October 30, 1994

The idea is that we have some control over maps in $\mathcal{M}$, but not much in $\mathcal{M}_t$, however under noetherian hypotheses - say restricting to finitely generated (resp. presented) modules - $\mathcal{M}$ and $\mathcal{M}_t$ maps might coincide.

For example if R is left noetherian, then $\mathrm{mod}(R)$ is a locally noetherian abelian category, the ind category of the full subcategory $\mathrm{modfg}(R)$ of noetherian objects. There should be a 1-1 correspondence between Serre subcategories of $\mathrm{modfg}(R)$ and Serre subcategories of $\mathrm{mod}(R)$ closed under direct sums. Then $\mathcal{M}_t(R,I)$ should be the locally noetherian category associated to the full subcategory $\mathrm{modfg}(R)/\mathrm{nilfg}(R,I) \subset \mathcal{M}(R,I)$. Note that for a noetherian module M, nil is equivalent to torsion since the increasing chain ${}_{I^n}M$ is stationary.

Let's proceed directly starting from

$$\mathrm{Hom}_{\mathcal{M}}(M,N) = \varinjlim_n \mathrm{Hom}_R(M, \mathrm{Hom}_R(I^{(n)},N))$$

If M is finitely presented we can take the $\varinjlim$ inside to get

$$\mathrm{Hom}_{\mathcal{M}}(M,N) = \mathrm{Hom}_R(M, \underbrace{\varinjlim_n \mathrm{Hom}_R(I^{(n)},N)}_{\text{call this } N^\#})$$

Notice that the map $N \to N^\#$ is the inductive limit of the nil isoms. $N \to \mathrm{Hom}_R(I^{(n)},N)$, and hence $N \longrightarrow N^{\#}$ <u>is a torsion-isom.</u>

Now if I is finitely presented, then

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{M}}(I, N) & = & \mathrm{Hom}_R(I, N^{\#}) & & \\ \uparrow \cong & & \uparrow & & \\ \mathrm{Hom}_{\mathcal{M}}(R, N) & = & \mathrm{Hom}_R(R, N^{\#}) & = & N^{\#} \end{array}$$

so [redacted] $N^{\#}$ is solid. Thus we've proved:

<u>Prop.</u> <u>When I is a finitely presented</u> R <u>module</u>, <u>the right adjoint</u> $j_*: \mathcal{M}_t(R, I) \longrightarrow \mathrm{mod}(R)$ (for the quotient functor j^*) <u>is given by</u>

$$j_*(j^*N) = \varinjlim_n \mathrm{Hom}_R(I^{(n)}, N)$$

(<u>Pf</u>: $j_*(j^*N)$ is characterized by the fact that it is solid and <u>comes with</u> a torsion isom $N \to j_*(j^*N)$.)

<u>Furthermore we have for</u> M <u>f.p.</u>

$$\mathrm{Hom}_{\mathcal{M}}(M, N) = \mathrm{Hom}_R(M, j_*(j^*N)) = \mathrm{Hom}_{\mathcal{M}_t}(j^*M, j^*N)$$

October 31, 1994

Let $S = \bigoplus_{n \geq 0} S_n$ be an $\mathbb{N}$-graded unital ring, and consider $\mathbb{Z}$-graded S-modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$. On M we have besides the multiplication operators by elements of S the projections [redacted] $e_n : M \longrightarrow M_n \subset M$ for $n \in \mathbb{Z}$. The following relations hold:

$$e_j e_k = \delta_{jk} e_k$$

$$e_n f = f e_{n-|f|}$$

Let $T = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z} e_n$ be the (non unital) ring generated by orthogonal idempotents e_n, $n \in \mathbb{Z}$, let be the ring

$$A = S \otimes_{\mathbb{Z}} T = \bigoplus S_j e_k$$

with multiplication $(f e_k)(g e_\ell) = fg\, e_{k-|g|} e_\ell = \begin{cases} fg e_\ell & \text{if } k = |g| + \ell \\ 0 & \text{otherwise} \end{cases}$

Note that A has "local" left and right identities, namely for any $a \in A$ one has $az = za = a$ with $z = \sum_{|n| \leq N} e_n$ for some N.

[redacted]

Let's embed A as ideal in the unital ring $R = S \otimes \tilde{T} = S \oplus A$ (semi-direct product). An R-module [redacted] is the same thing as an S-module N equipped with orthogonal projections e_n, $n \in \mathbb{Z}$, satisfying $e_n f = f e_{n-|f|}$, hence one has a diagram

$$\begin{tikzcd} & N \arrow[dr] & \\ \bigoplus_n e_n N \arrow[ur] \arrow[rr, hook] & & \prod_n e_n N \end{tikzcd}$$

(Suppose we wanted to consider $\mathbb{N}$-graded S-modules $M = \bigoplus_{n \geq 0} M_n$. These are the same as

$\mathbb{Z}$-graded S-modules killed by e_k for $k<0$. Note that the relation $e_n f = f e_{n-|f|}$ implies that $\bigoplus_{k<0} S e_k$ is an ideal in A. Thus to restrict to $\mathbb{N}$-graded S-modules we should replace A by

$$A/\bigoplus_{k<0} S e_k \;=\; \bigoplus_{n\geq 0} S e_n \;=\; \bigoplus_{0\leq j\leq k} e_k S_j \,.\Big)$$

Next consider the ideal $I = S^{>0}\otimes T$ in A. Recall associated to $I\subset A\subset R$ we have an 'exact' sequence

$$\bigcup_n \mathcal{M}(R/I^n, A/I^n) \hookrightarrow \mathcal{M}(R,A) \twoheadrightarrow \mathcal{M}(R,I)$$

or in nonunital ring terms

$$\bigcup_n \mathcal{M}(A/I^n) \hookrightarrow \mathcal{M}(A) \twoheadrightarrow \mathcal{M}(I)$$

Now $A = A^2$, in fact R/A is right R-flat since A has local left identities. So we know that $\mathcal{M}(A)$ can be identified with the category of R-modules M such that $AM = M$, i.e. such that $M = \sum e_n M$, as $A = ST = TS$. Thus $\mathcal{M}(A)$ can be identified with the category of graded S-modules

$$\boxed{\mathcal{M}(A) = \operatorname{grmod}(S)}$$

Now $I = S^{>0}T$ and $TS_j = \bigoplus_k e_k S_j = \bigoplus_k S_j e_{k-j} = S_j T$. Thus $I^2 = S^{>0}TS^{>0}T = (S^{>0})^2 T$ and similarly $I^n = (S^{>0})^n T$. Hence $A/I^n = (S/(S^{>0})^n)T$ and so $\boxed{\mathcal{M}(A/I^n) = \operatorname{grmod}(S/(S^{>0})^n)}$

So at this point, if we take S to be commutative (noetherian* to be safe) such that S is generated by S_1 over S_0, so that $(S^{>0})^n = S^{\geq n}$, then $\bigcup_n \mathcal{M}(A/I^n)$ is the category of graded S-modules such that every element is killed by a power of $S^{>0}$. It should follow then that

$$\mathcal{M}(I) \simeq \text{quasi coherent sheaves on } Proj(S)$$

* It's likely that the important condition is that S is generated by S_1 over S_0 and that S_1 is a finitely generated S-module (this implies that $S^{>0}$ is then a finitely generated S-module).

Return to $\displaystyle Hom_{a/s}(M,N) = \varinjlim_{\substack{M' \overset{s}{\hookrightarrow} M \\ N \overset{s}{\twoheadrightarrow} N'}} Hom_a(M',N')$

We've seen that a triple $(M' \subset M,\ N \twoheadrightarrow N',\ f: M' \to N')$ is equivalent to a subobject $W \subset M \oplus N$. Consider the diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & N \arrow[d] \\
M' \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\text{cocart}"] & N' \arrow[d, hook] \\
M \arrow[r] & V
\end{tikzcd}$$

obtained from such a triple (M',N',f). Thus $W = M' \times_{N'} N$ and $V = M \coprod^{M'} N'$. In fact these squares are bicartesian since $N \twoheadrightarrow N'$ and $M' \hookrightarrow M$.

This is clear because one has

$$0 \longrightarrow W \longrightarrow M' \oplus N \longrightarrow N' \longrightarrow 0$$

exact here by defn of W   exact here as $N \twoheadrightarrow N'$.

Thus the big square is bicartesian:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{bicart}"] & N \arrow[d] \\ M \arrow[r] & V \end{tikzcd}$$

which means W and V determine each other. In fact the exact sequence

$$0 \longrightarrow W \longrightarrow M \oplus N \longrightarrow V \longrightarrow 0$$

shows that V is the quotient of $M \oplus N$ by W (up to the autom $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $M \oplus N$).

I should check carefully that

$$\varinjlim_{M'' \xrightarrow{s} M} \mathrm{Hom}_a(M'', N) \xrightarrow{\ \sim\ } \varinjlim_{\substack{M' \overset{s}{\hookrightarrow} M \\ N \overset{s}{\twoheadrightarrow} N'}} \mathrm{Hom}_a(M', N')$$

$$fs^{-1} \longmapsto \mathrm{Im}\{(s,f) : M'' \longrightarrow M \oplus N\}$$

Because the [redacted] indexing categories are filtering, the $\varinjlim$'s are the same in Sets and Ab. [redacted] Now $\varinjlim F$ in Sets is just the set of components of the fibred category C/F. So we are comparing components of the category of pairs $(M \xleftarrow{s} M'' \longrightarrow N)$, or better the category of arrows $M'' \longrightarrow M \oplus N$ [redacted] s.t. $M'' \longrightarrow M$ is an S-isom., with the poset of $W \subset M \oplus N$ such that $W \longrightarrow M$ is an S-isom. The former category is fibred over the latter it seems by the functor

$$M'' \longrightarrow M \oplus N \longmapsto \mathrm{Im}(M'' \longrightarrow M \oplus N):$$

wrong ordering see p. 90

$$\begin{tikzcd} W' \times_W M'' \arrow[r, phantom, "\subset"] \arrow[d] & M'' \arrow[d, two heads] \\ W' \arrow[r, phantom, "\subset"] & W \end{tikzcd}$$

so it suffices to check that the fibre over a given W is connected. This fibre consists of all surjective S-isom. $M'' \twoheadrightarrow W$, and it is contractible because there's ~~[illegible]~~ a final object.

November 1, 1994

Let us see if there's something we can say about $D^+(\mathrm{tors}(R,I)) \longrightarrow D^+(R)$ being fully faithful. Recall in the Grothendieck category situation $\mathcal{T} \overset{\iota_*}{\rightleftarrows} \mathcal{A} \overset{j^*}{\longrightarrow} \mathcal{A}/\mathcal{T}$ the issue. I think we know that $\iota_* : D^+(\mathcal{T}) \longrightarrow D^+(\mathcal{A})$ is fully faithful (resp. an equivalence) iff for any injective E in $\mathcal{A}$ one has $R\iota^!(E/\iota_*\iota^!E) = 0$. Indeed, [redacted] assuming this condition and given X in $D^+(\mathcal{A})$ which wma is a complex E of injectives [redacted] [redacted] we then have an exact sequence

$$0 \longrightarrow \iota_*\iota^!E \longrightarrow E \longrightarrow E/\iota_*\iota^!E \longrightarrow 0$$

i.e. a triangle

$$\iota_* R\iota^!(X) \longrightarrow X \longrightarrow X^\# \longrightarrow$$

where $R\iota^!(X^\#) = 0$, and hence (by the minimal injective complex argument) $X^\# = Rj_*(j^*X^\#) = Rj_*(j^*X)$. (This is the criterion of Yao).

The difficulty in the case of $\mathcal{T} = \mathrm{tors}(R,I)$, $\mathcal{A} = \mathrm{mod}(R)$ is that we don't have much control over $\iota^!$ in general. One case in which we do know something about $\iota^!$ is when I is finitely generated as left R-module. In this case

$$\iota_*\iota^!(M) = \varinjlim_n \mathrm{Hom}_R(R/I^n, M) = \bigcup_n {}_{I^n}M$$

is the submodule of elements which are killed by some power of I. In effect

$$0 \longrightarrow \bigcup_n {}_{I^n}M \longrightarrow M \longrightarrow \varinjlim_n \mathrm{Hom}_R(I^n, M)$$

is exact and R/I is finitely presented so

$$\begin{aligned}\mathrm{Hom}_R\big(R/I, \varinjlim_n \mathrm{Hom}_R(I^n, M)\big) &= \varinjlim_n \mathrm{Hom}_R\big(R/I, \mathrm{Hom}_R(I^n, M)\big) \\ &= \varinjlim_n \mathrm{Hom}_R\big(\underbrace{I^n \otimes_R R/I}_{\substack{I^n/I^{n+1} \text{ an essentially zero} \\ \text{inverse system}}}, M\big) = 0\end{aligned}$$

showing that $\varinjlim_n \mathrm{Hom}_R(I^n, M)$ is torsion-free.

Let E be an injective R-module. Then

$$E/\iota_*\iota^! E = \varinjlim_n \mathrm{Hom}_R(I^n, E)$$

What's $R\iota^!$? Since

$$\iota^!(M) = \varinjlim_n \mathrm{Hom}_R(R/I^n, M)$$

if $X \in D^+(R)$ and $X \to E$ is an injective resolution, then

$$\begin{aligned} R\iota^!(X) = \iota^!(E) &= \varinjlim \mathrm{Hom}_R(R/I^n, E) \\ &= \varinjlim_n R\mathrm{Hom}_R(R/I^n, X). \end{aligned}$$

Let $P^{(n)} \to R/I^n$ be a projective resolution for each n, and choose maps $P^{(n)} \to P^{(n-1)}$ over $R/I^n \to R/I^{n-1}$ for each n. Then we have

$$\mathrm{Hom}_R(R/I^n, E) \xrightarrow[\text{quis}]{} \mathrm{Hom}_R(P^{(n)}, E) \xleftarrow[\text{quis}]{} \mathrm{Hom}_R(P^{(n)}, X)$$

so $$R\iota^!(X) = \varinjlim_n \mathrm{Hom}_R(P^{(n)}, X).$$

Return now to the injective module E.

Then

$$R\iota^!\left([redacted]E/\iota_*\iota^! E\right)$$
$$= \varinjlim_n \mathrm{Hom}_R\left(P^{(n)}, \varinjlim_k \mathrm{Hom}_R(I^k, E)\right)$$

Assume now that R/I^n is "∞-pseudo-coherent", i.e. $P^{(n)}$ can be chosen fin. gen. free in each degree. Then (note $P^{(n)}$ supported in $[-\infty, 0]$ while E is left bdd) we have

$$R\iota^!(E/\iota_*\iota^! E) = \varinjlim_n \varinjlim_k \mathrm{Hom}_R(P^{(n)}, \mathrm{Hom}_R(I^k, E))$$
$$= \varinjlim_n \varinjlim_k \mathrm{Hom}_R(\underbrace{I^k \otimes_R P^{(n)}}_{I^k \overset{L}{\otimes}_R R/I^n}, E)$$

so this vanishes when $I^\infty \overset{L}{\otimes}_R R/I^\infty = 0$, which is exactly approx. h-unitality.

So summarize: The first assumption is that R/I^n is ∞-pseudo-coherent, in particular I^n is of finite presentation. In this case we know

$$j_*(j^* M) = \varinjlim_n \mathrm{Hom}_R(I^{(n)}, M).$$

Also we assume I approx h-unital, [redacted] in particular $I^{(\infty)} \simeq I^\infty$. [redacted] The conclusion then is that we have the desired equivalence

$$D^+(\mathrm{tors}(R,I)) \xrightarrow{\sim} D^+(R)_{\mathrm{tors}(R,I)}$$

besides the one $D^b(\mathrm{nil}(R,I)) \xrightarrow{\sim} D^b(R)_{\mathrm{nil}(R,I)}$.

November 3, 1994

Let A be an h-unital ring: $A \overset{L}{\otimes}_{\tilde{A}} A \xrightarrow{\sim} A$, let M be an A module such that $AM = M$ but $A \otimes_{\tilde{A}} M \longrightarrow M$ is not an isomorphism. For example $A = C_0((0,1])$ $M = A/\tilde{A}x$, see p. 59. Consider the semi-direct product $A \oplus M$ where the right multiplication of A on M is zero. Put $R = \tilde{A} \oplus M$. Then
$RA = (\tilde{A} \oplus M)A = A$, $AR = A(\tilde{A} \oplus M) = A \oplus M$.

In this situation we have a Morita equivalence $\mathcal{M}(R, \underbrace{A \oplus M}_{AR}) \xrightarrow{\sim} \mathcal{M}(\tilde{A}, A)$ given by restriction of scalars. Now AR is not firm for $(\tilde{A}, A)$ since

$$
\begin{array}{ccccc}
A \otimes_{\tilde{A}} (A \oplus M) & = & A \otimes_{\tilde{A}} A & \oplus & A \otimes_{\tilde{A}} M \\
 & & \downarrow \wr & & \downarrow \text{not} \simeq \\
 & & A & & M
\end{array}
$$

and so AR is not firm for (R, AR). In particular AR is not h-unital.

November 6, 1994

Problem from long ago. Let $\mathcal{P}(A)$ be the category of f.g. projective A modules. The problem is to construct an infinite general linear group out of the groups $\mathrm{Aut}(P)$ for $P \in \mathcal{P}(A)$. The construction is to be as intrinsic as possible.

Note that given a split injection $P \underset{u}{\overset{u^*}{\rightleftarrows}} Q$, $\underbrace{u^*u=1}$ in $\mathcal{P}(A)$ there is an induced homom.

$$(u,u^*)_* : \mathrm{Aut}(P) \longrightarrow \mathrm{Aut}(Q) \qquad g \longmapsto ugu^* + 1 - uu^*$$

This gives us a functor from the category of f.g. proj. A-modules and split injections to the category of groups.

Now suppose one is given only an admissible injection $u : P \hookrightarrow Q$, i.e. such that $u(P)$ is a summand of Q. The possible splittings u^* of

$$0 \longrightarrow P \underset{u}{\overset{u^*}{\rightleftarrows}} Q \longrightarrow Q/P \longrightarrow 0$$

form a torsor under $1 + \mathrm{Hom}(Q/P, P) \subset \mathrm{Aut}(Q)$.

Put $Z = 1 + \mathrm{Hom}(Q/P, P) \subset \mathrm{Aut}(Q)$ and let $u^* : Q \longrightarrow P$ be a splitting: $u^*u = 1$. Let us now compare the homoms

$$g \mapsto ug(u^*z) + 1 - u(u^*z) \qquad g \longmapsto ugu^* + 1 - uu^*$$

associated to the two splittings u^* and u^*z.

Use matrix notation relative to $Q = P \oplus C$ where $C = \mathrm{Ker}(u^*)$. Then $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad u^* = \begin{pmatrix} 1 & 0 \end{pmatrix}$

and $\quad z = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad a \in \mathrm{Hom}(C, P)$

Then $\quad u^* z = (1 \ \ 0)\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = (1 \ \ a)$

$$ugu^* + (1 - uu^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} g (1 \ \ 0) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \ \ 0)$$

$$= \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

$$ug(u^*z) + 1 - u(u^*z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} g (1 \ \ a) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \ \ a)$$

$$= \begin{pmatrix} g & ga \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & ga - a \\ 0 & 1 \end{pmatrix}$$

$$z^{-1}\big(ugu^* + (1 - uu^*)\big) z = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & ga \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & ga - a \\ 0 & 1 \end{pmatrix}$$

Thus $\quad \boxed{z^{-1}\big(ugu^* + (1 - uu^*)\big) z = ug(u^*z) + \big(1 - u(u^*z)\big)}$

i.e. $\quad \boxed{(u, u^*z)(g) = z^{-1}\,(u, u^*)(g)\, z}$

We want to use this to construct a cofibred category over category of f.g. proj. A-modules and admissible injections with fibre $\mathrm{Aut}(P)$ at P. We already have such a cofibred category (which is coscinded) over the cat. of fg proj A-modules and split injections associated to the functor $P \mapsto \mathrm{Aut}(P)$. Let's describe it. The objects are the $P \in \mathrm{Ob}\, \mathcal{P}(A)$. A map from P to Q is a split injection $(u, u^*) : P \to Q$ together with an element $g \in \mathrm{Aut}(Q)$. Let us write (g, u, u^*) for this map. Then composition is given by

$$\begin{tikzcd} P \arrow[r, shift right, "u"'] & Q \arrow[l, shift right, "u^*"'] \arrow[loop above, "g"] \arrow[r, shift right, "v"'] & R \arrow[l, shift right, "v^*"'] \arrow[loop above, "h"] \end{tikzcd}$$

$$(h, v, v^*)(g, u, u^*) = (h\,(v, v^*)(g),\ vu,\ u^*v^*)$$

Let us now define an equivalence relation on maps by

$$(g, u, u^*) \sim (gz, u, u^*z)$$

where we recall that if $u : P \hookrightarrow Q$, then $z \in 1 + \mathrm{Hom}(Q/uP, uP)$. This group acts simply-trans. on the possible u^*, so upon choosing a u^* the equivalence classes are in one-one correspondence with elements of $\mathrm{Aut}(Q)$.

Let's now check compatibility of composition with this equivalence relation. NO

$$(h, v, v^*)(gz, u, u^*z) = (h(v, v^*)(gz), vu, u^*zv^*)$$

$$= (h(v, v^*)(g)\,(v, v^*)(z), vu, u^*zv^*)$$

$$= (h(v, v^*)(g)\,((v, v^*)(z)), vu, u^*\underbrace{v^*(v, v^*)(z)}_{\substack{v^*(vzv^* + 1 - vv^*) \\ = zv^*}})$$

$$\sim (h, v, v^*)(g, u, u^*)$$

Next let $\zeta \in 1 + \mathrm{Hom}(R/vQ, vQ)$. Then

$$(h\zeta, v, v^*\zeta)(g, u, u^*) = (h\zeta\underbrace{(v, v^*\zeta)(g)}_{\zeta^{-1}(v, v^*)(g)\zeta}, vu, u^*v^*\zeta)$$

$$= (h(v, v^*)(g)\zeta, vu, u^*v^*\zeta) \sim (h(v, v^*)(g), vu, u^*v^*)$$

You've forgotten to check $\zeta \in 1 + \mathrm{Hom}(R/vuP, vuP)$. Same mistake from April 30, 1971

November 9, 1994

Lundell review. Let G be a compact (Lie) group. Consider a family (smooth) of f.d. unitary representations of G parametrized by the manifold M, i.e. a hermitian vector bundle E over M with G action on E, G acting trivially on M. Then we have a canonical splitting

$$E = \bigoplus_{\chi} V_\chi \otimes \mathrm{Hom}_G(V_\chi, E)$$

where χ runs over the irreducible characters of G and V_χ is an irred. repn. with character χ.

Thus the family up to isomorphism is equivalent to a set of vector bundles $\{\mathrm{Hom}_G(V_\chi, E)\}$ indexed by the irred. repns. almost all zero.

Lundell situation. The Bott periodicity map $S^2 \wedge U_n \longrightarrow U_{2n}$ can be described as follows. Identify $\mathbb{C}^2 \otimes \mathbb{C}^n$ with $\mathbb{C}^{2n}$ as hermitian vector spaces. Let U_n act on $e_1 \otimes \mathbb{C}^n$ via the standard repn on $\mathbb{C}^n$ and let U_n act trivially on $e_2 \otimes \mathbb{C}^n$; here e_1, e_2 is the standard basis for $\mathbb{C}^2$. Note that the subgroup $\Delta U_1 \subset U_2 = U(\mathbb{C}^2) \subset U(\mathbb{C}^2 \otimes \mathbb{C}^n)$ centralizes this action of U_n, so conjugating by U_2 leads to a family of homomorphisms $U_n \to U_{2n}$ parametrized by $U^2/\Delta U_1 = \mathbb{C}P^1 = S^2$. Specifically $\varphi_L : U_n \to U_{2n}$ is the standard repn on $L \otimes \mathbb{C}^n$ and the trivial representation on $L^\perp \otimes \mathbb{C}^n$. If L_0 is the base point of $\mathbb{C}P^1$, then $L \mapsto \varphi_L \varphi_{L_0}^{-1}$ is the Bott map.

What we have here is a family of representations $\underbrace{\mathbb{C}^n}_{\text{standard repn}} \otimes \mathcal{O}(-1)$ of U_n parametrized by $\mathbb{CP}^1 = S^2$, which we have embedded in the trivial bundle with fibre $\mathbb{C}^{2n}$.

Now we know the vector bundle $\mathcal{O}(-1)^{\oplus n}$ over $\mathbb{CP}^1$ can be embedded in the trivial bundle with fibre $\mathbb{C}^{n+1}$, whence Lundell's theorem that the Bott map can be deformed to $S^2 \wedge U_n \longrightarrow U_{n+1}$.

November 10, 1994: (Jeem 54)

R commutative, $I = \sum_{i=1}^{s} Ra_i$ fin. gen.

In this case the localization of M [redacted] for the torsion theory $\mathrm{tors}(R,I)$ is, I claim,

$$M^{\#} = \Gamma(Sp(R) - Sp(R/I), \tilde{M}) = \check{H}^0(\mathcal{U}, \tilde{M})$$

where $\mathcal{U}$ is the affine open covering $\{Sp(R_{a_i})\}$ of $Sp(R) - Sp(R/I)$.

i.e. $$M^{\#} = \mathrm{Ker}\left\{ \prod_i M_{a_i} \rightrightarrows \prod_{i,j} M_{a_i a_j} \right\}$$

Why? We have to show $M^{\#}$ is solid and that the canonical map $M \longrightarrow M^{\#}$ is a $\mathrm{tors}(R,I)$ isomorphism. For $a \in I$ we have

$$\begin{array}{ccc} \mathrm{Hom}_R(I, M_a) & = & \mathrm{Hom}_{R_a}(R_a \otimes_R I, M_a) \\ \uparrow & & \uparrow \cong \\ \mathrm{Hom}_R(R, M_a) & = & \mathrm{Hom}_{R_a}(R_a \otimes_R R, M_a) \end{array}$$

since $0 \longrightarrow R_a \otimes_R I \longrightarrow R_a \otimes_R R \longrightarrow R_a \otimes_R R/I \longrightarrow 0$ is exact. [redacted] Thus M_a is solid,

$$R_a \otimes_R R/I = 0 \quad \text{as } a \in I$$

so $M^{\#}$ being the kernel of a map between solid modules is solid.

To show $M \longrightarrow M^{\#}$ is a torsion isomorphism it suffices to show [redacted] that $R_a \otimes_R -$ carries it into an isomorphism for any a in I, (in fact [redacted] for $a =$ any a_i). In effect, any element ξ of the kernel or [redacted] cokernel is killed by a_i^n for some n, which can be assumed independent of i since there are only finitely many i. Then $I^k \xi = 0$ since $\{I^k\}$ and $\{\sum_i Ra_i^n\}$ are cofinal.

But by exactness of localization we have $(M_a)^\# = (M^\#)_a$. Finally we use that $(M_a)^\# = M_a$ because of the basic calculation: $\Gamma(Sp(R), \tilde{M}) = M$ applied to R_a, M_a the point being that one has the open affine covering $Sp(R_{aa_i})$ of $Sp(R_a)$.

Recall that the other situation in which you understand the localization is the case where I is a finitely presented R-module.* Then we have

*R not nec. commutative.

$$j_*(j^*M) = \varinjlim_n \mathrm{Hom}_R(I^{(n)}, M)$$

When I is only finitely generated: $I = \sum_{j=1}^n Ra_j$ one can obtain the localization $j_*(j^*M)$ as the square of the functor

$$F(M) = \varinjlim_n \mathrm{Hom}_R(I^n, M)$$

To see this note that

$$\mathrm{Hom}_R(R/I, F(M)) = \varinjlim_n \mathrm{Hom}_R(R/I, \mathrm{Hom}_R(I^n, M))$$ because R/I is f.p.

$$= \varinjlim_n \mathrm{Hom}(\underbrace{I^n \otimes_R R/I}_{I^n/I^{n+1}}, M) = 0$$

essentially zero inverse system.

Thus $M \to F(M)$ is a torsion isomorphism with $F(M)$ torsion-free. In particular $F(M) = 0$ if M is torsion. Then for M torsion-free

$$0 \to M \to F(M) \to \overset{\text{torsion}}{T} \to 0$$

$$0 \to F(M) \xrightarrow{\sim} F^2(M) \to 0 \qquad F \text{ left exact}$$

so $F^2(M) \xrightarrow{\sim} F^3(M) \xrightarrow{\sim} \cdots$ for any M.

Also $\operatorname{Ker}\{M \to F(M)\} = tM$

Finally note that $M \xrightarrow{\sim} F(M)$ if M is solid is obvious from the definition. The exact sequences

$$0 \to M/tM \to F(M) \to T' \to 0$$

$$0 \to M/tM \to j_*j^*M \to T'' \to 0$$

T', T'' torsion

yield

$$F(M/tM) \xrightarrow{\sim} F^2(M)$$

$$F(M/tM) \xrightarrow{\sim} F(j_*j^*M) = j_*j^*M.$$

show that $j_*j^*M = F^2(M)$.

Another way to understand this is from the general fact that j_*j^*M for a torsion theory is obtained by squaring the functor

$$F(M) = \varinjlim_{\mathfrak{a} \in \mathcal{F}} \operatorname{Hom}_R(\mathfrak{a}, M)$$

where $\mathcal{F}$ is the Gabriel filter of cotorsion left ideals. In our case where I is f.g. the powers $\{I^n\}$ are cofinal in the Gabriel filter.

If you replace small category by its arrow ring the axioms for a site are formally similar to the axioms for a Gabriel filter. This suggests going through topos theory ideas (coherent topos?) for ideas about $\mathcal{M}_t$, e.g. good conditions that $\mathcal{M}_t$ is locally noetherian.

<u>November 17, 1994</u>

Recall in the $(a, \mathcal{S})$ situation that we identified elements of

$$\varinjlim_{\substack{M' \subseteq_s M \\ N \twoheadrightarrow_s N}} \operatorname{Hom}_a(M', N') = \operatorname{Hom}_{a/\mathcal{S}}(j^*M, j^*N)$$

with equivalence classes of subobjects $Z \subset M \oplus N$ such that $Z \to M$ is an $\mathcal{S}$-isomorphism. This statement is OK but there's an error on p. 76 about the partial ordering on the set of these correspondences.

Recall that there's a 1-1 correspondence between triples (M', N', f) with $M' \subseteq_s M$, $N \twoheadrightarrow_s N'$, and $f : M' \to N'$ and $Z \subset M \quad N$ such that $Z \to M$ is an $\mathcal{S}$-isom. This correspondence is given by

$$Z = M' \times_{N'} N$$

and

$$M' = Z + N/N$$
$$N' = N/Z \cap N$$
$$f : Z + N/N = Z/Z \cap N \longrightarrow N/Z \cap N$$

Put another way, M' is the domain of the correspondence Z from M to N, $N' = N/\text{indeterminacy}$, f is the map $M' \to N'$ given by the correspondence. The ~~partial ordering on~~ correspondences is given by shrinking the domain and increasing the indeterminacy. Thus

$Z_1 < Z$ means

$$Z_1 + N \subset Z + N$$
$$Z_1 \cap N \supset Z + N$$

$$\begin{array}{ccccc} Z+N/N & = & Z/Z\cap N & \xrightarrow{f} & N/Z\cap N \\ \cup & & & & \downarrow \\ Z_1+N/N & = & Z_1/Z_1\cap N & \xrightarrow{f_1} & N/Z_1\cap N \end{array}$$

Example: Suppose $\mathcal{A} = \mathrm{mod}(R)$, $\mathcal{B} = \mathrm{mod}(R/I)$ where $I = I^2$. In this situation we have

$$\mathrm{Hom}_{\mathcal{A}}(j_!j^*M, N) = \mathrm{Hom}_{\mathcal{A}}(IM, N/_IN) = \mathrm{Hom}_{\mathcal{A}}(M, j_*j^*M)$$

three descriptions of maps $j^*M \to j^*N$. A map $j^*M \xrightarrow{u} j^*N$ gives three [redacted] correspondences.

1) $\begin{array}{c} j_!j^*M \xrightarrow{f} N \\ \downarrow\alpha \\ M \end{array}$ is the initial object of the category of pairs (s,f) representing u. Let $Z_\ell = \mathrm{Im}(\alpha, f)$

$$\begin{tikzcd}
j_!j^*N \arrow[dr] \arrow[drr, bend left, "f"] \arrow[dddr, "\alpha"'] & & \\
& Z_\ell \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{bicart}"] & N \arrow[d] \\
& IM \arrow[r] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\text{bicart}"] & N' \arrow[d, phantom, "\cap"] \\
& M \arrow[r] & W_\ell
\end{tikzcd}$$

Z_ℓ has property

$IZ_\ell = Z_\ell$

2) $IM \longrightarrow N/_IN$ represents the initial object of $M \supset M' \longrightarrow N' \longleftarrow N$ representing u. Let Z_0 be the associated correspondence

$$\begin{tikzcd}
Z_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{bicart}"] & N \arrow[d] \\
IM \arrow[r] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\text{bicart}"] & N/_IN \arrow[d, phantom, "\cap"] \\
M \arrow[r] & W_r
\end{tikzcd}$$

properties.

$${}_IN \subset Z_0 \subset IM + N$$

3) $M \longrightarrow j_*j^*N$ initial among $\begin{array}{c} N \\ \downarrow s \\ M \xrightarrow{f} N' \end{array}$

$$\begin{tikzcd}
Z_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{bicart}"] & N \arrow[d] \arrow[dddr, "\beta"] & \\
M' \arrow[r] \arrow[dr, phantom, "\text{bicart}"] & N/_I N & \\
\cap & \cap & \\
M \arrow[r] \arrow[rr, bend right, "f"'] & W_r \arrow[r, hook] & j_* j^* N
\end{tikzcd}$$

property
$_I W_r = W_r$
This should be equivalent to $(Z_r : I) = Z_r$

The relationship between Z_ℓ, Z, Z_r is

$$Z_\ell = I Z_0 \qquad Z_r = (Z_0 : I)$$

$$Z_0 = Z_\ell + ({}_I N) \qquad Z_0 = Z_r \cap (IM \oplus N)$$

Check: $Z_0 = IZ_0 \oplus {}_I N$: $\supset$ clear. If $z \in Z_0$ then $p_1(z) \in IM$ and $IZ_0 \xrightarrow{p_1} IM$, so module IZ_0 can assume $p_1(z) = 0$ i.e. $z \in Z_0 \cap N = N_I$.

Similarly $(Z_0 : I) \cap (IM \oplus N) = Z_0$.

Here's the way to interpret this. The map $u: j^*M \longrightarrow j^*N$ is equivalent to a subobject $\Gamma \subset j^*M \oplus j^*N = j^*(M \oplus N)$ such that $\Gamma \xrightarrow{\sim} j^*M$, and Γ in turn is equivalent to a S-equivalence class of $Z \subset M \oplus N$ such that $Z \longrightarrow M$ is an S-isom. In the case $I = I^2$, there's a smallest Z, namely Z_ℓ, and a largest one, namely Z_r. Z_0 has the smallest image and largest kernel for $p_1: Z \longrightarrow M$.

Cayley transform idea: Think of $Z \subset M \oplus N$ have Cayley transform $g = F\varepsilon$ where $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and F is the involution belonging to Z. Then if $Z \Leftrightarrow (M', N', f)$, the complement $M \ominus M'$ and $\mathrm{Ker}(N \twoheadrightarrow N')$ are the ε eigenspaces of the -1 eigenspace for g.

<u>November 18, 1994</u>

Let $\mathcal{J}(M,N) = \{ Z \subset M \oplus N \mid \underset{\text{an } \mathcal{S}\text{-isom.}}{p_1 : Z \to M \text{ is}} \}$

Define $Z_1 \sim Z_2$ if equivalently

$$\begin{tikzcd}[column sep=small] & Z_1+Z_2 \arrow[dl, dash] \arrow[dr, dash] & \\ Z_1 \arrow[dr, dash] & & Z_2 \arrow[dl, dash] \\ & Z_1 \cap Z_2 \arrow[d, dotted, dash] & \\ & W & \end{tikzcd}$$

1) both $Z_1/Z_1 \cap Z_2$ and $Z_2/Z_1 \cap Z_2$ are in $\mathcal{S}$

2) both Z_1+Z_2/Z_1 and Z_1+Z_2/Z_2 in $\mathcal{S}$.

3) $\exists\ W \subset Z_1 \cap Z_2$ such that both Z_1/W and Z_2/W are in $\mathcal{S}$.

4) $\exists\ W \supset Z_1 + Z_2$ such that W/Z_1 and W/Z_2 are in $\mathcal{S}$.

These equivalences use $\mathcal{S}$ closed under sub and quotient objects. The relation $\sim$ is reflexive + symmetric. Given

$$\begin{tikzcd}[column sep=small] Z_1 \arrow[dr, dash] & & Z_2 \arrow[dl, dash] \arrow[dr, dash] & & Z_3 \arrow[dl, dash] \\ & W_1 \arrow[dr, dash] & & W_2 \arrow[dl, dash] & \\ & & W_1 \cap W_2 & & \end{tikzcd}$$

} these layers in $\mathcal{S}$

then $\quad W_1/W_1 \cap W_2 \hookrightarrow Z_2/W_2 \in \mathcal{S} \implies W_1/W_1 \cap W_2 \in \mathcal{S}$

and $\quad 0 \to \underset{\mathcal{S}}{\underset{\cap}{W_1/W_1 \cap W_2}} \to Z_1/W_1 \cap W_2 \to \underset{\mathcal{S}}{\underset{\cap}{Z_1/W_1}} \to 0$

$\implies Z_1/W_1 \cap W_2 \in \mathcal{S}$. Similarly $Z_3/W_1 \cap W_2 \in \mathcal{S}$. Thus the relation $\sim$ is transitive.

So $\sim$ is an equivalence relation, which we call $\mathcal{S}$-equivalence. Our objective now is to show

$$(*) \qquad \varinjlim_{\substack{M' \subseteq_{\mathcal{S}} M \\ N \twoheadrightarrow_{\mathcal{S}} N'}} \mathrm{Hom}_a(M', N') = \mathcal{J}(M,N)/\mathcal{S}\text{-equivalence}$$

Recall that we have identified $Z \in \mathcal{Z}(M,N)$ with triples (M', N', f):

$$\begin{tikzcd}
Z \arrow[r, "p_2"] \arrow[d] \arrow[dr, phantom, "\text{cart}"] \arrow[dd, bend right, "p_1"'] & N \arrow[d] \\
M' \arrow[r, "f"] \arrow[d, phantom, "\cap" description] & N' \\
M &
\end{tikzcd}$$

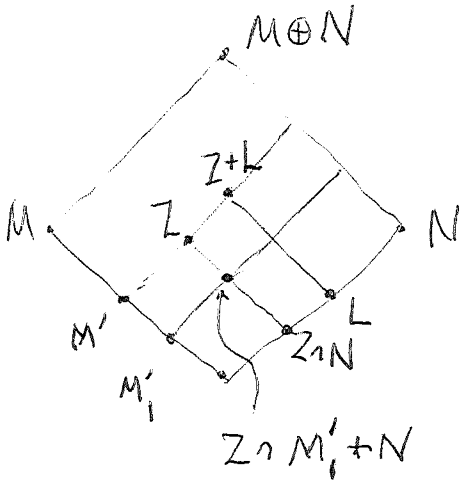
$$Z = M' \times_{N'} N$$

$$M' = p_1(Z) = Z + N/N$$

$$\mathrm{Ker}(N \twoheadrightarrow N') = \mathrm{Ker}(p_1) = Z \cap N$$

Both sides of ⊛ are quotient sets of $\mathcal{Z}(M,N) = \{(M', N', f)\}$ by certain equivalence relations. So we have to show these two equivalence coincide.

The equivalence relation on the left is generated by shrinking $M' = p_1(Z)$ and by expanding $\mathrm{Ker}(p_1) = Z \cap N$, these moves being within $\mathcal{S}$. Thus given $M_1' \subset M_1$ with cokernel in $\mathcal{S}$ we change Z to $M_1' \times_{M'} Z$, and given $Z \cap N \subset L \subset N$ with $L/Z \cap N \in \mathcal{S}$ (note these $\mathcal{S}$ conditions are the same as M/M_1' and L in $\mathcal{S}$), we change Z to $Z + L$:



These moves $Z \longmapsto Z + L$, $Z \longmapsto Z \cap M_1' + N$ do not change the $\mathcal{S}$-equivalence class of Z.

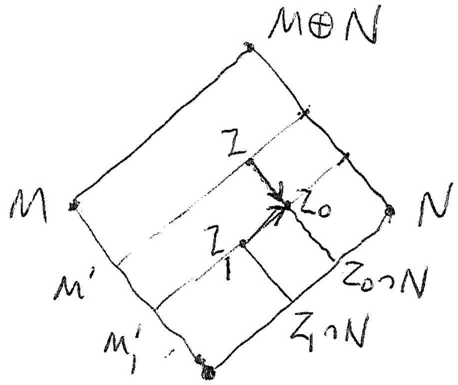
The equivalence relation on the left is generated by shrinking Z to Z_1 such that $Z/Z_1 \in \mathcal{S}$.

Put $Z_0 = Z_1 + Z \cap N \subset Z$, so $Z_1 \subset Z_0 \subset Z$.

Then $Z_0 + N = Z_1 + N$ and

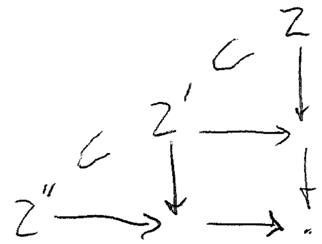
$$Z_0 \cap N = (Z_1 + Z \cap N) \cap N = Z \cap N$$

so the inclusion $Z_0 \subset Z$ is the type where $p_1(Z)$ is shrunk and the inclusion $Z_1 \subset Z_0$ is the type where $Z_1 \cap N$ is expanded:



So we conclude that equivalence relation is the same, which proves ⊛.

(Notice that the (M', N', f) picture yields a thing with two types of arrows – I'm reminded of Artin-Mazur composition:



)

TTF theories (see July 18, 1994 p734-7).

A TTF theory is a Serre subcategory $\mathcal{S}$ of mod(R) of the form $\mathcal{S} = \mathcal{T}_\tau = \mathcal{F}_{\tau'}$ for torsion theories τ, τ'. The claim is that $\mathcal{S} = \text{mod}(R/I)$ where R/I is right flat ([redacted] equiv. I has local left identities, in particular $I = I^2$).

More generally [redacted] suppose

$$\mathcal{S} = \{M \mid \text{Hom}(M,X) = 0 \quad \forall X \in \mathcal{X}\}$$
$$* \qquad = \{M \mid \text{Hom}(Y,M) = 0 \quad \forall Y \in \mathcal{Y}\}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are full subcats of modules. The former shows $\mathcal{S}$ is closed under quotients, extensions, and $\oplus$'s; the latter shows $\mathcal{S}$ is closed under submodules, extensions and Π's. Thus $\mathcal{S}$ is a Serre subcategory closed under Π's, hence of the form mod(R/I) with $I = I^2$. [redacted]

[redacted] Conversely suppose $\mathcal{S} = \text{mod}(R/I)$, $I = I^2$. Then we have $*$ with $\mathcal{X} = \{X \mid {}_I X = 0\}$ and $\mathcal{Y} = \{Y \mid IY = Y\}$. So far we don't get R/I to be right flat. However the assumption $\mathcal{S} = \mathcal{F}_{\tau'}$ for some torsion theory τ' implies that $\mathcal{S}$ is closed under injective hulls, equivalently the functor $tM = \text{Hom}_R(R/I, M)$ preserves injectives, equivalently its left adjoint $M \mapsto R/I \otimes_R M$ is exact.

Here's another way to understand this. The assumption $\mathcal{S} = \mathcal{F}_{\tau'}$ supplies the following extra information over $*$.

It tells us that $\mathcal{F}_{\tau'}$ is closed under injective hulls, which turns out to be equivalent to the class of torsion modules

$$\mathcal{S}_{\tau'} = \{ Y \mid \mathrm{Hom}(Y, M) = 0 \ \ \forall M \ni IM = 0 \}$$
$$= \{ Y \mid Y = IY \}$$

being closed under submodules. So I really want the implication

A) $Y' \subset Y$, $IY = Y \implies IY' = Y'$

$? \Downarrow$

B) R/I is flat

Suppose $M' \subset M$. $\mathrm{Ker}\{ R/I \otimes_R M' \to R/I \otimes_R M \} =$
$\mathrm{Ker}\{ M'/IM' \to M/IM \} = M' \cap IM / IM'$.
Take $Y = IM$, $Y' = M' \cap IM$. As $I = I^2$, we have $IY = Y$.
Assuming A) we get $I(M' \cap IM) = M' \cap IM$. But
$I(M' \cap IM) \subset IM' \cap I^2 M = IM' \subset M' \cap IM$, so $M' \cap IM = IM'$.

List the equivalent conditions (assume $I = I^2$).

A) $\{ Y \in \mathrm{mod}(R) \mid IY = Y \}$ is closed under submodules.

B) R/I is right flat.

C) If Q is an injective R-module, so is $\mathrm{Hom}(R/I, Q) = {}_I Q$.

D) $\mathrm{mod}(R/I)$ is closed under injective hulls in $\mathrm{mod}(R)$.

Notice that R/I right flat $\implies$

$$0 \to j_! j^* j_* j^* M \to j_* j^* M \to i_* i^* j_* j^* M \to 0$$
$$j_! j^* j_* j^* M = j_! j^* M$$

Thus $j_! j^* M \to j_* j^* M$ is injective, equivalently

$$IM \hookrightarrow \mathrm{Hom}_R(I, M)$$

equivalently $\;{}_IM \cap IM = 0$.

Observe the canonical adjunction maps

$$j_! j^* M \longrightarrow M \longrightarrow j_* j^* M$$

have composition an injection. Thus an R-module M lifting an object Z of $\mathcal{M}$ is equivalent to a factorization

$$\begin{tikzcd} & M \arrow[dr] & \\ j_! Z \arrow[ur] \arrow[rr, hook] & & j_* Z \end{tikzcd}$$

of the canonical injection for Z. This in turn is equivalent ~~to~~ by pull-back:

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[d, dashed] & M/j_! Z \arrow[d, "\varphi"] \\ j_* Z \arrow[r, two heads] & j_*Z/j_!Z = i_* i^* j_* Z \end{tikzcd}$$

to the nil-module $i_* N = M/j_! j^* M = i_* i^* M$ and map $\varphi : N \longrightarrow i^* j_* Z$.

<u>I'm reminded of dilation by this.</u>

<u>Example</u>. Suppose $I = eR$ where $e^2 = e$. Thus $ReR = eR$, i.e. $e^\perp R e = 0$.

$$R = \begin{pmatrix} eRe & eRe^\perp \\ 0 & e^\perp R e^\perp \end{pmatrix} \qquad I = \begin{pmatrix} eRe & eRe^\perp \\ 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IM & \longrightarrow & M & \longrightarrow & M/IM & \longrightarrow & 0 \\ & & \| & & \| & & \wr\| & & \\ & & eM & & M & & e^\perp M & & \end{array}$$

Firm modules ($IM = M$ when R/I is right flat) are equivalent to eRe-modules, and we can identify $j_!: \mathcal{M} \longrightarrow \mathrm{mod}(R)$ with sending an eRe module V to V with R acting via $R \to R/Re^{\perp} = eRe$.

We have

$$j_*(V) = \mathrm{Hom}_{eRe}(eR, V)$$

$$= \mathrm{Hom}_{eRe}(eRe \oplus eRe^{\perp}, V)$$

$$= V \oplus \mathrm{Hom}_{eRe}(eRe^{\perp}, V)$$

Choosing eRe and $eRe^{\perp}$ suitably one can arrange <u>$R^q j_*(V) \neq 0$ for arbitrarily large q.</u>

Also $\quad j_*(V)/j_!(V) = \mathrm{Hom}_{eRe}(eRe^{\perp}, V)$

so to extend V to an R-module M, i.e. $V = eM$, we must give $N = e^{\perp}M$ a nil module together with a map $\varphi: e^{\perp}M \longrightarrow \mathrm{Hom}_{eRe}(eRe^{\perp}, eM)$ of $e^{\perp}Re^{\perp}$-modules. Such a φ is equivalent to an eRe-module map

$$(*) \qquad eRe^{\perp} \otimes_{e^{\perp}Re^{\perp}} e^{\perp}M \longrightarrow eM.$$

So we get the expected picture, namely, an R-module is a pair $(e^{\perp}M, eM)$ together with an arbitrary multiplication $(*)$.

<u>November 24, 1994</u>

Suslin's [redacted] pseudo-free resolutions:

Let A be a nonunital ring such that $\mathrm{Tor}_p^{\tilde A}(\mathbb{Z}, A) = 0$ for all $p \leq k$. Suppose $k \geq 0$. Then $0 = \mathrm{Tor}_0^{\tilde A}(\mathbb{Z}, A) = \tilde A / A \otimes_{\tilde A} A = A/A^2$, so we [redacted] have $A = \sum_{i \in N_0} A a_i$. This gives a diagram of exact sequences

$$
(1) \qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & A^{(N_0)} \arrow[r, "\cdot(a_i)"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & \tilde A^{(N_0)} \arrow[r, "\cdot(a_i)"] \arrow[d] & A \arrow[r] & 0 \\
 & \mathbb{Z}^{(N_0)} \arrow[r, equal] \arrow[d] & \mathbb{Z}^{(N_0)} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

Then with $T_p(-) = \mathrm{Tor}_p^{\tilde A}(\mathbb{Z}, -)$ we have

$$
\begin{tikzcd}
 & T_0(M_1) \arrow[r] & T_0(A^{(N_0)}) \arrow[r] & T_0(A) \arrow[r] & 0 \\
 & T_1(M_1) \arrow[r] & T_1(A^{(N_0)}) \arrow[r] & T_1(A) \arrow[ulll, bend right] & \\
 & & \arrow[r] & T_2(A) \arrow[ulll, bend right] &
\end{tikzcd}
$$

Now I know that $\mathrm{Tor}_*^{\tilde A}(\mathbb{Z}, A) \cdot A = 0$. (This is [redacted] because

$$
0 \to \underset{\text{killed by } \cdot a}{\mathrm{Tor}_{p+1}^{\tilde A}(\mathbb{Z}, \mathbb{Z})} \longrightarrow \mathrm{Tor}_p^{\tilde A}(\mathbb{Z}, A) \longrightarrow \underset{\substack{\| \\ 0 \text{ for } p \geq 1}}{\mathrm{Tor}_p^{\tilde A}(\mathbb{Z}, \tilde A)} \xrightarrow[\uparrow\ \cong \text{ for } p=0.]{} \mathrm{Tor}_p^{\tilde A}(\mathbb{Z}, \mathbb{Z})
$$
)

Thus the maps $T_p(A^{(N_0)}) \to T_p(A)$ all zero for all p; so we have

$$* \qquad 0 \to T_{p+1}(A) \longrightarrow T_p(M_1) \longrightarrow T_p(A)^{(N_0)} \longrightarrow 0 \qquad \forall p \geq 0.$$

Hence if $k \geq 1$ $T_0(M_1) = 0$, i.e. $M_1 = \sum_{i \in N_1} A m_i$ and we get a diagram of exact sequences

$$(2) \qquad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & M_2 \arrow[r] \arrow[d] & A^{(N_1)} \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_2 \arrow[r] \arrow[d] & \tilde{A}^{(N_1)} \arrow[r] \arrow[d] & M_1 \arrow[r] & 0 \\ & \mathbb{Z}^{(N_1)} \arrow[r, equal] \arrow[d] & \mathbb{Z}^{(N_1)} \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

Actually I just noticed that $*$ implies the vanishing range of $T_p(M_1)$ is exactly one less than the vanishing range of $T_p(A)$. At the next stage we have

$$T_p(A)^{(N_1)} \longrightarrow T_p(M_1) \longrightarrow T_{p-1}(M_2) \longrightarrow T_{p-1}(A)^{(N_1)}$$

so the vanishing range of $T_p(M_2)$ is exactly one less than that of $T_p(M_1)$

When we splice (1) and (2) together we get

$$\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 \to & M_2 & \to & & A^{(N_1)} & \to & A^{(N_0)} & \to A \to & 0 \\
& \| & & & \downarrow & & \downarrow & \| & \\
0 \to & M_2 & \to & & \tilde{A}^{(N_1)} & \to & \tilde{A}^{(N_0)} & \to A \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & \mathbb{Z}^{(N_1)} & \xrightarrow{0} & \mathbb{Z}^{(N_0)} & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}$$

Supposing $k=\infty$, we ~~can~~ then construct a chain complex E of free $\tilde{A}$-modules together with an augmentation $E_0 = \tilde{A}^{(N_0)} \xrightarrow{\varepsilon} A$ such that

1) $A \otimes_{\tilde{A}} E$ is a resolution of A

2) $\varepsilon : E \to A$ is a nil-quis.

It might be better to take $E' = \text{Cone}(E \xrightarrow{\varepsilon} \tilde{A})$. Then $A \otimes_{\tilde{A}} E'$ is acyclic and E' has nil homology.

Let's try for an example of a Suslin pseudo-free resolution.

Let B be a unital ring, let ${}_BP$ and Q_B be ~~[illegible]~~ left and right B-modules respectively equipped with a surjective B-bimodule map

$$P \otimes_{\mathbb{Z}} Q \twoheadrightarrow B \qquad p \otimes q \mapsto pq$$

Finally let $A = Q \otimes_B P$ equipped with the multiplication

$$(q' \otimes p')(q'' \otimes p'') = q'(p'q'') \otimes p''$$

A is then a nonunital ring which is Morita equivalent to B. If $\exists\, p_0 \in P$, $q_0 \in Q$ such that $p_0 q_0 = 1 \in B$, then $e = q_0 \otimes p_0 \in A$ is idempotent and we have

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} AeA & Ae \\ eA & B \end{pmatrix}$$

We have the equivalence of categories (in general)

$$\mathcal{M}(A) \simeq \mathrm{mod}(B)$$

$$M \longmapsto P \otimes_A M$$

$$Q \otimes_B N \longleftarrow N$$

This identifies $j_! : \mathcal{M}(A) \longrightarrow \mathrm{mod}(\tilde{A})$ with the functor $N \longmapsto Q \otimes_B N$, hence

$$L_p j_!(Q \otimes_B N) = \mathrm{Tor}_p^B(Q, N).$$

We now ask when A is h-unital, i.e. $\mathbb{Z} \overset{\mathbb{L}}{\otimes}_{\tilde{A}} A = 0$. This is equivalent to the existence of a firm flat resolution: $\to E_1 \to E_0 \to A \to 0$. Any complex $E_\cdot$ of firm flat modules corresponds to a complex $F_\cdot$ of flat B-modules. We have $E_\cdot = Q \otimes_B F_\cdot$, $F_\cdot = P \otimes_{\tilde{A}} E_\cdot$. Moreover if $F_\cdot$ is a resolution of N, then $H_p(Q \otimes_B F) = L_p j_!(N)$.

So take the firm flat resolution $E_\cdot$ of $A = Q \otimes_B P$. Apply $P \otimes_{\tilde{A}} -$ and use that P is a f.g. projective

$\tilde{A}$ module, because one has $\sum p_i q_i = 1 \in B$. Thus $F_\bullet = P \otimes_{\tilde{A}} E_\bullet$ resolves $P \otimes_{\tilde{A}} A$, which should be P since $A = Q \otimes_B P$. So F is a flat resolution of the B-module P and

$$0 = H_*(E) = H_*(Q \otimes_B F) = \mathrm{Tor}_*^B(Q, P)$$

is the condition for A to be h-unital. In general we have the formula

$$L j_!(Q \otimes_B N) = Q \overset{L}{\otimes}_B N$$

as we mentioned, so A is h-unital $\iff$ $L j_!(A) = A$.

So now assume $\mathrm{Tor}_*^B(Q, P) = 0$ and try to construct Suslin's resolution. What we have to do is construct a resolution in $\mathrm{mod}(B)$

$$\xrightarrow{d} P^{(N_1)} \xrightarrow{d} P^{(N_0)} \xrightarrow{d} P \longrightarrow 0$$

where the differentials are given by matrices over A. Then applying $Q \otimes_B -$ to this complex gives the pseudo-free resolution

$$\longrightarrow A^{(N_1)} \longrightarrow A^{(N_0)} \longrightarrow A \longrightarrow 0$$

But we have $\sum_{i=1}^{n} p_i q_i = 1$ in B which means we have a B-module surjection

$$\begin{aligned} P^n &\twoheadrightarrow B \\ (p_i') &\longmapsto \sum p_i' q_i \end{aligned}$$

Thus the composition

$$\begin{aligned} P^n &\longrightarrow B \xrightarrow{\cdot p} P \\ (p_i') &\longmapsto \sum p_i' q_i \mapsto \sum p_i' q_i p \end{aligned}$$

is a B-module map $P^n \longrightarrow P$ whose matrix components $g_i p$ are in A.

Thus by using enough p to generate P over B we get an exact sequence

$$0 \longrightarrow N_1 \longrightarrow P^{(N_0)} \xrightarrow{\varphi} P \longrightarrow 0$$

where φ is given by a matrix over A. Then using generators for N_1 we get

$$\begin{tikzcd} P^{(N_1)} \arrow[r, two heads] \arrow[rr, bend left, "\psi"] & N_1 \arrow[r, hook] & P^{(N_0)} \end{tikzcd}$$

such that ψ has components in A, etc.

Suppose we have an exact sequence

$$0 \longrightarrow M_k \longrightarrow I^{(N_{k-1})} \longrightarrow \cdots \longrightarrow I^{(N_0)} \xrightarrow{\varphi} I \longrightarrow 0$$

of R-modules, where φ extends to $R^{(N_0)} \longrightarrow I$. Look at the spectral sequence with $T_p = \mathrm{Tor}_p^R(R/I, -)$:

picture for $k=2$
abutment is 0

E^1 term:

$$\begin{array}{|lllll} T_2(I) \xleftarrow{0} & & & & \\ T_1(I) \xleftarrow{0} T_1(I)^{(N_1)} & & & & \\ T_0(I) \xleftarrow{0} T_0(I)^{(N_0)} & \longleftarrow T_0(I)^{(N_1)} & \longleftarrow T_0(M_2) & \longleftarrow 0 & \\ \hline \end{array}$$

This immediately gives $T_0(I) = T_1(I) = \cdots = T_{k-1}(I) = 0$ and $T_0(M_k) \xrightarrow{\sim} T_k(I)$.

<u>November 28, 1994</u>

Suppose we start with

$$\mathcal{S} \xrightarrow{\iota_*} \mathcal{A} \xrightarrow{j^*} \mathcal{A}/\mathcal{S}$$

and assume that a left adjoint $L\iota^*$ for $\iota_* : D^-(\mathcal{S}) \longrightarrow D^-(\mathcal{A})$ exists satisfying $L\iota^* \iota_* \xrightarrow{\sim} 1$, equivalently $\iota_* : D^-(\mathcal{S}) \to D^-(\mathcal{A})$ is fully faithful.

Consider the full subcategories $\mathcal{X}^b = \operatorname{Ker} L\iota^*$, $\mathcal{X}^\# =$ ess. image of ι_* of $\mathcal{X} = D^-(\mathcal{A})$. Then we have orthogonality:

$$\operatorname{Hom}_{\mathcal{X}}(\mathcal{X}^b, \mathcal{X}^\#) = \operatorname{Hom}_{\mathcal{X}}(\mathcal{X}^b, \iota_* D^-(\mathcal{S}))$$

$$= \operatorname{Hom}_{D^-(\mathcal{S})}(L\iota^*(\mathcal{X}^b), D^-(\mathcal{S})) = 0$$

and for every X a triangle

$$X^b \longrightarrow X \xrightarrow{\beta_X} \iota_* L\iota^*(X) \longrightarrow$$

So we are in the BBG situation (I forgot to mention that $\mathcal{X}^b, \mathcal{X}^\#$ are closed under translation). Thus $\mathcal{X}^\#$ is a thick subcategory of $\mathcal{X}$ such that $\mathcal{X}/\mathcal{X}^\# \simeq \mathcal{X}^b$.

I would now like to identify $\mathcal{X}^\#$ with $D^-(\mathcal{A})_{\mathcal{S}} = \operatorname{Ker}\{j^* : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A}/\mathcal{S})\}$. The inclusion $\mathcal{X}^\# = \operatorname{ess\,Im}(\iota_*) \subset \operatorname{Ker}(j^*)$ is obvious from $j^* \iota_* = 0$. So let $X \in \operatorname{Ker}(j^*)$.

We form the Δ

$$X^b \longrightarrow X \xrightarrow{\beta_X} \iota_* L\iota^*(X) \longrightarrow$$

really we define X^b to be the fibre of β_X. Then $L\iota^*(X^b) = 0$, and $j^*(X^b) = 0$ because j^* kills both X and $\mathrm{Im}(\iota^*)$. Now $X^b \in D^-(R)$ so if $X^b \neq 0$ there is a least n such that $H_n(X^b) \neq 0$. We then have a nonzero map $X^b \longrightarrow H_n(X^b)[n]$ in $D^-(R)$. But $j^*(X) = 0 \implies H_n(X^b) \in \mathcal{S}$ so $H_n(X^b)[n] \in \mathrm{Im}(\iota^*)$ while $X^b \in \mathrm{Ker}(L\iota^*)$ contradicting orthogonality.

It thus seems that we have an equivalence of $\mathcal{X}^b$ with $\mathcal{X}/\mathcal{X}^\# = D^-(\mathcal{a})/D^-(\mathcal{a})_{\mathcal{S}} = D^-(\mathcal{a}/\mathcal{S})$, and in particular we get the existence of a left adjoint $Lj_!$ for $j^* : D^-(\mathcal{a}) \to D^-(\mathcal{a}/\mathcal{S})$.

This is a little surprising because we seem to be showing $\iota^* \exists \implies j_! \exists$ approximately, but it's probably OKAY since the exact sequence

$$0 \to \iota_* L_1 \iota^*(M) \longrightarrow j_!(j^*M) \longrightarrow M \longrightarrow \iota_*\iota^* M \longrightarrow 0$$

says roughly that $j_!$ should exist when ι^* and $L_1\iota^*$ do.

November 30, 1994

Let A be a ring such that $A^2 = A$. Can we associate to $\mathcal{M}(A)$ intrinsically a cyclic homology theory (type) and a K-theory such that when A is h-unital these are the K-theory and cyclic homology theory of A?

The following is necessary for this to work. Let A, B be h-unital rings such that $\exists$ an equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$. Then corresponding to this equivalence should be an isomorphism of algebraic K-theory and cyclic homology theory.

Let's try to establish this for cyclic homology, really for Hochschild homology. The composition

$$\mathcal{M}(A) \simeq \mathcal{M}(B) \hookrightarrow \text{mod}(B) \qquad \text{note } B = B^{(2)}$$

$$N \longmapsto B \otimes_A N$$

is right continuous, hence of the form $M \longmapsto P \otimes_A M$ where P, [redacted] the image of A, is a left $\tilde{B}$, right A bimodule, which is firm for A^{op} since the functor is defined on $\mathcal{M}(A)$, and firm for B since the image of the functor is contained in firm(B). Similarly for the inverse equivalence. Thus the equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$ should be given by a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ such that

$$P \otimes_A A \xrightarrow{\sim} P \qquad A \otimes_A Q \xrightarrow{\sim} Q$$

$$Q \otimes_B B \xrightarrow{\sim} Q \qquad B \otimes_B P \xrightarrow{\sim} P$$

$$P \otimes_A Q \xrightarrow{\sim} B \qquad Q \otimes_B P \xrightarrow{\sim} A$$

(the other two isoms. $A \otimes_A A \xrightarrow{\sim} A$, $B \otimes_B B \xrightarrow{\sim} B$ come from h-unitality.)

I would like to know that the corresponding isomorphisms hold on the derived category level, (assuming A, B h-unital, which means $A \overset{L}{\otimes}_A A \xrightarrow{\sim} A$, $B \overset{L}{\otimes}_B B \xrightarrow{\sim} B$).

Suppose I can show $Q \overset{L}{\otimes}_B P \xrightarrow{\sim} A$, $P \overset{L}{\otimes}_A Q \xrightarrow{\sim} B$. Then A and B have isomorphic Hochschild homology:

$$A \overset{L}{\otimes}_A \;\simeq\; Q \overset{L}{\otimes}_B P \overset{L}{\otimes}_A \;\simeq\; P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B \;\simeq\; B \overset{L}{\otimes}_B .$$

This argument seems to work provided we replace P, Q by their Dfirm equivalents, namely $P \overset{L}{\otimes}_A A$ instead of $P \otimes_A A = P$. I claim

1) $P \overset{L}{\otimes}_A A \longrightarrow P \otimes_A A = P$ is a nil(B)-quis, nil(A^{op})-quis

2) $P \overset{L}{\otimes}_A Q \longrightarrow P \otimes_A Q = B$ is a nil(B)-quis, nil(B^{op})-quis

3) $B \overset{L}{\otimes}_B P \longrightarrow B \otimes_B P = P$ is a $\left(\begin{array}{l}\text{nil}(B)\\ \text{nil}(A^{op})\end{array}\right.$ quis

4) $Q \overset{L}{\otimes}_B P \longrightarrow Q \otimes_B P = A$ is a $\left(\begin{array}{l}\text{nil}(A)\\ \text{nil}(A^{op})\end{array}\right.$ quis

Consider the second. We want to show $\text{Tor}_n^A(P, Q)$ $n \geq 1$ is killed by B on both left and right. Let $pq \in B$. Then right mult on Q by pq factors

$$Q \xrightarrow{\cdot p} A \subset \tilde{A} \xrightarrow{\cdot q} Q$$

so the induced map on $\text{Tor}_n^A(P, Q)$ factors through $\text{Tor}_n^A(P, \tilde{A}) = 0$, for $n \geq 1$. The rest is similar.

Since A, B are h-unital, we know that $B \overset{L}{\otimes}_B -$ maps nil(B)-quis into quis, and similarly for A. Thus from 1) + 3) we get quis

$$P\overset{L}{\otimes}_A A \xleftarrow{qu} B\overset{L}{\otimes}_B P\overset{L}{\otimes}_A A \xrightarrow{qu} B\overset{L}{\otimes}_B P$$

and from 2) + 4) we get quis

$$B\overset{L}{\otimes}_B P\overset{L}{\otimes}_A Q \xrightarrow{qu} B\overset{L}{\otimes}_B B \xrightarrow{qu} B$$

$$Q\overset{L}{\otimes}_B P\overset{L}{\otimes}_A A \xrightarrow{qu} A\overset{L}{\otimes}_A A \xrightarrow{qu} A$$

Then we get isomorphic Hochschild homology

$$\begin{aligned} A\overset{L}{\otimes}_A \quad &\xleftarrow{\sim} \quad Q\overset{L}{\otimes}_B P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A \\ &= P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A Q\overset{L}{\otimes}_B \\ &\xleftarrow{\sim} B\overset{L}{\otimes}_B P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A Q\overset{L}{\otimes}_B \\ &\xrightarrow{\sim} B\overset{L}{\otimes}_B P\overset{L}{\otimes}_A Q\overset{L}{\otimes}_B \quad \xrightarrow{\sim} B\overset{L}{\otimes}_B \end{aligned}$$

Let us now start with a ring A such that $A = A^2$. The question is whether we can find an h-unital ring B such that $\mathcal{M}(A) \simeq \mathcal{M}(B)$. A sufficient condition that B be h-unital is for B to be flat as right B-modules. In this case the composite

$$\begin{array}{ccccccc} \mathcal{M}(A) & \simeq & \mathcal{M}(B) & \xrightarrow{\sim} & \mathrm{firm}(B) & \subset & \mathrm{mod}(B) \\ & & N & \longmapsto & & & B\otimes_B N \end{array}$$

is exact. This composition is $M \mapsto P\otimes_A M$ where P is the firm B-module corresponding to $\tilde{A}$. Thus B is right B-flat $\Longleftrightarrow$ P is right A-flat.

I think we can always arrange this starting from any $A = A^2$. Choose $P \overset{\varepsilon}{\twoheadrightarrow} A$ where P is a right firm flat A-module, ε is a right A-module map. Then we have a surjective A-bimodule map

$$A \otimes_{\mathbb{Z}} P \twoheadrightarrow A \qquad (a, p) \mapsto a\varepsilon(p)$$

More generally we can replace A by any A-module Q equipped with a surjective A-bimodule map $Q \otimes_{\mathbb{Z}} P \overset{\langle\,,\,\rangle}{\twoheadrightarrow} A$, and such that $AQ = Q$. Then put

$$B = P \otimes_A Q \qquad (p_1, q_1)(p_2, q_2) = (p_1 \langle q_1, p_2\rangle, q_2)$$

Then $B^2 = P\langle Q, P\rangle \otimes_A Q = PA \otimes_A Q = P \otimes_A Q = B$. Also B acts on P and right acts on Q such that $BP = P\langle Q, P\rangle = PA = P$, etc. Thus we should have a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ such that P is right A-flat. I should check that P is firm for B. We have

$$0 \to K \to Q \otimes_B P \to A \to 0$$

with $AK = KA = 0$. Thus

$$\begin{array}{ccccccc} P \otimes_A K & \to & P \otimes_A Q \otimes_B P & \overset{\sim}{\to} & P \otimes_A A & \to & 0 \\ \| & & \| & & \| & & \\ PA \otimes_A K = 0 & & B \otimes_B P & & P & & \end{array}$$

December 1, 1994

I want to discuss some complements to yesterday's work. First let's consider a Morita equivalence $\mathcal{M}(A) = \mathcal{M}(B)$ where A, B are h-unital rings, which arises as a composition $\mathcal{M}(A) \simeq \mathcal{M}(C) \simeq \mathcal{M}(B)$ where C is unital. For simplicity suppose $A = AeA$, $B = BfB$ where e, f are idempotents in A, B resp., such that $eAe = C = fBf$. The functors are

$$\mathcal{M}(A) \simeq \mathcal{M}(C) \simeq \mathcal{M}(B)$$

$$M \longmapsto eM = eA\otimes_A M \longmapsto \underbrace{(Bf\otimes_C eA)}_{P}\otimes_A M$$

$$\underbrace{(Ae\otimes_C fB)}_{Q}\otimes_B N \longleftarrow fB\otimes_B N \longleftarrow N$$

One has $Q\otimes_B P = (Ae\otimes_C fB)\otimes_B (Bf\otimes_C eA) = Ae\otimes_C C\otimes_C eA$
$= Ae\otimes_C eA = A$, similarly $P\otimes_A Q = B$. Also
$B\otimes_B P = B\otimes_B (Bf\otimes_C eA) = Bf\otimes_C eA = P$, etc.

We know $A = Ae\otimes_C eA$ is h-unital $\iff$ $\mathrm{Tor}_n^C(Ae, eA) = 0$ for $n \geq 1$. Why? Because A-unital means that $\exists$ a firm flat (A-module) resolution $E \longrightarrow A$.

A complex E of firm flat A-modules corresponds to a complex of (flat) C-modules F by $F = eE$, $E = Ae\otimes_C F$. (Check: $W \mapsto W\otimes_A E = W\otimes_A Ae\otimes_C F = We\otimes_C F$. Since We is exact in W, this shows F C-flat $\Rightarrow$ E is A-flat.)

Now this correspondence makes flat C-module resolutions F of eA correspond to firm flat A-module complexes which are resolutions mod A-nil modules of A. The homology groups $H_n(Ae \otimes_C F) = \mathrm{Tor}_n^C(Ae, eA)$ are independent of the choice of F. Thus A is h-unital $\iff$ $\mathrm{Tor}_n^C(Ae, eA) = 0$ $\forall n \geq 1$.

Similarly B is h-unital $\iff$ $\mathrm{Tor}_n^C(Bf, fB) = 0$ $\forall n \geq 1$ (and of course $Bf \otimes_C fB \xrightarrow{\sim} B$).

Now consider whether $B \overset{L}{\otimes}_B P \longrightarrow B \otimes_B P = P$ is a qis. This is equivalent to P having a resolution by firm flat B-modules. A complex E of firm flat B-modules corresponds to a complex F of flat C-modules via: $E = Bf \otimes_C F$, $F = fE$.

If E resolves P, then $F = fE$ is a flat C-module resolution of $fP = f(Bf \otimes_C eA) = eA$ and we have

$$H_n(Bf \otimes_C F) = \mathrm{Tor}_n^C(Bf, eA) = 0 \qquad n \geq 1.$$

So we see for a Morita equivalence between h-unital rings $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where P, Q are firm on both sides, that P, Q are not Dfirm in general. (To construct a counterexample, start from C, choose Ae, eA, Bf, fB so that $\mathrm{Tor}_{>0}^C(Bf, eA) \neq 0$, but $\mathrm{Tor}_{>0}^C(Ae, eA) = 0$ and $\mathrm{Tor}_{>0}^C(Bf, fB) = 0$, e.g. take Ae and fB flat over C.)

Next: an observation on replacing A by a Morita equivalent h-unital ring.

In this construction we take P to be a firm flat right A-module mapping onto A, map $\varepsilon : P \twoheadrightarrow A$, and $Q = A$ and $\langle\, ,\, \rangle : A \otimes_{\mathbb{Z}} P \longrightarrow A$, $\langle a, p\rangle = a\,\varepsilon(p)$.

Then $B = P \otimes_A Q = P \otimes_A A = P$ with $p_1 p_2 = p_1\,\varepsilon(p_2)$.

Note $\varepsilon : B \longrightarrow A$ is a homomorphism:

$\varepsilon(p_1 p_2) = \varepsilon(p_1\,\varepsilon(p_2)) = \varepsilon(p_1)\,\varepsilon(p_2)$ since ε is a right A module map. Let $K = \mathrm{Ker}(\varepsilon)$. Then $K \subset B$ is an ideal such that $BK = 0$.

So we have an example of the situation $A = B/K$, $BKB = 0$, where we have a Morita equivalence given by $\begin{pmatrix} B/K & B/KB \\ B/BK & B \end{pmatrix}$.

A better Morita context for our purposes is obtained as follows. We assume $A = B/K$ with $BK = 0$. Then

$$\begin{pmatrix} \tilde{B} & \tilde{B} \\ B & \tilde{B} \end{pmatrix}\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} \tilde{B}K + \tilde{B}0 & \tilde{B}K + \tilde{B}0 \\ BK + \tilde{B}0 & BK + \tilde{B}0 \end{pmatrix} = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{B} & \tilde{B} \\ B & \tilde{B} \end{pmatrix} \subset \begin{pmatrix} K\tilde{B} + KB & K\tilde{B} + K\tilde{B} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$$

so $\begin{pmatrix} \tilde{B} & \tilde{B} \\ B & \tilde{B} \end{pmatrix} \Big/ \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{A} \\ B & \tilde{B} \end{pmatrix}$ is a Morita context.

left modules

$$M \longmapsto B \otimes_A M$$
$$\tilde{A} \otimes_B N \longleftarrow N$$

right modules:

$$M \longmapsto \underbrace{M \otimes_A \tilde{A}}_{} = M \quad \text{restriction of scalars for } B \to A.$$
$$B \otimes_B N \longleftarrow N$$

Thus we know that restriction of scalars for $B \twoheadrightarrow A$ gives an equivalence between right firm flat B-modules and right firm flat A-modules. In particular if $P = B$ is right A flat, then $P = B$ is right B flat, checking what we ~~[illegible]~~ said yesterday.

December 2, 1994

Let $\mathcal{A}$ be an abelian category, let U be an object of $\mathcal{A}$, let $R = \mathrm{Hom}_{\mathcal{A}}(U,U)$. We then have a functor

$$\mathcal{A} \longrightarrow \mathrm{mod}(R^{op})$$
$$M \longmapsto \mathrm{Hom}_{\mathcal{A}}(U, M)$$

which is additive and left continuous. We would like this functor to have a left adjoint:

$$\mathrm{Hom}_{\mathcal{A}}(W \otimes_R U, M) = \mathrm{Hom}_{R^{op}}(W, \mathrm{Hom}_{\mathcal{A}}(U, M))$$

Consider the class of right R-modules W such that $W \otimes_R U$ exists (i.e. such that the right side considered as functor of M is representable). This class is closed under cokernels and direct sums when they exist. So if $\mathcal{A}$ is closed under arbitrary $\oplus$'s we see $W \otimes_R U$ exists for all W in $\mathrm{mod}(R^{op})$.

So far I haven't used that $R = \mathrm{Hom}_{\mathcal{A}}(U,U)$, only that R acts on U.

Let's discuss examples.

1. Suppose $\mathcal{A} = \mathrm{mod}(A^{op})$ where A is unital, and $U = eA$ with e idempotent in A. Then $R = \mathrm{Hom}_{A^{op}}(eA, eA) = eAe$ and

$$\mathrm{Hom}_{\mathcal{A}}(U, M) = \mathrm{Hom}_{A^{op}}(eA, M) = Me = M \otimes_A Ae$$

$$N \otimes_R U = N \otimes_{eAe} eA$$

so we have the adjoint functors

$$\begin{array}{ccccc} \mathrm{mod}(R^{op}) & \simeq & \mathcal{M}(A^{op}, AeA) & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & \mathrm{mod}(A^{op}) \\ N & & \longmapsto & & N \otimes_{eAe} eA \\ Me = M \otimes_A Ae & & \longleftarrow\!\shortmid & & M \end{array}$$

In this situation the functor $N \mapsto N \otimes_R \mathcal{U}$ is fully faithful and right exact in general.

2. Take $\mathcal{a} = \mathcal{M}(\tilde{A}^{op}, A)$ where $\tilde{A} = A^2$ and let $\mathcal{U}$ be a generator for $\mathcal{a}$; we can suppose $\mathcal{U}$ is a firm A^{op} module. [redacted] Because $\mathcal{U}$ is firm we have $R = \mathrm{Hom}_{\mathcal{a}}(\mathcal{U}, \mathcal{U}) = \mathrm{Hom}_{\tilde{A}^{op}}(\mathcal{U}, \mathcal{U})$ and [redacted] $\mathrm{Hom}_{\mathcal{a}}(\mathcal{U}, M) = \mathrm{Hom}_{\tilde{A}^{op}}(\mathcal{U}, M)$, so from

$$\mathrm{Hom}_{R^{op}}(N, \mathrm{Hom}_{\tilde{A}^{op}}(\mathcal{U}, M)) = \mathrm{Hom}_{\tilde{A}^{op}}(N \otimes_R \mathcal{U}, M)$$

one sees the adjoint $N \mapsto N \otimes_R \mathcal{U}$ to $\mathrm{Hom}_{\mathcal{a}}(\mathcal{U}, -)$ is the usual tensor product module construction.

This right module notation I find confusing.

December 3, 1994

To understand Gabriel-Popescu theorem.

1. Let $\mathcal{A}$ be abelian, U in $\mathcal{A}$, $R \to \mathrm{Hom}_{\mathcal{A}}(U,U)^{op}$ a unital ring homomorphism. We have an additive functor $\mathrm{Hom}_{\mathcal{A}}(U,-) : \mathcal{A} \longrightarrow \mathrm{mod}(R)$. Denote by $N \longmapsto U \otimes_R N$ its left adjoint:

$$\mathrm{Hom}_{\mathcal{A}}(U \otimes_R N, M) = \underbrace{\mathrm{Hom}_R(N, \mathrm{Hom}_{\mathcal{A}}(U,M))}$$

for N in the class such that is a representable functor of M. This class is closed under cokernels and direct sums. Hence assuming $\mathcal{A}$ is closed under $\oplus$'s, this left adjoint exists and is right continuous.

Put $F = U \otimes_R -$ and $G = \mathrm{Hom}_{\mathcal{A}}(U,-)$.

2. TFAE:

a) Every M in $\mathcal{A}$ is a quotient of $U^{(\Sigma)} = \bigoplus_{\Sigma} U$ for some set Σ.

(Better to [illegible] put a) at the end)

b) $G = \mathrm{Hom}_{\mathcal{A}}(U,-)$ is faithful

c) The adjunction arrow $FG \xrightarrow{\alpha} 1$ is surjective.

Proof.

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{A}}(M,M') \arrow[dl, "G_*"'] \arrow[dr, "\alpha^t"] & \\
\mathrm{Hom}_R(GM,GM') \arrow[rr, equal] & & \mathrm{Hom}_{\mathcal{A}}(FGM,M')
\end{tikzcd}$$

The kernel of α^t is $\mathrm{Hom}_{\mathcal{A}}(M/\mathrm{Im}\,FGM, M')$, so b) $\Leftrightarrow$ c) is clear.

a) says $\forall M$ there exists a surjection $FL \twoheadrightarrow M$ where L is a free R-module. Assuming this we have a corresponding map $L \to GM$ such that $FL \to FGM \to M$ is surjective, hence c) holds. Conversely assuming $FGM \twoheadrightarrow M$ we choose $L \twoheadrightarrow GM$ and then $FL \twoheadrightarrow FGM \twoheadrightarrow M$, so a) holds. (Here we have used that F is right exact.)

Call U a <u>generator</u> for $\mathcal{A}$ when these equivalent conditions hold. (Actually b) is probably the definition.)

3. Next assume the AB5 axiom. We want to show F is exact. It suffices to show for any left ideal $\mathfrak{a} \subset R$ that $F(\mathfrak{a}) \to F(R) = U$ is injective. This is because any injection $N' \subset N$ is built up in a transfinite way from pushouts of such injections. (Alternatively, we know $\mathcal{A}$ has enough injectives so that F is exact $\iff$ G preserves injectives. One can test the injectivity of $G(E)$ using the inclusions $\mathfrak{a} \subset R$ for all $\mathfrak{a}$.)

Choose generators: $\mathfrak{a} = \sum_{i \in \Sigma} R r_i$, and define K by

$$0 \longrightarrow K \longrightarrow U^{(\Sigma)} \xrightarrow{(r_i)} U$$

By AB5 $\quad K = \bigcup_{\sigma \text{ finite} \subset \Sigma} (K \cap U^{(\sigma)})$, which means

$\exists \; U^{(\Sigma')} \twoheadrightarrow K$ such that $\forall j \in \Sigma'$ the image

$$U \xrightarrow{\text{in}_j} U^{(\Sigma')} \longrightarrow K \subset U^{(\Sigma)}$$

is contained in $U^{(\sigma)}$ for some σ. Thus the map $U^{(\Sigma')} \twoheadrightarrow K \subset U^{(\Sigma)}$ is given by a matrix (r_{ji}), $j \in \Sigma'$, $i \in \Sigma$ such that

$\forall j \ \{ i \in \Sigma \mid r_{ji} \neq 0 \}$ is finite.

Define N by

$$R^{(\Sigma')} \xrightarrow{(r_{ji})} R^{(\Sigma)} \longrightarrow N \longrightarrow 0$$

I forgot to say that I can assume $U^{(\Sigma')} \longrightarrow K$ chosen so that any map $U \longrightarrow K \cap U^{(\sigma)}$ comes from a map $U \longrightarrow U^{(\sigma')}$ for some $\sigma' \subset \Sigma'$.

$$\begin{tikzcd}
 & R^{(\Sigma')} \arrow[r, "(r_{ji})"] \arrow[d] & R^{(\Sigma)} \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & R^{(\Sigma)} \arrow[r] \arrow[dr, "(r_i)"'] & \mathfrak{a} \arrow[r] & 0 \\
 & & & R &
\end{tikzcd}$$

Since $\sum_i r_{ji} r_i = 0$ the vertical arrows exist.

By construction any element $\eta \in J$ (we can think of η as a map $U \longrightarrow K \cap U^{(\sigma)}$ for some finite $\sigma \subset \Sigma$) comes from an element of $R^{(\Sigma')}$. Thus $R^{(\Sigma')} \twoheadrightarrow J$ so $N \simeq \mathfrak{a}$.

Finally we have

$$\begin{array}{ccccccc}
F(R^{(\Sigma')}) & \longrightarrow & F(R^{(\Sigma)}) & \longrightarrow & F(\mathfrak{a}) & \longrightarrow & 0 \\
\| & & \| & & & & \\
U^{(\Sigma')} & \longrightarrow & U^{(\Sigma)} & \longrightarrow & U & &
\end{array}$$

so $F(\mathfrak{a}) \hookrightarrow U$. This concludes the proof that F is exact.

4. Let M be in $\mathcal{A}$, and choose

$$0 \to K \to \mathcal{U}^{(\Sigma)} \to M \to 0$$

As before we can choose $\mathcal{U}^{(\Sigma')} \twoheadrightarrow K$ such that $\forall$ finite $\sigma' < \Sigma'$ $\mathcal{U}^{(\sigma')}$ maps to $K \cap \mathcal{U}^{(\sigma)}$ for some finite $\sigma < \Sigma$, and moreover any $\mathcal{U} \to K \cap \mathcal{U}^{(\sigma)}$ factors through some $\mathcal{U}^{(\sigma')}$. Then we get

$$R^{(\Sigma')} \xrightarrow{(r_{ji})} R^{(\Sigma)} \longrightarrow N \longrightarrow 0 \qquad \text{defining } N$$

such that

$$\begin{array}{ccccccc} F(R^{(\Sigma')}) & \longrightarrow & F(R^{(\Sigma)}) & \longrightarrow & F(N) & \longrightarrow & 0 \\ \| & & \| & & & & \\ \mathcal{U}^{(\Sigma')} & \longrightarrow & \mathcal{U}^{(\Sigma)} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

hence $F(N) \xrightarrow{\sim} M$.

Now suppose we were to start with our original map $\mathcal{U}^{(\Sigma)} \to M$ such that the corresponding $R^{(\Sigma)} \to G(M)$ is surjective. We then have

$$\begin{array}{ccccccccc} & & R^{(\Sigma')} & \longrightarrow & R^{(\Sigma)} & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & R^{(\Sigma)} & \longrightarrow & G(M) & \longrightarrow & 0 \end{array}$$

where $R^{(\Sigma')}$ maps onto J. (Any η in J is the same as a map $\mathcal{U} \to \mathcal{U}^{(\sigma)}$ such that $\mathcal{U} \to \mathcal{U}^{(\sigma)} \to M$ is zero, i.e. $\mathcal{U} \to \mathcal{U}^{(\sigma)} \cap K$ and we know such things come from an element of $R^{(\Sigma')}$.) $\therefore N \xrightarrow{\sim} GM$

so $FGM \xrightarrow{\sim} M$.

At this point we know that
$F: \mathrm{mod}(R) \longrightarrow \mathcal{A}$ is exact and
has a right adjoint G such $FG \xrightarrow{\sim} 1$,
equivalently G is fully-faithful. From
this the Gabriel-Popescu theorem should be
clear.

(Dec 4, 1994)

Observation: In order to show a right exact
functor F is exact, it suffices to show $N' \subset L$ free
$\Longrightarrow F(N') \hookrightarrow F(L)$. In effect $L_1F(N) = \mathrm{Ker}\{F(N') \to F(L)\}$
where $N = L/N'$. Diagrammatically

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & & & 0 \arrow[d] & \text{because } L'' \text{ free} \\
 & F(K') \arrow[r] \arrow[d] & F(K) \arrow[r] \arrow[d] & F(K'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(N_1) \arrow[r] \arrow[d] & F(L) \arrow[r] \arrow[d] & F(L'') \arrow[r] \arrow[d] & 0 \\
 & F(N') \arrow[r] \arrow[d] & F(N) \arrow[r] \arrow[d] & F(N'') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

so $F(N') \to F(N)$ is injective.

Even simpler:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & & 0 \arrow[d] & \text{because } L \text{ free} & \\
 & F(K) \arrow[r, equal] \arrow[d] & F(K) \arrow[d] & & \\
0 \arrow[r] & F(N_1) \arrow[r] \arrow[d] & F(L) \arrow[r] \arrow[d] & F(N'') \arrow[r] \arrow[d, equal] & 0 \\
\text{because } L \text{ free} & F(N') \arrow[r] \arrow[d] & F(N) \arrow[r] \arrow[d] & F(N'') \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

so $F(N') \to F(N)$ is injective.

Even clearer:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & N_1 \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & L \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\ & & N'' \arrow[r, equal] \arrow[d] & N'' \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}
\qquad
\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d, dashed, "?"] & \\ F(K) \arrow[r] \arrow[d, equal] & F(N_1) \arrow[r] \arrow[d] & F(N') \arrow[r] \arrow[d] & 0 \\ 0 \to F(K) \arrow[r] \arrow[d] & F(L) \arrow[r] \arrow[d] & F(N) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[d] & F(N') \arrow[r, equal] \arrow[d] & F(N') \arrow[d] & \\ 0 & 0 & 0 & \end{tikzcd}$$

The serpent lemma shows $F(N') \hookrightarrow F(N)$.

I want to go over the GP thm. proof.

We have the functor $U\otimes_R -$ from mod(R) to $\mathcal{A}$ which is right exact. It has derived functors $\mathrm{Tor}_n^R(U,-)$, and we want to show $\mathrm{Tor}_1^R(U,-)$ vanishes. With the AB5 axiom, we should be able to argue that it's enough to prove $\mathrm{Tor}_1^R(U, R/\mathfrak{a}) = 0$ for any f.g. left ideal $\mathfrak{a}$.

Let's check this. $\mathrm{Tor}_1^R(U,N) = \mathrm{Ker}\{U\otimes_R K \to U\otimes_R L\}$ where $N = L/K$ with $L = R^{(\Sigma)}$ free. Then

$N = \bigcup N_\sigma$, $N_\sigma = R^{(\sigma)}/K\cap R^{(\sigma)}$ and

$$\mathrm{Tor}_1^R(U,N) = \mathrm{Ker}\{U\otimes_R K \longrightarrow U\otimes_R L\}$$

$$= \mathrm{Ker}\{U\otimes_R \varinjlim_\sigma (K\cap R^{(\sigma)}) \longrightarrow U\otimes_R \varinjlim_\sigma L^{(\sigma)}\}$$

$$= \mathrm{Ker}\{\varinjlim_\sigma U\otimes_R (K\cap R^{(\sigma)}) \to \varinjlim_\sigma U\otimes_R L^{(\sigma)}\}$$

$$= \varinjlim_\sigma \mathrm{Ker}\{U\otimes_R (K\cap R^{(\sigma)}) \to U\otimes_R L^{(\sigma)}\} = \varinjlim_\sigma \mathrm{Tor}_1^R(U, N_\sigma)$$

Thus reduce to showing $\mathrm{Tor}_1^R(U,N)=0$ for N finitely generated. Then $N=L/K$ with $L=R^n$ and we have $K=\bigcup K_\alpha$, where K_α runs over all f.g. submodules of K.

Again $\mathrm{Ker}\{U\otimes_R \bigcup K_\alpha \to U\otimes_R L\}$

$$= \varinjlim_\alpha \mathrm{Ker}\{U\otimes_R K_\alpha \to U\otimes_R L\}$$

so we can suppose N finitely presented. But first use that any f.g. module is a successive extension of cyclic modules $R/\mathfrak{a}$, then use the $\mathfrak{a}=\bigcup \mathfrak{a}_\alpha$ argument to reduce to $\mathfrak{a}$ f.g.

Let $\mathfrak{a}=\sum_{i=1}^n Rx_i$, $x_i\in R$, and ~~[illegible]~~ let

$$0\longrightarrow M \longrightarrow U^n \xrightarrow{(x_i)} U$$

define M. Choose generators $\Sigma' \to \mathrm{Hom}_a(U,M)$, whence we have an exact sequence

$$U^{(\Sigma')} \longrightarrow U^n \longrightarrow U$$

More precisely, since $R^{(\Sigma')} \twoheadrightarrow \mathrm{Hom}_a(U,M)$ the corresp. map $U^{(\Sigma')}\to M$ has the property that any $U\to M$ lifts to $U\to U^{(\Sigma')}$. In particular $U^{(\Sigma')}\twoheadrightarrow M$ as U is a generator. Next note that

$$R^{(\Sigma')} \longrightarrow R^n \xrightarrow{(x_i)} R$$

is exact since $\mathrm{Ker}\{R^n \xrightarrow{(x_i)} R\} = \mathrm{Hom}_a(U,M)$.

Thus
$$R^{(\Sigma')} \to R^n \longrightarrow \overbrace{\sum Rx_i}^{\mathfrak{a}} \longrightarrow 0 \quad \text{exact}$$

$$\begin{array}{ccccccc} U\otimes_R R^{(\Sigma')} & \longrightarrow & U\otimes_R R^n & \longrightarrow & U\otimes_R \mathfrak{a} & \longrightarrow & 0 \\ \| & & \| & & & & \\ U^{(\Sigma')} & \longrightarrow & U^n & \longrightarrow & U & & \end{array}$$

showing that $U \otimes_R \mathfrak{a} \hookrightarrow U \otimes_R R = U$,
i.e. $\mathrm{Tor}_1^R(U, R/\mathfrak{a}) = 0$.

Let's repeat this argument more carefully. Suppose we have a finitely presented module:

$$R^m \xrightarrow{\cdot(x_{ji})} R^n \longrightarrow N \longrightarrow 0$$

and define M by

$$0 \longrightarrow M \longrightarrow U^m \xrightarrow{\cdot(x_{ji})} U^n$$

Then $0 \longrightarrow \mathrm{Hom}_{\mathcal{a}}(U, M) \longrightarrow R^m \xrightarrow{\cdot(x_{ji})} R^n$

choose $R^{(\Sigma')} \twoheadrightarrow \mathrm{Hom}_{\mathcal{a}}(U, M)$. Then because U is a generator we have $U^{(\Sigma')} \twoheadrightarrow M$, hence

$$(*) \qquad U^{(\Sigma')} \longrightarrow U^m \longrightarrow U^n$$

is exact. Also

$$(**) \qquad R^{(\Sigma')} \longrightarrow R^m \longrightarrow R^n \longrightarrow N \longrightarrow 0$$

is exact. Since $(**)$ goes into $(*)$ upon tensoring with U, we conclude that

$$\mathrm{Tor}_1^R(U, N) = 0. \qquad N \text{ f.p.}$$

Observe that all we've used is that $\mathcal{a}$ is ~~[illegible]~~ closed under $\oplus$'s and U is a generator with $R = \mathrm{Hom}_{\mathcal{a}}(U,U)^{op}$. I can avoid the assumption that $\mathcal{a}$ is closed under $\oplus$'s when finitely presented modules are closed under kernels.

December 6, 1994

I now want to prove Roos' theorem characterizing abelian ~~ab~~ categories of the form $\mathcal{M}(A)$ with $A = A^2$.

Let $\mathcal{A}$ be a Grothendieck category with generator U, $R = \mathrm{Hom}_{\mathcal{A}}(U,U)^{op}$, and the adjoint functors of the Gabriel-Popescu thm:

$$1) \qquad \mathrm{mod}(R) \underset{G = \mathrm{Hom}_{\mathcal{A}}(U,-)}{\overset{F = U \otimes_R -}{\rightleftarrows}} \mathcal{A}$$

The kernel $\mathcal{S} \subset \mathrm{mod}(R)$ of F is a Serre subcategory closed under $\oplus$'s. We want conditions for $\mathcal{S}$ to be closed under Π's, in which case $\mathcal{S} = \mathrm{mod}(R/I)$, where I is an idempotent ideal in R. It suffices to show that F commutes with products, i.e.

$$2) \qquad F\left(\prod_{\alpha \in J} M_\alpha\right) \longrightarrow \prod_{\alpha \in J} F(M_\alpha)$$

is an isomorphism for all families (M_α).

If the index set J is fixed we can consider both sides as functors ~~▭~~ from $\mathrm{mod}(R)^J$ to $\mathcal{A}$. Because F is exact, the left side is exact and the right side is left exact. The right side will be exact if we assume $\mathcal{A}$ satisfies AB4*, namely

$$3) \qquad \forall \alpha \quad X_\alpha \twoheadrightarrow Y_\alpha \implies \prod X_\alpha \twoheadrightarrow \prod Y_\alpha$$

Assuming AB4* from now on, the families (M_α) for which 2) is an isomorphism ~~▭~~ form a ~~▭~~ full subcategory of $\mathrm{mod}(R)^J$ closed under cokernels (in fact kernels, cokernels, extensions). So it suffices to handle the case where M_α is a free R-module for each α.

The next observation is that 2) is an isom. when M_α is "solid", [redacted] i.e. in the image of G, for [redacted] all α. [redacted]

[redacted] In effect, if $M_\alpha = G(X_\alpha)$ then $F(M_\alpha) = FG(M_\alpha) \simeq M_\alpha$ and $F(\prod M_\alpha) = F(\prod G(X_\alpha))$ $= FG(\prod X_\alpha) = \prod X_\alpha = \prod F(M_\alpha)$.

We know $R = \mathrm{Hom}_a(U, U) = G(U)$ is solid, hence any f.g. free R-module is solid, so 2) is an isomorphism when all the M_α are f.g. free modules.

Grothendieck's AB6 axiom says for any family of directed systems $X_{\alpha\beta}$, $\beta \in D(\alpha)$ of subobjects (here [redacted] $D(\alpha)$ is a directed set for each $\alpha \in J$) that

4) $$\bigcup_{\phi \in \prod_\alpha D(\alpha)} \bigcap_\alpha X_{\alpha\phi(\alpha)} \xrightarrow{\sim} \bigcap_\alpha \bigcup_{\beta \in D(\alpha)} X_{\alpha\beta}$$

I need the following condition which should be equivalent (at least in the presence of AB4* and AB5)

5) $$\bigcup_{\phi \in \prod_\alpha D(\alpha)} \prod_\alpha X_{\alpha\phi(\alpha)} \xrightarrow{\sim} \prod_\alpha \bigcup_{\beta \in D(\alpha)} X_{\alpha\beta}$$

Here $X_{\alpha\beta}$, $\beta \in D(\alpha)$ is directed system of injections for each α.

So now take free modules M_α, $\alpha \in J$ and write then $M_\alpha = \bigcup_{\beta \in D(\alpha)} M_{\alpha\beta}$, where the $M_{\alpha\beta}$ are f.g. free modules. Then

$$
\begin{tikzcd}
F\Big(\bigcup_\phi \prod_\alpha M_{\alpha\phi(\alpha)}\Big) \arrow[r, "\sim", "\substack{\text{my AB6}\\ \text{in mod}(R)}"'] & F\Big(\prod_\alpha \bigcup_{\beta\in D(\alpha)} M_{\alpha\beta}\Big) \arrow[d] \arrow[r, phantom, "="] & F\Big(\prod_\alpha M_\alpha\Big) \arrow[d] \\
\bigcup_\phi F\Big(\prod_\alpha M_{\alpha\phi(\alpha)}\Big) \arrow[u, "\cong"] \arrow[d, "\cong"', "\substack{\text{by what}\\ \text{we've proved}\\ \text{for f.g. free}\\ \text{modules}}"] & \prod_\alpha F\Big(\bigcup_{\beta\in D(\alpha)} M_{\alpha\beta}\Big) \arrow[r, phantom, "="] & \prod_\alpha F(M_\alpha) \\
\bigcup_\phi \prod_\alpha F(M_{\alpha\phi(\alpha)}) \arrow[r, "\cong"', "\substack{\text{my AB6 condition}\\ 5)\ \text{in } \mathcal{A}}"'] & \prod_\alpha \bigcup_{\beta\in D(\alpha)} F(M_{\alpha\beta}) \arrow[u, "\cong\ \substack{F\ \text{rt}\\ \text{cont}}"'] &
\end{tikzcd}
$$

This [redacted] concludes the proof that F commutes with products.

Conversely if $\mathcal{S} = \text{mod}(R/I)$ with $I = I^2$ and $\mathcal{A} =$ [redacted] $\text{mod}(R)/\mathcal{S} = \mathcal{M}(R, I)$, then because $F = j^* : \text{mod}(R) \longrightarrow \mathcal{A}$ commutes with both inductive + projective limits, it follows that the axioms $AB4^*$ and (my) AB6 5) hold in $\mathcal{A}$.

At this point I understand Roos' theorem.

[redacted]

I now want to discuss a possible line of investigation. I know how to assign Hochschild homology groups intrinsically to a Roos-type abelian category, and I want to extend this to cyclic homology and K-theory, the latter being the most interesting. I need to make a list of ideas from the scratch paper.

1. Let $\mathcal{A}$ be a Roos type (maybe Jans-Roos) abelian category. Then if U is a generator and $R = \operatorname{Hom}_{\mathcal{A}}(U, U)^{op}$ we get an intrinsic idempotent ideal I in R which is the smallest left ideal $\mathfrak{a}$ such that $U \otimes_R R/\mathfrak{a} = 0$. Should we think of I as compact operators on U?

2. If $\mathcal{A}$ is a Grothendieck category do there exist non-trivial right continuous exact functors $F: \mathcal{A} \longrightarrow \mathcal{A}b$? These should be analogous to points in a topos, so probably there don't exist enough of them, however we know this is true for $\mathcal{A} = \mathcal{M}_t(R, I)$. Does this hold for torsion theories where the Gabriel filter has a basis of ideals.

3. There is the problem of the relation between $A \overset{L}{\otimes}_{\tilde{A}} A$ and $L j_!(j^* A)$

$$L j_!(j^* A) \longrightarrow A \longrightarrow A^{\#} \longrightarrow$$

$$A \overset{L}{\otimes}_{\tilde{A}} A \longrightarrow A \longrightarrow k \overset{L}{\otimes}_{\tilde{A}} A \longrightarrow$$

December 8, 1994

I want to review the relations among:

1) $K_1(R/I) \xrightarrow{\partial} K_0(I)$

2) Atiyah-Singer (proof of Bott periodicity) map from invertibles in the Calkin algebra to the restricted Grassmannian.

3) Cayley transform picture of the Grassmannian.

Description of A-S map. First in the ungraded case starting from nontrivial projections in the Calkin algebra. One lifts a projection to self-adjoint contractions A with essential spectrum $\{1,-1\}$; the fibre is contractible. Then

$$A = \frac{D}{\sqrt{1+D^2}}, \quad g = \frac{1+iD}{1-iD}, \quad g^{1/2} = \frac{1+iD}{\sqrt{1+D^2}} = \sqrt{1-A^2} + iA$$

gives a unitary g with essential spectrum -1.

In the graded case, deform invertibles to unitaries, identify a unitary u with an involution $\begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$ inverted by ε, lift this involution to a self adjoint contraction $A = \frac{1}{i}\begin{pmatrix} 0 & -\alpha^* \\ \alpha & 0 \end{pmatrix}$.

Then $$\sqrt{1-A^2} = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & 0 \\ 0 & \sqrt{1-\alpha\alpha^*} \end{pmatrix}$$

$$g^{1/2} = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & -\alpha^* \\ \alpha & \sqrt{1-\alpha\alpha^*} \end{pmatrix} \qquad g^{-1/2} = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & \alpha^* \\ -\alpha & \sqrt{1-\alpha\alpha^*} \end{pmatrix}$$

$$e = g^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g^{-1/2} = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & 0 \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} \sqrt{1-\alpha^*\alpha} & \alpha^* \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\alpha^*\alpha & \sqrt{1-\alpha^*\alpha}\,\alpha^* \\ \alpha\sqrt{1-\alpha^*\alpha} & \alpha\alpha^* \end{pmatrix}$$

Thus $\operatorname{Im}(e) = \operatorname{Im}\begin{pmatrix} \sqrt{1-\alpha^*\alpha} \\ \alpha \end{pmatrix} = \operatorname{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}$

where $T = \alpha(1-\alpha^*\alpha)^{-1/2}$. Then

$$1 + T^*T = 1 + (1-\alpha^*\alpha)^{-1/2}\alpha^*\alpha(1-\alpha^*\alpha)^{-1/2} = \frac{1}{1-\alpha^*\alpha}$$

so $$\alpha = T(1+T^*T)^{-1/2}.$$

Now for $\partial : K_1(R/I) \longrightarrow K_0(I)$. Start with an invertible matrix u over R/I, which I'll suppose is an invertible element of R/I to simplify. Lift u to p and u^{-1} to q. Let $x = 1 - qp$, $y = 1 - pq$. Then $q_2 = q + qy$ satisfies

$$1 - q_2 p = 1 - q(1+y)p = 1 - (1+x)qp = 1 - (1+x)(1-x) = x^2$$

$$1 - pq_2 = 1 - pq(1+y) = 1 - (1-y)(1+y) = y^2.$$ Thus we can arrange that $1 - qp = a^2$ with $a \in I$, so

$$(q \ \ a)\begin{pmatrix} p \\ a \end{pmatrix} = 1 \qquad \text{hence } e = \begin{pmatrix} p \\ a \end{pmatrix}(q \ \ a) \text{ is idempotent.}$$

As $e = \begin{pmatrix} pq & pa \\ aq & a^2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ mod I, e determines an element of $K_0(I)$.

Here's the method from Milnor's book. Identity

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -q \\ p & 1-pq \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1-qp & -(q+qy) \\ p & 1-pq \end{pmatrix} = \begin{pmatrix} p & 1-pq \\ -(1-qp) & q(1+y) \end{pmatrix} = \begin{pmatrix} p & y \\ -x & q(1+y) \end{pmatrix}$$

Observe this is $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ mod I.

In fact one has an invertible matrix:

$$\begin{pmatrix} p & y \\ -x & q(1+y) \end{pmatrix} \cdot \begin{pmatrix} q(1+y) & -x \\ y & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The idempotent $e = \begin{pmatrix} p \\ -x \end{pmatrix} \begin{pmatrix} q(1+y) & -x \end{pmatrix}$ projects onto $\mathrm{Im} \begin{pmatrix} p \\ -x \end{pmatrix}$.

December 11, 1994

Morita equivalence for d-firm perfect complexes.

Let U be a perfect complex of R-modules which is d-firm for (R,I): $R/I \overset{L}{\otimes}_R U = 0$. Given a Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ satisfying the usual conditions I would like to prove that $P \overset{L}{\otimes}_R U$ is a d-firm perfect complex of S-modules.

I know already that $P \overset{L}{\otimes}_R U$ is d-firm for (S,J). Recall a proof. We can suppose U is a right bdd complex of projective R-modules. Then $R/I \overset{L}{\otimes}_R U = 0$ means U/IU is acyclic, hence contractible since it's a $\overset{\text{right-bdd}}{\text{complex}}$ of projective R/I modules. Lifting a contraction k of U/IU:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d] & U \arrow[d] \\ U/IU \arrow[r, "k"] & U/IU \end{tikzcd}$$

then $f = 1 - [d,h]$ maps U into IU. Then

$$U_f = \varinjlim \left(U \xrightarrow{f} U \xrightarrow{f} \cdots \right)$$

is a firm flat module complex such that $U \longrightarrow U_f$ is a quis. $\therefore$ $P \overset{L}{\otimes}_R U \xrightarrow[\text{quis}]{} P \otimes_R U_f$ and we know $P \otimes_R -$ respects firm flat complexes. Thus

$S/J \overset{L}{\otimes}_S (P \overset{L}{\otimes}_R U) \simeq S/J \otimes_S (P \otimes_R U_f) = 0.$

I want to do the preceding argument in a more concrete fashion. Ultimately this argument amounts to $S/J \overset{L}{\otimes}_S (P \overset{L}{\otimes}_R U) = (S/J \overset{L}{\otimes}_S P) \overset{L}{\otimes}_R U = 0$, because

$S/J \overset{L}{\otimes}_S P$ has homology ~~[illegible]~~ which is nil as right R-module, and because $\mathcal{U}$ is d-firm.

Let's go back to $f = 1 - [d,h] : \mathcal{U} \to I\mathcal{U}$ and factor this

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "g"] \arrow[dr, "f"'] & I \otimes_R \mathcal{U} \arrow[d, "\mu"] \\
& \mathcal{U}
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "g"] \arrow[dr, equal] & I \otimes_R \mathcal{U} \arrow[r, "1\otimes g"] \arrow[d, "\mu"] \arrow[dr, equal] \arrow[l, phantom, "h"] & I \otimes_R I \otimes_R \mathcal{U} \arrow[r, "1^{(2)}\otimes g"] \arrow[d, "1\otimes\mu"] \arrow[l, phantom, "1\otimes h"] & I^{(3)} \otimes_R \mathcal{U} \arrow[d, "1^{(2)}\otimes\mu"] \arrow[l, phantom, "1^{(2)}\otimes h"] \arrow[r, dash] & {} \\
& \mathcal{U} \arrow[d, equal] & I \otimes_R \mathcal{U} \arrow[d, "\mu"] & I^{(2)} \otimes_R \mathcal{U} \arrow[d, "\mu^{(2)}"] & \\
& \mathcal{U} \arrow[r, equal] & \mathcal{U} \arrow[r, equal] & \mathcal{U} &
\end{tikzcd}$$

where the Δ's homotopy commute with the indicated homotopy. So

$$[d,h] = 1 - \mu g$$

$$[d, \mu(1\otimes h)g] = \mu(1\otimes(1-\mu g))g = \mu g - \underbrace{\mu(1\otimes\mu)}_{\mu^{(2)}}\underbrace{(1\otimes g)g}_{g^{(2)}}$$

$$[d, \mu^{(2)}(1^{(2)}\otimes h)g^{(2)}] = \mu^{(2)}(1^{(2)}\otimes(1-\mu g))g^{(2)}$$

$$= \mu^{(2)}g^{(2)} - \underbrace{\mu^{(2)}(1^{(2)}\otimes\mu)}_{\mu^{(3)}}\underbrace{(1^{(2)}\otimes g)g^{(2)}}_{g^{(3)}}$$

Thus for each n we have successive deformations of the identity map of U:

$$[d, h + \mu(1\otimes h)g + \cdots + \mu^{(n-1)}(1^{(n-1)}\otimes h)g^{(n-1)}]$$
$$= 1 - \mu^{(n)}g^{(n)} \quad \text{where} \quad \begin{tikzcd} U \arrow[r, "g^{(n)}"] & I^{(n)}\otimes_R U \arrow[d, "\mu^{(n)}"] \\ & U \end{tikzcd}$$

Actually a better proof might be as follows. We know because U is d-firm that $\mu : I\overset{L}{\otimes}_R U \longrightarrow U$ is an isomorphism in $D^-(R)$. Iterating we have isomorphism

$$\xrightarrow{1^{(2)}\otimes\mu} I\overset{L}{\otimes}_R I\overset{L}{\otimes}_R U \xrightarrow{1\otimes\mu} I\overset{L}{\otimes}_R U \xrightarrow{\mu} U$$

and hence there is a unique map in $D^-(R)$, $U \xrightarrow{g^{(n)}} [I\overset{L}{\otimes}_R]^{(n)} U$, which composes with $\mu^{(n)} : [I\otimes_R]^{(n)}U \to U$ to give the identity map on U. If U is a right bdd complex of projectives, $g^{(n)}$ is represented a map of complexes unique up to homotopy. Then one can compose with $[I\overset{L}{\otimes}_R]^{(n)}U \longrightarrow [I\otimes_R]^{(n)}U$ to get the above maps.

This kind of argument is superior to the deformation argument, provided one has $\overset{L}{\otimes}_R$ on the bimodule level – this is the basic transversality issue encountered when we try to do things over $\mathbb{Z}$ instead of a ground field.

Let's now take up the problem of showing $P\overset{L}{\otimes}_R U$ is perfect when U is perfect and d-firm. We can assume U strictly perfect (a bdd complex of f g projective R-modules). Let $U^* = \mathrm{Hom}^{\bullet}_R(U, R)$. This is a strictly perfect complex of right R-modules. We have a canonical isomorphism of $\mathbb{Z}$-mod. cxs.

$$(1) \qquad \mathrm{Hom}^{\bullet}_R(U, X) = U^* \otimes_R X$$

for all complexes X over R. Recall the adjunction formula

$$(2) \qquad \mathrm{Hom}^{\bullet}_R(U \otimes_{\mathbb{Z}} N, X) = \mathrm{Hom}^{\bullet}_{\mathbb{Z}}(N, \mathrm{Hom}_R(U, X))$$

on the level of mapping complexes. Passing to Z^0 we obtain the fact that

$$(3) \qquad C(\mathbb{Z}) \underset{\mathrm{Hom}^{\bullet}_R(U,-)}{\overset{U \otimes_{\mathbb{Z}} -}{\rightleftarrows}} C(R)$$

are adjoint. This adjunction fmla (2) holds for arbitrary U.

Now (1) [redacted] combined with this general adjunction fact says that $U^* \otimes_R -$ is [redacted] right adjoint to $U \otimes_{\mathbb{Z}} -$. In other words (1) means there are canonical maps [redacted]

$$(4) \qquad \begin{aligned} \alpha &: U \otimes_{\mathbb{Z}} U^* \longrightarrow R && \text{of } R\text{-bimodules complexes} \\ \beta &: \mathbb{Z} \longrightarrow U^* \otimes_R U && \text{of } \mathbb{Z}\text{-module complexes} \end{aligned}$$

such that

$$U = U \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow U \otimes_{\mathbb{Z}} U^* \otimes_R U \longrightarrow R \otimes_R U = U$$

$$U^* = \mathbb{Z} \otimes_{\mathbb{Z}} U^* \longrightarrow U^* \otimes_R U \otimes_{\mathbb{Z}} U^* \longrightarrow U^* \otimes_R R = U^*$$

are the identity.

Now I hope it's true that perfect complexes can be characterized by the nuclearity condition

$$R\,Hom_R(U, X) = U^* \overset{L}{\otimes}_R X$$

where $U^* = R\,Hom_R(U, R)$. Let's assume this is true and try to use it in the case of the S-module complex $P \overset{L}{\otimes}_R U$.

Put $V = P \overset{L}{\otimes}_R U$, $V^* = U^* \overset{L}{\otimes}_R Q$. Thus $V \in D^-(S)$, $V^* \in D^-(S^{op})$. We have an α-map

$$V \otimes_{\mathbb{Z}} V^* = P \overset{L}{\otimes}_R U \otimes_{\mathbb{Z}} U^* \overset{L}{\otimes}_R Q \longrightarrow P \overset{L}{\otimes}_R R \overset{L}{\otimes}_R Q = P \overset{L}{\otimes}_R Q \rightarrow S$$

when the last arrow is $P \overset{L}{\otimes}_R Q \longrightarrow P \otimes_R Q \rightarrow PQ \subset S$. Also we have

$$\begin{array}{c} V^* \overset{L}{\otimes}_S V = U^* \overset{L}{\otimes}_R Q \overset{L}{\otimes}_S P \overset{L}{\otimes}_R U \\ \big\downarrow \gamma \\ U^* \overset{L}{\otimes}_R R \overset{L}{\otimes}_R U = U^* \overset{L}{\otimes}_R U \end{array}$$

where the vertical map γ is induced by $Q \overset{L}{\otimes}_S P \longrightarrow Q \otimes_S P \longrightarrow QP \subset R$. This composition is a $nil(R^{op}, I^{op})$-quis, so the vertical map γ should be a quis since U is d-firm for (R, I). Thus we should obtain from $\beta : \mathbb{Z} \longrightarrow U^* \overset{L}{\otimes}_R U$ a β-map $\mathbb{Z} \longrightarrow V^* \overset{L}{\otimes}_S V$ for V.

We should notice that in the above

we haven't used that U is strictly perfect. It seems pretty clear that we should get Morita equivalence on the level of perfect d-firm complexes, provided perfect complexes are characterized by the condition that $R\mathrm{Hom}_R(U, -)$ is quis to $U^* \otimes_R -$ for some U^*. This seems to follow from Grothendieck's characterization of perfect complexes as those such that $R\mathrm{Hom}_R(U, -)$ commutes with filtered inductive limits (on the level of complexes).

I would like to check these ideas in the case $\begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix}$.

Here's an interesting point: suppose U strictly perfect + d-firm for (R,I) as above. Consider the functor

$$Y \longmapsto H^0 \mathrm{Hom}_S(P \otimes_R U, Y) \qquad Y \text{ in } D(S).$$

$$\| $$

$$H^0 \mathrm{Hom}_R\big(U, \mathrm{Hom}_S(P,Y)\big) \overset{U \text{ projective}}{=} R^0 \mathrm{Hom}_R\big(U, \mathrm{Hom}_S(P,Y)\big)$$

I think we know that $Y \mapsto \mathrm{Hom}_S(P,Y)$ from $K(S)$ to $K(R)$ is a triangulated functor carrying $K(S)_{nil(S,J)}$ into $K(R)_{nil(R,I)}$. Since U is d-firm for (R,I), $R^0\mathrm{Hom}_R(U,-)$ should kill $K(R)_{nil(R,I)}$. In particular $Y \mapsto H^0 \mathrm{Hom}_S(P \otimes_R U, Y)$ should descend to $K(S)/K(S)_0 = D(S)$. ?

Let's start again. Assume U is a right bounded complex of projective R-modules which is d-firm for (R,I): U/IU contractible. Then for Y in $K(S)$ we have

$$H^0 \mathrm{Hom}_S(P\otimes_R U, Y) = H^0 \mathrm{Hom}_R(U, \mathrm{Hom}_S(P,Y))$$
$$= R^0 \mathrm{Hom}_R(U, \mathrm{Hom}_S(P,Y)) \qquad \text{(bc } U \text{ proj.)}$$

Now $\mathrm{Hom}_S(P,-)$ is a triangulated functor $K(S)\to K(R)$ carrying $K(S)_{nil(S,J)}$ into $K(R)_{nil(R,I)}$, and because U is firm $K(R)_{nil(R,I)}$ is killed by $R^0\mathrm{Hom}(U,-)$. Thus $H^0\mathrm{Hom}_S(P\otimes_R U, -)$ descends from $K(S)$ to $K(S)/K(S)_{nil(S,J)} = D(\mathcal{M}(S,J))$, in particular this functor descends to $D(S)$. This should imply that $P\otimes_R U$ is homotopy equivalent to a right bounded complex of projective R-modules.

December 14, 1994

Let X be a right bounded complex of R-modules. Then we can construct a "free resolution" of X, i.e. a quis $W \to X$ where W is a right-bdd complex of free R-modules $W_n = R^{(\Sigma_n)}$. Consider graded submodules W' of W such that $W'_n = R^{(\sigma'_n)} \subset R^{(\Sigma_n)}$ with σ'_n a finite subset of Σ_n for each n, and such that $\sigma'_n \neq \phi$ for finitely many n. For any such W' there exists another W'' such that $W' \subset W''$ and such that W'' is a subcomplex. Indeed if n is largest such that $\sigma'_n \neq \phi$, then one successively enlarges $\sigma'_{n-1}, \sigma'_{n-2}, \ldots$ to $\sigma''_{n-1}, \sigma''_{n-2}, \ldots$ so that $d\left(R^{(\sigma''_p)}\right) \subset R^{(\sigma''_{p-1})}$; and this process stops because W is right-bounded.

Thus the subcomplexes W^α of W such that $W^\alpha = \bigoplus_n R^{(\sigma^\alpha_n)}$, σ^α_n finite $\subset \Sigma_n$ for all n, and $\sigma^\alpha_n = \phi$ for large n, form a direct set such that $W = \bigcup_\alpha W^\alpha$. This implies that X is up to quasi-isomorphism a filtered inductive limit of finitely generated free complexes, at least when X is right-bdd.

In the general case we consider the increasing Postnikov filtration

$$
\begin{array}{lcccccccccc}
F^{-p}: & \to & X_{p+1} & \to & Z_p & \to & 0 & \to & & & \\
\cap & & \| & & \cap & & \cap & & & & \\
F^{-p+1}: & \to & X_{p+1} & \to & X_p & \to & Z_{p-1} & \to & 0 & \to & \\
F^{-p+1}/F^{-p}: & \to & 0 & \to & X_p/Z_p & \to & Z_{p-1} & \to & 0 & \to &
\end{array}
$$

Then we can construct a compatible family of free resolutions $W^p \to F^p$ such that W^p is obtained by attaching a free resolution of $H^p(X)$ to W^{p-1}.

$$\begin{tikzcd} 0 \arrow[r] & W^{p-1} \arrow[r] \arrow[d, "\sim"] & W^p \arrow[r] \arrow[d, "\sim"] & W^p/W^{p-1} \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & F^{p-1} \arrow[r] & F^p \arrow[r] & F^p/F^{p-1} \arrow[r] & 0 \end{tikzcd}$$

$$F^p/F^{p-1} \simeq H^p(X)[-p]$$

The top row is an exact sequence of free modules with given basis elements in each degree. Now consider subcomplexes W' of $W = \bigcup W^p$ which are free subcomplexes with finitely many generators. Thus we are considering the union of the directed sets of such subcomplexes of W^p for all p. This gives W as the union of finitely generated free complexes forming a directed set.

Let's now consider complexes X, Y of R-modules, right bdd to fix the ideas, and a class

$$\omega \in H_0\left(R\mathrm{Hom}_R(X,R) \overset{L}{\otimes}_R Y\right)$$

suppose X is projective in each degree, so that $R\mathrm{Hom}_R(X,R) = \mathrm{Hom}_R(X,R)$. Let $W \to \mathrm{Hom}_R(X,R)$ be a free R^{op} resolution as above, let $\{W^\alpha\}$ be the directed set of f.g. free subcomplexes. Then

$$\omega \in H_0\left(\left(\bigcup W^\alpha\right) \overset{L}{\otimes}_R Y\right) = H_0\left(\left(\bigcup W^\alpha\right) \otimes_R Y\right) = \varinjlim_\alpha H_0(W^\alpha \otimes_R Y)$$

so we know that ω comes from a class

$$\omega' \in H_0(W' \otimes_R Y)$$

where W' is a f.g. free right module complex mapping to $\mathrm{Hom}_R(X, R)$.

Put $V = \mathrm{Hom}_{R^{op}}(W', R)$. Then

$$W' = \mathrm{Hom}_R(V, R) \quad \text{so} \quad W' \otimes_R Y = \mathrm{Hom}_R(V, Y)$$

and ω' is a homotopy class of maps $V \to Y$. On the other hand the map $W' \longrightarrow \mathrm{Hom}_R(X, R)$ is equivalent to a R-bimodule map $X \otimes_{\mathbb{Z}} W' \longrightarrow R$, which in turn is equivalent to an R-module map $X \to \mathrm{Hom}_{R^{op}}(W', R) = V$. Thus we have maps

$$X \longrightarrow V \longrightarrow Y$$

which should be a factorization of the image of ω under the canonical map

$$R\,\mathrm{Hom}_R(X, R) \overset{L}{\otimes}_R Y \longrightarrow R\,\mathrm{Hom}_R(X, Y).$$

We have not used anything about Y, but X has been assumed right bdd in order to handle $R\,\mathrm{Hom}_R(X, -)$. Perhaps this assumption is unnecessary, I mean, there might be a good $R\,\mathrm{Hom}_R(X, -)$ defined by the sort of free resolution $W \to X$ we have constructed.

December 15, 1994

Here's something I missed. Let U be a right bdd complex of projective R-modules which is d-firm, i.e. $R/I \otimes_R U = U/IU$ is acyclic. Then U/IU, being a right bdd complex of projective R/I modules is contractible. Let $k \in \mathrm{Hom}^{-1}_{R/I}(U/IU, U/IU)$ be a contraction. Then we can lift k to $h \in \mathrm{Hom}^{-1}_R(U,U)$. We have

$$\begin{tikzcd}
0 \arrow[r] & IU \arrow[r, "i"] \arrow[d, "h'"] & U \arrow[r] \arrow[d, "h"] & U/IU \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & IU \arrow[r, "i"] & U \arrow[r] & U/IU \arrow[r] & 0
\end{tikzcd}$$

and clearly h induces a homotopy operator h' on IU. Note that $1-[d,h]$ is zero on U/IU, so it induces $f: U \to IU$. Then $f: U \to IU$ is a homotopy inverse for the inclusion $i: IU \hookrightarrow U$. In effect, $[d,h] = 1 - if$ and $i[d,h'] = [d,hi] = (1-if)i = i(1-fi)$ $\Longrightarrow$ $[d,h'] = 1 - fi$ by the injectivity of i.

Thus it follows that all the canonical maps

$$\cdots \longrightarrow I \otimes_R I \otimes_R U \longrightarrow I \otimes_R U \longrightarrow U$$

are homotopy equivalences.

December 16, 1994

Let U be a complex of R-modules such that $\mu : I \otimes_R U \to U$ is a homotopy equivalence, that is, it becomes an isomorphism in $K(R)$. Let $N' \to N$ be an I-nil isomorphism of right modules. This we know means the dotted arrow exists in

$$\text{(1)} \qquad \begin{tikzcd} N' \otimes_R I^{(n)} \arrow[r] \arrow[d] & N \otimes_R I^{(n)} \arrow[d] \arrow[dl, dashed] \\ N' \arrow[r] & N \end{tikzcd}$$

for some n. Tensoring with U we obtain a commutative diagram

$$\text{(2)} \qquad \begin{tikzcd} N' \otimes_R I^{(n)} \otimes_R U \arrow[r] \arrow[d] & N \otimes_R I^{(n)} \otimes_R U \arrow[d] \arrow[dl] \\ N' \otimes_R U \arrow[r] & N \otimes_R U \end{tikzcd}$$

where the vertical maps are homotopy equivalences, hence $N' \otimes_R U \to N \otimes_R U$ is a homotopy equivalence. (More precisely the vertical arrows in (2) become isos. in $K(\mathbb{Z})$, hence so does $N' \otimes_R U \to N \otimes_R U$. Notice that if $N' \to N$ is a map of $S \otimes_{\mathbb{Z}} R^{op}$ bimodules which is also a right I-nil isomorphism, then the arrows in (2) and the homotopy inverses are compatible with the action of S.)

Before going further I should review what I know about homotopy equivalences. Let $f: X \to Y$ be a map of complexes. Its cone $C = \mathrm{Cone}(f)$ is

$$C_n = X_{n-1} \oplus Y_n \qquad d_C = \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}$$

By the Δated structure on $K(\mathcal{a})$ one knows that f is a homotopy equivalence iff C is contractible. A contraction h_C for C has the form $h_C = \begin{pmatrix} -h_X & g \\ u & h_Y \end{pmatrix}$ where

$$\left[\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}, \begin{pmatrix} -h & g \\ u & h \end{pmatrix} \right] = \begin{pmatrix} dh + hd + gf & -dg + gd \\ -fh + du \\ -ud + hf & fg + dh + hd \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $g: Y \to X$ is a map of complexes, and h_X, h_Y are homotopy operators such that $[d, h_X] = 1 - gf$, $[d, h_Y] = 1 - fg$, and u is such that $[d,u] = [f,h]$, i.e. it relates the homotopies under f.

Now tensor product $X \mapsto N \otimes_R X$ is a triangulated functor from $K(R)$ to $K(S)$; here N is an $S \otimes R^{op}$-bimodule complex. [redacted] This means essentially that it commutes with forming cones which is pretty clear. Thus a contraction for $\mathrm{Cone}(X' \to X)$ yields a contraction on $\mathrm{Cone}(N \otimes_R X' \to N \otimes_R X)$.

This is what I was using above, e.g.

$I \otimes_R U \to U$ a heq $\Rightarrow$ $I \otimes_R (I \otimes_R U) \to I \otimes_R U$ a heq, etc.

Let's return to the Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$. Suppose U is a complex of R-modules such that $I \otimes_R U \to U$ is a homotopy equivalence, i.e. isom in $K(R)$. (These are the "K-firm" complexes.)

Equivalently $\mathrm{Cone}(I \to R) \otimes_R U$ is contractible.

I want to show that $P \otimes_R U$ is K-firm for (S,J).

i.e. $\mathrm{Cone}(J \to S) \otimes_S P \otimes_R U = \mathrm{Cone}(J \otimes_S P \to S \otimes_S P) \otimes_R U$ is contractible. Actually this already covered the argument on p144. We know that $J \otimes_S P \to S \otimes_S P = P$ is a right nil(R,I)-isomorphism, i.e. we have

$$\begin{tikzcd} J \otimes_S P \otimes_R I^{(n)} \arrow[r] \arrow[d] & P \otimes_R I^{(n)} \arrow[d] \arrow[dl, dashed] \\ J \otimes_S P \arrow[r] & P \end{tikzcd}$$

so tensoring with U on the right yields a similar diagram in $K(S)$ with vertical arrows being isomorphisms.

Another argument. Let $C = \mathrm{Cone}(J \to S) \otimes_S P = \mathrm{Cone}(J \otimes_S P \to P)$. Suppose we can show that for some n the multiplication map $C \otimes_R I^{(n)} \to C$ is homotopic to zero. Then the map

$$C \otimes_R I^{(n)} \otimes_R U \longrightarrow C \otimes_R U$$

$\alpha = (a_1, \dots, a_n)$
$\bar{\alpha} = a_1 \cdots a_n$

$$c \otimes \alpha \otimes u \longmapsto c\bar{\alpha} \otimes u = c \otimes \bar{\alpha} u$$

is both homotopic to zero and a homotopy equivalence, hence $C \otimes_R U$ is contractible.

Recall what we know about C: From

$$\begin{tikzcd} J\otimes_S P \arrow[r, "\cdot q"] \arrow[d] & J \arrow[r, "\cdot a"] \arrow[d] & J \arrow[r, "\cdot p"] \arrow[d] & J\otimes_S P \arrow[d] \\ P \arrow[r, "\cdot q"] & S \arrow[r, "\cdot a"] & S \arrow[r, "\cdot p"] & P \end{tikzcd}$$

we get

$$\mathrm{Cone}\begin{pmatrix} J\otimes_S P \\ \downarrow \\ P \end{pmatrix} \xrightarrow{\cdot q} \mathrm{Cone}\begin{pmatrix} J \\ \downarrow \\ S \end{pmatrix} \xrightarrow{\cdot a} \mathrm{Cone}\begin{pmatrix} J \\ \downarrow \\ S \end{pmatrix} \xrightarrow{\cdot p} \mathrm{Cone}\begin{pmatrix} J\otimes_S P \\ \downarrow \\ P \end{pmatrix}$$

or better, maps

$$\mathrm{Cone}\begin{pmatrix} J\otimes_S P \\ \downarrow \\ P \end{pmatrix}\otimes_R Q\otimes_S J\otimes_S P \longrightarrow \mathrm{Cone}\begin{pmatrix} J \\ \downarrow \\ S \end{pmatrix}\otimes_S J\otimes_S P \to \mathrm{Cone}\begin{pmatrix} J \\ \downarrow \\ S \end{pmatrix}\otimes_S P \to \mathrm{Cone}\begin{pmatrix} J\otimes_S P \\ \downarrow \\ P \end{pmatrix}$$

The point is that the map

$$\mathrm{Cone}\begin{pmatrix} J \\ \downarrow \\ S \end{pmatrix}\otimes_S J \longrightarrow \mathrm{Cone}\begin{pmatrix} J \\ \downarrow \\ S \end{pmatrix}$$

is canonically null-homotopic:

$$\left[\begin{pmatrix} -d & \\ f & d \end{pmatrix}, \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} gf & -dg+gd \\ & fg \end{pmatrix}$$

Here $f: J \to S$ is the inclusion, $g: S \to J$ is $\cdot a$.

Thus we find that $C\otimes_R (Q\otimes_S J\otimes_S P) \longrightarrow C$ is null-homotopic. On the other hand one should have a $\mathrm{nil}(R^{op}, I^{op})$-isom $Q\otimes_S J\otimes_R P \longrightarrow R$, so that $(Q\otimes_S J\otimes_R P)\cancel{\otimes_R U} \longrightarrow U$ is a h.eq.

At this point I've checked that U K-firm for (R, I) $\Longrightarrow$ <u>$P\otimes_R U$ is K-firm</u> for (S, J).

Let's review what we have learned. Suppose $U \in K(R)$ is K-firm for I, i.e. $I \otimes_R U \xrightarrow{\mu} U$ is an isomorphism in $K(R)$. Then we know for any $S \otimes_{\mathbb{Z}} R^{op}$-module map $N' \to N$ which is a right I-nil isom. that $N' \otimes_R U \to N \otimes_R U$ is an isom. in $K(S)$.

What actually seem to happen is that because $N' \to N$ is a right I-nil isom one has

$$\begin{tikzcd} N' \otimes_R I^{(n)} \arrow[r] \arrow[d] & N \otimes_R I^{(n)} \arrow[d] \arrow[dl, dashed] \\ N' \arrow[r] & N \end{tikzcd}$$

which implies that

$$\mathrm{Cone}\begin{pmatrix} N' \\ \downarrow \\ N \end{pmatrix} \otimes_R I^{(n)} \longrightarrow \mathrm{Cone}\begin{pmatrix} N' \\ \downarrow \\ N \end{pmatrix}$$

is null-homotopic. Then after applying $- \otimes_R U$ we obtain a null-homotopic homotopy equivalence, whence $\mathrm{Cone}\begin{pmatrix} N' \\ \downarrow \\ N \end{pmatrix} \otimes_R U = 0$, hence $N' \otimes_R U \to N \otimes_R U$ is a h.eq.

This raises the question as to what complexes W, like $\mathrm{Cone}\begin{pmatrix} N' \\ \downarrow \\ N \end{pmatrix}$ have the property that $W \otimes_R I^{(n)} \to W$ is null homotopic for some n. The obvious conjecture is that this is true if W is bounded and $j^* W$ is contractible. Here $j^* : \mathrm{mod}(R^{op}) \to \mathcal{M}(R^{op}, I^{op})$ is extended to complexes in the obvious way.

This conjecture is obviously true, for a contraction for j^*W lifts to a homotopy operator $h : W \otimes_R I^{(n)} \longrightarrow W$ such that $[d,h] = \mu^n$, whence μ^n is null-homotopy. Here we use the fact that W is bounded to get an n working in all degrees. We could also work modulo the Serre subcategory $\bigcup_n C(\mathrm{mod}(R/I^n)) \subset C(\mathrm{mod}(R))$ of uniformly nil complexes and drop the bdd-ness condition.

Given U in K-firm(R,I), we have that $P \otimes_R U$ is in K-firm(S,J). In effect we know $(J \longrightarrow S) \otimes_S P = (J \otimes_S P \longrightarrow P)$ is an I^{op}-null isomorphism, thus $J \otimes_S P \otimes_R U \longrightarrow P \otimes_R U$ is a homotopy equivalence.

Finally $Q \otimes_S P \longrightarrow R$ is an I^{op}-null isom., so $Q \otimes_S P \otimes_R U \longrightarrow U$ is a homotopy equivalence.

Thus Morita equivalence holds for K-firm:

$$\text{K-firm}(R,I) \simeq \text{K-firm}(S,J)$$
$$U \longmapsto P \otimes_R U$$
$$Q \otimes_S V \longleftarrow V$$

Next we define U to be K-solid when

$$U = \mathrm{Hom}_R(R,U) \xrightarrow{\mu'} \mathrm{Hom}_R(I,U)$$

is a homotopy equivalence. If C is a complex such that $I^{(n)} \otimes_R C \longrightarrow C$ is null-homotopic for some n, then

$$\begin{tikzcd}[column sep=small] & \mathrm{Hom}^{\cdot}_R(C,U) \arrow[dl, "\sim 0"'] \arrow[dr, "\text{heq}"] & \\ \mathrm{Hom}^{\cdot}_R(I^{(n)}\otimes_R C, U) & \simeq & \mathrm{Hom}^{\cdot}_R(C, \mathrm{Hom}_R(I^{(n)},U)) \end{tikzcd}$$

so $\mathrm{Hom}^{\cdot}_R(C,U)$ is contractible for such C, U.

It's clear we should have Morita equivalence for K-solid categories

$$\text{K-sol}(R,I) \simeq \text{K-sol}(S,J)$$

$$U \longmapsto \mathrm{Hom}_R(Q,U)$$

$$\mathrm{Hom}_S(P,V) \longleftarrow\!\shortmid V$$

This structure is interesting and needs to be clarified. Observe that in $K(R)$ we have

K-nil : $\quad I^{(n)}\otimes_R C \longrightarrow C$ is 0 for some n.

K-firm : $\quad I \otimes_R U \xrightarrow{\sim} U$

K-sol : $\quad U \xrightarrow{\sim} \mathrm{Hom}_R(I, U)$

December 19, 1994

Consider $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ and concentrate upon right bounded complexes. Notions of firmness:

firm: $I \otimes_R X \xrightarrow{\sim} X$ in $C(R)$

K-firm: ——— in $K(R)$

D-firm: $I \overset{L}{\otimes}_R X \xrightarrow{\sim} X$ in $D(R)$

We have seen that for X right-bdd projective, we have D-firm $\Rightarrow$ K-firm.

We also have Morita invariance for these three firm-type categories, which is proved in a formal way using the fact that an (S, R^{op})-bimodules P determined a functor $P \otimes_R -$ on the corresponding module categories.

For the past few days I have been trying to understand, or better to check, the proof that if U is projective D-firm ~~for~~ (R, I), then $P \otimes_R U$ is homotopy equivalent to a V which is projective D-firm for (S, J): Choose $V \xrightarrow[\text{qis}]{} P \otimes_R U$ with V projective. We have

$$H^0 \mathrm{Hom}_S(P \otimes_R U, Y) = \underbrace{H^0 \mathrm{Hom}_R}_{\substack{\text{inverts} \\ I\text{-nil qis}}}(U, \underbrace{\mathrm{Hom}_S}_{\substack{\text{carries } J\text{-nil qis} \\ \text{into } I\text{-nil qis}}}(P, Y))$$

Thus $[P \otimes_R U, -]$ inverts qis in particular, and we have

$$\begin{array}{ccc} [P\otimes_R U, V] & \xrightarrow{\sim} & [P\otimes_R U, P\otimes_R U] \\ \downarrow & & \downarrow \\ [V, V] & \xrightarrow{\sim} & [V, P\otimes_R U] \end{array}$$

where horizontal arrows are $u \mapsto fu$ with $f: V \to P\otimes_R U$ the given map, and the vertical arrows are $v \mapsto vf$. Thus we have a unique $g: P\otimes_R U \to V$ such that $fg = 1_{P\otimes_R U}$, and $fgf = f \cdot 1_V \Rightarrow gf = 1_V$, so f is an isom. in $K(S)$.

Consider the example where $U: \to 0 \to U_1 \xrightarrow{d} U_0 \to 0 \to$ has length one. We have the quis

$$\begin{array}{ccccc} \to V_2 & \to & V_1 & \to & V_0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & P\otimes_R U_1 & \to & P\otimes_R U_0 \end{array}$$

hence the square

$$\begin{array}{ccc} V_1/dV_2 & \to & V_0 \\ \downarrow & & \downarrow \\ P\otimes_R U_1 & \to & P\otimes_R U_0 \end{array}$$

is bicartesian. Thus

$$\begin{array}{ccc} \mathrm{Hom}_S(P, V_1/dV_2) & \longrightarrow & \mathrm{Hom}_S(P, V_0) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_S(P, P\otimes_1 U_1) & \longrightarrow & \mathrm{Hom}_S(P, P\otimes_R U_0) \end{array}$$

is cartesian.

It would be better to say that the sequence

$$0 \to V_1/dV_2 \to P\otimes_R U_1 \oplus V_0 \to P\otimes_R U_0 \to 0$$

is exact, hence

$$0 \to \mathrm{Hom}_S(P, V_1/dV_2) \to \mathrm{Hom}_S(P, P\otimes_R U_1) \oplus \mathrm{Hom}_S(P, V_0) \to \mathrm{Hom}_S(P, P\otimes_R V_0)$$

is exact and the cokernel of the last map is killed by QP:

$$f \longmapsto f(p) \longmapsto (p' \longmapsto (p'q)f(p)) = f(p'(qp)) = (qp)f.$$

$$\begin{tikzcd}
\mathrm{Hom}_S(P, N') \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & \mathrm{Hom}_S(P, N') \arrow[d] \\
\mathrm{Hom}_S(P, N) \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & \mathrm{Hom}_S(P, N) \arrow[d] \\
\mathrm{Coker} \arrow[d] & 0 & \mathrm{Coker} \arrow[d] \\
 & & 0
\end{tikzcd}$$

So we have an obvious map

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(P, V_1/dV_2) \arrow[r] & \mathrm{Hom}_S(P, P\otimes_R U_1) \oplus \mathrm{Hom}_S(P, V_0) \arrow[r, "\pi"] & \mathrm{Hom}_S(P, P\otimes_R U_0) \\
 & & U_1 \arrow[u] \arrow[r] & U_0 \arrow[u]
\end{tikzcd}$$

because the cokernel of π is killed by I and $U_1/IU_1 \xrightarrow{\sim} U_0/IU_0$ so $(U_0/dU_1) = I(U_0/dU_1)$, it follows that the image of U_0 lies in the image of π. ~~cutting~~ $\mathrm{Hom}_S(P, P\otimes_S U_0)$ down to $\mathrm{Im}(\pi)$ makes the top row above acyclic. Then as U_1, U_0 are projective the above map is null-homotopic.

This should translate into a lifting

$$\begin{tikzcd}
 & \mathrm{Hom}_S(P, V_1/dV_2) \arrow[r] \arrow[d] & \mathrm{Hom}_S(P, V_0) \arrow[d] \\
 & \mathrm{Hom}_S(P, P\otimes_R U_1) \arrow[r] & \mathrm{Hom}_S(P, P\otimes_R U_0) \\
U_1 \arrow[r] \arrow[uur, dashed] \arrow[ur] & U_0 \arrow[uur, dashed] \arrow[ur] &
\end{tikzcd}$$

Unfortunately, this gets too hard to follow. Instead I should use the cone on $\mathrm{Hom}_S(P,V) \longrightarrow \mathrm{Hom}_S(P, P\otimes_R U)$; call this X and consider the decreasing Postnikov filtration

$$\begin{array}{lccccccccc}
X & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & 0 \\
\cup & & \| & & \cup & & & & \\
F_{\geq 1}X & \longrightarrow & X_2 & \longrightarrow & Z_1 & \longrightarrow & 0 & & \\
\cup & & \cup & & & & & & \\
F_{\geq 2}X & \longrightarrow & Z_2 & \longrightarrow & 0 & & & &
\end{array}$$

We start with $U \xrightarrow{g} X$. Since $H_0(U) = I H_0(U)$ and $H_0(X)$ is killed by I^n we see $g(U_0) \subset dX_1$, so by lifting $g: U_0 \longrightarrow dX_1$ to $h: U_0 \longrightarrow X_1$, we deform g to a map such that $g(U_0) = 0$. Then g maps U to $F_{\geq 1}X$. Consider the composition $U \longrightarrow F_{\geq 1}X \longrightarrow H_1(X)[1]$. This factors through U/I^nU which is contractible, hence the map g can be deformed until $U \to F_{\geq 1}X \to H_1(X)[1]$ is zero, whence $g: U_1 \longrightarrow dX_1$, and we can lift this to $U_1 \to X_2$ and so achieve a deformation of g to zero.

Here's another way to see that U projective D-firm $\Rightarrow$ $P \otimes_R U$ h.eq. to a projective D-firm complex. Again choose $V \xrightarrow{\text{quis}} P \otimes_R U$ with V projective. We know by Morita equivalence of the D-firm categories that $P \overset{L}{\otimes}_R U = P \otimes_R U$ is D-firm, so V is D-firm and projective. Now

$$\begin{tikzcd} Q \overset{L}{\otimes}_S V \arrow[r, "\text{quis}"] \arrow[d, equal] & Q \overset{L}{\otimes}_S P \otimes_R U \arrow[d, "\text{quis}"] \\ Q \otimes_S V \arrow[r] & U \end{tikzcd}$$

because $Q \overset{L}{\otimes}_S P \to R$ has cone with right nil homology

so we get that $Q \otimes_S V \longrightarrow U$ is a quis, hence there's a map $U \longrightarrow Q \otimes_S V$ such that the appropriate composition is homotopic to the identity of U. So we have

$$U \xrightarrow{\text{quis}} Q \otimes_S V \xrightarrow{\text{quis}} U$$

$$\begin{tikzcd} P \overset{L}{\otimes}_R U \arrow[r] \arrow[d, equal] & P \overset{L}{\otimes}_R Q \otimes_S V \arrow[r] \arrow[d, "\text{quis}"] & P \overset{L}{\otimes}_R U \arrow[d, equal] \\ P \otimes_R U \arrow[r, "\text{quis}"] \arrow[rr, bend right, "\text{homotopic to } 1"'] & V \arrow[r, "\text{quis}"] & P \otimes_R U \end{tikzcd}$$

then V being projective implies the other composition $V \longrightarrow P \otimes_R U \longrightarrow V$ is an isomorphism in the homotopy category, whence we have a homotopy equivalence.

December 20, 1994

Yesterday, I wrote another argument that U projective D-firm $\Rightarrow P\otimes_R U$ is homotopy equivalent to a projective D-firm complex. This argument was based on the result that $Q\overset{L}{\otimes}_S P\otimes_R U \longrightarrow U$ is a quis.

Here's another place this result is (or might be) used.

Suppose U is f.g projective & D-firm, and let $U^* = \mathrm{Hom}_R(U,R)$. We wish to show $P\otimes_R U$ is a perfect complex over S. The idea is to exhibit the appropriate adjunction maps:

$$\alpha : \quad (P\otimes_R U)\otimes_{\mathbb{Z}}(U^*\otimes_R Q) \longrightarrow S$$

$$\beta : \quad \mathbb{Z} \longrightarrow (U^*\otimes_R Q)\overset{L}{\otimes}_S (P\otimes_R U)$$

and then hopefully this is sufficient. α is obvious. For β we use

$$\begin{array}{ccc} & & U^*\otimes_R Q\overset{L}{\otimes}_S P\otimes_R U \\ & & \| \\ & & U^*\otimes_R (Q\overset{L}{\otimes}_S P)\otimes_R U \\ & & \Big\downarrow \text{ quis by the cited result} \\ \mathbb{Z} & \longrightarrow & U^*\otimes_R U \end{array}$$

At this point it seems worthwhile to go over Grothendieck's theory of perfect complexes in order to gain insight about Morita equivalence for perfect D-firm complexes.

Statements: 1) Given $\omega \in H_0(X \overset{L}{\otimes}_R Y)$, $\exists$ a f.g. free complex U and maps $f : U^* \to X$, $g : U \to Y$ such that $\omega = (f \otimes g)(1_U)$, where $1_U \in U^* \otimes_R U = \mathrm{Hom}_R(U, U)$ is the identity elt.

2) Given $\omega \in H_0(\mathrm{Hom}_R(X,R) \overset{L}{\otimes}_R Y)$, $\exists$ a f.g. free complex U and maps $f' : X \to U$, $g : U \to Y$ such that $gf' = \mathrm{Im}(\omega)$ in $H^0(\mathrm{Hom}_R(X,Y))$.

Proof of 2) from 1).

$$\mathrm{Hom}_{R^{op}}(U^*, \mathrm{Hom}_R(X,R)) = \mathrm{Hom}_{R \otimes R^{op}}(X \otimes U^*, R)$$

$$= \mathrm{Hom}_R(X, \underbrace{\mathrm{Hom}_{R^{op}}(U^*, R)}_{U})$$

so $f : U^* \to \mathrm{Hom}_R(X, R)$ is equivalent to a map $f' : X \to U$. Then

$$\begin{tikzcd} U^* \otimes_R U \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(X,R) \overset{L}{\otimes}_R Y \arrow[d] \\ \mathrm{Hom}_R(U,U) \arrow[r] & \mathrm{Hom}_R(X,Y) \\ \mathrm{id}_U \arrow[r, mapsto] & gf' \end{tikzcd}$$

December 21, 1994

From Lang's Algebra p451 there is a theorem of Morita as follows: Let P be a generator of mod(S), $R^{op} = \mathrm{End}_S(P)$. Then $\mathrm{End}_{R^{op}}(P) = S$ (i.e. P is a balanced S-module) and P is a finitely generated R^{op}-module.

In fact it turns out that P is a f.g. projective R^{op}-module.

Why? By the Gabriel-Popescu theorem we have adjoint functors

$$\mathrm{mod}(R) \underset{\mathrm{Hom}_S(P,-)}{\overset{P\otimes_R -}{\rightleftarrows}} \mathrm{mod}(S)$$

where $P\otimes_R -$ is exact and identifies mod(S) with a quotient Grothendieck category of mod(R). By Roos' theorem (applies as mod(S) satisfies $AB4^*$ and $AB6$) the kernel of $P\otimes_R -$ is closed under Π's, hence corresponds to an idempotent ideal I in R. Then one has left-adjoint $Q\otimes_S -$ for $P\otimes_R -$. One has a Morita equivalence $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ of $M(R,I)$ and mod(S). From the pairing $Q\otimes_S P \xrightarrow{\alpha} R$ and fact that $PQ = S$, we conclude that P is f.g. projective over R^{op} with dual Q, and that $S = P\otimes_R Q = Q\otimes_{R^{op}} P = \mathrm{End}_{R^{op}}(P)$.

Actually it seems that the Faith proof in Lang's book also shows P is proj over R^{op}.

Here's a refinement, better: improvement, in the previous work on showing $P \otimes_R U$ is Perfect and D-firm when U is.
We have maps assuming U strictly perfect

$$\mathrm{Hom}_S(P \otimes_R U, Y) = \mathrm{Hom}_R(U, \mathrm{Hom}_S(P, Y))$$

$$\begin{array}{ll} (1) & = U^* \otimes_R \mathrm{Hom}_S(P, Y) \\ & \quad \uparrow 1 \otimes v \\ & \quad U^* \otimes_R Q \otimes_S Y \end{array}$$

Now v is a nil isomorphism in a specific canonical way:

(2)
$$\begin{tikzcd} Q \otimes_S P \otimes_R Q \otimes_S Y \arrow[r] \arrow[d] & Q \otimes_S P \otimes_R \mathrm{Hom}_S(P, Y) \arrow[d] \arrow[dl, dashed] \\ Q \otimes_S Y \arrow[r] & \mathrm{Hom}_S(P, Y) \end{tikzcd}$$

As U^* is strictly perfect and D-firm we know that $U^* \otimes_R (Q \otimes_S P) \longrightarrow U^*$ is a homotopy equivalence. Thus on tensoring the above diagram with U^*, the vertical arrows become homotopy equivalences, and we concludes

(3) $$U^* \otimes_R Q \otimes_S Y \xrightarrow{1 \otimes v} U^* \otimes_R \mathrm{Hom}_S(P, Y)$$

is a homotopy equivalence. Thus

(4) $$H^0(\mathrm{Hom}_S(P \otimes_R U, Y)) = H_0((U^* \otimes_R Q) \otimes_S Y)$$

Actually I see this argument can be modified to work in the case where U is projective & 0-firm. We consider the adjoint diagram to (2):

(5)
$$\begin{tikzcd}
Q\otimes_S Y \arrow[r] \arrow[d] & \mathrm{Hom}_S(P,Y) \arrow[d] \arrow[dl, dashed] \\
\mathrm{Hom}_R(Q\otimes_S P, Q\otimes_S Y) \arrow[r] & \mathrm{Hom}_R(Q\otimes_S P, \mathrm{Hom}_S(P,Y))
\end{tikzcd}$$

Apply $\mathrm{Hom}_R(U,-)$ and use that $Q\otimes_S P\otimes_R U \to U$ is a homotopy equivalence; this gives from (5) a diagram where the vertical arrows are heq's, hence

(6)
$$\begin{array}{ccc}
\mathrm{Hom}_R(U, Q\otimes_S Y) & \longrightarrow & \mathrm{Hom}_R(U, \mathrm{Hom}_S(P,Y)) \\
 & & \parallel \\
 & & \mathrm{Hom}_R(P\otimes_R U, Y)
\end{array}$$

is a homotopy equivalence and

(7)
$$\begin{array}{ccc}
H^0(\mathrm{Hom}_S(P\otimes_R U, Y)) & = & H^0(\mathrm{Hom}_R(U, \mathrm{Hom}_S(P,Y))) \\
 & & \parallel \\
 & & H^0(\mathrm{Hom}_R(U, Q\otimes_S Y))
\end{array}$$

So far ~~the~~ constructions take place on the level of complexes and complexes up to homotopy. Next I want to move to the derived category. The point is that either (4) or (7) show that $Y \mapsto H^0(\mathrm{Hom}_S(P\otimes_R U, Y))$ kills complexes with nil homology, in particular acyclic complexes. I think of this as a nonconstructive step, in that

I don't see how to obtain this result from a <u>deformation</u> of U.

Return to U strictly perfect and D-fin.

$$H^0(\mathrm{Hom}_S(P\otimes_R U, Y)) = H_0(U^*\otimes_R Q\otimes_S Y)$$

Since this kills acyclic complexes we have that

$$R^0\,\mathrm{Hom}_S(P\otimes_R U, Y) = H_0(U^*\otimes_R Q\otimes_S Y).$$

This implies ↑ commutes with filtered $\varinjlim$'s, and so by Grothendieck $P\otimes_R U$ is perfect.

Specifically we choose $V \xrightarrow[\text{qis}]{} P\otimes_R U$ with V free and consider the directed set of free f.g. subcomplexes V^α of V.

$$\begin{tikzcd} R^0\,\mathrm{Hom}_S(P\otimes_R U, V) \arrow[d, "\cong"'] & \varinjlim R^q\,\mathrm{Hom}_S(P\otimes_R U, V^\alpha) \arrow[l, "\sim"'] \\ R^0\,\mathrm{Hom}_S(P\otimes_R U, P\otimes_R U) & \end{tikzcd}$$

i.e.
$$\begin{tikzcd} V^\alpha \arrow[d] \\ P\otimes_R U \arrow[u, bend left, dashed] \end{tikzcd}$$
$\exists$ a section up to homotopy.

We have just shown that $P\otimes_R U$ is perfect using Grothendieck's criterion and the fact that $H_0(U^*\otimes_R Q\otimes_S Y)$ kills acyclic complexes Y, which is one of these "nonconstructive" results we seem unable to avoid.

Here's another version: The isom.

$$H^0\left(\mathrm{Hom}_S(P\otimes_R U, Y)\right) = H_0\left(U^*\otimes_R Q\otimes_S Y\right)$$

really the map $\to$ is given by a canonical element in $H_0(U^*\otimes_R Q\otimes_S P\otimes_R U)$ which in principle I can write down starting from a deformation of U into IU. But I really want an element in $H_0(U^*\otimes_R Q\overset{L}{\otimes}_S P\otimes_R U)$, because then it lifts to an element of

$$\begin{array}{ccc} H_0(U^*\otimes_R Q\otimes_S V^\alpha) & \longrightarrow & H_0(U^*\otimes_R Q\overset{L}{\otimes}_S P\otimes_R U) \\ \| & & \| \\ H^0\left(\mathrm{Hom}_S(P\otimes_R U, V^\alpha)\right) & \longrightarrow & H^0\left(\mathrm{Hom}_S(P\otimes_S U, P\otimes_S U)\right) \end{array}$$

for some fg free V^α mapping to $P\otimes_S U$.

Thus the nonconstructive part involves the fact that $Q\overset{L}{\otimes}_S P\otimes_R U \longrightarrow U$ is a qis.

Notice that given $\omega\in H_0\left(\mathrm{Hom}_R(X,R)\overset{L}{\otimes}_R Y\right)$, we can choose $V \xrightarrow{\text{qis}} Y$ with V free, then

$$\begin{array}{ccc} \omega' & \longmapsto & \omega \\ H_0(\mathrm{Hom}_R(X,R)\otimes_R V^\alpha) & \longrightarrow & H_0(\mathrm{Hom}_R(X,R)\overset{L}{\otimes}_R Y) \\ \downarrow & & \downarrow \\ H_0(\mathrm{Hom}_R(X,V^\alpha)) & \longrightarrow & H_0(\mathrm{Hom}_R(X,Y)) \end{array}$$

so the map $X\to Y$ assoc. to ω can be factored $X\longrightarrow V^\alpha\longrightarrow Y$ with V^α f.g. free.

December 28, 1994

Suppose given idempotent rings A, B and an equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$. We know such an equivalence is given by a Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where P, Q are firm on both sides and $P\otimes_A Q \xleftarrow{\sim} P\otimes_A Q\otimes_B P\otimes_A Q \xrightarrow{\sim} B\otimes_B B$ and similarly $Q\otimes_B P \simeq A\otimes_A A$. **Moreover** this Morita context is unique up to canonical isomorphism. Then one has a canonical isomorphism

$$A\otimes_A A\otimes_A \simeq Q\otimes_B P\otimes_A \simeq P\otimes_A Q\otimes_B \simeq B\otimes_B B\otimes_B .$$

This shows that we have a trace group associated to a Grothendieck-Roos category $\mathcal{M}$, which is canonically isom to $A\otimes_A A\otimes_A$ whenever one is given an equivalence $\mathcal{M}(A) \simeq \mathcal{M}$.

I now want to construct an example of an idempotent ring A such that $A\otimes_A A\otimes_A \longrightarrow A\otimes_A$ is not an isomorphism. The example will be Morita equivalent to a unital algebra, the Morita context being $\begin{pmatrix} A=AeA & Ae \\ eA & eAe=B \end{pmatrix}$. We have

$$A = \begin{pmatrix} eAe = B & eAe^\perp \\ e^\perp Ae & e^\perp Ae^\perp = e^\perp AeAe^\perp \end{pmatrix}$$

Put $V = e^\perp Ae$, $W = eAe^\perp$ whence

$$A = \begin{pmatrix} B & W \\ V & V\otimes_B W/K \end{pmatrix}$$

and the product in A is calculated from the right + left B-module structure on $V+W$ and a pairing $W\otimes V \xrightarrow{\langle,\rangle} B$.

We know that

$$A^{(2)} = Ae \otimes_B eA = \begin{pmatrix} B \\ V \end{pmatrix} \otimes_B \begin{pmatrix} B & W \end{pmatrix} = \begin{pmatrix} B & W \\ V & V\otimes_B W \end{pmatrix}$$

Let's calculate $A^{(2)}/[A^{(2)}, A^{(2)}]$. We have

$$\left[\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b' & w \\ v & v_1\otimes w_1 \end{pmatrix}\right] = \begin{pmatrix} [b,b'] & bw \\ -vb & 0 \end{pmatrix}$$

so that the commutator quotient space is a quotient of $B/[B,B] \oplus V\otimes_B W$.

$$\left[\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b' & w' \\ v & v_1\otimes w_1 \end{pmatrix}\right] = \begin{pmatrix} \langle w, v\rangle & \langle w, v_1\rangle w_1 - b'w \\ & -v\otimes w \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}, \begin{pmatrix} b' & w \\ v' & v_1\otimes w_1 \end{pmatrix}\right] = \begin{pmatrix} -\langle w, v\rangle & 0 \\ vb' - v_1\langle w_1, v\rangle & v\otimes w \end{pmatrix}$$

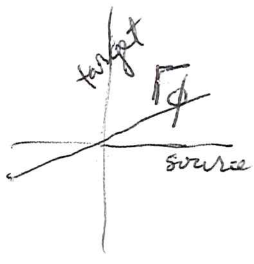
$$\left[\begin{pmatrix} 0 & 0 \\ 0 & v_1\otimes w_1 \end{pmatrix}, \begin{pmatrix} b & w \\ v & v_2\otimes w_2 \end{pmatrix}\right] = \begin{pmatrix} 0 & -\langle w, v_1\rangle w_1 \\ v_1\langle w_1, v\rangle & v_1\langle w_1, v_2\rangle \otimes w_2 - v_2\langle w_2, v_1\rangle \otimes w_1 \end{pmatrix}$$

Notice that we have a well defined map

$$V\otimes_B W \xrightarrow{\phi} B/[B,B] \qquad v\otimes w \overset{\phi}{\longmapsto} \langle w, v\rangle$$

Thus

$$\left(B/[B,B] \oplus V\otimes_B W\right) \Big/ \underbrace{\{\langle w, v\rangle \;\underline{0}\; v\otimes w\}}_{\text{graph of } \phi}$$



is canonically isom. to $B/[B,B]$.

Moreover ϕ sends $v_1\langle w_1, v_2\rangle \otimes w_2 - v_2\langle w_2, v_1\rangle\otimes w_1$ to $\left(\langle w_2, v_1\rangle\langle w_1, v_2\rangle - \langle w_1, v_2\rangle\langle w_2, v_1\rangle\right) = 0$. Thus we conclude $A^{(2)}/[A^{(2)}, A^{(2)}] = B/[B,B]$ as it should.

We now want to construct ~~the~~ ideal K in $A^{(2)}$ such that

$$A = \begin{pmatrix} B & W \\ V & V\otimes_B W/K \end{pmatrix}$$

has a smaller commutator quotient space. We want to arrange things so that there is an element $\xi = \sum v_i \otimes w_i \in V \otimes_B W$ such that $A^{(2)}\xi = \xi A^{(2)} = 0$, i.e.

$$\forall v \qquad \xi v = \sum v_i \langle w_i, v\rangle = 0$$

$$\forall w \qquad w\xi = \sum \langle w, v_i\rangle w_i = 0$$

(*)

$$\forall v, w \qquad (v\otimes w)\xi = \sum v\langle w, v_i\rangle \otimes_B w_i = 0$$

$$\xi(v\otimes w) = \sum v_i \langle w_i, v\rangle \otimes w = 0$$

and also such that $\langle w, v\rangle \neq 0$ in $B/[B,B]$.

Let's take $B = k + k\varepsilon$, $\varepsilon^2 = 0$ and $V \simeq B/k\varepsilon$, $W \simeq B/k\varepsilon$ with generators $v_1 \in V$ and $w_1 \in W$ satisfying $v_1\varepsilon = 0$, $\varepsilon w_1 = 0$. Define the pairing $W \otimes_k V \longrightarrow B$ by $\langle w_1, v_1\rangle = \varepsilon$. The equations (*) clearly hold, yet $\langle w_1, v_1\rangle \notin [B,B]$ which is zero as B is commutative.

Let $\mathcal{M} = \mathcal{M}(B)$, B idempotent. We would like to express the idea of an equivalence $\mathcal{M}(A) \simeq \mathcal{M}(B)$ without mentioning B, only the Grothendieck-Roos category $\mathcal{M}$. Let $\begin{pmatrix} A & Q \\ P & B\end{pmatrix}$ be the good

Morita context belonging to this equivalence. Then P is a generator for $\mathcal{M}$ and Q can be identified with the right continuous functor $N \longmapsto Q \otimes_B N$ from $\mathcal{M}$ to $\mathcal{Ab}$. Now it makes sense to tensor P with an abelian group, so $N \longmapsto P \otimes_{\mathbb{Z}} Q \otimes_B N$ is a right continuous functor from $\mathcal{M}$ to itself. In this case there is a surjection $P \otimes_{\mathbb{Z}} (Q \otimes_B N) \longrightarrow N$ from this functor to the identity.

~~January~~ December 29, 1994

Consider a Morita context $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ and ideals $I \subset R$, $J \subset S$ such that

$PIQ \supset J^s$, $QJP \supset I^t$ for some $s, t \in \mathbb{N}$.

Then

$$J^{sn} \subset (PIQ)^n \subset P(IQP)^n Q \subset PI^n Q$$

$$I^{tn} \subset (QJP)^n \subset QJ^n P$$

$$I^{tsn} \overbrace{\subset}^{(QJP)^{sn} \subset} QJ^{sn}P \subset QPI^nQP \subset I^n$$

(Note if either s or t is zero we get $R \subset I^n$ so $I = R$ and similarly $J = S$.)

Thus we have

$$\boxed{I^\infty = (QJP)^\infty = QJ^\infty P}$$

and $\boxed{J^\infty = (PIQ)^\infty = PI^\infty Q}$ by symmetry.

Let us now examine the idempotent case: $I = I^2$, $J = J^2$, whence

$$\boxed{\begin{aligned} I &= QJP \\ J &= PIQ \end{aligned}}$$

(Check: Assume only $PIQ \supset J$ and $QJP \supset I$. Then $I \supset QPIQP \supset QJP \supset I$, so $I = QJP$ etc.)

Next $\begin{aligned} PI &= PQJP \subset JP \\ JP &= PIQP \subset PI \end{aligned}$ $\Longrightarrow$ $\boxed{PI = JP}$

and similarly $\boxed{QJ = IQ}$

$$\begin{aligned} S/J \otimes_S P \otimes_R I &= P/JP \otimes_R I \\ &= P/PI \otimes_R I \\ &= P \otimes_R (R/I \otimes_R I) = P \otimes_R (I/I^2) = 0 \end{aligned}$$

Thus $P \otimes_R I$ is J-cnil-free

hence $J \otimes_S P \otimes_R I$ is J-firm.

$P/JP = S/J \otimes_S P$

Similarly $J \otimes_S P \otimes_R R/I = J \otimes_S (P/PI) = J \otimes_S S/J \otimes_S P = 0$

so $J \otimes_S P \otimes_R I$ is I^{op}-firm.

By symmetry $I \otimes_R Q \otimes_S J$ is I-firm and J^{op}-firm

Recall that $J \otimes_S P \longrightarrow S \otimes_S P = P$ is an I^{op}-nil isomorphism. Thus,

$$J \otimes_S P \otimes_R I \xleftarrow{\ \cong\ } J \otimes_S P \otimes_R I \otimes_R I \xrightarrow{\ \cong\ } P \otimes_R I \otimes_R I$$

and we have canonical isomorphisms

$$J \otimes_S J \otimes_S P = J \otimes_S P \otimes_R I = P \otimes_R I \otimes_R I$$

$$I \otimes_R I \otimes_R Q = I \otimes_R Q \otimes_S J = Q \otimes_S J \otimes_S J$$

Denote the above bimodules by P^f, Q^f respectively; these are firm versions of P, Q. Recall that $P \otimes_R Q \longrightarrow S$ is a J-nil isomorphism (also J^{op}-nil isom.) Thus we have

$$J \otimes_S J \otimes_S P \otimes_R Q \xrightarrow{\ \sim\ } J \otimes_S J \otimes_S S = J \otimes_S J$$

$$J \otimes_S J \otimes_S P \otimes_R Q \otimes_S J \otimes_S J \xrightarrow{\ \sim\ } J \otimes_S J \otimes_S J \otimes_S J = J \otimes_S J$$

whence a canonical isomorphism

$$\boxed{P^{f} \otimes_R Q^{f} = J \otimes_S J}$$ and

$$\boxed{Q^{f} \otimes_S P^{f} = I \otimes_R I}$$ by symmetry.

These then yield a canonical isom.

$$\boxed{I \otimes_R I \otimes_R \; = \; J \otimes_S J \otimes_S}$$

The point of the above discussion is that it should extend straightforwardly to the general case provided we replace I, J by the idempotent pro-ideals I^{∞}, J^{∞}.

We have seen that $I^{(\infty)}$ = the inverse system $I^{(n)} = I \otimes_R \overset{n}{\cdots} \otimes_R I$ generalizes $I^{(2)}$ in the idempotent case. Let's check now that $I^{(\infty)}$ is the same as $I^{\infty} \otimes_R I^{\infty}$.

1st proof. We have an exact sequence

$$(1) \qquad 0 \longrightarrow K^n \longrightarrow I^{(n)} \longrightarrow I^n \longrightarrow 0$$

where $I^m \cdot K^n = 0$ for some m; in fact we can take $m = n$. Why?

$$\begin{tikzcd}
 & I^{(n)} \otimes K^n \arrow[r] \arrow[d] & I^{(n)} \otimes_R I^{(n)} \arrow[r] \arrow[d] & I^{(n)} \arrow[d] \arrow[dl, dashed, "1"'] \\
0 \arrow[r] & K^n \arrow[r] & I^{(n)} \arrow[r] & R
\end{tikzcd}$$

Tensoring (1) with I^m yields an exact sequence

$$I^m \otimes_R K^n \longrightarrow I^m \otimes_R I^{(n)} \longrightarrow I^m \otimes_R I^n \longrightarrow 0$$

$$\parallel$$

$$\underbrace{I^m/I^{m+n}}_{\text{essentially zero in } m.} \otimes_R K^n$$

Thus $$I^\infty \otimes_R I^{(n)} \xrightarrow{\sim} I^\infty \otimes_R I^n$$

hence $$\begin{array}{ccc} I^\infty \otimes_R I^{(\infty)} & \xrightarrow{\sim} & I^\infty \otimes_R I^\infty \\ \downarrow \cong & & \\ I^{(\infty)} = R \otimes_R I^{(\infty)} & & \end{array}$$

yielding $I^{(\infty)} = I^\infty \otimes_R I^\infty$.

2nd proof.

$$0 \longrightarrow \operatorname{Tor}_1^R(R/I^m, I^n) \longrightarrow I^m \otimes_R I^n \longrightarrow I^{m+n} \longrightarrow 0$$

$$\underbrace{I^\ell/I^{\ell+m}}_{\text{essentially zero as } \ell \to \infty} \otimes_R \operatorname{Tor}_1^R(R/I^m, I^n) \to I^\ell \otimes_R I^m \otimes_R I^n \to I^\ell \otimes_R I^{m+n} \to 0$$

Thus $$I^\infty \otimes_R I^m \otimes_R I^n \xrightarrow{\sim} I^\infty \otimes_R I^{m+n}$$

hence $I^\infty \otimes_R I^\infty \otimes_R I^n \xrightarrow{\sim} I^\infty \otimes_R I^\infty$, which shows that $I^\infty \otimes_R I^\infty \otimes_R -$ inverts $I^n \hookrightarrow R$ and hence all I-nil isos. The rest is clear.

Both proofs really amount to the fact that $I^\infty \otimes_R -$ inverts surjective nil-isos.

Let's now consider $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$, $I \subset R$, $J \subset S$ such that $PI^\infty Q \supset J^\infty$, $QJ^\infty P \supset I^\infty$.

Then $J^\infty \supset PQJ^\infty PQ \supset PI^\infty \supset J^\infty$

$\Rightarrow$ $\boxed{J^\infty = PI^\infty Q}$ and sim. $\boxed{I^\infty = QJ^\infty P.}$

Also $PI^\infty = PQJ^\infty P \subset J^\infty P = PI^\infty Q P$

$\subset PI^\infty \Rightarrow \boxed{PI^\infty = J^\infty P}$ and sim $\boxed{QJ^\infty = I^\infty Q}$

Then $\boxed{PI^\infty \cdot QJ^\infty = (J^\infty)^2 = J^\infty}$

$\boxed{QJ^\infty \cdot PI^\infty = (I^\infty)^2 = I^\infty}$

I now want to check that $J^\infty \otimes_S P \otimes_R I^\infty$ is J- ferm.

$$0 \to \mathrm{Tor}_2^S(S/J, S/J^m) \to J \otimes_S J^m \longrightarrow J^{m+1} \to 0$$

$$\mathrm{Tor}_2^S(S/J, S/J^m) \otimes_S P \otimes_R I^n \longrightarrow J \otimes_S J^m \otimes_S P \otimes_R I^n \longrightarrow J^{m+1} \otimes_S P \otimes_R I^n \to 0$$

$$\overset{\shortparallel}{\mathrm{Tor}_2^S(S/J, S/J^m) \otimes_S \underbrace{P/J^m P \otimes_R I^n}}$$

essentially zero as $n \to \infty$ as $J^\infty P = PI^\infty$

$$\therefore \quad J \otimes_S J^m \otimes_S P \otimes_R I^\infty \xrightarrow{\sim} J^{m+1} \otimes_S P \otimes_R I^\infty$$

Now let $m \to \infty$ and we see that $J^\infty \otimes_S P \otimes_R I^\infty$ is J-ferm. Similarly get it is I^{op}- ferm.

Then we should get

$$J^\infty \otimes_S P \otimes_R I^\infty \xleftarrow{\sim} J^\infty \otimes_S P \otimes_R I^\infty \otimes_R I^\infty \xrightarrow{\sim} P \otimes_R I^\infty \otimes_R I^\infty$$

because $J^\infty \otimes P \otimes I^\infty$ is I^{op}-ferm

because $J^m \otimes_S P \to P$ is I^{op}-nil isom.

Thus

$$P^f = J^\infty \otimes_S P \otimes_R I^\infty = J^\infty \otimes_S J^\infty \otimes_S P = P \otimes_R I^\infty \otimes_R I^\infty$$

and similarly for Q^f. As before we have the J-nil iso $P \otimes_R Q \longrightarrow S$ so

$$J^\infty \otimes_S J^\infty \otimes_S P \otimes_R Q \xrightarrow{\sim} J^\infty \otimes_S J^\infty$$

$$J^\infty \otimes_S J^\infty \otimes_S P \otimes_R Q \otimes_S J^\infty \otimes_S J^\infty \xrightarrow{\sim} J^\infty \otimes_S J^\infty$$

i.e. $\quad P^f \otimes_R Q^f \simeq J^\infty \otimes_S J^\infty$

similarly $\quad Q^f \otimes_S P^f \simeq I^\infty \otimes_R I^\infty$

whence a canonical isomorphism

$$\boxed{I^\infty \otimes_R I^\infty \otimes_R \simeq J^\infty \otimes_S J^\infty \otimes_S}$$