April 12, 1994

Prop. Let A be a unital ring, let M be an A-module. TFAE:

1) M is flat

2) For all A modules N of finite presentation one has

$$(*) \qquad \mathrm{Hom}_A(N,A)\otimes_A M \xrightarrow{\sim} \mathrm{Hom}_A(N,M)$$

Proof of 1) $\Rightarrow$ 2) $(*)$ is the map $f\otimes m \mapsto (n \mapsto f(n)m)$, and it a morphism of functors of N, which is an isomorphism for $N=A$, hence for $N = A^n$ for any n (thus for $N \in P(A)$ and any module M). When M is flat the functor on the left is left exact, hence both functors are left exact. Thus $(*)$ is an isomorphism for any N of finite presentation.

(Note that for M flat $(*)$ is injective for N of finite type, since if we choose $A^n \twoheadrightarrow N$, then

$$\begin{tikzcd}
\mathrm{Hom}_A(N,A)\otimes_A M \arrow[r] \arrow[d, hook] & \mathrm{Hom}_A(N,M) \arrow[d, hook] \\
\mathrm{Hom}_A(A^n,A)\otimes_A M \arrow[r, "\sim"] & \mathrm{Hom}_A(A^n,M)
\end{tikzcd}$$
)

Proof of 2) $\Rightarrow$ 1). We show first that $(*)$ implies the category of finite type free A-modules over M is a filtering category. We must show given solid arrows in the diagram

$$
(**)\qquad
\begin{tikzcd}
A^p \arrow[r, "\cdot a"] \arrow[dr, "0"'] \arrow[rr, bend left, dashed, "0"] & A^q \arrow[r, dashed, "\cdot a'"] \arrow[d, "\cdot m"] & A^r \arrow[dl, dashed, "\cdot m'"] \\
 & M &
\end{tikzcd}
$$

there exist r and the dotted arrows. But if $N = A^q/A^p.a$, then $\cdot m \in \mathrm{Hom}_A(N, M)$, and to express this in the form $\sum f_i \otimes m_i \in N^* \otimes_A M$ is the same as factoring: $m = a'm'$ where $aa' = 0$.

We recognize in $(**)$ the linear equations criterion for flatness of Cartan–Eilenberg: Any A linear relations $a_{ij} m_j = 0$ in M can be written in terms of linear relations in A: $m_j = a'_{jk} m'_k$ where $a_{ij} a'_{jk} = 0$.

[redacted] But $M = \varinjlim_{\{A^n \to M\}} A^n$ and a filtered inductive limit of flat modules is flat. Thus 2) $\Rightarrow$ 1).

Alternative: To show M flat it suffices to show $\mathrm{Tor}_1^A(X, M) = 0$ for any right A-module of finite presentation X. So given

$$0 \longrightarrow K \longrightarrow A^p_r \xrightarrow{a\cdot} A^q_r \longrightarrow X \longrightarrow 0$$

we want to show that

$$0 \longrightarrow K \otimes_A M \longrightarrow M^p \xrightarrow{a\cdot} M^q \longrightarrow X \otimes_A M \longrightarrow 0$$

is exact at M^p. Define N by

$$A^q \xrightarrow{\cdot a} A^p \longrightarrow N \longrightarrow 0$$

so that we have

$$0 \longrightarrow N^* \longrightarrow A_r^p \xrightarrow{a\cdot} A_r^q$$

identifying N^* with K. Now we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(N,M) & \longrightarrow & M^p & \xrightarrow{a\cdot} & M^q \\ & & \cong\uparrow \text{ by hypothesis} & & & & \\ & & N^* \otimes_A M & & & & \end{array}$$

so we conclude the exactness of

$$0 \longrightarrow K \otimes_A M \longrightarrow M^p \xrightarrow{a\cdot} M^q.$$

Corollary: M flat and of fin. pres. $\Longrightarrow$ M projective.

Because in this case we have

$$\mathrm{Hom}_A(M,M) \xleftarrow{\sim} M^* \otimes M$$

and writing the identity as $f_i \otimes m_i$ gives a map $M \xrightarrow{(f_i)} A^n \xrightarrow{(m_i)} M$ with composition id_M.

Ultimately I want to examine whether there is a good analogue of flat and finite presentation modules in the case of a non unital algebra, say $Ae \otimes_B eA$.

The observation is that A is H-unital iff $A \overset{!}{\otimes}_A A \longrightarrow A$ is a quis. In view of $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$ we have a triangle

$$\begin{tikzcd}
A \overset{!}{\otimes}_A A \arrow[rr] & & \underset{\tilde{A} \overset{!}{\otimes}_A A}{\overset{A}{\shortparallel}} \arrow[dl] \\
& \mathbb{C} \overset{!}{\otimes}_A A \arrow[ul] &
\end{tikzcd}$$

so $A \overset{!}{\otimes}_A A \longrightarrow A$ is a quis $\Longleftrightarrow$ $\mathbb{C} \overset{!}{\otimes}_A A = 0$ i.e. $\mathrm{Tor}_*^A(\mathbb{C}, A) = 0$, and this is equivalent to the reduced bar construction being acyclic.

For an A-module M, define M to be an H-unital A-module when $A \overset{!}{\otimes}_A M \longrightarrow M$ is a quis, equivalently $\mathbb{C} \overset{!}{\otimes}_A M = 0$. The condition $\mathbb{C} \overset{!}{\otimes}_A M = 0$ is nice to work with, e.g. if M is A-flat, then M is H-unital iff $M = AM$.

In particular if A is A-flat, then A is an H-unital algebra iff $A^2 = A$.

Let's now consider Morita invariance of Hochschild homology in the situation $A = Ae \otimes_B eA$. Then

$$A = Ae \otimes_B eA \underset{\text{quis}}{\overset{\sim}{\longleftarrow}} Ae \overset{!}{\otimes}_B eA$$

provided we assume eA is a flat B-module. Then

$$A \overset{!}{\otimes}_A \simeq Ae \overset{!}{\otimes}_B eA \overset{!}{\otimes}_A = eA \overset{!}{\otimes}_A Ae \overset{!}{\otimes}_B \xrightarrow{\sim} B \overset{!}{\otimes}_B$$

where we use that

$$eA \overset{!}{\otimes}_A Ae \simeq eA \otimes_A Ae = B$$

as Ae(eA) is a flat left (right) A-module.

The same assumption: eA is a flat B module implies A is H-unital

$$A \overset{!}{\otimes}_A A \simeq A \overset{!}{\otimes}_A \left(Ae \overset{!}{\otimes}_B eA\right) \simeq \left(A \overset{!}{\otimes}_A Ae\right) \overset{!}{\otimes}_B eA$$

$$\simeq Ae \overset{!}{\otimes}_B eA \simeq A.$$

(here we use that Ae is always H-unital as A-module). The hypothesis eA flat$/B$ can be weakened to $Ae \overset{!}{\otimes}_B eA \simeq Ae \otimes_B eA$, i.e. $\mathrm{Tor}_n^B(Ae, eA) = 0$ for $n > 0$.

However one reason for [redacted] liking the hypothesis that eA is B flat is that it implies $A = Ae \otimes_B eA$ is A flat:

$$\begin{array}{ccccc} M_1 & \xmapsto{\text{exact}} & M_2 \otimes_A Ae & \xmapsto{\text{exact}} & (M \otimes_A Ae) \otimes_B eA \\ & & \| & & \| \\ & & Me & & M \otimes_A A \end{array}$$

Thus A is A flat and such that $A^2 = A$, so A is H-unital.

Continuing with the assumption that eA is B-flat, we have that M A-flat $\Longrightarrow$ $eM = eA \otimes_A M$ is B-flat $\Longrightarrow$ $Ae \otimes_B eM = A \otimes_A M$ is A flat. Thus we have an equivalence between flat B-modules and flat A-modules M such that $AM = M$.

April 14, 1994

Let's describe all (noncommutative) algebras of the form $A = Ae \otimes_B eA$ where $B = eAe$ is the groundfield $\mathbb{C}$.

First note that if V and W are vector spaces and if $(v,w) \mapsto \langle v,w \rangle$, $V \otimes W \to \mathbb{C}$ is a bilinear map, then we obtain an associative product on $V \otimes W$ given by

$$(v_1, w_1)(v_2, w_2) = (v_1 \langle w_1, v_2 \rangle, w_2) = (v_1, \langle w_1, v_2 \rangle w_2)$$

Moreover V (resp. W) is naturally a left (resp. right) module over this algebra, which we denote A.

Suppose now that $e' \in V$, $e'' \in W$ are elements such that $\langle e'', e' \rangle = 1$. Then $e = (e', e'')$ is an idempotent in A. One has

$$(e', e'')(v, w) = (e' \langle e'', v \rangle, w)$$
$$(v, w)(e', e'') = (v, \langle w, e' \rangle e'')$$

so that $eA \simeq W$, $Ae \simeq V$, $eAe = \mathbb{C}e$.

Put

$$V_1 = \{ v \in V \mid \langle e'', v \rangle = 0 \}$$
$$W_1 = \{ w \in W \mid \langle e', w \rangle = 0 \}$$

Then $V = \mathbb{C}e' \oplus V_1$, $W = \mathbb{C}e'' \oplus W_1$. Then we can write A in matrix form

$$A = \begin{pmatrix} \mathbb{C} & W_1 \\ V_1 & V_1 \otimes W_1 \end{pmatrix} \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Here the product in A is given by the product in $A_1 = V_1 \otimes W_1$ obtained by restricting $\langle , \rangle$ to $W_1 \otimes V_1$, as well as the natural left (resp. right) A_1 module structure on V_1 (resp. W_1).

This discussion makes clear the following

Prop. A pair (A, e) consisting of a nonunital algebra A and an idempotent e in A such that

$$Ae \otimes_{eAe} eA \xrightarrow{\sim} A$$

[redacted] has the form

$$A = \begin{pmatrix} B & W_1 \\ V_1 & A_1 \end{pmatrix} \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $B = eAe$ can be any unital algebra, $V_1 = Ae$ can be any right [redacted] unital B-module, $W_1 = eA$ ——— left unital B-module, $A_1 = V_1 \otimes W_1$ equipped with the product arising [redacted] from a B-bimodule map $W_1 \otimes V_1 \longrightarrow B$ which can be arbitrary. The product in A is given by the products in B and A_1, [redacted] the B-module structures on V_1, W_1, and the A_1-module structures on V_1, W_1.

Let's go back to the case [redacted] where B is the groundfield $\mathbb{C}$. [redacted] Then we find that [redacted] pair (A, e) such that $eAe = \mathbb{C}$ [redacted] [redacted] such that $A = Ae \otimes_{\mathbb{C}} eA$ has the form

$$A = \begin{pmatrix} \mathbb{C} & W_1 \\ V_1 & V_1 \otimes W_1 \end{pmatrix}$$

where the product is associated to any pairing $W_1 \otimes V_1 \to \mathbb{C}$. The most degenerate case is where the pairing $\langle\,,\rangle$ is identically zero. The most nondegenerate case is where V_1, W_1 are finite dimensional and in duality via the pairing (equivalently A_1 is unital). In this case A is a matrix algebra.

Consider the general situation $A = Ae \otimes_B eA$, $B = eAe$. We know that $A^2 = A$ and that there is an equivalence between B-modules and good A modules.

Assum[redacted] that eA is a flat B-module. Then we know that A is a flat A-module, hence (since $A^2=A$) that A is H-unital: $A \overset{!}{\otimes}_A A \xrightarrow{\sim} A$. Furthermore we have equivalence of Hochschild (probably also cyclic) homology: $A \overset{!}{\otimes}_A \simeq B \overset{!}{\otimes}_B$.

Now I claim that one also has equivalence on the level of K-theory, more precisely that the bimodule Ae (which is a representation of B over A) give rise to an isomorphism

$$K_*(B) \xrightarrow{\sim} K_*(A)$$

The reason is that K theory commutes with filtered inductive limits of algebras. Thus $eAe^{\perp}$ is a flat B module, so it is an inductive limit of ft free B modules. But recall that the algebra

A has the form

$$A = \begin{pmatrix} B & W_1 \\ V_1 & V_1 \otimes W_1 \end{pmatrix} .$$

where V_1 (resp. W_1) is a right (resp. left) B-module and the product depends only on a B-bimodule map $W_1 \otimes V_1 \longrightarrow B$. Thus I can suppose, or better reduce to the case where $V_1 = e^{\perp}Ae$ is finitely presented over B_R and $W_1 = eAe^{\perp}$ is finite type free. Then we can apply the Dawydov result which tells us that assuming $eA \in P(B)$, and A an ideal in R, we have

$$K_*(R) = K_*(B) \oplus K_*(R/A)$$

In the present situation we take $R = \tilde{A}$.

I now want to understand concretely why $K_0(B) \xrightarrow{\sim} K_0(A)$ when (A, e) is such that $A \xleftarrow{\sim} Ae \otimes_B eA$, $B = eAe$ and eA is a flat B module. Recall that

$$K_0(A) \overset{\text{defn}}{=} \operatorname{Ker}\left(K_0(\tilde{A}) \longrightarrow K_0(\mathbb{C})\right) \simeq K_0(\tilde{A}) / K_0(\mathbb{C})$$

Consider $A = Ae \otimes_B eA$, $B = eAe$. Call an [redacted] A module M [redacted] of *finite type* (resp of *finite presentation*) when it is so as unital $\tilde{A}$ module, i.e when $\exists$ $\tilde{A}^p \twoheadrightarrow M$ (resp. $\tilde{A}^p \to \tilde{A}^q \to M \to 0$).

Prop. 1) $\exists$ surjection $Ae^p \twoheadrightarrow M \iff M$ is f.t. and $AM = M$.

2) $\exists$ presentation $Ae^p \to Ae^q \to M \to 0$ $\iff M$ is f.p. and $A \otimes_A M \xrightarrow{\sim} M$.

Proof. The direction $\Rightarrow$ is easy, so consider $\Leftarrow$. Suppose $\exists$ $\tilde{A}^p \twoheadrightarrow M$, i.e. M is generated by elements m_i, $1 \le i \le p$. Since $M = AM = AeAM \subset AeM$ there exist $a_{ij} \in A$, $m'_j \in M$ such that

$$m_i = a_{ij} e m'_j \qquad (\text{here } 1 \le j \le q)$$

Then one has maps

$$Ae^q \subset \tilde{A}^q \xrightarrow{\cdot m'} M$$

$$(a_j) \longmapsto (a_1 \ \ a_q)\begin{pmatrix} m'_1 \\ m'_q \end{pmatrix} = a_i m'_i$$

and the composition $Ae^q \to M$ is surjective since $(a_{ij} e)_{1 \le j \le q} \longmapsto m_i$. This proves 1) $\Leftarrow$

Next suppose M f.p. [redacted] and $A \otimes_A M = M$. [redacted] Choose a surjection $Ae^p \twoheadrightarrow M$ which is possible by 1) and let K be the kernel:

$$0 \to K \to Ae^p \to M \to 0$$

Then M f.p. and Ae f.t. $\Rightarrow K$ f.t.

[redacted] One has

$$\begin{array}{ccccccc} A\otimes_A K & \to & A\otimes_A Ae^p & \to & A\otimes_A M & \to & 0 \\ \downarrow & & \| & & \| & & \\ 0 \to K & \to & Ae^p & \to & M & \to & 0 \end{array}$$

whence $A\otimes_A K \twoheadrightarrow K$ so $AK = K$. Then we know $\exists$ $Ae^q \twoheadrightarrow K$ proving 2) $\Leftarrow$.

Sorites about f.t. and f.p. modules

$$0 \to M' \to M \to M'' \to 0$$

M f.t. $\Rightarrow$ M'' f.t.

M', M'' f.t. $\Rightarrow$ M f.t.

M'' f.p., M f.t. $\Rightarrow$ M' f.t.

M f.p., M' f.t. $\Rightarrow$ M'' f.p.

M', M'' f.p. $\Rightarrow$ M f.p.

[redacted]

Let's now consider a flat A-module M such that $AM = M$. We wish to show that it corresponds to a flat B-module, i.e. that eM is B-flat. Recall [redacted] that if N is a f.p. B-module then $Ae\otimes_B N$ is a f.p. A-module which is good and conversely. Thus

$$\mathrm{Hom}_B(N, eM) = \mathrm{Hom}_A(Ae \otimes_B N, M)$$

$$= \mathrm{Hom}_A(Ae \otimes_B N, \tilde{A}) \otimes_A M \qquad \left(\begin{array}{ll} M & \text{flat} \\ Ae \otimes_B N & \text{f.p.} \end{array} \right.$$

$$= \mathrm{Hom}_A(Ae \otimes_B N, \tilde{A}) \otimes_A Ae \otimes_B eM$$

$$= \mathrm{Hom}_A(Ae \otimes_B N, \underbrace{\tilde{A}e}_{Ae}) \otimes_B eM$$

$$= \mathrm{Hom}_B(N, B) \otimes_B eM$$

which means that eM is B-flat. Thus we have proved that good flat A-modules correspond to flat B-modules and also (p. 506) that good f.p. A-modules correspond to f.p. B modules

Putting these together we see good f.t. projective A-modules correspond to f.t. projective B-modules

Suppose B a nonunital alg, V a B_r-module, W a B_ℓ module, and $\langle -,- \rangle : W \otimes V \longrightarrow B$ a B-bimodule map.

Let $A = V \otimes_B W$. Define a product on A by
$$(v_1, w_1)(v_2, w_2) = (v_1 \langle w_1, v_2 \rangle, w_2)$$
and left (resp. right) mult. by A on V (resp W) by
$$(v_1, w_1) \circ v = v_1 \langle w_1, v \rangle$$
$$w \circ (v_1, w_1) = \langle w, v_1 \rangle w_1$$
Then A becomes a (nonunital) alg and V, W becomes bimodules: ${}_A V_B$, ${}_B W_A$ such that $\langle -,- \rangle$ descends to an A-bimodule map
$$p : W \otimes_A V \longrightarrow B \qquad p(w,v) = \langle w, v \rangle$$
satisfying
$$p(x)\, y = x\, p(y) \qquad \forall x, y \in W \otimes_A V.$$

Check that $\langle w \circ a, v \rangle = \langle w, a \circ v \rangle$: Let $a = (v_1, w_1)$. Then
$$\langle w \circ (v_1, w_1), v \rangle = \langle \langle w, v_1 \rangle w_1, v \rangle = \langle w, v_1 \rangle \langle w_1, v \rangle$$
$$\langle w, (v_1, w_1) \circ v \rangle = \langle w, v_1 \langle w_1, v \rangle \rangle = \langle w, v_1 \rangle \langle w_1, v \rangle. \checkmark$$
Next let $x = (v, w)$, $y = (v_1, w_1)$. Then
$$p(x) y = \langle v, w \rangle (v_1, w_1) = (\langle v, w \rangle v_1, w_1) = (v \circ (w, v_1), w_1)$$
$$x\, p(y) = (v, w) \langle v_1, w_1 \rangle = (v, w \langle v_1, w_1 \rangle) = (v, (w, v_1) \circ w_1)$$
and these agree in $W \otimes_A V$.

Assume now that B is unital, that V, W are unital B modules, and that p is surjective.

Then p is an isomorphism because if $x = (w_i, v_i)$ is such that $p(x) = \langle w_i, v_i \rangle = 1$ and if $y \in \operatorname{Ker} p$, then $y = p(x)y = x\,p(y) = 0$. (Here have used $1(w, v) = (1w, v) = (w, v)$, i.e. the fact that W is a unital module.)

Furthermore the elements $w_i \in W$, $v_i \in V$ and the A-bimodule map

$$V \otimes W \longrightarrow A$$
$$v \otimes w \longmapsto (v, w)$$

satisfy

$$(v, w_i) \cdot v_i = v \langle w_i, v_i \rangle = v 1_B = v$$
$$w_i \cdot (v_i, w) = \langle w_i, v_i \rangle w = 1_B w = w$$

which means that $V \in \mathcal{P}(\tilde{A}_\ell)$, $W \in \mathcal{P}(\tilde{A}_r)$ and V, W are dual to each other: $V = \operatorname{Hom}_{A_r}(W, A)$, $W = \operatorname{Hom}_{A_\ell}(V, A)$.

Notice that A is good:

$$A \otimes_A A = (V \otimes_B W) \otimes_A (V \otimes_B W) = V \otimes_B B \otimes_B W = V \otimes_B W$$

(using that W is a unital B module). In fact we have maps

$$V \xrightarrow{((-, w_i))} A^n \xrightarrow{(\cdot v_i)} V$$
$$W \xrightarrow{((v_i, -))} A^n \xrightarrow{(w_i \cdot)} W$$

so that V is a ft proj $\tilde{A}_\ell$ module such that $AV = V$, and W is a ft proj $\tilde{A}_r$ module such that $WA = W$.

The present situation reduces to the case $Ae \otimes_B eA = A$, when $\exists\ w_1 \in W$ $v_1 \in V$ such that $\langle w_1, v_1 \rangle = 1$. In this case $e = (v_1, w_1) \in A$ is an idempotent: $e^2 = (v_1, w_1)(v_1, w_1) = (v_1 \langle w_1, v_1 \rangle, w_1) = e$

Note that

$$A \xrightarrow{\cdot v_1} V \xrightarrow{(-, w_1)} A$$

$$(v, w) \longmapsto v\langle w, v_1 \rangle \longmapsto (v \langle w, v_1 \rangle, w_1) = (v, w)\underbrace{(v_1, w_1)}_{e}$$

showing $V = Ae$. Similarly $W = eA$.

In general given $\langle w_i, v_i \rangle = 1$ the matrix $e_{ij} = (v_i, w_j) \in M_n A$ is idempotent.

$$e_{ij} e_{jk} = (v_i, w_j)(v_j, w_k) = (v_i \langle w_j, v_j \rangle, w_k)$$
$$= (v_i, w_k) = e_{ik}$$

I think this means that if we replace B, V_B, W_B, A by B, , $\mathbb{C}^n \otimes V$, $W \otimes \mathbb{C}^{n*}$, $M_n A = (\mathbb{C}^n \otimes V) \otimes_B (W \otimes \mathbb{C}^{n*})$, then we effectively reduce to the situation $A = Ae \otimes_B eA$. Thus the results about finite presentation and flat modules should carry about to the present situation.

<u>April 18, 1994</u>

Basic construction: suppose given B, V_B, ${}_BW$, and $\langle -,- \rangle : W \otimes V \to B$ bimod map $/B$.

Then one has on $A = V \otimes_B W$ an $\overset{(nu)}{\text{alg}}$ structure given by $(v_1, w_1)(v_2, w_2) = (v_1 \langle w_1, v_2 \rangle, w_2) = (v_1, \langle w_1, v_2 \rangle w_2)$

a left A-module structure on V and a right A-module structure on W given by

$$(v_1, w_1) \cdot v = v_1 \langle w_1, v \rangle$$
$$w \cdot (v_1, w_1) = \langle w, v_1 \rangle w_1$$

One thereby gets bimodules ${}_AV_B$, ${}_BW_A$ [redacted] and bimodule maps

$$V \otimes_B W \xrightarrow{\sim} A \qquad \text{this is the identity}$$

$$W \otimes_A V \xrightarrow{p} B \qquad p((w, v)) = \langle w, v \rangle$$

over A, B resp. The following squares are commutative

$$(*) \quad \begin{tikzcd} V \otimes_B W \otimes_A V \arrow[r] \arrow[d, "\cong"] & V \otimes_B B \arrow[d] \\ A \otimes_A V \arrow[r] & V \end{tikzcd} \qquad \begin{tikzcd} W \otimes_A V \otimes_B W \arrow[r, "\cong"] \arrow[d] & W \otimes_A A \arrow[d] \\ B \otimes_B W \arrow[r] & W \end{tikzcd}$$

and p satisfies $\boxed{p(x)\,y = x\,p(y)}$

Now make the assumptions

1) B, V_B, ${}_BW$ are good: $B \otimes_B B \xrightarrow{\sim} B$, $V \otimes_B B \xrightarrow{\sim} V$, $B \otimes_B W \xrightarrow{\sim} W$. e.g. all unital

2) $W \otimes V \to B$ $\quad (w, v) \mapsto \langle w, v \rangle$ is surjective.

Then we claim that
$$p: W \otimes_A V \xrightarrow{\sim} B$$
and that A, ${}_AV$, W_A are good.

Proof. In general consider a B-bimodule map $M \xrightarrow{u} B$ which is surjective and such that $u(x)y = x\,u(y)$. From the exact sequence $0 \to K \to M \to B \to 0$ where $K = \mathrm{Ker}(u)$ we obtain the exact sequence
$$B \otimes_B K \longrightarrow B \otimes_B M \longrightarrow B \otimes_B B \longrightarrow 0$$
Now K is null B-bimodule: $BK = KB = 0$, so if $B = B^2$, then $B \otimes_B K = B^2 \otimes_B K = B \otimes_B BK = 0$. Thus $B \otimes_B M \xrightarrow{\sim} B \otimes_B B$, so if M and B are both good B modules we have $M \xrightarrow{\sim} B$. This applies to our B and to $M = W \otimes_A V$, because B and ${}_BW$ are assumed good.

Next we have

$$\begin{tikzcd} V \otimes_B W \otimes_A V \arrow[r, "\sim"] \arrow[d, "\cong"] & V \otimes_B B \arrow[d, "\cong"] \\ A \otimes_A V \arrow[r] & V \end{tikzcd} \qquad \begin{tikzcd} W \otimes_A V \otimes_B W \arrow[r, "\cong"] \arrow[d, "\cong"] & W \otimes_A A \arrow[d] \\ B \otimes_B W \arrow[r, "\cong"] & W \end{tikzcd}$$

showing that ${}_AV$, W_A are good. This implies $A = \underline{V \otimes_B W}$ is a good A module.

Remark: The above argument shows B, V_B good + $\langle\,,\,\rangle$ surjective $\Longrightarrow$ $W \otimes_A V \simeq B$, ${}_AV$, A good. But one does not get ${}_BW$ good, which would be nice if one wants $V \otimes_B -$ and $W \otimes_A -$ to map into good modules.

Way to think: In a Morita equivalence:
$A, B, {}_AV_B, {}_BW_A$, $V\otimes_B W = A$, $W\otimes_A V = B$
you probably want ${}_AV$, ${}_BW$ to be good,
and then this implies that A, B, V_B, ${}_BW_A$
are also good.

Question: When is an algebra Morita equivalent to a unital algebra?

Suppose $A, B, {}_AV_B, {}_BW_A$, $V\otimes_B W = A$, $W\otimes_A V = B$ is a (good as above) Morita equivalence, where B is unital. Then V_B, ${}_BW$ are good $\Rightarrow$ they are unital modules.

First we show that $V \in \mathcal{P}(\tilde{A})$ and that $W = \mathrm{Hom}_A(V, A)$ is the dual of V in $\mathcal{P}(\tilde{A}_s)$. Let $\langle -,-\rangle_A$, $\langle -,-\rangle_B$ denote the isomorphisms $V\otimes_B W \xrightarrow{\sim} A$, $W\otimes_A V \xrightarrow{\sim} B$.

Let $w_i \in W$, $v_i \in V$, $1 \le i \le n$ be such that $\langle w_i, v_i\rangle_B = 1$.

From

$$\begin{tikzcd}
(v, w_i, v_i) \arrow[r, mapsto] \arrow[d, mapsto] & (v, 1) \arrow[d, mapsto] \\
V\otimes_B W\otimes_A V \arrow[r, "\sim"] \arrow[d, "\wr"] & V\otimes_B B \arrow[d, "\wr"] \\
A\otimes_A V \arrow[r, "\sim"] & V \\
(\langle v, w_i\rangle_A, v_i) \arrow[r, mapsto] & \langle v, w_i\rangle_A v_i
\end{tikzcd}$$

and

$$\begin{tikzcd}
W\otimes_A V\otimes_B W \arrow[r, "\sim"] \arrow[d, "\wr"] & W\otimes_A A \arrow[d, "\wr"] & (w_i, v_i, w) \arrow[r, mapsto] \arrow[d, mapsto] & (w_i, \langle v_i, w\rangle_A) \arrow[d, mapsto] \\
B\otimes_B W \arrow[r, "\sim"] & W & (1, w) \arrow[r, mapsto] & w
\end{tikzcd}$$

yielding

$$v = \langle v, w_i\rangle_A v_i \qquad w = w_i \langle v_i, w\rangle_A$$

and this implies $V \in \mathcal{P}(\tilde{A})$ and that W is its dual.

Note also $v = \langle v, w_i \rangle_A v_i \implies V = AV$ so that V is a finitely generated projective good A-module.

Let's study ~~fg~~proj good A-modules. If $V \in \mathcal{P}(\tilde{A})$, then V has the form $\tilde{A}^n e$ where $e^2 = e \in M_n(\tilde{A})$. Then $V = \sum_i \tilde{A} v_i$, $v_i = (e_{ij})_j =$ the i-th row of e.

Assume $V = AV$, i.e. $v_i = a_{ik} v_k$ with $a_{ik} \in A$, i.e. $(e_{ij})_j = a_{ik}(e_{kj})_j$, or $e_{ij} = a_{ik} e_{jk}$. Then $e \in M_n A$. Conversely if $e \in M_n A$ is idempotent, then $\tilde{A}^n e \subset \tilde{A}^n e e \subset A^n e$, so $A^n e = \tilde{A}^n e \in \mathcal{P}(\tilde{A})$ and $A A^n e = A^n e$. Thus we have the first part of

Prop: If $A^2 = A$, then an A module V is a good fgproj module $\iff$ $V = A^n e$ ~~[illegible]~~ for some n and idempotent $e \in M_n A$. If so then the dual $W = \mathrm{Hom}_A(V, \tilde{A})$ is a fg proj good right module.

The last assertion follows from the fact that the dual is given by the transposes ~~matrix~~:

$$\mathrm{Hom}_A(\tilde{A}^n e, \tilde{A}) = e \tilde{A}^n$$

I think now the following is clear

<u>Prop.</u> <u>A good algebra A is Morita equivalent to a unital algebra iff there exists a fg projective good A module V such that if $W = \mathrm{Hom}_A(V, A)$, then the obvious pairing $V \otimes W \longrightarrow A$ is surjective.</u>

I should have noted that when $AV = V$ we have

$$0 \to \mathrm{Hom}_A(V, A) \to \mathrm{Hom}_A(V, \tilde{A}) \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}_A(V, \mathbb{C})}$$

Proof. $(\Longrightarrow)$ this we've done.

$(\Longleftarrow)$. With W defined this way we have A, ${}_AV$, W_A all good and $V \otimes W \twoheadrightarrow A$. So we [redacted] a good Morita equivalence with $B = W \otimes_A V$. But because $V \in \mathcal{P}(\tilde{A})$ and W is its dual one knows that $W \otimes_A V = \mathrm{Hom}_A(V,V)^{op}$ $= \mathrm{Hom}_A(W,W)$. Thus B is unital.

Summarize a few ideas from scratch paper the past few days.

If A is good, then $M \mapsto A \otimes_A M$ is right adjoint to the inclusion of good modules in modules. However if only $A^2 = A$, then the right adjoint should be $M \mapsto A \otimes_A A \otimes_A M$. $A \otimes_A M$ need not be good in general, e.g. $A \otimes_A \tilde{A} = A$. But if $AM = M$, then $A \otimes_A M$ is good because

$$\underset{\substack{\| \\ }}{K \otimes_A M} \longrightarrow A \otimes_A A \otimes_A M \longrightarrow A \otimes_A M \longrightarrow 0$$

$$K \otimes_A AM = KA \otimes_A M = 0$$

Thus the functor $A \otimes_A -$ has to be applied twice to get the adjoint. This is [redacted] similar to sheaves + presheaves, and it might be worthwhile to explore this example.

You want to work on Wodzicki's result that if $A \subset R$ is a left ideal such that $A^2 = A$, then A is A-flat iff A is R-flat. In this vein note that if M is a good A-module $A \otimes_A M = M$, then it has a unique R-module structure extending the A-module structure.

April 21, 1994

Let A be a nonunital ring such that $A^2 = A$, let $Mod(A)$ be the abelian category of its (left) modules, let $Modg(A)$ be the full subcategory of modules which are good: $A \otimes_A M \xrightarrow{\sim} M$, let $\mathcal{N}$ be the full subcategory of $Mod(A)$ consisting of <u>null</u> modules: $AM = 0$. Let $M \in Mod(A)$

1) $AM = 0 \iff A \otimes_A M = 0$.

Pf. $\Leftarrow$ clear since $A \otimes_A M \twoheadrightarrow AM$.

$\Rightarrow$ Assuming $AM = 0$, to show any $(a, m) \in A \otimes_A M$ is zero. Since $A^2 = A$, one can assume $a = a_1 a_2$. Then $(a_1 a_2, m) = (a_1, a_2 m) = (a_1, 0) = 0$.

2) <u>The kernel and cokernel of</u> $A \otimes_A M \xrightarrow{\mu} M$, $\mu : (a, m) \longmapsto am$ <u>are null modules.</u>

Pf. The cokernel is M/AM which is killed by A. If $\sum(a_i, m_i) \in$ the kernel, i.e. $\sum a_i m_i = 0$, then $a \sum(a_i, m_i) = \sum(a a_i, m_i) = (a, \sum a_i m_i) = 0$.

3) <u>If</u> $AM = M$, <u>then</u> $A \otimes_A M$ <u>is good.</u>

Pf. [redacted] One has the exact sequence

$$0 \longrightarrow K \longrightarrow A \otimes_A M \xrightarrow{\mu} M \longrightarrow 0$$

where $AK = 0$ by 2). This gives the exact sequence

$$A \otimes_A K \longrightarrow A \otimes_A A \otimes_A M \xrightarrow{1 \otimes \mu} A \otimes_A M \longrightarrow 0$$

where $A \otimes_A K = 0$ by 1). Now $1 \otimes \mu : (a_1, a_2, m) \mapsto (a_1, a_2 m) = (a_1 a_2, m)$ is the same as μ for the module $A \otimes_A M$. Thus $A \otimes_A A \otimes_A M \xrightarrow{\sim} A \otimes_A M$, $(a, (a_1, m)) \longmapsto (a a_1, m)$.

4) $A \otimes_A A \otimes_A M$ <u>is good.</u>

Pf. Either because $A(A\otimes_A M) = A\otimes_A M$ and 3), or because $A\otimes_A A$ is good by 3) hence
$$A\otimes_A(A\otimes_A A\otimes_A M) = \underline{A\otimes_A(A\otimes_A A)}\otimes_A M = A\otimes_A A\otimes_A M.$$

Suppose $M \in Mod(A)$, $N \in Modg(A)$.

5) $\quad Hom_A(N, A\otimes_A M) \xrightarrow{\sim} Hom_A(N, M)$

Proof. Consider
$$0 \to K \to A\otimes_A M \to M$$
This gives
$$0 \to Hom_A(N,K) \to Hom_A(N, A\otimes_A M) \to Hom_A(N,M)$$
where $Hom_A(N,K) = Hom_{\mathbb{Z}}(N/AN, K) = 0$, since N good $\Rightarrow AN = N$. It this suffices to show that any module map $N \xrightarrow{f} M$ lifts to $N \to A\otimes_A M$. But this is clear from
$$\begin{array}{ccc} A\otimes_A N & \xrightarrow{1\otimes f} & A\otimes_A M \\ \cong\downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

6) $\quad Hom_A(N, A\otimes_A A\otimes_A M) \xrightarrow{\sim} Hom_A(N, M)$

Pf. Combine 5) for M and $A\otimes_A M$.

7) <u>One has adjoint functors</u>
$$Modg(A) \underset{A\otimes_A A\otimes_A -}{\overset{}{\rightleftarrows}} Mod(A)$$
<u>where the upper is left adjoint to the lower.</u>

Immediate from 6).

Our goal next is to show the additive category of good modules is abelian. For this we want to calculate the kernel + cokernel of a map of good modules in the category $\mathrm{Modg}(A)$.

Let $f : N_1 \to N_2$ be a map in $\mathrm{Modg}(A)$, form the exact sequence in $\mathrm{Mod}(A)$

$$0 \longrightarrow K \xrightarrow{i} N_1 \xrightarrow{f} N_2 \xrightarrow{p} C \longrightarrow 0$$

where (K, i) and (C, p) are the kernel + cokernel of f in $\mathrm{Mod}(A)$. Let $A' = A \otimes_A A$.

8) <u>C is good and</u> (C, p) <u>is the cokernel of f in</u> $\mathrm{Modg}(A)$.

Proof.
$$\begin{tikzcd}
A \otimes_A N_1 \arrow[r] \arrow[d, "\cong"] & A \otimes_A N_2 \arrow[r] \arrow[d, "\cong"] & A \otimes_A C \arrow[r] \arrow[d] & 0 \\
N_1 \arrow[r] & N_2 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

implies C is good. One has the exact sequence

$$0 \to \mathrm{Hom}_A(C, M) \to \mathrm{Hom}_A(N_1, M) \to \mathrm{Hom}_A(N_2, M)$$

for all modules M, in particular, for all good modules, showing C is the cokernel of f in $\mathrm{Modg}(A)$.

9) <u>The kernel of</u> f <u>in</u> $\mathrm{Modg}(A)$ <u>is the composite map</u> $A' \otimes_A K \longrightarrow K \xrightarrow{i} N_1$.

Pf. For all good modules N we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(N, K) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(N, N_1) \arrow[r] & \mathrm{Hom}_A(N, N_2) \\
& \mathrm{Hom}_A(N, A' \otimes_A K) & &
\end{tikzcd}$$

using 6). qed

Next let I be the image of f in $\mathrm{Mod}(A)$ so that we have exact sequences

$$0 \longrightarrow K \xrightarrow{i} N_1 \xrightarrow{g} I \longrightarrow 0$$

$$0 \longrightarrow I \xrightarrow{j} N_2 \xrightarrow{p} C \longrightarrow 0$$

with $jg = f$.

From 8) one has $\mathrm{Cok}_g(f) = C$, and from 9) one has $\mathrm{Ker}_g(f) = A' \otimes_A K$, where the g subscript denotes Ker, Cok in $\mathrm{Mod}_g(A)$. Recall that the image and coimage are defined to be

$$\mathrm{Im}_g(f) = \mathrm{Ker}_g\{N_2 \longrightarrow \mathrm{Cok}_g(f)\}$$

$$\mathrm{Coim}_g(f) = \mathrm{Cok}_g\{\mathrm{Ker}_g(f) \longrightarrow N_1\}$$

and that an additive category (assumes existence of 0 and finite direct sums = finite direct products) is abelian when $\mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$ for all f.

We have

$$\mathrm{Im}_g(f) = \mathrm{Ker}_g\{N_2 \longrightarrow C\} = A' \otimes_A I$$

$$\mathrm{Coim}_g(f) = \mathrm{Cok}_g\{A' \otimes_A K \longrightarrow N_1\} = \mathrm{Cok}\{A' \otimes_A K \to N_1\}$$

But from

$$\begin{tikzcd}
 & A' \otimes_A K \arrow[r] \arrow[d] & A' \otimes_A N_1 \arrow[r] \arrow[d, "\cong"] & A' \otimes_A I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & N_1 \arrow[r] & I \arrow[r] & 0
\end{tikzcd}$$

we see $\mathrm{Cok}\{A' \otimes_A K \to N_1\} \simeq A' \otimes_A I$. Thus $\mathrm{Coim}_g(f) \xrightarrow{\sim} \mathrm{Im}_g(f)$ proving

10) $\mathrm{Mod}_g(A)$ <u>is an abelian category</u>.

11) <u>The functor $M \mapsto A' \otimes_A M$ from $Mod(A)$ to $Modg(A)$ is exact.</u>

Pf. Because this functor is a right adjoint we know it commutes with arbitrary projective limits, so we only have to see that if $M_1 \twoheadrightarrow M_2$ is a surjection in $Mod(A)$, then $A' \otimes_A M_1 \to A' \otimes_A M_2$ is a surjection in $Modg(A)$. But this is clear because the cokernel in $Modg(A)$ of $A' \otimes_A M_1 \to A' \otimes_A M_2$ is its cokernel in $Mod(A)$.

<u>April 22, 1994:</u>

Let's now discuss limits in $Modg(A)$.

Let $i \mapsto M_i, N_i$ denote functors from a small category I to $Mod(A)$, $Modg(A)$ resp., write $\varinjlim_g N_i$, $\varprojlim_g N_i$ for the inductive and projective limits in $Modg(A)$ when these exist.

Let $F : Modg(A) \hookrightarrow Modg(A)$ be the inclusion functor and $G(M) = A' \otimes_A M$ its right adjoint.

12) <u>$Modg(A)$ is closed under projective limits and G respects projective limits. One has</u>

$$\varprojlim_g N_i = A' \otimes_A \varprojlim N_i$$

$$\varprojlim_g A' \otimes_A M_i = A' \otimes_A \varprojlim M_i$$

<u>Proof:</u> $\mathrm{Hom}(N, G(\varprojlim M_i)) = \mathrm{Hom}(F(N), \varprojlim M_i)$

$= \varprojlim \mathrm{Hom}(F(N), M_i) = \varprojlim \mathrm{Hom}(N, G(M_i))$

$= \mathrm{Hom}(N, \varprojlim_g G(M_i))$. This shows the existence of $\varprojlim_g G(M_i)$, and that it is $G(\varprojlim M_i)$. Taking

$M_i = F(N_i)$ gives the existence of $\varprojlim_g N_i$ and that it is $G(\varprojlim F(N_i))$.

13) Modg(A) <u>is closed under inductive limits</u> <u>and</u> F <u>respects inductive limits</u>:

$$\varinjlim_g N_i = \varinjlim N_i$$

Pf. Existence of $\varinjlim_g N_i$ follows from existence of direct sums, since cokernels exist in any abelian category:

$$\bigoplus_{i \to j}{}_g N_i \longrightarrow \bigoplus_{i}{}_g N_i \longrightarrow \varinjlim_g N_i \longrightarrow 0$$

But $\bigoplus_i N_i$ is clearly good, so

$$\bigoplus_i{}_g N_i = \bigoplus_i N_i$$

The fact that F respects inductive limits follows because it is a <u>left</u> adjoint functor.

Remark: The only content to 12) + 13) is the existence of the limits. The rest is obvious properties of adjoint functors. The real surprize is the following

14) G <u>respects arbitrary inductive limits</u>:

$$A' \otimes_A \varinjlim M_i \xleftarrow{\sim} \varinjlim_g A' \otimes_A M_i$$

<u>Proof</u>: Since $\varinjlim_g A' \otimes_A M_i = \varinjlim A' \otimes_A M_i$ by 13), this is clear from the fact that $X \otimes_A -$ is a left adjoint

---

Next let's show Modg(A) satisfies

Grothendieck's AB5 axiom. I think this says that if N_i is a filtering family of subobjects of N and if N' is a subobject of N, then

$$15) \qquad N' \cap \bigcup_i N_i = \bigcup_i (N' \cap N_i)$$

I know that AB5 $\Longleftrightarrow$ filtered inductive limits are exact. Observe that 15) is a consequence of filtered $\varinjlim$'s exact:

$$N'/N' \cap N_i \longrightarrow N/N_i \quad \text{is a monom.}$$

$$\Longrightarrow \quad \varinjlim N'/N' \cap N_i \longrightarrow \varinjlim N/N_i \quad \text{is a monom.}$$

$$\begin{array}{ccc} \| & & \| \\ N'/\bigcup (N' \cap N_i) & & N/\bigcup N_i \end{array}$$

$$\Longrightarrow \quad N' \cap \bigcup N_i = \bigcup (N' \cap N_i)$$

We claim

16) <u>In Modg(A) filtered inductive limits are exact, i.e. one has axiom AB5.</u>

<u>Proof</u>: Let $N_i' \longrightarrow N_i$ be a filtered system of ~~[illegible]~~ maps in $\mathrm{Modg}(A)$, and let K_i be the kernel of the map $N_i' \to N_i$ in $\mathrm{Mod}(A)$, so that

$$0 \longrightarrow K_i \longrightarrow N_i' \longrightarrow N_i$$

is exact in $\mathrm{Mod}(A)$. Then $N_i' \to N_i$ is a monom. in $\mathrm{Modg}(A)$ iff $AK_i = 0$. Assuming this for all i we have

$$\begin{array}{ccccccc} & & & & \varinjlim_g N_i' & \longrightarrow & \varinjlim_g N_i \\ & & & & \| & & \| \\ 0 & \longrightarrow & \varinjlim_i K_i & \longrightarrow & \varinjlim N_i' & \longrightarrow & \varinjlim N_i \end{array}$$

The point is that $AK_i = 0 \Rightarrow A \varinjlim K_i = 0$, so that $\varinjlim_g N_i' \to \varinjlim_g N_i$ is a monom. in $\mathrm{Modg}(A)$.

---

17) A' <u>is a generator for</u> $\mathrm{Modg}(A)$.

Indeed if N is good, then choose a surjection $\bigoplus_I \tilde{A} \twoheadrightarrow N$ in $\mathrm{Mod}(A)$. Since $AN=0$, one has a surjection of good modules

$$\bigoplus_I A' = A' \otimes_A \Big(\bigoplus_I \tilde{A}\Big) \twoheadrightarrow A' \otimes_A N = N$$

which we know is an epimorphism in $\mathrm{Modg}(A)$.

18) $\mathrm{Modg}(A)$ <u>has sufficiently many injectives.</u>

Follows from Grothendieck AB5 + generators $\Rightarrow \exists$ enough injectives.

Next we would like to give ~~[illegible]~~ another proof of 18) using existence of injective hulls in $\mathrm{Mod}(A)$. Recall the basic facts about these, fixing the setting to be the category (unital) modules over a unital ring.

An injection $M \hookrightarrow N$ is called <u>essential</u> ~~[illegible]~~ when for any submodule $Y \subset N$ one has $Y \neq 0 \Rightarrow Y \cap M \neq 0$. Composition of essential injections is an essential injection, as well the inductive limit of a filtering family $M \hookrightarrow N_i$.

M <u>is injective</u> $\Leftrightarrow$ <u>every essential injection $M \hookrightarrow N$ is an isomorphism.</u> $\Leftarrow$ because $N = M \oplus Y$. $\Rightarrow$ choose an embedding $M \hookrightarrow I$ with I injective; ~~[illegible]~~ by Zorn $\exists\, Y \subset I$ maximal such that $Y \cap M = 0$ (this uses AB5). Then $M \hookrightarrow I/Y$ is essential, so $M \simeq I/Y$, i.e. $M \oplus Y = I$ so M is injective.

Given M choose embedding $M \hookrightarrow I$ with I injective. By Zorn $\exists$ a maximal N with $M \subset N \subset I$ such that $M \hookrightarrow N$ is essential. If $N \hookrightarrow N_1$ is an essential injection, then $\exists$ a comm. triangle

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, phantom, "\cap"] & N_1 \arrow[dl, dashed, "f"] \\ I & \end{tikzcd}$$

since I is injective. The dotted arrow f is injective since its kernel intersects N trivially and $N \to N_1$ is essential. Then $N \subset f(N_1)$ is essential and by maximality of N one has $N = f(N_1)$, and so $N \xrightarrow{\sim} N_1$. Thus N is an injective module.

This shows the existence for any M of an <u>injective hull</u> $M \hookrightarrow I$, which can be characterized either as a maximal essential injection, or as a minimal embedding into an injective. The injective hull is determined up to non-canonical isomorphism.

Return now to $\mathrm{Modg}(A)$.

19) <u>Let I be an injective A-module such that</u> $\mathrm{Hom}_A(\mathbb{Z}, I) = \{y \in I \mid Ay = 0\} = 0$. <u>Then</u> $A' \otimes_A I$ <u>is injective in</u> $\mathrm{Modg}(A)$.

<u>Proof.</u> The functor $M \mapsto \mathrm{Hom}_A(M, I)$ is exact and it kills null modules, so it gives rise to an <u>exact</u> functor on $\mathrm{Modg}(A)$. Thus

$$\mathrm{Hom}_A(N, I) = \mathrm{Hom}_A(N, A' \otimes_A I)$$

is an exact functor of $N \in \mathrm{Modg}(A)$, which means that $A' \otimes_A I$ is injective in $\mathrm{Modg}(A)$.

Let's another proof of the existence of enough injectives in $Modg(A)$. Let N be a good module, let $\mathrm{ann}_A(N) =$ $\{n \in N \mid An = 0\}$, let I be "the" injective hull of $N/\mathrm{ann}_A(N)$:

$$N/\mathrm{ann}_A(N) \hookrightarrow I \qquad \text{essential injection with } I \text{ injective } A\text{-module.}$$

Then $\mathrm{ann}_A(I) \cap N/\mathrm{ann}_A(N) = 0 \implies \mathrm{ann}_A(I) = 0$, so $A' \otimes_A I$ is injective in $Modg(A)$. Moreover the kernel of $N \longrightarrow A' \otimes_A I$ is contained in $\mathrm{ann}_A(N)$, so this map is a monom. in $Modg(A)$.

Actually it seems we can improve 19) to a description of injectives in $Modg(A)$.

20) <u>Let J be an injective in</u> $Modg(A)$. <u>Then</u> $\mathrm{Hom}_A(A', J)$ <u>is an injective</u> A-<u>module whose A-annihilator is zero and one has</u> $J \simeq A' \otimes_A \mathrm{Hom}(A', J)$.

Pf. Let Φ be the functor on $Mod(A)$ defined by $\Phi(M) = \mathrm{Hom}_A(A' \otimes_A M, J)$. This is the composite of the exact functor $M \longmapsto A' \otimes_A M$ from $Mod(A)$ to $Modg(A)$ and the exact functor $\mathrm{Hom}(-, J)$ on $Modg(A)$. Since

$$\Phi(M) = \mathrm{Hom}_A(A' \otimes_A M, J) = \mathrm{Hom}_A(M, \mathrm{Hom}_A(A', J))$$

it follows that $\mathrm{Hom}_A(A', J)$ is an injective A-module. Since $\Phi(M) = 0$ when M is a null module, the A-annihilator of $\mathrm{Hom}_A(A', J)$ is zero. Finally restricting to good modules we have

$$\Psi(N) = \mathrm{Hom}_A(N, J)$$

$$\Phi(N) = \mathrm{Hom}_A(N, \mathrm{Hom}_A(A', J))$$

$$= \mathrm{Hom}_A(N, A' \otimes_A \mathrm{Hom}_A(A', J))$$

so $J \simeq A' \otimes_A \mathrm{Hom}_A(A', J)$.

What is happening here is that there is another adjoint:

$$\begin{tikzcd} \mathrm{Modg}(A) \arrow[r, hook, shift left=2] & \mathrm{Mod}(A) \arrow[l, "A' \otimes_A -"] \\ & \end{tikzcd}$$

$$\xrightarrow[\mathrm{Hom}_A(A', -)]{}$$

such that 2) $\boxed{A' \otimes_A \mathrm{Hom}_A(A', N) \xrightarrow{\sim} N}$. In effect

$$\mathrm{Hom}_A(N_1, A' \otimes_A \mathrm{Hom}_A(A', N)) = \mathrm{Hom}_A(N_1, \mathrm{Hom}_A(A', N))$$

$$= \mathrm{Hom}_A(A' \otimes_A N_1, N) = \mathrm{Hom}_A(N_1, N).$$

The point is that instead of the functor $M \mapsto A \otimes_A M$ one can also use $M \mapsto \mathrm{Hom}_A(A, M)$ to give a parallel treatment.

1)' M is null $\iff$ $\mathrm{Hom}_A(A, M) = 0$

Pf: One has the exact sequence

$$0 \longrightarrow \mathrm{ann}_A(M) \longrightarrow M \xrightarrow{\phi} \mathrm{Hom}_A(A, M)$$

$$\phi(m)(a) = am$$

M is null $\iff$ $\mathrm{ann}_A(M) = M$, so the implication $\Leftarrow$ is clear, Conversely if M is null, and if $f \in \mathrm{Hom}_A(A, M)$, then $f(a_1 a_2) = a_1 f(a_2) = 0$ so $f = 0$ as $A = A^2$.

2') <u>The kernel and cokernel of $\underbrace{\phi}_{M \to \mathrm{Hom}_A(A,M)}$ are null.</u>

Pf. $\mathrm{ann}_A(M)$ is null obviously. If $f \in \mathrm{Hom}_A(A,M)$, then af is by definition $a' \mapsto f(a'a)$. But $f(a'a) = a'f(a) = \phi(f(a))(a')$. Thus $af = \phi(f(a))$ showing that multiplication by a is zero on the cokernel of ϕ.

Digress to point out that from

$$0 \to A \to A^+ \to \mathbb{Z} \to 0$$

one gets

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^A(A,M) \arrow[r] & A \otimes_A M \arrow[r] \arrow[dr, "\mu"'] & A^+ \otimes_A M \arrow[r] \arrow[d, equal] & \mathbb{Z} \otimes_A M \arrow[r] \arrow[d, equal] & 0 \\
& & & M \arrow[r] & M/AM \arrow[r] & 0
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(\mathbb{Z}, M) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(A^+, M) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(A, M) \arrow[r] & \mathrm{Ext}^1_A(\mathbb{Z}, M) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{ann}_A(M) \arrow[r] & M \arrow[ur, "\phi"'] & & &
\end{tikzcd}$$

Call M r-good when ϕ is an isomorphism

3') $\mathrm{ann}_A(M) = 0 \implies \mathrm{Hom}_A(A, M)$ is r-good

Pf. $0 \to M \to \mathrm{Hom}_A(A, M) \to \underset{\text{null}}{C} \to 0$

yields

$$0 \to \mathrm{Hom}_A(A, M) \to \mathrm{Hom}_A(A, \mathrm{Hom}_A(A, M)) \to \underset{\overset{\parallel}{0} \text{ by } 1')}{\mathrm{Hom}_A(A, C)}$$

4') $\mathrm{Hom}_A(A, \mathrm{Hom}_A(A, M)) = \mathrm{Hom}_A(A \otimes_A A, M)$ is r-good.

Why? Let $f \in \mathrm{Hom}_A(A, M)$ be such that $Af = 0$. Then $(af)(a') = f(a'a) = 0$ for all $a, a' \Rightarrow f = 0$, as $A^2 = A$.

5') Assume N is r-good. Then

$$\mathrm{Hom}_A(\mathrm{Hom}_A(A, M), N) \xrightarrow{\sim} \mathrm{Hom}_A(M, N)$$

Proof: One has

$$0 \longrightarrow K \longrightarrow M \longrightarrow \mathrm{Hom}_A(A, M) \longrightarrow C \longrightarrow 0$$

where K, C are null. This yields

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{Hom}_A(C, N) & \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(A, M), N) \longrightarrow \mathrm{Hom}_A(M, N) \\ & \| & \\ & \mathrm{Hom}_{\mathbb{Z}}(C, \underset{\substack{\| \\ 0}}{\mathrm{ann}_A(N)}) & \end{array}$$

so it suffices to extend any $u \in \mathrm{Hom}_A(M, N)$ to $\mathrm{Hom}_A(A, M) \longrightarrow N$. But clear from

$$\begin{tikzcd} M \arrow[r] \arrow[d, "u"'] & \mathrm{Hom}_A(A, M) \arrow[d, "u_*"] \\ N \arrow[r, "\sim"] & \mathrm{Hom}_A(A, N) \end{tikzcd}$$

6') $\mathrm{Hom}_A(\underbrace{\mathrm{Hom}_A(A, \mathrm{Hom}_A(A, M))}_{\mathrm{Hom}_A(A', M)}, N) = \mathrm{Hom}_A(M, N)$

Existence of enough good flat modules.

Let M be an A-module such that $AM = M$. I want to show M is a quotient of a good flat A-module. Starting from a finite subset $m^0_{i_0}$, $1 \le i_0 \le n_0$ we can using $AM=M$ construct successive factorizations

$$m^0_{i_0} = a^1_{i_0 i_1} m^1_{i_1} \qquad 1 \le i_1 \le n_1$$

$$m^1_{i_1} = a^2_{i_1 i_2} m^2_{i_2} \qquad 1 \le i_2 \le n_2$$

This gives the following diagram

$$\begin{tikzcd}
A^{n_0} \subset \tilde{A}^{n_0} \arrow[r, "\cdot a^1"] \arrow[dr, "\cdot m^0"'] & A^{n_1} \subset \tilde{A}^{n_1} \arrow[r, "\cdot a^2"] \arrow[d, "\cdot m^1"] & A^{n_2} \subset \tilde{A}^{n_2} \arrow[r] \arrow[dl, "\cdot m^2"] & \cdots \\
& M & &
\end{tikzcd}$$

The inductive limit E of the sequence at the top is flat since $E = \varinjlim \tilde{A}^{n_i}$, and it satisfies $AE = E$ since $E = \varinjlim A^{n_i}$. One has a map $E \longrightarrow M$ whose image contains the submodule generated by the $m^0_{i_0}$. Then it's clear that by starting from enough finite subsets to generate M and taking the direct sum of the E's we can write M as a quotient of a flat A module E such that $AE = E$. Then E is good since $0 \to A \to \tilde{A} \to \mathbb{Z} \to 0$ yields

$$0 \to \underset{\substack{\| \\ 0}}{\mathrm{Tor}_1^A(\mathbb{Z}, E)} \to A \otimes_A E \to E \to \underset{\substack{\| \\ 0}}{E/AE} \to 0$$

I next want to use the existence of enough flat good modules to construct the left-derived functors L_nF of the inclusion

$$F : \mathrm{Modg}(A) \hookrightarrow \mathrm{Mod}(A)$$

1) <u>Let</u> E <u>be a flat good</u> A^{op} <u>module</u>. <u>Then</u> $M \mapsto E \otimes_A M$, $\mathrm{Mod}(A) \to \mathrm{Mod}(\mathbb{Z})$ <u>is exact, and it kills</u> $\mathcal{N}$, <u>hence it induces an exact functor</u> $\mathrm{Modg}(A) \to \mathrm{Modu}(\mathbb{Z})$, $N \mapsto E \otimes_A N$.

The point is that if $AM = 0$, then $E \otimes_A M = EA \otimes_A M = E \otimes_A AM = 0$, and the rest is obvious.

2) <u>Let</u> $X \in \mathrm{Mod}(A^{op})$. <u>Then</u> $M \mapsto X \otimes_A M$, $\mathrm{Mod}(A) \to \mathrm{Modu}(\mathbb{Z})$ <u>descends to</u> $\mathrm{Mod}(A)/\mathcal{N} \iff X \otimes_A A \simeq X$.

Pf. $(\Longrightarrow)$ $A \hookrightarrow \tilde{A}$ becomes an isom. in $\mathrm{Mod}(A)/\mathcal{N}$, so $X \otimes_A A \xrightarrow{\sim} X \otimes_A \tilde{A} = X$.

$(\Longleftarrow)$ one has $X \otimes_A A \simeq X \Rightarrow X \otimes_A A' \simeq X$, so $X \otimes_A M = X \otimes_A (A' \otimes_A M)$. Now use the fact that the inverse of $\mathrm{Modg}(A) \xrightarrow{\sim} \mathrm{Mod}(A)/\mathcal{N}$ is $M \mapsto A' \otimes_A M$.

Recall that for a unital algebra R then an additive functor $F : \mathrm{Modu}(R) \to \mathrm{Modu}(\mathbb{Z})$ has the form $F(M) = X \otimes_R M$ for some unital R^{op} module iff F commutes with arbitrary $\varinjlim$'s. (equivalently F is right exact and commutes with direct sums.) In effect for any F we have a morphism of functors

$$3) \qquad \begin{array}{ccc} F(R) \otimes_R M & \longrightarrow & F(M) \\ \xi \otimes m & \longmapsto & \xi(\cdot m)_* \end{array}$$

where $\cdot m : R \to M$ is $r \mapsto rm$ and $\xi \mapsto \xi(\cdot m)_* : F(R) \to F(M)$ denotes the induced map.

The map 3) is an isomorphism for $M = R$, hence if F commutes with $\oplus$'s it is an isom. for free M, and if F also is right exact, it is an isomorphism for any M which is a cokernel of a map between free modules, i.e. any M.

At this point, given two rings A, B such that $A^2 = A$ and $B^2 = B$, we can describe all additive functors

$$\mathrm{Mod}(A)/\mathcal{N}_A \longrightarrow \mathrm{Mod}(B)/\mathcal{N}_B$$

commuting with $\varinjlim$'s as follows. Compose with the equivalence $\mathrm{Mod}(B)/\mathcal{N}_B \xrightarrow{\sim} \mathrm{Modg}(B)$, $X \mapsto B' \otimes_B X$, and then with the inclusion $\mathrm{Modg}(B) \hookrightarrow \mathrm{Mod}(B)$ which we know commutes with $\varinjlim$'s (because its left adjoint to $Y \mapsto B' \otimes_B Y$). Then we have a functor

$$\mathrm{Mod}(A) \twoheadrightarrow \mathrm{Mod}(A)/\mathcal{N}_A \longrightarrow \mathrm{Mod}(B)/\mathcal{N}_B \xrightarrow{B' \otimes_B -} \mathrm{Mod}(B)$$

which commutes with $\varinjlim$'s. By the above remark this functor has the form

$$M \longmapsto X \otimes_A M$$

where X is a bimodule ${}_B X_A$; X is the image of $\tilde{A} \in Mod(A)$. The fact that this functor descends to $Mod(A)/\mathcal{N}_A$ is equivalent by 2) above to $X \otimes_A A \simeq X$. The fact that it has values in $Modg(B)$ means that $B \otimes_B X \xrightarrow{\sim} B$.

a simple example:

The bimodule ${}_A A'_A$ yields the inverse

$$Mod(A)/\mathcal{N}_A \longrightarrow Modg(A)$$

of the equivalence going the other way. Notice that this functor is exact, but the bimodule is not necessarily A^{op} flat. (So I don't understand yet exact functors compatible with $\varinjlim$'s from $Mod(A)/\mathcal{N}_A$ to $Mod(B)/\mathcal{N}_B$.)

Let's return to the problem of the left derived functors L_nF, where $F : Modg(A) \hookrightarrow Mod(A)$ is the inclusion.

Note that the composition

$$Mod(A) \xrightarrow{G = A' \otimes_A -} Mod/\mathcal{N}_A \simeq Modg(A) \overset{F}{\hookrightarrow} Mod(A)$$

is given by the bimodule ${}_A A'_A$ so the first guess would be that L_nF is given by $Tor_n^A(A', -)$. But the difficulty here is that the composite functor situation FG is not good, G does not take projectives to F-acyclic objects, e.g. $G(\tilde{A}) = A'$ is not F-acyclic, as we should see eventually.

However we can construct flat resolutions for objects in $\mathrm{Modg}(A)$.

Let $M \in \mathrm{Mod}(A)$, put $M_0 = \overbrace{A\otimes_A M \text{ or }}\, AM$

Then $A \otimes_A M_0$ is good, by p. 531
there is a surjection $P_0 \twoheadrightarrow A\otimes_A M_0$ with P_0 good and flat. Let M_1 be the kernel of $P_0 \twoheadrightarrow A\otimes_A M_0$ in $\mathrm{Mod}(A)$, so that we have

$$\begin{tikzcd}
 & A\otimes_A M_1 \arrow[r] \arrow[d] & A\otimes_A P_0 \arrow[r] \arrow[d, "\cong"] & A\otimes_A M_0 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & P_0 \arrow[r] & A\otimes_A M_0 \arrow[r] & 0
\end{tikzcd}$$

This shows $AM_1 = M_1$, so that we can repeat the process to construct exact sequences

$$0 \longrightarrow M_2 \longrightarrow P_1 \longrightarrow A\otimes_A M_1 \longrightarrow 0$$

$$0 \longrightarrow M_3 \longrightarrow P_2 \longrightarrow A\otimes_A M_2 \longrightarrow 0$$

$$\cdots$$

where $AM_n = M_n$ and P_n are good and flat $\forall n$. We then have a complex in $\mathrm{Mod}(A)$

$$\longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

consisting of good flat modules, such that

$$H_0(P_\bullet) = A\otimes_A M_0 = A'\otimes_A M$$

$$H_n(P_\bullet) = \mathrm{Ker}\{A\otimes_A M_n \twoheadrightarrow M_n\} \qquad n \geq 1$$

Thus $P_\bullet$ is a flat resolution of $A'\otimes_A M$ in $\mathrm{Modg}(A)$; or $P_\bullet$ is a flat resolution of M in $\mathrm{Mod}(A)/\mathcal{N}_A$.

Similarly if $X \in \mathrm{Mod}(A^{op})$, let $X_0 = XA$ or $X \otimes_A A$ and construct

$$0 \longrightarrow X_1 \longrightarrow E_0 \longrightarrow X_0 \otimes_A A \longrightarrow 0$$

$$0 \longrightarrow X_2 \longrightarrow E_1 \longrightarrow X_1 \otimes_A A \longrightarrow 0$$

$$\cdots$$

inductively such that E_n is A^{op} flat + good and $X_n A = X_n$ for all $n \geq 0$. This gives a complex of right flat good modules $E_\bullet$ such that

$$H_0(E_\bullet) = X_0 \otimes_A A = X \otimes_A A'$$

$$H_n(E_\bullet) = \mathrm{Ker}\{X_n \otimes_A A \twoheadrightarrow X_n\} \qquad n \geq 1.$$

Take $X = A$ above and consider the bicomplex $E_p \otimes_A P_q$. For p fixed

$$H_q(E_p \otimes_A P_\bullet) = E_p \otimes_A H_q(P_\bullet) \qquad \text{because } E_p \text{ flat}$$

$$= \begin{cases} E_p \otimes_A A' \otimes_A M = E_p \otimes_A M & q = 0 \\ 0 & q > 0 \end{cases}$$

because for $q > 0$ $H_q(P_\bullet) \in \mathcal{N}$ and $E_p A = E_p$.

For q fixed

$$H_p(E_\bullet \otimes_A P_q) = H_p(E_\bullet) \otimes_A P_q$$

$$= \begin{cases} A' \otimes_A P_q = P_q & p = 0 \\ 0 & p > 0 \end{cases}$$

Thus we get canonical isomorphisms

$$H_n(P_\bullet) = H_n(E_\bullet \otimes_A P_\bullet) = H_n(E_\bullet \otimes_A M)$$

Where
$$H_0(P_\cdot) = H_0(E_\cdot \otimes_A M) = A' \otimes_A M .$$

At this point we have constructed the left derived functors of the inclusion
$$\mathrm{Mod}(A)/\mathcal{N}_A \simeq \mathrm{Mod}_g(A) \hookrightarrow \mathrm{Mod}(A).$$
$$M \longmapsto A' \otimes_A M$$

Namely $M \longmapsto E_\cdot \otimes_A M$ is an exact functor from $\mathrm{Mod}(A)/\mathcal{N}_A$ to complexes of abelian groups, so the homology $H_n(E_\cdot \otimes_A M)$ is a connected sequence of functors on $\mathrm{Mod}(A)/\mathcal{N}_A$ reducing to the above inclusion for $n = 0$. If M is good flat, then for $n > 0$
$$H_n(E_\cdot \otimes_A M) = H_n(E_\cdot) \otimes_A M = 0$$
because $H_n(E_\cdot)$ is null and $M = AM$. Thus $H_n(E_\cdot \otimes_A M)$ is effaceable for $n > 0$.

(Notice that M flat in $\mathrm{Mod}(A) \not\Rightarrow$ M flat in $\mathrm{Mod}(A)/\mathcal{N}_A$ in the sense that the good module $A' \otimes_A M$ corresponding to M is flat.)

Here's another way (to see the existence of L_nF); suppose one is given a short exact sequence in $\mathrm{Mod}_g(A)$: $N_0 \rightarrowtail N_1 \twoheadrightarrow N_2$

This means that $N_1 \to N_2$ is a surjective map of good modules and $N_0 = A \otimes_A K$ where K is the kernel of $N_1 \to N_2$ in $\mathrm{Mod}(A)$.

From

$$0 \to K \to N_1 \to N_2 \to 0$$

we get

$$\mathrm{Tor}_1^A(A, N_1) \to \mathrm{Tor}_1^A(A, N_2) \to \overbrace{A \otimes_A K}^{N_0} \to N_1 \to N_2 \to 0$$

If N_2 is flat then

$$0 \to \overbrace{A \otimes_A K}^{N_0} \to N_1 \to N_2 \to 0$$

is exact. This shows that flat good ~~[illegible]~~ modules are acyclic for F. Moreover if N_1 is flat, then we have

$$0 \to \mathrm{Tor}_1^A(A, N_2) \to \overset{N_0}{\overset{\shortparallel}{A \otimes_A K}} \to N_1 \to N_2 \to 0$$

This shows that we have

$$\boxed{L_1 F(N) = \mathrm{Tor}_1^A(A, N)}$$

Now that we know good flat modules are acyclic for F and that there are enough of them, we get the existence of $L_n F$ for all n and the fact that $L_n F(N) = H_n(P_\cdot)$, where $P_\cdot$ is the sort of resolution constructed before:

$$0 \to K_1 \to P_0 \to N \to 0$$

$$0 \to K_2 \to P_1 \to A \otimes_A K_1 \to 0$$

Thus $$L_n F(N) = \mathrm{Ker}\{A \otimes_A K_n \to K_n\} \qquad n \geq 1.$$

April 24, 1994

Book: Torsion Theories by Jonathan Golan.
R unital, R-mod unital left R-modules
mod-R unital right R-modules

A torsion theory $\mathcal{T}$ over R can be defined as a Serre subcategory of R-mod which is closed under direct sums.

A torsion theory $\mathcal{T}$ is called jansian if it is closed under arbitrary products.

Prop: Jansian torsion theories correspond bijectively to ideals $A \subset R$ such that $A^2 = A$.

Proof: Given A let $\mathcal{T}_A = \{M \in R\text{-mod} \mid AM = 0\}$. This is a jansian torsion theory: note that if $AM_j = 0$, then $A \cdot \prod M_j = 0$, because if $a \in A$, then $a \prod M_j \subset \prod a M_j = 0$.

Given a jansian $\mathcal{T}$, let S be the set of cyclic modules $R/\mathfrak{a}$, $\mathfrak{a}$ a left ideal, such that $R/\mathfrak{a} \in \mathcal{T}$. Let $A = \bigcap_{\mathfrak{a} \in S} \mathfrak{a}$. Then we have

$$R/A \hookrightarrow \prod_{\mathfrak{a} \in S} R/\mathfrak{a} \in \mathcal{T}$$

so $R/A \in \mathcal{T}$. Given $m \in M \in \mathcal{T}$ we have $R/\mathfrak{a} \simeq Rm \subset M$, so $Am = 0$. Thus $A \subset$ the intersection of all annihilators of modules in $\mathcal{T}$. On the other hand the this intersection kills R/A so it is $\subset A$. Since the annihilator of module is an ideal, A is an ideal in R. One has $\mathcal{T} \subset \{M \mid AM = 0\}$, and the other inclusion $\{M \mid AM = 0\} \subset \mathcal{T}$ because any such M is an R/A module, thus a quotient of $\oplus R/A$, thus in $\mathcal{T}$. Finally $A^2 = A$, because the exact sequence

$$0 \to A/A^2 \to R/A^2 \to R/A \to 0$$

shows $R/A^2 \in \mathcal{T}$, so $A(R/A^2) = 0 \Rightarrow A \subset A^2$.

In general a torsion theory $\mathcal{T}$ corresponds bijectively to a family of left ideals:

$$\mathcal{T} \longleftrightarrow \{\mathfrak{a} \mid R/\mathfrak{a} \in \mathcal{T}\}$$

called a <u>Gabriel filter</u>.

<u>Gabriel-Popescu theorem</u>: <u>Let $\mathcal{A}$ be a Grothendieck abelian category</u> (AB5 + generator), <u>let</u> U <u>be a generator, let</u> $R = \mathrm{End}(U)^{op}$. <u>Then</u> $h^U = \mathrm{Hom}(U, -) : \mathcal{A} \longrightarrow R\text{-mod}$ <u>has an exact adjoint, and this yields an equivalence</u>

$$R\text{-mod}/\mathcal{T} \simeq \mathcal{A}$$

<u>where</u> $\mathcal{T}$ <u>is a torsion theory on</u> R.

In other words Grothendieck categories are exactly those of the form $R\text{-mod}/\mathcal{T}$ corresponding to a torsion theory. Now I would like to describe those which correspond to Jansian $\mathcal{T}$. These should be the good categories of modules belonging to non unital A such that $A^2 = A$.

Here's an example of an A such that $A^2 = A$ such that there are no finitely generated good modules $\neq 0$. Take A to be germs of continuous functions on $\mathbb{R}$ at the origin which vanish there. In other words A is the maximal ideal in a local ring $R = \mathbb{C} \oplus A$ where $A^2 = A$. Then if M is finitely generated and $AM = M$, Nakayama's lemma $\Longrightarrow M = 0$. Recall the proof: finitely generated $\Longrightarrow$ $\exists$ maximal submodule in M, so we can suppose M simple, whence $M = \mathbb{C} = R/A$, which contradicts $AM = M$.

April 25, 1994

Let $A \subset I$ be nonunital rings, assume $A^2 = A$, $IA \subset A$ (whence $IA = A$), (thus A is a left ideal in I), $AI = I$, (I is the ideal in I generated by A), [redacted] whence $I^2 = I$.

<u>Claim the category</u> A-modg <u>of good</u> A-<u>modules and the category</u> I-<u>modg are canonically equivalent. In fact there is a canonical Morita equivalence between these categories.</u>

Pf. First note that if $N \in A\text{-modg}$: $A \otimes_A N \xrightarrow{\sim} N$ then one has a well-defined I-module structure on N defined by
$$\underset{I}{\overset{\cap}{x}}(\underset{A}{\overset{\cap}{a}} n) = (x \underset{A}{\overset{\cap}{a}}) n.$$

Also $AN = N \implies IN \supset N$ so $IN = N$, and consequently $I \otimes_I N$ is a good I-module.

If $M \in I\text{-modg}$, then $IM = M$, so [redacted] $AM = AIM = IM = M$. Consequently $A \otimes_A M$ is a good A-module.

Thus we have functors

$$\begin{array}{ccc} A\text{-modg} & \underset{G}{\overset{F}{\rightleftarrows}} & I\text{-modg} \\ N & \longmapsto & I \otimes_I N \\ A \otimes_A M & \longleftarrow & M \end{array}$$

Define [redacted] natural transformations

$$GF \longrightarrow 1 \qquad A \otimes_A I \otimes_I N \longrightarrow N$$
$$(a, x, n) \longmapsto axn$$

$$FG \longrightarrow 1 \qquad I \otimes_I A \otimes_A M \longrightarrow M$$
$$(x, a, m) \longmapsto xam$$

We show $GF \xrightarrow{\sim} 1$. One has an exact sequence

$$0 \longrightarrow K \longrightarrow I \otimes_I N \longrightarrow N \longrightarrow 0$$

of I-modules. We know $IK = 0$, hence $AK = 0$. This yields the right exact sequence

$$\begin{array}{ccccccc} A \otimes_A K & \longrightarrow & A \otimes_A I \otimes_I N & \xrightarrow{\sim} & A \otimes_A N & \longrightarrow & 0 \\ \| & & & & & & \\ 0 & & (a, x, n) & \longmapsto & (a, xn) & & \end{array}$$

But N good $\Longrightarrow$ $A \otimes_A N \xrightarrow{\sim} N$, done.

We show $FG \to 1$. One has an exact sequence

$$0 \longrightarrow K' \longrightarrow A \otimes_A M \longrightarrow M \longrightarrow 0$$

of I-modules. We know $AK' = 0$; now $\operatorname{ann}_I(K')$ is an ideal in I containing A; this ideal is $AI = I$, so $IK' = 0$. This yields

$$\begin{array}{ccccccc} & & (x, a, m) & & (x, am) & & \\ I \otimes_I K' & \longrightarrow & I \otimes_I A \otimes_A M & \longrightarrow & I \otimes_I M & \longrightarrow & 0 \\ \| & & & & \downarrow\wr & & \\ 0 & & & & M & xam & \end{array}$$

showing $FG \xrightarrow{\sim} 1$.

Finally check that $FGF \rightrightarrows F$, $GFG \rightrightarrows G$ coincide so the isos. $FG \xrightarrow{\sim} 1$, $GF \xrightarrow{\sim} 1$ are compatible:

$$\begin{array}{ccc} (x,\ a,\ x',\ m) & \longmapsto & (x, ax'm) \\ I \otimes_I A \otimes_A I \otimes_I N & \longrightarrow & I \otimes_I M \\ \downarrow & & \\ I \otimes_I N \quad (xax', n) & & \end{array}$$

and $(xax', n) = (x, ax'n)$ as $ax' \in AI = I$

$$A \otimes_A I \otimes_I A \otimes_A M \rightrightarrows A \otimes_A M$$

$$(a, x, a', m) \longmapsto (a, xa'm)$$
$$ \longmapsto (axa', m)$$

and these agree as $xa' \in IA = A$.

Finally the Morita equivalence is given by the bimodules:

$$A\text{-modg} \underset{(A \otimes_A I) \otimes_I -}{\overset{(I \otimes_I A) \otimes_A -}{\rightleftarrows}} I\text{-modg}$$

i.e. the bimodules $I \otimes_I A$, $A \otimes_A I$ which are a good I-module and A-module resp. The composites are

$$(a, x, a') \longmapsto (a, xa')$$
$$(A \otimes_A I) \otimes_I (I \otimes_I A) = A \otimes_A I \otimes_I A \xrightarrow{\sim} A \otimes_A A$$

$$(x, a, x') \longmapsto (x, ax')$$
$$(I \otimes_I A) \otimes_A (A \otimes_A I) = I \otimes_I A \otimes_A I \xrightarrow{\sim} I \otimes_I I$$

where the isomorphisms are proved as above.

April 27, 1994

Let $B \subset A$ be rings, assume $B^2 = B$, $B = BAB$, $A = ABA$. Then $A = ABA \subset A^3 \subset A^2 \subset A$ so $A = A^2$. Claim we have a Morita equivalence

$$\text{B-gmod} \rightleftarrows \text{A-gmod}$$

1) $$N \longmapsto (A \otimes_A AB) \otimes_B N$$
$$(B \otimes_B BA) \otimes_A M \longleftarrow M$$

Note that, as $A(AB) = AB$, $A \otimes_A AB$ is a good A-mod. Similarly $B \otimes_B BA$ is a good B-mod, so the above functors are defined.

We have maps joining the two composites to the identity given by the bimodule maps:

2) $$(B \otimes_B BA) \otimes_A (A \otimes_A AB) \longrightarrow B \otimes_B B$$
$$(b, w, a, v) \longmapsto (b, wav)$$

3) $$(A \otimes_A AB) \otimes_B (B \otimes_B BA) \longrightarrow A \otimes_A A$$
$$(a, v, b, w) \longmapsto (a, vbw)$$

Compatibility

$$(b, w, a, v, b', w') \longmapsto (b, w, a, vb'w')$$

$$\begin{tikzcd}
(B\otimes_B BA)\otimes_A (A\otimes_A AB)\otimes_B (B\otimes_B BA) \arrow[r] \arrow[d] & (B\otimes_B BA)\otimes_A (A\otimes A) \arrow[d, "{(b,\, wavb'w')}"] \\
(B\otimes_B B)\otimes_B (B\otimes_B BA) \arrow[r] & B\otimes_B BA
\end{tikzcd}$$

$$(b, wav, b', w') \qquad\qquad (bwavb', w') \quad \| \text{ since } wavb' \in B$$

$$\begin{tikzcd}
(A\otimes_A AB)\otimes_B(B\otimes_B BA)\otimes_A(A\otimes_A AB) \arrow[r] \arrow[d] & (A\otimes_A AB)\otimes_B(B\otimes_B B) \arrow[d] \\
(A\otimes_A A)\otimes_A(A\otimes_A AB) \arrow[r, mapsto] & A\otimes_A AB
\end{tikzcd}$$

$(a, v, b, w, a', v') \mapsto (a, v, b, wa'v')$

$(a, vbw, a', v') \longmapsto (avbwa', v)$

$(a, vbwa'v') = (avbwa', v')$ as $vbwa' \in A$

We now show that 2), 3) are isoms.

2): First have $A\otimes_A AB \longrightarrow AB$ surjective and its kernel K is such that $AK = 0$.

Then $BA\otimes_A K = 0$, so

$$BA\otimes_A A\otimes_A AB \xrightarrow{\sim} BA\otimes_A AB$$

$(w, a, v) \mapsto (w, av)$

Consider now the multiplication map

$$BA\otimes_A AB \longrightarrow B$$

which is surjective as $BA^2B = BAB = B$. Its kernel is killed by B: if $(w_i, v_i) \longmapsto w_iv_i = 0$, then $b(w_i, v_i) = (bw_i, v_i) = (b, w_iv_i) = 0$, using $w_i \in A$.

Thus

$$B\otimes_B BA\otimes_A AB \xrightarrow{\sim} B\otimes_B B$$

$(b, w, v) \longmapsto (b, wv)$

Combining these two isos, we see 2) is an isom.

3): First $B\otimes_B BA \longrightarrow BA$ is surjective and its kernel K is killed by B, so using $AB = (AB)B$ we have $AB\otimes_B K = 0$, yielding

$$AB\otimes_B B\otimes_B BA \xrightarrow{\sim} AB\otimes_B BA$$

Consider the multiplication map

$$AB\otimes_B BA \longrightarrow A$$

This is surjective as $AB^2A = ABA = A$; let K be the kernel. If $(v_i, w_i) \in K$, then $b_1 b_2 (v_i, w_i) = (b_1 b_2 v_i, w_i) = (b_1, b_2 v_i w_i) = 0$ in $AB \otimes_B BA$, using the fact that $b_2 v_i \in BAB = B$. Thus $BK = B^2 K = 0$. But K is an A-module so $AK = ABAK \subset ABK = 0$. We conclude then

$$A \otimes_A AB \otimes_B BA \xrightarrow{\sim} A \otimes_A A$$

and 3) follows by combining the above two isos.

Now suppose $B \subset R$ idempotent rings such that $BRB = B$. Let $A = RBR$. Then $B \subset A$,

$BA = BRBR = BR$, $\quad AB = RBRB = RB$,

$B = BBB \subset BAB \subset BRB = B \implies BAB = B$.

$A = RBR = RBBR = ABBA = ABA$.

Thus one has $B \subset A$, $B^2 = B$, $BAB = B$, $ABA = A$ so we see from the above that B-gmod is equivalent to A-gmod, where A is the ideal RBR generated by B in R.

Consider functors from [redacted] A-mod to $\mathcal{Ab}$ and ask when they descend to $\mathcal{a} = A\text{-mod}/A\text{-null}$.

[redacted] 1) $X \in \text{mod-}A$. Then $X \otimes_A - : A\text{-mod} \longrightarrow \mathcal{Ab}$ descends to $\mathcal{a} \iff X \in \text{gmod-}A : X \otimes_A A \xrightarrow{\sim} X$.

Pf. Recall $X \in \text{gmod-}A \iff X \otimes_A A^g \xrightarrow{\sim} X$ where $A^g = A \otimes_A A$. One a commutative triangle

$$\begin{tikzcd}[column sep=small] (X \otimes_A A^g) \otimes_A M \arrow[rr, equal] \arrow[dr, "\alpha"'] & & X \otimes_A (A^g \otimes_A M) \arrow[dl, "\beta"] \\ & X \otimes_A M & \end{tikzcd}$$

X is good $\iff \alpha$ is an isomorphism (for $\Leftarrow$ take $M = \tilde{A}$).
$X \otimes_A -$ descends to $\mathcal{a} \iff \beta$ is an isomorphism for all M,
($\Rightarrow$ because $A^g \otimes_A M \to M$ becomes an isom. in $\mathcal{a}$,
$\Leftarrow$ because $A^g \otimes_A -$ descends to $\mathcal{a}$).

2) Let $N \in A\text{-mod}$. Then $\text{Hom}_A(N, -)$ descends to $\mathcal{a} \iff N \in A\text{-gmod}$.

Pf. Comm. triangle:

$$\begin{tikzcd}[column sep=small] \text{Hom}_A(A^g \otimes_A N, M) \arrow[rr, equal] & & \text{Hom}_A(N, \text{Hom}_A(A^g, M)) \\ & \text{Hom}_A(N, M) \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

N is good $\iff \alpha$ is an isomorphism $\forall M$.
$\text{Hom}_A(N, -)$ descends to $\mathcal{a} \iff \beta$ is an isom,
($\Rightarrow$ because $M \longrightarrow \text{Hom}_A(A^g, M)$ becomes an isom. in $\mathcal{a}$,
$\Leftarrow$ because $\text{Hom}_A(A^g, -)$ descends to $\mathcal{a}$)

3) *Let* $Q \in A\text{-mod}$. *Then* $\operatorname{Hom}_A(-,Q)$ *descends to* $\mathcal{A}$ *iff* $Q \in A\text{-g'mod}$: $Q \xrightarrow{\sim} \operatorname{Hom}_A(A, Q)$.

Pf. Comm. triangle

$$\begin{tikzcd}
\operatorname{Hom}_A(A^g \otimes_A M, Q) \arrow[rr, equal] & & \operatorname{Hom}_A(M, \operatorname{Hom}_A(A^g, Q)) \\
& \operatorname{Hom}_A(M,Q) \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

β is an isom $\forall M \iff Q$ is good'

α is an isom $\forall M \iff \operatorname{Hom}_A(-,Q)$ descends to $\mathcal{A}$;

($\Leftarrow$ because $A^g \otimes_A M \to M$ becomes an isom in $\mathcal{A}$,

$\Leftarrow$ because $A^g \otimes_A -$ descends to $\mathcal{A}$)

Write $\operatorname{Homg}_A(M,N)$ for $\operatorname{Hom}_{\mathcal{A}}(M,N)$ where $\mathcal{A} = A\text{-mod}/A\text{-null}$. Similarly write $X \overset{g}{\otimes}_A M$ for the tensor product functor on

$$(\text{mod-}A/\text{null-}A) \times (A\text{-mod}/A\text{-null})$$

~~From~~ 1)-3) above one has

$$X \overset{g}{\otimes}_A M = X \otimes_A M \quad \text{if either } \begin{cases} X \text{ is } A^{op}\text{-good or} \\ M \text{ is } A\text{-good} \end{cases}$$

$$\operatorname{Homg}_A(M,N) = \operatorname{Hom}_A(M,N) \quad \text{if either } \begin{cases} M \text{ is } A\text{-good or} \\ N \text{ is } A\text{-good}' \end{cases}$$

Now let $u: A \to B$ be a homomorphism of idempotent rings. One then has functors

$$\mathcal{A} = A\text{-mod}/A\text{-null} \quad \begin{matrix} \xrightarrow{\quad u_! \quad} \\ \xleftarrow{\quad u^* \quad} \\ \xrightarrow{\quad u_* \quad} \end{matrix} \quad B\text{-mod}/B\text{-null} = \mathcal{B}$$

where each functor is left adjoint to the one immediately below. u^* is the restriction functor; it is induced by restriction of scalars from B-mod to A-mod; this is exact and carries null B-modules into null A-modules, so it descends to an exact functor between the quotient categories.

One has

$$\begin{aligned} \mathrm{Homg}_A(N, u^*(M)) &= \mathrm{Hom}_A(N, M) && \text{if } N \text{ is } A\text{-good} \\ &= \mathrm{Hom}_A(N, \mathrm{Hom}_B(B, M)) && \text{if } M \text{ is } B\text{-good} \\ &= \mathrm{Hom}_B(B \otimes_A N, M) \\ &= \mathrm{Homg}_B(B \otimes_A N, M) \end{aligned}$$

Thus u^* has the left adjoint $u_!$ given by $u_!(N) = B \otimes_A N$, when N is A-good. Note for N arbitrary that $B(B \otimes_A N) = B \otimes_A N$, hence $B \otimes_B B \otimes_A N = B^g \otimes_A N$ is the B-module arising from N. ~~B-good. Thus~~ we have the formula

$$u_!(N) = B^g \otimes_A A^g \otimes_A N$$

for any $N \in \mathcal{A}$, where the right side is B-good.

Next one has

$$\begin{aligned} \mathrm{Homg}_A(u^*(M), N) &= \mathrm{Hom}_A(M, N) && \text{if } N \text{ is } A\text{-good}' \\ &\simeq \mathrm{Hom}_A(B \otimes_B M, N) && \text{if } M \text{ is } B\text{-good} \end{aligned}$$

$$= \mathrm{Hom}_B(M, \mathrm{Hom}_A(B,N))$$
$$= \mathrm{Homg}_B(M, \mathrm{Hom}_A(B,N))$$

Thus u^* has the right adjoint u_* given by $u_*(N) = \mathrm{Hom}_A(B,N)$ for N A-good'. Note that for N-arbitrary $\mathrm{Hom}_B(Z, \mathrm{Hom}_A(B,N)) = \mathrm{Hom}_A(B\otimes_B Z, N) = 0$, so that $\mathrm{Hom}_B(B, \mathrm{Hom}_A(B,N)) = \mathrm{Hom}_A(B^g, N)$ is the B-good' module arising from $\mathrm{Hom}_A(B,N)$. Thus one has the formula

$$u_*(N) = \mathrm{Hom}_A(B^g, \mathrm{Hom}_A(A^g, N))$$
$$= \mathrm{Hom}_A(A^g \otimes_A B^g, N)$$

for any $N \in \mathcal{A}$, where the right side is B-good'.

Think as follows. The usual left adjoint for the restriction of scalars is $N \mapsto B\otimes_A N$. To get $u_!$ on the level of good modules, we make it B-good: $B\otimes_A B\otimes_A N = B^g\otimes_A N$, and then $B^g\otimes_A A^g\otimes_A N$ if we want $N \in \mathcal{A}$. Similarly the usual right adjoint for the restriction of scalars is $N \longmapsto \mathrm{Hom}_A(B,N)$. To get u_* on the level of good' modules we make it B^g-good': $\mathrm{Hom}_B(B, \mathrm{Hom}_A(B,N)) = \mathrm{Hom}_A(B^g, N)$ and then $\mathrm{Hom}_A(A^g\otimes_A B^g, N)$ if we want N to range over $\mathcal{A}$.

Let's consider the 'good' tensor product

$$X, M \longmapsto X \overset{g}{\otimes}_A M \overset{\text{defn}}{=} X \otimes_A A^g \otimes_A M$$

from $(\text{mod-}A/\text{null-}A) \times (A\text{-mod}/A\text{-null}) \longrightarrow \mathcal{A}b$

Note that if X is good, then $X \overset{g}{\otimes}_A M = X \otimes_A M$. Moreover if X is good and flat, then this is an exact functor of M. However if X is just flat as an A^{op}-module, then $X \overset{g}{\otimes}_A M = X \otimes_A A^g \otimes_A M$ need not be exact in M, e.g. for $X = \tilde{A}$ we have $A^g \otimes_A -$, unless A^g is a flat A^{op}-module.

Recall that there exist enough flat good modules in A-gmod and gmod-A. Thus the left derived functors of $- \overset{g}{\otimes}_A -$, denoted $\mathrm{Torg}_n^A(-,-)$ are defined. Let's recall their construction in analogy with $\mathrm{Tor}_n^A(-,-)$.

Let X be a right A-module. Choose

$$0 \to K_1 \to E_0 \to X \qquad E_0 \text{ flat good } \Rightarrow K_1 A = K_1$$

$$0 \to K_2 \to E_1 \to K_1 \otimes_A A \to 0 \qquad E_1 \text{ ——— } \qquad K_2 A = K_2$$

$$\cdots\cdots$$

Then $E_\bullet$ is a complex of flat good A^{op}-modules which is a resolution of $X \otimes_A A^g$ in gmod-A.

Similarly we can construct a complex $F_\bullet$ of good flat A-modules which is a resolution of $A^g \otimes_A M$ in the category A-gmod. Then we have quis

$$X \otimes_A F_\bullet \longleftarrow E_\bullet \otimes_A F_\bullet \longrightarrow E_\bullet \otimes_A M$$

In effect $\quad H_p(E_\bullet \otimes_A F_q) = H_p(E_\bullet) \otimes_A F_q \quad$ as F_q is flat

$$= \begin{cases} X \otimes_A A^g \otimes_A F_q = X \otimes_A F_q & q = 0 \\ 0 & q > 0 \end{cases}$$

as F_q is good and $H_p(E_\cdot)$ is null-A for $p > 0$. Then we [redacted] define

$$\mathrm{Torg}_n^A(X, M) = H_n(X \otimes_A F_\cdot) = H_n(E_\cdot \otimes_A F_\cdot) = H_n(E_\cdot \otimes_A M)$$

This shows the independence of the choice of the resolutions $E_\cdot$, $F_\cdot$. [redacted]

$\{\mathrm{Torg}_n^A(X, -)\}$ are the derived functors of $X \overset{g}{\otimes}_A$ [redacted] $-$. For $X = A$ one has

$$\mathrm{Torg}_0^A(A, M) = A \otimes_A A^g \otimes_A M = A^g \otimes_A M$$

which is the canonical left exact embedding

$$A\text{-mod}/A\text{-null} \xrightarrow{\sim} A\text{-gmod} \subset A\text{-mod}$$
$$M \longmapsto A^g \otimes_A M.$$

Call this functor F, so that we have

$$\boxed{L_n F(M) = \mathrm{Torg}_n^A(A, M)}$$

I next want to relate [redacted] these $\mathrm{Tor}_n^A(A, M)$. Consider $\mathrm{Torg}_n^A(A, M) = H_n(E_\cdot \otimes_A M)$ where $E_\cdot$ is a flat resolution of A^g in [redacted] gmod-A. Note that

$$H_n(E_\cdot) = \mathrm{Torg}_n^A(A, A) = L_n F(A)$$

where $L_n F(A^g) \xrightarrow{\sim} L_n F(A) \xrightarrow{\sim} L_n F(\tilde{A})$ since $L_n F$ is defined on the quotient category A-mod/A-null. Because $E_\cdot$ is a [redacted] complex of flat A^{op}-modules ~~one~~ one has a spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^A(H_q(E_\cdot), M) \Rightarrow H_n(E_\cdot \otimes_A M)$$

yielding the spectral sequence

$$\boxed{E^2_{pq} = \mathrm{Tor}^A_p\big(L_q F(A), M\big) \Longrightarrow L_n F(M)}$$

Picture

$$\begin{array}{lll}
L_1F(A)\otimes_A M & & \\
A^g \otimes_A M & \mathrm{Tor}_1^A(A^g, M) & \mathrm{Tor}_2^A(A^g, M)
\end{array}$$

We get the 5-term sequence

$$\boxed{L_2F(M) \to \mathrm{Tor}_2^A(A^g, M) \longrightarrow L_1F(A)\otimes_A M \longrightarrow L_1F(M) \to \mathrm{Tor}_1^A(A^g, M) \to 0}$$

$$L_1F(A)\otimes_A M = L_1F(A)\otimes_{\mathbb{Z}} M/AM$$

Thus we get $\boxed{L_1F(M) = \mathrm{Tor}_1^A(A^g, M) \quad \text{if} \quad M = AM}$

On the other hand suppose we resolve M:

$$0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow A^g\otimes_A M \longrightarrow 0$$

$$0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow A\otimes_A K_1 \longrightarrow 0$$

where $F_\cdot$ is flat and good. Then

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Tor}_1^A(A, A^g\otimes_A M) & \to & A\otimes_A K_1 & \to & F_0 & \to & A^g\otimes_A M & \to 0 \\
 & & & \| & & \| & & \| & \\
0 \to & L_1F(A^g\otimes_A M) & \to & F(K_1) & \to & F(F_0) & \to & F(A^g\otimes_A M) & \to 0
\end{array}$$

Thus we get

$$\boxed{\mathrm{Tor}_1^A(A, A^g\otimes_A M) = L_1F(M)}$$

So we get various expressions for $L_1F(M)$, namely

$$\boxed{L_1F(M) = \mathrm{Tor}_1^A(A^g, A\otimes_A M) = \mathrm{Tor}_1^A(A, A^g\otimes_A M)}$$

~~The only reasonable~~ [redacted] ~~condition sufficient for F to be exact is that A^g be flat as right A-module.~~

Claim F: A-mod/A-null $\xrightarrow{\sim}$ A-gmod $\subset$ A-mod

$$M \longmapsto A^g\otimes_A M \mapsto A^g\otimes_A M$$

is exact iff A^g is a flat A^{op}-module.

Why? One has an equivalence between exact functors on A-mod/A-null and exact functors on A-mod killing A-null. Thus $F(M) = A^g\otimes_A M \in$ A-mod is exact on A-mod/A-null $\iff$ $M \mapsto A^g\otimes_A M$ from A-mod to A-mod is exact $\iff$ A^g is right A-flat.

The same holds for $M \longmapsto X\otimes_A M$, where X is A^{op} good, from A-mod/A-null $\longrightarrow$ A-mod.

Put another way, a good A^{op}-module X is flat in [redacted] gmod-A $\iff$ it is flat in mod-A.

[redacted]

Prop: Let $X \in$ mod-A, $M \in$ A-mod satisfy $XA = X$, $AM = M$. Then $X \overset{g}{\otimes}_A M \xrightarrow{\sim} X\otimes_A M$.

Pf. $0 \to K \to A\otimes_A M \to M \to 0$, $AK = 0 \Rightarrow X\otimes_A K = 0$

Thus $X\otimes_A A\otimes_A M \xrightarrow{\sim} X\otimes_A M$. Applying this to $A\otimes_A M$ in place of M yields

$$X\otimes_A A\otimes_A A\otimes_A M \xrightarrow{\sim} X\otimes_A A\otimes_A M \xrightarrow{\sim} X\otimes_A M$$

$\square$

April 30, 1994

Morita equivalence - general case.

A Morita equivalence between rings A, B is given by bimodules ${}_BP_A$, ${}_AQ_B$ together with pairings $P \otimes_A Q \longrightarrow B$, $Q \otimes_B P \longrightarrow A$ satisfying certain conditions of compatibility. These can be expressed by saying one has a ring R with block decomposition

$$1) \qquad R = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

Now, it should turns out that the Morita equivalence between A, B is the composition of Morita equivalences $A \sim R$ and $B \sim R$. We have ~~[illegible]~~ previously obtained a M.eq. $A \sim R$ assuming that $ARA = A$, $RAR = R$, $A \subset R$ and $A = A^2$. Let's calculate

$$RA = \begin{pmatrix} A^2 & 0 \\ PA & 0 \end{pmatrix} \qquad ARA = \begin{pmatrix} A^3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$RAR = \begin{pmatrix} A^2 & 0 \\ PA & 0 \end{pmatrix}\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A^3 & A^2Q \\ PA^2 & PAQ \end{pmatrix}$$

Thus we want to have

$$2) \qquad A = A^2 \qquad P = PA \qquad Q = AQ \qquad PQ = B$$

similarly for $B \sim R$ we want

$$3) \qquad B = B^2 \qquad P = BP \qquad Q = QB \qquad QP = A$$

These 8 conditions reduce to four:

$$4) \qquad QP = A \qquad PQ = B \qquad PQP = P \qquad QPQ = Q$$

Now we should have an equivalence of categories

5) $$\text{A-mod}/\text{A-null} \underset{Q\overset{g}{\otimes}_B N \,\leftarrow\!\shortmid\, N}{\overset{M \mapsto P\overset{g}{\otimes}_A M}{\rightleftarrows}} \text{B-mod}/\text{B-null}$$

Let's note first that because $PA = P$ we have $P \otimes_A A^g \xrightarrow{\sim} P \otimes_A A$, and similarly $Q \otimes_B B^g \xrightarrow{\sim} Q \otimes_B B$. Thus $P \overset{g}{\otimes}_A M = P \otimes_A A \otimes_A M$, $Q \overset{g}{\otimes}_B N = Q \otimes_B B \otimes_B N$. We now want to see that the canonical map

$$(P \otimes_A A) \otimes_A (Q \otimes_B B) \otimes_B N \longrightarrow (B \otimes_B B) \otimes_B N$$

$$(p, a, q, b, n) \longmapsto (paq, b, n)$$

is an isomorphism of functors of N. It's suffices to show that

6) $$P \otimes_A A \otimes_A Q \otimes_B B \longrightarrow B \otimes_B B$$

$$(p, a, q, b) \longmapsto (paq, b)$$

is an isomorphism. Now $PA = P$, $AQ = Q \implies$ $P \otimes_A A \otimes_A Q \xrightarrow{\sim} P \otimes_A Q$. Consider

7) $$P \otimes_A Q \xrightarrow{\pi} B \qquad (p,q) \longmapsto pq$$

This is a map of B-bimodules which is surjective as $PQ = B$ by hypothesis. Also

$$p'q'(p, q) = (p'q'p, q) = (p', q'pq)$$

$$(p,q)p'q' = (p, qp'q') = (pqp', q')$$

as $QP = A$ and the tensor product $P \otimes_A Q$ is over A. This implies that the kernel of π is null as both left + right B-module. Thus if $(p_i, q_i) \in \mathrm{Ker}(\pi)$, i.e. $p_i q_i = 0$, then

$$(p_i, q_i)\, p'q' = (p_i q_i p', q') = 0$$

whence $(p_i, q_i)B = 0$ as $PQ = B$.

Since π is surjective with kernel killed by B we conclude

$$P \otimes_A Q \otimes_B B \xrightarrow{\sim} B \otimes_B B$$

Thus 6) which is the composite

$$P \otimes_A A \otimes_A Q \otimes_B B \xrightarrow{\sim} P \otimes_A Q \otimes_B B \xrightarrow{\sim} B \otimes_B B$$

is an isomorphism.

It's clear now that 5) is an equivalence.

If instead of the quotient categories we use the good module categories we have the equivalence

$$A\text{-gmod} \underset{A \otimes_A Q \otimes_B -}{\overset{B \otimes_B P \otimes_A -}{\rightleftarrows}} B\text{-gmod}$$

Note that $BP = P \Longrightarrow B \otimes_B P$ is a good left B module
$AQ = Q \Longrightarrow A \otimes_A Q$ ——— A ———.

May 1, 1994

Concept of a generator U in an abelian category $\mathcal{A}$: By definition U is a generator when the functor $h^U = \mathrm{Hom}(U,-)$ from $\mathcal{A}$ to $\mathcal{Ab}$ is faithful, i.e. $\forall X, Y$

$$\mathrm{Hom}(X,Y) \hookrightarrow \mathrm{Hom}(h^U(X), h^U(Y))$$

equivalently $X \xrightarrow[\neq 0]{} Y \Rightarrow \exists\, U \to X$ such that $U \to X \to Y$ is $\neq 0$. Observe this last condition depends only on the image of $X \to Y$, so that U is a generator $\Longleftrightarrow$ $\forall\, X \twoheadrightarrow Y \neq 0$ $\exists\, P \to X$ such that $P \to X \to Y$ is $\neq 0$, equivalently $\forall\, X' \subsetneq X$, $\exists\, P \to X$ with image not contained in X'.

Thus U is a generator $\Longleftrightarrow$ $\forall X$ the smallest subobject of X containing the images of all $U \to X$ is X. If $\mathcal{A}$ has direct sums this means X is a quotient of $\bigoplus_I U$ for some set I.

Note that any generator for A-mod (i.e. U such that $\tilde{A}$ is a summand of $\bigoplus_I U$ for some I) is a generator for $\mathcal{A} = A\text{-mod}/A\text{-null}$. However A is not a generator for A-mod, but it is a generator for $\mathcal{A}$, since given any M such that $AM = M$, one has $A \otimes_A M \twoheadrightarrow M$, so a set of generators m_i for M gives rise to a surjection $\bigoplus A \twoheadrightarrow M$, $(a_i) \mapsto a_i m_i$.

<u>Prop.</u> [redacted] An A-module $\mathcal{U}$ is a generator for $\mathcal{A}$ iff A is a quotient of $\bigoplus_I A^g \otimes_A \mathcal{U}$ for some set I.

<u>Proof</u>: [redacted] As A generates $\mathcal{A}$, $\mathcal{U}$ generates iff there is an epim. $\bigoplus_I \mathcal{U} \twoheadrightarrow A$ in $\mathcal{A}$, equivalently there is an epim [redacted] $A^g \otimes_A (\bigoplus_I \mathcal{U}) = \bigoplus_I A^g \otimes_A \mathcal{U} \twoheadrightarrow A^g$ in A-gmod. Since $A^g \twoheadrightarrow A$ we get $(\Longrightarrow)$. Conversely if we have $\bigoplus_I A^g \otimes_A \mathcal{U} \twoheadrightarrow A$, then tensoring with A yields $\bigoplus_I A^g \otimes_A \mathcal{U} =$ [redacted] $A \otimes (\bigoplus_I A^g \otimes_A \mathcal{U}) \twoheadrightarrow A \otimes_A A = A^g$, so $\bigoplus_I \mathcal{U} \twoheadrightarrow A$ in $\mathcal{A}$.

Consider a Morita equivalence

The condition $QP = A \Longrightarrow$ [redacted] $\exists$ surjection $\bigoplus_I Q \twoheadrightarrow A$ in A-mod for some I. Thus Q is a generator for A-mod/A-null, and similarly P is a generator for mod-A/null-A.

1) $$\boxed{R = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \qquad \begin{matrix} A = A^2 = QP & Q = AQ = QB \\ P = PA = BP & B = B^2 = PQ \end{matrix}}$$

Another way to write this: $R_+ = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $R_- = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$

Then we have a superalgebra such that

$$(R_-)^2 = R_+ \qquad (R_-)^3 = R_-$$

Proof of the equivalence of categories

$$2) \qquad (A\text{-gmod}) \underset{A \otimes_A Q \otimes_B -}{\overset{B \otimes_B P \otimes_A -}{\rightleftarrows}} (B\text{-gmod})$$

Note: $BP = P \implies B \otimes_B P$ is a good B-module $\implies B \otimes_B P \otimes_A M$ is a good B-module $\forall M$. Thus these functors are defined.

Next have canonical surjection of B-bimodules

$$3) \qquad P \otimes_A Q \twoheadrightarrow B$$

whose kernel is killed by B on either side: [redacted]
if $\sum (p_i, q_i) \longmapsto \sum p_i q_i = 0$, then

$$pq \sum (p_i, q_i) = \sum (pqp_i, q_i) = \sum (p, qp_iq_i) = (p, q \textstyle\sum p_iq_i) = 0$$

Thus get

$$4) \qquad B \otimes_B P \otimes_A Q \xrightarrow{\sim} B \otimes_B B$$

and hence

$$5) \quad (B \otimes_B P) \otimes_A (A \otimes_A Q) \xrightarrow{\sim} B \otimes_B P \otimes_A Q \xrightarrow{\sim} B \otimes_B B$$

where the first isom results from $X \otimes_A A \otimes_A M \xrightarrow{\sim} X \otimes_A M$ if $XA = X$, $AM = M$.

Similarly have canonical surjection of A-bimodules

$$6) \qquad Q \otimes_B P \twoheadrightarrow A$$

whose kernel is killed by A on either side. Thus

$$7) \qquad A \otimes_A Q \otimes_B P \xrightarrow{\sim} A \otimes_A A$$

$$8) \quad (A \otimes_A Q) \otimes_B (B \otimes_B P) \xrightarrow{\sim} A \otimes_A Q \otimes_B P \xrightarrow{\sim} A \otimes_A A.$$

The isos. 5), 8) prove the equivalence of categories 2).

From 3) we also get

$$(P\otimes_A Q)\otimes_B P \xrightarrow{\sim} B\otimes_B P \qquad Q\otimes_B(P\otimes_A Q) \xrightarrow{\sim} Q\otimes_B B$$

and from 6) we get

$$(Q\otimes_B P)\otimes_A Q \simeq A\otimes_A Q \qquad P\otimes_A(Q\otimes_B P) \simeq P\otimes_A A$$

Thus we have comm. squares

9)
$$\begin{tikzcd} P\otimes_A Q\otimes_B P \arrow[r, "\cong"] \arrow[d, "\cong"] & P\otimes_A A \arrow[d] \\ B\otimes_B P \arrow[r] & P \end{tikzcd} \qquad \begin{tikzcd} Q\otimes_B P\otimes_A Q \arrow[r, "\sim"] \arrow[d, "\sim"] & Q\otimes_B B \arrow[d] \\ A\otimes_A Q \arrow[r] & Q \end{tikzcd}$$

~~At the same time~~ In a similar way we can start with

10) $$P\otimes_A A \longrightarrow P$$

which is a surjection of B_ℓ, A_r bimodules whose kernel is null on both sides: If $(p_i, q_i) \in P\otimes_A A$ is such that $p_i q_i = 0$, and $pq \in B$, then

$$pq\,(p_i, q_i) = (pqp_i, q_i) = (p, qp_iq_i) = 0.$$

Then 10) yields

11) $$B\otimes_B P\otimes_A A \xrightarrow{\sim} B\otimes_B P$$

showing that $B\otimes_B P$ is right A-good.

Similarly

12)
$$B \otimes_B P \longrightarrow P$$
is a surjection of B_ℓ, A_r bimodules whose kernel is null on both sides, whence
$$B \otimes_B P \otimes_A A \xrightarrow{\sim} P \otimes_A A$$
showing $P \otimes_A A$ is left B-good.

Thus we learn that $P \otimes_A A \simeq B \otimes_B P$ is the good form of the bimodule P, while $A \otimes_A Q \simeq Q \otimes_B B$ is the good form of the bimodule Q, so that we can write the equivalences as follows.

$$\text{A-gmod} \underset{A \otimes_A Q \otimes_B -}{\overset{P \otimes_A -}{\rightleftarrows}} \text{B-mod/B-null}$$

$$\text{A-mod/A-null} \underset{Q \otimes_B -}{\overset{B \otimes_B P \otimes_A -}{\rightleftarrows}} \text{B-gmod}$$

Notice that this implies that the B-bimodule $P \otimes_A A \otimes_A Q = P \otimes_A Q$ gives a well-defined functor on B-mod/B-null, and this functor is the identity. This suggests that $P \otimes_A Q$ is B_r-good. Check:
$$P \otimes_A Q \otimes_B B = P \otimes_A A \otimes_A Q = P \otimes_A Q$$
But then since $P \otimes_A Q \longrightarrow B$ has null kernel we must have $P \otimes_A Q = B \otimes_B B$.

Check this:

$$\cancel{\begin{array}{ccc} B\otimes_B (P\otimes_A Q) & \xrightarrow{\simeq} & B\otimes_B B \\ \uparrow\simeq & & \\ B\otimes_B (P\otimes_A A\otimes_A Q) & \xrightarrow{\simeq} & P\otimes_A A\otimes_A Q \end{array}}$$

~~uses~~

$$\begin{tikzcd} B\otimes_B B & B\otimes_B P\otimes_A Q \arrow[l, "\simeq"'] & P\otimes_A A\otimes_A Q \arrow[l, phantom, "\simeq"] \arrow[r, "\simeq"] & P\otimes_A Q \\ & & B\otimes_B P\otimes_A A\otimes_A Q \arrow[ul, "\simeq"] \arrow[u, "\simeq"'] & \end{tikzcd}$$

because $B\cdot \mathrm{Ker}\{P\otimes_A A\to P\}=0$
or $\mathrm{Ker}\{P\otimes_A A\to P\}\cdot A=0$

because $\mathrm{Ker}(B\otimes_B P\twoheadrightarrow P)\cdot A=0$

So we find for any Morita equivalence data $\begin{pmatrix} A & Q \\ P & B\end{pmatrix}$ that

$$P\otimes_A Q = B\otimes_B B \qquad Q\otimes_B P = A\otimes_A A$$

$$P\otimes_A A = B\otimes_B P \qquad Q\otimes_B B = A\otimes_A Q$$

Let us say that Morita equivalence data $\begin{pmatrix} A & Q \\ P & B\end{pmatrix}$ is good when instead of just the relations

$$\begin{pmatrix} A=A^2=QP & Q=AQ=QB \\ P=PA=BP & B=B^2=PQ \end{pmatrix}$$

we have the stronger relations

$$\begin{pmatrix} A=A\otimes_A A=Q\otimes_B P & Q=A\otimes_A Q=Q\otimes_B B \\ P=P\otimes_A A=B\otimes_B P & B=B\otimes_B B=P\otimes_A Q \end{pmatrix}$$

Then it's pretty clear that given any Morita equiv. data $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$, then it can be replaced by good Morita equiv. data:

$$\begin{pmatrix} A \otimes_A A & A \otimes_A Q = Q \otimes_B B \\ P \otimes_A A = B \otimes_B P & B \otimes_B B \end{pmatrix}$$

In terms of the superalgebra R the Morita equivalence is good when

$$R^- \otimes_{R^+} R^- \xrightarrow{\sim} R^+ \qquad R^- \otimes_{R^+} R^- \otimes_{R^+} R^- \xrightarrow{\sim} R^-$$

May 3, 1994

Woedzicki result: Let A be a left ideal in a ring R and let M be an R-module such that $AM = M$. Then M is flat over $A \iff M$ is flat over R.

Proof. Easy direction ($\Longrightarrow$): If $X \in \text{mod-}R$ one has

$$X \otimes_A M \xrightarrow{\sim} X \otimes_R M$$

because

$$xr \otimes_A am = xra \otimes_A m = x \otimes_A ram .$$

Assuming M is A-flat the functor $X \mapsto X \otimes_A M = X \otimes_R M$ from mod-R to $\mathcal{Ab}$ is exact, hense M is R-flat.

($\Longleftarrow$): Use the Cartan-Eilenberg linear equation criterion for flatness: Given solid arrows

$$
1)\quad
\begin{tikzcd}
\tilde{A}^p \arrow[r, "a"] \arrow[rr, bend left, dashed, "0"] \arrow[dr, "0"'] & \tilde{A}^q \arrow[r, dashed, "a'"] \arrow[d, "m"] & \tilde{A}^r \arrow[d, dashed, "m'"] \\
& M \arrow[r, equal] & M
\end{tikzcd}
$$

the dotted arrows exist. Here a, a' are matrices over $\tilde{A}$ and m, m' are column vectors with entries in M. We can look at the linear equations $am = 0$ over R and use the fact that M is R-flat to obtain the solid arrows:

$$
2)\quad
\begin{tikzcd}
\tilde{R}^p \arrow[r, "a"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & \tilde{R}^q \arrow[r, "r"] \arrow[d, "m"] & \tilde{R}^s \arrow[r, dashed, "\alpha"] \arrow[d, "m''"] & A^t \arrow[d, dashed, "m'"] \\
& M \arrow[r, equal] & M \arrow[r, equal] & M
\end{tikzcd}
$$

Using $AM = M$ the dotted arrows exist, where α has entries in A. ~~Then we obtain~~

Note here we use the fact that A is a left ideal to conclude that $\tilde{R}^s \ni x \mapsto x\alpha \in \tilde{R}^k$ has its image in A^s.

Then putting $a' = r\alpha$ we get the desired completion of 1).