April 20, 1995.

Recall $U = \mathrm{Cone}\left(k[z]\otimes T \xrightarrow{1-ze} k[z]\otimes T\right)$. Suppose we change z to $1-z$, so that $1-ze$ becomes $1-(1-z)e = 1-e+ze$. Then 0-cocycles on U should be described by $(u_0, u_0', u_1, u_1', \ldots)$ satisfying $[d,u_n]=0$, $[d,u_n'] = u_n(1-e)+u_{n+1}e$. Moreover if $v: T\to U$ is a cocycle on U, then vi should be $(ve, vh, 0, 0, \ldots)$. Check:

$$[d,ve]=0, \quad [d,vh] = v(e-e^2) = (ve)(1-e)+(0)e.$$

New description of U: $U = k[z,\sigma]\otimes T$, the operators e, h on T are extended to $e = 1\otimes e$, $h = 1\otimes h$ (super sense) on U. The differential on U is $d = d'+d''$, where $d'' = 1\otimes d$ and $d' = (1-e+ze)\partial$, $\partial =$ (super) derivative wrt σ = degree -1 derivation on $k[z,\sigma]$ such that $\partial(\sigma)=1$, $\partial(z)=0$.

Then $[d,\sigma] = [(1-e+ze)\partial, \sigma] = 1-e+ze$

$$\begin{aligned}[d,h] &= [(1-e+ze)\partial, h] + [d'',h] \\ &= -[1-e+ze,h]\partial + e-e^2 \\ &= e-e^2 + (1-z)[e,h]\partial\end{aligned}$$

The nice thing about the conditions

$$[d,u_n]=0 \qquad [d,u_n'] = u_n(1-e)+u_{n+1}e \qquad n\geq 0$$

is that $u_n(1-e) \sim u_n(1-e)^2 \sim u_{n+1}e(1-e) \sim 0 \quad \forall n\geq 0$ and also $u_{n+1}e\sim 0 \quad \forall n\geq 0$ so that we have $u_0(1-e)\sim 0$, $u_n\sim 0 \quad n\geq 1$. We proceed as in the old notation to find s_n satisfying $[d,s_n]=u_n$ for $n\geq 1$ and $[d,s_0] = u_0(1-e)$. We have

$$s_0 = u_0'(1-e) + (u_0 - u_1)h$$

$$s_n = u_{n-1}'e + u_n'(1-e) - (u_{n-1} - 2u_n + u_{n+1})h \qquad n \geq 1$$

Next we compute the coboundary of $(s_0, 0, s_1, 0, \dots)$. For n large ($n \geq 1$ should do) we find

$$s_n(1-e) + s_{n+1}e = u_n' + [d, (-u_{n-1}' + 2u_n' - u_{n+1}')h]$$

(*)

$$+ \underbrace{(-u_{n-1} + 3u_n - 3u_{n+1} + u_{n+2})[e,h]}_{[d,\, (-s_{n-1} + 3s_n - 3s_{n+1} + s_{n+2})[e,h]]}$$

At this point I want to shift from operations on cochains to operators on U. If $\phi : U \to Z$ corresponds to $(u_0, u_0', u_1, u_1', \dots)$ then $u_n = \phi z^n y$, $u_n' = \phi \sigma z^n y$. Thus the operator cocycles $(u_0, u_0', \dots) \longmapsto s_n$ corresponds to the operator

$$\sigma z^{n-1} e y + \sigma z^n (1-e) y - (z^{n-1} - 2z^n + z^{n+1}) h y.$$

This means I should examine the operator

$$\boxed{k = \sigma(z^{-1}e + 1 - e) - (z^{-1} - 2 + z)h} \qquad (\text{say on } U[z^{-1}])$$

Then

$$[d,k] = (1 - e + ze)(1 - e + z^{-1}e) - (z^{-1} - 2 + z)(e - e^2 + (1-z)[e,h]\partial)$$

$$= 1 - \cancel{2e} + \cancel{e^2} + z\cancel{e(1-e)} + z^{-1}\cancel{(1-e)e} + \cancel{e^2} - \cancel{z^{-1}(e-e^2)} + \cancel{2(e-e^2)} + \cancel{z(e-e^2)}$$
$$- (z^{-1} - 2 + z)(1-z)[e,h]\partial$$

i.e.

$$\boxed{[d,k] = 1 - z^{-1}(1-z)^3[e,h]\partial}$$

So far I've done the stable (large n) calculation, and one can see how nicely it correlates to (*). It appears that writing the last term in (*) as $[d, -]$ amounts to the contraction:

$$[d,\, k(1 + z^{-1}(1-z)^3[e,h]\partial)] = 1$$

resulting from the fact that $(z^{-1}(1-z)^3[e,h]\partial)^2 = 0$.

Next examine the transient behavior.

Let z^* be the Toeplitz operator corresponding to z^{-1}, so that $z^*z = 1$, $1 - zz^* =$ projection onto $z^0 T \oplus \sigma z^0 T$. We have

$$\boxed{[d, z^*] = [(1-e+ze)\partial, z^*] = [z, z^*]e\partial}$$

Put

$$k = \sigma(1-e+z^*e) - (z^*-2+z)h$$

$$[d,k] = (1-e+ze)(1-e+z^*e) - \sigma[d,z^*]e - [d,z^*]h$$
$$- (z^*-2+z)\big(e-e^2 + (1-z)[e,h]\partial\big)$$

$$= 1-2e+e^2 + \cancel{z(e-e^2)} + \cancel{z^*(e-e^2)} + zz^*e^2 - e^2 + e^2$$
$$- \sigma[z,z^*]e\partial e - [z,z^*]e\partial h$$
$$- \cancel{z^*(e-e^2)} + 2(e-e^2) - \cancel{z(e-e^2)} - \overbrace{(z^*-2+z)}^{z^*(1-z)^2}(1-z)[e,h]\partial$$

$$\boxed{[d,k] = 1 + [z,z^*]\big(e^2 - e^2\sigma\partial + eh\partial\big) - z^*(1-z)^3[e,h]\partial}$$

Our choice for k is probably not correct 'at the bottom', because $s_0 = u_0'(1-e) + (u_0-u_1)h$ not $u_0'(1-e) + (2u_0-u_1)h$.

April 21, 1995

I propose to find a homotopy: $1-ji=[d,k]$ at least in the case $[e,h]=0$. Recall

$$\phi j i = (u_0 e, u_0 h, 0, 0, \dots)$$

when $\phi=(u_0,u_0',u_1,\dots)$. Note that $[z^*,z]=1-zz^*$ projects onto $(j\oplus\sigma j)(T)$. We have

$$[z^*,z]\big((1-\sigma\partial)e+h\partial\big) \qquad \begin{cases} j & = je \\ \sigma j & = jh \\ z^n j & = 0 \\ z^n\sigma j & = 0 \end{cases} \quad n\geq 1$$

so that

$$\boxed{ji = [z^*,z]\big(e(1-\sigma\partial)+h\partial\big)}$$

Some formulas:

$$[e,h]e+e[e,h] = [e^2,h] = [e-[d,h],h]$$

$$[e,h]e+e[e,h] = [e,h]-[d,h^2]$$

so

$$\boxed{\begin{aligned}(1-e)[e,h] &= [e,h]e+[d,h^2]\\ [e,h](1-e) &= e[e,h]+[d,h^2]\end{aligned}}$$

$$\begin{aligned} e[e,h]e &= [e,h](e-e^2)-[d,h^2e] \\ &= [e,h][d,h]-[d,h^2e] \\ &= [d,-[e,h]h-h^2e] = [d,-eh^2+heh-h^2e] \end{aligned}$$

$$\boxed{e[e,h]e = [d,-eh^2+heh-h^2e]}$$

$$[d, u_n] = 0$$

$$[d, u'_n] = u_n(1-e) + u_{n+1} e$$

$$\boxed{s_0 = u'_0(1-e) + (u_0 - u_1) h}$$

$$[d, s_0] = u_0(1-e)^2 + u_1(e-e^2) + (u_0-u_1)(e-e^2)$$

$$\boxed{[d, s_0] = u_0(1-e)}$$

$$[d, u'_n(1-e) + (u_n - u_{n+1})h] = u_n(1-e)$$

$$[d, u'_n] = u_n(1-e) + u_{n+1} e$$

$$\boxed{[d, u'_n e + (-u_n + u_{n+1})h] = u_{n+1} e}$$

$$[d, u'_{n-1} e + (-u_{n-1} + u_n)h = u_n e] \qquad n \geq 1$$

$$\boxed{s_n = u'_{n-1} e + u'_n(1-e) + (-u_{n-1} + 2u_n - u_{n+1})h} \qquad n \geq 1$$

$$\boxed{[d, s_n] = u_n} \qquad n \geq 1$$

$$\begin{aligned}
s_n(1-e) + s_{n+1} e &= u'_{n-1}(e-e^2) + u'_n(1-2e+e^2) + (-u_{n-1} + 2u_n - u_{n+1})h(1-e) \\
&\quad + u'_n e^2 + u'_{n+1}(e-e^2) + (-u_n + 2u_{n+1} - u_{n+2})he \\
&= u'_n + (u'_{n-1} - 2u'_n + u'_{n+1})[d,h] + (-u_{n-1} + 2u_n - u_{n+1})h \\
&\quad + (u_{n-1} - 3u_n + 3u_{n+1} - u_{n+2})he \\
&= u'_n + [d, (-u'_{n-1} + 2u'_n - u'_{n+1})h] + \big(-u_{n-1}(1-e) - u_n e + 2u_n(1-e) + 2u_{n+1} e \\
&\qquad - u_{n+1}(1-e) - u_{n+2} e\big) h \\
&\quad + (-u_{n-1} + 2u_n - u_{n+1})h + (u_{n-1} - 3u_n + 3u_{n-1} - u_{n+2})he \\
&= u'_n + [d, (-u'_{n-1} + 2u'_n - u'_{n+1})h] + (-u_{n-1} + 3u_n - 3u_{n+1} + u_{n+2})[e,h]
\end{aligned}$$

$$\boxed{\begin{aligned} s_n(1-e)+s_{n+1}e &= u_n' + [d,(-u'_{n-1}+2u'_n-u'_{n+1})h] \\ n\geq 1. \qquad & \quad + (-u_{n-1}+3u_n-3u_{n+1}+u_{n+2})[e,h] \end{aligned}}$$

$$\begin{aligned} s_0(1-e)+s_1e &= u_0'(1-2e+e^2)+(u_0-u_1)h(1-e) \\ &\quad + u_0'e^2 + u_1'(e-e^2) + (-u_0+2u_1-u_2)he \\ &= u_0' + (-2u_0'+u_1')[d,h] + (u_0-u_1)h \\ &\quad + (-2u_0+3u_1-u_2)he \\ &= u_0' + [d,(2u_0'-u_1')h] - (2u_0(1-e)+2u_1e-u_1(1-e)-u_2e)h \\ &\quad + (u_0-u_1)h + (-2u_0+3u_1-u_2)he \\ &= u_0' + [d,(2u_0'-u_1')h] + (+2u_0-3u_1+u_2)[e,h] \\ &\quad -u_0h \end{aligned}$$

$$\boxed{\begin{aligned} s_0(1-e)+s_1e &= u_0'-u_0h + [d,(2u_0'-u_1')h] \\ &\quad + (2u_0-3u_1+u_2)[e,h] \end{aligned}}$$

Put $\boxed{s_n' = (u'_{n-1}-2u'_n+u'_{n+1})h + (s_{n-1}-3s_n+3s_{n+1}-s_{n+2})[e,h] \qquad n\geq 1}$

Then for we have

$$\begin{aligned} [d,s_n'] + s_n(1-e)+s_{n+1}e &= \\ [d,(u'_{n-1}-2u'_n+u'_{n+1})h] &+ ([d,s_{n-1}]-3[d,s_n]+3[d,s_{n+1}]-[d,s_{n+2}])[e,h] \\ + u_n' + [d,(-u'_{n-1}+2u'_n-u'_{n+1})h] &+ (-u_{n-1}+3u_n-3u_{n+1}+3u_{n+2})[e,h] \end{aligned}$$

$$\boxed{[d,s_n']+s_n(1-e)+s_{n+1}e = \begin{cases} u_n' & \text{if } n\geq 2 \\ u_1' - u_0e[e,h] & \text{if } n=1 \end{cases}}$$

At this point we have to adjust things to work well at $n=0,1$.

Put $\boxed{s_0' = (-2u_0'+u_1')h + (-2s_0+3s_1-s_2)[e,h]}$

Then $[d, s_0'] + s_0(1-e) + s_1 e =$

$$[d, (-2u_0'+u_1')h] + (-2[d,s_0]+3[d,s_1]-[d,s_2])[e,h]$$

$$+ u_0' - u_0 h + [d, (2u_0'-u_1')h] + (2u_0 - 3u_1 + u_2)[e,h]$$

$$= u_0' - u_0 h + 2u_0 e[e,h]$$

$$\boxed{\begin{aligned} [d, s_0] &= u_0 - u_0 e \\ [d, s_0'] + s_0(1-e) + s_1 e &= u_0' - u_0 h + 2u_0 e[e,h] \\ [d, s_1] &= u_1 \\ [d, s_1'] + s_1(1-e) + s_2 e &= u_1' - u_0 e[e,h] \\ [d, s_2] &= u_2 \\ [d, s_2'] + s_2(1-e) + s_3 e &= u_2' \\ &\ldots \end{aligned}}$$

There are two ways to proceed. First note that the cocycle $u_0 e[e,h]$ is reproduced by $1-e$.

$$[d, -e^2h + heh - he^2] = e[e,h]e$$

$$[d, u_0(-e^2h+heh-he^2)] + u_0 e[e,h](1-e) = u_0 e[e,h]$$

Put $\boxed{\begin{aligned} \tilde{s}_0 &= s_0 - 2u_0 e[e,h] \\ \tilde{s}_0' &= s_0' - 2u_0(-e^2h+heh-he^2) \\ \tilde{s}_1 &= s_1 + u_0 e[e,h] \\ \tilde{s}_1' &= s_1' + u_0(-e^2h+heh-he^2) \end{aligned}}$

and $\tilde{s}_n = s_n$, $\tilde{s}'_n = s'_n$ for $n \geq 2$.

Then we get

$$\boxed{\begin{aligned} [d, \tilde{s}_0] &= u_0 - u_0 e \\ [d, \tilde{s}'_0] + \tilde{s}_0(1-e) + \tilde{s}_1 e &= u'_0 - u_0 h \\ [d, \tilde{s}_1] &= u_1 \\ [d, \tilde{s}'_1] + \tilde{s}_1(1-e) + \tilde{s}_2 e &= u'_1 \\ \cdots \end{aligned}}$$

Now that we can contract cocycles we get the required homotopy operator k such that $[d,k] = 1 - ji$. Let's compute the modified h given by ikj. Think of $i : U \to T$ as the cocycle $(e, h, 0, 0, \cdots)$, then ik should be the cochain $(\tilde{s}_0, \tilde{s}'_0, \cdots)$ for $u_0 = e$, $u'_0 = h$, $u_n = u'_n = 0$ for $n \geq 1$. Then $ikj = \tilde{s}_0$.

$$\tilde{s}_0 = -2u_0 e[e,h] + u'_0(1-e) + (u_0 - u_1)h \quad \text{in general}$$

$$= -2e^2[e,h] + h(1-e) + (e-0)h$$

$$= h + [e,h] - 2e^2[e,h]$$

$$\therefore \boxed{ikj = h + [e,h] - 2e^2[e,h]}$$

$$\sim h + [e,h] - e[e,h] - [e,h](1-e)$$

$$= h - [e,[e,h]]$$

The second way is to note that $(s_0, s'_0, \cdots)$ gives a homotopy between 1 and ji for a different i namely $(e, h - 2e[e,h], 0, e[e,h], 0, \cdots)$

The corresponding modified h is

$$
\begin{aligned}
s_0 &= u_0'(1-e) + (u_0 - u_1)h \\
&= (h - 2e[e,h])(1-e) + eh \\
&= h + [e,h] - 2e[e,h](1-e)
\end{aligned}
$$

which is also equivalent to $h - [e,[e,h]]$.

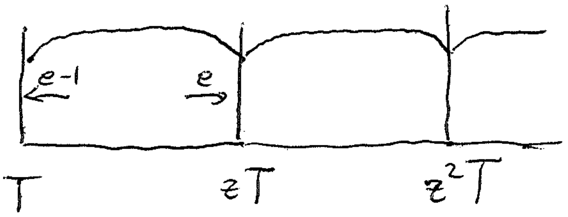
Let's see how hard it is to compute $\iota k^2 j$. We need s_0 for the cocycle ιk which means we need s_0, s_0', s_1 for ι. Seems too hard.

<u>April 23, 1995</u>

New idea: Note that $U = k[z,\sigma] \otimes T$ with the differential $\tilde{d} = (ze + 1 - e)\partial + d$ (here d is $1 \otimes d_T$ and I have changed notation to avoid the mistake of leaving out the second term in

$$[\tilde{d}, h] = [d,h] + [(ze+1-e)\partial, h]$$
$$= e - e^2 + (1-z)[e,h]\partial \quad)$$

corresponds geometrically to the telescope



Let's consider the analogue of the infinite telescope in both directions. Put

$$W = k[\sigma] \otimes k[z, z^{-1}] \otimes T = k[z, z^{-1}] \otimes T \oplus \sigma k[z, z^{-1}] \otimes T$$

$$W^+ = k[z] \otimes T \oplus \sigma k[z] \otimes T = U \text{ above}$$

$$W^- = k[z^{-1}] \otimes T \oplus \sigma z^{-1} k[z^{-1}] \otimes T$$

Then W^+ and W^- are subcomplexes of W equipped with the differential $\tilde{d}$:

$$(ze+1-e)\partial \begin{pmatrix} k[z^{-1}] \otimes T \\ \oplus \\ \sigma z^{-1} k[z^{-1}] \otimes T \end{pmatrix} \subset \begin{pmatrix} 0 \\ \oplus \\ (ze+1-e) z^{-1} k[z^{-1}] \otimes T \end{pmatrix} \subset W^-$$

We have $W^+ \cap W^- = T$, $W^+ + W^- = W$, whence an exact sequence of complexes

$$0 \longrightarrow T \longrightarrow W^+ \oplus W^- \longrightarrow W \longrightarrow 0$$

which is clearly locally split. We now show that W is contractible and it then follows that $T \sim W^+ \oplus W^-$.

We have $[\tilde{d}, \sigma] = ze + 1 - e$ so

$$[\tilde{d}, \sigma(z^{-1}e + 1 - e)] = (ze + 1 - e)(z^{-1}e + 1 - e)$$
$$= 1 - 2e + e^2 + (z^{-1} + z)(e - e^2)$$
$$= 1 + (z^{-1} - 2 + z)(e - e^2)$$

Then $[\tilde{d}, \sigma(z^{-1}e + 1 - e) + (-z^{-1} + 2 - z)h]$

$$= 1 + (z^{-1} - 2 + z)(e - e^2) + (-z^{-1} + 2 - z)(e - e^2 + (1-z)[e,h]\partial)$$
$$= 1 + (-z^{-1} + 2 - z)(1 - z)[e,h]\partial = 1 - z^{-1}(1 - 2z + z^2)(1-z)[e,h]\partial$$
$$= 1 - \underbrace{z^{-1}(1-z)^3[e,h]\partial}_{\text{square zero}}$$

Thus $[\tilde{d}, k] = 1$ where

$$\boxed{k = \left(\sigma(z^{-1}e + 1 - e) - z^{-1}(1-z)^2 h\right)\left(1 + z^{-1}(1-z)^3[e,h]\partial\right)}$$

I think this is the operator corresponding to the formulas on 269-270:

$$s_n = u'_{n-1} e + u'_n(1-e) + (-u_{n-1} + 2u_n - u_{n+1})h$$

$$s'_n = (u'_{n-1} - 2u'_n + u'_{n+1})h + (s_{n-1} - 3s_n + 3s_{n+1} - s_{n+2})[e,h]$$

Thus if $\phi \leftrightarrow (u_0, u'_0, \dots)$, i.e. $\phi z^n y = u_n, \phi\sigma z^n y = u'_n$,

then $\phi k z^n y = \phi(\sigma(z^{-1}e + 1 - e) - z^{-1}(1-z)^2 h) z^n y$

$$= u'_{n-1}e + u_n(1-e) + (-u_{n-1} + 2u_n - u_{n+1})h = s_n$$

$\phi k \sigma z^n y = \phi(\sigma(z^{-1}e + 1 - e) - z^{-1}(1-z)^2 h)\sigma z^n y = + \phi\sigma z(1-z)^2 h$

$+ \phi(\qquad\qquad) z^{-1}(1 - z^3)[e,h] z^n y \qquad \therefore$ clear.

Let's now analyze the diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, shift right, "a=\binom{j^+}{-j^-}"'] & W^+ \oplus W^- \arrow[l, shift right, "b=(\iota^+ - \iota^-)"'] \arrow[r, shift right, "(in^+\ in^-)=p"'] \arrow[d, shift right, "\eta=(1\ 0)"'] & W \arrow[l, shift right, dashed, "\ell"'] \arrow[r] & 0 \\ & & W^+ \arrow[u, shift right, "\varepsilon=\binom{1}{0}"'] & & \end{tikzcd}$$

Here j^+, j^- are the obvious inclusions of $T = z^0 T$ in W^+, W^-. $\iota^\pm : W^\pm \to T$ are the maps corresponding to the cocycles $|e, h, 0, ..)$ on W^+ and $(..0, h, 1-e|$ on W^-. In general a cocycle on W^+ is a sequence $|u_0^+, u_0', u_1, u_1', ...)$ $\ni$ $[d, u_n] = 0$, $[d, u_n'] = u_n(1-e) + u_n e$ and a cocycle on W^- is a sequence $(.., u_{-2}', u_{-1}, u_{-1}', u_0^-|$.

Check $\iota^+ = |e, h, 0, ..)$, $\iota^- = (.., 0, h, 1-e|$ are cocycles:
$[d, e] = 0$, $[d, h] = e(1-e) + 0e$; $[d, 1-e] = 0$, $[d, h] = 0.(1-e) + (1-e)e$.
Thus $b = (\iota^+, -\iota^-)$ is a map of complexes. One has $ba = \iota^+ j^+ + \iota^- j^- = e + 1 - e = 1$ so that the the short exact exact sequence of complexes splits.
ℓ is defined by $\ell p = 1 - ab$.

Calculate ℓ. Given a cocycle on $W^+ \oplus W^-$:

$$* \qquad (.., u_{-1}', u_0^- | u_0^+, u_0', u_1, ..)$$

pull back via $a = \binom{j^+}{-j^-}$ to get $u_0^+ - u_0^- = v$
then pull back via $b : (.0, -h, -1+e | e, h, 0, 0..)$ to get $(\ ,0, -vh, -v(1-e) | ve, vh, 0, ..)$. Remove from $*$ to get

$$(.., u_{-1}' + u_0^+ h - u_0^- h,\ u_0^+(1-e) + u_0^- e \ |\ u_0^+(1-e) + u_0^- e,\ u_0' - u_0^+ h + u_0^- h,\ u_1, ...).$$

(Note $u_0^+ - (u_0^+ - u_0^-)e = u_0^+(1-e) + u_0^- e$, $u_0^- + (u_0^+ - u_0^-)(1-e) = u_0^+(1-e) + u_0^- e$).

whence we have

$$(\ldots, u'_{-1}, u_0^- \,|\, u_0^+, u'_0, \ldots)\,l = \ldots$$

$$(\ldots, u_{-1}, u'_{-1} + u_0^+ h - u_0^- h, u_0^+(1-e) + u_0^- e, u'_0 - u_0^+ h + u_0^- h, u_1, \ldots)$$

Return to

$$\begin{tikzcd} T \arrow[r, shift right, "a"'] & W^+ \oplus W^- \arrow[l, shift right, "b"'] \arrow[loop right, "lkp"] \arrow[d, shift right, "\eta"'] \\ & U = W^+ \arrow[u, shift right, "\varepsilon"'] \end{tikzcd} \qquad 1 - ab = [d, lkp]$$

let $i = b\varepsilon$, $j = \eta a$, $k^+ = \eta\, lkp\, \varepsilon$. Then we have

$[d, k^+] = \eta[d, lkp]\varepsilon = \eta(1-ab)\varepsilon = 1 - ji$. [redacted]

We also have

$$i = b\varepsilon = (i^+ \;\; -i)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i^+$$

$$j = \eta a = (1 \;\; 0)\begin{pmatrix} j^+ \\ -j^- \end{pmatrix} = j^+$$

and we know that $i^+ j^+ = e$. Thus we get our A^∞ idempotent [redacted] $i^+(k^+)^n j^+$ extending e.

Calculate $i^+ k^+ j^+ = b\varepsilon\eta\, lkp\, \varepsilon\eta a$. Start with the identity on T, pullback via b to get $(0, -h, -(1-e) | e, h, 0, \ldots)$ then by $\varepsilon\eta$ getting $(\ , 0, 0 | e, h, 0, \ldots)$, then by l to get

$$\begin{matrix} 0, & eh, & e(1-e), & h - eh, & 0 \\ \ldots u'_{-1}, & u_0, & u'_0, & \ldots \end{matrix}$$

Next we pull back via k and then

$$p\varepsilon\eta a = in^+ j^+ = j^+ : T \longrightarrow W^+$$

This means all we have to do is to calculate the s_0 belonging to $u'_{-1} = eh$, $u_0 = e - e^2$, $u'_0 = h - eh$ and the rest 0.

$$
\begin{aligned}
s_0 &= u'_{-1}e + u'_0(1-e) + (-u_{-1} + 2u_0 - u_1)h \\
&= ehe + (h-eh)(1-e) + 2(e-e^2)h \\
&= ehe + h - he - eh + ehe + 2eh - 2e^2h \\
&= h + [e,h] + 2ehe - 2e^2h \\
&= h + [e,h] - 2e[e,h] \\
&\equiv h + [e,h] - e[e,h] - [e,h](1-e) \\
&= h - [e,[e,h]].
\end{aligned}
$$

Let's straighten out the relation between solutions of

$$
\begin{aligned}
[d, e_0] &= 0 \\
[d, e_1] &= e_0 - e_0^2 \\
[d, e_2] &= -e_0e_1 + e_1e_0 \\
[d, e_3] &= e_2 - e_0e_2 + e_1^2 - e_2e_0 \\
[d, e_4] &= -e_0e_3 + e_1e_2 - e_2e_1 + e_3e_0 \\
[d, e_5] &= e_4 - e_0e_4 + e_1e_3 - e_2^2 + e_3e_1 - e_4e_0 \\
&\cdots\cdots
\end{aligned}
$$

and twisting cochains on the bar construction of the nonunital algebra ke with $e = e^2$, with values in a DG algebra Γ. The dual of $\mathrm{Bar}(ke)$ is a poly ring $k[w]$ where $|w| = 1$ and $dw = -w^2$; note that $w \mapsto -w$ changes $dw = -w^2$ to $dw = w^2$. A twisting cochain from $\mathrm{Bar}(ke)$ to Γ is an element $\theta = \sum_{n\geq 0} \theta_n w^{n+1}$ of the tensor product DG algebra $\Gamma \otimes k[w]$ satisfying $[d,\theta] + \theta^2 = 0$. I should mention that $\theta_n \in \Gamma_n = \Gamma^{-n}$ and that $|w|=1$ for the upper indexing. One has

$$[d,\theta] = \sum_{n\geq 0} [d,\theta_n]\, w^{n+1} + \sum_{n\geq 0} (-1)^n \theta_n \underbrace{[d, w^{n+1}]}_{=\begin{cases} 0 & n \text{ odd} \\ -w^{n+2} & n \text{ even}\end{cases}}$$

$$= \sum_{n\geq 0} [d,\theta_n]\, w^{n+1} + \sum_{n\geq 0} -\theta_{2m}\, w^{2m+2}$$

$$\theta^2 = \sum_{k,l\geq 0} \overset{(-1)^{(k+1)l}}{\wedge} \theta_k \theta_l\, w^{k+l+2}$$

$$[d,\theta] + \theta^2 = [d,\theta_0]\, w + \left([d,\theta_1] - \theta_0 + \theta_0^2\right) w^2$$
$$+ \left([d,\theta_2] - \theta_0\theta_1 + \theta_1\theta_0\right) w^3$$
$$+ \left([d,\theta_3] - \theta_2 + \theta_0\theta_2 + \theta_1^2 + \theta_2\theta_0\right) w^4$$
$$+ \left([d,\theta_4] - \theta_0\theta_3 + \theta_1\theta_2 - \theta_2\theta_1 + \theta_3\theta_0\right) w^5$$
$$+ \left([d,\theta_5] - \theta_4 + \theta_0\theta_4 + \theta_1\theta_3 + \theta_2^2 + \theta_3\theta_1 + \theta_4\theta_0\right) w^6$$

yielding

$$[d,\theta_0] = 0 \qquad \theta_0 = e_0$$
$$[d,\theta_1] = \theta_0 - \theta_0^2 \qquad \theta_1 = e_1$$
$$[d,\theta_2] = \theta_0\theta_1 - \theta_1\theta_0 \qquad \theta_2 = -e_2$$
$$[d,\theta_3] = \theta_2 - \theta_0\theta_2 - \theta_1^2 - \theta_2\theta_0 \qquad \theta_3 = -e_3$$
$$[d,\theta_4] = -\theta_0\theta_3 - \theta_1\theta_2 + \theta_2\theta_1 - \theta_3\theta_0 \qquad \theta_4 = e_4$$
$$[d,\theta_5] = \theta_4 - \theta_0\theta_4 - \theta_1\theta_3 - \theta_2^2 - \theta_3\theta_1 - \theta_4\theta_0 \qquad \theta_5 = e_5$$

Thus it would appear that changing the signs of θ_n for $n \equiv 2,3 \pmod 4$ converts the θ_n-equations into the e_n equations.

Consider now homotopy equivalences (earlier work: July 25, 1992 pp 14–19), which yield a similar system of equations. The idea is that we have maps of complexes and homotopies

$$h_X \circlearrowright X \underset{f}{\overset{g}{\rightleftarrows}} Y \circlearrowleft h_Y$$

such that

$$1 - gf = [d, h_X]$$
$$1 - fg = [d, h_Y]$$

and compatibilities between these homotopies

$$fh_X - h_Y f = [d, u], \quad gh_Y - h_X g = [d, v] \quad \text{etc.}$$

Introduce the operators on $X \oplus Y$:

$$\alpha_0 = \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix} \qquad \alpha_1 = \begin{pmatrix} h_X & 0 \\ 0 & h_Y \end{pmatrix} \qquad \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$$

Then

$$[d, \alpha_0] = 0 \qquad [d, \alpha_1] = \begin{pmatrix} 1-gf & 0 \\ 0 & 1-fg \end{pmatrix} = 1 - \alpha_0^2$$

$$\left[d, \begin{pmatrix} & v \\ u & \end{pmatrix}\right] = \begin{pmatrix} 0 & gh_Y - h_X g \\ fh_X - h_Y f & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix}, \begin{pmatrix} h_X & 0 \\ 0 & h_Y \end{pmatrix}\right]$$

$$= [\alpha_0, \alpha_1] = \alpha_0\alpha_1 - \alpha_1\alpha_0$$

To find the higher equations, consider the case where X, Y are both SDR's of the same complex E:

$$\begin{tikzcd} & E \arrow[loop left, "k"] \arrow[loop right, "c"] \arrow[dl, shift right, "j"'] \arrow[dr, shift left, "b"] & \\ X \arrow[ur, shift right, "i"'] & & Y \arrow[ul, shift left, "a"] \end{tikzcd}$$

$[d,a]=[d,b]=0 \qquad ba=1$
$1-ab=[d,c]$
$c^2=ca=bc=0$
$[d,i]=[d,j]=0 \qquad ji=1$
$1-ij=[d,k] \qquad k^2=ki=jk=0$

Let $f=bi$, $g=ja$. Then

$$1-gf = ji - jabi = j[d,c]i = [d, \overbrace{jci}^{h_X}]$$
$$1-fg = ba - bija = b[d,k]a = [d, \underbrace{bka}_{h_Y}].$$

$$fh_X - h_Y f = bijci - bkabi$$
$$= b(1-[d,k])ci - bk(1-[d,c])i$$
$$= b(-[d,k]c + k[d,c])\,i$$
$$= -b[d,kc]i = [d,-bkci]$$

$$[d, jcka] = j([d,c]k - c[d,k])a$$
$$= j((1-ab)k - c(1-ij))a$$
$$= -(ja)(bka) + (jci)(ja)$$
$$= -gh_Y + h_X g$$

So put

$$\alpha_0 = \begin{pmatrix} 0 & ja \\ bi & 0 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} jci & \\ & bka \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & jcka \\ bkci & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} jckci & 0 \\ 0 & bkcka \end{pmatrix}$$

Then $[d, \alpha_0] = 0$

$$[d,\alpha_1] = 1-\alpha_0^2$$
$$[d,\alpha_2] = \begin{pmatrix} 0 & -(ja)(bka)+(jci)(ja) \\ -(bi)(jci)+(bk)(abi) & 0 \end{pmatrix} = -\alpha_0\alpha_1 + \alpha_1\alpha_0$$

$$[d,\alpha_3] = \begin{pmatrix} [d, jcki] & 0 \\ 0 & [d, bkcka] \end{pmatrix}$$

$$= \begin{pmatrix} j(-abkc + cijc - ckab)i & \\ 0 & b(-ijck + kabk - kcij)a \end{pmatrix}$$

$$= -\alpha_0\alpha_2 + \alpha_1^2 - \alpha_2\alpha_0$$

April 24, 1995

Analysis of the preceding formulas.

$$\begin{pmatrix} j & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i & \\ & a \end{pmatrix} = \begin{pmatrix} 0 & j \\ b & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & ja \\ bi & 0 \end{pmatrix} = \alpha_0$$

$$\begin{pmatrix} j & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i & \\ & a \end{pmatrix} = \begin{pmatrix} j & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & a \end{pmatrix}$$

$$= \begin{pmatrix} jci & 0 \\ 0 & bka \end{pmatrix} = \alpha_1$$

Let us put $\alpha_n = \begin{pmatrix} j & 0 \\ 0 & b \end{pmatrix}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & c \end{pmatrix}\right)^n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & a \end{pmatrix}$.

Then
$$\alpha_2 = \begin{pmatrix} j & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} ck & 0 \\ 0 & kc \end{pmatrix}\begin{pmatrix} 0 & a \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & jcka \\ bkci & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} j & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} ckc & 0 \\ 0 & kck \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} jckci & 0 \\ 0 & bkcka \end{pmatrix}$$

Here is the general pattern. We have an SDR situation

$$\begin{array}{c} X \\ \oplus \\ Y \end{array} \underset{\varepsilon = \begin{pmatrix} i & 0 \\ 0 & a \end{pmatrix}}{\overset{\eta = \begin{pmatrix} j & 0 \\ 0 & b \end{pmatrix}}{\rightleftarrows}} \begin{array}{c} E \\ \oplus \\ E \end{array} \circlearrowleft \xi = \begin{pmatrix} k & 0 \\ 0 & c \end{pmatrix}$$

and an odd involution $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $E \oplus E$.

Then
$$\alpha_n = \eta (F\xi)^n F\varepsilon$$
satisfies
$$[d, \alpha_n] = \eta \sum_{i=1}^{n} (-1)^{i-1} (F\xi)^{i-1} F[d,\xi] (F\xi)^{n-i} F\varepsilon$$
$=$ [redacted]
$$\eta F(1-\varepsilon\eta)(F\xi)^{n-1}F\varepsilon + \eta \sum_{i=2}^{n-1} (-1)^{i-1} (F\xi)^{i-1} F(1-\varepsilon\eta)(F\xi)^{n-i} F\varepsilon$$
$$+ \eta (-1)^{n-1} (F\xi)^{n-1} F(1-\varepsilon\eta) F\varepsilon$$

Note $\eta F(1)(F\xi)^{i-1} = \eta\xi(F\xi)^{i-2} = 0$ as $\eta\xi = 0$.

$$[d, \alpha_n] = -(\eta F\varepsilon)(\eta (F\xi)^{n-1} F\varepsilon)$$
[redacted]
$$- \sum_{i=2}^{n-1} (-1)^{i-1} (\eta (F\xi)^{i-1} F\varepsilon)(\eta (F\xi)^{n-i} F\varepsilon)$$
$$- (-1)^{n-1} (\eta (F\xi)^{n-1} F\varepsilon)(\eta F\varepsilon)$$
$$= -\alpha_0 \alpha_{n-1} + \alpha_1 \alpha_{n-2} - \cdots - (-1)^{n-1} \alpha_{n-1} \alpha_0$$

Suppose we start with the equations
$$[d, \alpha_0] = 0$$
$$[d, \alpha_1] = 1 - \alpha_0^2$$
$$[d, \alpha_2] = -\alpha_0\alpha_1 + \alpha_1\alpha_0$$
$$[d, \alpha_3] = -\alpha_0\alpha_2 + \alpha_1^2 - \alpha_2\alpha_0$$
$$\cdots$$

and put $\alpha_0 = 2e_0 - 1$, $\alpha_1 = 4e_1$, $\alpha_2 = 8e_2$, $\alpha_3 = 16e_3$, ...

Then $[d, 2e_0 - 1] = 0 \Longrightarrow [d, e_0] = 0$.

$$[d, 4e_1] = 1 - \alpha_0^2 = 1 - (2e_0 - 1)^2 = 4(e_0 - e_0^2) \Longrightarrow [d, e_1] = e_0 - e_0^2$$
$$[d, 8e_2] = -(2e_0 - 1)4e_1 + 4e_1(2e_0 - 1) = -8e_0e_1 + \cancel{4e_1} + 8e_1e_0 - \cancel{4e_1}$$
$$\Longrightarrow [d, e_2] = -e_0e_1 + e_1e_0$$

$$[d, 16e_3] = -(2e_0-1)8e_2 + (4e_1)^2 - (8e_2)(2e_0-1)$$
$$= +8e_2 - 16e_0e_2 + 16e_1^2 - 16e_2e_0 + 8e_2$$
$$\Rightarrow [d, e_3] = e_2 - e_0e_2 + e_1^2 - e_2e_0$$

. . . .

Another way to see these powers of 2 is from the formulas:

$$\alpha_0 = \eta F \varepsilon \ , \quad \alpha_1 = \eta F \xi F \varepsilon \ , \quad \alpha_2 = \eta F \xi F \xi F \varepsilon$$

First observe that since $\xi^2 = \eta\xi = \xi\varepsilon = 0$, and $\eta\varepsilon = 1$, we have on setting $p = \frac{F+1}{2}$ or $F = 2p-1$

$$\alpha_0 = 2\eta p \varepsilon - 1 \ , \quad \alpha_1 = 4\eta p \xi p \varepsilon \ , \quad \alpha_2 = 8\eta p \xi p \xi p \varepsilon ;$$

Moreover $e_0 = \eta p \varepsilon$, $e_1 = \eta p \xi p \varepsilon$, $e_2 = \eta p \xi p \xi p \varepsilon, \ldots$ can be seen to satisfy the e-equations. In fact the e-equations hold for $e_n = j h^n i$, where $1 - ij = [d, h]$. So in the case of interest one has taken $1 - \varepsilon\eta = [d, \xi]$ and applied p on both sides $[d, p\xi p] = p(1-\varepsilon\eta)p = p - (p\varepsilon)(\eta p)$. Visually one has

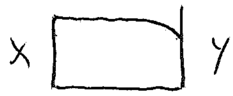
$$\begin{tikzcd} T \arrow[r, shift right, "\varepsilon"'] & U \arrow[l, shift right, "\eta"'] \arrow[r, shift right, two heads, "p"'] & pU \arrow[l, shift right, "p"'] \end{tikzcd}$$

so U is up to homotopy a direct summand of P and pU is a direct summand of U.

April 25, 1995.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of complexes.

Let $M(f)_n = \begin{matrix} X_n \\ \oplus \\ X_{n-1} \\ \oplus \\ Y_n \end{matrix}$ $\quad d = \begin{pmatrix} d & -1 & \\ & -d & \\ & f & d \end{pmatrix}$ be

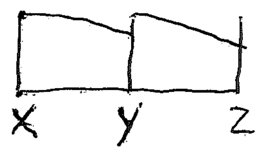
the mapping cylinder of f; picture 

Recall the maps

$$\begin{tikzcd} X \arrow[r, "i"] & M(f) \arrow[loop above, "k"] \arrow[d, shift right, "p"'] \\ & Y \arrow[u, shift right, "j"'] \end{tikzcd}$$

$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $j = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $p = (f \ \ 0 \ \ 1)$ $k = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

satisfying: $[d,i] = [d,p] = [d,j] = 0$ $\quad pi = f$ and the SDR relations: $pj = 1$ $\quad 1 - jp = [d,h]$ $\quad h^2 = ph = hj = 0$.

Let $N = M(f) \underset{Y}{+} M(g)$; picture: 

$N_n = X_n \oplus X_{n-1} \oplus Y_n \oplus Y_{n-1} \oplus Z_n$

$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & 1 \end{pmatrix}$

$d = \begin{pmatrix} d & -1 & & & \\ & -d & & & \\ & f & d & -1 & \\ & & & -d & \\ & & & g & d \end{pmatrix}$ $\quad k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then $M(gf) \underset{a}{\overset{b}{\leftrightarrows}} N \circlearrowleft^{k}$ is an SDR.

April 29, 1995

Miscellaneous comments.

1. Double mapping cylinder for the maps $T \xleftarrow{1-f} T \xrightarrow{f} T$ is the h-pushout

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "1-f"'] & T \arrow[d] \\ T \arrow[r] \arrow[ur, phantom, "\nearrow"] & M \end{tikzcd}$$

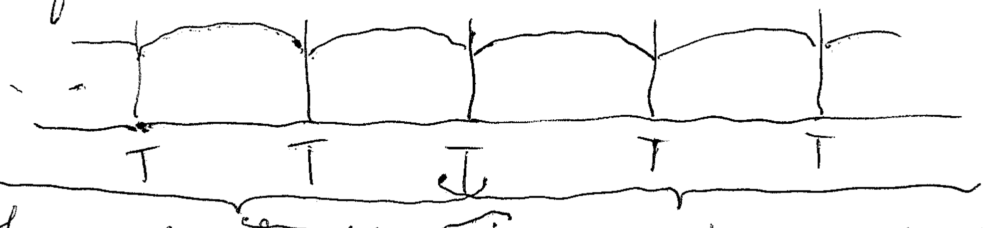
$M_n = T_n \oplus T_{n-1} \oplus T_n$ with $d = \begin{pmatrix} d & f-1 & \\ & -d & \\ & f & d \end{pmatrix}$. This is homotopy equivalent to T, in fact one has an SDR of M onto T given by $\overset{h}{\circlearrowleft} M \underset{i}{\overset{p}{\rightleftarrows}} T$

$$i = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad p = \begin{pmatrix} f & 0 & 1-f \end{pmatrix} \qquad h = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The embeddings

$$\begin{tikzcd} T \arrow[d, shift right=2, "\begin{pmatrix}1\\0\\0\end{pmatrix}"'] \arrow[d, shift left=2, "\begin{pmatrix}0\\0\\1\end{pmatrix}"] \arrow[dd, bend right=60, "f"'] \arrow[dd, bend left=60, "1-f"] \\ M \arrow[d, "p"] \\ T \end{tikzcd}$$

correspond to f and $1-f$ respectively. The doubly infinite iteration W:



is the sum of subcomplexes $W^- + W^+$, such that $W^- \cap W^+ = T$.

One gets then an exact sequence

$$0 \longrightarrow T \longrightarrow W^{*} \oplus W^{+} \longrightarrow W \longrightarrow 0$$

which realizes a Δ of "descent"

$$T \longrightarrow \underrightarrow{\text{holim}}(T, 1-f) \oplus \underbrace{\underrightarrow{\text{holim}}(T, f)}_{\underrightarrow{\text{holim}}(T \xrightarrow{f} T \xrightarrow{f} T \to \cdots)} \longrightarrow \underrightarrow{\text{holim}}(T, f(1-f)) \longrightarrow$$

Suppose $f = e$ where $e - e^2 = [d, h]$. In note in general that

$$W = k[z, z^{-1}] \otimes T \oplus \sigma k[z, z^{-1}] T$$

$$\tilde{d} = 1 \otimes d + (f - 1 + fz)\partial \qquad \begin{matrix} \sigma^2 = 0 = \partial^2 \\ [\partial, \sigma] = 1. \end{matrix}$$

When $f = e$ we have the near homotopy

$$k = \sigma(z^{-1}e + e - 1) + (z^{-1} + 2 + z)h$$

$$[\tilde{d}, k] = [1 \otimes d + (e - 1 + ez)\partial \,,\ \sigma(z^{-1}e + e - 1) + (z^{-1} + 2 + z)h]$$

$$= (z^{-1} + 2 + z)\overbrace{[d, h]}^{e-e^2} + (e - 1 + ez)\overbrace{[\partial, \sigma]}^{1}(z^{-1}e + e - 1)$$

$$- (z^{-1} + 2 + z)[e - 1 + ez, h]\partial$$

$$= z^{-1}(e - e^2) + 2(e - e^2) + z(e - e^2) + \overbrace{(e-1)^2 + e^2}^{2e^2 - 2e + 1} + z(e^2 - e) + z^{-1}(e^2 - e)$$

$$- (z^{-1} + 2 + z)(1 + z)[e, h]\partial.$$

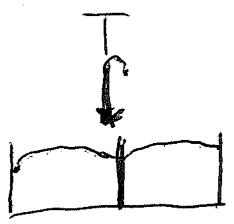
$$[\tilde{d}, k] = 1 - z^{-1}(1 + z)^3 [e, h]\partial$$

There are fewer signs than before. The matrix of k relative to $\ldots, z^{-1}\sigma j, j, \sigma j, zj, z\sigma j, \ldots$ is

$$\begin{pmatrix} h &&&&& \\ e & -h &&&& \\ 2h & 0 & h &&& \\ e-1 & -2h & e & \cdot && \\ h & 0 & 2h & \cdot && \\ & -h & e-1 & \cdot && \\ && h & \cdot && \end{pmatrix}$$

Another point is that the term $z^{-1}(1+z)^3[e,h]\partial$ is symmetric in the sense that if you think of $|z|=2$ and $|\sigma|=1$, then $|\partial|=-1$ so $|z^{-1}\partial|=-3$ and $|z^2\partial|=+3$.

It seems to be a waste of time to continue with these calculations as things get much too complicated.

Note the map  is ~ 0, a homotopy being $\begin{pmatrix} h \\ e \\ 2h \\ e-1 \\ h \end{pmatrix}$.

2. Homology of the DGA $\Gamma = k\langle e, h\rangle$ where $|e|=0$, $|h|=1$, $[d,e]=0$, $[d,h]=e-e^2$. See p. 263 for background.

$$\Gamma: \quad \begin{array}{ccccc} AhAhA & \xrightarrow{d} & AhA & \xrightarrow{d} & A \\ \| & & \wr\| & & \| \\ k[x,y,z] & & A\otimes A = k[x,y] & & k[x] \end{array} \qquad A = k[e]$$

$Z_1 = (x-y)k[x,y]$, B_1 = ideal in $k[x,y]$ generated by $g(x)-g(y)$ with $g \in (x^2-x)k[x]$. Put $g = (x-x^2)f(x)$, then

$$g(x)-g(y) = \big(f(x)-f(y)\big)(x-x^2) + f(y)\big(x-x^2-y+y^2\big)$$

$$\in k[x,y]\underbrace{(x-y)(x-x^2)}_{\in B_1^{\#}} + k[x,y]\underbrace{(x-y)(1-x-y)}_{\text{in } B_1^{\#}}$$

Why? $(x-y)(x-x^2) \longleftrightarrow (e-e^2)[e,h] = [d,h][e,h] = [d,h[e,h]]$

$(x-y)(1-x-y) \longleftrightarrow [e,h] - e[e,h] - [e,h]e = [e-e^2,h]$
$= [[d,h],h] = [d,h^2]$

Thus $B_1^{\#} = \big((x-y)(x-x^2),\ (x-y)(1-x-y)\big)$

$$H_1^{\#} = Z_1^{\#}/B_1^{\#} \simeq k[x,y]/(x-x^2,\ 1-x-y) = k[x]/(x-x^2) = k\oplus ke$$

Conclusion is that $H_1^{\#}$ is 2-dimensional, really free of rank 2 over k. It is generated as a bimodule over $H_0 = k[x]/(x-x^2) = k\oplus ke$ by $[e,h] \longleftrightarrow x-y$.
Now we know that H_1 is spanned by the four elements $e[e,h]e$, $e^\perp[e,h]e$, $e[e,h]e^\perp$, $e^\perp[e,h]e^\perp$.
Also $e[e,h]e$, $e^\perp[e,h]e^\perp$ are boundaries, so we get a canonical isomorphism

$$\Omega^1(k\oplus ke) \xrightarrow{\sim} H_1$$
$$de \longmapsto \text{class of } [e,h]$$

<u>Question</u>: Does the isomorphism $\Omega^n(k\oplus ke) = H_n$ for $n=0,1$ extend to an isomorphism of graded algebras $\Omega^\bullet(k\oplus ke) \xrightarrow{\sim} H_*(\Gamma)$?

Change notation: Let d be replaced by ∂. Then ∂ is the degree -1 derivation of $\Gamma = A\langle h\rangle$, $A = k[e]$ such that $\partial(h) = e-e^2$, $\partial(a) = 0$. Let d be the degree $+1$ derivation such that $d(a) = [h,a]$, $d(h) = h^2$.

Then $d^2(a) = d[h,a] = [h^2,a] - [h,[h,a]] = 0$
$d^2(h) = d(h^2) = h^2 \cdot h - h \cdot h^2 = 0$
so $d^2 = 0$. Also $[d,\partial](a) = 0 + \partial[h,a] = [e-e^2,a] = 0$
$[d,\partial](h) = d(e-e^2) + \partial(h^2) = [h,e-e^2] + (e-e^2)h - h(e-e^2) = 0.$

Thus $[d,\partial] = 0$. Thus d induces a degree $+1$ derivation on $H_*(\Gamma)$ such that $d^2 = 0$. This then induces a map of DGA's $\Omega(k \oplus ke) \longrightarrow H_*(\Gamma)$ extending the identity in degree 0.

April 30, 1995

Notation: $\Gamma = k\langle e,h\rangle = A\langle h\rangle$, $A = k[e]$,
$B = k[e]/(e-e^2) = k + k\bar{e}$. One has a surjective homom.

$$(1) \qquad A\langle h\rangle \longrightarrow\!\!\!\!\!\rightarrow B\langle h\rangle$$

compatible with the differentials ∂, d defined by $\partial(e) = 0$, $\partial(h) = e - e^2$ on $A\langle h\rangle$
$d(e) = [h,e]$, $d(h) = h^2$
resp. $\partial = 0$, $d(\bar{e}) = [h,\bar{e}]$, $d(h) = h^2$ on $B\langle h\rangle$.
Thus we get a homomorphism on homology wrt ∂

$$(2) \qquad H_*(\Gamma) \longrightarrow B\langle h\rangle$$

In degree zero this is the obvious iso $H_0(\Gamma) \simeq B$. In degree one, since $H_1(\Gamma)$ is generated by the class of $[h,e]$ as B-bimodule the image is contained in $B(h\bar{e} - \bar{e}h) = \Omega^1 B \subset B \otimes B = BhB$. From our computation of $H_1(\Gamma)$ we see that (2) is surjective in degree 1.

It's now clear that $H_*(\Gamma)$ is at least as big as ΩB, namely our homomorphism $\Omega B \to H_*(\Gamma)$ is onto.

Further evidence for $H_*(\Gamma) \simeq \Omega B$.

Consider the adic filtration of $\Gamma = A\langle h\rangle$ with respect to the ideal J generated by h and $\partial(h) = e - e^2$. Then

$$\Gamma/J = k\langle e, h\rangle/(h, e-e^2) = B$$

and because Γ is a tensor algebra it should be true that

$$\bigoplus_{n\geq 0} J^n/J^{n+1} = T_B(J/J^2).$$

Picture:

$$\begin{array}{rcl}
 & & A \\
 & AhA \xrightarrow{\partial} & I \\
AhAhA \xrightarrow{\partial} & IhA + AhI \xrightarrow{\partial} & I^2
\end{array}$$

J/J^2 should be the complex

$$\begin{array}{ccc}
BhB & \xrightarrow{\partial} & I/I^2 \\
\wr\wr & & \uparrow \cdot(e-e^2) \\
B\otimes B & \xrightarrow{\mu} & B
\end{array}$$

Thus J/J^2 should be quis $\Omega^1 B[1]$ and so $gr^J(\Gamma)$ quis $\bigoplus_{n\geq 0} \Omega^n B[n]$.

Conclude that if the spectral sequence for this filtration converges, then $H_*(\Gamma) \simeq \Omega B$.

May 1, 1995

Let's start with a htpy retract situation

(1) $$U \underset{i}{\overset{j}{\rightleftarrows}} T \circlearrowleft h \qquad \begin{aligned}[d,i] &= [d,j] = 0 \\ 1 - ji &= [d,h].\end{aligned}$$

Set $\quad e_n = i h^n j; \quad e_n \in \mathrm{Hom}(T,T)_n$.

(2) $$\begin{aligned}
[d,e_0] &= 0 \\
[d,e_1] &= e_0 - e_0^2 \\
[d,e_2] &= -e_0e_1 + e_1e_0 \\
&\cdots\cdots \\
[d,e_{n+1}] &= \begin{Bmatrix} e_n & n \text{ even} \\ 0 & n \text{ odd}\end{Bmatrix} - \sum_{j=0}^{n}(-1)^j e_j e_{n-j}
\end{aligned}$$

Let

(3) $$\tilde{d} = \begin{pmatrix} d & 1-e_0 & -e_1 & -e_2 & -e_3 & \cdots \\ & -d & e_0 & e_1 & e_2 & \cdots \\ & & d & 1-e_0 & -e_1 & \cdots \\ & & & -d & e_0 & \cdots \\ & & & & d & \cdots \\ & & & & & \ddots \end{pmatrix}$$

operate on $\tilde{T} = T \oplus T[1] \oplus T[2] \oplus \cdots\cdots$. $\quad |\tilde{d}| = -1$.

The identities (2) [redacted] are equivalent to $\tilde{d}^2 = 0$.

Let

$$k = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \\ & & & \ddots \end{pmatrix}$$

operate on $\tilde{T}$. $\quad |k| = +1$.

Then

$$\tilde{d}k = \begin{pmatrix} 1-e_0 & -e_1 & -e_2 & -e_3 & \cdots \\ -d & e_0 & e_1 & e_2 & \cdots \\ & d & 1-e_0 & -e_1 & \cdots \\ & & -d & e_0 & \cdots \\ & & & d & \cdots \end{pmatrix}$$

$$k\tilde{d} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ d & 1-e_0 & -e_1 & -e_2 & -e_3 & \cdots \\ & -d & e_0 & e_1 & e_2 & \\ & & d & 1-e_0 & -e_1 & \\ & & & -d & e_0 & \end{pmatrix}$$

$$\therefore \quad [\tilde{d}, k] = \begin{pmatrix} 1-e_0 & -e_1 & -e_2 & -e_3 & \cdots \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{pmatrix}$$

$$= I - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} e_0 & e_1 & e_2 & \cdots \end{pmatrix}$$

Set $a = \;$: $T \longrightarrow \tilde{T}$ $\quad b = \;$: $\tilde{T} \longrightarrow T$. Note $\tilde{d}a = ad$ and that a is <u>injective</u>. Since ab commutes with $\tilde{d}$ $\left([[\tilde{d}, [\tilde{d}, k]] = [\tilde{d}^2, k] = 0 \right)$, we have $a(b\tilde{d} - \tilde{d}b) = ab\tilde{d} - \tilde{d}ab = 0 \Rightarrow b\tilde{d} = db$ by the injectivity.

Actually the last step should have been done differently, namely:

$$[\tilde{d}, k] = I - \underbrace{\begin{pmatrix} e_0 & e_1 & e_2 & \cdots \\ & & 0 & \end{pmatrix}}$$

where

$$= \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}}_{\tilde{i}} \underbrace{\begin{pmatrix} j & hj & h^2j & h^3j & \cdots \end{pmatrix}}_{\tilde{j}}$$

Now clearly one has $\tilde{d}\tilde{i} = \tilde{i} d$ so $\tilde{i} : U \longrightarrow \tilde{T}$ is a map of complexes. I claim $\tilde{j} : \tilde{T} \longrightarrow U$ is also a map of complexes.

$$\tilde{j}\tilde{d} = \begin{pmatrix} j & hj & h^2j & \cdots \end{pmatrix} \begin{pmatrix} d & 1-ij & -ihj & -ih^2j \\ & -d & ij & ihj \\ & & d & 1-ij \\ & & & -d \end{pmatrix}$$

$$= \begin{pmatrix} jd & \begin{matrix} j(1-ij) \\ -hjd \end{matrix} & \begin{matrix} -jihj \\ +hjij \\ +h^2jd \end{matrix} & \begin{matrix} -jih^2j \\ +hjihj \\ +h^2j(1-ij) \\ -h^3jd \end{matrix} \end{pmatrix} = d\tilde{j}.$$

$$[d, hj] = (1-ji)j - hjd$$

$$[d, h^2j] = (1-ji)hj - h(1-ji)j + h^2jd$$

$$d\,h^3j = (1-ji)h^2j - h(1-ji)hj + h^2(1-ji)j - h^3jd$$

Thus we have maps of complexes

$$U \underset{\tilde i}{\overset{\tilde j}{\leftrightarrows}} \tilde T$$

and a k on $\tilde T$ such that $1 - \tilde i \tilde j = [\tilde d, k]$

Also $\tilde j \tilde i = (j \ \ h_j \ \cdots)\begin{pmatrix} i \\ 0 \end{pmatrix} = ji = 1-[d,h]$.

Thus $\tilde j, \tilde i$ give a homotopy equivalence of U and $\tilde T$.

Misc. ① If e is an operator a module M, then the two canonical dilations of it to an idempotent are $\begin{pmatrix} e & e-e^2 \\ 1 & 1-e \end{pmatrix}$ and $\begin{pmatrix} e & 1 \\ e-e^2 & 1-e \end{pmatrix}$ on $M \oplus M$.

② If $e - e^2 \in I \qquad I^2 = 0$, then

$(e+\delta e)^2 = e + \delta e$ where $\delta e = (2e-1)(e-e^2)$.

③ Given $e_0 = e, h$ with $[d,e]=0 \quad [d,h] = e-e^2$, we know that $e_1 = h - ad(e)^2 h = h - e^2h + 2heh - he^2$ satisfies $[d, e_1] = e_0 - e_0^2$ and $[d, e_2] = -e_0 e_1 + e_1 e_0$ for some e_2. I have calculated that

$\boxed{e_2 = -h^2 + 3eh^2 - 4heh + 3h^2 e}$ works.

May 2, 1995

The problem is to show how a homotopy idempotent can be refined to an A_∞ idempotent. I hope to do this by studying the complexes $T^{(n)} = T \oplus \sigma T \oplus \cdots \oplus \sigma^n T$ with differential

$$\left[\underbrace{\begin{pmatrix} d & 1-e_0 & -e_1 & \cdots & -e_n \\ & -d & e_0 & \cdots & +e_{n-1} \\ & & d & \cdots & -e_{n-2} \\ & & & & \vdots \end{pmatrix}}, \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \right]$$

$$= I - \begin{pmatrix} e_0 & +e_1 & \cdots & +e_n & 0 \\ & & & & +e_n \\ & & & & -e_{n-1} \\ & & & & \vdots \\ & & & & \vdots \end{pmatrix}$$

$$= I - \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} e_0 & e_1 & \cdots & e_{n+1} \end{pmatrix} - \begin{pmatrix} -e_{n+1} \\ e_n \\ \vdots \\ \vdots \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

this ends with $1-e_0$ for $n+1$ even, e_0 for $n+1$ odd

Examples

$$\left[\begin{pmatrix} d & 1-e_0 \\ & -d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1-e_0 & \\ & 1-e_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} e_0 & e_1 \end{pmatrix} - \begin{pmatrix} -e_1 \\ e_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} d & 1-e_0 & -e_1 \\ & -d & e_0 \\ & & d \end{pmatrix}, \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1-e_0 & -e_1 & 0 \\ \cancel{d} & 1\cancel{e_0} & -e_1 \\ & \cancel{d} & e_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} e_0 & e_1 & e_2 \end{pmatrix} - \begin{pmatrix} -e_2 \\ +e_1 \\ 1-e_0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Let's examine the case of $T^{(1)}$. Note that we have maps of complexes

$$(*) \qquad T \underset{\binom{1}{0}}{\overset{(e_0\ e_1)}{\rightleftarrows}} T^{(1)} \underset{(0\ 1)}{\overset{\binom{-e_1}{e_0}}{\rightleftarrows}} T[1]$$

Because

$$1 \sim \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} e_0 & e_1 \end{pmatrix}}_{\alpha} + \underbrace{\begin{pmatrix} -e_1 \\ e_0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}}_{\beta}$$

and $\beta\alpha = 0$, it follows that α, β are orthogonal idempotents up to homotopy.

Note that $T^{(1)}$ depends only on e_0, but the upper operators in $(*)$ depend on e_1. The hope I have is that by constructing a suitable homotopy splitting of $T^{(1)}$ I am forced to modify e_1 so that e_2 exists. In $(*)$ the composition of the top arrows $-e_0e_1 + e_1e_0$ is ~ 0 iff $e_2\ \exists$.

Abstract question. Suppose X, Y objects in a category having ~~some~~ the same object Z as retract

$$\begin{tikzcd} X \arrow[rr, shift left, "u=aj"] \arrow[dr, shift left, "j"] & & Y \arrow[ll, shift left, "v=ib"] \arrow[dl, shift left, "b"] \\ & Z \arrow[ul, shift left, "i"] \arrow[ur, shift left, "a"] & \end{tikzcd} \qquad ji=1 \quad ba=1.$$

Let $u = aj$, $v = ib$. Then

$$uv = ajib = ab = \text{the projector on } Y \text{ with image } Z$$

$$vu = ibaj = ij = \text{the projector on } X \text{ with image } Z$$

and

$$uvu = abaj = aj = u$$

$$vuv = ijib = ib = v.$$

I think the way to summarize the preceding is to say that an isomorphism between two objects (X,e), (Y,e') in the Karoubian envelope of a category is specified by a pair of maps $X \underset{v}{\overset{u}{\rightleftarrows}} Y$ such that $uvu=u$, $vuv=v$, $vu=e$, $uv=e'$.

Let's apply this to

$$T \underset{u=\binom{1}{0}}{\overset{v=(e_0\ e_1)}{\rightleftarrows}} T^{(1)}$$

Then

$$uvu = \begin{pmatrix}1\\0\end{pmatrix}(e_0\ e_1)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e_0\\0\end{pmatrix} \quad \text{so}$$

$$u - uvu = \begin{pmatrix}1-e_0\\0\end{pmatrix}$$

$$\begin{pmatrix} 1-e_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-e_0 & \\ & 1-e_0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \left[\begin{pmatrix} d & 1-e_0 \\ & -d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} d & 1-e_0 \\ & -d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} d$$

showing that $u \sim uvu$.

Next $vuv = (e_0 \;\; e_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} (e_0 \;\; e_1) = (e_0 e_0 \;\; e_0 e_1)$

so $v - vuv = ((1-e_0)e_0 \;\; (1-e_0)e_1)$. Now

$$(e_0 \;\; e_1) \left[\begin{pmatrix} d & 1-e_0 \\ & -d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] = (e_0 \;\; e_1) \begin{pmatrix} 1-e_0 & \\ & 1-e_0 \end{pmatrix}$$

$$= (e_0(1-e_0) \;\; e_1(1-e_0))$$

is ~ 0. Subtracting this from $v - vuv$ we find

$$v - vuv \sim (0 \;\; -e_0 e_1 + e_1 e_0)$$

We want this to be ~ 0 i.e. of the form $(\alpha \;\; \beta) \begin{pmatrix} d & 1-e_0 \\ & -d \end{pmatrix} - d(\alpha \;\; \beta)$

$$= (-[d,\alpha] \;\; \alpha(1-e_0) - [d,\beta])$$

Thus we want to find α, β such that

$$[d,\alpha] = 0$$

$$\alpha(1-e_0) - [d,\beta] = -[e_0, e_1]$$

I believe we know this is possible iff $[e_0,e_1]e_0 \sim 0$

Similarly consider

$$T^{(1)} \underset{v' = \begin{pmatrix} -e_1 \\ e_0 \end{pmatrix}}{\overset{u' = (0 \;\; 1)}{\rightleftarrows}} T[1]$$

Then I've calculated that $u' - u'v'u' \sim 0$ but that $v' - v'u'v' \sim 0$ iff $e_0[e_0, e_1]$ is a boundary.

Thus I want $e_0[e_0, e_1] \sim 0$ and $[e_0, e_1]e_0 \sim 0$, which I believe happens iff $[e_0, e_1] \sim 0$.

May 4, 1995

I can now give the inductive construction which refines a homotopy idempotent to an A_∞-idempotent.

We start with e_0 on T such that $[d,e_0]=0$ and $e_0 \sim e_0^2$. Let e_1 be such that $[d,e_1] = e_0 - e_0^2$. Form $T^{(1)} = T \oplus \sigma T$ with differential $\begin{pmatrix} d & 1-e_0 \\ & -d \end{pmatrix}$. We have maps of complexes

$$\begin{tikzcd} T \arrow[r, shift right, "u=\binom{1}{0}"'] & T^{(1)} \arrow[l, shift right, "v=(e_0\ e_1)"'] \arrow[r, shift right, "u'=(0\ 1)"'] & T[1] \arrow[l, shift right, "v'=\binom{-e_1}{e_0}"'] \end{tikzcd}$$

The arrows at the bottom are part of a Δ of complexes. One has

$$I = \left[\begin{pmatrix} d & 1-e_0 \\ & -d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(e_0\ e_1) + \begin{pmatrix} e_1 \\ e_0 \end{pmatrix}(0\ 1) \quad \text{i.e.}$$

$$1 = [d,h] + uv + v'u'$$

$$u'u = 0, \quad vu = e_0, \quad u'v' = e_0, \quad vv' = -e_0e_1 + e_1e_0$$

Let $z = vv' = -e_0e_1 + e_1e_0$. The homology class of z is an obstruction to finding e_2 such that $[d,e_2] = -e_0e_1 + e_1e_0$.

We have

$$vv' = v\big([d,h] + uv + v'u'\big)v'$$
$$= vu\,v'v + vv'\,u'v' + [d, vhv']$$

i.e. $$\boxed{z = e_0 z + z e_0 + [d, vhv']}$$

This implies $(1-e)z \sim ze_0$, $z(1-e) \sim e_0 z$ hence

$$(1-e_0)z(1-e_0) \sim 0 \quad \text{and} \quad e_0 z e_0 \sim 0, \text{ so}$$

$$z \sim e_0 z(1-e_0) + (1-e_0)z e_0 .$$

Let's consider now a change from e_1 to $\tilde{e}_1 = e_1 + \delta e_1$, where $[d, \delta e_1] = 0$ so that $[d, \tilde{e}_1] = e_0 - e_0^2$. Then $\delta z = -e_0 \delta e_1 + \delta e_1 e_0$ and we would like to arrange $z + \delta z \sim 0$ i.e.

$$z = e_0 \delta e_1 - \delta e_1 e_0$$

This implies

$$e_0 z(1-e_0) \sim e_0 \delta e_1 (1-e_0)$$

$$(1-e_0) z e_0 \sim -(1-e_0)\delta e_1 e_0$$

so if we put $\delta e_1 = e_0 z(1-e_0) - (1-e_0) z e_0$ we have

$$e_0 \delta e_1 - \delta e_1 e_0 = e_0^2 z(1-e_0) - e_0(1-e_0) z e_0$$
$$- e_0 z(1-e_0) e_0 + (1-e_0) z e_0^2$$
$$\sim e_0 z(1-e_0) + (1-e_0) z e_0 \sim z$$

as desired. Actually a simpler choice is

$$\boxed{\delta e_1 = z(1-e_0) - (1-e_0) z = [e_0, z]}$$

since

$$\begin{aligned} e_0 \delta e_1 &= e_0 z(1-e_0) - e_0(1-e_0) z \sim e_0 z(1-e_0) \\ -\delta e_1 e_0 &\quad -z(1-e_0) e_0 + (1-e_0) z e_0 \quad +(1-e_0) z e_0 \end{aligned} \sim z.$$

For this choice of δe_1 we have the choice

$$\boxed{\tilde{e}_1 = e_1 + \delta e_1 = e_1 + [e_0, z] = e_1 - [e_0, [e_0, e_1]]}$$

which was found before.

Next stage. Suppose e_0, e_1, e_2 given satisfying the first 3 A_∞-idemp equations we have maps.

$$\begin{tikzcd}[column sep=huge] T \arrow[r, shift right, "u=\begin{pmatrix}1\\0\\0\end{pmatrix}"'] & T^{(2)} \arrow[l, shift right, "v=(e_0\ e_1\ e_2)"'] \arrow[r, shift right, "u'=(0\ 0\ 1)"'] & T[2] \arrow[l, shift right, "v'=\begin{pmatrix}-e_2\\ e_1\\ 1-e_0\end{pmatrix}"'] \end{tikzcd}$$

$$1 = \left[\begin{pmatrix} d & 1-e_0 & -e_1 \\ & -d & e_0 \\ & & d \end{pmatrix}, \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix}\right] + \begin{pmatrix}1\\0\\0\end{pmatrix}(e_0\ e_1\ e_2) + \begin{pmatrix}-e_2\\ e_1\\ 1-e_0\end{pmatrix}(0\ 0\ 1)$$

$$1 = uv + v'u' + [d,h]$$

$u'u = 0$, $\quad vu = e_0$, $\quad u'v' = 1-e_0$, $\quad vv' = z$ where

$$z = -e_0e_2 + e_1^2 + e_2(1-e_0)$$

One then has

$$z = vv' = \underbrace{vu}_{e_0}vv' + vv'\underbrace{u'v'}_{1-e_0} + v[d,h]v'$$

so $$\boxed{z = e_0 z + z(1-e_0) + [d, vhv']}$$

This implies $(1-e_0)z \sim z(1-e_0)$ and

$$z \sim e_0 z e_0 + (1-e_0)z(1-e_0)$$

Let's now consider a change δe_2 such that $[d,\delta e_2]=0$. Then $\delta z = -e_0\delta e_2 + \delta e_2(1-e_0)$, and we would like $z+\delta z \sim 0$ i.e.

$$z = e_0\delta e_2 - \delta e_2(1-e_0)$$

The simplest choice appears to be

$$\boxed{\delta e_2 = z e_0 - (1-e_0)z}$$

for
$$\begin{aligned} e_0\delta e_2 &= e_0 z e_0 - e_0\cancel{(1-e_0)}z + \\ -\delta e_2(1-e_2) &\quad - z\cancel{e_0(1-e_0)} + (1-e_0)z(1-e_0) \sim z \end{aligned}$$

Next consider $T^{(3)}$ whose diffl is
$$\begin{pmatrix} d & 1-e_0 & -e_1 & \vline & -e_2 \\ & -d & e_0 & \vline & e_1 \\ & & d & \vline & 1-e_0 \\ \hline & & & \vline & -d \end{pmatrix}.$$
(We recognize this as the cone on the map $\begin{pmatrix} -e_2 \\ e_1 \\ 1-e_0 \end{pmatrix} : T[2] \longrightarrow T^{(2)}$, but this doesn't seem useful.)

We assume e_2 has been modified so that there exists an e_3 such that $[d, e_3] = -e_0 e_2 + e_1^2 + e_2(1-e_0)$. Then we have the identity

$$1 = \Bigg[\begin{pmatrix} d & 1-e_0 & -e_1 & -e_2 \\ & -d & e_0 & e_1 \\ & & d & 1-e_0 \\ & & & -d \end{pmatrix}, \underset{h}{\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{pmatrix}} \Bigg] + \underset{u}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} \underset{v}{\begin{pmatrix} e_0 & e_1 & e_2 & e_3 \end{pmatrix}} + \underset{v'}{\begin{pmatrix} -e_3 \\ e_2 \\ -e_1 \\ e_0 \end{pmatrix}} \underset{u'}{\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}}$$

leading to maps

$$T \underset{u}{\overset{v}{\rightleftarrows}} T^{(3)} \underset{u'}{\overset{v'}{\rightleftarrows}} T[3]$$

such that
$$1 = uv + v'u' + [d, h]$$

$$u'u = 0, \quad vu = e_0, \quad u'v' = e_0, \quad vv' = z \quad \text{where}$$
$$z = -e_0 e_3 + e_1 e_2 - e_2 e_1 + e_3 e_0$$

One has
$$vv' = \overbrace{vu}^{e_0} vv' + vv' \overbrace{u'v'}^{e_0} + [d, vhv']$$
or
$$z \sim e_0 z + z e_0.$$
As before for $T^{(1)}$ $\delta z = -e_0 \delta e_3 + \delta e_3 e_0$, so we can take
$$\delta e_3 = [e_0, z] \qquad \tilde{e}_3 = e_3 + [e_0, z]$$
to kill the class of z, and then e_4].

May 5, 1995

Construction: Let $\tilde{T} = T \oplus \sigma T \oplus \sigma^2 T \oplus \cdots$ equipped with the twisted differential given by the A_∞-idempotent $e_0, e_1, \ldots$. We have the identity

$$[\text{redacted}]\ I = \left[\begin{pmatrix} d & 1-e_0 & \cdots & \\ & -d & \ddots & \\ & & & \ddots \end{pmatrix}, \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & & \ddots \end{pmatrix} \right] + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} e_0 & e_1 & e_2 & \cdots \end{pmatrix}$$

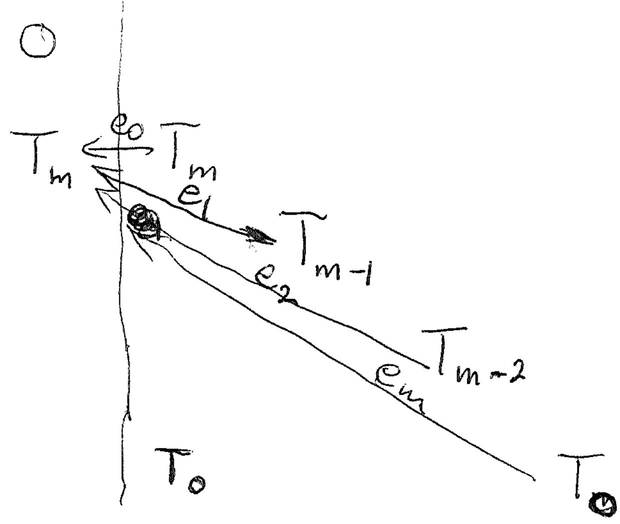
We can view $\tilde{T}$ as a kind of double complex [redacted] with the columns all equal to T

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
T_2 \arrow[d, "d"'] & T_2 \arrow[l, "1-e_0"'] \arrow[d, "-d"] & T_2 \arrow[l, "e_0"'] \arrow[d, "d"] & \arrow[l] \\
T_1 \arrow[d, "d"'] & T_1 \arrow[l, "1-e_0"'] \arrow[d, "-d"] & T_1 \arrow[l, "e_0"'] \arrow[d, "d"] & \arrow[l] \\
T_0 & T_0 \arrow[l, "1-e_0"'] & T_0 \arrow[l, "e_0"'] & T_0 \arrow[l]
\end{tikzcd}$$

except that the total differential has higher compenents like the differentials in a spectral sequence.

Now suppose T supported in $[0, m]$. Then $e_{m+1} = e_{m+2} = \cdots = 0$ for obvious reasons. I claim that $(e_0\ e_1 \cdots): \tilde{T} \longrightarrow T$ is zero in degrees $> m$.

In effect



this shows what $(e_0, \ldots)$ does to $(\tilde{T})_m$ and it's evident that $(\tilde{T})_{m+1} \longrightarrow T_{m+1}$ is zero. In fact this picture is unnecessary at

$(e_0, e_1, \dots) : \tilde{T} \longrightarrow T$ is of degree zero and hence zero on $(\tilde{T})_n$ for $n > m$ since $T_n = 0$ there.

Thus on $\tilde{T}$ we have $[d,h] = 1$ in degrees $> m$.

Now consider more generally a complex E with a homotopy h satisfying $[d,h]=1$ in degrees $>m$:

$$
\begin{tikzcd}
E_{m+2} \arrow[r, shift right, "d"'] & E_{m+1} \arrow[l, shift right, "h"'] \arrow[r, shift right, "d_{m+1}"'] & E_m \arrow[l, shift right, "h_m"']
\end{tikzcd}
$$

Let $h' = h$ on E_n for $n \geq m+1$ and $h' = 0$ on E_n for $n \leq m$. Then

$$
[d, h'] = \begin{cases} 1 & \text{on } E_n \quad n \geq m+1 \\ d_{m+1} h_m & \text{on } E_m \\ 0 & \text{on } E_n \quad n \leq m \end{cases}
$$

Note that $d_{m+1} h_m$ on E_m is a projector, since $d_{m+1} h_m d_{m+1} = d_{m+1}(h_m d_{m+1} + d_{m+2} h_{m+1}) = d_{m+1}$.

Thus $[d,h']$ is a projector on the complex E, and splits E into $[d,h']E \oplus (1 - [d,h'])E$. The former is contractible, so the latter is h.eq to E. The latter is

$$
0 \longrightarrow 0 \longrightarrow (1 - d_{m+1} h_m) E_m \longrightarrow E_{m-1} \longrightarrow \cdots
$$

This argument applied to $\tilde{T}$ shows that $\tilde{T}$ is h. equiv. to a subcomplex $= \tilde{T}$ in degrees $< m$, to a direct summand of $(\tilde{T})_m$ in degree m, and to zero in degrees $> m$.

May 6, 1995

Consider a Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ and let $A = QP \subset R$, $B = PQ \subset S$. Let U be a s. perfect complex over R such that U/AU is contractible as a complex of R/A modules, i.e. $\exists\, \bar{h}$ on U/AU such that $[d, \bar{h}] = 1$, and $\bar{h}$ commutes with R-multiplication. Since U is projective in each degree $\exists\, h$ on U compatible with R-multiplication [redacted] which lifts $\bar{h}$:

$$\begin{tikzcd} U \arrow[r, "\exists h"] \arrow[d] & U \arrow[d] \\ U/AU \arrow[r, "\bar{h}"] & U/AU \end{tikzcd}$$

[redacted] Put $f = 1 - [d,h]$ on U. One has

$$\begin{tikzcd} 0 \arrow[r] & A \otimes_R U \arrow[r, "\mu"] \arrow[d, "1\otimes f"] & U \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\varphi"] & U/AU \arrow[r] \arrow[d, "1-[d,\bar{h}]=0"] & 0 \\ 0 \arrow[r] & A \otimes_R U \arrow[r, "\mu"] & U \arrow[r] & U/AU \arrow[r] & 0 \end{tikzcd}$$

so there is a unique $\varphi : U \to A \otimes_R U$ such that $f = \mu\varphi$. Moreover $\varphi\mu = 1 \otimes f$. Here we have used flatness of U for the exactness of the rows above.

We then have

$$\begin{tikzcd} A \otimes_R U \arrow[loop above, "1\otimes h"] \arrow[r, shift left, "\mu"] & U \arrow[loop above, "h"] \arrow[l, shift left, "\varphi"] \end{tikzcd}$$

satisfying
$$[d, 1\otimes h] = 1 \otimes [d,h] = 1\otimes 1 - 1 \otimes f = 1_{A\otimes_R U} - \varphi\mu$$
$$[d, h] = 1 - f = 1 - \mu\varphi$$

so that φ is a homotopy inverse for $\mu : A \otimes_R U \longrightarrow U$. (Notice also the compatibility of the homotopies with μ: $\mu(1\otimes h) = h\mu$ which means we have a contraction on Cone(μ):

$$\left[\begin{pmatrix} d & \mu \\ & -d \end{pmatrix}, \begin{pmatrix} h & 0 \\ \varphi & -1\otimes h \end{pmatrix}\right] = \begin{pmatrix} dh+hd+\mu\varphi & -\mu(1\otimes h)+h\mu \\ -d\varphi+\varphi d & d(1\otimes h)+(1\otimes h)d+\varphi\mu \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1\otimes 1 \end{pmatrix}.)$$

We can iterate this htpy equiv. to higher order:

$$\begin{tikzcd} \cdots \arrow[r, shift left] & A^{(2)}\otimes_R U \arrow[l, shift left, "1\otimes 1\otimes\varphi"] \arrow[r, shift left, "1\otimes\mu = \mu\otimes 1"] & A\otimes_R U \arrow[l, shift left, "1\otimes\varphi"] \arrow[r, shift left, "\mu"] & U \arrow[l, shift left, "\varphi"] \end{tikzcd}$$

Let's call a complex E of R-modules <u>h-firm</u> (wrt the ideal A) when $\mu : A\otimes_R E \longrightarrow E$ is a homotopy equivalence. Formally it then follows that $M \longmapsto M\otimes_R E$ from right modules to complexes carries a nil-isom. into a h. equivalence.

For example consider $Q\otimes_S P \twoheadrightarrow A$, whose kernel K is killed by A

$$\begin{tikzcd} K\otimes_R A \arrow[r] \arrow[dr, "0"'] & Q\otimes_S P\otimes_R A \arrow[r, two heads] \arrow[d, "1\otimes\mu"] & A\otimes_R A \arrow[d, "\mu"] \arrow[dl, dashed, "\exists"'] \\ & Q\otimes_S P \arrow[r] & A \end{tikzcd}$$

because $\Big(\underbrace{\sum q_1\otimes p_1}_{\in K}\Big)\otimes(q_2p_2) \xmapsto{1\otimes\mu} \sum q_1\otimes p_1q_2p_2 = \left(\sum q_1p_1\right)q_2\otimes p_2 = 0$

Better method is to observe that the two maps $Q\otimes_S P\otimes_R Q\otimes_S P \longrightarrow Q\otimes_S P$ sending $q_1\otimes p_1\otimes q_2\otimes p_2$ to $q_1p_1q_2\otimes p_2$ and $q_1\otimes p_1q_2p_2$ resp. coincide. The former factors through $A\otimes_R Q\otimes_S P = \underbrace{Q\otimes P\otimes Q\otimes P}_{\lambda}/K\otimes Q\otimes P$, the latter factors through $Q\otimes_S P\otimes_R A = \lambda/Q\otimes P\otimes K$, hence we get a well-defined map

$$A\otimes_R A \longrightarrow Q\otimes_S P$$

such that $q_1p_1\otimes q_2p_2 \longmapsto q_1p_1q_2\otimes p_2 = q_1\otimes p_1q_2p_2$.

Anyway from the commutative diagram of R-bimodules

$$\begin{tikzcd}
Q\otimes_S P\otimes_R A \arrow[r] \arrow[d, "1\otimes\mu"] & A\otimes_R A \arrow[dl] \arrow[d, "\mu"] \\
Q\otimes_S P \arrow[r] & A
\end{tikzcd}$$

we get a comm. diagram of complexes

$$\begin{tikzcd}
Q\otimes_S P\otimes_R A\otimes_R U \arrow[r] \arrow[d] & A\otimes_R A\otimes_R U \arrow[dl] \arrow[d] \\
Q\otimes_S P\otimes_R U \arrow[r] & A\otimes_R U
\end{tikzcd}$$

where the vertical maps are h.eqs since U is h-firm. Conclude $Q\otimes_S P\otimes_R U \longrightarrow A\otimes_R U$ is a h. equiv.

Next consider $P\otimes_R U$. We know this is h-firm, since $-\otimes_S P$ carries B-nil isos to A-nil-isos.)

and $-\otimes_R U$ carries A-nil isos to h. equivs. We want to show that $P\otimes_R U$ is h. eq. to a s-perfect complex.

We know the obvious map $Q\otimes_S P\otimes_R U \xrightarrow{v} U$ is a homotopy equivalence. Let $\varphi : U \longrightarrow Q\otimes_S P\otimes_R U$ be a homotopy inverse so that in particular $v\varphi \sim 1_U$. Because U is strictly perfect, φ is equivalent to a 0-cycle in

$$(U^*\otimes_R Q)\otimes_S (P\otimes_R U) \simeq \mathrm{Hom}_R(U, Q\otimes_S P\otimes_R U)$$

Thus for each degree n we can write

$$\varphi_n = \sum_{j=1}^{\nu} \xi_j \otimes v_j \qquad \begin{array}{l} \xi_j \in \mathrm{Hom}_R(U_n, Q) \\ v_j \in P\otimes_R U_n \end{array}$$

so φ_n factors

$$\begin{tikzcd}
U_n \arrow[rr, "\varphi_n"] \arrow[dr, "(\xi_j)"'] & & Q\otimes_S P\otimes_R U_n \\
& Q^\nu \arrow[ur, "(1\otimes v_j)"'] &
\end{tikzcd}$$

and $1\otimes v\varphi_n$ factors

$$\begin{tikzcd}
P\otimes_R U_n \arrow[rr, "1\otimes\varphi_n"] \arrow[dr, "(1\otimes\xi_j)"'] & & P\otimes_R Q\otimes_S P\otimes_R U_n \arrow[rr, "{1\otimes_P v = \text{mult}\otimes 1_{U_n}}"] & & P\otimes_R U_n \\
& (P\otimes_R Q)^\nu \arrow[ur, "(1\otimes 1\otimes v_j)"'] \arrow[rr, "\text{mult.}"] & & S^\nu \arrow[ur, "(v_j)"'] &
\end{tikzcd}$$

Thus $1\otimes v\varphi_n$ is nuclear. This shows that the identity map of $P\otimes_R U$ has a deformation $1\otimes v\varphi$ which is nuclear. (I should have mentioned that U being bdd, there are only finitely many $\varphi_n \neq 0$.)

So we see that $P \otimes_R U$ is h. eq to a s-perfect complex.

[redacted] We have now shown that if U is s-perfect + h-firm, then [redacted] $P \otimes_R U$ is h-perfect + h-firm. It follows that U is h-perfect + h-firm $\Longleftrightarrow$ $P \otimes_R U$ is h-perfect and h-firm.

Additional comments.

1. Suppose we only assume that U is an h-firm complex of projective modules. Then we should still be able to factor φ_n

$$\begin{tikzcd}
U_n \arrow[rr, "\varphi_n"] \arrow[dr, "(\xi_j)"'] & & Q \otimes_S P \otimes_R U_n \\
& \bigoplus_J Q \arrow[ur, "(1 \otimes v_j)"'] &
\end{tikzcd}$$

In effect we can add [redacted] $U_n^\perp$ [redacted] to U_n to make it free, + replace (U_n, φ_n) by $(U_n + U_n^\perp, \varphi_n + 0)$. [redacted] If $U_n = \bigoplus_I R$, then factor each summand R as above and take the direct sums. J is then a disjoint union of finite sets indexed by I.

Hence it's clear that $1 \otimes v\varphi_n : P \otimes_R U_n \to P \otimes_R U_n$ factors through a free S-module for each n. This means that $P \otimes_R U$ is an h-retract of a complex of free S-modules, hence h. eq. to a complex of free S-modules which is right bdd if U is.

This checks the previous result that $P \otimes_R U$ is heq to a complex of projectives when U is an h-firm complex of projectives. (Things appear slightly better since the complexes need not be right bdd.)

2. Suppose 1_U has a deformation φ which factors degreewise as follows:

$$\begin{tikzcd} U_n \arrow[rr, "\varphi_n"] \arrow[dr, "i_n"'] & & U_n \\ & A^{\nu_n} \subset R^{\nu_n} \arrow[ur, "j_n"'] & \end{tikzcd}$$

Then we know that U is an h-retract of $T = \bigoplus T_n$, $T_n = R^{\nu_n}$, with differential idj. Moreover $\tilde{i} : U \longrightarrow T$ has image contained in AT since $\tilde{i} = i(1-dh)$. Thus $1 - j\tilde{i} = [d, \tilde{h}]$ on U reduces to $1 = [d, \bar{h}]$ on U/AU, so that U/AU is contractible. But $\bar{h}$ has the lifting $\tilde{h}$, so it's clear that U is h-firm. <u>NO</u>, we don't have $A \otimes_R U = AU$.

May 11, 1995

It seems to be worthwhile to understand length one complexes better. Reasons: Any perfect firm complex can be taken apart into length one perfect firm complexes in a certain sense. Also the Atiyah-Bott-Shapiro treatment of relative K.

I recall that a map of length one complexes

$$\begin{array}{ccc} X_1 & \xrightarrow{d_X} & X_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ Y_1 & \xrightarrow{d_Y} & Y_0 \end{array}$$

is a quis iff

$$(*) \qquad 0 \longrightarrow X_1 \xrightarrow[i = \begin{pmatrix} -d_X \\ f_1 \end{pmatrix}]{} X_0 \oplus Y_1 \xrightarrow[j = (f_0 \; d_Y)]{} Y_0 \longrightarrow 0$$

is exact, and it is a homotopy equivalence iff this sequence is split exact. In fact a splitting of the sequence is equivalent to homotopy inverse data for f, that is $g: Y \to X$ and homotopy operators h_X, h_Y such that $1-gf = [d,h_X]$, $1-fg = [d,h_Y]$. (more needed; see below.) The correspondence is given by

$$0 \longrightarrow X_1 \underset{i = \begin{pmatrix} -d_X \\ f_1 \end{pmatrix}}{\overset{r = (-h_X \; g_1)}{\rightleftarrows}} X_0 \oplus Y_1 \underset{j = (f_0 \; d_Y)}{\overset{s = \begin{pmatrix} g_0 \\ h_Y \end{pmatrix}}{\rightleftarrows}} Y_0 \longrightarrow 0$$

Let's check this statement. It seems you have to assume the homotopies h_X, h_Y are compatible with

respect to either f or g, i.e.
either $-f_1 h_x + h_y f_0 \sim 0$ or

$$-h_x g_0 + g_1 h_y \sim 0$$

Note that for degree reasons $\sim 0 \Rightarrow = 0$ in these cases, i.e. $[f,h] : X \longrightarrow Y$ has degree $+1$, so $[f,h] = [d,k]$ means $[f,h]=0$ since k has degree 2.

Observe that

$$ri = h_x d + g_1 f_1 = 1$$

$$js = f_0 g_0 + d h_y = 1$$

$$ir + sj = \begin{pmatrix} -d \\ f_1 \end{pmatrix} (h_x \ \ g_1) + \begin{pmatrix} g_0 \\ h_y \end{pmatrix} (f_0 \ \ d_y)$$

$$= \begin{pmatrix} dh_x + g_0 f_0 & -dg_1 + g_0 d \\ -f_1 h_x + h_y f_0 & f_1 g_1 + h_y d_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f_1 h_x + h_y f_0 & 1 \end{pmatrix}$$

$$rs = -h_x g_0 + g_1 h_y$$

Now

$$ir + sj = 1 \iff -f_1 h_x + h_y f_0 = 0$$

$$\Downarrow \ irs + s\overbrace{js}^{1} = s$$

$$\Rightarrow \ irs = 0 \Rightarrow rs = rirs = 0$$

and $rs = 0 \iff -h_x g_0 + g_1 h_y = 0$.

Conversely $\overbrace{\ }^{\text{assume}}$ $rs = 0$, and note that $ir + sj = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ is invertible. But

$$(1-ir-sj)i = i - iri = i - i = 0$$

$$(1-ir-sj)s = s - \cancel{irs} - sjs = s - s = 0$$

so $(1-ir-sj)(ir+sj) = 0 \implies 1 = ir+sj$.

Now recall what a contraction for Cone(f) looks like:

$$\left[\begin{pmatrix} -d_X & \\ f & d_Y \end{pmatrix}, \begin{pmatrix} -h_X & g \\ u & h_Y \end{pmatrix}\right] = \begin{pmatrix} [d,h_X]+gf & -[d,g] \\ -[f,h]+[d,u] & [d,h_Y]+fg \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So a contraction for Cone(f) satisfies $[f,h]=0$. Similarly a contraction for Cone(g) satisfies $[g,h]=0$.

It would have been simpler to point out that $(*)$ is Cone(f). Then a contraction for Cone(f) satisfies $[f,h]=0$, so one doesn't have to compute $ir+sj$.

The problem with the cone is that it is a complex of length 2. Here's a criterion inside length one complexes.

<u>Prop.</u> X and Y are homotopy equivalent $\iff$ $\exists$ an isomorphism $X \oplus (Y_0 \xrightarrow{1} Y_0) \cong (X_0 \xrightarrow{1} X_0) \oplus Y$

$\Leftarrow$ is obvious

$\Rightarrow$ we prove by a formula for this isom:

$$\begin{tikzcd}
X_1 \oplus Y_0 \arrow[rr, "\begin{pmatrix} d_x & \\ & 1 \end{pmatrix}"] \arrow[d, shift right, "\begin{pmatrix} -d_x & g_0 \\ f_1 & h_y \end{pmatrix}"'] & & X_0 \oplus Y_0 \arrow[d, shift right, "\begin{pmatrix} -1 & g_0 \\ f_0 & d_y h_y \end{pmatrix}"'] \\
X_0 \oplus Y_1 \arrow[rr, "\begin{pmatrix} 1 & \\ & d_y \end{pmatrix}"'] \arrow[u, shift right, "\begin{pmatrix} -h_x & g_1 \\ f_0 & d_y \end{pmatrix}"'] & & X_0 \oplus Y_0 \arrow[u, shift right, "\begin{pmatrix} -d_x h_x & g_0 \\ f_0 & 1 \end{pmatrix}"']
\end{tikzcd}$$

$$\begin{pmatrix} -d_x & g_0 \\ f_1 & h_y \end{pmatrix}\begin{pmatrix} -h_x & g_1 \\ f_0 & d_y \end{pmatrix} = \begin{pmatrix} d_x h_x + g_0 f_0 & -d_x g_1 + g_0 d_y \\ -f_1 h_x + h_y f_0 & f_1 g_1 + h_y d_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -d_x h_x & g_0 \\ f_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & d_y \end{pmatrix} = \begin{pmatrix} -d_x h_x & g_0 d_y \\ f_0 & d_y \end{pmatrix} = \begin{pmatrix} d_x & \\ & 1 \end{pmatrix}\begin{pmatrix} -h_x & g_1 \\ f_0 & d_y \end{pmatrix}$$

$$\begin{pmatrix} -d_x h_x & g_0 \\ f_0 & 1 \end{pmatrix}\begin{pmatrix} -1 & g_0 \\ f_0 & d_y h_y \end{pmatrix} = \begin{pmatrix} d_x h_x + g_0 f_0 & -d_x h_x g_0 + g_0 d_y h_y \\ 0 & f_0 g_0 + d_y h_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

etc.

<u>May 16, 1995</u>

A - nonunital ring. sphf(A) = category of strictly perfect complexes U over $\tilde{A}$ which a homotopy firm wrt A, i.e. $U/AU \sim 0$. Define $K_0(\text{sphf}(A))$ to be the abelian group generated by $[U]$ depending on the isom. class of U subject to the relations

i) $[U \oplus U'] = [U] + [U']$

ii) $U \sim 0 \implies [U] = 0$

iii) $0 \to U' \to U \to U'' \to 0$

exact $\implies [U] = [U'] + [U'']$.

Define $K_0'(A)$ to be the abelian group with generators $[U]$, for each sphf complex of length 1 : $U_1 \xrightarrow{d} U_0$ s.t. $U_1/AU_1 \xrightarrow{\sim} U_0/AU_0$, where $[U]$ depends only on U up to isomorphism, with the same relations:

1) $[U \oplus U'] = [U] + [U']$

2) $U \sim 0$ (i.e. $U_1 \xrightarrow[\sim]{d} U_0$) $\implies [U] = 0$.

3) $0 \to U' \to U \to U'' \to 0$ exact $\implies$

$[U] = [U'] + [U'']$.

restricted to length one complexes.

I would like to check that in the presence of 1) + 2), the relation 3) is equivalent to

3') If $U_1 \xrightarrow{f} U_0$, $U_0 \xrightarrow{g} V_0$ are length one sphf complexes, then $[U_1 \xrightarrow{f} U_0] + [U_0 \xrightarrow{g} V_0] = [U_1 \xrightarrow{gf} V_0]$.

$3) \Rightarrow 3')$ follows from the general result relating $\mathrm{Cone}(gf)$ to $\mathrm{Cone}(f)$, $\mathrm{Cone}(g)$ in the case of maps of complexes $X \xrightarrow{f} Y \xrightarrow{g} Z$.

$\mathrm{Cone}(f) \qquad \mathrm{Cone}\, f \cup_Y \mathrm{Cyl}(g) \qquad \mathrm{Cone}(g)$

namely we have an exact sequence

$$0 \to \mathrm{Cone}(f) \longrightarrow \underbrace{\mathrm{Cone}(f) \cup_Y \mathrm{Cyl}(g)} \longrightarrow \mathrm{Cone}(g) \longrightarrow 0$$

together with a SDR of $\uparrow$ onto $\mathrm{Cone}(gf)$. $1)+2)$ together with this SDR gives $[\mathrm{Cone}(f) \cup_Y \mathrm{Cyl}(g)] = [\mathrm{Cone}(gf)]$

$3') \Rightarrow 3)$. Suppose given an exact sequence of length 1 split complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_1 & \longrightarrow & V_1 & \longrightarrow & W_1 & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \\
0 & \longrightarrow & U_0 & \longrightarrow & V_0 & \longrightarrow & W_0 & \longrightarrow & 0
\end{array}$$

Because W_n is projective this sequence splits locally, so we can assume $V_i = U_i \oplus W_i$, $i = 0, 1$ and $f = \begin{pmatrix} f' & u \\ & f'' \end{pmatrix} : U_1 \oplus W_1 \longrightarrow U_0 \oplus W_0$.

But f factors

$$\begin{pmatrix} f' & u \\ & f'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & f'' \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} f' & 0 \\ 0 & 1 \end{pmatrix}$$

i.e.

$$\begin{array}{c} U_1 \\ \oplus \\ W_1 \end{array} \xrightarrow{\begin{pmatrix} f' & 0 \\ 0 & 1 \end{pmatrix}} \begin{array}{c} U_0 \\ \oplus \\ W_1 \end{array} \xrightarrow[\sim]{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}} \begin{array}{c} U_0 \\ \oplus \\ W_1 \end{array} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & f'' \end{pmatrix}} \begin{array}{c} U_0 \\ \oplus \\ W_0 \end{array}$$

By 3') we have $[f] = \left[\begin{pmatrix} f' & 0 \\ 0 & 1 \end{pmatrix}\right] + \left[\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right] + \left[\begin{pmatrix} 1 & 0 \\ 0 & f'' \end{pmatrix}\right]$

$$= [f'] + [f'']$$

There's an obvious map

$$K_0'(A) \longrightarrow K_0(\mathrm{sphf}(A))$$

induced by the inclusion of length 1 complexes.

We next define a map in the opposite direction. Given U in $\mathrm{sphf}(A)$, let $\bar{U} = U/AU$ and $U^\# = \tilde{A} \otimes_{\mathbb{Z}} \bar{U}$. $\bar{U}$ is contractible over $\mathbb{Z}$, hence $U^\#$ is contractible over $\tilde{A}$. One has a canonical iso $\overline{U^\#} = \bar{U}$. This isomorphism can be lifted to a map $U^\# \xrightarrow{f} U$ unique up to homotopy. In effect one has

$$\begin{tikzcd} & 0 \arrow[d] \\ & AU \arrow[d] \\ U^\# \arrow[r, dashed, "f"] \arrow[dr] & U \arrow[d] \\ & \overline{U^\#} = \bar{U} \arrow[d] \\ & 0 \end{tikzcd}$$

and because $U^\#$ is projective, the obstruction to the existence + h-uniqueness lie in $H_* \mathrm{Hom}_{\tilde{A}}(U^\#, AU)$, which is zero as $U^\#$ is contractible.

On the other hand there is also a lifting $g : \mathcal{U} \to \mathcal{U}^\#$ unique up to homotopy

$$\begin{tikzcd}
 & A\mathcal{U}^\# \arrow[d] \\
\mathcal{U} \arrow[r, "g"] \arrow[dr] & \mathcal{U}^\# \arrow[d] \\
 & \overline{\mathcal{U}^\#}
\end{tikzcd}$$

for the obstructions lie in $H_* \operatorname{Hom}_{\tilde{A}}(\mathcal{U}, A\mathcal{U}^\#)$, which is zero as $A\mathcal{U}^\# = A \otimes_{\mathbb{Z}} \tilde{\mathcal{U}}$ is contractible.

If $\mathcal{U}, V$ are two strictly perfect cxs of $\tilde{A}$ modules and $f : \mathcal{U} \to V$ is a map such that $\bar{f} : \bar{\mathcal{U}} \xrightarrow{\sim} \bar{V}$, then the cone on f is the total complex of

$$\begin{array}{ccccccc}
\to & \mathcal{U}_2 & \xrightarrow{-d} & \mathcal{U}_1 & \xrightarrow{-d} & \mathcal{U}_0 \\
 & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\
\to & V_2 & \xrightarrow{d} & V_1 & \xrightarrow{d} & V_0
\end{array}$$

The columns are length one sphf complexes, so

$$\chi(\mathcal{U} \xrightarrow{f} V) = \sum_q (-1)^q [f_q : \mathcal{U}_q \to V_q] \in K_0'(A)$$

is defined.

Properties: a) $\chi(\mathcal{U} \xrightarrow{f} V) + \chi(V \xrightarrow{g} W) = \chi(\mathcal{U} \xrightarrow{gf} W)$

b) If $\mathcal{U}, V$ are contractible and $f : \mathcal{U} \to V \ni \bar{f} \cong$, then $\chi(\mathcal{U} \xrightarrow{f} V) = 0$.

a) is clear from 3') property above.

For b) use the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_q U \arrow[r] \arrow[d, "Z_q f"] & U_q \arrow[r] \arrow[d, "f_q"] & Z_{q-1} U \arrow[r] \arrow[d, "Z_{q-1} f"] & 0 \\ 0 \arrow[r] & Z_q V \arrow[r] & V_q \arrow[r] & Z_{q-1} V \arrow[r] & 0 \end{tikzcd}$$

to get $\forall q$, $[f_q] = [Z_q f] + [Z_{q-1} f]$ in $K_0'(A)$.

whence $\sum (-1)^q [f_q] = 0$.

Now restrict to U in sphf(A), and choose liftings $f: U^\# \to U$, $g: U \to U^\#$ as above. We have

$$\chi(f: U^\# \to U) + \chi(g: U^\# \to U) = \underset{\substack{\| \\ 0}}{\chi(gf: U^\# \to U^\#)}$$

This shows $\chi(f: U^\# \to U)$ is independent of the choice of f.

We now show there is a well-defined map $K_0(\mathrm{sphf}(A)) \longrightarrow K_0'(A)$ sending $[U]$ to $\chi(f: U^\# \to U)$. We have to check the relations i) - iii) are satisfied.

i) $\chi(f: U^\# \to U) + \chi(f': U'^\# \to U') = \chi(f \oplus f': U^\# \oplus U'^\# \to U \oplus U')$.

ii) $U \sim 0 \Rightarrow$ both $U, U^\# \sim 0$ so $\chi(f: U^\# \to U) = 0$.

iii) Suppose $0 \to U' \to U \to U'' \to 0$ a given exact sequence in sphf(A). Then $0 \to \bar{U}' \to \bar{U} \to \bar{U}'' \to 0$ is an exact sequence of contractible complexes over $\mathbb{Z}$, so it splits. Thus one has

$$\begin{tikzcd} 0 \arrow[r] & U'^{\#} \arrow[r, shift right, "i^{\#}"'] \arrow[d, "f'"'] & U^{\#} \arrow[l, shift right, "r"'] \arrow[r, shift right, "p^{\#}"'] \arrow[d, dashed, "f"] & U''^{\#} \arrow[l, shift right, "\ell"'] \arrow[r] \arrow[d, "f''"] \arrow[dl, dashed, "\varphi"] & 0 \\ 0 \arrow[r] & U' \arrow[r, "i"'] & U \arrow[r, "p"'] & U'' \arrow[r] & 0 \end{tikzcd}$$

where we've chosen f', f'' lifting the canonical isos modulo A. ~~Define~~

Because $U''^{\#} \sim 0$ the lifting φ of f'' $\exists$.

Define f to be $i f' r + \varphi p^{\#}$. Then we have a map of exact sequences of complexes

$$\begin{tikzcd} 0 \arrow[r] & U'^{\#} \arrow[r] \arrow[d, "f'"] & U^{\#} \arrow[r] \arrow[d, "f"] & U''^{\#} \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & U' \arrow[r] & U \arrow[r] & U'' \arrow[r] & 0 \end{tikzcd}$$

where the vertical arrows are isos modulo A, hence $[f_q] = [f'_q] + [f''_q]$ in $K'_0(A)$. $\therefore$

$$\chi(f : U^{\#} \to U) = \chi(f' : U'^{\#} \to U') + \chi(f'' : U''^{\#} \to U'')$$

proving iii).

Final step is to check the maps

$$K'_0(A) \longrightarrow K_0(\mathrm{sphp}(A)) \longrightarrow K'_0(A)$$

are inverse. The first map is onto since given U we know

$$[U] = \underset{\substack{\| \\ 0}}{[U^{\#}]} + \underset{\substack{\| \\ \sum (-1)^q [U_q^{\#} \xrightarrow{f_q} U_q]}}{[\mathrm{Cone}(f : U^{\#} \to U)]} \in \text{Image of } K'_0 A,$$

So it remains to show the composition is 1.

Take ~~acyclic~~ a length one split complex U: $P \xrightarrow{d} Q$. Then we have

$$\begin{tikzcd} P^{\#} \arrow[r, phantom, "\simeq"] \arrow[d, "f_1"'] & Q^{\#} \arrow[d, "f_0"] \\ P \arrow[r, "d"] & Q \end{tikzcd}$$

so $\chi(f : U^{\#} \to U) = [f_0 : Q^{\#} \to Q] - [f_1 : P^{\#} \to P]$

in $K_0'(A)$. But

$$\left[P^{\#} \xrightarrow{f_0 d = d f_1} Q\right] = [f_0] + \overbrace{[d : P^{\#} \xrightarrow{\simeq} Q^{\#}]}^{0}$$

$$= [d : P \to Q] + [f_1]$$

$\therefore$ $[U] = [f_0] - [f_1] = \chi(f : U^{\#} \to U)$ as desired.

July 1, 1995

There is a serious problem with linking $K_1 A$, for A nonunital & $A = A^2$, to $K_0' A$ defined via perfect firm complexes. [redacted] For example when A is a radical ring $(A = J(A))$, [redacted] there are no such complexes except contractible ones while $K_1 A$ can be non-trivial.

A better way to say this is that the idea of obtaining the higher $K_n' A$ from the category of perfect firm complexes up to homotopy, [redacted] say via Waldhausen theory, seems not to work for a radical ring. One might still hope to use perfect firm complexes, [redacted] but some new ideas are needed.

It seems $K_0 R$ can be obtained as K_1 of the suspension of R for R unital. [redacted] Also $K_0 R$ sits naturally as a summand of $K_1(R[z,z^{-1}])$ (Bass). Note that $A = A^2 \Longrightarrow A[z,z^{-1}]$ has the same property. So perhaps (under some extra h-unital conditions on A) one can find the "good" $K_0 A$ sitting as a summand of the Vaserstein $K_1(A[z,z^{-1}])$.

So it seems worthwhile to understand Bass' theory for $K_1(A[z,z^{-1}])$. There ~~[redacted]~~ are canonical maps

$$K_0 R \longrightarrow K_1(R[z,z^{-1}]) \longrightarrow K_0 R$$

with composition equal to the identity. These [redacted] maps come from the Atiyah-Bott elementary proof of the periodicity theorem. The former assigns to an idempotent matrix e over R the invertible matrix $zp+1-p$ over $R[z,z^{-1}]$. The latter ~~assigns to~~ constructs from $g \in GL_n(R[z,z^{-1}])$ the "rank n vector bundle" E_g over $\mathbb{P}^1_R$ obtained by clutching and assigns to $[g] \in K_1(R[z,z^{-1}])$ the class $[R\Gamma(E_g(-1))] \in K_0R$ (roughly).

An important step in the Atiyah-Bott proof is studying linear families of maps

$$az+b : V \longrightarrow W \qquad z \in \mathbb{C}$$

(V, W two f.d. vector spaces or bundles) such that $az+b$ is invertible for $z \in S^1$. Such a family splits canonically into two pieces where on the first $az+b$ is invertible inside S^1 and on the second it is invertible outside (including $z=\infty$). The projection operator is

$$\frac{1}{2\pi i}\oint (az+b)^{-1}\,a\,dz \qquad \text{on } V$$

(and probably $\displaystyle \frac{1}{2\pi i}\oint a\,dz\,\frac{1}{(az+b)}$ on W).

Let's discuss this splitting result algebraically. Suppose given $az+b \in M_n(R[z]) \cap GL_n(R[z,z^{-1}])$. Replace R by M_nR to reduce to the case $az+b \in GL(R[z,z^{-1}])$ with $a, b \in R$.

Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & R[z] \oplus R[z^{-1}]z^{-1} \arrow[r, "\sim"] \arrow[d, "{(az+b\cdot,\ az+b\cdot)}"] & R[z,z^{-1}] \arrow[d, "\cong"', "az+b\cdot"] & \\
0 \arrow[r] & R \arrow[r, "\Delta"] \arrow[dr, "\sim"'] & R[z] \oplus R[z^{-1}] \arrow[r, "in_+ - in_-"] \arrow[d] & R[z,z^{-1}] \arrow[r] & 0 \\
 & & R[z]/(az+b)R[z] \oplus R[z^{-1}]/(a+bz^{-1})R[z^{-1}] \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Work with right module structure so that left multiplication $az+b\cdot$ is a module map. The above diagram leads to a canonical isomorphism

$$R \xrightarrow{\sim} R[z]/(az+b)R[z] \oplus R[z^{-1}]/(a+bz^{-1})R[z^{-1}]$$

of right R-modules. This means we have a decomposition $R = eR \oplus e^{\perp}R$ where e is an idempotent in R. Now we find e. This corresponds to $(1,0)$ in the right above.

Lift to $(1,0)$ in $R[z] \oplus R[z^{-1}]$, which goes to $1 \in R[z,z^{-1}]$ and corresponds under the isomorphisms to

$$\left((az+b)^{-1}_{+} ,\ (az+b)^{-1}_{-} \right) \in R[z] \oplus R[z^{-1}]z^{-1}$$

Let $(az+b)^{-1} = \sum c_k z^k$ so that

$$(az+b)\sum c_k z^k = \sum c_k z^k (az+b) = 1 \quad \text{i.e.}$$

$$a c_{k-1} + b c_k = c_{k-1} a + c_k b = \delta_{k0}.$$

Then

$$(az+b)^{-1}_{+} = \sum_{k\geq 0} c_k z^k \qquad (az+b)^{-1}_{-} = \sum_{k<0} c_k z^k$$

$$(az+b)(az+b)^{-1}_{+} = \sum_{k\geq 0} ac_k z^{k+1} + bc_k z^k$$

$$= bc_0 + \sum_{k\geq 1} \overbrace{(ac_{k-1}+bc_k)}^{\delta_{k0}=0} z^k = bc_0$$

$$(az+b)(az+b)^{-1}_{-} = \sum_{k<0} ac_k z^{k+1} + bc_k z^k$$

$$= ac_{-1} + \sum_{k\leq -1} (ac_{k-1}+bc_k) z^k = ac_{-1}$$

Then $(1,0) - (bc_0, ac_{-1}) = (1-bc_0, ac_{-1}) = \Delta(ac_{-1})$.

Thus $\boxed{e = ac_{-1} = \operatorname{Res}\left(a(az+b)^{-1}dz\right)}$

Let's interpret the preceding argument. Go back to a family $az+b : V \longrightarrow W$ with V, W f.d vector spaces over $\mathbb{C}$, and assume $(az+b)^{-1}$ $\exists$ for all $z \neq 0, \infty$. Then we obtain a coherent sheaf F over $\mathbb{CP}^1$ defined by

$$0 \longrightarrow \mathcal{O}(-1)\otimes V \xrightarrow{az+b} \mathcal{O}\otimes W \longrightarrow F \longrightarrow 0$$

where F has support $\subset \{0,\infty\}$. F is regular (Mumford) and we have canonical isos.

$$W \xrightarrow{\sim} \Gamma(F) \qquad \Gamma(F(-1)) \xrightarrow{\sim} H^1(\mathcal{O}(-2)\otimes V) \simeq V.$$

The obvious decomposition of F into $\underbrace{F^+}_{\text{supp}\subset\{0\}} \oplus \underbrace{F^-}_{\text{supp}\subset\{\infty\}}$ then induces the corresponding splitting of the family $az+b$.

Interesting point: One has managed to extend the notions of spectrum and characteristic polynomial from operators on a f.d. vector space W to certain correspondences

$$\begin{array}{ccc} V & \xrightarrow{b} & W \\ {\scriptstyle a}\downarrow & & \\ W & & \end{array}$$

, namely those which are transverse to the graph of multiplication by z for generic z. Thus $az+b : V \xrightarrow{\sim} W \iff (a,b)V \subset W$ is transversal to $(1,-z)W$ = kernel of $(w_1, w_2) \longmapsto zw_1 + w_2$. The spectral sheaf is F as above (which generalizes V as a $\mathbb{C}[z]$-module with z acting as the operator). The characteristic polynomial of the correspondence is the divisor of F, which should resulting from the map

$$\begin{array}{ccc} \Lambda^{max}(\mathcal{O}(-1)\otimes V) & \longrightarrow & \Lambda^{max}(\mathcal{O}\otimes W) \\ \| & & \\ \mathcal{O}(-d)\otimes \Lambda^d V & \longrightarrow & \mathcal{O}\otimes \Lambda^d W \end{array}$$

This gives a homogeneous poly of degree d in the homogeneous coords (z_0, z_1) well-defined up to a scalar factor.

$z = \infty$ is not an eigenvalue $\iff$ a^{-1} exists in which case we have the char poly of $-a^{-1}b$ (or $-ba^{-1}$) (conjugate)

$z = 0$ is not an eigenvalue $\iff$ b^{-1} exists in which case the char poly $= |z - b^{-1}a|$.

Changes of coordinates: assume $az+b$ is invertible at $z=1$. Then

$$(a+b)^{-1}(az+b)$$

has the same spectrum, which reduces to the case of $za+(1-a)$. Now put $z=1-\frac{1}{w}$ so that $z=0,\infty$ corresp. to $w=1,0$ resp. Then

$$za+(1-a) = 1-\frac{1}{w}a = \frac{1}{w}(w-a)$$

so that we're looking for the spectrum of a.

Note that $za+(1-a) \in GL(R[z,z^{-1}]) \iff a(1-a)$ is nilpotent.

Proof. ($\Leftarrow$) $\big(za+(1-a)\big)\big(z^{-1}a+(1-a)\big) =$

$$a^2+(1-a)^2+(z+z^{-1})a(1-a) = 1+\underbrace{(z-2+z^{-1})a(1-a)}_{\text{nilpotent}}$$

is invertible showing $za+(1-a)$ is invertible over $R[z,z^{-1}]$.

($\Rightarrow$) If $\big(za+(1-a)\big)h(z)=1$ with h invertible over $R[z,z^{-1}]$, then $h(z)h(z^{-1})$ is an inverse for $1+(z-2+z^{-1})a(1-a)$. Now $h(z)h(z^{-1})$ is a matrix over the subring of $R[z,z^{-1}]$ consisting of Laurent series invariant under $z\mapsto z^{-1}$. This is a poly ring $R[x]$, where $x=z-2+z^{-1}$. (I should have pointed out that $za+(1-a)$ and $z^{-1}a+(1-a)$ commute, so that their inverse $h(z)$ and $h(z^{-1})$ also commute.) Thus $1+x\,a(1-a)$ is invertible in $R[x]$, which implies $a(1-a)$ is nilpotent.

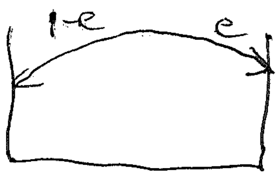
Question: Can any of this $\mathbb{P}^1$ stuff shed light on homotopy idempotents?

Note that

$$\begin{tikzcd}
 & T[z] \oplus z^{-1}T[z^{-1}] \arrow[d, "{(az+b,\, az+b)}"] \arrow[r, equal] & T[z,z^{-1}] \arrow[d, "az+b", "\sim"'] \\
T \arrow[r] & T[z] \oplus T[z] \arrow[r] \arrow[d] & T[z,z^{-1}] \\
 & \mathrm{Cone}\left(T[z] \circlearrowleft^{az+b}\right) \oplus \mathrm{Cone}\left(T[z^{-1}] \circlearrowleft^{a+bz^{-1}}\right) &
\end{tikzcd}$$

$b = 1-a$

is closely related to things we examined before: telescopes made of



At this point I want to leave $az+b$ and look at a more general $g \in GL(R[z,z^{-1}])$. Again take $g \in GL_1(R[z,z^{-1}])$. Associate to g the length 1 complex

$$\circledast \qquad R[z] \oplus g z^{-1} R[z^{-1}] \xrightarrow{(in,\,-in)} R[z,z^{-1}]$$

Note that if $g = 1$ this complex is acyclic and if $g = z$ it has homology R on the left only.

This complex should be the Čech complex for $E_g(-1)$ over $\mathbb{P}^1$. Observe it's isomorphic to

$$zR[z] \oplus g\, R[z^{-1}] \xrightarrow{(in,\,-in)} R[z,z^{-1}]$$

If $g \in R[z^{-1}]$, then $gR[z^{-1}] \subset R[z^{-1}]$ so that the above complex is h.equiv to

$$gR[z^{-1}] \hookrightarrow R[z,z^{-1}]/zR[z] = R[z^{-1}]$$

The homology on the right is

$$R[z^{-1}]/gR[z^{-1}].$$

Actually $g \in R[z^{-1}]$ is unnecessary. One has $zR[z] \xrightarrow{1} zR[z]$ as a contractible subcomplex so that $\circledast$ is quis to

$$\begin{array}{ccc} gR[z^{-1}] & \longrightarrow & R[z,z^{-1}]/zR[z] \\ \uparrow g & & \| \\ R[z^{-1}] & & R[z^{-1}] \end{array}$$

which is a (kind of) Toeplitz operator on $R[z^{-1}]$ associated to g.

Consider

$$g^{-1}R[z] \oplus z^{-1}R[z^{-1}] \xrightarrow{(-\mathrm{in},+\mathrm{in})} R[z,z^{-1}]$$

which is isomorphic to $\circledast$. This is quis to

$$R[z] \xrightarrow{g^{-1}} g^{-1}R[z] \subset R[z,z^{-1}] \twoheadrightarrow R[z]$$

where the map is the Toeplitz operator on $R[z]$ associated to g^{-1}. To get the correct sign I should replace g by g^{-1} and consider the above complex as a chain complex with the left of degree $+1$.

So our $R\Gamma(E_g(-1))$ becomes (essentially) the Toeplitz operator

$$R[z] \xrightarrow{\rho(g)} R[z]$$

Different version: Introduce the Toeplitz algebra $R\langle z, z^*\rangle/(1-z^*z) \simeq R[z] \otimes_R R[z^*]$, and the toeplitz extension

$$0 \longrightarrow J \longrightarrow T\ddot{o}p \longrightarrow R[z, z^{-1}] \longrightarrow 0$$

This gives $K_1(R[z,z^{-1}]) \longrightarrow K_0(J) \simeq K_0(R)$

where the latter comes from Morita invariance.

Concretely given $g \in GL_1(R[z,z^{-1}])$ we lift g, g^{-1} to $\rho(g)$ and $\rho(g^{-1})$ over Töp, then form the complex

1) $$T\ddot{o}p \xrightarrow{\rho(g)} T\ddot{o}p$$

Put $T = T\ddot{o}p$, $e = 1 - zz^*$. We have a Morita context.

$$\begin{pmatrix} T & Te \\ eT & eTe \end{pmatrix} = \begin{pmatrix} T & R[z] \\ R[z^*] & R \end{pmatrix}$$

The image under the Morita context of the complex 1) is

2) $$R[z] \xrightarrow{\rho(g)} R[z]$$

since $T \otimes_T Q = Q = R[z]$.

The other point is that if $g \in R[z]$, then

$\rho(g) = g$ so that the only homology is $R[z]/gR[z]$ in degree 0.

Now we also know that $\rho(g^{-1})$ gives a parametrix for 2). Observe that

$$\rho(z^k)\rho(z^\ell) = \rho(z^{k+\ell}) \qquad \text{if } \ell \geq 0$$

This is obvious for $k \geq 0$, so suppose $\ell \leq 0$ and put $\varepsilon = -\ell \geq 0$. Then

$$\rho(z^k)\rho(z^\ell) = (z^*)^{\varepsilon} z^\ell = \begin{cases} (z^*)^{\varepsilon - \ell} & \text{if } \varepsilon \geq \ell \\ z^{\ell - \varepsilon} & \text{if } \varepsilon \leq \ell \end{cases}$$

$$= \rho(z^{-\varepsilon+\ell}) = \rho(z^{k+\ell})$$

Thus if $g \in R[z]$ we have $\rho(g^{-1})\rho(g) = \rho(g^{-1}g) = 1$. So we have

$$R[z] \underset{\rho(g)}{\overset{\rho(g^{-1})}{\rightleftarrows}} R[z]$$

and $1 - \rho(g)\rho(g^{-1})$ projects onto H_0.

Unfortunately when $g = az+b$, this projection $1 - \rho(g)\rho(g^{-1})$ does not seem to be the one studied on p. 326-7.

July 18, 1995

Program: To understand the following and the relations between them.

1. Pedersen-Weibel deloopings.

2. Cone and suspension of a ring (used by Karoubi, Wagoner to deloop).

3. John Roe's finite propagation C^* algebras.

Rough background: Roe told me at Lancaster that his stuff and Pedersen's ideas are closely related. Ranicki in his book on Lower K+L theory discusses Pedersen-Weibel, at least a metric space version using open cones instead of the integer lattices in the original PW paper. I assumed 1.+2. were very similar, but this now seems naive.

Related ideas. Negative K groups via (Laurent) polynomial extensions (Bass-Heller-Swan ?). Toeplitz algebras, periodicity proofs. Controlled K-theory

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Let's begin with the cone on a unital ring A.

The key idea here is to embed $\mathcal{P}(A)$ in a $\mathcal{P}(R)$ having trivial K-theory because of an infinite direct sum argument.

First examine when $K_0(R) = 0$.

Claim $K_0(R) = 0 \iff \exists k \in \mathbb{N}$ such that $P \oplus R^k \simeq R^k$ for all P in $\mathcal{P}(R)$.

Proof: $(\Leftarrow)$ obvious.

$(\Rightarrow)$ I know that $K_0(R) = \mathrm{Iso}(\mathcal{P}(R)) \times \mathbb{N} / \sim$

where $([P], n) \sim ([P'], n') \iff \exists k \; P \oplus R^{n'} \oplus R^k \simeq P' \oplus R^n \oplus R^k$.

Hence $K_0(R) = 0 \iff \forall P \; \exists k$ st. $P \oplus R^k \simeq R^k$.

In particular $\exists k$ such that $R \oplus R^k \simeq R^k$, whence $R^k \simeq R^{k+1} \simeq R^{k+2} \simeq \cdots$

Now given P we know $\exists \ell$ such that $P \oplus R^\ell \simeq R^\ell$ and then $P \oplus R^k \simeq R^k$ follows because either $\ell < k$ and you can add $R^{k-\ell}$ or $\ell \geq k$ and you have $R^\ell \simeq R^k$.

Recall the Eilenberg swindle. Assume $P \oplus Q = F$ free, then

$$\underbrace{P \oplus Q}\oplus\underbrace{P \oplus Q} \oplus \cdots \quad = (F \oplus F \oplus \cdots \quad)$$

$$\wr\!\wr$$

$$P \oplus \underbrace{Q \oplus P} \oplus Q \oplus P \oplus \cdots \quad = P \oplus (F \oplus F \oplus \cdots \quad)$$

(This might be useful in connection with $ze + 1 - e$ and Bott maps.)

Here's a simpler argument. Given $P \in \mathcal{P}(R)$ suppose that $\Sigma P = \bigoplus_{n \in \mathbb{N}} P$ is in $\mathcal{P}(R)$. Since there's a canonical isom $P \oplus \Sigma P \simeq \Sigma P$ it follows that $[P] = 0$ in $K_0(R)$.

Thus we have $K_0(\mathcal{A}) = 0$, $\mathcal{A}$ additive category if there exists a functor $\Sigma : \mathcal{A} \longrightarrow \mathcal{A}$ together with an isomorphism $id \oplus \Sigma = \Sigma$.

Question: Suppose $\exists\, \Sigma R$ in $\mathcal{P}(R)$ such that $R \oplus \Sigma R \simeq \Sigma R$. Does it follow that $K_0(R) = 0$?

Note that ΣR is a summand of R^k for some k, hence $R \oplus R^k \simeq R^k$. Thus the question becomes whether R stably trivial $\Longrightarrow K_0(R) = 0$.

Perhaps Cuntz's algebra O_2 should be examined.

Let's focus ~~attention~~ on the situation where the K-theory is trivial because of an infinite direct sum functor Σ.

First example: Consider the additive category $\mathcal{A}$ of vector spaces of countable dimension. This is of the form $\mathcal{P}(R)$ where R is the ring of endos of $k^{(\infty)} = k^{(\mathbb{N})}$. R is the ring of matrices $(a_{ij})_{i,j\geq 0}$ with finite columns (finite means a.e. 0).

Because the countable direct sum is defined in this category the K-theory of R is trivial.

I think the cone $C(A)$ on a ring is the ring of matrices $(a_{ij})_{i,j\geq 0}$ with both rows and columns finite. Here's an interesting way to get this ring. (Recall A is a unital ring).

Consider the following category. The objects are triples $\{{}_AQ, P_A, \langle\, ,\, \rangle : Q \otimes_{\mathbb{Z}} P \longrightarrow A\}$, where $\langle\, ,\, \rangle$ is an A-bimodule map. A map

$$(Q, P, \langle\, ,\, \rangle) \longrightarrow (Q', P', \langle\, ,\, \rangle)$$

is a pair of maps $Q \xrightarrow{u} Q'$, $P' \xrightarrow{u^*} P$ satisfying $\langle u(q), p' \rangle = \langle q, u^*(p') \rangle$.

I think this is an additive category which is Karoubian. There's an obvious direct sum for families of triples, although whether it gives the categorical direct sum is not clear.

Now consider the triples $(A_\ell^{(\infty)}, A_r^{(\infty)}, \langle , \rangle)$ where $A_\ell^{(\infty)}$ is the left A-module of finite row vectors, $A_r^{(\infty)}$ is the right A-module of finite column vectors, and $\langle , \rangle$ is dot product. Here $A_\ell^{(\infty)} = \bigoplus_{n \in \mathbb{N}} A$, etc. Then the ring of endos of this triple is the ring of row + column finite matrices, i.e. the cone $C(A)$.

Let's discuss Pedersen-Weibel. Given an additive category $\mathcal{C}$ their first delooping $\mathcal{C}^1$ is defined as follows. The objects are family $(P_n)_{n\in\mathbb{Z}}$ of objects in $\mathcal{C}$, and a map $(P_n)\to(Q_n)$ in $\mathcal{C}'$ is a matrix (φ_{mn}), with $\varphi_{mn}\in \mathrm{Hom}_{\mathcal{C}}(P_n,Q_m)$, such that the support of (φ_{mn}) is contained in $\{(m,n) \mid |m-n|\le r\}$ for some r. The k-th delooping $\mathcal{C}^k$ is defined similarly using $\mathbb{Z}^k$ instead of $\mathcal{C}$.

Note that $\mathrm{Hom}_{\mathcal{C}'}(P,Q)$ has a natural filtration indexed by $\mathbb{N}$ which is exhaustive and compatible with composition. One can extend the definition of $\mathcal{C}^1$ to similarly filtered additive categories $\mathcal{C}$. One then has $\mathcal{C}^k=(\mathcal{C}^{k-1})'$, allowing inductive proofs.

A key point in the proofs I think is to break $\mathbb{Z}$ up into $(-\mathbb{N})\cup(\mathbb{N})$, then argue that objects supported over $\mathbb{N}$ form a subcategory whose K-theory is trivial because of an infinite direct sum argument. Thus given $P=(P_n)_{n\in\mathbb{N}}$ ($P_n=0$ for $n<0$), we can shift: $\sigma^k P=(P_{n-k})_{n\in\mathbb{Z}}$ and form

$$\begin{array}{lllllll} \Sigma P = \bigoplus_{k\geq 0}\sigma^k P = & P_0 \oplus & P_1 \oplus & P_2 \oplus & \cdots \\ & & P_0 \oplus & P_1 \oplus & \cdots \\ & & & P_2 \oplus & \cdots \end{array}$$

The canonical isom. $P\oplus\Sigma P\simeq\Sigma P$ holds in $\mathcal{C}^1$. More precisely one has a decomposition

$$\Sigma P = P\oplus\sigma\Sigma P$$

and an isomorphism $\sigma \Sigma P \simeq \Sigma P$ of finite propagation.

Suppose now that $\mathcal{C} = \mathcal{P}(A)$. It's clear that $\mathcal{C}^1$ is not of the form $\mathcal{P}(R)$. In effect you would need a generator $P = (P_n)$ for $\mathcal{C}^1$ and you can always manufacture a (Q_n) which grows too fast to be a summand of some P^k.

This means the Pedersen-Weibel construction leaves the K-theory of rings (maybe just unital rings). I find this surprising in view of what Roe told me and also the theory of $C(A)$.

One thing we can do is specify a growth rate. To fix the ideas work over $\mathbb{N}$ and consider those objects $P = (P_n)_{n \in \mathbb{Z}}$ such that P_n is a summand of $A^{r(n)}$, where $r(n) \leq C f(n)$, f being the growth function. Then $(A^{f(n)})_n$ should be a projective generator for this subcategory.

If we take $f(n) = 2^n$, then this subcategory is closed under the functor Σ. Since

$$rk(P_0 \oplus \cdots \oplus P_n) = \sum_{k=0}^{n} rk(P_k) \leq C \sum_{k=0}^{n} 2^k \leq C 2^{n+1}$$

July 28, 1995

Let $\mathcal{L}_n(R, A)$ be the category of fg proj R-module complexes which [redacted] become contractible modulo A and which are n-diml chain complexes (supported in degrees $0 \le k \le n$).

Form the Grothendieck group $L_n(R, A)$ generated by objects of $\mathcal{L}_n(R, A)$ where the relations are additivity for locally split (short) exact sequences and [redacted] equality for homotopy equivalent complexes.

We should have have Morita invariance for $L_n(R, A)$, in particular $L_n(\tilde{A}, A) \xrightarrow{\sim} L_n(R, A)$. In any case suppose [redacted] from now on that $R = \tilde{A}$.

We have obvious maps

[redacted]
$$\begin{array}{ccccc} L_1 & \longrightarrow & L_2 & \longrightarrow \cdots\cdots & \varinjlim_n L_n \\ \| & & & & \| \\ K_0'(A) & & & & K_0(\mathcal{Pf}(A)) \end{array}$$

perfect homotopy-firm cxs.

and propose to show these are isomorphisms. I know $K_0'(A) \longrightarrow K_0(\mathcal{Pf}(A))$ is an isomorphism, so it's enough to show $L_{n-1} \to L_n$ is surjective.

Given

$$U: \quad U_n \longrightarrow U_{n-1} \longrightarrow \cdots\cdots \longrightarrow U_0 \qquad \text{in } \mathcal{L}_n(\tilde{A}, A)$$

~~[illegible] is split locally [illegible]~~

we choose $f: U^\# \longrightarrow U$ inducing the identity mod A.

We know $U^\#$ splits into elementary complexes, hence we can restrict f to the "bottom" summand to get a map $f': C(U_0^\#) \to U$ reducing to the identity mod A in degree 0. Form the cone.

$$C(f'): \quad \begin{tikzcd} & & & U_0^\# \arrow[r] \arrow[d] & U_0^\# \arrow[d, "f_0"] \\ U_n \arrow[r] & U_{n-1} \arrow[r] & \cdots \arrow[r] & U_1 \arrow[r] & U_0 \end{tikzcd}$$

If $n \geq 2$ this is in $\mathcal{L}_n$, and it is h.eq. to U.

On the other hand $U_0^\# \xrightarrow{f_0} U_0 \in \mathcal{L}_1$ is a subcomplex and the quotient is the suspension of a complex V in $\mathcal{L}_{n-1}$. So we have

$$[U] = [U_0^\# \xrightarrow{f_0} U_0] - [V] \in \operatorname{Im}\{L_{n-1} \to L_n\}$$

Another point is that for $\mathcal{L}_n$ one obtains the same Grothendieck group if one weakens $U \sim V \Rightarrow [U] = [V]$ to $U \sim 0 \Rightarrow [U] = 0$. This is not obvious. The argument when the dimension of the chain complexes increases is as follows. Given a h.eq. $f: U \to V$ one forms the mapping cylinder $M(f)$ which satisfies

$$M(f) = V \oplus C(1_U) \qquad \text{in general}$$
$$= U \oplus \underbrace{C(f)}_{\text{contract.}} \qquad \text{if } f \text{ a heq}$$

The problem is that $M(f)$ is $n+1$ dimensional in general.

However one can modify things as follows.

We have $M(f)_k = \mathcal{U}_k \oplus \mathcal{U}_{k-1} \oplus V_k$ $\quad d_M = \begin{pmatrix} d & -1 & \\ & -d & \\ & f & d \end{pmatrix}$

$$\begin{tikzcd} & & 0 & & \\ & & C(\mathcal{U}) \arrow[u] & & \\ 0 \arrow[r] & \mathcal{U} \arrow[r] & M(f) \arrow[u] \arrow[r] & C(f) \arrow[r] & 0 \\ & & V \arrow[u] & & \\ & & 0 \arrow[u] & & \end{tikzcd}$$

Picture of $M(f)$ at the top:

$$\begin{tikzcd} \mathcal{U}_n \arrow[r, "-d"] \arrow[d, "{(-1,f)}"] & \mathcal{U}_{n-1} \arrow[r] \arrow[d, "{(-1,f)}"] & {} \\ \mathcal{U}_n \oplus V_n \arrow[r, "d\oplus d"'] & \mathcal{U}_{n-1} \oplus V_{n-1} \arrow[r] & {} \end{tikzcd}$$

Replace $M(f)$ by $M(f)/d_M(\mathcal{U}_n)$. Now $d_M(\mathcal{U}_n)$ is an elementary complex, hence contractible. We want to see that it maps isomorphically onto a ~~locally split~~ direct summand of both $C(\mathcal{U})$ and $C(f)$ (ignoring differentials). ~~It~~ It will then follow that we have locally split exact sequences

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & d_M(U_n) \arrow[r, equal] \arrow[d] & d_M(U_n) \arrow[d] & \\
0 \arrow[r] & U \arrow[r] & M(f) \arrow[r] \arrow[d] & C(f) \arrow[r] \arrow[d] & 0 \\
 & & M'(f) \arrow[r] \arrow[d] & C'(f) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

whence $M'(f) \cong U \oplus C'(f)$ with $C'(f) \sim 0$. Similarly $M'(f) \cong V \oplus C'(U)$ with $C'(U) \sim 0$.

Now $d_M(U_n)$ is obviously contained as summand in $C(f)$:

$$
\begin{tikzcd}
U_n \arrow[r, "-d"] \arrow[d, "-1"'] & U_{n-1} \arrow[r] \arrow[d, "-1"] & {} \\
U_{n-1} \arrow[r, "d"] & U_{n-2} \arrow[r] & {}
\end{tikzcd}
$$

. Now

$$
C(f): \quad
\begin{tikzcd}
U_n \arrow[r, "-d"] \arrow[d, "f"] & U_{n-1} \arrow[r] \arrow[d, "f"] & {} \\
V_n \arrow[r, "d"'] & V_{n-1} \arrow[r] & {}
\end{tikzcd}
$$

is contractible so $0 \to U_n \to V_n \oplus U_{n-1}$ is a split injection and we win.

August 7, 1995

I want to derive a formula for the inverse of $K_0\mathcal{L}^1(R,A) \xrightarrow{\sim} K_0\mathcal{L}(R,A)$, that is, to express $[U] \in K_0\mathcal{L}(R,A)$ in terms of classes of length one complexes.

The initial idea goes as follows. Suppose U a chain complex, and let h be a htpy operator such that $dh + hd \equiv 1 \mod A$. We can attach an elementary complex using the map

$$\begin{tikzcd} \arrow[r] & U_2 \arrow[r] & U_1 \arrow[r, "d_1"] & U_0 \\ & & U_0 \arrow[u, "h_0"] \arrow[r, "1"] & U_0 \arrow[u, "d_1 h_0"'] \end{tikzcd}$$

This gives the complex heq U

$$\longrightarrow U_3 \xrightarrow{\begin{pmatrix} d_3 \\ 0 \end{pmatrix}} \begin{matrix} U_2 \\ \oplus \\ U_0 \end{matrix} \xrightarrow{\begin{pmatrix} d_2 & h_0 \\ 0 & -1 \end{pmatrix}} \begin{matrix} U_1 \\ \oplus \\ U_0 \end{matrix} \xrightarrow{(d_1 \;\; d_1 h_0)} U_0$$

and the dim 1 subcomplex such that the quotient is U':

$$\longrightarrow U_3 \xrightarrow{\begin{pmatrix} d_3 \\ 0 \end{pmatrix}} \begin{matrix} U_2 \\ \oplus \\ U_0 \end{matrix} \xrightarrow{(d_2 \;\; h_0)} U_1 \longrightarrow 0$$

Thus in the Grothendieck group we have

$$[U] = [d_1 h_0 : U_0 \to U_0] - [U']$$

Now note in the transition $U \mapsto U'$, U_0 has been shifted to be with U_2. Repeating this process should lead to a dim 1 complex made of U^{even}, U^{odd} with differential constructed from d, h — hopefully $d+h$.

Consider $\tilde{U} = U \oplus U[1] \oplus U[2] \oplus \cdots\cdots$ with the differential and homotopy

$$\tilde{d} = \begin{pmatrix} d & 0 & & & & \\ & -d & -1 & & & \\ & & d & 0 & & \\ & & & -d & -1 & \\ & & & & d & \ddots \\ & & & & & \ddots \end{pmatrix} \qquad k = \begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ & -1 & 0 & & & \\ & & 0 & 0 & & \\ & & & -1 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

Since $\left[\begin{pmatrix} -d & -1 \\ & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ -d+d & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

it's clear that we have the complex

$$\tilde{U} = U \oplus C(U[1]) \oplus C(U[3]) \oplus \cdots$$

Now conjugate by the invertible operator

$\begin{pmatrix} 1 & h \\ & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & h \\ & 1 \end{pmatrix} \oplus \cdots$ on $(U \oplus U[1]) \oplus (U[2] \oplus U[3]) \oplus \cdots$

$$\begin{pmatrix} 1 & -h & & & & \\ & 1 & & & & \\ & & 1 & -h & & \\ & & & 1 & & \\ & & & & 1 & -h \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} d & 0 & & & & \\ & -d & -1 & & & \\ & & d & 0 & & \\ & & & -d & -1 & \\ & & & & d & 0 \\ & & & & & -d \end{pmatrix} =$$

$$\begin{pmatrix} d & hd & h & 0 & & \\ & -d & -1 & & & \\ & & d & hd & h & \\ & & & -d & -1 & \\ & & & & d & hd \\ & & & & & -d \end{pmatrix} \begin{pmatrix} 1 & h & & & & \\ & 1 & & & & \\ & & 1 & h & & \\ & & & 1 & & \\ & & & & 1 & h \\ & & & & & 1 \end{pmatrix}$$

$$\tilde{d} = \begin{pmatrix} d & [d,h] & h & h^2 & 0 & 0 \\ 0 & -d & -1 & -h & 0 & 0 \\ 0 & 0 & d & [d,h] & h & h^2 \\ 0 & 0 & 0 & -d & -1 & -h \\ 0 & 0 & 0 & 0 & d & [d,h] \\ 0 & 0 & 0 & 0 & 0 & -d \end{pmatrix}$$

Conjugation doesn't change k since

$$\left(\begin{array}{cc|cc} 1 & +h & & \\ & 1 & & \\ \hline & & 1 & +h \\ & & & 1 \end{array}\right), \quad \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & -1 & 1 & \\ & & & 1 \end{array}\right) \qquad \text{commute.}$$

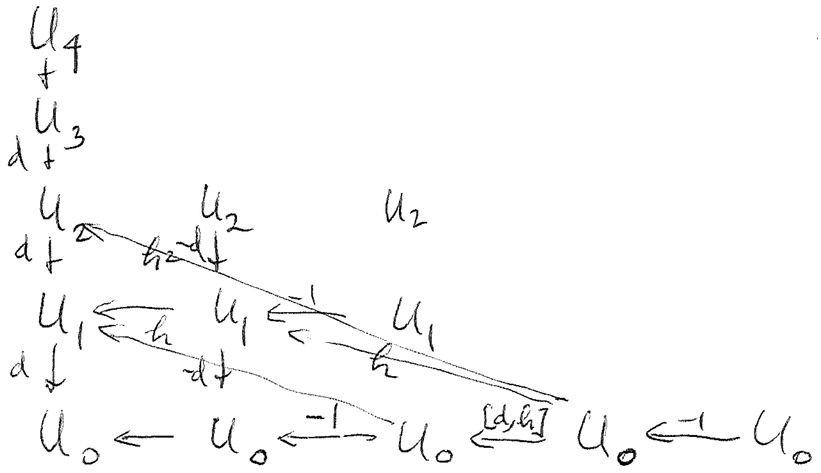
You should be more careful but the point is that

$$\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & & \\ b & 0 & & \\ & b & 0 & \\ & & b & 0 \end{pmatrix}$$

commute if a and b do, and this holds for

$$a = \begin{pmatrix} 1 & +h \\ & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Here's a picture of $(\tilde{u}, \tilde{d})$:

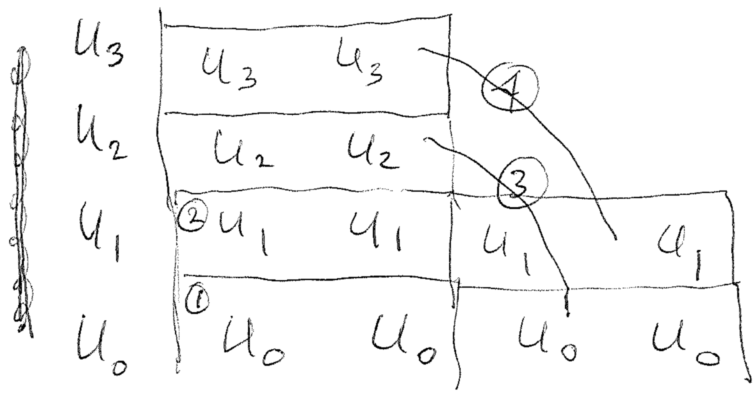


col no: 0 1 2 3 4

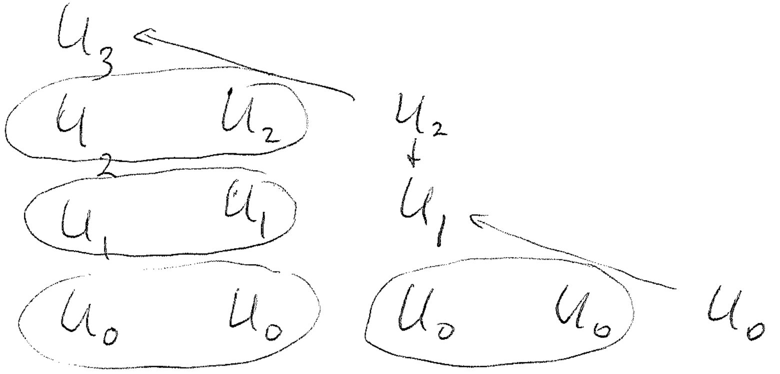
From an even column the differential is the sum of $+d, -1, +h$.

From an odd column the differential is the sum of $-d, [d,h], -h, h^2$.

Basic steps of the attaching construction are $U_0, U_1, U_0+U_2, U_1+U_3, U_0+U_2+U_4, U_1+U_3+U_5, \ldots$



Suppose U 3-dimensional and consider the resulting cx.



One can see this complex is $\sim U$ since it's closed under the homotopy operator k. The circled dim 1 subcomplexes make up the steps of a filtration.

From this complex we should get the formula

$$[U] = 2[U_0 \xrightarrow{[d,h]_0} U_0] - [U_1 \xrightarrow{[d,h]_1} U_1] + [U_2 \xrightarrow{[d,h]_2} U_2]$$

$$- \left[d+h : \begin{matrix} U_2 \\ \oplus \\ U_0 \end{matrix} \longrightarrow \begin{matrix} U_3 \\ \oplus \\ U_1 \end{matrix} \right]$$

Actually before this complex I could have done the simpler

$$\begin{array}{ccc} U_3 & & \\ + & & \\ U_2 & & \\ U_1 & U_1 & U_1 \\ U_0 & U_0 & U_0 \end{array}$$

which gives

$$[U] = [U_0 \xrightarrow{1-f_0} U_0] - [U_1 \xrightarrow{1-f_1} U_1]$$

$$+ \left[d+h : \begin{matrix} U_3 \\ \oplus \\ U_1 \end{matrix} \longrightarrow \begin{matrix} U_2 \\ \oplus \\ U_0 \end{matrix} \right]$$

Observe that

$$[d+h : U^+ \to U^-] + [d+h : U^- \to U^-]$$

$$= [\underbrace{(d+h)^2}_{[d,h]+h^2 = 1-f+h^2} : U^+ \to U^+] = [(d+h)^2 : U^- \to U^-]$$

But $1-f+h^2$ is triangular: $\begin{matrix} U_2 \\ \oplus \\ U_0 \end{matrix} \xleftarrow{\begin{pmatrix} 1-f_2 & h^2 \\ & 1-f_0 \end{pmatrix}} \begin{matrix} U_2 \\ \oplus \\ U_0 \end{matrix}$

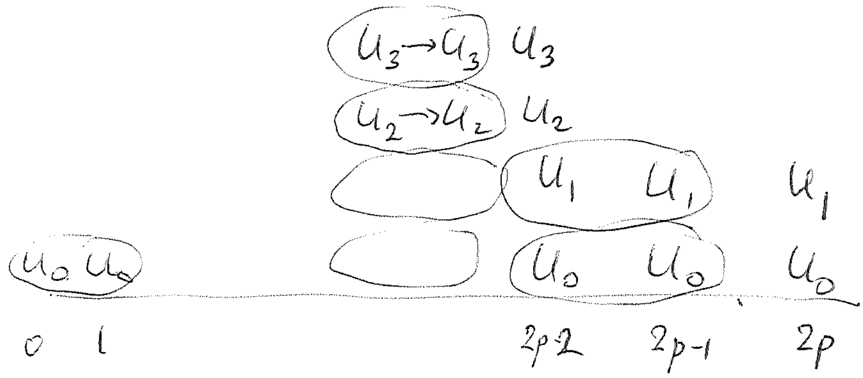
so we get

$$\sum_i [1-f_{2i}] = \sum_i [1-f_{2i+1}]$$

or

$$\boxed{\sum (-1)^n [1-f_n] = 0}$$

General formula:



$$[U] = [d+h : U^- \longrightarrow U^+]$$
$$+ \sum_{i \geq 0} (p-i)[1-f_{2i}] - \sum_{i \geq 0} (p-i)[1-f_{2i+1}]$$

$$\boxed{\begin{aligned}[U] = {} & [d+h : U^- \longrightarrow U^+] \\ & - \sum_{i \geq 0} i[1-f_{2i}] + \sum_{i \geq 0} i[1-f_{2i+1}]\end{aligned}}$$

Note that under the map $K_0'A \longrightarrow K_0A$, the class $[1-f_n]$ go to zero so we get

$$\mathrm{Im}\,[U] \text{ in } K_0A = [d+h : U^- \longrightarrow U^+]$$

which is a Whitehead type formula.

August 26, 1995

In trying to establish Morita invariance for Hochschild homology of h-unital rings in general I seem to encounter the problem that a free bimodule over R need not be flat as left or right R-module. Let's discuss aspects of this problem.

Let's start with the formula

$$X \otimes_R M = (X \otimes_{\mathbb{Z}} M) \otimes_{R \otimes_{\mathbb{Z}} R^{op}} R$$

$$= (X \otimes_{\mathbb{Z}} M) / \mathfrak{a}(X \otimes_{\mathbb{Z}} M)$$

where $\mathfrak{a}$ is the left ideal in $R \otimes_{\mathbb{Z}} R^{op}$ generated by $a \otimes 1 - 1 \otimes a$, $a \in R$. Let $E \longrightarrow R$ be a free (flat should be enough) R-bimodule resolution of R. Suppose $\hat{X}, \hat{M}$ are projective resolutions of X and M over R^{op} and R resp. Then

$$\hat{X} \otimes_R \hat{M} \longleftarrow (\hat{X} \otimes_{\mathbb{Z}} \hat{M}) \otimes_{R \otimes_{\mathbb{Z}} R^{op}} E = \hat{X} \otimes_R E \otimes_R \hat{M}$$

should be a quis. Why:

$$\hat{X} \otimes_R \hat{M} = (\hat{X} \otimes_{\mathbb{Z}} \hat{M}) \otimes_{R \otimes_{\mathbb{Z}} R^{op}} R \longleftarrow \underbrace{(\hat{X} \otimes_{\mathbb{Z}} \hat{M})}_{\text{complex of proj } R \otimes_{\mathbb{Z}} R^{op} \text{ modules}} \otimes_{R \otimes_{\mathbb{Z}} R^{op}} E$$

because tensoring with a proj. cx respects quis.

The point however is that
$\hat{X} \otimes_{\mathbb{Z}} \hat{M} \longrightarrow X \otimes_{\mathbb{Z}} M$ is not necessarily a quis. Thus we have

$$\begin{array}{ccc} X \overset{L}{\otimes}_R M & \xleftarrow{qu} & \left(\hat{X} \otimes_{\mathbb{Z}} \hat{M}\right) \overset{L}{\otimes}_{R \otimes_{\mathbb{Z}} R^{op}} R \\ & & \downarrow \\ & & \left(X \otimes_{\mathbb{Z}} M\right) \overset{L}{\otimes}_{R \otimes_{\mathbb{Z}} R^{op}} R = X \otimes_R E \otimes_R M \end{array}$$

but it's not clear what the homology of $\hat{X} \otimes_{\mathbb{Z}} \hat{M}$ is.

Consider a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with everything firm and flat on both sides. Let
$E \longrightarrow \tilde{A}$ be a projective $\tilde{A}$-bimodule resolution and
$F \longrightarrow \tilde{B}$ ——— $\tilde{B}$ ———.
Consider

$$A \otimes_{\tilde{A}} E \otimes_{\tilde{A}} \xleftarrow{①} Q \otimes_{\tilde{B}} F \otimes_{\tilde{B}} P \otimes_{\tilde{A}} E \otimes_{\tilde{A}}$$
$$\parallel$$
$$P \otimes_{\tilde{A}} E \otimes_{\tilde{A}} Q \otimes_{\tilde{B}} F \otimes_{\tilde{B}} \xrightarrow{②} B \otimes_{\tilde{B}} F \otimes_{\tilde{B}}$$

Claim ① is a quis. Since E consists of proj $\tilde{A}$ bimodules we know $- \otimes_{\tilde{A}} E \otimes_{\tilde{A}}$ ~~accept~~ respects quis. Check: $T \otimes_{\tilde{A}} (\tilde{A} \otimes_{\mathbb{Z}} \tilde{A}) \otimes_{\tilde{A}} = \tilde{A} \otimes_{\tilde{A}} T \otimes_{\tilde{A}} \tilde{A} = T$
Now $F \longrightarrow \tilde{B}$ quis and Q flat over $\tilde{B}^{op}$,
P flat over $\tilde{B}$ imply $Q \otimes_{\tilde{B}} F \otimes_{\tilde{B}} P \longrightarrow Q \otimes_{\tilde{B}} \tilde{B} \otimes_{\tilde{B}} P = Q \otimes_{\tilde{B}} P = A$ is a quis. Thus ① is a quis.
A similar argument holds in the case

of ② which uses P, Q flat over $\tilde{A}^{op}$ and $\tilde{A}$ resp.

Now I think we know that in an everything firm Morita context that

P firm flat over $\tilde{A}^{op} \iff P \otimes_{\tilde{A}} Q = B$ is firm flat over $\tilde{B}^{op}$

Q firm flat over $\tilde{A} \iff P \otimes_{\tilde{A}} Q = B$ is firm flat over $\tilde{B}$

So in an everything firm situation, if B is flat on both sides then we have a canonical map (after inverting quis)

$$\begin{tikzcd} B \otimes_{\tilde{B}} F \otimes_{\tilde{B}} \arrow[r] \arrow[d, equal] & A \otimes_{\tilde{A}} E \otimes_{\tilde{A}} \arrow[d, equal] \\ B \overset{L}{\otimes}_{\tilde{B}} & A \overset{L}{\otimes}_{\tilde{A}} \end{tikzcd}$$

Apparently I can define HH for a Rooz category using biflat coordinates. But I don't see how to go further. For example suppose I assume P, Q $\tilde{A}$ flat (equiv. B is $\tilde{B}$ biflat), this is a biflat coordinatization. Further assume A $\tilde{A}$ flat (equiv. P is $\tilde{B}$ flat). I want ① to be a quis, and it suffices that

$$Q \otimes_{\tilde{B}} F \otimes_{\tilde{B}} P \longrightarrow Q \otimes_{\tilde{B}} P = A$$

because P is $\tilde{B}$ flat

be a quis., and it further suffices that $Q \otimes_{\tilde{B}} F \to Q \otimes_{\tilde{B}} \tilde{B} = Q$ be a quis. Can this be done by a heq of $F \to \tilde{B}$ as a map over $\tilde{B}$. Certainly there's a section $F \rightleftarrows \tilde{B}$.

August 28, 1995

Let $A \longrightarrow B$ be a homomorphism of nonunital rings. We have the restriction of scalars functor

$$\mathrm{mod}(\tilde{B}) \longrightarrow \mathrm{mod}(\tilde{A})$$

which is exact and carries B-nil modules into A-nil modules, hence it induces an exact functor

$$\mathcal{M}(B) \longrightarrow \mathcal{M}(A)$$

Suppose A, B idempotent. Then for N a $\tilde{B}$-module, M an $\tilde{A}$-module we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{M}(A)}(M, N) &= \mathrm{Hom}_{\mathcal{M}(A)}(M, \mathrm{Hom}_B(B^{(2)}, N)) \\
&= \mathrm{Hom}_A(A^{(2)} \otimes_A M, \mathrm{Hom}_B(B^{(2)}, N)) \\
&= \mathrm{Hom}_B(\tilde{B} \otimes_A A^{(2)} \otimes_A M, \mathrm{Hom}_B(B^{(2)}, N)) \\
&= \mathrm{Hom}_B(B^{(2)} \otimes_B \tilde{B} \otimes_A A^{(2)} \otimes_A M, N) \\
&= \mathrm{Hom}_B(B^{(2)} \otimes_A A^{(2)} \otimes_A M, N) \\
&= \mathrm{Hom}_{\mathcal{M}(B)}(B^{(2)} \otimes_A A^{(2)} \otimes_A M, N)
\end{aligned}$$

you could have done this directly

In other words there's an extension of scalars functors $\mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ given by

$$M \longmapsto A^{(2)} \otimes_A M \longmapsto \tilde{B} \otimes_A A^{(2)} \otimes_A M \longmapsto B^{(2)} \otimes_A A^{(2)} \otimes_A M$$

makes M firm — basechange wrt $\tilde{A} \to \tilde{B}$ — if you want.

For example, let $R \longrightarrow T$ be a nonunital ring homom. where R, T happen to be unital. Then we get the adjoint functors

$$\begin{array}{ccc} \mathcal{M}(R) & \overset{\longrightarrow}{\longleftarrow} & \mathcal{M}(T) \\ \| & & \| \\ \mathrm{mod}(R) & & \mathrm{mod}(T) \end{array}$$

$$M \longmapsto T \otimes_R M = (Te \oplus Te^{\perp}) \otimes_R M$$
$$= Te \otimes_R M$$

since $Te^{\perp} \otimes_R M = Te^{\perp} \otimes_R eM = 0$

rest. of scalars then made firm over R: $eN = R \otimes_R N \longleftarrow N$

where e is the image of 1_R in T.

Consider now a firm ring A and a triple $({}_AQ, P_A, Q \otimes P \xrightarrow{\psi} A)$ with Q, P firm and ψ arbitrary. [redacted] Let $C = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A, Q) = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$.
We have the Morita context

[redacted]

$$(*) \qquad \left(\begin{array}{c|cc} A & A & Q \\ \hline A & A & Q \\ P & P & B \end{array}\right)$$

with ideals
$$(A \;\; Q)\begin{pmatrix} A \\ P \end{pmatrix} = A$$
$$\begin{pmatrix} A \\ P \end{pmatrix}(A \;\; Q) = C.$$

So this context gives a Morita equivalence

$$\mathcal{M}(A) \longrightarrow \mathcal{M}(C)$$

$$M \longmapsto \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A M$$

On the other hand we have a nonunital ring homomorphism $A \hookrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix} = C$ which induces a functor $\mathcal{M}(A) \longrightarrow \mathcal{M}(C)$ as above. It sends M to the firm version $A \otimes_A M$ followed by extension of scalars

$$C \otimes_A A \otimes_A M = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q) \otimes_A A \otimes_A M$$

$$= \begin{pmatrix} A & 0 \\ P & 0 \end{pmatrix} \otimes_A M$$

since $Q \otimes_A A = Q \otimes_A A^2 = QA \otimes_A A = 0$.

Thus we see that the Morita equivalence associated to the context $(*)$ coincides with extension and restriction of scalars wrt $A \subset \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. Let's check the other functor

$$N \longmapsto (A \quad Q) \otimes_C N \qquad \mathcal{M}(C) \longrightarrow \mathcal{M}(A)$$

$$N \longmapsto A \otimes_A N = A \otimes_A C \otimes_C N$$

$$= \underbrace{A \otimes_A \begin{pmatrix} A \\ P \end{pmatrix}}_{\begin{pmatrix} A \\ 0 \end{pmatrix}} \otimes_A (A \quad Q) \otimes_C N$$

$$= \begin{pmatrix} A & Q \\ 0 & 0 \end{pmatrix} \otimes_C N$$