is a homotopy equivalence for U in $\mathcal{U}(R, A)$. Consider the commutative diagram

$$\begin{array}{ccc} Q\otimes_S P\otimes_R A & \to & A\otimes_R A \\ \downarrow & \swarrow & \downarrow \\ Q\otimes_S P & \to & A \end{array}$$

where the horizontal and vertical arrows given in the obvious way by multiplication in the Morita context. To define the diagonal arrow, let $a, a' \in A$ and choose representations $a = \sum q_i p_i$, $a' = \sum q'_j p'_j$. From $q_i p_i q'_j \otimes p'_j = q_i \otimes p_i q'_j p'_j$ we obtain

$$a(\sum_j q'_j \otimes p'_j) = (\sum_i q_i \otimes p_i)a'$$

in $Q\otimes_S P$. The first (resp. second) expression shows this element is independent of the choice of representation of a (resp. a'). We thus get a well-defined map $A \times A \to Q\otimes_S P$, which extends to give the diagonal arrow.

Now apply the functor $-\otimes_R U$ to this diagram. The vertical maps become homotopy equivalences as $A\otimes_R U \to U$ is a homotopy equivalence. Here we use the equality $aa'\otimes u = a\otimes a'u$ of the two possible multiplication maps from $A\otimes_R A\otimes_R U$ to $A\otimes_R U$, and similarly in the case of the left vertical arrow. It then follows formally that the horizontal maps become homotopy equivalences. In effect, the diagonal arrow provides the required homotopy inverses. Thus we have homotopy equivalences $Q\otimes_S P\otimes_R U \to A\otimes_R U \to U$, yielding the desired result.

Next we show that if U is in $\mathcal{U}(R, A)$, then $V = P\otimes_R U$ is in $\mathcal{U}(S, B)$. We consider commutative diagram of canonical maps

$$\begin{array}{ccccc} \mathrm{Hom}_R(U,R)\otimes_R Q\otimes_S P\otimes_R U & \xrightarrow{w} & \mathrm{Hom}_R(U,R)\otimes_R U & \xrightarrow{x} & \mathrm{Hom}_R(U,U) \\ \downarrow & & & & \downarrow \\ \mathrm{Hom}_S(V,P)\otimes_R Q\otimes_S V & \xrightarrow{y} & \mathrm{Hom}_S(V,B)\otimes_S V & \xrightarrow{z} & \mathrm{Hom}_S(V,V) \end{array}$$

where the vertical maps arise from effect of the functor $P\otimes_R-$. The map y is induced the
by sending $\lambda\otimes q \in \mathrm{Hom}_S(V, P)\otimes_R Q$ to $v \mapsto \lambda(v)q$ in $\mathrm{Hom}_S(V, B)$, and w is induced by the map $Q\otimes_S P\otimes_R U \to U$, which we have just shown to be a homotopy equivalence. Hence w is a homotopy equivalence, and x is a homotopy equivalence by 1.1, 2). Thus

1_U is homotopic to a map coming from the upper left corner. It follows then that 1_V is homotopic to a map in the image of z, i.e. a B-nuclear map, so V is in $\mathcal{U}(S,B)$ by 2.4.

By symmetry it follows that if V is in $\mathcal{U}(S,B)$, then $Q\otimes_S V$ is in $\mathcal{U}(R,A)$, and $P\otimes_R Q\otimes_S V\to V$ is a homotopy equivalence. □

Let $\mathrm{Ho}\,\mathcal{U}(R,A)$ be the homotopy category having the same objects as $\mathcal{U}(R,A)$ and homotopy classes of maps for its morphisms. It is a triangulated category where the distinguished triangles arise from short exact sequences of complexes which are split exact in each degree. Theorem 3.1 yields immediately the following Morita equivalence for the homotopy categories.

Corollary 3.2 *The functors $P\otimes_R -$, $Q\otimes_S -$ give inverse equivalences of triangulated categories between $\mathrm{Ho}\,\mathcal{U}(R,A)$ and $\mathrm{Ho}\,\mathcal{U}(S,B)$.*

§4. The Grothendieck group $K_0\mathcal{U}(R,A)$ is defined to be the abelian group generated by elements $[U]$ for each object U in $\mathcal{U}(R,A)$ subject to the relations:
a) $[U]=[U']+[U'']$ when there is a short exact sequence $U'\to U\to U''$ which is locally split, i.e. split exact in each degree.
b) $[U]=[U']$ when U and U' are homotopy equivalent.

We define $K_0\mathcal{L}(R,A)$ similarly. Note that any short exact sequence in $\mathcal{L}(R,A)$ is locally split.

Proposition 4.1 *On has an isomorphism $K_0\mathcal{L}(R,A)\xrightarrow{\sim} K_0\mathcal{U}(R,A)$ induced by the inclusion of $\mathcal{L}(R,A)$ in $\mathcal{U}(R,A)$.*

Proof. Given U in $\mathcal{U}(R,A)$ we can choose a T in $\mathcal{L}(R,A)$ which is homotopy equivalent to U. The element $[T]\in K_0\mathcal{L}(R,A)$ is clearly independent of the choice of T, so we have a well-defined map $\varphi: U\mapsto [T]$ from objects of $\mathcal{U}(R,A)$ to $K_0\mathcal{L}(R,A)$.

The map φ clearly equalizes homotopy equivalent complexes, and we now check that it is additive for a locally split short exact sequence $U'\to U\to U''$. We can assume $U=U'\oplus U''$ with differential $d'+d''+\theta$, where $\theta: U''\to U'$ has degree -1 and satisfies $d'\theta+\theta d''=0$, Choose homotopy equivalences $a: T''\to U''$, $b: U'\to T'$, and let T be $T'\oplus T''$ with differential $d'+d''+b\theta a$. We then have a locally split short exact sequence $T'\to T\to T''$ in $\mathcal{L}(R,A)$ such that T homotopy equivalent to U, so $\varphi(U)=\varphi(U')+\varphi(U'')$.

Consequently φ induces a homomorphism $K_0\mathcal{U}(R,A)\to K_0\mathcal{L}(R,A)$, which is easily seen to be inverse to the obvious map going the other way. □

Theorem 4.2 1) *$K_0\mathcal{L}(R,A)$ is Morita invariant, i.e. it depends only on the adic topology associated to A and in the situation of* 3.1 *there is a canonical isomorphism $K_0\mathcal{L}(R,A)\simeq K_0\mathcal{L}(S,B)$.*
2) *Let $\tilde{A}=\mathbf{Z}\oplus A$ be the ring obtained by adjoining an identity to A. Then we have*

an isomorphism $K_0\mathcal{L}(\tilde{A}, A) \xrightarrow{\sim} K_0\mathcal{L}(R, A)$ *induced by extension of scalars with respect to the canonical unital ring homomorphism* $\tilde{A} \to R$. *In particular, up to canonical isomorphism* $K_0\mathcal{L}(R, A)$ *depends only on the nonunital ring* A.

Proof. 1) The part concerning adic topologies is clear from 2.3. For the rest it suffices by 4.1 to produce a canonical isomorphism $K_0\mathcal{U}(R, A) \simeq K_0\mathcal{U}(S, B)$ in the situation of 3.1. This follows from 3.1 and the definition of K_0, using the fact that the functors $P \otimes_R -$, $Q \otimes_S -$ respect homotopy equivalences and locally short exact split
sequences.

2) We apply 3.1 in the case of the Morita context

$$\begin{pmatrix} R & R \\ A & \tilde{A} \end{pmatrix}$$

sitting inside 2×2 matrices over $\tilde{R}$. The functor $Q \otimes_{\tilde{A}} -$ is extension of scalars with respect to the canonical homomorphism $\tilde{A} \to R$. Thus we have $K_0\mathcal{U}(\tilde{A}, A) \xrightarrow{\sim} K_0\mathcal{U}(R, A)$ induced by extension of scalars. Then 2) follows using 4.1 and the fact that extension of scalars preserves finitely generated projective modules. □

We now briefly mention the analogous result for n-dimensional chain complexes. Let $\mathcal{L}^n(R, A)$, $\mathcal{U}^n(R, A)$ be the full subcategories of $\mathcal{L}(R, A)$, $\mathcal{U}(R, A)$ respectively consisting of complexes U such that $U_k = 0$ for $k < 0$ and $k > n$. Corresponding to 4.1 we have

$$K_0\mathcal{L}^n(R, A) \xrightarrow{\sim} K_0\mathcal{U}^n(R, A)$$

proved in the same way. The only new point is to observe that if U is an n-dimensional chain complex which is homotopy equivalent to a finite projective complex T, then on T there is a homotopy operator h such that $[d, h] = 1$ in degrees $k < 0$ and $k > n$, and consequently we can split off contractible complexes from T and assume T is a finite projective n-dimensional chain complex.

Corresponding to 4.2 we have the following with the same proof.

Theorem 4.3 1) $K_0\mathcal{L}^n(R, A)$ *is Morita invariant.*
2) *There is an isomorphism* $K_0\mathcal{L}^n(\tilde{A}, A) \xrightarrow{\sim} K_0\mathcal{L}^n(R, A)$ *induced by extension of scalars with respect to the canonical homomorphism* $\tilde{A} \to R$.

§5. The category of all finite projective complexes over R is known to have the same Grothendieck group as the subcategory of 0-dimensional chain complexes, i.e. $\mathcal{P}(R)$. The analogous result for $\mathcal{L}(R, A)$ requires 1-dimensional chain complexes in general. We shall prove

Theorem 5.1 *The inclusion of* $\mathcal{L}^1(R, A)$ *in* $\mathcal{L}(R, A)$ *induces an isomorphism* $K_0\mathcal{L}^1(R, A) \xrightarrow{\sim} K_0\mathcal{L}(R, A)$.

By 4.2, 4.3 it suffices to treat the case $R = \tilde{A}$. Let $\mathcal{L}^1$ stand for $\mathcal{L}^1(\tilde{A}, A)$ and similarly for $\mathcal{L}$. We begin by establishing surjectivity.

Given U in $\mathcal{L}$, we put $U^\# = \tilde{A} \otimes_{\mathbf{Z}} (U/AU)$. Since U/AU is a contractible finite projective complex over $\mathbf{Z}$, $U^\#$ is a contractible finite projective complex over $\tilde{A}$. Note that there is a canonical isomorphism $U^\#/AU^\# = U/AU$.

Lemma 5.2 *There exist maps of module complexes $f : U^\# \to U$, $g : U \to U^\#$ which cover the canonical isomorphism modulo A.*

Proof. Since U is projective, the exact sequence

$$0 \to AU^\# \to U^\# \to U^\#/AU^\# \to 0$$

gives rise to an exact sequence of mapping complexes

$$0 \to \mathrm{Hom}_{\tilde{A}}(U, AU^\#) \to \mathrm{Hom}_{\tilde{A}}(U, U^\#) \to \mathrm{Hom}_{\tilde{A}}(U, U^\#/AU^\#) \to 0$$

The mapping complex on the left is contractible because $AU^\#$ is contractible. It follows that the cycle in the mapping complex on the right, which is represented by the map $U \to U/AU = U^\#/A^\# U$, can be lifted to a cycle in the middle complex. This yields the desired map g, and f is obtained by a similar argument. □

Let $f : U^\# \to U$ as above, and let C be the mapping cone on f, that is, $C_n = U^\#_{n-1} \oplus U_n$ with differential $\begin{pmatrix} -d & f \\ 0 & d \end{pmatrix}$.

Let $F_n C$ be the subcomplex made of $U^\#_k$ and U_k for $k \leq n$. These subcomplexes form a locally split filtration of C such that $F_n C / F_{n-1} C$ is the n-th suspension of $f_n : U^\#_n \to U_n$ considered as an object of $\mathcal{L}^1$. Hence we have $[C] = \sum (-1)^n [f_n]$ in $K_0 \mathcal{L}$, showing that $[C]$ lies in the image of $K_0 \mathcal{L}^1$. On the other hand, U and C are homotopy equivalent because $U^\#$ is contractible, so $[U] = [C]$, proving surjectivity.

Our next task is to refine the preceding argument to produce a homomorphism $K_0 \mathcal{L} \to K_0 \mathcal{L}^1$. We first prove two lemmas which are valid when R is any unital ring containing A as ideal.

Observe that an object U of $\mathcal{L}^1(R, A)$ is the same as a map $f : P \to Q$ in $\mathcal{P}(R)$, which is an isomorphism modulo A. We write $[f]$ for the element $[U]$ of $K_0 \mathcal{L}^1(R, A)$.

Lemma 5.3 *If $f : P' \to P$, $g : P \to Q$ are maps in $\mathcal{P}(R)$ which are isomorphisms modulo A, then $[gf] = [f] + [g]$ in $K_0 \mathcal{L}^1(R, A)$.*

Proof. There are short exact sequence in $\mathcal{L}^1(R, A)$

$$\begin{array}{ccccccccc} 0 & \to & P' & \xrightarrow{i} & P' \oplus P & \xrightarrow{j} & P & \to & 0 \\ & & f \downarrow & & gf \oplus 1 \downarrow & & g \downarrow & & \\ 0 & \to & P & \xrightarrow{i'} & Q \oplus P & \xrightarrow{j'} & Q & \to & 0 \end{array}$$

with $i = \binom{1}{f}$, $j = (f \ \ -1)$, $i' = \binom{g}{1}$, and $j' = (1 \ \ -g)$. Hence $[f] + [g] = [gf \oplus 1] = [gf]$. □

Lemma 5.4 *Let $f : U \to V$, $g : V \to W$ be maps of finite projective complexes over R which are isomorphisms modulo A, and define*

$$\chi(f) = \sum_n (-1)^n [f_n] \in K_0\mathcal{L}^1(R, A)$$

and similarly for g. Then
1) $\chi(gf) = \chi(f) + \chi(g)$,
2) If both U and V are contractible, then $\chi(f) = 0$.

Proof. 1) follows immediately from the preceding lemma.
2) As U, V are contractible, we have a map of short exact sequences in $\mathcal{P}(R)$

$$\begin{array}{ccccccccc} 0 & \to & Z_nU & \to & U_n & \to & Z_{n-1}U & \to & 0 \\ & & Z_nf \downarrow & & f_n \downarrow & & Z_{n-1}f \downarrow & & \\ 0 & \to & Z_nV & \to & V_n & \to & Z_{n-1}V & \to & 0 \end{array}$$

where Z_n denotes the kernel of d in degree n. Since each f_n is an isomorphism modulo A, the same holds for each Z_nf by induction. Thus $[f_n] = [Z_nf] + [Z_{n-1}f]$ in $K_0\mathcal{L}^1(R, A)$, so $\sum_n (-1)^n [f_n] = 0$. □

Returning now to the situation $R = \tilde{A}$, let $f : U^\# \to U$, $g : U \to U^\#$ as in 5.2. By 5.4 the elements $\chi(f), \chi(g)$ in $K_0\mathcal{L}^1$ satisfy $\chi(f) + \chi(g) = \chi(gf) = 0$, since $U^\#$ is contractible. Thus $\chi(f) = -\chi(g)$ is independent of the choice of f.

So for any U in $\mathcal{L}$ we have a well-defined element $\chi(U)$ of $K_0\mathcal{L}^1$ given by $\chi(U) = \chi(f)$, where $f : U^\# \to U$ is any lifting of the canonical isomorphism modulo A. We will show that $U \mapsto \chi(U)$ satisfies the defining relations for $K_0(\mathcal{L})$, and hence induces a homomorphism $K_0\mathcal{L} \to K_0\mathcal{L}^1$.

First we verify that χ is additive for short exact sequences in $\mathcal{L}$. Consider

$$\begin{array}{ccccccccc} 0 & \to & U'^\# & \to & U^\# & \to & U''^\# & \to & 0 \\ & & f' \downarrow & & f \downarrow & & f'' \downarrow & & \\ 0 & \to & U' & \to & U & \to & U'' & \to & 0 \end{array}$$

where the bottom row is exact in $\mathcal{L}$, and the top row is obtained from it by applying $\#$. We want to construct liftings f', f, f'' of the canonical isomorphisms modulo A such that this diagram commutes. In this case $[f_n] = [f'_n] + [f''_n]$ in $K_0\mathcal{L}^1$ for each n, and so $\chi(U) = \chi(U') + \chi(U'')$.

Now the top row splits by the contractibility of either $U'^\#$ or $U''^\#$, hence there is a subcomplex K such that $U'^\# \oplus K \xrightarrow{\sim} U^\#$. We choose f', f and then replace f by the sum f' on U' and the restriction of f to K. Then f' and f are compatible, and they induce the desired map f''.

Next we check that $\chi(U) = \chi(V)$ when U, V are homotopy equivalent. If $s : U \to V$ is a homotopy equivalence, then one knows that the mapping cylinder M of s splits: $M = U \oplus C(s) = C(1_U) \oplus V$, where the mapping cones $C(s), C(1_U)$ are contractible.

Now 5.4 implies $\chi(C) = 0$ when C is contractible, so using the additivity of χ we have $\chi(U) = \chi(V)$.

At this point we have defined a homomorphism $K_0\mathcal{L} \to K_0\mathcal{L}^1$ sending $[U]$ to $\chi(U)$. We consider

$$K_0\mathcal{L}^1 \to K_0\mathcal{L} \to K_0\mathcal{L}^1 \tag{3}$$

where the left map is already known to be surjective. In order to show that both maps are isomorphisms inverse to each other, it suffices to check that the composition is the identity.

Let $U = (d : U_1 \to U_0)$ be in $\mathcal{L}^1$, and choose f:

$$\begin{array}{ccc} U_1^\# & \xrightarrow{d^\#} & U_0^\# \\ f_1 \downarrow & & \downarrow f_0 \\ U_1 & \xrightarrow{d} & U_0 \end{array}$$

lifting the canonical isomorphism modulo A. The composition (3) sends $[U] = [d]$ to $[f_0] - [f_1]$. Now $[f_0] = [f_0 d^\#] = [df_1] = [f_1] + [d]$, since $d^\#$ is an isomorphism. This shows the composition is the identity, finishing the proof of Theorem 5.1. □

We next derive the analogous result for n-dimensional chain complexes.

Theorem 5.5 *One has* $K_0\mathcal{L}^1(R, A) \xrightarrow{\sim} K_0\mathcal{L}^n(R, A)$ *for* $n \geq 1$.

Proof. As before we can suppose $R = \tilde{A}$. The injectivity is already clear from 5.1, so we need only prove surjectivity. We will modify our surjectivity argument above. Let U be in $\mathcal{L}^n(R, A)$, and let $f : U^\# \to U$ be as in 5.2. Since $U^\#$ is contractible, it splits into the direct sum of the elementary complex consisting of $U_n^\#$ in degrees $n-1, n$ and an $(n-1)$-dimensional contractible chain complex V. Let $f' : V \to U$ be the restriction of f, let C' be its mapping cone. Then C' is in $\mathcal{L}^n(R, A)$ and is homotopy equivalent to U.

Furthermore C' has a filtration F_pC', $0 \leq p \leq n$, where $F_nC' = C'$ and F_pC' for $p < n$ is the subcomplex made of V_k, U_k for $k < p$. The successive quotients of this filtration are suspensions of complexes in $\mathcal{L}^1(R, A)$, hence $[U] = [C']$ in $K_0\mathcal{L}(R, A)$ is a sum of elements coming from $K_0\mathcal{L}^1(R, A)$. □

§6. Define the group $K_0'A$ for a nonunital ring A by

$$K_0'A = K_0\mathcal{L}^1(\tilde{A}, A)$$

We have seen that $K_0'A$ maps isomorphically to $K_0\mathcal{L}^n(R, A)$ for $n \geq 1$ and to $K_0\mathcal{L}(R, A)$ whenever A is embedded as an ideal in the unital ring R.

Recall that K_0A can be defined using K_0 for unital rings by

$$K_0A = K_0\tilde{A}/K_0\mathbf{Z}$$

It is the group of stable isomorphism classes of finitely generated projective $\tilde{A}$-modules.

There is a canonical surjection

$$K_0'A \to K_0A$$

sending $[U]$, where U is a complex $U_1 \to U_0$ in $\mathcal{L}(\tilde{A}, A)$, to $[U_0] - [U_1]$ in K_0A. To see that this map is surjective, let P be in $\mathcal{P}(\tilde{A})$, and choose $f : P^\# \to P$ lifting the canonical isomorphism modulo A, where $P^\# = \tilde{A} \otimes_{\mathbf{Z}} (P/AP)$. Since P/AP is in $\mathcal{P}(\mathbf{Z})$, we have $P/AP \cong \mathbf{Z}^n$ and $P^\# \cong \tilde{A}^n$ for some n, hence $[P^\#] = 0$ in K_0A. Thus $[f]$ in $K_0'A$ maps to $[P]$ in K_0A.

In general K_0A does not have the Morita invariance property we have established for $K_0'A$, e.g. $K_0(A^2)$ can be different from K_0A. So it is of interest to examine when the canonical map above is an isomorphism.

Proposition 6.1 *One has $K_0'A \xrightarrow{\sim} K_0A$ iff for all n and $a \in M_nA$ the map $1 + a : \tilde{A}^n \to \tilde{A}^n$ represents zero in $K_0'A$.*

Proof. The necessity is clear. Conversely, assuming the second condition we will define a homomorphism $K_0A \to K_0'A$ inverse to the canonical map going the other way. Given P in $\mathcal{P}(\tilde{A})$, we choose $f : P^\# \to P$ and $g : P \to P^\#$ lifting the canonical isomorphism modulo A. Since $P^\# \cong \tilde{A}^n$ for some n, gf has the form $1+a : \tilde{A}^n \to \tilde{A}^n$, and so $[f] + [g] = [1 + a] = 0$ in $K_0'A$ by our hypothesis. Consequently $[f] = -[g]$ is independent of the choice of f and depends only on P. It is then easy to check that $P \mapsto [f]$ induces a homomorphism $K_0A \to K_0'A$ inverse to the canonical map. □

We now discuss some situations where $K_0'A \xrightarrow{\sim} K_0A$. The first covers the case where A has 'local identities'.

Proposition 6.2 *Assume for any $a \in A$ there exists $b \in A$ such that $ba = a$, (resp. $ab = a$). Then $K_0'A \xrightarrow{\sim} K_0A$.*

Proof. Using the fact that the set of $1 - b$ in $\tilde{A}$ with $b \in A$ is closed under multiplication, one sees by induction that for any $a_1, \cdots, a_n$ in A there is such a $b \in A$ with $(1 - b)a_i = 0$ for all i. Now let $a = (a_{ij}) \in M_nA$, and choose b so that $ba_{ij} = a_{ij}$ for all i, j. Then we have $(1-b)(1+a) = 1-b$ in $1+M_nA$, hence $[1-b]+[1+a] = [1-b]$ in $K_0'A$, so $[1 + a] = 0$. □

Proposition 6.3 *If A is a C^*-algebra, then $K_0'A \xrightarrow{\sim} K_0A$.*

Proof. Given $a \in M_nA$, we will show that $1 + a$ represents zero in $K_0'A$ by using the functional calculus for self-adjoint elements in a C^* algebra.

Let $\rho(t)$ be the continuous function on the nonnegative reals equal to 1 on $[0, 1]$, $2 - t$ on $[1, 2]$, and zero for $t \geq 2$. Note that $\rho(2t)\rho(t) = \rho(2t)$ and $\rho(t)^2t \leq 2$.

Since $\rho(0) = 1$, the functional calculus applied to aa^* in the C^* algebra M_nA defines elements $\rho(saa^*) \in 1 + M_nA$ for $s \geq 0$ satisfying

$$\rho(2saa^*)(1 + a) = \rho(2saa^*)(1 + \rho(saa^*)a)$$

It suffices to show for suitable s that the last factor is invertible, for it then represents zero in $\cancel{K'_0}$, hence $1 + a$ also represents zero. K'_0A

Now with $x_s = \rho(saa^*)a$ we have

$$||x_s||^2 = ||x_s x_s^*|| = ||\rho(saa^*)^2(saa^*)||s^{-1} \leq 2s^{-1}$$

Thus for $s > 2$, we have $||x_s|| < 1$, hence $1 + x_s$ is invertible, concluding the proof. □

As an application we deduce the following variant of ([A], 2.6.1).

Theorem 6.4 *Let X be a compact Hausdorff space, and Y a closed subspace. Then the Grothendieck group of complexes of complex vector bundles on X which are acyclic over Y can be identified with the topological K-theory group $K^0(X,Y) = \tilde{K}^0(X/Y)$.*

Proof. Let $R = C(X)$ the ring of continuous complex-valued functions on X, and let A be the ideal of functions vanishing on Y, so that $R/A = C(Y)$ by the Tietze extension theorem. By the Serre-Swan theorem $\mathcal{P}(R)$ is equivalent to the category of complex vector bundles on X, and similarly for R/A and Y. Moreover $P \mapsto P/AP$ corresponds to restricting a vector bundle to Y. We see then that $\mathcal{L}(R, A)$ is equivalent to the category of complexes of vector bundles on X which are acyclic over Y. Thus K'_0A is the Grothendieck group of the latter category up to canonical isomorphism.

Now K_0A can be identified with $\tilde{K}^0(X/Y)$, starting with $K_0C(X/Y) = K^0(X/Y)$ by Serre-Swan, and splitting off the summand $K_0\mathbf{C} = K^0(pt)$ to obtain the reduced groups. Finally we have $K'_0A = K_0A$, since A is a C^*-algebra. □

§7. We next present an example where $K'_0A \neq K_0A$.

Proposition 7.1 *Let A an ideal in a regular noetherian commutative ring R. Then K'_0A is isomorphic to the Grothendieck group of the abelian category $\mathcal{C}$ of finitely generated R-modules M such that $M/AM = 0$.*

For example, let A be the ideal $m\mathbf{Z}$ in $R = \mathbf{Z}$. Then $\mathcal{C}$ is the category of finite abelian groups of order prime to m, so $K'_0(m\mathbf{Z})$ is a free abelian group with one generator for each prime number not dividing m. On the other hand from

$$K_1\mathbf{Z} \to K_1(\mathbf{Z}/m\mathbf{Z}) \to K_0(m\mathbf{Z}) \to K_0\mathbf{Z} \to K_0(\mathbf{Z}/m\mathbf{Z})$$

one obtains

$$K_0(m\mathbf{Z}) = (\mathbf{Z}/m\mathbf{Z})^\times / \{\pm 1\}$$

Using 8.1 below one can show that $K_0'(m\mathbf{Z}) \to K_0(m\mathbf{Z})$ sends the generator corresponding to the prime p to the image of p in the latter group.

We now prove the proposition. Let M_x denote the localization of the R module M at a prime ideal x of R, and let Z be the closed subset of $Spec(R)$ consisting of all prime ideals containing A. We recall that $M = 0$ iff $M_x = 0$ for all prime ideals x, hence $M/AM = 0$ iff $(M/AM)_x = M_x/A_xM_x = 0$ for all $x \in Z$. Moreover, for M finitely generated and $x \in Z$, we have $M_x/A_xM_x = 0$ iff $M_x = 0$ by Nakyama's lemma. Thus $\mathcal{C}$ consists of all finitely generated M such that $M_x = 0$ for all $x \in Z$ (in other words, the support of M is disjoint from Z). Since $M \mapsto M_x$ is exact and R is noetherian, it follows that $\mathcal{C}$ is an abelian category.

Lemma 7.2 *A finite projective complex U over R is in $\mathcal{L}(R,A)$ iff its homology $H(U) = \bigoplus_n H_n(U)$ is in $\mathcal{C}$.*

Proof. As $H(U)$ is finitely generated we know that $H(U)$ is in $\mathcal{C}$ iff $H(U)_x = H(U_x)$ vanishes for all $x \in Z$. On the other hand $H(U/AU) = 0$ iff $H(U/AU)_x = H(U_x/A_xU_x)$ vanishes for all $x \in Z$. Thus it suffices to show for any $x \in Z$ that U_x is acyclic iff U_x/A_xU_x is acyclic.

If U_x is acyclic, then being right-bounded projective it is contractible, so U_x/A_xU_x is also contractible and hence acyclic. Conversely, if U_x/A_xU_x is acyclic, then 2.2 in the case of R_x, A_x, U_x yields a homotopy operator h on U_x such that $[d,h] = 1 - f$ with $f(U_x) \subset A_xU_x$. As A_x is contained in the maximal ideal of the local ring R_x, it follows that $1-f$ is invertible, so U_x is contractible with contraction $(1-f)^{-1}h$, and hence acyclic. □

From this lemma we obtain a homomorphism

$$\alpha : K_0'A = K_0\mathcal{L}(R,A) \to K_0\mathcal{C} \qquad\qquad [U] \mapsto \sum(-1)^n[H_n(U)]$$

On the other hand, because R is regular noetherian any M in $\mathcal{C}$ has a finite projective resolution V which is unique up to homotopy equivalence. This gives rise to a homomorphism

$$\beta : K_0\mathcal{C} \to K_0'A$$

sending $[M]$ to $[V]$.

Clearly $\alpha\beta = 1$. To prove $\beta\alpha = 1$, it is enough to check this relation on classes $[U]$ with U in $\mathcal{L}^1(R,A)$. Let P be a finite projective resolution of $H_1(U)$. Then Q:

$$\cdots \to P_1 \to P_0 \to U_1 \to U_0$$

is a finite projective resolution of $H_0(U)$, and there is a short exact sequence $U \to Q \to P[2]$, so that $[U] = [Q] - [P]$ in $K_0'A$. Then $\beta\alpha[U] = \beta[H_0(U)] - \beta[H_1(U)] = [Q] - [P] = [U]$, finishing the proof. □

§8. So far we have described the canonical surjection $K_0'A \to K_0A$ by means of finitely generated projective modules over $\tilde{A}$. Suppose now that A is embedded as an ideal in a unital ring R. We would like to understand this map in terms of finitely generated projective R-modules. We begin by reviewing the usual R-module description of K_0A.

Recall ([M], §4) that associated to the canonical surjection $R \to R/A$ there is a relative Grothendieck group $K_0(R \to R/A)$ fitting into an exact sequence

$$K_1R \to K_1(R/A) \to K_0(R \to R/A) \to K_0R \to K_0(R/A) \tag{4}$$

which is defined by

$$K_0(R \to R/A) = K_0D/\Delta_*(K_0R)$$

where $D = R \times_{R/A} R$ is the double of R along A, and $\Delta : R \to D$ is the diagonal embedding.

By [M], §2, $\mathcal{P}(D)$ is equivalent to the category of triples (P, Q, w) such that P, Q are in $\mathcal{P}(R)$ and $w : P/AP \xrightarrow{\sim} Q/AQ$ is an isomorphism in $\mathcal{P}(R/A)$. The finitely generated projective D-module corresponding to (P, Q, w) is given by the fibre product

$$M(P, Q, w) = P \times_{Q/AQ} Q$$

of the maps $P \to P/AP \xrightarrow{w} Q/AQ$ and $Q \to Q/AQ$. We write $[(P, Q, w)]$ for the class in $K_0(R \to R/A)$ represented by $M(P, Q, w)$.

It is then clear that $K_0(R \to R/A)$ is the abelian group generated by the elements $[(P, Q, w)]$ for each such triple, subject to the relations guaranteeing that the function $(P, Q, w) \mapsto [(P, Q, w)]$ equalizes isomorphic triples, is additive with respect to direct sum, and is such that $[(P, P, 1_{P/AP})] = 0$ for all P in $\mathcal{P}(R)$.

Applying [B], IX, 5.4 to the cartesian square

$$\begin{array}{ccc} \tilde{A} & \to & R \\ \downarrow & & \downarrow \\ \mathbf{Z} & \to & R/A \end{array}$$

we obtain an excision isomorphism

$$K_0(\tilde{A} \to \mathbf{Z}) \xrightarrow{\sim} K_0(R \to R/A)$$

On the other hand, (4) gives an exact sequence

$$0 \to K_0(\tilde{A} \to \mathbf{Z}) \to K_0\tilde{A} \to K_0\mathbf{Z} \to 0$$

hence

$$K_0(\tilde{A} \to \mathbf{Z}) \xrightarrow{\sim} K_0\tilde{A}/K_0\mathbf{Z} = K_0A$$

Thus K_0A and $K_0(R \to R/A)$ are canonically isomorphic, which achieves the desired description of K_0A in terms of finitely generated projective R-modules.

We have seen that $K_0'A$ and $K_0\mathcal{L}^1(R, A)$ are canonically isomorphic, and that $\mathcal{L}^1(R, A)$ is the category of maps $f : P \to Q$ in $\mathcal{P}(R)$ such that the induced map $\bar{f} : P/AP \to Q/AQ$ is an isomorphism.

Proposition 8.1 *The functor sending $f : P \to Q$ to $(P, Q, \bar{f})$ induces a surjection*

$$K_0\mathcal{L}^1(R, A) \to K_0(R \to R/A) \tag{5}$$

which agrees up to canonical isomorphism with the map $K_0'A \to K_0A$ of §6.

We first prove two lemmas.

Lemma 8.2 *If $(P', Q', w') \to (P, Q, w) \to (P'', Q'', w'')$ is a short exact sequence of triples as above, then $[(P, Q, w)] = [(P', Q', w')] + [(P'', Q'', w'')]$.*

Proof. The fibre product $M(P, Q, w)$ is part of an evident short exact sequence

$$M(P, Q, w) \to P \oplus Q \to Q/AQ$$

and similarly with $'$ and $''$. The nine lemma then gives a short exact sequence in $\mathcal{P}(D)$

$$M(P', Q', w') \to M(P, Q, w) \to M(P'', Q'', w'')$$

which necessarily splits, whence the result. □

Lemma 8.3 *Let $U = (d : U_1 \to U_0)$ and $U' = (d' : U_1' \to U_0')$ be homotopy equivalent 1-dimensional chain complexes. Then $U \oplus C' \cong C \oplus U'$, where $C = (1_{U_0} : U_0 \to U_0)$ and similarly for C'.*

Proof. Let $x : U \to U'$ be a homotopy equivalence. Then the mapping cone on x is contractible, i.e. the sequence

$$0 \to U_1 \xrightarrow{i} U_0 \oplus U_1' \xrightarrow{j} U_0' \to 0$$

$$i = \binom{-d}{x_1} \qquad j = (x_0 \;\; d')$$

is split exact. A splitting of this sequence has the form

$$U_1 \xleftarrow{r} U_0 \oplus U_1' \xleftarrow{l} U_0'$$

$$l = \binom{y_0}{h'} \qquad r = (h \;\; y_1)$$

where $y : U' \to U$ is a map of complexes and h, h' are homotopy operators such that $1 - yx = [d, h]$, $1 - xy = [d', h']$, and $hy_0 + y_1h' = -x_1h + h'x_0 = 0$. Then

$$\begin{array}{ccc} U_1 \oplus U_0' & \xrightarrow{\alpha_1} & U_0 \oplus U_1' \\ d \oplus 1 \downarrow & & \downarrow 1 \oplus d' \\ U_0 \oplus U_0' & \xrightarrow{\alpha_0} & U_0 \oplus U_0' \end{array}$$

where

$$\alpha_1 = \begin{pmatrix} -d & y_0 \\ x_1 & h' \end{pmatrix} \qquad \alpha_1^{-1} = \begin{pmatrix} -h & y_1 \\ x_0 & d' \end{pmatrix}$$

$$\alpha_0 = \begin{pmatrix} -1 & y_0 \\ x_0 & d'h' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -x_0 & 1 \end{pmatrix} \begin{pmatrix} -1 & y_0 \\ 0 & 1 \end{pmatrix}$$

is the desired isomorphism of complexes. □

We now prove the proposition. Let φ be the map sending $f : P \to Q$ to $[(P, Q, \bar{f})]$. A short exact sequence $U' \to U \to U''$ in $\mathcal{L}^1(R, A)$ is clearly carried to a short exact sequence of triples, so $\varphi(U) = \varphi(U') + \varphi(U'')$ by the first lemma. In particular, φ is additive for direct sums. Using the second lemma and the relation $[(P, P, 1_{P/AP})] = 0$ we deduce that $\varphi(U) = \varphi(U')$ when U, U' are homotopy equivalent. Thus φ satisfies the relations defining $K_0\mathcal{L}^1(R, A)$, so we obtain the desired map (5).

The surjectivity is clear since $w : P/AP \xrightarrow{\sim} Q/AQ$ lifts to a map $f : P \to Q$ as P is projective.

Finally the identification of (5) with the map $K_0'A \to K_0A$ is easily checked in the case $R = \tilde{A}$, and follows in general by naturality with respect to $\tilde{A} \to R$ together with the excision assertions already mentioned. □

We end this section with another description of $K_0'A = K_0\mathcal{L}^1(R, A)$, which seems interesting because of the similarity to K_1; compare Ranicki's isomorphism torsion [R2].

Theorem 8.4 *$K_0\mathcal{L}^1(R, A)$ is the abelian group generated by elements $[f]$, where f is any map in $\mathcal{P}(R)$ which is an isomorphism modulo A, subject to the relations:*
1) $[f : P \to Q] + [f' : P' \to Q'] = [f \oplus f' : P \oplus P' \to Q \oplus Q']$
2) $[f : P' \to P] + [g : P \to Q] = [gf : P' \to Q]$
3) $[f] = 0$ *when f is an isomorphism.*

Proof. Let G be the abelian group defined by these generators and relations, and put $K_0' = K_0\mathcal{L}^1(R, A)$. Note that K_0' and G have the same generators. Now 1) and 3) clearly hold in K_0', and 2) also does by 5.3. Thus we have to show that the relations defining K_0', namely:
a) additivity for short exact sequences
b) $[U] = [U']$ when U, U' are homotopy equivalent
hold in G.

First we observe that 2) and 3) imply that $[U] = [V]$ in G when U, V are isomorphic. Using 8.3 we see that b) holds in G.

Now consider a short exact sequence $U' \to U \to U''$ where U' is $f' : P' \to Q'$, and similarly for U, U''. To prove additivity in G for this sequence, we can replace U by an isomorphic complex and assume $P = P' \oplus P''$, $Q = Q' \oplus Q''$, where

$$f = \begin{pmatrix} f' & \theta \\ 0 & f'' \end{pmatrix}$$

with $\theta : P'' \to Q'$. Then f factors

$$f = \begin{pmatrix} 1_{Q'} & 0 \\ 0 & f'' \end{pmatrix} \begin{pmatrix} 1_{Q'} & \theta \\ 0 & 1_{P''} \end{pmatrix} \begin{pmatrix} f' & 0 \\ 0 & 1_{P''} \end{pmatrix}$$

and the middle factor is an isomorphism. Using 1)-3) we obtain $[f] = [1 \oplus f''] + [f' \oplus 1] = [f''] + [f']$ in G as desired. $\square$

§9. According to 5.1 any element of $K_0\mathcal{L}(R, A)$ can be expressed in terms of classes of 1-dimensional chain complexes. We now derive a Whitehead type formula doing this explicitly.

Theorem 9.1 *Let U be a complex in $\mathcal{L}(R, A)$, and let h be a homotopy operator on U such that $1 - [d, h]$ maps U into AU. Let $\Delta_n = [d, h]_n : U_n \to U_n$, and put $U^+ = \bigoplus_j U_{2j}$, $U^- = \bigoplus_j U_{2j+1}$. Then in $K_0'A = K_0\mathcal{L}(R, A)$ we have*

$$\sum_n (-1)^n [\Delta_n] = 0 \tag{6}$$

$$[U] = [d + h : U^- \to U^+] - \sum_j j([\Delta_{2j}] - [\Delta_{2j+1}]) \tag{7}$$

Furthermore, the image of $[U]$ in $K_0A = K_0(R \to R/A)$ is $[(U^-, U^+, \bar{d} + \bar{h})]$.

Proof. The last assertion follows immediately from (7) and the fact that Δ_n induces the identity on U_n/AU_n.

Let C be the mapping cone of $\Delta = [d, h] : U \to U$. Then C has an increasing filtration F_nC such that $F_nC/F_{n-1}C$ is the n-th suspension of the 1-dimensional chain complex $\Delta_n : U_n \to U_n$. This lies in $\mathcal{L}^1(R, A)$ for all n, so we have $[C] = \sum_n (-1)^n [\Delta_n]$. On the other hand, there is a short exact sequence $U \to C \to U[1]$, so $[C] = [U] - [U] = 0$, proving (6).

To prove the second formula we construct a suitable contractible complex W. Let V be the graded module defined by

$$V = U[1] \oplus U[2] \oplus U[3] \oplus \dots$$

where $U[p]$ denotes the graded module with $U[p]_n = U_{n-p}$. Thus V is the 'total' graded module associated to the bigraded module (V_{pq}) such that $V_{pq} = U_q$ for all $p \geq 1$ and $q \in \mathbf{Z}$.

Define operators δ and s on V by

$$\delta = \begin{pmatrix} -d & -1 & -h & & & & \\ & d & \Delta & h & h^2 & & \\ & & -d & -1 & -h & & \\ & & & d & \Delta & h & h^2 \\ & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$s = \begin{pmatrix} 0 & & & & \\ -1 & 0 & & & \\ & & 0 & & \\ & & -1 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

Here the i,j-th entry gives the component $U[j] \to U[i]$ of the operator with respect to the above direct sum decomposition of V.

Lemma 9.2 *The operators δ, s have degrees -1, $+1$ respectively, and they satisfy $\delta^2 = 0$, $[\delta, s] = 1$.*

Define operators on V by

$$\tilde{d} = \begin{pmatrix} -d & -1 & & & & \\ & d & & & & \\ & & -d & -1 & & \\ & & & d & \ddots & \\ & & & & & \ddots \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & & & & & & \\ & 1 & h & & & & \\ & & 1 & & & & \\ & & & 1 & h & & \\ & & & & 1 & \ddots & \\ & & & & & & \ddots \end{pmatrix}$$

Then $\tilde{d}$, s have degrees -1, $+1$ respectively, and they satisfy $\tilde{d}^2 = 0$, $[\tilde{d}, s] = 1$. Also g has degree 0 and is invertible. One readily checks that $g^{-1}\tilde{d}g = \delta$, $g^{-1}sg = s$, which proves the lemma. □

We now regard V as an unbounded complex equipped with the differential δ. It is contractible with the contraction s.

Choose m so that $U_n = 0$ for $n > 2m+1$, and let W be the bigraded submodule of V consisting of the modules $V_{pq} = U_q$ such that $q \leq 2m+1-2k$ when $p = 2k+1, 2k+2$, $k \geq 0$. Thus W can be viewed as the following staircase region in V:

$$\begin{array}{ccccccccccccc}
U_{2m+1} & \xleftarrow{-1} & U_{2m+1} & & & & & & & & & \\
\downarrow -d & & \downarrow d & & & & & & & & & \\
U_{2m} & \xleftarrow{-1} & U_{2m} & & & & & & & & & \\
\downarrow -d & & \downarrow d & & & & & & & & & \\
U_{2m-1} & \xleftarrow{-1} & U_{2m-1} & \xleftarrow{\Delta} & U_{2m-1} & \xleftarrow{-1} & U_{2m-1} & & & & & \\
\downarrow -d & & \downarrow d & & \downarrow -d & & \downarrow d & & & & & \\
U_{2m-2} & \xleftarrow{-1} & U_{2m-2} & \xleftarrow{\Delta} & U_{2m-2} & \xleftarrow{-1} & U_{2m-2} & & & & & \\
\downarrow -d & & \downarrow d & & \downarrow -d & & \downarrow d & & & & & \\
U_{2m-3} & \xleftarrow{-1} & U_{2m-3} & \xleftarrow{\Delta} & U_{2m-3} & \xleftarrow{-1} & U_{2m-3} & \xleftarrow{\Delta} & U_{2m-3} & \xleftarrow{-1} & U_{2m-3} \\
\downarrow -d & & \downarrow d & & \downarrow -d & & \downarrow d & & \downarrow -d & & \downarrow d \\
 & & & & & & & & & & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array}$$

$p:$ 1 2 3 4 5 6

Corresponding to the nontrivial diagonals in the matrix defining δ we can write $\delta = d^0 + d^1 + d^2 + d^3$, where $d^r_{pq} : V_{pq} \to V_{p-r,q+r-1}$ like differentials in a spectral sequence. Now d^0 and d^1 are given by the vertical and horizontal arrows respectively in the above diagram, while $d^2_{pq} = (-1)^p h$, and d^3_{pq} is equal to h^2 for p odd and 0 for p even. We see then that W is closed under δ, hence W is a subcomplex of V.

Moreover, the operator s is given by -1 in the opposite direction to each of the -1 arrows above. Thus W is closed under s, so W is a contractible finite projective complex.

Next let W' be the bigraded submodule of V consisting of the modules $V_{pq} = U_q$ such that $q \le 2m+1$ for $p = 1$, and such that $q \le 2m+1-2k$ for $p = 2k, 2k+1$, $k \ge 1$. W' can be viewed is the staircase region obtained by shifting W horizontally one step to the left, and it is a subcomplex of W.

Now observe that W' is made up of the column $p = 1$ and all the Δ arrows inside W. In fact we obtain an increasing filtration of W' by subcomplexes starting with the column $p = 1$, which is the suspension of U, then adding in order of increasing q the Δ arrows between the $p = 2, 3$ columns, then the Δ arrows between the $p = 4, 5$ columns, etc.

As the successive quotients of this filtration are in $\mathcal{L}(R, A)$ we get

$$\begin{aligned}[W'] &= -[U] + \sum_{q \le 2m-1} (-1)^q [\Delta_q] + \sum_{q \le 2m-3} (-1)^q [\Delta_q] + \cdots \\ &= -[U] + \sum_{j \le m-1} (m-j)([\Delta_{2j}] - [\Delta_{2j+1}])\end{aligned}$$

in $K_0\mathcal{L}(R, A)$. By the choice of m the last sum can be taken over all j, hence we have

$$[W'] = -[U] - \sum_j j([\Delta_{2j}] - [\Delta_{2j+1}])$$

using (6).

Finally W/W' consists of the even submodule U^+ in degree $2m$, the odd submodule U^- in degree $2m+1$, and the map $d + h$ between them, so

$$0 = [W] = [W'] + [d + h : U^- \to U^+]$$

proving (7). $\square$

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