January 10, 1995

To Define $K_1 A$ for A nonunital idempotent:
Let $GL_n(A) = (1+M_n(A))^\times \subset GL_n(\tilde{A})$, i.e. an element of $GL_n(A)$ is a matrix $1+x$ with x an $n\times n$ matrix over A which is invertible. One has $(1+x)^{-1}(1+x)=1$ so $(1+x)^{-1} = 1-(1+x)^{-1}x$ has the form $1+y$ with $y \in M_n(A)$. Thus we have $GL_n(A) \simeq \{x \in M_n(A) \mid \exists\, y \in M_n(A)$ such that $xy = yx = -x-y\}$.

Let $E_n(A)$ be the subgroup of $GL_n(A)$ generated by the elements $1+ae_{ij}$ with $a \in A$ and $i \neq j$; here e_{ij} is the matrix with 1 in the (i,j)th position and zero elsewhere. We have for i,j,k distinct indices in $\{1,\dots,n\}$.

$$(1+ae_{ij})(1+be_{jk})(1-ae_{ij})(1-be_{jk})$$

$$= (1+ae_{ij}+be_{jk}+abe_{ik})(1-ae_{ij}-be_{jk}+abe_{ik})$$

$$\begin{aligned} = \ & 1 - ae_{ij} - be_{jk} + abe_{ik} \\ + \ & ae_{ij} \quad\quad - abe_{ik} \quad = 1 + abe_{jk} \\ & \quad + be_{jk} \\ & \quad\quad + abe_{ik} \end{aligned}$$

This shows that $E_n(A^2) \subset [E_n(A), E_n(A)]$ for $n \geq 3$, hence $E_n(A)$ is perfect for $n \geq 3$ assuming $A^2 = A$.

The next step is to prove the appropriate analogue of the Whitehead lemma, i.e. $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ is a product of elementary matrices. (This analogue is perhaps Vaserstein's lemma.) Start with

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y \\ x & 1+xy \end{pmatrix}$$

Suppose $(1+xy)^{-1}$ exists. Then

$$\begin{pmatrix} 1 & -y(1+xy)^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ x & 1+xy \end{pmatrix} = \begin{pmatrix} 1-y(1+xy)^{-1}x & 0 \\ x & 1+xy \end{pmatrix}$$

Note that $(1+yx)\left(1-y(1+xy)^{-1}x\right)$

$$= (1+yx) - \underbrace{(1+yx)y}_{y\cancel{(1+xy)}}\cancel{(1+xy)^{-1}}x = 1$$

and similarly in the reverse order, whence $(1+yx)^{-1}\ \exists$ and $(1+yx)^{-1} = 1-y(1+xy)^{-1}x$. Then

$$\begin{pmatrix} 1 & 0 \\ -x(1+yx) & 1 \end{pmatrix}\begin{pmatrix} (1+yx)^{-1} & 0 \\ x & 1+xy \end{pmatrix} = \begin{pmatrix} (1+yx)^{-1} & 0 \\ 0 & 1+xy \end{pmatrix}$$

so we have proved the identity

$$\boxed{\begin{pmatrix} 1 & 0 \\ -x(1+yx) & 1 \end{pmatrix}\begin{pmatrix} 1 & -y(1+xy)^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1+yx)^{-1} & 0 \\ 0 & 1+xy \end{pmatrix}}$$

assuming $(1+xy)^{-1}\ \exists$.

Now take $g \in GL_n(A)$. Replacing A by $M_n(A)$ we can suppose g belongs to $GL_1(A) = (1+A)^\times$. Since $A = A^2$ we have $g = 1 + \sum_{i=1}^{m} a_i b_i$

Now take $x = (a_1, \dots, a_m)$, $y = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Then

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} = \left(\begin{array}{c|c} 1_m & y \\ \hline x & g \end{array}\right) \qquad g = 1 + xy$$

and the identity on the preceding page shows that $\begin{pmatrix} (1+yx)^{-1} & 0 \\ 0 & 1+xy \end{pmatrix} \in E_{m+1}(A)$

Thus we have

$$\begin{pmatrix} (1+yx)^{-1} & 0 & 0 \\ 0 & 1+xy & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} (1+yx)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+xy \end{pmatrix}$$

are in $E_{m+2}(A)$ yielding $\begin{pmatrix} 1_m & & \\ & g^{-1} & \\ & & g \end{pmatrix} \in E_{m+2}(A)$

Now permutation of coordinates preserve $E_n(A)$ so we have the Whitehead lemma. Actually I should be careful because I don't want to assume $E_n(A)$ is normal in $GL_n(A)$. The above argument shows that for any g in $GL_n(A)$, there is an ~~[illegible]~~ m such that

$$\begin{pmatrix} g & & \\ & g^{-1} & \\ & & 1_m \end{pmatrix} \in E_{2n+m}(A).$$

Then

$$\begin{pmatrix} g_1g_2g_1^{-1}g_2^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} g_1 & & & \\ & g_1^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} g_2 & & & \\ & 1 & & \\ & & g_2^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} g_1 & & & \\ & g_1^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{-1} \begin{pmatrix} g_2 & & & \\ & 1 & & \\ & & g_2^{-1} & \\ & & & 1 \end{pmatrix}^{-1}$$

belongs to $E_N(A)$, so we conclude that $[GL(A), GL(A)] \subset E(A)$, hence $[GL(A), GL(A)] = E(A)$, as $E(A)$ is perfect.

January 11, 1995

Consider an [redacted] equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$ where A, B are firm. Then we know this is given by a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where all 8 maps $A \otimes_A A \longrightarrow A$, $Q \otimes_B P \longrightarrow A$, etc. are isos. I want to factor this Morita equivalence into simpler steps.

The basic point of view I want to adopt is think of having the Roos category $\mathcal{M}(A)$ fixed and to regard an equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$ as a parameterization, or coordinatization, of $\mathcal{M}(A)$. Since $P \otimes_A Q \xrightarrow{\sim} B$, [redacted] the ring B together with the Morita context can be expressed in terms of ~~the~~ the triple (Q, P, ϕ), where Q is a firm A-mod, P is a firm A^{op}-module, and $\phi : Q \otimes_{\mathbb{Z}} P \twoheadrightarrow A$ is a surjective A-bimodule map. The ring structure on $P \otimes_A Q$ is given by

$$(p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \otimes \phi(q_1, p_2) q_2$$

while

$$(p_1 \otimes q_1) p = p_1 \phi(q_1, p)$$

$$q (p_1 \otimes q_1) = \phi(q, p_1) q_1$$

give the left and right B module structure on P, Q respectively.

We want to get from $(A, A, \mu : A \otimes A \to A)$ (which represents the identity equivalence $\mathcal{M}(A) = \mathcal{M}(A)$) to (Q, P, ϕ) which represents $\mathcal{M}(A) \simeq \mathcal{M}(B)$.

Here are possible paths

$\mu = \text{mult}: A \otimes A \to A$

$$\begin{tikzcd}[column sep=small]
& (A, A, \mu) \arrow[dl, dash] \arrow[dr, dash] & \\
(A \oplus Q, A, \mu \oplus 0) \arrow[dr, dash] & & (A, A \oplus P, \mu \oplus 0) \arrow[dl, dash] \\
& (A \oplus Q, A \oplus P, \mu \oplus \phi) \arrow[dl, dash] \arrow[dr, dash] & \\
(A \oplus Q, P, 0 \oplus \phi) \arrow[dr, dash] & & (Q, A \oplus P, 0 \oplus \phi) \arrow[dl, dash] \\
& (Q, P, \phi) &
\end{tikzcd}$$

In terms of the corresponding rings we have

$$\begin{tikzcd}[column sep=small]
& A \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{pmatrix} A & Q \\ 0 & 0 \end{pmatrix} \arrow[dr, dash] & & \begin{pmatrix} A & 0 \\ P & 0 \end{pmatrix} \arrow[dl, dash] \\
& \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{pmatrix} 0 & 0 \\ P & B \end{pmatrix} \arrow[dr, dash] & & \begin{pmatrix} 0 & Q \\ 0 & B \end{pmatrix} \arrow[dl, dash] \\
& B &
\end{tikzcd}$$

It seems that the simple transitions are of the form

$$(Q, P, \phi) \to (Q \oplus X, P, \phi \oplus \psi) \qquad \psi : X \otimes P \to A$$

$$(Q, P, \phi) \to (Q, P \oplus Y, \phi \oplus \psi') \qquad \psi' : Q \otimes Y \to A$$

where X (resp. Y) is a firm A (resp. A^{op}) module, and the ψ's are arbitrary bimodule map.

Notice that if we change from the A-picture to the B-picture, we have the transitions

$$(A, A, \mu) \longrightarrow (A \oplus X, A, \mu \oplus \psi)$$
$$(A, A, \mu) \longrightarrow (A, A \oplus Y, \mu \oplus \psi')$$

Here $\psi : X \otimes A \longrightarrow A$ (resp. $\psi' : A \otimes Y \to A$) can be any A-bimodule map.

<u>Mistake to avoid</u>: Do not pretend that ψ is equivalent to an A-module map $f : X \to A$, more precisely that ψ has the form

$$\mu(f \otimes 1) : X \otimes A \xrightarrow{f \otimes 1} A \otimes A \xrightarrow{\mu} A$$

If ψ has this form, then by means of the automorphism $\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ of $A \oplus X$ we can make $\psi = 0$. Put another way, $\mu \oplus \psi : (A \oplus X) \otimes A \longrightarrow A$ is the composition $(A \oplus X) \otimes A \xrightarrow{(1+f)\otimes 1} A \otimes A \xrightarrow{\mu \oplus 0} A$.

A reason for trying to reduce to the case $\psi = 0$ is that the corresponding ring assoc. to $(A \oplus X, A, \mu \oplus 0)$ is $A \otimes_A (A \oplus X) = A \oplus X$ with the semi-direct product multiplication such that $X \cdot A = 0$.

January 13, 1995.

1. Let's calculate $[M_n(A), M_n(A)]$, $n \geq 2$.

$$[ae_{ij}, a'e_{kl}] = aa' e_{ij}e_{kl} - a'a\, e_{kl}e_{ij}$$

$= 0$ if $j \neq k$ and $l \neq i$ [redacted]

$= aa' e_{il}$ if $j = k$ and $l \neq i$

$= -a'a\, e_{kj}$ if $j \neq k$ and $l = i$

$= aa' e_{ii} - a'a\, e_{jj}$ if $j = k$ and $i = l$.

For $n = 2$ we have

$$[M_2(A), M_2(A)] = \begin{pmatrix} [A,A] & A^2 \\ A^2 & [A,A] \end{pmatrix} + \left\{ \begin{pmatrix} a_i a_i' & 0 \\ 0 & -a_i' a_i \end{pmatrix} \right\}$$

$$= \begin{pmatrix} [A,A] & A^2 \\ A^2 & [A,A] \end{pmatrix} + \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \;\middle|\; \alpha \in A^2 \right\}$$

Similarly

$$[M_n(A), M_n(A)] = \begin{pmatrix} [A,A] & A^2 & A^2 & \cdots \\ A^2 & [A,A] & A^2 & \cdots \\ A^2 & A^2 & [A,A] & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} + \left\{ \operatorname{diag}(\alpha_1, \dots, \alpha_n) \;\middle|\; \begin{matrix} \alpha_i \in A^2 \\ \sum \alpha_i = 0 \end{matrix} \right\}$$

and so $\boxed{M_n(A)/[M_n(A), M_n(A)] = A/[A,A] \quad \text{iff } A = A^2}$

2. Here's a mechanism for Morita invariance of K_1, hopefully.

First, recall that if [redacted] x, y are two elements of a ring A, then $(1+xy)^{-1}$ exists $\Leftrightarrow$ $(1+yx)^{-1}$ exists.

Pf: $$\left(1-y(1+xy)^{-1}x\right)(1+yx)$$ [redacted]

$$= (1+yx) - y(1+xy)^{-1}\underbrace{x(1+yx)}_{(1+xy)x} = 1+yx-yx = 1$$

and similarly the other way [redacted] round.

Secondly recall the identity

$$\begin{pmatrix} 1 & 0 \\ -x(1+yx) & 1 \end{pmatrix}\begin{pmatrix} 1 & -y(1+xy)^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1+yx)^{-1} & 0 \\ 0 & 1+xy \end{pmatrix}$$

assuming $1+xy$ invertible. This shows that (assuming $A = A^2$ so that $K_1(A)$ is defined) the elements $1+xy$ and $1+yx$ represent the same element of $K_1(A)$.

Now suppose that we have a triple $({}_AV, U_A, \psi : V \otimes U \to A)$ with ψ an A-bimodule map. Let $B =$ [redacted] $U \otimes_A V$ and suppose $1 + u \otimes v$ is invertible. (Assume to be safe that A is idempotent and V, U are firm over A.)

Put $C = \begin{pmatrix} A & V \\ U & B \end{pmatrix}$, and write uv for $u \otimes v$.

Then $1 + uv$ invertible in $\tilde{B}$ $\Longrightarrow$

$\begin{pmatrix} 1 & 0 \\ 0 & 1+uv \end{pmatrix}$ invertible in $\tilde{C}$. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1+uv \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = \begin{pmatrix} 1+vu & 0 \\ 0 & 1 \end{pmatrix}$$

we see $1 + vu$ is invertible in $\tilde{A}$.

Moreover we know that the classes $[1+uv] \in K_1(B)$ and $[1+vu] \in K_1(A)$ become equal in $K_1(C)$.

Now suppose we have an arbitrary invertible element $1 + \sum_{i=1}^{r} u_i v_i$ of $\tilde{B}$. Take $r=2$ and $C = \begin{pmatrix} A & A & V \\ A & A & V \\ U & U & B \end{pmatrix}$. Then

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} & & \\ & & \\ u_1 & u_2 & \end{pmatrix}\begin{pmatrix} & & v_1 \\ & & v_2 \\ & & \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1+\sum u_i v_i \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} & & v_1 \\ & & v_2 \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & & \\ u_1 & u_2 & \end{pmatrix} = \begin{pmatrix} 1+\langle v_1, u_1\rangle & \langle v_1, u_2\rangle & 0 \\ \langle v_2, u_1\rangle & 1+\langle v_2, u_2\rangle & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $(\delta_{ij} + \langle v_i, u_j \rangle) \in M_n(A)^\sim$ is invertible and the classes

$$[(\delta_{ij} + \langle v_i, u_j\rangle)] \in K_1(A) \qquad [1 + \sum u_i v_i] \in K_1(B)$$

agree in $K_1(C)$.

3. <u>Claim</u>: Let A be h-unital, let $({}_A Q, P_A, \phi : Q \otimes P \to A)$ be such that Q, P are firm over A and ϕ is surjective. Then $B = P \otimes_A Q$ is h-unital $\iff$ the canonical map $P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \to B$ is a quis.

<u>Pf.</u> ($\Leftarrow$) We have a comm. square

$$\begin{tikzcd}
P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B B \arrow[r, "a"] \arrow[d, "b"'] & B \overset{L}{\otimes}_B B \arrow[d, "c"] \\
P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \arrow[r, "d"] & B
\end{tikzcd}$$

a, d are quis by hypothesis

We know $Q \overset{L}{\otimes}_B B \to Q$ is an A-nil quis, hence as A is h-unital (hence $A \overset{L}{\otimes}_A -$ takes A-nil quis into quis), we have $A \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B B \to A \overset{L}{\otimes}_A Q$ is a quis, whence b is a quis. Thus c is a quis and B is h-unital.

($\Rightarrow$) We have a comm. diagram

$$\begin{tikzcd}
P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B B \arrow[r, "a"] \arrow[d, "b"'] & P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B B \arrow[d, "c"] \\
P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \arrow[dr, "e"'] & B \overset{L}{\otimes}_B B \arrow[d, "d"] \\
 & B
\end{tikzcd}$$

a is a quis because $A \overset{L}{\otimes}_A Q \longrightarrow Q$ is a right B-mil quis and B is h-unital.

b is a quis because $Q \overset{L}{\otimes}_B B \longrightarrow B$ is an A-mil quis and A is h-unital.

c is a quis because $P \overset{L}{\otimes}_A Q \longrightarrow B$ is a right B-mil quis and B is h-unital.

d is a quis because B is h-unital

$\therefore$ e is a quis. qed

January 14, 1995

Consider the question of Morita invariance of $K_1(A)$ for A firm. Given a triple $({}_AQ, P_A, \phi : Q \otimes P \to A)$ with Q and P firm over A and ϕ an arbitrary A-bimodule map, we must prove that $K_1 A \xrightarrow{\sim} K_1 C$, where $C = \begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$. Another way to put this is to consider the functor on such triples

$$F(Q, P, \phi) = K_1(P \otimes_A Q)$$

Then the result we need is that

$$\circledast \quad F(Q,P,\phi) \xrightarrow{\sim} F(Q \oplus Q', P \oplus P', \phi \oplus \phi')$$

when ϕ is surjective.

The above possibly does not include the appropriate h-unital hypothesis.

I feel the real issue involves extending from the usual matrix situation:

$$K_1(A) \xrightarrow{\sim} K_1(M_{mn}(A))$$

to the general case, i.e. general (Q, P, ϕ).

One approach I've been trying is consider separately increasing Q and increasing P. This leads to examing triples

$$\left({}_A Q,\ P_A = A,\ \phi : Q \otimes A \longrightarrow A \right).$$

Now ϕ is equivalent to a A-module map

$$Q \longrightarrow \mathrm{Hom}_{A^{op}}(A, A) \quad \Big(\overset{\text{since } A \text{ firm}}{=} \mathrm{Hom}_{\mathcal{M}(A^{op})}(A, A) \Big).$$

The latter we recognize as the endomorphism ring of the generator A of $\mathcal{M}(A^{op})$, and this suggests looking at Roos' theorem.

Let's put $R = \mathrm{Hom}_{A^{op}}(A, A)$; it's the ring of left multipliers. We have adjoint functors

$$\mathcal{M}(R^{op}) = \mathrm{mod}(R^{op}) \underset{\mathrm{Hom}_{A^{op}}(A, -)}{\overset{- \otimes_R A}{\rightleftarrows}} \mathcal{M}(A^{op})$$

According to the proof of Roos' theorem the functor $- \otimes_R A$ is exact & surjective and it's kernel is $\mathrm{mod}((R/I)^{op})$ for some idempotent ideal I. We ~~shall~~ will be able to obtain I by finding a Morita context $\begin{pmatrix} \tilde{A} & ? \\ A & R \end{pmatrix}$ giving ~~the~~ equivalence

$$\mathcal{M}(R^{op}, I^{op}) \simeq \mathcal{M}(A^{op}).$$

Put $P = {}_R A_A$, $Q = {}_A R_R$. Explain: $R = \mathrm{Hom}_{A^{op}}(A, A)$ so $f \in R$ acts on A obviously. But we also have a map

$$\lambda : A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A) = R$$

given by $\lambda_a(a') = aa'$.

We have

$$(\lambda_{a_1}\lambda_{a_2})(a') = \lambda_{a_1}(a_2 a') = a_1 a_2 a' = \lambda_{a_1 a_2}(a')$$

$$(\lambda_a f)(a') = a\, f(a')$$

$$(f\lambda_a)(a') = f(aa') = f(a)a' = \lambda_{f(a)}(a').$$

Thus λ is a homomorphism $A \to R$ and the image of λ, $\lambda(A)$, is a left ideal in R. The Morita context we want is

$$\begin{pmatrix} A & Q = \lambda(A)R \\ P = A & R \end{pmatrix}$$

ideals

$PQ = \lambda(A)\lambda(A)R =$ [redacted] $\lambda(A)R$

$QP = \lambda(A)R \cdot A =$ [redacted]

$= \lambda(A) \cdot A = A^2 = A$

Actually the completely firm Morita context should be

$$\begin{pmatrix} A & A \otimes_A R \\ A & A \otimes_A R \end{pmatrix}.$$

$$A \otimes_A R = A \otimes_A \mathrm{Hom}_{A^{op}}(A, A) \longrightarrow \mathrm{Hom}_{A^{op}}(A, A) = R$$

is some sort of finite rank operator ring.

So we learn roughly that upon picking the generator A for $\mathcal{M}(A^{op})$, that $\mathcal{M}(A^{op}) \simeq$ [redacted] $\mathcal{M}(R^{op}, I^{op})$, where I = Image of $A \otimes_A \mathrm{Hom}_{A^{op}}(A,A)$ in $\mathrm{Hom}_{A^{op}}(A,A)$.
Note R = all <u>operators</u> and I = finite rank operators.

Let's consider a ring B, a left B-module A and a B-module map $u: A \longrightarrow B$. For example, take u to be the inclusion of a left ideal A of B.

In this situation we can define a product on A by

$$a_1 a_2 = u(a_1) \cdot a_2$$

Associativity $(a_1 a_2) a_3 = u(a_1 a_2) \cdot a_3 = u(u(a_1) \cdot a_2) \cdot a_3$
$= (u(a_1) u(a_2)) \cdot a_3$. $a_1(a_2 a_3) = u(a_1) \cdot (u(a_2) \cdot a_3)$
$= u(a_1) u(a_2)) \cdot a_3$. Also $u(a_1 a_2) = u(u(a_1) \cdot a_2)$
$= u(a_1) u(a_2)$, so $u : A \to B$ is a homomorphism.

Next check that $b \cdot$ is a left multiplier on A. $b \cdot (a_1 a_2) = b \cdot (u(a_1) \cdot a_2) = (b u(a_1)) \cdot a_2$
$= (u(b \cdot a_1)) \cdot a_2 = (b \cdot a_1) a_2$. Thus we have a homomorphism v such that

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \| & & \downarrow v \\ A & \xrightarrow{\lambda} & \mathrm{Hom}_{A^{op}}(A,A) \end{array}$$

commutes: $(v\, u(a))(a') = u(a) \cdot a' = aa' = \lambda_a(a')$.

This seems to give a sort of universal property for the left multiplier algebra.

You should check Kassel's letter to see if his student considered $A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$.

January 25, 1995

Given $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ as usual let $\begin{pmatrix} e & q \\ p & f \end{pmatrix}$ be an element of this ring such that

$$e = e^2 = qp \qquad q = eq = qf$$
$$p = pe = fp \qquad f = f^2 = pq$$

For example, suppose $p_0 \in P$, $q_0 \in Q$ such that $e = q_0 p_0 \in A$, $f = p_0 q_0 \in B$ are idempotent. Then we obtain such an $\begin{pmatrix} e & q \\ p & f \end{pmatrix}$ with $p = p_0 q_0 p_0$, $q = q_0 p_0 q_0$. E.g. $pq = (p_0q_0p_0)(q_0p_0q_0) = (p_0q_0)^3 = p_0q_0 = e$.

We know Ae is a firm projective A-module: $Ae = \tilde{A}e$ is projective and

$$Ae \simeq (A \otimes_A \tilde{A})e = A \otimes_A \tilde{A}e = A \otimes_A Ae.$$

Hence $P \otimes_A Ae = Pe$ is a firm projective B-module. In fact, we claim Pe is canonically isomorphic to Bf.

Define $\phi : B \longrightarrow Pe$ by $\phi(b) = bp$. Since $(bp)e = bp$, ϕ is well-defined. Since $fp = p$, ϕ kills $\{b - bf \mid b \in B\}$, so we get

$$Bf \simeq B/\{b - bf\} \xrightarrow{\bar{\phi}} Pe$$

$\bar{\phi}$ is surjective as $p'e = p'qp = bp$ with $b = p'q$.
$\bar{\phi}$ is injective as $bp = 0 \Rightarrow bf = bpq = 0$, so $b = b - bf$. Therefore $Bf \simeq Pe$ as claimed.

January 29, 1995

Consider the Morita invariance problem for K-theory: Given a Morita context
$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$
which is completely firm $A = A\otimes_A A = Q\otimes_B P$, etc. to construct a canonical isomorphism $K_*(A) \simeq K_*(B)$ under suitable flatness or h-unital assumptions.

Let's study the case where A is a field. If A is fixed, then the rest of the Morita context depends on the triple $(Q, P, \psi: Q\otimes P \to A)$ where Q, P are vector spaces and ψ is surjective. We can then choose $q_0 \in Q$, $p_0 \in P$ such that $\psi(q_0, p_0) = 1$. Then $\underset{aq_0 \leftarrow\!\shortmid\, a}{Q \cong A} \oplus W$, $W = \{q \mid \psi(q, p_0) = 0\}$

and $\underset{p_0 a \leftarrow\!\shortmid\, a}{P \xleftarrow{\sim} A} \oplus V$, $V = \{p \in P \mid \psi(q_0, p) = 0\}$, and
$$B = P\otimes_A Q = \begin{pmatrix} A \\ V \end{pmatrix} \otimes_A (A \ \ W) = \begin{pmatrix} A & W \\ V & V\otimes_A W \end{pmatrix}$$

In this way we are led to study triples $(W, V, \psi: W\otimes V \to A)$, where W, V are vector spaces and ψ is an arbitrary pairing, and to prove the canonical homom.
$$K_*(A) \longrightarrow K_*\left(\underbrace{\begin{pmatrix} A & W \\ V & V\otimes_A W \end{pmatrix}}\right)$$
is an isomorphism. Note that is idempotent, in fact firm, even h-unital, probably flat on both sides, so its K_*

are defined.

Put

$$C(W, V, \varphi) = \begin{pmatrix} A & W \\ V & V \otimes_A W \end{pmatrix}$$

and note that it is functorial in the triple (W, V, φ). Let's now analyze cases. We are trying to show $K_*(A) \xrightarrow{\sim} K_*(C(W, V, \varphi))$ for all triples, and this implies $K_*(C(W, V, \varphi)) \xrightarrow{\sim} K_*(C(W', V', \varphi'))$ whenever we have a map $(W, V, \varphi) \longrightarrow (W', V', \varphi')$.

The first point is that K_* is compatible with filtered inductive limits of rings. Hence we can restrict attention to the situation where W, V are finite dimensional.

Take $V = 0$. Then $C = \begin{pmatrix} A & W \\ 0 & 0 \end{pmatrix}$ and the result is true. What's here is something like Suslin's affine group theory. From my old viewpoint there is a multiplicative group of $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \in \tilde{C} = \begin{pmatrix} A & W \\ 0 & A \end{pmatrix}$, whose actions on $H^*(GL(C))$ is trivial, and whose action on $H^*(M(W))$ completely reducible & highly non-trivial.

Suppose $V \neq 0$. The pairing $\varphi : W \otimes V \to A$ is equivalent to a map $W \longrightarrow V^*$. The category of triples (W, V, φ) with V fixed is equivalent to vector spaces over V^*. It has initial object $W = 0$, final objects $W = V^*$.

January 30, 1995

Review: Take A a field, put

$$C(W,V,\psi) = \begin{pmatrix} A & W \\ V & V\otimes_A W \end{pmatrix}$$

where $\psi : W\otimes V \longrightarrow A$ is any pairing. To prove $K_*(A) \xrightarrow{\sim} K_*(C(W,V,\psi))$ it suffices to treat the case where W, V are finite dimensional.

Set $W_0 = \{w \in W \mid \psi(w,V)=0\}$

$V_0 = \{v \in V \mid \psi(W,v)=0\}$.

Then ψ induces a nondegenerate pairing $W/W_0 \otimes V/V_0 \to A$.

Choose complements: $W = W_1 \oplus W_0$, $V = V_1 \oplus V_0$.

Then

$$C(W,V,\psi) = \begin{pmatrix} A \\ V_1 \\ V_0 \end{pmatrix} \otimes_A \begin{pmatrix} A & W_1 & W_0 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} A \\ V_1 \end{pmatrix} \otimes_A (A \;\; W_1) & \begin{pmatrix} A \\ V_1 \end{pmatrix} \otimes_A W_0 \\ V_0 \otimes_A (A \;\; W_1) & V_0 \otimes_A W_0 \end{pmatrix} = \begin{pmatrix} A' & Q' \\ P' & P'\otimes_{A'} Q' \end{pmatrix}$$

where A' is a matrix algebra and the pairing

$$Q'\otimes P' = \left(\begin{pmatrix} A \\ V_1 \end{pmatrix} \otimes_A W_0\right) \otimes \left(V_0 \otimes_A (A \;\; W_1)\right)$$

$$\longrightarrow \begin{pmatrix} A \\ V_1 \end{pmatrix} \otimes_A (A \;\; W_1) = A'$$

is zero.

February 5, 1995

Consider $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ completely firm and suppose A is h-unital. Recall (p 183) that B is h-unital $\iff$ $P \overset{L}{\otimes}_A A \overset{L}{\otimes}_A Q \longrightarrow P \otimes_A Q = B$ is a quis.

First remark is that we get a map on Hochschild homology as follows:

$$A \overset{L}{\otimes}_A \xleftarrow{\alpha} Q \overset{L}{\otimes}_B P \overset{L}{\otimes}_A = P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B \longrightarrow B \overset{L}{\otimes}_B$$

I claim α is a quis when A is h-unital. Why? We know the cone on the $Q \overset{L}{\otimes}_B P \longrightarrow A$ has its homology killed by $QP = A$ on either side. So we reduce to showing that if M is an A-bimodule nil on both sides, then $M \overset{L}{\otimes}_A = 0$. But $M \overset{L}{\otimes}_A = M \otimes_{-\tau} B(A) \otimes_{\tau}$ which equals $M \otimes B(A)$ when $AM = MA = 0$, and $B(A)$ is acyclic for A h-unital. NO $B(A)$ quis $\mathbb{Z}$.

(This argument assumes we can calculate $\overset{L}{\otimes}_A$ using $B(A)$, which is the case when A is flat over the ground ring k.)

A lesson to be learned from the above is that probably $K_1(A)$ maps ~~naturally~~ to $K_1(B)$, and not the other direction as I have been trying to do, the idea being that $B = P \otimes_A Q$ is some sort of ring of compact operators over A.

Another idea is to use the evident homomorphisms

$$A \subset \underbrace{\begin{pmatrix} A & Q \\ P & B \end{pmatrix}}_{C} \supset B$$

Now $C = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A \begin{pmatrix} A & Q \end{pmatrix}$ so that C is h-unital

$\iff$

$$\underbrace{\begin{pmatrix} A \\ P \end{pmatrix} \overset{L}{\otimes}_A A \overset{L}{\otimes}_A \begin{pmatrix} A & Q \end{pmatrix}}_{\begin{pmatrix} A\overset{L}{\otimes}_A A \overset{L}{\otimes}_A A & A\overset{L}{\otimes}_A A\overset{L}{\otimes}_A Q \\ P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A A & P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A Q \end{pmatrix}}, \longrightarrow C \quad \text{is a quis}$$

So ~~[illegible]~~ for C to be h-unital, we need besides $P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A Q \simeq B$ (i.e. B is h-unital), that $P\overset{L}{\otimes}_A A \simeq P$ and $A\overset{L}{\otimes}_A Q \simeq Q$ i.e. P, Q are d-firm over A. (We already have a counterexample for this always holding; p. 113.)

February 6, 1995

Let's try to prove Morita invariance for K_1 under suitable flatness assumptions. Consider a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where everything is firm: $A \otimes_A A \xrightarrow{\sim} A$, $A \otimes_A Q \xrightarrow{\sim} Q$, etc. and everything is left flat; i.e. ${}_A A$, ${}_A Q$, ${}_B P$, ${}_B B$ are flat modules. (I recall that it suffices to assume that A is firm and left flat, ${}_A Q$ is firm flat, P_A is firm, and $Q \otimes P \to A$ is surjective. In effect ${}_A Q$ flat $\Longrightarrow$ $P \otimes_A Q = B$ is left B-flat, and ${}_A A$ flat $\Longrightarrow$ $P = P \otimes_A A$ is left B-flat.)

Applying these facts just recalled in the case of the Morita context

$$\left(\begin{array}{c|cc} A & A & Q \\ \hline A & A & Q \\ P & P & B \end{array}\right)$$

we see it also has everything firm + left flat.

I want to establish a canonical isomorphism $K_1(A) \simeq K_1(B)$ in the case of an "everything-firm-and-left-flat" Morita context. We have homomorphisms

$$A \subset \underbrace{\begin{pmatrix} A & Q \\ P & B \end{pmatrix}}_{C} \supset B$$

which induce maps $K_1(A) \longrightarrow K_1(C) \longleftarrow K_1(B)$.

By Vaserstein's lemma: $\quad q \in M_{nk}(P)$, $p \in M_{kn}(Q) \Rightarrow 1 + qp \in GL_n(A)$

$$\Longrightarrow \begin{pmatrix} (1+qp)^{-1} & 0 \\ 0 & (1+pq) \end{pmatrix} \in E_{n+k}(C) \qquad \text{we know}$$

that $\mathrm{Im}\{K_1(A) \to K_1(C)\} = \mathrm{Im}\{K_1(B) \to K_1(C)\}$.

I now want to show that $K_1(A) \to K_1(C)$ is surjective. Consider the homomorphisms

$$
\begin{tikzcd}
A \arrow[d] \arrow[dr, "\text{(upper left)}"] \arrow[dr, phantom, shift right=4, "*"] & & \\
\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[r, "\text{(lower right)}"'] & \left(\begin{array}{c|cc} A & A & Q \\ \hline A & A & Q \\ P & P & B \end{array}\right) \arrow[r] & \left(\begin{array}{c|c} C & C \\ \hline C & C \end{array}\right)
\end{tikzcd}
$$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = C \qquad \left(\begin{array}{c|cc} A & A & Q \\ \hline A & A & Q \\ P & P & B \end{array}\right) = D \qquad \left(\begin{array}{c|c} C & C \\ \hline C & C \end{array}\right) = M_2(C)$$

These induce maps on K_1

$$
\begin{tikzcd}
K_1(A) \arrow[d, "1"'] \arrow[dr, "2"] \arrow[dr, phantom, shift right=4, "*"] & & \\
K_1(C) \arrow[r, "3"] \arrow[rr, bend right, "\cong"'] & K_1(D) \arrow[r, "4"] & K_1(M_2(C))
\end{tikzcd}
$$

where the Δ $*$ commutes because the two obvious embeddings $A \rightrightarrows D$ factor $A \rightrightarrows M_2(A) \subset D$ and both embeddings $A \rightrightarrows M_2(A)$ have the same effect on K_1. We know by Vaserstein above that $\mathrm{Im}\,2 = \mathrm{Im}\,3$. Given $\gamma \in K_1(C)$ $\exists\, \alpha \in K_1(A)$ such that $2\alpha = 3\gamma$. Then $43\gamma = 42\alpha = 431\alpha$ $\Rightarrow \gamma = 1\alpha$ as 43 is an isom. Thus 1 is surjective as claimed.

Up to now I haven't used flatness, and probably everything holds with everything idempotent: $A = A^2 = QP$ etc.

Next I want to show that $K_1(A) \to K_1(C)$ is injective. By assumption Q is a flat A-module, hence it is a filtered inductive limit of finitely generated free modules: $Q = \varinjlim F_\alpha$, where $F_\alpha = \tilde{A}^{n_\alpha}$.

We also have

$$Q = A \otimes_A Q = \varinjlim_\alpha A \otimes_A F_\alpha = \varinjlim_\alpha AF_\alpha$$

with $AF_\alpha = A^{n_\alpha}$. Hence

$$C = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q) = \varinjlim_\alpha \underbrace{\begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad AF_\alpha)}_{C_\alpha}$$

Note that C_α is idempotent since $\begin{pmatrix} A \\ P \end{pmatrix}$, $(A \quad AF_\alpha)$ are fin A-modules and the pairing

$$(A \quad AF_\alpha) \otimes \begin{pmatrix} A \\ P \end{pmatrix} \longrightarrow A$$

is surjective.

We have $K_1(C) = GL(C)_{ab} = \varinjlim_\alpha K_1(C_\alpha)$, and since this is a filtered inductive limit, any $\alpha \in \operatorname{Ker}\{K_1(A) \to K_1(C)\}$ goes to zero in some $K_1(C_\alpha)$.

From the map $F_\alpha \longrightarrow Q$ we get an induced pairing $F_\alpha \otimes P \longrightarrow Q \otimes P \longrightarrow A$, i.e. an A^{op}-module map $P \longrightarrow \operatorname{Hom}_A(F_\alpha, A) \simeq A^{n_\alpha}$

This gives us a homomorphism

$$C_\alpha = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad AF_\alpha) \longrightarrow \begin{pmatrix} A \\ \operatorname{Hom}_A(F_\alpha, A) \end{pmatrix} \otimes_A (A \quad AF_\alpha)$$

But the latter is $M_{n_\alpha+1}(A)$. The composition

$$A \longrightarrow C_\alpha \longrightarrow M_{n_\alpha+1}(A)$$

is the upper left inclusion, which induces an isomorphism on K_1. Thus $K_1(A) \to K_1(C_\alpha)$ is injective.

Next I would like to check this injectivity result holds for all K_n. Review the assumptions. Given A, ${}_AQ$, P_A, $\psi : Q \otimes P \longrightarrow A$, where A, ${}_AQ$, P_A are firm and ψ is an arbitrary A-bimodule map. Then $C = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \;\; Q) = \begin{pmatrix} A & Q \\ P & P\otimes_A Q \end{pmatrix}$ is firm. This is not enough because I want $K_n(A)$ and $K_n(C)$ to be defined. So I assume that A is left flat and that ${}_AQ$ is flat. Then I know that C is left flat. Thus, A and C, being left flat idempotent rings, are h-unital, so $K_n(A)$ and $K_n(C)$ are defined.

Because ${}_AQ$ is flat we have $\underbrace{Q = \varinjlim_\alpha F_\alpha}_{\text{a filtered inductive limit}}$ where the F_α are f.g. free A-modules. Also $Q = A \otimes_A Q = \varinjlim_\alpha AF_\alpha$, where $AF_\alpha \simeq A\tilde{A}^{n_\alpha} = A^{n_\alpha}$ is a firm flat A-module. Thus $C_\alpha = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \;\; AF_\alpha)$ is firm & left flat, and its K_n is defined. Moreover

$$K_n(C) = \varinjlim_\alpha K_n(C_\alpha)$$

so to prove $K_n(A) \longrightarrow K_n(C)$ is injective, it's enough to show $K_n(A) \longrightarrow K_n(C_\alpha)$ is injective.

So can assume $Q = AF$ with F f.g. free and where the pairing $AF \otimes P \longrightarrow A$ is the composition $AF \otimes P \twoheadrightarrow F \otimes P \longrightarrow A$. This means we have a map $P \longrightarrow \mathrm{Hom}_A(F, A)$ compatible with pairings. More precisely

$$AF \otimes P \longrightarrow F \otimes P \longrightarrow F \otimes \mathrm{Hom}_A(F, A) \xrightarrow{\text{canon}} A$$

better:

$$\begin{array}{ccccc} AF \otimes P & \longrightarrow & AF \otimes \mathrm{Hom}_A(F, A) & \xrightarrow{\text{canon}} & A \\ & & \wr\!\wr \qquad\quad \wr\!\wr & & \| \\ & & A^n \;\otimes\; A^n & \longrightarrow & A \end{array}$$

dot product

$$(a_1 \cdots a_n) \otimes \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix} \longmapsto \sum a_i a_i'$$

Then we get a map

$$A \longrightarrow \begin{pmatrix} A & AF \\ P & P \otimes_A AF \end{pmatrix} \longrightarrow \begin{pmatrix} A & A^n \\ A^n & M_n(A) \end{pmatrix} = M_{n+1}(A)$$

$$A \longrightarrow C_\alpha \longrightarrow M_{n+1}(A)$$

which implies $K_*(A) \hookrightarrow K_*(C_\alpha)$. $\therefore$ get

<u>Prop.</u> Assume A firm and left flat, P_A firm, ${}_AQ$ firm and flat, $\psi : Q \otimes P \to A$ arbitrary A-bimod map. Then $C = \begin{pmatrix} A & Q \\ P & P \otimes_A Q \end{pmatrix}$ is firm and left flat, and the obvious map $K_*(A) \to K_*(C)$ is injective.

February 16, 1995

From Kucerovsky's thesis (Cohen's theorem in the case of C^* algebras + Hilbert modules).

Lemma 1. E a right Hilbert A-module, then any $x \in E$ can be written $x = y\langle y, y\rangle$ for some $y \in E$.

Proof. Consider

$$\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \in \mathcal{K}_A(A \oplus E)$$

where $x^*(\xi) = \langle x, \xi \rangle$ for $\xi \in E$. This is self-adjoint. Apply the continuous function $t \mapsto t^{1/3}$ to get

$$\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}^{1/3} = \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}$$

where the diagonal entries must vanish, because the cube root must anti-commute with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (Recall $a \mapsto a^{1/3}$ is obtained by polynomial approximation, and the polynomials can be taken to be odd.) Then $x = yy^*y$ where $y^*y = \langle y, y \rangle$.

In particular any element of A can be factored xx^*x or xx^*xx^*x etc.

Lemma 2: If $x^*x \le b^3$, where $b \ge 0$, in a C^*-algebra A, then $x = cb$ with $\|c\| \le \|b\|^{1/2}$.

Proof. Take $E = A \oplus A$. By hypothesis $\exists y$ such that $x^*x + y^*y = b^3$. Consider $x \oplus y \in A \oplus A$, apply lemma 1 to get $c \oplus d$ in $A \oplus A$ with $x \oplus y = (c \oplus d)\langle c\oplus d, c \oplus d\rangle$, where

$$\begin{array}{ccc} \langle x\oplus y, x \oplus y\rangle & = & \langle c \oplus d, c \oplus d\rangle^3 \\ \| & & \| \\ b^3 = x^*x + y^*y & & (c^*c + d^*d)^3 . \end{array}$$

Thus as $b \geq 0$ we have $b = c^*c + d^*d$ and $x \oplus y = (c\oplus d)(c^*c + d^*d) = (c \oplus d)\, b$. Thus $x = cb$ where $\|c\|^2 = \|c^*c\| \leq \|c^*c + d^*d\| = \|b\|$.

<u>Cohen theorem:</u> E Hilbert B-module, $A \xrightarrow{\phi} \mathcal{L}_B(E)$ a $*$ homomorphism, then any $\xi \in \overline{AE}$ factors $\xi = a \cdot \xi'$ with $a \in A$, $\xi' \in E$.

<u>*Proof.*</u> Let $\xi \in \overline{AE}$. There is an approx unit $\{a_\lambda\}$ for A and it acts as a left approximate unit on $\overline{AE}$ (meaning $\|a_\lambda y - y\| \to 0$, $y \in \overline{AE}$).

Define inductively $x_0 = \xi$, $x_{n+1} = x_n - a_n x_n$ where a_n is chosen using an approximate unit such that $\|x_{n+1}\| \leq 4^{-n-1}$ and $\|a_n\| \leq 1$. Then

$$\sum_{n=0}^{\infty} a_n x_n = \sum_{n=0}^{\infty} x_n - x_{n+1} = \xi$$

Now let $b_n = a_n 2^{-n}$ and $b = \left(\sum b_n b_n^*\right)^{1/3}$. Since $b_n b_n^* \leq b^3$, lemma 2 yields $b_n^* = c_n b$ with $\|c_n\| \leq \|b\|^{1/2}$.

Then $\xi = \sum_1 a_n x_n^{*} = \sum b_n (2^n x_n)$

$= b \sum_1 c_n^* 2^n x_n = b\,\xi'$, where $\xi' = \sum c_n^* 2^n x_n \in \overline{AE}$

$\subset E$.

Here's a proof of triple factorization by the technique used in the first lemma.

Given $a_1, \dots, a_n \in A$ we consider the s.a. matrix

$$\begin{pmatrix} 0 & a_1^* \cdots a_n^* \\ a_1 & \\ \vdots & 0 \\ a_n & \end{pmatrix} \in M_{n+1}(A)$$

and take its 5th root which has the form

$$\begin{pmatrix} 0 & x_1^* \cdots x_n^* \\ x_1 & \\ \vdots & 0 \\ x_n & \end{pmatrix}.$$

Then $a_j = x_j \left(\sum_1 x_i^* x_i\right)^2$. In particular $a_j = b_j cd$ $\forall j$ where the left annihilator of cd equals the left annihilator of c.

February 18, 1995

I would like to understand Kucerovsky's functional calculus for unbounded regular normal operators. Recall the basic definition due to Baaj. A closed densely-defined operator $T : E_1 \to E_2$ between Hilbert modules is <u>regular</u> when $E_1 \oplus E_2 = \Gamma_T \oplus \Gamma_T^\perp$ and $\Gamma_T^\perp$ is ε the graph of a densely defined operator $T^* : E_2 \to E_1$:

$$\Gamma_T^\perp = \left\{ \begin{pmatrix} -T^*\xi \\ \xi \end{pmatrix} \,\middle|\, \xi \in \mathcal{D}_{T^*} \subset E_2 \right\}.$$

I would like to reduce as much as possible to unitaries via the Cayley transform.

Recall the basic formulas for bounded adjointable operators T first. Put $X_1 = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$. X_1 is skew-adjoint in the C^*-alg $\mathcal{L}(E_1 \oplus E_2)$, hence $1 + X_1 = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}$ is invertible, showing that $E_1 \oplus E_2 = \Gamma_T \oplus \varepsilon \Gamma_{T^*}$. Let $g_1 = \frac{1+X_1}{1-X_1}$ be the Cayley transform of X and $F = g_1 \varepsilon$.

$$F \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = g_1 \varepsilon (1 + X_1) = \frac{1+X_1}{1-X_1}(1-X_1)\varepsilon = (1+X_1)\varepsilon$$

i.e.

$$F \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which means that F is the involution such that $F = +1$ on Γ_T and $F = -1$ on $\varepsilon \Gamma_{T^*}$.

Let's now restrict to $E_1 = E_2 = E$ and $T = X$ skew adjoint on E. Let $g = \frac{1+X}{1-X}$, so that g is a unitary on E. We have the decomposition

$$E \oplus E = \Gamma_X \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_X$$

~~[illegible]~~ since $\begin{pmatrix}-T^*\\ 1\end{pmatrix}E = \begin{pmatrix}X\\ 1\end{pmatrix}E = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Gamma_X$.

Now $\quad g = \dfrac{1+X}{1-X} \quad \Longrightarrow \quad g = -1+\dfrac{2}{1-X} = 1+\dfrac{2X}{1-X}$

$\Longrightarrow \quad \dfrac{g-1}{g+1} = X$. Thus

$$\Gamma_X = \begin{pmatrix}1\\ \frac{g-1}{g+1}\end{pmatrix}E = \begin{pmatrix}g+1\\ g-1\end{pmatrix}E, \qquad \sigma\Gamma_X = \begin{pmatrix}g-1\\ g+1\end{pmatrix}E$$

where $\sigma = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Observe that $h = \begin{pmatrix}\frac{g+1}{2} & \frac{g-1}{2}\\ \frac{g-1}{2} & \frac{g+1}{2}\end{pmatrix}$

is unitary since

$$hh^* = \begin{pmatrix}\frac{g+1}{2} & \frac{g-1}{2}\\ \frac{g-1}{2} & \frac{g+1}{2}\end{pmatrix}\begin{pmatrix}\frac{g^{-1}+1}{2} & \frac{g^{-1}-1}{2}\\ \frac{g^{-1}-1}{2} & \frac{g^{-1}+1}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\dfrac{1+1+g+g^{-1}+1+1-g-g^{-1}}{4} & \dfrac{1-1+g^{-1}-g+1-1-g^{-1}+g}{4}\\[2ex] \dfrac{1-1+g-g^{-1}+1-1-g+g^{-1}}{4} & \dfrac{1+1-g-g^{-1}+1+1+g+g^{-1}}{4}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and similarly $h^*h = 1$. We have

$$hE_+ = \Gamma_X \qquad hE_- = \Gamma_X^{\perp}$$

E_+ where $\varepsilon = +1$
E_- ——— -1

so that $h\varepsilon h^{-1} = F$, the involution corresponding the decomposition $E_+ \oplus E_- = \Gamma_X \oplus \Gamma_X^{\perp}$. Thus

$$F = \begin{pmatrix}\frac{g+1}{2} & \frac{g-1}{2}\\ \frac{g-1}{2} & \frac{g+1}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{g^{-1}+1}{2} & \frac{g^{-1}-1}{2}\\ \frac{g^{-1}-1}{2} & \frac{g^{-1}+1}{2}\end{pmatrix} = \begin{pmatrix}\frac{g+1}{2} & \frac{-g+1}{2}\\ \frac{g-1}{2} & \frac{-g-1}{2}\end{pmatrix}\begin{pmatrix}\frac{g^{-1}+1}{2} & \frac{g^{-1}-1}{2}\\ \frac{g^{-1}-1}{2} & \frac{g^{-1}+1}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\dfrac{1+1+g+g^{-1}-1-1+g+g^{-1}}{4} & \dfrac{1-1-g+g^{-1}+1+1-g+g^{-1}}{4}\\[2ex] \dfrac{1-1+g-g^{-1}-1+1+g-g^{-1}}{4} & \dfrac{1+1-g^{-1}-g-1-1-g^{-1}-g}{4}\end{pmatrix} = \begin{pmatrix}\dfrac{g+g^{-1}}{2} & \dfrac{g^{-1}-g}{2}\\[2ex] \dfrac{g-g^{-1}}{2} & -\dfrac{g+g^{-1}}{2}\end{pmatrix}$$

So far we have been assuming that $g = \frac{1+X}{1-X}$ with X bdd skew adjoint in $\mathcal{L}(E)$. I would now like to check that the formula

$$F = \begin{pmatrix} \frac{g+g^{-1}}{2} & \frac{g^{-1}-g}{2} \\ \frac{g-g^{-1}}{2} & -\frac{g+g^{-1}}{2} \end{pmatrix}$$

gives a 1-1 correspondence between [redacted] unitaries g in $\mathcal{L}(E)$ and [redacted] splittings

$$E \oplus E = \Gamma \oplus \Gamma^{\perp} \quad \text{such that } \Gamma^{\perp} = \sigma\Gamma,$$

where of course $F = \pm 1$ on $\Gamma, \Gamma^{\perp}$ resp.. It's equivalent to say F is an involution (self-adjoint) in $\mathcal{L}(E)$ anticommuting with σ.

Let u be the unitary $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and note that

$$u^{-1}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} u = \frac{1}{2}\begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} g^{-1} & g^{-1} \\ g & -g \end{pmatrix} = \begin{pmatrix} \frac{g+g^{-1}}{2} & \frac{g^{-1}-g}{2} \\ \frac{g-g^{-1}}{2} & -\frac{g+g^{-1}}{2} \end{pmatrix}$$

$$u^{-1}\varepsilon u = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma$$

Now $\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$, as g ranges [redacted] over unitaries in $\mathcal{L}(E)$, yields all involutions in $\mathcal{L}(E \oplus E)$ anticommuting with $-\varepsilon$, hence $u^{-1}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} u = \begin{pmatrix} \frac{g+g^{-1}}{2} & \frac{g^{-1}-g}{2} \\ \frac{g-g^{-1}}{2} & -\frac{g+g^{-1}}{2} \end{pmatrix}$ gives all involutions anti-commuting with σ, as claimed.

The Cayley transform picture which identifies splittings $E\oplus E = \Gamma \oplus \Gamma^{\perp}$ such that $\Gamma^{\perp} = \sigma\Gamma$ with unitaries g is nice, but somehow the interesting stuff begins when we ask Γ to be the graph of a densely defined operator X. X is then a closed densely-defined skew-adjoint operator on E.

This condition means that $\Gamma \subset E\oplus E \xrightarrow{pr_1} E$ is injective with dense image. Let g be the unitary corresponding to Γ. Since

$$\Gamma = \begin{pmatrix} g+1 \\ g-1 \end{pmatrix} E$$

we have a bijection $E \xrightarrow{\frac{1}{2}\begin{pmatrix} g+1 \\ g-1 \end{pmatrix}} \Gamma$. (Recall this is the restriction of the unitary h to $E = E_+ \subset E\oplus E$). Thus

$$\begin{tikzcd}
\Gamma \arrow[d, equal] & \subset & E\oplus E \arrow[r, "pr_1"] & E \\
E \arrow[urr, "\frac{1}{2}\begin{pmatrix} g+1 \\ g-1 \end{pmatrix}"] \arrow[urrr, bend right, "\frac{g+1}{2}"'] & & &
\end{tikzcd}$$

so we want $\frac{g+1}{2}$ to be injective and hence dense image. Note that

$$(g+1)E = (g+1)g^{-1}E = (g^{-1}+1)E$$

and $\langle \xi, (g+1)\eta \rangle = \langle (g^{-1}+1)\xi, \eta \rangle$, so if $\overline{(g+1)E} = E$, then $\overline{(g^{-1}+1)E} = E \Longrightarrow \big((g+1)\eta = 0 \Longrightarrow \eta = 0 \big)$. Thus the condition $\overline{(g+1)E} = E$ implies $g+1$ and $g^{-1}+1$ are injective. Note that $(g+1)E$ is the domain $\mathcal{D}_X$.

At this point we have identified unbdd regular skew-adjoint operators X on the Hilbert module E with unitaries g in $\mathcal{L}(E)$ such that $\overline{(g+1)E} = E$.

February 19, 1995

Yesterday we identified regular (possibly) unbounded skew adjoint operators X on a Hilbert module E with unitaries g in $\mathcal{L}(E)$ such that $\overline{(g+1)E} = E$. The correspondence is as follows:

1) $g \longmapsto \Gamma = \begin{pmatrix} g+1 \\ g-1 \end{pmatrix} E$ is a bijection from $U(\mathcal{L}(E))$ to submodules $\Gamma \subset E \oplus E$ such that $\Gamma \oplus \Gamma^{\perp} = E \oplus E$ and $\Gamma^{\perp} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma$.

2) $\Gamma = \begin{pmatrix} g+1 \\ g-1 \end{pmatrix} E$ is the graph of a regular unbdd skew-adjoint operator $X \iff \overline{(g+1)E} = E$.

In this situation $g = \frac{1+X}{1-X}$, $X = \frac{g-1}{g+1}$.

Next comes Kucerovsky's functional calculus for regular self-adjoint operators D say. Corresponding to $X = iD$ is the unitary $g = \frac{1+iD}{1-iD}$, and g is equivalent to a unital $*$ homomorphism

$$C(S^1) \xrightarrow{\phi} \mathcal{L}(E) \qquad z \longmapsto g$$

The condition $\overline{(g+1)E} = E$ means exactly that setting $A = C_0(S^1 - \{-1\}) \subset C(S^1)$, so that $\tilde{A} = C(S^1)$, then the restriction

$$\phi : A \longrightarrow \mathcal{L}(E)$$

is approximately unital which means the following equivalent conditions hold:

1) $\overline{\phi(A)E} = E$.

2) There is an approx identity $\{a_\lambda\}$ for A such that $\phi(a_\lambda)\xi \to \xi$ for all $\xi \in E$.

3) Condition 2) holds for all approximate identities.

There's a general criterion for $\phi : A \to \mathcal{L}(E)$ to extend to a strictly continuous $*$ homomorphism $\underline{\phi} : \mathcal{M}(A) \to \mathcal{L}(E)$, namely this happens iff $\exists$ a projection p in $\mathcal{L}(E)$ such that $\overline{AE} = pE$. (see Jensen + Thomsen Elements of KK-Theory, 1.1.13). Using this criterion we see our approx. unital $*$ hom

$$A = C_0(S^1 - \{-1\}) \xrightarrow{\phi} \mathcal{L}(E)$$

has a unique strictly continuous extn to $\mathcal{M}(A)$. Recall that $\mathcal{M}(A)$ consists of all bdd continuous functions on $S^1 - \{-1\}$.

Let's now use the identification

$$C_0(S^1 - \{-1\}) = C_0(\mathbb{R})$$

given by the homeom. $z \longleftrightarrow \dfrac{1+it}{1-it}$

March 8, 1995

I will attempt to describe [redacted] the things I have been learning about on this trip. [redacted] I want to throw away the scratch work which is on the papers from the Mods exams, but I don't have time to work out the details.

Book, Jensen + Thomsen,

[redacted] This book starts with [redacted] Hilbert modules E over B, the C^* algebra $L_B(E)$ and the closed $*$ideal $K_B(E)$. An important idea is the multiplier algebra $\mathcal{M}(A)$ of a C^*-algebra A. [redacted]

The picture is the following. A sits inside $\mathcal{M}(A)$ as a norm closed $*$ideal, which is dense in the strict [redacted] topology. More precisely if $m \in \mathcal{M}(A)$ and (u_α) is an approx identity, then $m = \lim m u_\alpha = \lim u_\alpha m$ in the strict topology. The first result is that a $*$-homom. $\phi : A \longrightarrow K_B(E)$ extends to a strictly continuous $*$-hom $\bar{\phi} : \mathcal{M}(A) \longrightarrow L_B(E)$ iff $\overline{\phi(A)E} = pE$ for some projection (s.a. understood) p in $L_B(E)$. From this follows that $\mathcal{M}(K_B(E)) = L_B(E)$.

~~I would like to understand the multiplier algebra much better. One thing that I feel can be done is to develop the C^* algebra theory~~

It might be useful to develop the theory of C^*-algebras making extensive use of Hilbert modules. At the [redacted] moment one defines [redacted] $L_B(E_1, E_2)$ as the set of adjointable operators $T : E_1 \longrightarrow E_2$ commuting with the right B-action, then uses the closed graph theorem (or Banach-Steinhaus) to conclude T is bounded. I think one might be able to [redacted] directly [redacted] define [redacted] the C^* algebra $L_B(E_1 \oplus E_2)$ in terms of $K_B(E_1 \oplus E_2)$:.

Note the formulas

$$K_B(B,E) \xleftarrow{\sim} E$$

(*)

$$K_B(E,B) \xleftarrow{\sim} {}_B E^{*}$$

more precisely the latter is given by the map $x \longmapsto \langle x, - \rangle$; ${}_B E^{*}$ is E considered as left B module via $b.x = xb^*$.

~~Idea: Place yourself in your module theory for idempotent rings and try to~~

<u>Idea</u>: Start with your module theory for an idempotent ring, and examine the Hilbert module theory. A Hilbert B-module E gives a triple $({}_B E, E, {}_B E \otimes E \longrightarrow B)$ like the ones $({}_B P, Q_B, P \otimes Q \longrightarrow B)$ you consider.

In the proof of (*) above [redacted] one needs [redacted] that $E = \overline{EB}$ for any Hilbert B-module E. In fact one has $E = \overline{E\langle E,E\rangle}$. This is based on completeness: If (u_α) is an approximate identity on $\langle E,E\rangle$ in the sense that $bu_\alpha \to b$ and $bu_\alpha \to b$ for $b \in \langle E,E\rangle$, [redacted] (also $\|u_\alpha\| \leq 1$), then

$$\langle \xi u_\alpha - \xi, \xi u_\alpha - \xi \rangle \longrightarrow 0$$

so $\xi = \lim \xi u_\alpha \in \overline{EB}$; then take $B = \overline{\langle E,E\rangle}$.

<u>Idea</u>: When we consider $\langle {}_B P, Q_B, P \otimes Q \xrightarrow{\varphi} B \rangle$ with φ not surjective, then up to Morita equivalence we are looking at a <u>quotient category</u> of $\mathcal{M}(B)$, namely $\mathcal{M}(\langle P,Q\rangle)$.

$$K_B(B \oplus E) = \begin{pmatrix} B & {}_B E \\ E & K_B(E) \end{pmatrix}$$

Statement from Pimsner's paper I don't understand (rather see the proof of).

$$\mathcal{E}_+ = \bigoplus_{n \geq 0} E^{\otimes n}$$ Hilbert module $\oplus$

$$L(\mathcal{E}_+) \supset J(\mathcal{E}_+) = C^*\text{-subalg gen. by } L(\mathcal{E}_+^{\text{finite}})$$

Then $$M(J(\mathcal{E}_+)) = \{ T \in L(\mathcal{E}_+) \mid TJ(\mathcal{E}_+), J(\mathcal{E}_+)T \subset J(\mathcal{E}_+) \}$$

Facts about KK theory:

Kasparov bimodule (or cycle) is a triple (E, ϕ, F) where E is a Hilbert B-module (right) countably generated, $\phi : A \longrightarrow L_B(E)$ a $*$ hom., $F \in L_B(E)$ such that $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$, $(F - F^*)\phi(a) \in K_B(E)$ for $a \in A$.

Kasparov defines two versions of $KK(A,B)$:

cycles/htpy cycles/ ~~stabilizing~~ wrt degenerate cycles and operator htpy.

Then shows the obvious map $\leftarrow$ is bijective. Here A is assumed separable (i.e. countably generated over itself. This is needed for applying Kasparov's stabilization theorem: $E \oplus H_B \approx H_B$ for any countably generated Hilbert B-module E.

In the Jensen-Thomsen book Kasparov's theory with the cup product is introduced first, then simpler models are shown to be equivalent to it.

Example: Extensions. Extensions of A by $K_B(H_B) = \mathcal{K} \otimes B$ are equivalent to homomorphisms

$$\phi : A \longrightarrow L_B(H_B)/K_B(H_B) = \mathcal{Q}_B(H_B) \qquad \text{"Calkin alg".}$$

Now identify extensions which are conjugate under unitaries in $L_B(H_B)$ and you get an abelian monoid under Whitney sum. Invert, i.e. stabilize with respect to degenerate extensions (where ϕ lifts to $A \longrightarrow L_B(H_B)$) and take the invertible elements. This gives $\mathrm{Ext}^{-1}(A, \mathcal{K}\otimes B)$ ($^{-1}$ means invertible)

Result: ϕ invertible $\Longleftrightarrow$ ϕ lifts to a completely positive $\rho : A \longrightarrow L_B(H_B)$. Proof: Dilating ρ gives another extension whose Whitney sum with ϕ is degenerate.

Another result identifies $\mathrm{Ext}^{-1}(A, \mathcal{K}\otimes B)$ with $KK^1(A,B)$. Here one starts with (E, ϕ, F) (ungraded), applies Kasparov stabilization to assume up to adding degenerate cycles that $E = H_B$. Next up to operator homotopy can assume $F = F^*$ and $\|F\| \leq 1$. Then by dilating F to an involution one gets the situation of the Whitney sum of two extensions being a degenerate extension.

Apparently similar arguments work in the even case leading to quasi-homomorphisms.

The rest problem for generalizing this stuff is how to handle homotopies.

There is work of Higson on the half-exactness of KK which uses another infinite process. Apparently excision requires another hypothesis like nuclearity.

Homotopy for cycles means ~~[illegible]~~ specializing a cycle for A, $IB = C([0,1], B)$. A degenerate cycle is homotopic to the zero cycle. The point is that if E is a Hilbert module over $C_0([0,1), B)$, then specializing at $t=1$ yields the zero module, because one completes the the algebraic tensor product.

~~[illegible]~~

The equivalence of the two KK definitions by Kasparov can be interpreted as a homotopy invariance theorem for the KK groups defined by stabilizing wrt degenerate cycles and using operator homotopy.

Question: In the case of $E = C_0(X, \ell_2)$ can you study $L_B(E)$, show $L_B(E) =$ strongly cont $f : X \longrightarrow L(\ell_2)$, by graph methods? It is easy to identify an orthogonal splitting of E with a strongly continuous family F_x of involutions on ℓ^2. Can we extend this to handle graphs of adjointable operators?

Idea: Try to link the Atiyah-Singer periodicity proof ~~[illegible]~~ which involves projections and unitaries to Kasparov's ideas. One difference is the norm versus the strong topology. Different types of projections in $\mathcal{L}(H)$.

Let E be a Hilbert B-module. Form $B_{(1)} = B \times B$ and $E_{(1)} = E \times E$ and write an element of $E_{(1)}$ as $\begin{pmatrix} x \\ y \end{pmatrix}$, and an element of $B_{(1)}$ as $\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$; the multiplication is

$$\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} xb_1 \\ yb_2 \end{pmatrix}$$

and the inner product is

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \begin{pmatrix} \langle x_1, x_2 \rangle & 0 \\ 0 & \langle y_1, y_2 \rangle \end{pmatrix}$$

Now introduce the $\mathbb{Z}/2$ gradings

$$\varepsilon\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \qquad \varepsilon\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} b_2 & 0 \\ 0 & b_1 \end{pmatrix}.$$

These are compatible with the multiplication and inner product. For example

$$\left\langle \underbrace{\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}}_{\text{even}}, \underbrace{\begin{pmatrix} x_2 \\ -x_2 \end{pmatrix}}_{\text{odd}} \right\rangle = \underbrace{\begin{pmatrix} \langle x_1, x_2 \rangle & 0 \\ 0 & -\langle x_1, x_2 \rangle \end{pmatrix}}_{\text{odd.}}$$

Let F be an operator on E. Then we can extend F' uniquely to an odd operator on $E_{(1)}$ reducing to F on the first components.

$$F'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x) \\ -F(y) \end{pmatrix}$$

Clearly this is a one-one correspondence between operators F on E and odd operators F' on $E_{(1)}$ given in this way.

Let's now calculate F'^*

$$\left\langle F'\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} F(x_1) \\ -F(y_1) \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle$$

$$= \begin{pmatrix} \langle Fx_1, x_2\rangle & 0 \\ 0 & -\langle Fy_1, y_2\rangle \end{pmatrix} = \begin{pmatrix} \langle x_1, F^*x_2\rangle & 0 \\ 0 & -\langle y_1, F^*y_2\rangle \end{pmatrix}$$

$$= \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} F^*x_2 \\ -F^*y_2 \end{pmatrix} \right\rangle$$

Thus $\boxed{F'^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F^*x \\ -F^*y \end{pmatrix}}$ so $F' = F'^* \Leftrightarrow F = F^*$

March 10, 1995

I would like to summarize stuff on the "polar decomposition", by which I mean a circle of ideas including the

Lemma: ~~[illegible]~~ Assume $b \geq 0$. Then $x^*x \leq b^3 \iff \exists\, c$ such that $x = cb$ and $c^*c \leq b$.

I learned from Kučerovský's thesis.

~~[illegible]~~

The polar decomposition of x is

$$x = x(x^*x)^{-1/2} \cdot (x^*x)^{1/2}$$

when x is invertible. When x is not invertible then $x(x^*x)^{-1/2}$ is not necessarily defined. However one always has the decomposition

$$x = x(x^*x)^{-\varepsilon} \cdot (x^*x)^{\varepsilon} \qquad \text{for } 0 \leq \varepsilon < \tfrac{1}{2}.$$

More generally one has

$$x = x\, f(x^*x)^{-1} \cdot f(x^*x)$$

where $f: \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is any continuous function such that $t f(t^2)^{-1}$ and $f(t^2)$ extend to continuous functions on $\mathbb{R}$. The proof is to apply functional calculus for self-adjoint ~~[illegible]~~ elements to $x' = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}$. Note $x'^2 = \begin{pmatrix} x^*x & 0 \\ 0 & xx^* \end{pmatrix}$

Approximating $g_1(t) = t f(t^2)^{-1}$ and $g_2(t) = f(t^2)$ over $[-\|x\|, \|x\|]$ by odd and even polynomials respectively, one gets the existence of $g_2(x'^2) = \begin{pmatrix} f(x^*x) & 0 \\ 0 & f(xx^*) \end{pmatrix}$ and

$$g_1(x') = \text{[redacted]} \begin{pmatrix} 0 & x^* f(xx^*)^{-1} \\ x f(x^*x)^{-1} & 0 \end{pmatrix}$$

Another point is that if one has $x^*x \le a$, then [redacted] one has the decomposition

$$x = x f(a)^{-1} \cdot f(a)$$

The proof is to choose y such that $x^*x + y^*y = a$, e.g. $y = (a - x^*x)^{1/2}$, and apply the functional calculus to $x' = \begin{pmatrix} 0 & x^* & y^* \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$; in this case

$$x' f(x'^2)^{-1} = \begin{pmatrix} 0 & (xf(a)^{-1})^* & (yf(a)^{-1})^* \\ xf(a)^{-1} & 0 & 0 \\ yf(a)^{-1} & 0 & 0 \end{pmatrix}.$$

[redacted]

In general one would like to know when one can factor x into ub with $b \ge 0$ prescribed. A necessary condition is that $xb^t \to x$ as $t \downarrow 0$ and this condition [redacted] is equivalent to $x \in \overline{Ab}$. Check: $x \in \overline{Ab} \Rightarrow \forall \varepsilon > 0 \ \exists u \ni \|x - ub\| \le \varepsilon \Rightarrow \|xb^t - ubb^t\| \le \varepsilon \|b\|^t \Rightarrow \overline{\lim} \|xb^t - x\| \le 2\varepsilon$. $\therefore xb^t \to x$.

It seems likely that $x \in \overline{Ab} \iff x \in A f(b)$ where f is a suitable unbounded function of b.

Return to $x^*x \le b^3 \Rightarrow x = ub$ with $u^*u \le b$. I claim in this situation that u is unique.

Indeed suppose $x = u_1 b = u_2 b$ where $u_i^* u_i \le b$. Then

$$\left(\frac{u_1-u_2}{2}\right)^*\left(\frac{u_1-u_2}{2}\right) + \left(\frac{u_1+u_2}{2}\right)^*\left(\frac{u_1+u_2}{2}\right) = \frac{u_1^*u_1+u_2^*u_2}{2} \le b$$

so $\Delta u^* \Delta u \le 2b$. But $(\Delta u) b = 0$, so

$$(\Delta u^* \Delta u)^3 \le (\Delta u^* \Delta u)\, 2b\, (\Delta u^* \Delta u) = 0$$

$$\Longrightarrow \quad \Delta u^* \Delta u = 0 \quad \Longrightarrow \quad \Delta u = 0 .$$

There is a lot of work one can do here to clean up things. I can mention some things:

To what extent is replacing (a_i) by $a_i \left(\sum a_j^* a_j\right)^{-\varepsilon}$ like working with non commutative projective coordinates, or noncomm. partitions of 1.

Meaning of the proof of Cohen's theorem:

Let $x_0 \in \overline{EB} \subset E$. Then

$$x_0 = x_0 b_0 + x_1 = x_0 b_0 + x_1 b_1 + x_2 = \cdots = \sum x_n b_n$$

$$= \sum x_n \lambda_n \, b_n/\lambda_n = \left(\sum x_n \lambda_n u_n\right)\left(\sum \frac{b_n^* b_n}{\lambda_n^2}\right)^{\frac{1}{2}} = xb$$

with $x \in E$, $b \in B$. Here b_n is a "fast" approximate unit so that $x_n \to 0$ very fast, and that λ_n are like 2^n so that $\sum \frac{b_n^* b_n}{\lambda_n^2}$ converges.

March 16, 1995

I want to make a few notes about C^*-algebras before leaving MIT. This is mostly from Pedersen's book.

On multipliers. Terminology: left centralizer for A is a map $x \longmapsto Tx$ on A s.t. $T'(xa) = (T'x)a$. right centralizer similar; and a double centralizer is a pair (T', T'') consisting of a left + right centralizer such that $x(T'y) = (xT'')y$. Using multiplier notation we have

$$T(xy) = (Tx)y$$
$$(xy)T = x(Ty)$$
$$x(Ty) = (xT)y.$$

(i.e. double centralizer = multiplier in Hochschild's sense)

Notice that the latter implies the earlier two when A has zero left + right annihilators:

$$z\Big(T(xy)\Big) = (zT)(xy) = \Big((zT)x\Big)y$$
$$= \Big(z(Tx)\Big)y = z\Big((Tx)y\Big)$$

for all $z \implies T(xy) = (Tx)y$

There's also a notion of quasi-centralizer, namely a map $A \otimes A \longrightarrow A$ of A-bimodules.

Result 1. Any left, right, double, quasi-centralizer on a C^* algebra is bdd.

Proof in the right case. Assume $f : A \longrightarrow A$ satisfies $f(aa') = af(a')$. If f is not bounded we can find $x_n \in A$ such that $\|x_n\| \le 1$ and $\|f(x_n)\| \uparrow \infty$. Then passing to a subsequence and rescaling we can suppose $\sum x_n^* x_n = a \in A$ and that $\|f(x_n)\| \uparrow \infty$. Then factor

$$x_n = u_n a^{\varepsilon}$$

$u_n = x_n a^{-\varepsilon}$ here $0 < \varepsilon < \frac{1}{2}$

$u_n^* u_n \le \sum_i u_n^* u_n = a^{1-2\varepsilon}$

and you have

$$\|f(x_n)\| = \|f(u_n a^{\varepsilon})\| = \|u_n f(a^{\varepsilon})\| \le \|u_n\| \, \|f(a^{\varepsilon})\|$$

which is a contradiction as $\|u_n\| \le \|a\|^{\frac{1}{2}-\varepsilon}$.

Similarly for the 'left' case. In the quasi-case use the uniform bddness principle together with the left and right case.

<u>Result 2.</u> Suppose $A \subset L(H)$ nondegenerate in the sense that $\overline{AH} = H$. Then the left, right, double, quasi-centralizers on A can be identified respectively with the families of operators $T \in L(H)$ satisfying $TA \subset A$, $AT \subset A$, both of these, $ATA \subset A$.

It seems that there can exist left centralizers which do not extend to double centralizers, although I haven't seen a counterexample. Notice that if $T : A \longrightarrow A$ is a left-centralizer: $T(xy) = (Tx)y$, which has an adjoint T^* when A is viewed as right Hilbert A-module, i.e.

$$(T^* x)^* y = x^* (Ty)$$

then T has a compatible right centralizer given by

$$xT = \left(T^*(x^*)\right)^*$$

Conversely $(xT)y = x(Ty)$ can be changed to $\left((x^*T)^*\right)^* y = x^*(Ty)$ so that $y \longmapsto Ty$ has the adjoint $T^*(x) = (x^*T)^*$, equivalently $xT = \left(T^*(x^*)\right)^*$.

$\therefore$ adjointable for a one-sided multiplier means we have a multiplier.

<u>Result 3.</u> There's an equivalence between:

1) closed left ideals L in A
2) closed cones M in A_+ which are hereditary: $0 \le y \le x \in M \Rightarrow y \in M$.
3) C^* subalgebras $B \subset A$ which are hereditary: B_+ hereditary in A_+.

The equivalence is based on the following arguments. Given M let $L(M) = \{x \in A \mid x^*x \in M\}$. Then $L(M)$ is closed and $(ax)^*(ax) = x^*a^*ax \le \|a\|^2 x^*x \in M$ where $x^*x \in M$.
Also $(x+y)^*(x+y) + (x-y)^*(x-y) = 2(x^*x + y^*y) \in M$ when $x^*x, y^*y \in M$. $\therefore$ $L(M)$ is a closed left ideal in A.
Given L, $B = L \cap L^*$ is a C^* subalgebra of A such that $B_+ = L_+$. If $x \in L$, then $x = \left(x(x^*x)^{-1/3}\right)(x^*x)^{1/3} \in AL_+$. The rest should be clear.

<u>Result 4:</u> $A \to B$ surjective map of <u>separable</u> C^*-algebras $\Longrightarrow$ the induced map $M(A) \to M(B)$ is surjective.

In Jensen-Thomsen this result is refined so as to allow for commutation with a separable closed self adjoint subspace of $M(A)$, and this result is used to prove Cuntz's KK picture.

These ~~thms~~ results are non commutative versions of the Tietze extension theorem.

The multiplier alg $M(A)$ sits inside the von Neumann algebra A'' = double commutant in the universal Hilbert space representation.

Another result is that A'' as a top. vector space is the double dual of A.

In Blackadar's book, one finds an account of the Pimsner-Voiculescu calculation of K-groups of a cross product, proved via Connes' Thom isomorphism theorem. It is mentioned that the original PV proof uses arguments similar to those employed earlier by Cuntz to calculate K groups of $\mathcal{O}_n$.

March 25, 1995

Let A be a nonunital ring, let E be a left (resp. E^* a right) A-module, let

$$E \otimes_{\mathbb{Z}} E^* \longrightarrow A \qquad \xi \otimes \eta \longmapsto \langle \xi, \eta \rangle$$

be an A-bimodule map, let

$$\sum \eta_i \otimes \xi_i \in E^* \otimes_A E$$

be such that
$$\xi = \sum \langle \xi, \eta_i \rangle \xi_i \qquad \forall \xi \in E$$
$$(*) \qquad \eta = \sum \eta_i \langle \xi_i, \eta \rangle \qquad \forall \eta \in E^*.$$

Then we know that $E \in \mathcal{P}(\tilde{A})$, that

$$E^* \simeq \mathrm{Hom}_A(E, \tilde{A}) \in \mathcal{P}(\tilde{A}^\circ)$$

Moreover, because $\langle \, , \, \rangle$ has values in A, we have $E = AE$ and $E^* = E^* A$, so that E is a A-firm f.g. projective $\tilde{A}$-module.

Conversely if $E \in \mathcal{P}(A)$, $E^* = \mathrm{Hom}_A(E, \tilde{A})$ then one has a canonical pairing $E^* \otimes_{\mathbb{Z}} E \longrightarrow \tilde{A}$ and identity element $\sum \eta_i \otimes \xi_i \in E^* \otimes_A E \simeq \mathrm{Hom}_A(E,E)$ such that $(*)$ holds. If E is also firm, i.e. $E = AE$ since E is A-flat, then $\langle E, E^* \rangle = \langle AE, E^* \rangle = A \langle E, E^* \rangle \subset A$, so the pairing $\langle \xi, \eta \rangle$ has values in A. Thus we have the above situation.

Consider now a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$. Given $(E, E^*, \langle\,,\,\rangle, \sum \eta_i \otimes \xi_i)$ as above, consider the left B-module $P \otimes_A E$, the right B-module $E^* \otimes_A Q$ and the pairing

$$(P \otimes_A E) \otimes_Z (E^* \otimes_A Q) \longrightarrow B$$

$$\langle p \otimes \xi \,,\, \eta \otimes q \rangle = p \langle \xi, \eta \rangle q .$$

We wish to have an element lifting $\sum \eta_i \otimes \xi_i$ wrt

$$(E^* \otimes_A Q) \otimes_B (P \otimes_A E) \longrightarrow E^* \otimes_A E$$

$$(\eta \otimes q) \otimes (p \otimes \xi) \longmapsto \eta \otimes (qp) \xi$$

We have
$$\sum_i \eta_i \otimes \xi_i = \sum_{i,j} \eta_i \otimes \langle \xi_i, \eta_j \rangle \xi_j .$$

Assuming $A = QP$ we can write

$$\langle \xi_i, \eta_j \rangle = \sum_k q_{ijk} p_{ijk}$$

and then we get the element

$$\sum_{i,j,k} (\eta_i \otimes q_{ijk}) \otimes (p_{ijk} \otimes \xi_j) \in (E^* \otimes_A Q) \otimes_B (P \otimes_A E)$$

lifting $\sum \eta_i \otimes \xi_i \in E^* \otimes_A E$. We have

$$\sum_{ijk} \eta_i \otimes q_{ijk} \langle p_{ijk} \otimes \xi_j , \eta \otimes q \rangle = \sum_{ijk} \eta_i \otimes q_{ijk} p_{ijk} \langle \xi_j, \eta \rangle q$$

$$= \sum_{ij} \eta_i \otimes \langle \xi_i, \eta_j \rangle \langle \xi_j, \eta \rangle q = \sum \eta_i \otimes \langle \xi_i, \eta \rangle q = \eta \otimes q$$

and similarly

$$\sum_{ijk} \langle p\otimes\xi, \eta_i \otimes q_{ijk}\rangle p_{ijk}\otimes \xi_j = p\otimes \xi$$

Thus it follows that $P\otimes_A E$ is a firm fg proj. B-module with dual $E^*\otimes_A Q$.

We have proved:

<u>Prop:</u> <u>If $\begin{pmatrix} A & Q \\ P & B\end{pmatrix}$ is a Morita context such that $QP = A$ and $PQ = B$, then one has an equivalence between the [redacted] categories of [redacted] firm fg projective modules over A and B.</u>

[redacted] Remark that $\langle E, E^*\rangle$ is an idempotent ideal in A, so that if we only assume $QP \supset A^k$ $PQ \supset B^k$, then $QP \supset A^k \supset \langle E,E^*\rangle^k = \langle E, E^*\rangle$ and the preceding holds.

The next thing I would like to do is extend the [redacted] Morita equivalence above from firm fg projective modules to firm perfect complexes.

I want to study [redacted] length one complexes of fgproj modules which are firm. Suppose A ideal in R unital. Let

$$U: \quad U_1 \xrightarrow{d} U_0$$

be a length one complex of fg proj R-modules, which is A-firm: $U_1/AU_1 \xrightarrow{\sim} U_0/AU_0$. I eventually hope to [redacted] understand in [redacted] a very concrete way

Morita equivalence for such complexes.

I should first describe the problems I would like to solve. The main problem is Morita invariance for higher K-groups $K_n A$ assuming suitable h-unitality of A. I have made some progress on this for K_1. The next case to consider is K_0.

One approach to Morita invariance for K_n would be to show

(a) Morita equivalence for the Δated categories of firm perfect complexes.

(b) K_n can be calculated in some Waldhausen manner from the Δated category of firm perfect complexes.

I think can prove (a), although there are still some details to be written out.

On the other hand (b) is probably false at least when interpreted naively. If A is the maximal ideal in nondiscrete rank 1 valuation ring, then A is h-unital and equal to its Jacobson radical, hence every firm perfect complex is quis to 0. Thus the Waldhausen K-theory of firm perfect complexes over A should be zero, while $K_1 A$ should contain $1+A$ as a direct summand.

But it still might be true that $K_0 A$ is the K_0 of the triangulated category of firm perfect

complexes. Here are some indications.
Recall the exact sequence of Bass

$$K_1 R \longrightarrow K_1(R/A) \longrightarrow K_0 A \longrightarrow K_0 R \longrightarrow K_0(R/A)$$

Let U be a finite length complex of f.g. proj R-modules which is firm i.e. $R/A \otimes_R U$ is acyclic. Then the class $\sum_i (-1)^i [U_i] \in K_0 R$ goes to zero in $K_0(R/A)$, so it comes from an element of $K_0 A$. In fact the topological arguments of Atiyah-Bott-Shapiro (Clifford modules) might allow one to identify $K_0 A$ with the Grothendieck group of the Δated category of firm perfect complexes.

March 27, 1995

Let R be unital, $\mathcal{P}(R)$ = the additive category of f.g.proj R-modules. A complex of R-modules will be called <u>strictly perfect</u> when it is a bounded complex of f.g.proj modules. A complex of R-modules is perfect if it is quasi-isomorphic to (isomorphic in the derived category of R-modules to) a strictly perfect complex.

Let $D(\mathrm{mod}(R))$ be the derived category of R-modules, $D_{perf}(\mathrm{mod}(R))$ the full subcategory of perfect complexes. One has an equivalence of Δated cats:

$$\underline{K}^b(\mathcal{P}(R)) \xrightarrow{\sim} D_{perf}(\mathrm{mod}(R))$$

where $\underline{K}^b(\mathcal{A})$ denotes the Δated cat of bdd. complexes in the additive cat $\mathcal{A}$ with homotopy classes of maps for morphisms.

Consider the map sending $E \in \underline{K}^b(\mathcal{A})$ to

$$\chi(E) = \sum_n (-1)^n [E_n] \in K_0(\mathcal{A}).$$

Properties:

i) If $E' \to E \to E''$ are maps in $K^b(\mathcal{A})$ such that $\forall n$ $0 \to E'_n \to E_n \to E''_n \to 0$ is split exact, then $\chi(E) = \chi(E') + \chi(E'')$.

ii) If E is contractible, then $\chi(E) = 0$.

iii) If E, E' are homotopy equivalent, then $\chi(E) = \chi(E')$.

i)′ If $E' \to E \to E'' \to E'[1]$ is a Δ in $\underline{K}^b(\mathcal{A})$ then $\chi(E) = \chi(E') + \chi(E'')$.

Actually ii) seems to require a Karoubian type hypothesis on $\mathcal{A}$. Suppose one has maps

$$P' \underset{i}{\overset{p}{\rightleftarrows}} P$$ in $\mathcal{A}$ such that $pi = 1$, but such that the kernel of p doesn't exist.

Alternatively suppose e is an idempotent operator on the object P such that eP exists but $(1-e)P$ does not exist. Then

$$0 \longrightarrow eP \longrightarrow P \xrightarrow{e} P \longrightarrow eP \longrightarrow 0$$

is an acyclic complex one can't split into elementary complexes. (This is not a counterexample, but it indicates the problems).

Proofs of these properties: (i) is obvious.

(ii): Let h be of degree $+1$ $\ni$ $1 = dh + hd$ on E. Replacing h by hdh we can suppose $h^2 = 0$. Then hd, dh are idempotents, $E = hdE \oplus dhE$, E is the direct sum of the elementary complexes $hdE_{n+1} \xrightarrow[\cong]{d} dhE_n$

iii) It's obvious that if $f: E' \longrightarrow E$ is a map of complexes, then $\boxed{\chi(\mathrm{Cone}(f)) = \chi(E) - \chi(E')}$ $*$; recall that $\mathrm{Cone}(f)_n = E_n \oplus E'_{n-1}$. If f is a h. eq. then $\mathrm{Cone}(f)$ is contractible so $\chi(E) - \chi(E') = 0$ by (ii).

i) Given the Δ: $E' \xrightarrow{f} E \to E'' \to E'[1]$ one knows that E'' is htpy eq. to $\mathrm{Cone}(f)$, so this follows from iii) and $*$.

What should I remember from the preceding? I think the important properties of χ are that it is additive on Δ's and it vanishes for elementary complexes.

It follows from the skeletal filtration:

$$F_p(E) = \{E_p \to E_{p-1} \to \cdots\} \subset E$$

that any χ additive on Δ's satisfies

$$\chi(E) = \sum_n{}' \chi(E_n[n])$$

If further χ vanishes on elementary complexes, then we have $\chi(E_n[n]) = - \chi(E_n[n-1]) = \cdots$ $= (-1)^n \chi(E_n[0])$. Thus these two properties characterize the Euler characteristic from $Ob(K^b(\mathcal{A}))$ to $K_0(\mathcal{A})$.

Suppose now A is an ideal in R. Let us consider strictly perfect complexes of R-modules E which are A-fermi, i.e. E/AE is acyclic (hence contractible as a complex of R/A modules).

Suppose the extension $A \to R \to R/A$ is split, i.e. lifting homomorphism $R/A \to R$. Then because E/AE is a direct sum of elementary complexes $P_{n+1} \xrightarrow{\sim} P_n$ with P_n a projective R/A module, there exists a section $s : E/AE \to E$ of the obvious surjection the other way which is an R/A-module complex map. We can extend s uniquely to a R-module complex map $R \otimes_{R/A} E/AE \xrightarrow{t} E$, which is clearly an isomorphism modulo A. Put $E' = R \otimes_{R/A} E/AE$ and let $C = \mathrm{Cone}(t : E' \to E)$. The skeletal filtrations of E, E' induce a filtration of C

$$\begin{array}{ccccccc}
\to & E'_{p+1} & \to & E'_p & \to & E'_{p-1} & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & E_{p+1} & \to & \underbrace{E_p \to E_{p-1} \to \qquad}_{F_p C} & & &
\end{array}$$

such that $\quad F_pC/F_{p-1}C = \mathrm{Cone}\,(t : E_p'[0] \to E_p[0])[p].$

This [redacted] filtration gives rise to exact sequences

$$0 \longrightarrow F_{p-1}C \longrightarrow F_pC \longrightarrow F_pC/F_{p-1}C \longrightarrow 0$$

of A-firm strictly perfect R-module complexes.

[redacted]

Recall that there is an exact sequence

$$K_1(R) \longrightarrow K_1(R/A) \longrightarrow K_0(A) \longrightarrow K_0(R) \longrightarrow K_0(R/A)$$

due to Bass, for any ideal A in a unital ring R. Taking $R = \tilde{A}$ gives a split exact sequence

$$0 \longrightarrow K_0(A) \longrightarrow K_0(\tilde{A}) \rightleftarrows K_0(\mathbb{Z}) \longrightarrow 0$$

which we can use as the definition of $K_0(A)$.

I would like to identify the Grothendieck group of A-firm [redacted] strictly perfect complexes of R-modules with $K_0(A)$ for a general extension $A \to R \to R/A$.

March 30, 1995

Let's restrict attention to bounded complexes of R-modules. Call a complex strictly perfect when it is a bdd cx of fg projective modules; perfect means quasi-isomorphic to a strictly perfect complex. For proving Morita invariance for K_0 it appears I need to consider complexes which are homotopy equivalent to a perfect complex.

Example: Let $A = ReR$, e idempotent in R. Let $d: U_1 \to U_0$ be a map of fg proj R-modules such that d induces $U_1/AU_1 \xrightarrow{\sim} U_0/AU_0$. This is analogous to a ψDO. We expect an "index" in $K_0(A)$ – this ~~should~~ be the class of ~~the~~ A-firm strictly perfect complex $U_1 \xrightarrow{d} U_0$. Under Morita equivalence for firm complexes this ~~complex~~ corresponds to the complex $eU_1 \to eU_0$ of modules over the initial ring B. But eU_i is not necessarily a fg proj B-module. Nevertheless it seems that $eU_1 \to eU_0$ is homotopy equivalent to a length one complex in $\mathcal{P}(B)$.

Let's describe the sort of argument I propose for Morita equivalence. Consider $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ a Morita context with ideals $A = QP$, $B = PQ$. Let U be a strictly perfect complex of R-modules which is A-firm – this means U/AU is contractible. Then we know that by lifting a contraction for U/AU

to h on $\mathcal{U}$ we obtain an operator $f = 1 - [d,h]$ on $\mathcal{U}$, which is a deformation of the identity, such that

$$f(\mathcal{U}) \subset A\,\mathcal{U} = A \otimes_R \mathcal{U}.$$

Assume to simplify that $A = Q \otimes_S P$. Then f gives a 0-cycle in

$$\mathrm{Hom}_R(\mathcal{U}, A \otimes_R \mathcal{U}) = \overbrace{\mathrm{Hom}_R(\mathcal{U}, R)}^{\mathcal{U}^*} \otimes_R A \otimes_R \mathcal{U}$$
$$= (\mathcal{U}^* \otimes_R Q) \otimes_S (P \otimes_R \mathcal{U}).$$

Put $V = P \otimes_R \mathcal{U}$, $V' = \mathcal{U}^* \otimes_R Q = \mathrm{Hom}_R(\mathcal{U}, Q)$ and observe there is an obvious pairing (S-bimodule map)

$$\langle -,- \rangle : V \otimes V' \longrightarrow S \qquad \text{i.e.}$$

$$P \otimes_R \mathcal{U} \otimes \mathrm{Hom}_R(\mathcal{U}, Q) \longrightarrow P \otimes_R Q \to S$$
$$p \otimes u \otimes f \longmapsto p f(u)$$

Consider $f_n : V_n \longrightarrow V_n$. Then we can write $f_n = \sum_1^{\nu} v_i' \otimes v_i \in V' \otimes_S V$, which implies that f_n is 'nuclear' - it factors

$$V_n \xrightarrow{(v_i')} S^{\nu} \xrightarrow{(v_i)} V_n$$
$$v \longmapsto (\langle v, v_i' \rangle) \longmapsto \sum_i \langle v, v_i' \rangle v_i$$

From the above discussion it is more or less clear that for $V = P \otimes_R \mathcal{U}$ the identity has the deformation f (really $1 \otimes f$) with the property that in each degree f is nuclear. I now want to show such a complex is h.eq. to a strictly perfect complex.

Let now U be a complex of R+modules (bounded) such that there exists a deformation $f = 1 - [d,h]$ of the identity map which in each degree is nuclear, i.e. factors

$$U_n \xrightarrow{\iota_n} R^{\nu_n} \xrightarrow{j_n} U_n \qquad\qquad f_n = j_n \iota_n$$

To simplify suppose U supported in $[0,2]$. We have maps of length 2 complexes:

$$\begin{tikzcd}
U: \arrow[d] & U_2 \arrow[r, "d"] \arrow[d, "\iota_2"] & U_1 \arrow[r, "d"] \arrow[d, "\iota_1 f_1"] & U_0 \arrow[d, "\iota_0 f_0^2"] \\
T: \arrow[d] & R^{\nu_2} \arrow[r, "\iota_1 d j_2"] \arrow[d, "f_2^2 j_2"] & R^{\nu_1} \arrow[r, "\iota_0 d j_1"] \arrow[d, "f_1 j_1"] & R^{\nu_0} \arrow[d, "j_0"] \\
U: & U_2 \arrow[r, "d"] & U_1 \arrow[r, "d"] & U_0
\end{tikzcd}$$

with composition f^3 on U. (Check:
$$(\iota_1 d j_2)\iota_2 = \iota_1 d f_2 = \iota_1 f_1 d \;, \quad d f_1 j_1 = f_0 d j_1 = j_0(\iota_0 d j_1).)$$

Thus we conclude that up to homotopy U is a retract of the bounded complex T of fg free modules.

Conversely suppose U is a homotopy retract of a strictly perfect complex T, i.e. there are maps $U \xrightarrow{\iota} T \xrightarrow{j} U$ such that $j\iota \sim 1$. Then $f = j\iota$ is a deformation of the identity which factors

$$\begin{tikzcd}
U_n \arrow[r, "\iota"] & T_n \arrow[r, "1"] \arrow[dr] & T_n \arrow[r, "j"] & U_n \\
 & & R^{\nu_n} \arrow[u] &
\end{tikzcd}$$

showing that f in each degree is nuclear.

I would like to show that any complex U which is a homotopy retract of a strictly perfect complex T is homotopy equivalent to a perfect complex. This should follow from Grothendieck's $\varinjlim$ characterization of perfect complexes. Why? First U being an h-retract of T perfect $\Rightarrow$ $[U,-]$ is a direct summand of $[T,-]$ $\Rightarrow$ $[U,-]$ takes quis to heq $\Rightarrow$ U is homotopy equivalent to a complex of projectives. Also $[U,-]$ being a direct summand of $[T,-]$ $\Rightarrow$ $[U,-]$ satisfies the $\varinjlim$ characterization, and so U is perfect + projective, thus it should be heq to a strictly perfect complex.

April 2, 1995

Let X be a compact space, Y a closed subspace, X/Y the quotient space of X obtained by collapsing Y to a point. Geometrically one knows that a vector bundle on X/Y with fibre W over the basepoint $* = Y/Y$ is equivalent to a vector bundle E over X equipped with an isomorphism $E|_Y \simeq W_Y$, (W_Y = trivial v. bundle with fibre W.) The proof uses the fact that the isom $E|_Y \simeq W_Y$ extends to a nbd of Y.

The corresponding algebraic K-theory statement is Milnor's patching theorem in the case of the cartesian square

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ R \arrow[r] & R/A \end{tikzcd}$$

i.e.

$$\mathcal{P}(\tilde{A}) \longrightarrow \mathcal{P}(R) \overset{2}{\underset{\mathcal{P}(R/A)}{\times}} \mathcal{P}(\mathbb{Z})$$

is an equivalence, in other words a fg projective $\tilde{A}$-module V is equivalent to a triple (U, W, α) with $U \in \mathcal{P}(R)$, $W \in \mathcal{P}(\mathbb{Z})$, and $\alpha : U/AU \simeq R/A \otimes_{\mathbb{Z}} W$. More precisely, given such a triple the corresponding fg proj $\tilde{A}$ module is given by the following fibre product

$$\begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[d] \\ U \arrow[r] & U/AU = R/A \otimes_{\mathbb{Z}} W \end{tikzcd}$$

Recall Milnor's proof that V defined this way is fgproj proceeds by taking the direct sum of (U, W, α) with a complementary triple (U', W', α') such that $U \oplus U'$, $W \oplus W'$ are free. This uses the fact that two complements for $U/AU = R/A \otimes_{\mathbb{Z}} W$ are stably-isom hence by adding free modules may be assumed isomorphic. Thus one can suppose $U = R^n$, $W = \mathbb{Z}^n$, $\alpha \in GL_n(R/A)$. Taking direct sum $(R^n, \mathbb{Z}^n, \alpha) \oplus (R^n, \mathbb{Z}^n, \alpha^{-1})$, using Whitehead to write $\alpha \oplus \alpha^{-1}$ as a product of elementaries, and lifting the elementaries wrt $R \twoheadrightarrow R/A$, one reduces to $(R^{2n}, \mathbb{Z}^{2n}, 1)$ whence $V = \tilde{A}^{2n}$.

I propose to give a different proof that

$$V = U \times_{(U/AU)} W \in P(\tilde{A}).$$

Suppose $W = \mathbb{Z}^n$, so that $\alpha : U/AU \xrightarrow{\sim} (R/A)^n$. Lift α to a R-module map $d : U \to R^n$ (possible as U proj.), lift h^{-1} to an R-mod map $h : R^n \to U$. Then we have map

$$U \underset{h}{\overset{d}{\rightleftarrows}} R^n$$

such that

$1 - hd : U \to AU$, $1 - dh : R^n \to A^n$. Replace h by $h_1 = (1 + 1 - hd)h = 2h - hdh$ so that

$$1 - dh_1 = 1 - 2dh + (dh)^2 = (1 - dh)^2 : R^n \to A^2(R^n)$$
$$1 - h_1 d = 1 - 2hd + (hd)^2 = (1 - hd)^2 : U \to A^2(U).$$

Since U is fgproj we can factor

$$1 - h_1 d = \sum_1 \psi_j \xi_j$$

with $\psi_j \in \operatorname{Hom}_R(U, A)$ and $\xi_j \in AU$.

We have the diagram

$$\begin{tikzcd}
U \arrow[r, "\begin{pmatrix}\alpha\\ \psi\end{pmatrix}"] \arrow[d] & R^n \oplus R^\nu \arrow[r, "(h,\ \xi)"] \arrow[d] & U \arrow[d] \\
U/AU \arrow[r, "\begin{pmatrix}\alpha\\ 0\end{pmatrix}"] & (R/A)^n \oplus (R/A)^\nu \arrow[r, "(\alpha^{-1}\ 0)"] & U/AU \\
\mathbb{Z}^n \arrow[u] \arrow[r, "\begin{pmatrix}1\\ 0\end{pmatrix}"] & \mathbb{Z}^n \oplus \mathbb{Z}^\nu \arrow[u] \arrow[r, "(1\ 0)"] & \mathbb{Z}^n \arrow[u]
\end{tikzcd}$$

where the horizontal arrows compose to give the identity. One has $V = U \times_{(U/AU)} \mathbb{Z}^n$. Taking the fibre products associated to the sets of vertical arrows in the three columns above, one gets maps of $\tilde{A}$ modules

$$V \longrightarrow \tilde{A}^{(n+\nu)} \longrightarrow V$$

with composition the identity. This shows V is a fgproj $\tilde{A}$-module.

April 5, 1995

Consider a complex U such that the identity operator has a deformation $1-[d,h]$ which is nuclear, i.e. it factors

$$\begin{tikzcd} 1-[d,h]: U \arrow[rr] \arrow[dr, "i"'] & & U \\ & T \arrow[ur, "j"'] & \end{tikzcd}$$

where T is a graded module and i, j are graded module maps. We make T into a complex with the differential idj. Note that $dj\,i\,d = d(1-[d,h])d = (1-[d,h])d^2 = 0$.

Next we have maps of complexes

$$U \xrightarrow{i(1-dh)} T \xrightarrow{(1-hd)j} U$$

because

$$idj \cdot i(1-dh) = iji\,d(1-dh) = ijid$$
$$i(1-dh)\cdot d = i(1-dh-hd)d = ijid$$

Also

$$(1-hd)j \cdot idj = (1-hd)dj\,ij = dj\,ij$$
$$d\cdot(1-hd)j = d(1-hd-dh)j = dj\,ij.$$

The composition of these two maps is

$$(1-hd)ji(1-dh) = ji - jidh - hdji + hdjidh$$
$$= 1 - dh - hd - (1-dh-\cancel{hd})dh - hd(1-\cancel{dh}-hd)$$
$$= 1 - 2dh - 2hd + (dh)^2 + (hd)^2$$
$$= 1 - 2[d,h] + [d, hdh]$$

which is homotopic to the identity, homotopy $2h - hdh$.

April 10, 1995

Start with $U \xrightarrow{i} T \xrightarrow{j} U$, maps of complexes: $[d, i] = [d, j] = 0$ such that $ji = 1 - [d, h]$. In the special case where $h = 0$ we have a double complex which is a resolution of U (actually h eq to U)

$$U \xleftarrow{j} T \xleftarrow{1-ij} T \xleftarrow{ij} T \xleftarrow{1-ij} T \xleftarrow{ij} \cdots$$

I propose to use HPT to construct in the general case a complex h.eq to U, which is given by $T \oplus T[1] \oplus T[2] \oplus \cdots$ with a twisted differential. Let

$$M = \mathrm{Cyl}(U \xrightarrow{i} T) \qquad M_n = \begin{array}{c} T_n \\ \oplus \\ U_{n-1} \\ \oplus \\ U_n \end{array} \qquad d = \begin{pmatrix} d & i & \\ & -d & \\ & -1 & d \end{pmatrix}$$

be the mapping cylinder, and let $C = \mathrm{Cone}(U \xrightarrow{i} T) = M/T$ be the mapping cone.

Then (i) T is SDR of M (in general).

(ii) U is a retract of M (because the pair $j : T \to U$ and homotopy h from 1 to ji gives rise to a retraction of M onto U).

If $U \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} M$ are the retraction and inclusion we have an idempotent operator $e = \eta\varepsilon$ on M with image $\simeq U$, hence a double complex as above

$$U \xleftarrow{\varepsilon} M \xleftarrow{1-e} M \xleftarrow{e} M \xleftarrow{1-e} \cdots$$

Then because of the SDR of M onto T we get from HPT a twisted version of $T \oplus T[1] \oplus \cdots$ which is a SDR of the associated total complex of $M \xleftarrow{1-e} M \xleftarrow{e}$; hence h.equiv. to U.

Let's find the formulas. We have the arrows

$$\begin{tikzcd} U \arrow[r, shift right, "\eta"'] & M \arrow[l, shift right, "\varepsilon"'] \arrow[loop above, "k"] \arrow[r] \arrow[d, shift right, "b"'] & C \\ & T \arrow[u, shift right, "a"'] & \end{tikzcd}$$

$$1-[d,k] = ab$$
$$k^2 = ka = bk = 0$$
$$\varepsilon\eta = 1, \quad \eta\varepsilon = e.$$

which are in degree n

$$k = \begin{pmatrix} & & -1 \\ & & \\ & & \end{pmatrix}$$

$$\begin{tikzcd} U_n \arrow[r, shift right, "{\eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}"'] & T_n \oplus U_{n-1} \oplus U_n \arrow[l, shift right, "{\varepsilon = (g\ \ -h\ \ 1)}"'] \arrow[r, "{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}"] \arrow[d, shift right, "{b = (1\ \ 0\ \ i)}"'] & T_n \oplus U_{n-1} \\ & T_n \arrow[u, shift right, "{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a}"'] & \end{tikzcd}$$

$$ka = \begin{pmatrix} & & -1 \\ & & \\ & & \end{pmatrix}\begin{pmatrix} 1 \\ \\ \end{pmatrix} = 0$$

$$bk = (1\ \ 0\ \ i)\begin{pmatrix} & & -1 \\ & & \\ & & \end{pmatrix} = 0$$

$$k^2 = 0.$$

Check the SDR first:

$$[d,k] = \left[\begin{pmatrix} d & x & \\ & -d & \\ & -1 & d \end{pmatrix}, \begin{pmatrix} & & -1 \\ & & \\ & & \end{pmatrix}\right] = \begin{pmatrix} & & -i \\ & 1 & \cancel{d-d} \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(1\ \ 0\ \ i) = 1 - ab$$

$$\varepsilon d = (j \;\; -h \;\; 1)\begin{pmatrix} d & \iota & \\ & -d & \\ & -1 & d \end{pmatrix} = (jd \;\; \underbrace{\iota j + hd - 1}_{-dh} \;\; d)$$

$$= d(j \;\; -h \;\; 1) = d\varepsilon$$

$$d\eta = \begin{pmatrix} d & \iota & \\ & -1 & \\ & -1 & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix} = \eta d.$$

$$\varepsilon\eta = (j \;\; -h \;\; 1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1.$$

$$e = \eta\varepsilon = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(j \;\; -h \;\; 1) = \begin{pmatrix} & & \\ & & \\ j & -h & 1 \end{pmatrix}$$

$$1 - e = \begin{pmatrix} 1 & & \\ & 1 & \\ -j & h & \end{pmatrix}$$

Recall the formulas from HPT:

$$\begin{matrix} E^{\circlearrowleft k, \theta} & [d,b] = [d,a] = 0 & \\ b \downarrow \uparrow a & ba = 1,\ 1 - [d,k] = ab & \\ E' & d\theta = \theta^2. & \end{matrix} \quad \left.\begin{matrix} k^2 = ka = bk = 0 \\ \\ \end{matrix}\right\} \text{ SDR conditions}$$

Then get perturbed operators

$$\tilde{k} = k\frac{1}{1-\theta k} = \frac{1}{1-k\theta}k, \quad \tilde{a} = \frac{1}{1-k\theta}a, \quad \tilde{b} = b\frac{1}{1-\theta k}$$

giving an SDR wrt the differential $d - \theta$ on E and perturbed diff $d - \theta'$, $\theta' = b\frac{1}{1-\theta k}\theta a = \tilde{b}\theta a = b\theta\tilde{a}$, on E'.

I now want the perturbed diff

$$d - \theta' = d - b\theta a - b\theta k\theta a - b(\theta k)^2\theta a - \cdots$$

on $T \oplus T[1] \oplus T[2] \oplus \cdots$

The calculations have to be done more carefully to avoid sign problems stemming from the fact that d on $M \oplus M[1] \oplus \cdots$ is given by the matrix D:

$$D = \begin{pmatrix} d & & & & \\ & -d & & & \\ & & d & & \\ & & & -d & \\ & & & & \ddots \end{pmatrix} \qquad K = \begin{pmatrix} k & & & & \\ & -k & & & \\ & & k & & \\ & & & -k & \\ & & & & \ddots \end{pmatrix}$$

Similarly k on $M \oplus M[1] \oplus \cdots$ is given by K above.

Let

$$A = \begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} b & & \\ & b & \\ & & \ddots \end{pmatrix}$$

give $T \oplus T[1] \oplus \cdots \underset{a}{\overset{b}{\rightleftarrows}} M \oplus M[1] \oplus \cdots$

so that we have the SDR relations $BA = 1$

$[D, K] = 1 - AB$, $\quad KA = BK = K^2 = 0$.

Let

$$\Theta = \begin{pmatrix} 0 & -1+e & & & \\ & 0 & -e & & \\ & & 0 & -1+e & \\ & & & 0 & \ddots \end{pmatrix}$$

so that $D - \Theta$ is the total differential of the bicomplex associated to

$$M \xleftarrow{1-e} M \xleftarrow{e} M \xleftarrow{1-e} \cdots$$

The perturbed differential on $T \oplus T[1] \oplus \cdots$

is then

$$D - \theta' = D - B\theta \frac{1}{1 - K\theta} A$$

$$= D - B\theta A - B\theta K \theta B - \cdots$$

$$\cancel{}\ -B\theta = \begin{pmatrix} 0 & b(1-e) & & & \\ & 0 & be & & \\ & & 0 & b(1-e) & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$b(1-e) = \begin{pmatrix} 1 & 0 & i \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ -j & h & \end{pmatrix} = \begin{pmatrix} 1-ij & ih & 0 \end{pmatrix}$$

$$be = \begin{pmatrix} 1 & 0 & i \end{pmatrix} \begin{pmatrix} & & \\ & & \\ j & -h & 1 \end{pmatrix} = \begin{pmatrix} ij & -ih & i \end{pmatrix}$$

$$b(1-e)a = \begin{pmatrix} 1-ij & ih & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 - ij$$

$$bea = \begin{pmatrix} ij & -ih & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = ij$$

$$\therefore\ -B\theta A = \begin{pmatrix} 0 & 1-ij & & & \\ & 0 & ij & & \\ & & 0 & 1-ij & \\ & & & 0 & \end{pmatrix}$$

$$K\theta = \begin{pmatrix} 0 & -k(1-e) & & \\ & 0 & +ke & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

$$-k(1-e) = \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ -j & h & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -j & h & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ke = \begin{pmatrix} & & \\ & & -1 \\ & & \end{pmatrix} \begin{pmatrix} & & \\ & & \\ j & -h & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -j & +h & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-B\Theta K\Theta A = \begin{pmatrix} 0 & b(1-e) & & \\ & 0 & be & \\ & & & \end{pmatrix}\begin{pmatrix} 0 & -k(1-e)a & & \\ & 0 & kea & \\ & & 0 & \end{pmatrix}$$

$$b(1-e)kea = \begin{pmatrix} 1-ij & ih & 0 \end{pmatrix}\begin{pmatrix} -j & h & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-ij & ih & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -j \\ 0 \end{pmatrix} = -ihj$$

$$be(-k)(1-e)a = \begin{pmatrix} ij & -ih & i \end{pmatrix}\begin{pmatrix} -j & h & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} ij & -ih & i \end{pmatrix}\begin{pmatrix} 0 \\ -j \\ 0 \end{pmatrix} = ihj$$

$$\therefore \quad -B\Theta K\Theta A = \begin{pmatrix} 0 & 0 & -ihj & & & \\ & 0 & 0 & ihj & & \\ & & & 0 & 0 & -ihj \\ & & & & & \ddots \end{pmatrix}$$

$$b(1-e)ke(-k)(1-e)a = \begin{pmatrix} 1-ij & ih & 0 \end{pmatrix}\begin{pmatrix} -j & h & -1 \end{pmatrix}\begin{pmatrix} -j & h & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-ij & ih & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -hj & h^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -ih^2j & ih^3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -ih^2j$$

$$be(-k)(1-e)kea = \begin{pmatrix} ij & -ih & i \end{pmatrix}\begin{pmatrix} -j & h & 0 \end{pmatrix}\begin{pmatrix} -j & h & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} ihj & -ih^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -j & h & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ih^2j & -ih^3 & -ih^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= ih^2j.$$

$$\therefore -B(\theta K)^2\theta A = \begin{pmatrix} 0 & 0 & 0 & -ih^2j & & & \\ & 0 & 0 & 0 & ih^2j & & \\ & & 0 & 0 & 0 & -ih^2j & \\ & & & \cdot & \cdot & \cdot & \end{pmatrix}$$

So it seems that the perturbed differential on $T \oplus T[1] \oplus \cdots$ is

$$\begin{pmatrix} d & 1-ij & -ihj & -ih^2j & -ih^3j & \cdots & \\ & -d & ij & ihj & ih^2j & \vdots & \\ & & d & 1-ij & -ihj & \vdots & \\ & & & -d & ij & \vdots & \\ & & & & d & \vdots & \end{pmatrix}$$

Check the square of this is zero.

$$d(-ihj) + \overset{i[d,h]j}{(1-ij)\,ij} + (-ihj)d$$

$$= i(-dh + [d,h] - hd) = 0$$

$$d(-ih^2j) + (1-ij)ihj + (-ihj)(1-ij) + (-ih^2j)d$$

$$= i\{-dh^2 + [d,h]h - h[d,h] - h^2d\}j = 0$$

$$d(-ih^3j) + (1-ij)ih^2j + (+ihj)(+ihj) + (-ih^2j)ij + (-ih^3j)d$$

$$= i\{-dh^3 + [d,h]h^2 + h(1-[d,h])h - h^2(1-[d,h]) - h^3d\}j$$

$$= i\{-[d,h^3] + [d,h]h^2 - h[d,h]h + h^2[d,h]\}j = 0$$

April 15, 1995

$C = \mathrm{Cone}(f : X \longrightarrow Y)$ is given by

$$C_n = Y_n \oplus X_{n-1} \qquad d = \begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}$$

For any complex Z

$$\mathrm{Hom}(C,Z)_n = \mathrm{Hom}(Y,Z)_n \oplus \mathrm{Hom}(X,Z)_{n+1}$$

with differential

$$[d,(u\ \ u')] = d(u\ \ u') - (-1)^n (u\ \ u')\begin{pmatrix} d & f \\ & -d \end{pmatrix}$$

$$= \big([d,u] \quad [d,u'] - (-1)^n uf \big)$$

$\therefore$ A map of complexes $C \longrightarrow Z$ is a pair (u,u') with $u : Y \to Z$, $u' : X \to Z$ of degrees $0, 1$ resp. such that $\boxed{[d,u] = 0 \text{ and } [d,u'] = uf.}$

A map of cxs. $(u\ \ u') : C \longrightarrow Z$ is nullhomotopic when $\exists$ operators $s : Y \to Z$, $s' : X \longrightarrow Z$ of degree $1, 2$ resp. such that $\boxed{u = [d,s] \quad \text{and} \quad u' = [d,s'] + sf.}$

Apply this to

$$k[z] \otimes T \xrightarrow{1-ze} k[z] \otimes T \longrightarrow \mathcal{U} \longrightarrow$$

A map of complexes $\mathcal{U} \longrightarrow Z$ is given by $(u_0, u_0', u_1, u_1', \dots)$ with $u_n : T \to Z$, $u_n' : T \to Z$ of degrees $0, 1$ resp. such that

$$\boxed{[d,u_n] = 0 \quad \text{and} \quad [d,u_n'] = u_n - u_{n+1}e \qquad \forall n}$$

This map is nullhomotopic when $\exists\ (s_0, s_0', s_1, s_1', \dots)$

with $s_n, s'_n : T \to Z$ of degrees 1, 2 resp. such that

$$\boxed{u_n = [d, s_n] \quad \text{and} \quad u'_n = [d, s'_n] + s_n - s_{n+1}e}$$

Check: Go back to $\text{Cone}(X \xrightarrow{f} Y)$ and write the pair $(y, x) \in C_n = Y_n \oplus X_{n-1}$ as $y + \sigma x$.

Then $\quad d(y + \sigma x) = dy + f(x) - \sigma(dx).$

If $s, s' : T \to Z$ are of degrees 1, 2 then the operator $(s\ s') : C \to Z$ is $y + \sigma x \longmapsto s(y) + s'(x)$.

$$[d, (s\ s')](y + \sigma x) = d\big(s(y) + s'(x)\big) + \underbrace{(s\ s')\big(dy + f(x) - \sigma dx\big)}_{s(dy + fx) - s'(dx)}$$

$$= [d, s]y + [d, s']x + sfx = \big([d, s] \quad [d, s'] + sf\big)(y + \sigma x)$$

so that $\quad [d, (s\ s')] = \big([d, s] \quad [d, s'] + sf\big) \quad$ as claimed.

Now in the case of $f = 1 - ze : k[z] \otimes T \to k[z] \otimes T$, an element of U can be written $\sum z^n \xi_n + \sigma z^n \xi'_n$ and an operator $(s\ s') : U \to Z$ is given by $(s_0, s'_0, s_1, s'_1, \dots)$ where $s_n(\xi) = s(z^n \xi)$, $s'_n(\xi') = s'(z^n \xi')$. Then

$$\big\{[d, s'] + s(1 - ze)\big\}(z^n \xi') = d\big(s'(z^n\xi')\big) - s'\big(d(z^n\xi')\big) + s(z^n \xi') - s(z^{n+1} e \xi')$$

$$= ds'_n(\xi') - s'_n(d\xi') + s_n(\xi') - s_{n+1}(e\xi')$$

$$= \big([d, s'_n] + s_n - s_{n+1}e\big)(\xi').$$

At this point I have a way to calculate homotopy classes of maps $U \to Z$ for any Z. I want to show that U is the image of the idempotent operator e on T in the homotopy category.

The abstract argument that this is true uses the long exact sequence

$$\longrightarrow [U, Z] \longrightarrow [k[z]\otimes T, Z] \xrightarrow{1-ze} [k[z]\otimes T, Z]$$

which should result in a short exact sequence

$$0 \longrightarrow R^1\varprojlim\left([\Sigma T, Z] \xrightarrow{e} [\Sigma T, Z] \cdots\right) \longrightarrow [U, Z] \longrightarrow \varprojlim\left([T,Z] \xrightarrow{e} [T,Z] \to \cdots\right) \to 0$$

Because e is homotopic idempotent the inverse system $[\Sigma T, Z] \xrightarrow{e} [\Sigma T, Z] \xrightarrow{e} \cdots$ is ML, so the $R^1\varprojlim$ term vanishes yielding

$$\circledast \qquad [U, Z] \xrightarrow{\sim} \{\alpha \in [T, Z] \mid \alpha e = \alpha\}.$$

Concretely the map $[U, Z] \to [T, U]$ is induced by $j : T \to U$, the inclusion of T as $z^0 \otimes T$. One has $(u_0, u_0', u_1, u_1', \ldots) j = u_0$.
To prove $\circledast$ is bijective, let's show it is surjective and injective using our formulas for $[U, Z]$.

Given $w : T \to Z$, $[d,w]=0$ note that

$$(we, wh, we, wh, \ldots)$$

is a cocycle: $[d, we] = 0$, $[d, wh] = w(e-e^2) = we - (we)e$.
This proves the surjectivity of $\circledast$. Alternative formulation: There is a map $\iota : U \to T$ arising from

$$\begin{tikzcd}
T \arrow[r, "e"] \arrow[d, "z^0e"'] & T \arrow[r, "e"] \arrow[d, "z^1e"] & T \arrow[r] \arrow[d, "z^2e"] & {} \\
U \arrow[r, equal] \arrow[ur, "h"] & U \arrow[r, equal] \arrow[ur, "h"] & U \arrow[r] & {}
\end{tikzcd}$$

Clearly $ij : T \to U \to T$ is e.

We want to prove that $ji : U \to T \to U$ is homotopic to the identity. This is equivalent to injectivity of $j^* : [U, Z] \to [T, Z]$. In effect $1 - ji \in [U, U]$ goes under j^* to $(1-ji)j = j - je$. Now $j - je$ sends $\phi : U \to Z$ to $\phi j - \phi j e = u_0 - u_0 e$ if $\phi = (u_0, u_0', u_1, u_1', \ldots)$.

We want to see for any cocycle ϕ that $u_0 - u_0 e$ is homotopic to zero. Now

$$u_0 \sim u_1 e \sim u_1 e^2 = (u_1 e) e \sim u_0 e$$

concretely
$$[d, u_0'(1-e) + u_1 h] = (u_0 - u_1 e)(1-e) + u_1(e - e^2)$$
$$= u_0 - u_0 e.$$

Thus $(1-ji)j = j - je \sim 0$, so $1 - ji \sim 0$ if $j^* : [U, Z] \to [T, Z]$ is injective always. Note also that $1 - ji \sim 0 \Rightarrow (\phi j \sim 0 \Rightarrow \phi \sim \phi j i \sim 0)$.

So I want to show that any cocycle $\phi = (u_0, u_0', u_1, \ldots)$ with $u_0 e \sim 0$ (equiv. $u_0 \sim 0$) is a coboundary.

Start with a cocycle $\phi = (u_0, u_0', u_1, u_1', \ldots)$ remove $\phi j i = (u_0 e, u_0 h, u_0 e, u_1 h, \ldots)$ to obtain

$$\phi - \phi j i = (u_0 - u_0 e,\ u_0' - u_0 h,\ u_1 - u_0 e, \ldots$$

Next let $s_0 = u_0'(1-e) + u_1 h$, whence $[d, s_0] = (u_0 - u_1 e)(1-e) + u_1(e - e^2) = u_0 - u_0 e$.

$$\text{Cobdry}\,(s_0, 0, 0, \ldots) = ([d, s_0], s_0, 0, 0, \ldots)$$

Removing this from $\phi - \phi j$ we get a cocycle

$$(0,\ u_0'-u_0h-s_0,\ u_1-u_0e,\ u_1'-u_0h, ..)$$

(1) I have now reduced to showing that any cocycle $(0, u_0', u_1, u_1', \cdots)$ is a coboundary.

Put $t_1 = -u_0' + u_1'(1-e) + u_2h$. We know already that $[d, u_n'(1-e)+u_{n+1}h] = u_n - u_ne$ for any cocycle. Since $[d, u_0'] = u_0 - u_1e = -u_1e$ when $u_0 = 0$, it follows that $[d, t_1] = u_1$.

$$\text{Cobdry}(0, 0, t_1, 0, 0) = (0, -t_1e, [d,t_1], t_1, 0, ..)$$

Removing this from $(0, u_0', u_1, \cdots)$ we get

$$(0,\ u_0'+t_1e,\ 0,\ u_1'-t_1,\ u_2, ..)$$

which reduces us to the case

(2) $$(0,\ u_0',\ 0,\ u_1',\ u_2, \ldots)$$

Next we use

$$\text{Cobdry}(u_0'(1-e), 0, -u_0', 0, 0, 0 \ldots)$$
$$= ([d, u_0'(1-e)], [d,0] + u_0'(1-e) + u_0'e, [d, -u_0'], -u_0', 0, 0,)$$
$$= (0,\ u_0',\ 0,\ -u_0',\ 0,\ 0, ..)$$

Subtracting from (2) yields

$$(0,\ 0,\ 0,\ u_1+u_0',\ u_2,\ u_2', \ldots)$$

reducing to case

(3) $$(0,\ 0,\ 0,\ u_1',\ u_2,\ u_2', \ldots)$$

Put $t_2 = -u_1' + u_2'(1-e) + u_3 h$, whence

$$[d,t_2] = -(\overset{0}{\overset{\shortparallel}{u_1'}} - u_2 e) + (u_2 - u_3 e)(1-e) + u_3(e-e^2) = u_2.$$

Subtract Cobdry$(0,0,0,0,t_2,0,\ldots)$

$= (0,0,0,-t_2 e, u_2, t_2, 0, 0, \ldots)$ from (3)

to get $(0,0,0,u_1'+t_2 e, 0, u_2'-t_2, u_3', \ldots)$

reducing to the case

(4) $(0,0,0,u_1',0,u_2',u_3,u_3',\ldots$

Subtract Cobdry$(0, -u_1'h, u_1'(1-e), 0, -u_1', 0, 0, \ldots)$

$= (0, [d,-u_1'h] + u_1'(1-\overset{e}{\wedge}), [d,u_1'(1-e)], u_1'(1-e)-(-u_1')e, [d,-u_1'], -u_1', 0 \ldots)$

$= (0, 0, 0, u_1', 0, -u_1', 0, 0, 0 \ldots)$

from (4) to get

$(0,0,0,0,0,u_2'+u_1', u_3, u_3', \ldots)$

reducing to the case

$(0,0,0,0,0,u_2', u_3, u_3', \ldots)$

Thus we can successively kill $u_0, u_1, u_0', u_2, u_1'$, ... in this order by subtracting coboundaries. Moreover the connectivity of the coboundaries increases, so the sum of the homotopies makes sense.

Note that there are two basic processes. First given $0, 0, u_j', u_{j+1}, u_{j+1}'$ we can kill u_{j+1} using $t_{j+1} = \overset{-u_j'}{\wedge} + u_{j+1}'(1-e) + u_{j+2}' h$. Second given

$(\ 0, 0, u_j', 0, u_{j+1}', u_{j+2}, \dots)$ we know that

$$, 0, 0, u_j', 0, -u_j', 0, 0, 0 \quad \text{is the coboundary}$$

of $\dots, 0, -u_j' h,\ u_j'(1-e), 0, -u_j', 0, \dots$

Another way to proceed is to note that when $u_0 \sim 0$, then

$$u_n \sim u_{n+1} e \sim u_{n+1} e^2 \sim \dots \sim u_{n+1} e^{n+1} \sim u_n e^n \sim \dots \sim u_0$$

is also htpic to 0. ~~[illegible]~~ Thus choosing $u_n = [d, s_n]$ for all n and removing

$$\text{Cobdry}\,(s_0, 0, s_1, 0, s_2, 0, \dots)$$

$$= (\ u_0, s_0 - s_1 e, u_1, s_1 - s_2 e, u_2, \dots\ )$$

we reduce to the case

$$(0, u_0', 0, u_1', 0, u_2', 0, \dots)$$

which we can handle since we know $(0, u_0', 0, -u_0', 0, 0, \dots 0)$, etc. are coboundaries.

Other ideas. Observe that e or T extends to an endomorphism on $\mathcal{U}$. In terms of cocycles we have $(u_0, u_0', \dots) \longmapsto (u_0 e, u_0' e, u_1 e, \dots)$. I want to show this endomorphism is homotopic to the identity.

First show it's homotopy idempotent, i.e. that $(u_0 e - u_0 e^2, u_0' e - u_0' e^2, \dots)$ is always a coboundary. One has

$$\mathrm{Cobdry}(u_0 h, 0, u_1 h, 0, \dots)$$
$$= (u_0(e-e^2), u_0 h - u_1 h e, u_1(e-e^2), u_1 h - u_2 h e, \dots)$$

Better is

$$\mathrm{Cobdry}(u_0 h, -u_0' h, u_1 h, -u_1' h, \dots)$$
$$= ([d, u_0 h], [d, u_0' h] + u_0 h - u_1 h e, \dots$$
$$= \Big(u_0(e-e^2),\ u_0'(e-e^2) + \cancel{u_0} h - u_1 h e,\ u_1(e-e^2), \qquad -(\cancel{u}_0 - u_1 e)h$$
$$= (u_0(e-e^2),\ u_0'(e-e^2) - u_1[h,e],\ u_1(e-e^2), \dots$$

Subtracting we get the cocycle

$$(0,\ u_1[h,e],\ 0,\ u_2[h,e],\ 0, \dots)$$

which we know is a coboundary in a reasonably simple, but infinite, way.

Now we also know that $1 - ze$ in $\mathcal{U}$ is homotopic to zero in a very simple way

$$\mathrm{Cobdry}(u_0', 0, u_1', 0, u_2', 0 \dots)$$
$$= ([d, u_0'], u_0' - u_1' e, [d, u_1'], u_1' - u_2' e, \dots)$$
$$= (u_0 - u_1 e, u_0' - u_1' e, u_1 - u_2 e, u_1' - u_2' e, \dots)$$
$$= \phi(1 - ze).$$

Thus one has the following relations among operators on $\mathcal{U}$:

$$1 \sim ze \sim ze^2 \sim (ze)e \sim e$$

Alternatively z, e commute as $1 \sim ze = ez$, so e and z are invertible + mutually inverse. Since

$e(1-e) \sim 0$ it follows that $1 \sim e$.

April 16, 1995

Consider a 0 cocycle $\phi = (u_0, u_0', u_1, u_1', \cdots)$. I want to deform this to $(u_0 e, u_0 h, u_0 e, \cdots) = \phi j i$. The idea will be to construct $s_0, s_1, \cdots$ in $\mathrm{Hom}(T, X)$, such that $[d, s_n] = u_n - u_0 e$, $\forall n \geq 0$. Then we remove the coboundary of $(s_0, 0, s_1, 0, s_2, 0 \cdots)$, which is $([d, s_0], s_0 - e s_1, [d, s_1], s_1 - e s_2, \cdots)$, from $\phi - \phi j i$ to obtain $(0, u_0' - u_0 h - s_0 + e s_1, 0, u_1' - u_0 h - s_1 + s_2 e, 0, \cdots)$, which we know how to write as a coboundary.

Let's recall how the last step is done. Consider a cocycle of the form $(0, u_0', 0, u_1', 0, u_2', 0, \cdot)$. There are two formulas we can use to modify this. To simplify suppose only $u_n' \neq 0$, where $n \geq 1$. Then

$$\text{Coboundary}(0, -u_n' h, u_n'(1-e), 0, -u_n', 0, 0, 0 \cdots)$$
$$= (\cdots 0, 0, 0, u_n', 0, -u_n', 0, 0, \cdots)$$

$$\text{Coboundary}(\cdots 0, -u_n' h, u_n'(1-e), -u_n' h, -u_n' e, 0, 0 \cdots)$$
$$= (0, 0, 0, u_n', 0, -u_n' e, 0, \cdots)$$

Check: $[d, -u_n' h] - u_n'(1-e)e = u_n'(e-e^2) - u_n'(e-e^2) = 0$

$u_n'(1-e) - (-u_n')e = u_n'$

$[d, -u_n' h] + u_n'(1-e) - (-u_n' e)e = u_n'(e - \cancel{e^2}) + u_n' - \cancel{u_n' e} + \cancel{u_n' e^2}$
$= u_n'$

Actually it's better to replace the latter with

$$\boxed{\begin{aligned}&\mathrm{Cobdry}(\cdot\cdot,0,-u'_n h,\ u'_n(1-e),-u'_n h,-u'_n e,-u'_n h,-u'_n e,\dots)\\ &=(\cdot\cdot,0,\ 0,\ u'_n,\ 0,\ 0,\ 0,\ 0,\dots)\end{aligned}}$$

This is also what you obtain by iterating the first formula, i.e. adding

$$\begin{array}{cccccccccc} 0, & 0, & -u'_n h, & u'_n(1-e), & 0, & -u'_n, & 0, & 0, & 0, & 0 \\ & & & & -u'_n h, & u'_n(1-e), & 0, & -u'_n, & 0, & 0 \\ & & & & & & -u'_n h, & u'_n(1-e), & 0, & -u'_n \\ \hline (0, & 0, & -u'_n h, & u'_n(1-e), & -u'_n h, & -u'_n e, & -u'_n h, & -u'_n e, & \dots &) \end{array}$$

Now let's return to $\phi=(u_0,u'_0,u_1,u'_1,u_2,\dots)$ and construct s_n such that $[d,s_n]=u_n-u_0e$. The idea here is that $u_0\sim u_1e\sim u_1e^2\sim u_0e$,

$$u_1\sim u_2e\sim u_2e^2\sim u_1e\sim u_0$$

$$\begin{array}{ccccccccc} u_n & \sim & u_{n+1}e & \sim & u_{n+1}e^2 & \sim & u_n e & \sim & u_{n-1} \\ & u'_n & & u_{n+1}h & & -u'_n e & & -u'_{n-1} & \end{array}$$

Put

$$t_n = -u'_{n-1}+u'_n(1-e)+u_{n+1}h \qquad n\geq 1$$

$$t_0 = u'_0(1-e)+u_1h$$

Then

$$[d,t_0] = (u_0-\cancel{u_1e})(1-e)+\cancel{u_1(e-e^2)} = u_0-u_0e$$

$$[d,t_n] = -u_{n-1}+u_ne+(u_n-\cancel{u_{n+1}e})(1-e)+\cancel{u_{n+1}(e-e^2)}$$

$$= u_n-u_{n-1}$$

Put $s_n = t_0 + t_1 + \cdots + t_n$ so that

$$[d, s_n] = (u_0 - u_0 e) + (u_1 - u_0) + \cdots + (u_n - u_{n-1}) = u_n - u_0 e$$

as desired. Clearly

$$s_n = -\sum_{k=0}^{n-1} u'_k + \sum_{k=0}^{n} (u'_k - u'_k e) + \sum_{k=0}^{n} u_{k+1} h$$

$$\boxed{s_n = u'_n - \sum_{k=0}^{n} u'_k e + \sum_{k=0}^{n} u_{k+1} h} \qquad \boxed{[d, s_n] = u_n - u_0 e}$$

Next

$$s_n - s_{n+1} e = u'_n - \sum_0^{n(+1)} u'_k e + \sum_0^n u_{k+1} h$$
$$-u'_{n+1} e + \sum_0^{n+1} u'_k e^2 - \sum_0^{n+1} u_{k+1} h e$$

$$= \underbrace{u'_n - \sum_0^{n+1} u'_k [d, h]}_{[d, \sum_0^{n+1} u'_k h]} + \boxed{\sum_0^n u_{k+1} h} - \sum_0^{n+1} u_{k+1} h e$$

cancels except for $k=0$

$$- \sum_0^{n+1} (u_k - u_{k+1} e) h$$

$$= u'_n - u_0 h + \sum_0^{n+1} u_{k+1} [e, h] + [d, \sum_0^{n+1} u'_k h]$$

Put $\boxed{s'_n = \sum_0^{n+1} -u'_k h}$, then

$$\boxed{[d, s_n] = u_n - u_0 e}$$

$$\boxed{[d, s'_n] + s_n - s_{n+1} e = u'_n - u_0 h + \sum_0^{n+1} u_{k+1} [e, h]}$$

Thus $\phi - \phi g i = (u_0 - u_0 e, u'_0 - u_0 h, u_1 - u_0 e, \ldots)$
is cohomologous to $(0, u_1 [h, e], 0, (u_1 + u_2)[h, e], 0, \ldots)$

What's interesting is maybe to examine the case where $[h, e] = [d, h_1]$ for some h_1.

Special case where all $u_n = 0$. Then

$$s_n = u_n' - \sum_0^n u_k' e \quad , \quad s_n' = -\sum_0^{n+1} u_k' h$$

satisfy $\quad [d, s_n] = 0 \quad , \quad [d, s_n'] + s_n - s_{n+1} e = u_n'$

We can apply this to write the cocycle

$$(0, u_1[e,h], 0, (u_1+u_2)[e,h], 0, \ldots)$$

as a coboundary.

<u>April 18, 1995</u>

Program: Analysis of a homotopy idempotent, eventually (with luck) also a homotopy equivalence (the 'odd' version of an idempotent).

Let e be an operator on the complex T which is h-idempotent: $[d,e]=0$, $[d,h]=e-e^2$ for some h of degree 1. There is the problem of constructing a twisted differential on $T \oplus T[1] \oplus T[2] \oplus \cdots$ which generalizes the total differential on the double complex given by the sequence of complexes

$$T \xleftarrow{1-e} T \xleftarrow{e} T \xleftarrow{1-e} T \xleftarrow{e} \cdots$$

in the case $e=e^2$. This twisted differential is what one should mean by an A_∞-idempotent.

~~[illegible]~~

Example: $T \underset{j}{\overset{\iota}{\leftrightarrows}} U$ $\quad [d,\iota]=[d,j]=0$, $\quad 1-j\iota=[d,k]$

Put $e_n = \iota k^n j$, $n \geq 0$, so that $e_n \in \mathrm{Hom}(T,T)_n$.

Then

$$[d,e_0]=0$$

$$[d,e_1] = i[d,k]j = i(1-ji)j$$

$$= e_0 - e_0^2$$

$$[d,e_2] = i[d,k^2]j = \iota([d,k]k - k[d,k])j$$

$$= i((1-ji)k - k(1-ji))j = -e_0e_1 + e_1e_0$$

$$[d,e_3] = i((1-ji)k^2 - k(1-ji)k + k^2(1-ji))j$$

$$= e_2 - e_0e_2 + e_1^2 - e_2e_0$$

$$[d,e_4] = \quad -e_0e_3 + e_1e_2 - e_2e_1 + e_3e_0$$

$$[d,e_5] = e_4 - e_0e_4 + e_1e_3 - e_2^2 + e_3e_1 - e_4e_0$$

The twisted differential on $T \oplus T[1] \oplus T[2] \oplus \cdots$ is

$$\tilde{d} = \begin{pmatrix} d & 1-e_0 & -e_1 & -e_2 & -e_3 & \cdots \\ & -d & e_0 & e_1 & e_2 & \cdots \\ & & d & 1-e_1 & -e_2 & \cdots \\ & & & -d & e_1 & \cdots \\ & & & & d & \vdots \\ & & & & & \ddots \end{pmatrix}$$

the above equations being equivalent to $\tilde{d}^2 = 0$.

~~I think its true that an A_∞-idempotent, i.e. family of e_n of degree $n \geq 0$ in a DGA Γ, is equivalent to a twisting cochain from the nonunital algebra ke, $e=e^2$ to Γ~~

Let's define an A_∞-idempotent in a DGA Γ to be a family $e_n \in \Gamma_n$, $n \geq 0$ satisfying the above relations. I think this is the same as a twisting cochain from the bar construction of the nonunital algebra ke, $e = e^2$ to Γ. (The sort of data are the same: $\mathrm{Bar}(ke)_n$ has the bases $e^{\otimes n}$, $n \geq 1$ so $\pm\tau(e^{\otimes n})$ is an element $e_{n-1} \in \Gamma_{n-1}$. The differential in $\mathrm{Bar}(ke)$ is $b'(e^{\otimes n}) = \begin{cases} e^{\otimes n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ and this fits with the even e_n occurring in the formulas. It's only necessary to get the signs straight.)

Granted this, we deduce that an A_∞-idempotent in Γ is equivalent to a DGA map

$$\mathrm{Cobar}(\mathrm{Bar}(ke)) \longrightarrow \Gamma$$

The former should be a free DG algebra resolution of ke.

Now we have seen in the case $\Gamma = \mathrm{Hom}(T,T)$ that any homotopy idempotent e can be represented $e = ij$ where $U \underset{i}{\overset{j}{\leftrightarrows}} T$, $[d,i]=[d,j]=0$, $[d,k]=1-ji$. Hence any homotopy idempotent e can be extended to an A_∞-idempotent $(e_0=e, e_1, \dots)$. This is puzzling because $\mathrm{Hom}(T,T)$ is not acyclic and we seem to always have a lifting

$$\begin{tikzcd}
\mathrm{Cobar}(\mathrm{Bar}(ke)) \arrow[r, "\exists"] \arrow[d] & \mathrm{trunc}_{\geq 0} \mathrm{Hom}(T,T) \arrow[d] \\
ke \arrow[r] & H_0 \mathrm{Hom}(T,T)
\end{tikzcd}
\qquad
\mathrm{trunc}_{\geq 0} \mathrm{Hom}(T,T) = \begin{cases} 0 & \text{in deg } n<0 \\ Z_0\Gamma & \text{in deg } n=0 \\ \Gamma_n & \text{for } n>0. \end{cases}$$

If we want to study whether a homotopy idempotent e lifts to an A_∞-idempotent in a general DGA Γ, then we might as well suppose $\Gamma = k\langle e,h\rangle$, with $|e|=0$, $|h|=1$, $[d,e]=0$, $[d,h]=e-e^2$. Presumably the 1-cycle $[h,e]$ in Γ is not a bdry, (see next page) and we would like to modify h to $e_1 = h+\delta h$ where δh is a cycle (so that $[d,e_1]=[d,h]=e-e^2$) such that $[e_1,e]$ is a boundary.

I tried to find this e_1 using $U \underset{i}{\overset{j}{\leftrightarrows}} T$ with $U = \mathrm{Cone}\left(k[z]\otimes T \xrightarrow{1-ze} k[z]\otimes T\right)$, but this was too complicated. Instead let's now consider the homology of Γ.

Let $A = k[e]$, e an ~~an~~ indeterminate.

Then Γ is

$$f(e)\,h\,g(e) \longmapsto f(e)(e-e^2)g(e)$$

$$A h A h A \xrightarrow{\ d\ } A h A \xrightarrow{\ d\ } A$$

$$\begin{array}{c} A h A \\ \wr\!\mid \\ A\otimes A = k[x,y] \ni f(x)g(y) \end{array}$$

1-cycles: $\quad Z^1 \cong \{\varphi(x,y) \mid (e-e^2)\varphi(e,e) = 0 \ \text{ in } A\}$

$$\Updownarrow$$

$$\varphi(e,e) = 0 .$$

$\therefore \quad Z^1 \simeq (x-y)\,k[x,y] \qquad$ ideal of the diagonal

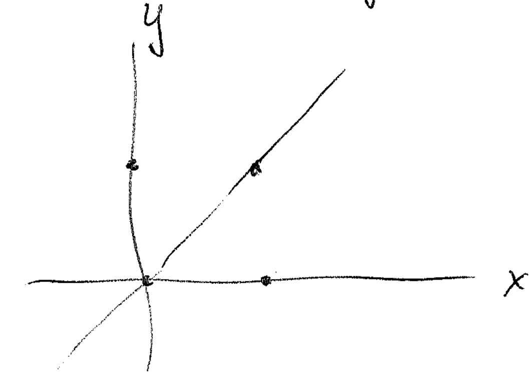
$$d\big(h\, f(e)\, h\big) = (e-e^2)f(e)\,h - h\,f(e)(e-e^2)$$

$$\updownarrow$$

$$(x-x^2)f(x) - (y-y^2)f(y)$$

$\therefore \quad B^1 \simeq$ ideal in $k[x,y]$ generated by $g(x)-g(y)$ where $g \in (e-e^2)\,k[e]$.

Observe that if $g(x) \in (x-x^2)\,k[x]$, then $g(x)-g(y)$ vanishes on the diagonal but also at $(1,0), (0,1)$



Thus $x-y$ is a 1-cycle which is not a 1-bdry.

$\therefore\ H_1(\Gamma) \neq 0$. Note that $x-y$ corresponds to the cycle $[e,h] \in \Gamma_1$.

We want to modify h by a cycle δh so that $[e, h+\delta h]$ is a boundary. In particular we want to modify $1 \in k[x,y]$ to $1 + (x-y)\varphi(x,y)$ so that $(x-y)(1+(x-y)\varphi(x,y))$ vanishes at $(1,0)$, $(0,1)$. The simplest choice is $\varphi(x,y) = -(x-y)$, giving $(x-y)(1-(x-y)^2)$. Corresponding to $1-(x-y)^2$ is the element

$$\tilde{h} = h - [e,[e,h]]$$

which satisfies $[d,\tilde{h}] = [d,h] = e - e^2$.

$$[e,\tilde{h}] = [e,h] - [e,[e,[e,h]]]$$

$$[e,[e,[e,h]]] = e^2[e,h] - 2e[e,h]e + [e,h]e^2$$

Note $e[e,h] + [e,h]e = [e^2,h] = [e,h] - \underbrace{[[d,h],h]}_{[d,h^2]}$

$$\Rightarrow (1-e)[e,h] \equiv [e,h]e$$

$$e[e,h] \equiv [e,h](1-e) \qquad \text{mod } B^1$$

Also $\quad e^2[e,h] = e[e,h] - \underbrace{[d,h][e,h]}_{[d,h[e,h]]}$

Thus $\quad e^2[e,h] - 2e[e,h]e + [e,h]e^2$

$$\equiv e[e,h] - 2e(1-e)[e,h] + [e,h]e$$

$$\equiv e[e,h] + (1-e)[e,h] = [e,h]$$

showing that $[e,\tilde{h}] \in B^1$ as desired.