March 23, 1994 (Carl 29 yesterday)

List of V. Jones formulas.

$$A_0 = B$$

$A_1 = A$ equipped with

$A_0 = B \subset A = A_1$ and

$\rho_1 : A_1 \to A_0$ given by $\rho : A \to B$

$x_i^{(1)} = x_i,\ y_i^{(1)} = y_i \in A$ $\ni$ $\rho(ax_i)y_i = x_i\rho(y_i a) = 0.$

$$A_2 = A \otimes_B A$$

product $(a_1, a_2)(a_3, a_4) = (a_1\rho(a_2a_3), a_4)$

identity (x_i, y_i)

$A_1 \hookrightarrow A_2$: $a \longmapsto (ax_i, y_i) = (x_i, y_i a)$

$\rho_2 : A_2 \to A_1$: $(a_1, a_2) \longmapsto a_1a_2$

$x_i^{(2)} = (x_i, 1)$ $y_i^{(2)} = (1, y_i)$

$$A_3 = A_2 \otimes_{A_1} A_2 = A \otimes_B A \otimes_B A$$

$((a_1, a_2), (a_3, a_4)) \longmapsto (a_1, a_2a_3, a_4)$

product $(a_1, a_2, a_3)(a_4, a_5, a_6) = (a_1, a_2\rho(a_3a_4)a_5, a_6)$

identity $(x_i, 1, y_i)$

$A_2 \hookrightarrow A_3$: $(a_1, a_2) \longmapsto (a_1, 1, a_2)$

$\rho_3 : A_3 \to A_2$: $(a_1, a_2, a_3) \longmapsto (a_1\rho(a_2), a_3)$

$x_i^{(3)} = (x_i, x_j, y_j)$ $y_i^{(3)} = (x_j, y_j, y_i)$

$$A_4 = A_3 \otimes_{A_2} A_3 \xrightarrow{\sim} [A\otimes_B]^{(3)} A$$

$$((a_1,a_2,a_3),(a_4,a_5,a_6)) \xrightarrow{\sim} (a_1, a_2\rho(a_3a_4), a_5, a_6)$$

product: $(a_1,a_2,a_3,a_4)(a_5,a_6,a_7,a_8)$

$$= (a_1, a_2\rho(a_3\rho(a_4a_5)a_6), a_7, a_8)$$

identity (x_i, x_j, y_j, y_i)

$A_3 \hookrightarrow A_4$: $(a_1,a_2,a_3) \longmapsto (a_1, a_2x_i, y_i, a_3)$

ρ_4 : $(a_1,a_2,a_3,a_4) \longmapsto (a_1, a_2a_3, a_4)$

$x_i^{(4)} = (x_i, x_j, 1, y_j)$ $\qquad y_i^{(4)} = (x_k, 1, y_k, y_i)$

$$A_5 = A_4 \otimes_{A_3} A_4 \xrightarrow{\sim} [A\otimes_B]^{(4)} A$$

$$((a_1,a_2,a_3,a_4),(a_5,a_6,a_7,a_8)) \xmapsto{\sim} (a_1, a_2, a_3\rho(a_4a_5)a_6, a_7, a_8)$$

prod: $(a_1,a_2,a_3,a_4,a_5)(a_6,a_7,a_8,a_9,a_{10})$

$$= (a_1, a_2, a_3\rho(a_4\rho(a_5a_6)a_7)a_8, a_9, a_{10})$$

id : $(x_i, x_j, 1, y_j, y_i)$

$A_4 \hookrightarrow A_5$: $(a_1,a_2,a_3,a_4) \longmapsto (a_1, a_2, 1, a_3, a_4)$

ρ_5 : $(a_1,a_2,a_3,a_4,a_5) \longmapsto (a_1, a_2\rho(a_3), a_4, a_5)$

$x_i^{(5)} = (x_i, x_j, x_k, y_k, y_j)$

$y_i^{(5)} = (x_l, x_m, y_m, y_l, y_i)$

$$A_6 = A_5 \otimes_{A_4} A_5 \simeq [A \otimes_B]^{(5)} A$$

$$((a_1, a_2, a_3, a_4, a_5), (a_6, a_7, a_8, a_9, a_{10}))$$
$$\longmapsto (a_1, a_2, a_3\rho(a_4\rho(a_5 a_6)a_7), a_8, a_9, a_{10})$$

product: $(a_1, \dots, a_6)(a_7, \dots, a_{12})$
$$= (a_1, a_2, a_3\rho(a_4\rho(a_5\rho(a_6 a_7)a_8)a_9), a_{10}, a_{11}, a_{12})$$

identity $= (x_i, x_j, x_k, y_k, y_j, y_i)$

$A_5 \hookrightarrow A_6 : (a_1, \cancel{a_2} a_3, a_4, a_5) \longmapsto (a_1, a_2, a_3 \cancel{x_i}, y_i, a_4, a_5)$

$\rho_6 : (a_1, a_2, a_3, a_4, a_5, a_6) \longmapsto (a_1, a_2, a_3 a_4, a_5, a_6)$

$$x_i^{(6)} = (x_i, x_j, x_k, 1, y_k, y_j)$$
$$y_i^{(6)} = (x_j, x_k, 1, y_k, y_j, y_i)$$

March 28, 1994

Background: I am presently studying papers of Lars Kadison and Michael Pimsner related to the V Jones construction. From Kadison's preprints:

Separability and the Jones polynomial,
Algebraic aspects of the Jones basic construction

I learn that [redacted] many things makes sense in a purely algebraic context, more precisely, they are independent of the $*$ algebra setting and Hilbert module inner product. From Pimsner:

A class of C^*-algs generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$.

I learn that certain things hold more generally for a bimodule over an algebra rather than for a pair of algebras.

[redacted] My goal is now to work out [redacted] a purely algebraic version of Pimsner's theory. A price one must pay for leaving the C^* algebra - Hilbert module setting is [redacted] the failure of strong Morita equivalence. This means restricting to ftp (finite type projective) modules.

To conform to standard notation in Hilbert module theory a module over an algebra will be a right module unless stated otherwise, and we write $Hom_A(M, N)$ for right module [redacted] maps. Use A^l, A^r to distinguish left and right module structures.

First I go over duality for ftp modules.

Let B be an algebra, let P be a right module and Q a left module over B. Suppose given a pairing

$$Q \otimes P \longrightarrow B \qquad (q,p) \longmapsto \langle q,p \rangle$$

which is a B-bimodule map: $\langle b_1 q, p b_2 \rangle = b_1 \langle q,p \rangle b_2$.
This is equivalent to a left B-module map

$$Q \longrightarrow \mathrm{Hom}_B(P,B) \qquad q \mapsto (p \mapsto \langle q,p \rangle)$$

and also equivalent to a right module map

$$P \longrightarrow \mathrm{Hom}_{B^e}(Q,B)$$

The following conditions are equivalent:

① $\exists\ (x_i, y_i) \in P \otimes_B Q$ such that

$$p = x_i \langle y_i, p \rangle \qquad \forall p \in P$$
$$q = \langle q, x_i \rangle y_i \qquad \forall q \in Q$$

② $\forall\ M_B$ (rt B-modules) the map

$$M \otimes_B Q \longrightarrow \mathrm{Hom}_B(P, M)$$
$$(m, q) \longmapsto (p \longmapsto m \langle q, p \rangle)$$

is a bijection

②′ $\forall\ {}_B N$ the map

$$P \otimes_B N \longrightarrow \mathrm{Hom}_{B^e}(Q, N)$$
$$(p, n) \longmapsto (q \longmapsto \langle q, p \rangle n)$$

is a bijection.

③ P is a ftp B-module and

$$Q \xrightarrow{\sim} \mathrm{Hom}_B(P, B)$$

③′ Q is a ftp B^e-module and

$$P \xrightarrow{\sim} \mathrm{Hom}_{B^e}(Q, B)$$

Why? ① ⟹ ②

$$\begin{array}{ccc} (m, q) & \longmapsto & (p \mapsto m\langle q, p\rangle) \\ M \otimes_B Q & \rightleftarrows & \mathrm{Hom}_B(P, M) \\ (f(x_i), y_i) & \longleftarrow & f \end{array}$$

are mutually inverse:

$$(m\langle q, x_i\rangle, y_i) = (m, \langle q, x_i\rangle y_i) = (m, q)$$

$$f(x_i)\langle y_i, p\rangle = f(x_i\langle y_i, p\rangle) = f(p).$$

② ⟹ ① Since $P \otimes_B Q \xrightarrow{\sim} \mathrm{Hom}_B(P, P)$ there is a unique $(x_i, y_i) \in P \otimes_B Q$ mapping to id, i.e. $x_i\langle y_i, p\rangle = p$, $\forall p \in P$. By Yoneda we get a map of functors $\mathrm{Hom}_B(P, M) \to M \otimes_B Q$ such that $f \longmapsto (f(x_i), y_i)$.

Moreover this functor is inverse to $(m, q) \mapsto (p \mapsto m\langle q, p\rangle)$, i.e. $(m\langle q, x_i\rangle, y_i) = (m, q)$. Thus $(m, \langle q, x_i\rangle y_i) = (m, q)$ for all M_B, $m \in M$, $q \in Q$ so taking $M = B$, $m = 1$ we get (using $B \otimes_B Q \xrightarrow{\sim} Q$) that $\langle q, x_i\rangle y_i = q$, $\forall q \in Q$.

Similarly ① ⟺ ②′.

The equivalences ② ⟺ ③, ②′ ⟺ ③′ are

standard. Recall

$$M \otimes_B \mathrm{Hom}_B(P, B) \longrightarrow \mathrm{Hom}_B(P, M)$$

is an isomorphism for P ft free, hence for P ftp, as P is a summand of some $B^{\oplus n}$. Thus ③ $\Rightarrow$ ②. Conversely assuming ② we get $Q \xrightarrow{\sim} \mathrm{Hom}_B(P, B)$ taking $M = B$, so we need only show B is ftp. But if $(x_i, y_i) \mapsto id_P$ i.e. $x_i \langle y_i, p\rangle = p$, $\forall p$ it follows that P is a summand of $B^{\oplus n}$:

$$P \xrightarrow{\begin{pmatrix} \vdots \\ \langle y_i, -\rangle \\ \vdots \end{pmatrix}} B^{\oplus n} \xrightarrow{(x_1 \cdots x_n)} P .$$

Combining ③ and ③$'$ we get

$$Q \xrightarrow{\sim} \mathrm{Hom}_B(P, B)$$

$$P \xrightarrow{\sim} \mathrm{Hom}_{B^e}(Q, B) \xleftarrow{\sim} \mathrm{Hom}_{B^e}(\mathrm{Hom}_B(P, B), B)$$

showing P coincides with its double dual.

Now we move on to pairs of algebras and "correspondences", i.e. bimodules.

Let A, B be algebras, let ${}_A P_B$ be a bimodule and consider the functor

$$F : \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(B)$$
$$X \longmapsto X \otimes_A P$$

We ask when F has left and right adjoint functors which are also given by bimodules. Since

$$\mathrm{Hom}_B(X\otimes_A P, Y) = \mathrm{Hom}_A(X, \mathrm{Hom}_B(P,Y))$$

it follows that the right adjoint functor of F exists always and is $Y \longmapsto \mathrm{Hom}_B(P,Y)$. This is given by a bimodule ${}_BQ_A$ i.e.

$$Y\otimes_B Q = \mathrm{Hom}_B(P,Y)$$

iff (see above) P is ftp over B, and in this case $Q = \mathrm{Hom}_B(P,B)$.

Suppose now that F has a left adjoint. Then F commutes with arbitrary projective limits, in particular P is A-flat. It's probable that P must have finite presentation, in which case it is ftp over A. Then $F(X) = X\otimes_A P = \mathrm{Hom}_A(P^\vee, X)$, where ${}_B\check{P}_A = \mathrm{Hom}_{A^e}(P,A)$, and the left adjoint functor of F is $Y \longmapsto Y\otimes_B \check{P}$.

We can proceed instead without worrying about general existence results for module categories as follows. Suppose $F(X) = X\otimes_A P$ has the left adjoint functor $G(Y) = Y\otimes_B Q$, where Q is a bimodule ${}_BQ_A$. This means we are given adjunction arrows $\alpha: FG \longrightarrow 1$, $\beta: 1 \longrightarrow GF$ such that $F \xrightarrow{F.\beta} FGF \xrightarrow{\alpha.F} F$, $G \xrightarrow{\beta.G} GFG \xrightarrow{G.\alpha} G$ are the identity maps. Now

$$FG(Y) = F(Y \otimes_B Q) = Y \otimes_B Q \otimes_A P$$

$$GF(X) = G(X \otimes_A P) = X \otimes_A P \otimes_B Q$$

so α and β amount to bimodule maps

$$Q \otimes_A P \xrightarrow{\alpha} B \qquad (q, p) \longmapsto \langle q, p \rangle$$

$$A \xrightarrow{\beta} P \otimes_B Q \qquad 1 \longmapsto (x_i, y_i)$$

such that

$$P = A \otimes_A P \xrightarrow{\beta \otimes 1} P \otimes_B Q \otimes_A P \xrightarrow{1 \otimes \alpha} P \otimes_B B = B$$

$$p \mapsto (x_i, y_i, p) \longmapsto x_i \langle y_i, p \rangle$$

and

$$Q = Q \otimes_A A \xrightarrow{1 \otimes \beta} Q \otimes_A P \otimes_B Q \xrightarrow{\alpha \otimes 1} B \otimes_B Q = Q$$

$$q \longmapsto (q, x_i, y_i) \longmapsto \langle q, x_i \rangle y_i$$

are the identity maps. By preceding discussion it follows that P, Q are ftp over B and are B duals ~~of~~ each other.

Similarly if $F(X) = X \otimes_A P$ has the left adjoint $Y \longmapsto Y \otimes_B N$, where N is a bimodule ${}_B N_A$, then P, N are ftp over A and are A duals of each other.

Alternative notation Q_l, Q_r instead of N, Q to indicate the bimodules giving the left and right adjoints to $X \to X \otimes_A P$.

Suppose again given A, B, ${}_A P_B$ as above. Assume P is ftp both over A and B so that we have (B,A)-bimodules

$$Q_l = \mathrm{Hom}_{A^e}(P, A)$$

$$Q_r = \mathrm{Hom}_B(P, B)$$

yielding the left and right adjoint functors for $M \longmapsto M \otimes_A P$. We have adjunction arrows

$$\begin{cases} P \otimes_B Q_l \longrightarrow A \\ B \longrightarrow Q_l \otimes_A P \end{cases}$$

$$\begin{cases} Q_r \otimes_A P \longrightarrow B \\ A \longrightarrow P \otimes_B Q_r \end{cases}$$

Kadison, inspired by the example of restriction and transfer for finite groups, looks at the situation where the left and right adjoints coincide. This means we have an isomorphism $Q_l \simeq Q_r$ of (B,A)-bimodules. One might further ask that the compositions

$$A \longrightarrow P \otimes_B Q_r \simeq P \otimes_B Q_l \longrightarrow A$$

$$B \longrightarrow Q_l \otimes_A P \simeq Q_r \otimes_A P \longrightarrow B$$

be (scalar multiples of) the identity.

Let's discuss the example of a homomorphism $u : B \to A$ and the bimodule ${}_AP_B = {}_AA_B$ corresponding to restriction of scalars:

$$M \longmapsto M \otimes_A A_B = M_B$$

Then $Q_\ell = \mathrm{Hom}_{A^e}({}_AA_B, A) = {}_BA_A$ and the left adjoint is extension of scalars

$$N \longmapsto N \otimes_B Q_\ell = N \otimes_B A$$

The adjunction arrows for the pair $({}_BA_A, {}_AA_B)$ are

$$M \otimes_B A \longrightarrow M \qquad (m, a) \longmapsto ma$$

$$N \longrightarrow N \otimes_B A \qquad n \longmapsto (n, 1)$$

corresponding the A (resp B) bimodule maps

1) $$\boxed{\begin{array}{ll} A \otimes_B A \longrightarrow A & (a_1, a_2) \longmapsto a_1 a_2 \\ B \longrightarrow B \otimes_B A \to A & b \mapsto (b \otimes 1) \Leftrightarrow u(b) \end{array}}$$

Next we have the right adjoint

$N \longmapsto \mathrm{Hom}_B(A, N)$ for restriction of scalars:

$$\mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)) = \mathrm{Hom}_B(M, N)$$

The adjunction arrows are

$$\mathrm{Hom}_B(A, N) \longrightarrow N \qquad f \longmapsto f(1)$$

$$M \longrightarrow \mathrm{Hom}_B(A, M) \qquad m \longmapsto (a \longmapsto ma)$$

Assume now that we are given an isomorphism between the left and right adjoints:

$$\circledast \quad N \otimes_B A \simeq \mathrm{Hom}_B(A, N)$$

This means that A is ftp over B and that one has an isomorphism

$$A \xrightarrow{\sim} \mathrm{Hom}_B(A, B)$$

of (B, A)-bimodules. Because this map is compatible with right A-multiplication it has the form $a \longmapsto (\alpha \longmapsto \rho(a\alpha))$, where $\rho : A \longrightarrow B$ is a right B-module map. Because it is compatible with left B-multiplication, we have $\rho(ba\alpha) = b\rho(a\alpha)$, so ρ is a left B-module map.

Thus $\circledast$ is given by

$$(n, a) \longmapsto (\alpha \longmapsto n\rho(a\alpha))$$
$$(f(x_i), y_i) \longleftarrow f$$

where $(x_i, y_i) \in A \otimes_B A$ yields the identity map

The adjunction arrows for the restriction of scalars and its right adjoint $\circledast$ are then

$$N \otimes_B A \simeq \mathrm{Hom}_B(A, N) \xrightarrow[\text{at } 1]{\text{evaluation}} N$$
$$(n, a) \longmapsto (\alpha \longmapsto n\rho(a\alpha)) \longmapsto n\rho(a)$$

and

$$M \longrightarrow \mathrm{Hom}_B(A, M) \simeq M \otimes_B A$$
$$m \longmapsto (a \longmapsto ma) \longmapsto (mx_i, y_i)$$

which correspond the bimodule maps

$$2) \quad \boxed{\begin{array}{lll} A = B \otimes_B A \longrightarrow B & & a \longmapsto \rho(a) \\ A \longrightarrow A \otimes_B A & & a \longmapsto (ax_i, y_i) \end{array}}$$

over B, A respectively.

The compositions are then

$$B \xrightarrow{u} A \xrightarrow{\rho} B \qquad\qquad 1 \longmapsto \rho(1)$$

$$A \longrightarrow A \otimes_B A \longrightarrow A \qquad 1 \longmapsto (x_i, y_i) \longmapsto x_i y_i$$

The purpose of the above calculation was to ~~[illegible]~~ explore Kadison's notion for [illegible] functors (F, G) which are adjoint in either order such that the adjunction counits $FG \longrightarrow 1$, $GF \longrightarrow 1$ are split epis. I learned that the splittings are not usually given by the unit maps $1 \to GF$, $1 \to FG$.

March 29, 1994

Let's record an important result from yesterday's work:

Given algebras A, B bimodules ${}_BP_A$, ${}_AQ_B$ such that the functors $F(M) = P\otimes_A M$, $G(N) = Q\otimes_B N$ are adjoint, i.e. one has a functorial isom.

$$(*) \qquad \mathrm{Hom}_{B^e}(P\otimes_A M, N) \simeq \mathrm{Hom}_{A^e}(M, Q\otimes_B N).$$

The claim is that the adjunction data $FG \xrightarrow{\alpha} 1$, $1 \xrightarrow{\beta} GF$ yielding $(*)$ amount to bimodule maps

$$P\otimes_A Q \longrightarrow B \qquad (p,q) \longmapsto \langle p,q\rangle$$

$$A \longrightarrow Q\otimes_B P \qquad 1 \longmapsto (q_i, p_i)$$

over B and A respectively which satisfy

$$p = \langle p, q_i\rangle p_i \qquad q = q_i\langle p_i, q\rangle \qquad p\in P, q\in Q$$

In this case Q is an ftp B^r module, P is an ftp B^ℓ module and they are dual.

Suppose now ${}_BP_A$ is an <u>invertible</u> bimodule. This means there is a bimodule ${}_AQ_B$ together with bimodule isomorphisms

$$(*) \qquad P\otimes_A Q \simeq B \qquad Q\otimes_B P \simeq A$$

over B, A respectively which are compatible in the sense that the compositions

$$P = B\otimes_B P \simeq P\otimes_A Q\otimes_B P \simeq P\otimes_A A = P$$

$$Q = A\otimes_A Q \simeq Q\otimes_B P\otimes_A Q \simeq Q\otimes_B B = Q$$

are the identity.

This is the same as a <u>Morita equivalence</u> between A, B. Thus $M \longmapsto P \otimes_A M$ is an equivalence of categories from $\mathrm{Mod}(A^l)$ to $\mathrm{Mod}(B^l)$ [redacted] with quasi-inverse $N \longmapsto Q \otimes_B N$.

We can view the isomorphisms $(*)$ as adjunction arrows

$$P \otimes_A Q \longrightarrow B \qquad\qquad A \longrightarrow Q \otimes_B P$$

[redacted] with the additional property of being isomorphisms. Thus we know Q is a ftp B^r module and $P \simeq \mathrm{Hom}_{B^r}(Q, B)$ is its dual. We can also view $(*)$ as adjunctions arrows

$$P \otimes_A Q \longleftarrow B \qquad\qquad A \longleftarrow Q \otimes_B P$$

which are isomorphisms, whence P is a ftp A^r module and $Q \simeq \mathrm{Hom}_{A^r}(P, A)$ is its dual. Finally

$$B \simeq P \otimes_A Q = P \otimes_A \mathrm{Hom}_{A^r}(P, A) = \mathrm{Hom}_{A^r}(P, P)$$

$$B \simeq P \otimes_A Q = \mathrm{Hom}_{A^l}(P, A) \otimes_A Q = \mathrm{Hom}_{A^l}(Q, Q)$$

The latter says that [redacted] $Q \otimes_B B = Q$, which is the image under $\mathrm{Mod}(B^l) \longrightarrow \mathrm{Mod}(A^l)$, $N \mapsto Q \otimes_B N$, has the same endomorphisms as B.

I now want to discuss Pimsner's construction, an algebraic (ftp) version.

Let A be an algebra, let E be an A bimodule which is ftp as A^2 module, let $E^* = \mathrm{Hom}_{A^2}(E, A)$. Then we have canonical bimodule maps

$$E^* \otimes_A E \longrightarrow A \qquad (y, x) \mapsto \langle y, x \rangle$$

$$A \longrightarrow E \otimes_A E^* \qquad 1 \mapsto (x_i, y_i)$$

such that $\langle y, x_i \rangle y_i = y$, $x_i \langle y_i, x \rangle = x$ $\forall y \in E^*, x \in E$.

Define the <u>Toeplitz algebra</u> $\mathcal{T}_E$ corresponding to this data to be the tensor algebra over A of $E \oplus E^*$ divided by the ideal generated by the relations $yx = \langle y, x \rangle$ for $x \in E$, $y \in E^*$. Thus $\mathcal{T}_E$ is generated $\overset{\text{over } A}{\text{by}}$ elements T_x, T_y^* satisfying $T_{x+x'} = T_x + T_{x'}$, $T_{axa'} = a T_x a'$, similarly for T_y^*, and finally the key relation

$$T_y^* T_x = \langle y, x \rangle$$

$\mathcal{T}_E$ acts on $T(E) = A \oplus E \oplus E^{\otimes 2} \oplus \cdots$ (where here $E^{\otimes 2}$ means $E \otimes_A E$) with T_x left multiplication by $x \in E$ and T_y^* interior product by y:

$$T_x(x_1, \ldots, x_n) = (x, x_1, \ldots, x_n)$$

$$T_y^*(x_1, \ldots, x_n) = (\langle y, x_1 \rangle x_2, x_3, \ldots, x_n)$$

Thus $\mathcal{T}_E$ is a kind of noncommutative Clifford algebra in the sense that $\mathrm{Cliff}(V \oplus V^*) = \mathrm{End}(\Lambda V)$.

The Cuntz-Krieger algebra $\mathcal{O}_E$ should be the quotient of $\mathcal{T}_E$ by the additional relation

$$1 = T_{x_i} T_{y_i}^*$$

Note that $T_{x_i} T_{y_i}^*$ is the identity on $E^{\otimes n}$ for $n \geq 1$ and zero on $E^{\otimes 0} = A$. Thus $1 - T_{x_i} T_{y_i}^*$ is the projection onto A and

$$\left(1 - T_{x_i} T_{y_i}^*\right) T_x = 0$$

$$T_y^* \left(1 - T_{x_i} T_{y_i}^*\right) = 0.$$

Now $\mathcal{T}_E$ is spanned by products

$$T_{\xi_1} \cdots T_{\xi_p} T_{\eta_q}^* \cdots T_{\eta_1}^* \qquad \begin{array}{l} \xi_i \in E \\ \eta_j \in E^* \end{array}$$

so if $\mathcal{K}_E$ is the ideal $\mathcal{T}_E \left(1 - T_{x_i} T_{y_i}^*\right) \mathcal{T}_E$, then $\mathcal{K}_E$ is spanned by products

$$T_{\xi_1} \cdots T_{\xi_p} \left(1 - T_{x_i} T_{y_i}^*\right) T_{\eta_q}^* \cdots T_{\eta_1}^*$$

These act as finite rank operators on $T(E)$.

We have the Toeplitz extension

$$0 \longrightarrow \mathcal{K}_E \longrightarrow \mathcal{T}_E \longrightarrow \mathcal{O}_E \longrightarrow 0$$

Next note that on $\mathcal{T}_E$ there is a $\mathbb{Z}$ grading where $|a| = 0$, $|T_x| = 1$, $|T_y^*| = -1$. This grading is compatible with the Toeplitz extension. In $\mathcal{O}_E$ we have $1 = T_{x_i} T_{y_i}^*$ where $|T_{x_i}| = 1$, $|T_y^*| = -1$. Let's examine this sort of situation.

Suppose we have a $\mathbb{Z}$-graded algebra $R = \bigoplus_{n\in\mathbb{Z}} R_n$ and put $B = R_0$, $P = R_1$ and $Q = R_{-1}$. Then P, Q are B-bimodules, and PQ, QP are ideals in B. Assume $PQ = B$ i.e. $p_i q_i = 1$. Then considering Q as B^ℓ module and P as B^r module we have a B-bimodule map

$$Q \otimes P \longrightarrow B \qquad (q,p) \longmapsto \langle q,p\rangle = \underset{\text{in } R}{qp} \in B$$

such that

$$p = p_i q_i p = p_i \langle q_i, p\rangle$$

$$q = q p_i q_i = \langle q, p_i\rangle q_i$$

so we know $\overset{(p.440)}{\text{that}}$ P is a ftp B^r-module, Q is a ftp B^ℓ-module and $Q = \operatorname{Hom}_{B^r}(P, B)$.

Moreover we have for any B^r module M:

$$M \otimes_B Q \xrightarrow{\sim} \operatorname{Hom}_{B^r}(P, M)$$

$$(m, q) \longmapsto (p \mapsto m\langle q,p\rangle)$$

so

$$P \otimes_B Q \xrightarrow{\sim} \operatorname{Hom}_{B^r}(P, P)$$

$$(p_1, q) \longmapsto (p \mapsto \underbrace{p_1\langle q,p\rangle}_{p_1 q p})$$

showing that

$$\begin{tikzcd} P\otimes_B Q \arrow[rr, "\sim"] \arrow[dr] & & \operatorname{Hom}_{B^r}(P,P) \\ & B \arrow[ur, "\text{left mult action}"'] & \end{tikzcd}$$

$(p,q) \mapsto pq$

onto since $p_i q_i = 1$

Thus

$$P \otimes_B Q \xrightarrow{\sim} B \xrightarrow{\sim} \operatorname{Hom}_{B^r}(P,P)$$

So far we have used only that $PQ = B$. But now suppose further that $QP = B$. Then we should have also $Q \otimes_B P \xrightarrow{\sim} B$ also given by multiplication. Thus P is an invertible bimodule and Q is its inverse.

Short proof. Assume $PQ = B$, and let $p_i q_i = 1$. Consider

$$B \longrightarrow P \otimes_B Q \longrightarrow B$$

$$b \longmapsto (bp_i, q_i), \quad (p, q) \longmapsto pq$$

Then $b \longmapsto (bp_i, q_i) \longmapsto bp_i q_i = b$, and

$$(p, q) \longmapsto pq \longmapsto (\underbrace{pqp_i}_{\in B}, q_i) = (p, qp_i q_i) = (p, q)$$

Thus $P \otimes_B Q \xrightarrow{\sim} B$ and similarly $QP = B \Rightarrow Q \otimes_B P \xrightarrow{\sim} B$

Finally the compatibility holds:

$$P = B \otimes_B P \xleftarrow{\sim} P \otimes_B Q \otimes_B P \xrightarrow{\sim} P \otimes_B B = P$$

$$p_1 q p_2 \longleftarrow (p_1 q, p_2) \longleftarrow (p_1, q, p_2) \longmapsto (p_1, qp_2) \longmapsto p_1 q p_2$$

In $\mathcal{O}_E$ we have $T_y^* T_x = \langle y, x \rangle$ and $1 = T_{x_i} T_{y_i}^*$. The latter shows that if $Q = (\mathcal{O}_E)_{(-1)}$, $B = (\mathcal{O}_E)_{(0)}$, $P = (\mathcal{O}_E)_{(1)}$, then $PQ = B$.

To get $QP = B$ it suffices that $E^* \otimes_A E \xrightarrow{\langle\, ,\, \rangle} A$ be surjective.

<u>March 31, 1994</u>

A alg, E an A-bimodule, $E^* = \mathrm{Hom}_{A^r}(E, A)$. Define the <u>Toeplitz-Clifford algebra</u> $\mathcal{T}_E$ to be

$$\mathcal{T}_E = T_A(E \oplus E^*) \big/ (yx - \langle y, x\rangle)$$

where $(y,x) \longmapsto \langle y, x\rangle$ is the canonical pairing $E^* \otimes_A E \longrightarrow A$, and y runs over E^*, x runs over E. Write T_x, T_y^* for the images of $x \in E$, $y \in E^*$ in $\mathcal{T}_E$. Then $\mathcal{T}_E$ is the algebra over A generated by the elements T_x, T_y^* subject to the relations of A-bilinearity: $T_{x_1+x_2} = T_{x_1} + T_{x_2}$, $T_{a_1 x a_2} = a_1 T_x a_2$, similarly for T_y^*, and the key relation $T_y^* T_x = \langle y, x\rangle$.

Concrete construction. First note that for any A^l bimodule M we have a canonical map

$$E^* = \mathrm{Hom}_{A^r}(E, A) \longrightarrow \mathrm{Hom}_{A^r}(E \otimes_A M, M)$$
$$y \longmapsto (x \otimes m \longmapsto \langle y, x\rangle m).$$

Write this $\iota_y(x \otimes m) = \langle y, x\rangle m$. Then

$$(a_1 \iota_y a_2)(x \otimes m) = a_1 \langle y, a_2 x\rangle m = \langle a_1 y a_2, x\rangle m = \iota_{a_1 y a_2}(x \otimes m).$$

Define a left action of $\mathcal{T}_E$ on $T_A(E) = \bigoplus_{n \geq 0} E^{\otimes n}$ by T_x = left mult by x and $T_y^* = \iota_y$, where ι_y is defined to be zero on $A = E^{\otimes 0}$. Let's check the relation $T_y^* T_x = \langle y, x\rangle$ in degree 0. Write $(\cdot) \in E^0$ for the identity of A. Then

$$T_y^* T_x (\cdot) = T_y^*(x) = \langle y, x\rangle (\cdot)$$

Next define a left action of $\mathcal{T}_E$ on $T_A(E) \otimes_A T_A(E^*)$ by

$$T_x(x_1,\dots,x_n,y_m,\dots,y_1) = (x,x_1,\dots,x_n,y_m,\dots,y_1)$$

$$T_y^*(x_1,\dots,x_n,y_m,\dots,y_1) = \begin{cases} \langle y,x_1\rangle (x_2,\dots,x_n,y_m,\dots,y_1) & \text{if } n \geq 1 \\ (y,y_m,\dots,y_1) & \text{if } n=0. \end{cases}$$

The relation $T_y^* T_x = \langle y,x \rangle$ is clear.

So now define two maps

$$\mathcal{T}_E \underset{\alpha}{\overset{\sigma}{\rightleftarrows}} T_A(E) \otimes_A T_A(E^*)$$

α is given by acting on $(\cdot) = 1 \otimes 1$ and

$$\sigma(x_1,\dots,x_n,y_m,\dots,y_1) = T_{x_1}\cdots T_{x_n} T_{y_m}^* \cdots T_{y_1}^*.$$

Since $T_{y_m}^* \cdots T_{y_2}^* T_{y_1}^* (\cdot) = T_{y_m}^* \cdots T_{y_2}^*(y_1) = \cdots = (y_m,\dots,y_1)$ it is clear that σ is a section of α: $\alpha\sigma = 1$.

Finally the image of σ is closed under left multiplication by T_x, T_y^*, so it's a left ideal containing 1 and σ is surjective.

Thus σ, α are inverses of each other.

Let E be a ftp A^2 module, $E^* = \mathrm{Hom}_{A^2}(E, A)$.
Recall that there is a canonical A-bimodule map
$$E^* \otimes_{\mathbb{C}} E \xrightarrow{\langle\,,\,\rangle} A$$
and a canonical element $(x_i, y_i) \in E \otimes_A E^*$ such that
$$x = x_i \langle y_i, x\rangle \qquad \forall x \in E$$
$$y = \langle y, x_i\rangle y_i \qquad \forall y \in E^*.$$

Suppose E is a generator for $\mathcal{P}(A^2)$, i.e. A is a direct summand of $E^{\oplus n}$ for some n. Then we have maps
$$A \xrightarrow{a \mapsto \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} a} E^{\oplus n} \xrightarrow{(y_1', \dots, y_n')} A$$
with $x_i' \in E$ and $y_i' \in E^*$ such that $\langle y_i', x_i'\rangle = 1$
Thus $\langle\,,\,\rangle : E^* \otimes E \longrightarrow A$ is surjective.
Conversely if this map is surjective we get $\langle y_i', x_i'\rangle = 1$ and so A is a direct factor of $E^{\oplus n}$.

In general let $I = \mathrm{Im}\{E^* \otimes E \xrightarrow{\langle\,,\,\rangle} A\}$. This is an ideal in A such that $I^2 = I$. In effect, if $\langle y_j', x_j'\rangle \in I$, then with $(x_i, y_i) \in E \otimes_A E^*$ as above we have
$$\langle y_j', x_j'\rangle = \langle y_j', x_i \langle y_i, x_j'\rangle\rangle = \langle y_j', x_i\rangle \langle y_i, x_j'\rangle$$
showing $\langle y_j', x_j'\rangle \in I^2$. Alternatively if $E = e\underbrace{A^{\oplus n}}_{\text{column vectors}}$ with $e^2 = e$ in $M_n(A)$, then $E^* = \underbrace{(A^{\oplus n})^t}_{\text{row vectors}} e$

and $\langle E^*, E\rangle = (A^{\oplus n})^t e e A^{\oplus n} = \{ a^i e_i^j a_j \mid (a^i) \in (A^{\oplus n})^t, (a_j) \in A^{\oplus n} \}$

Thus I is the ideal $\sum_i Ae_i^j A$ and $I = I^2$, because $e_i^k = e_i^j e_j^k$ shows $e_i^k \in I^2$.

Repeat: E an ftp A^r-module, $E^* = \mathrm{Hom}_{A^r}(E, A)$; $B = \mathrm{Hom}_{A^r}(E,E)$. Then we have a canonical map

$$E^* \otimes_B E \xrightarrow{\langle\, , \,\rangle} A$$

of A-bimodules which is surjective iff E is a generator for $\mathcal{P}(A^r)$. In this case this map is an isomorphism.

In general We also have a canonical isom.

$$E \otimes_A E^* \xrightarrow{\sim} \mathrm{Hom}_{A^r}(E,E) = B$$

$$(x\, ,\, y) \longmapsto (x' \mapsto x\langle y, x'\rangle).$$

April 1, 1994

In studying Pimsner I have encountered a puzzle, which concerns his restriction to A-bimodules E such that A is the span of all inner products $\langle y, x\rangle$, $y \in E^*$, $x \in E$.

Let's recall the algebraic setting I use to try to understand his paper. E is a bimodule over A which is fp as A^r-module, $E^* = \mathrm{Hom}_{A^r}(E, A)$. One has bimodule maps

$$E^* \otimes E \xrightarrow{\langle\,,\,\rangle} A \qquad (y,x) \longmapsto \langle y, x\rangle$$

$$A \longrightarrow E \otimes_A E^* = \mathrm{Hom}_{A^r}(E,E)$$

We consider the Toeplitz-Clifford algebra

$$\mathcal{T}_E = T_A(E \oplus E^*) \big/ (T_y^* T_x - \langle y, x\rangle) \simeq T_A(E) \otimes_A T_A(E^*)$$

which is a $\mathbb{Z}$-graded algebra:

$$\begin{array}{ccc}
E^* & A & E \\
E \otimes E^{*\otimes 2} & E \otimes E^* & E^{\otimes 2} \otimes E^* \\
\vdots & E^{\otimes 2} \otimes E^{*\otimes 2} & E^{\otimes 3} \otimes E^{*\otimes 2} \\
\vdots & \vdots & \vdots \\
\underbrace{\qquad\qquad}_{\mathcal{T}_E^{(-1)}} & \underbrace{\qquad\qquad}_{\mathcal{T}_E^{(0)}} & \underbrace{\qquad\qquad}_{\mathcal{T}_E^{(1)}}
\end{array}$$

The Cuntz-Krieger alg $\mathcal{O}_E$ is the quotient of $\mathcal{T}_E$ by the relation $1 = T_{x_i} T_{y_i}^*$ where $(x_i, y_i) \in E \otimes_A E^*$ gives the identity operator on E. This means one takes the direct limit vertically in the above picture using the maps

$$E^{\otimes p} \otimes E^{*\otimes q} \longrightarrow E^{\otimes p+1} \otimes E^{*\otimes q+1}$$

$$T_{x_1} \cdots T_{x_p} T^*_{y_q} \cdots T^*_{y_1} \longmapsto T_{x_1} \cdots T_{x_p} T_{x_i} T^*_{y_i} T^*_{y_q} \cdots T^*_{y_1} .$$

This gives the picture for $\mathcal{O}_E$ as follows:

$$\begin{array}{ccc}
\mathcal{O}_E^{(-1)} & \mathcal{O}_E^{(0)} & \mathcal{O}_E^{(1)} \\
\| & \| & \| \\
 & \varinjlim E^{\otimes p} \otimes E^{*\otimes p} & \\
 & \| & \\
\underbrace{A_\infty \otimes_A E^*}_{E^*_\infty} & A_\infty & \underbrace{E \otimes_A A_\infty}_{E_\infty}
\end{array}$$

Then we reach a situation of a bimodule E_∞ over A_∞ such that $A_\infty \xrightarrow{\sim} \mathrm{Hom}_{A_\infty}(E_\infty, E_\infty)$. Notice that it does not seem to be true in general that E_∞ is an invertible bimodule, because

$$\langle E^*_\infty , E_\infty \rangle = A_\infty \langle E^*, E \rangle A_\infty$$

and $\langle E^*, E \rangle \subset A$ is an ideal which is equal to A iff E is a generator for $\mathcal{P}(A^r)$.

Pimsner's comments suggest that it might be possible to replace A by the ideal $\langle E^*, E \rangle$ in some way.

Problem: We have seen that the VJones construction (as we have discussed it) concerns a homomorphism $B \to A$ such that the two adjoints of the restriction of scalars functor are isomorphic. Let's examine whether B and $A_2 = A \otimes_B A$ are Morita equivalent, also A, and A_3.

Review: Restriction is given by the bimodule ${}_{A}P_{B} = {}_{A}A_{B}$, so that $M \otimes_A A_B = M_B$; in the case of right modules. We need to use right modules to fit the VJones pattern: $N \otimes_B A \simeq \mathrm{Hom}_{B^r}(A, N)$

First adjunction formula

$$\mathrm{Hom}_{A^r}(N \otimes_B A, M) = \mathrm{Hom}_{B^r}(N, {}_B M)$$

(here $Q_\ell = \mathrm{Hom}_{A^\ell}(P, A) = \mathrm{Hom}_{A^\ell}(A, A) = {}_B A_A$).

Adjunction arrows

$$M \otimes_B A \longrightarrow M \qquad \text{mult.}$$

$$N \longrightarrow N \otimes_B A \qquad n \mapsto n \otimes 1$$

Second adjunction fmla

$$\mathrm{Hom}_{A^r}(M, \mathrm{Hom}_{B^r}(A, N)) = \mathrm{Hom}_{B^r}(M, N)$$

Adjunction arrows

$$\mathrm{Hom}_{B^r}(A, N) \longrightarrow N \qquad \text{eval at } 1$$

$$M \longmapsto \mathrm{Hom}_{B^r}(A, M) \qquad m \mapsto (a \mapsto ma)$$

Suppose now given an isomorphism between the adjoints

$$N \otimes_B A \simeq \mathrm{Hom}_{B^r}(A, N)$$

The map $\longrightarrow$ given by a B-bimodule map $\rho : A \longrightarrow B$, namely $(n, a) \mapsto (\alpha \mapsto n\rho(a\alpha))$

The map $\longleftarrow$ given by $(x_i, y_i) \in A \otimes_B A$, the identity element, namely $\phi \mapsto (\phi(x_i), y_i)$.

The adjunction arrow for $\mathrm{Hom}_{B^e}(A, N) = N \otimes_B A$ are

$$N \otimes_B A \longrightarrow N \qquad (n, a) \longmapsto n\rho(a)$$

$$M \longrightarrow M \otimes_B A \qquad m \longmapsto (mx_i, y_i)$$

We now look at Morita invariance between B and $A_2 = A \otimes_B A \simeq \mathrm{Hom}_{B^e}(A, A)$. We have $P = {}_A A_B$ which is fp as B^e module, and its dual is $P^* = \mathrm{Hom}_{B^e}(A, B) \simeq A$, the isom. being $a \mapsto (\alpha \mapsto \rho(a\alpha))$. The pairing $P^* \otimes P \longrightarrow B$ is

$$A \otimes A \longrightarrow \mathrm{Hom}_{B^e}(A, B) \otimes A \longrightarrow B$$
$$(a_1, a_2) \longmapsto ((\alpha \mapsto \rho(a_1\alpha)), a_2) \longmapsto \rho(a_1 a_2)$$

So the ideal in B of scalar products is the image of $\rho : A \longrightarrow B$. It's not clear in general that ρ has to be onto.

In the case of $A_1 \longrightarrow A_2$ the analogue of ρ is the multiplication $\rho(2) : A \otimes_B A \longrightarrow A$, $(a_1, a_2) \mapsto a_1 a_2$, and this is onto as $(1, 1) \longmapsto 1$. Then it follows that $\rho(n) : A_n \longrightarrow A_{n-1}$ is onto for all $n \geq 2$. That is $\rho(3)$ is onto because it's a $\rho(2)$ ~~starting~~ from $A_1 \to A_2$. This is clear also from the formulas and $x_i \rho(y_i) = 1$.

Next examine instead of a homom. $B \to A$ the case of the nonunital homomorphism $B = eAe \subset A$. The "restriction" functor in this case is $M \mapsto Me \simeq M \otimes_A Ae$, so it is given by the bimodule ${}_A P_B = Ae$. This is fp as A^l module with dual $Q_l \simeq \mathrm{Hom}_{A^l}(Ae, A) = eA$. The first adjunction formula is

$$\mathrm{Hom}_{A^r}(N \otimes_B eA, M) = \mathrm{Hom}_{B^r}(N, Me)$$

The left adjoint of $M \mapsto Me = M \otimes_A Ae$ is $N \mapsto N \otimes_B eA$. The right adjoint is $N \mapsto \mathrm{Hom}_{B^r}(Ae, N)$.

Assume these adjoints are isomorphic. Then Ae is a fp B^r module with dual eA. Now we have an obvious pairing

$$eA \otimes_A Ae \xrightarrow{\langle\,,\,\rangle} B$$
$$ea_1, a_2e \longmapsto ea_1a_2e$$

which gives a map $eA \longrightarrow \mathrm{Hom}_{B^r}(Ae, B)$. Let's assume this pairing gives the isomorphism

$$N \otimes_B eA \xrightarrow{\sim} \mathrm{Hom}_{B^r}(Ae, N)$$
$$(n, ea) \longmapsto (\alpha e \longmapsto n(ea\alpha e))$$

This means we have $(x_ie, ey_i) \in Ae \otimes_B eA$ such that

$$ea = \langle ea, x_ie\rangle ey_i = ea(x_iey_i)$$
$$ae = x_ie\langle ey_i, ae\rangle = (x_iey_i)\,ae.$$

Then $a_1 e a = (a_1 e a)(x_i e y_i)$

$a e a_1 = (x_i e y_i)(a e a_1)$

which means that $x_i e y_i$ is an identity element for $I = AeA$. So putting $p = x_i e y_i$, we have $pa = \underset{\in I}{(pa)}\,p = p\,\underset{\in I}{(ap)} = ap$. Thus p is a central idempotent, which is a rather special situation.

Davydov result (seen briefly at MPI visit).

Let e be an idempotent in A, and consider the fp A^r module eA. The dual fp A^l module is Ae and $\mathrm{Hom}_{A^r}(eA, eA) = eA \otimes_A Ae = eAe = B$. The functor $N \longmapsto N \otimes_B eA$ maps $\mathcal{P}(B^r) \longrightarrow \mathcal{P}(A^r)$ and hence induces a map of K-groups $K_*(B) \longrightarrow K_*(A)$.

If we <u>assume</u> Ae is fp as B^r module, then $M \longmapsto M \otimes_A Ae = Me$ maps $\mathcal{P}(A^r) \longrightarrow \mathcal{P}(B^r)$, yielding a map $K_*(A) \longrightarrow K_*(B)$. Since

$$N \otimes_B eA \otimes_A Ae \simeq N$$

The composition $K_*(eAe) \longrightarrow K_*(A) \longrightarrow K_*(eAe)$ is the identity, so that we have

$$K_*(A) = K_*(eAe) \oplus (?)$$

Davydov I think has a formula for the complement, which should be something like the K-theory of $A/AeA = e^\perp A e^\perp / e^\perp AeAe^\perp$.

What is intriguing here is that the non unital algebra $\mathcal{A} = Ae \otimes_B eA$ seems to have

K-theory the same as B.

<u>April 2, 1994</u>

In the VJones construction: $B \subset A \xrightarrow{\rho} B$, x_i, y_i etc., let $e \in A$ be an idempotent such that $\rho(e) = 1$. Then $(e,e) \in A_2$ is idempotent:

$$(e,e)(e,e) = (e\rho(e^2), e) = (e\rho(e), e) = (e,e)$$

and $\rho_2(e,e) = e^2 = e$. Actually I want to be able to iterate this so suppose only then $e\rho(e) = e$ (or $\rho(e)e = e$). Then $e_2 = (e,e)$ is an idempotent in A_2 such that $e_2\rho_2(e_2) = (e,e)e$ $= (e,e^2) = (e,e) = e_2$.

Assume now $\rho(1) = 1$, take $e_1 = 1 \in A$, then $e_2 = (1,1) \in A_2$, $e_3 = ((1,1),(1,1)) \leftrightarrow (1,1,1)$, $e_4 = (1,1,1,1)$ are all idempotents. What's the relation to VJones' idempotents?

Davydov's result seems to be the following. Let e be an idempotent in A. Assume $eA \in \mathcal{P}(eAe)$ and the canonical map

$$Ae \otimes_{eAe} eA \longrightarrow A$$

is injective. Then one has a canonical isom.

$$K_*(A) = K_*(eAe) \oplus K_*\big(\underbrace{e^\perp A e^\perp / e^\perp A e A e^\perp}_{A/AeA}\big).$$

Why? Put $B = eAe$. One has adjoint functors

$$\mathrm{Mod}(B) \rightleftarrows \mathrm{Mod}(A) \qquad \begin{array}{c} N \longmapsto Ae\otimes_B N \\ eM = eA\otimes_A M \longleftarrow\!\shortmid\, M \end{array}$$

which are the extension & restriction of scalars resp. associated to the nonunital homom. $eAe \longrightarrow A$. The upper is the left adjoint and the composition

$$N \longmapsto e(Ae\otimes_B N) = eAe \otimes_B N = N$$

is isomorphic to the identity via the adjunction arrow. Hence the extension of scalars $N \mapsto Ae\otimes_B N$ is a fully faithful functor $\mathrm{Mod}(B) \hookrightarrow \mathrm{Mod}(A)$. In this way one can identify $\mathrm{Mod}(B)$ with the full subcategory of A-modules generated by Ae.

Extension of scalars also gives a fully faithful functor $\mathcal{P}(B) \hookrightarrow \mathcal{P}(A)$ identifying $\mathcal{P}(B)$ with the Karoubian subcategory of $\mathcal{P}(A)$ with generators Ae.

The assumption that $eA \in \mathcal{P}(B)$ means that restriction of scalars $M \longmapsto eM = eA\otimes_A M$ maps $\mathcal{P}(A)$ into $\mathcal{P}(B)$. Thus we have adjoint functors

$$\mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

composing to give the identity on $\mathcal{P}(B)$. These induce maps on higher K groups so $K_*(B)$ is naturally a (direct) summand of $K_*(A)$.

Next let $C = A/AeA = e^{\perp}Ae^{\perp}/e^{\perp}AeAe^{\perp}$ (since $(eA)^2 = eA$ and $(Ae)^2 = Ae$). The homomorphism $A \twoheadrightarrow C$ induces $\mathcal{P}(A) \to \mathcal{P}(C)$ killing the subcategory $(\simeq)\ \mathcal{P}(B)$ since $(A/AeA)\otimes_A Ae = Ae/(Ae)^2 = 0$. Thus one has a map $K_*(A)/K_*(B) \longrightarrow K_*(C)$.

We now want to get a map $K_*(C) \to K_*(A)$ using restriction of scalars for the homom. $A \to C$. For this we want to know C is a perfect complex of A-modules, so we can apply resolution.

Now by the assumption that $Ae\otimes_B eA \longrightarrow A$ is injective we have an exact sequence

$$0 \longrightarrow Ae\otimes_B eA \longrightarrow A \longrightarrow \underset{\overset{\parallel}{C}}{A/AeA} \longrightarrow 0$$

where $Ae\otimes_B eA$ is in $\mathcal{P}(A)$ as $eA \in \mathcal{P}(B)$. More generally for $M \in \mathcal{P}(A)$ we have an exact sequence

$$0 \longrightarrow Ae\otimes_B eM \longrightarrow M \longrightarrow C\otimes_A M \longrightarrow 0$$

where $Ae\otimes_B eM$ and $M \in \mathcal{P}(A)$.

~~At this point I want to replace $\mathcal{P}(C)$ by the image of $\mathcal{P}(A) \to \mathcal{P}(C)$~~

Let us now introduce $\mathcal{P}' \subset Mod(A)$, the full subcategory of M such that i) $\exists$ a length one

resolution
$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$
with $P_0, P_1 \in \mathcal{P}(A)$, and

ii) $\mathrm{Tor}_1^A(C, M) = 0$.

I think the resolution theorem shows that the inclusion $\mathcal{P}(A) \hookrightarrow \mathcal{P}'$ induces $K_*(A) \simeq K_*(\mathcal{P}')$.

I should check carefully that the restriction functor $\mathrm{Mod}(C) \longrightarrow \mathrm{Mod}(A)$ associated to $A \twoheadrightarrow C$ carries $\mathcal{P}(C)$ into $\mathcal{P}'$. Suppose that $X \in \mathcal{P}(C)$ and choose $0 \to K \to P \to X \to 0$ with $P \in \mathcal{P}(A)$. We want to show that $K \in \mathcal{P}(A)$. Let X' be such that $X \oplus X' = C^m$, choose $0 \to K' \to P' \to X' \to 0$ with $P' \in \mathcal{P}(A)$. Then, if $I = AeA = Ae \otimes_B eA$, upon comparing the resolutions
$$\begin{array}{ccccccccc} 0 & \longrightarrow & K \oplus K' & \longrightarrow & P \oplus P' & \longrightarrow & X \oplus X' & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & I^m & \longrightarrow & A^m & \longrightarrow & C^m & \longrightarrow & 0 \end{array}$$
Shanuel's lemma gives $P \oplus P' \oplus I^m \simeq A^m \oplus K \oplus K'$ showing that $K, K' \in \mathcal{P}(A)$. On the other hand
$$\mathrm{Tor}_1^A(C, C) = \mathrm{Tor}_1^A(A/I, A/I) = I/I^2 = 0.$$
so also $\mathrm{Tor}_1^A(C, X) = 0$ for X a summand of the C-module C^m.

At this point we have exact functors
$$\mathcal{P}(C) \hookrightarrow \mathcal{P}' \longrightarrow \mathcal{P}(C)$$
$$M \longmapsto C \otimes_A M$$
with composition the identity. So we have the following maps

$$\begin{tikzcd} K(B) \arrow[r, shift left, "Ae\otimes_B -"] & K(A) \arrow[l, shift left, "e-"] \arrow[d, "s"] & \\ & K(P') \arrow[r, shift left, "C\otimes_A -"] & K(C) \arrow[l, shift left, "\iota_*"] \end{tikzcd}$$

Note that the composition $(C\otimes_A -)\iota_*$ is the identity, whence $K(A) \longrightarrow K(C)$ is onto. Finally the exact sequence

$$0 \longrightarrow Ae\otimes_B eM \longrightarrow M \longrightarrow C\otimes_A M \longrightarrow 0$$

shows that one has

$$1 = (Ae\otimes_B -)(e-) + \iota_*(C\otimes_A -)$$

Since $(C\otimes_A -)(Ae\otimes_B -) = 0$, the two summands commute and annihilate each other, so

$$\underbrace{(Ae\otimes_B -)}_{\text{injective}}(e-)(\iota_*)\underbrace{(C\otimes_A -)}_{\text{surjective}} = 0$$

so $(e-)(\iota_*) = 0$. This concludes the proof of Davydov's result.

Next discuss an example. Suppose $eAe^\perp = 0$, i.e.

$$A = \begin{pmatrix} eAe & 0 \\ e^\perp Ae & e^\perp Ae^\perp \end{pmatrix}$$

Then $Ae = AeA$ and $e^\perp A = Ae^\perp A$ are ideals.

Let's first note that the hypotheses of Davidov's result hold. One has $eA = eAe \in \mathcal{P}(eAe)$ obviously. On the other hand

$$Ae \otimes_{eAe} eA = Ae \otimes_{eAe} eAe \xrightarrow{\sim} Ae \subset A$$

Thus one has by his thm.

$$K_*(A) = K_*(eAe) \oplus K_*(e^\perp A e^\perp)$$

The proof in this case is simpler it seems. One has the exact sequence of A-bimodules

$$0 \longrightarrow Ae \longrightarrow A \longrightarrow A/Ae \longrightarrow 0$$

which as left A-modules are in $\mathcal{P}(A)$. Thus the identity map of $K_*(A)$ is the sum of the maps

$$[M] \longmapsto [Ae \otimes_A M]$$

$$[M] \longmapsto [A/Ae \otimes_A M]$$

Now $\quad Ae \otimes_A M = Ae \otimes_A M/e^\perp M = Ae \otimes_{A/e^\perp A} M/e^\perp M$

and $A/e^\perp A$ is $B = eAe$ considered as quotient algebra of A. Thus $M \mapsto Ae \otimes_A M$ is the composite of

$$\mathcal{P}(A) \longrightarrow \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$$

$$M \longmapsto B \otimes_A M$$

$$N \longmapsto Ae \otimes_B N$$

and yields the map

$$K(A) \underset{\substack{\uparrow \\ \text{induced by} \\ \text{the surjection } A \to B}}{\longrightarrow} K(B) \underset{\substack{\uparrow \\ \text{induced by the} \\ \text{nonunital homom. } B \hookrightarrow A}}{\longrightarrow} K(A)$$

The second functor

$$M \longmapsto A/Ae \otimes_A M$$

is

$$\begin{array}{ccccc} \mathcal{P}(A) & \longrightarrow & \mathcal{P}(A/Ae) & \longrightarrow & \mathcal{P}(A) \\ M & \longmapsto & A/Ae \otimes_A M & & \\ & & N & \longmapsto & N \end{array}$$

where the second functor is well-defined as the bimodule ${}_A(A/Ae)_{A/Ae}$ is fp as left A-module. The second functor yields the map

$$K(A) \xrightarrow[\substack{\text{extension of}\\ \text{scalars relative}\\ \text{to } A \twoheadrightarrow A/Ae}]{} K(\underbrace{A/Ae}_{A/AeA \,=\, e^{\perp}Ae^{\perp}}) \xrightarrow[\substack{\text{restriction of}\\ \text{scalars relative}\\ \text{to } A \to A/Ae}]{} K(A)$$

~~Since~~ since

$$\begin{array}{ccccc} \mathcal{P}(A/Ae) & \xrightarrow{\text{res}} & \mathcal{P}(A) & \xrightarrow{\text{extn}} & \mathcal{P}(A/Ae) \\ N & \longmapsto & N & \longmapsto & A/Ae \otimes_A N \end{array}$$

is the identity, things check.

Next we consider an example which might show $Ae \otimes_A eA \xrightarrow{\sim} AeA$ is necessary. Take

$$A = \begin{pmatrix} \mathbb{C}e & \mathbb{C}x \\ \mathbb{C}y & \mathbb{C}e^{\perp} \end{pmatrix} \qquad \begin{array}{l} e^2 = e \\ e^{\perp} = 1 - e \end{array}$$

where $exe^{\perp} = x$, $e^{\perp}ye = y$, $xy = 0$, $yx = 0$.

In this example

$$Ae \otimes_{eAe} eA = \begin{pmatrix} \mathbb{C}e \\ \mathbb{C}y \end{pmatrix} \otimes_{\mathbb{C}e} \begin{pmatrix} \mathbb{C}e & \mathbb{C}x \end{pmatrix} = \begin{pmatrix} \mathbb{C}e & \mathbb{C}x \\ \mathbb{C}y & \mathbb{C}y\otimes x \end{pmatrix}$$

so the map $Ae \otimes_{eAe} eA \longrightarrow AeA$ is not injective.

This algebra A is a semi-direct product of the bimodule $M = \mathbb{C}x \oplus \mathbb{C}y$ over the separable algebra $S = \mathbb{C}e \oplus \mathbb{C}e^{\perp}$ so we can calculate the cyclic homology rather easily. Because S is separable there are no higher derived tensor products. So

$$HC_{\cdot}(A) = HC(S) \oplus (M\otimes_S) \oplus \Sigma\left([M\otimes_S]^{(2)}_{\lambda}\right) \oplus \cdots$$

Note that
$$M \otimes_S M = (\mathbb{C}x \oplus \mathbb{C}y) \otimes_S (\mathbb{C}x \oplus \mathbb{C}y)$$
$$= \mathbb{C}(x\otimes y) \oplus \mathbb{C}(y \otimes x)$$

It's clear that $[M\otimes_S]^{(2i+1)} = 0$, and that $[M\otimes_S]^{(2i)}$ is 2 dimensional with basis $v_1 = (x, y, \cdots x, y)$ and $v_2 = (y, x, \cdots, y, x)$. Then $\lambda v_1 = -v_2$ and $\lambda v_2 = -v_1$ so $\lambda(v_1 - v_2) = -v_2 + v_1$, and $v_1 - v_2$ is λ-invariant.

It thus appears that the relative cyclic homology of $A \longrightarrow S$ is $\mathbb{C}$ in odd degrees ≥ 1. If we did the same computation over $\mathbb{Q}$, then Goodwillie's big theorem would say that the relative rational K-theory is non-trivial.

Notice that A modules are the same as supercomplexes. If M is an A-module, then e gives a grading $M_+ = eM$, $M_- = e^{\perp}M$, and $y : M_+ \longrightarrow M_-$, $x : M_- \longrightarrow M_+$ are the differential. Projective A-modules are the same as supercomplexes which are acyclic.

April 5, 1994

Let $\mathcal{a}$ be a nonunital algebra. Recall that a multiplier μ on $\mathcal{a}$ is a pair of operators on $\mathcal{a}$: $a \mapsto \mu \times a$, $a \times \mu$ such that

$$\mu \times (a_1 a_2) = (\mu \times a_1) a_2$$

$$a_1(\mu \times a_2) \text{[redacted]} = (a_1 \times \mu) a_2$$

$$(a_1 a_2) \times \mu = a_1 (a_2 \times \mu)$$

The set $\mathcal{M}(\mathcal{a})$ of multipliers on $\mathcal{a}$ is an algebra with product $(\mu\nu) \times a = \mu \times (\nu \times a)$, $a \times (\mu\nu) = (a \times \mu) \times \nu$. One has a homomorphism

$$\mathcal{a} \xrightarrow{\phi} \mathcal{M}(\mathcal{a})$$

where $\phi(a) \times a' = aa'$, $a' \times \phi(a) = a'a$. The image of ϕ is an ideal in $\mathcal{M}(\mathcal{a})$, in fact $\mu\phi(a) = \phi(\mu \times a)$, $\phi(a)\mu = \phi(a \times \mu)$. The kernel of ϕ is the subset of $a \in \mathcal{a}$ such that $aa' = a'a = 0$, $\forall a' \in A$.

If $\mathcal{a}$ has an A-bimodule structure such that the product is an A-bimodule map $\mathcal{a} \otimes_A \mathcal{a} \to \mathcal{a}$, then one has an obvious homomorphism $A \to \mathcal{M}(\mathcal{a})$. In general $\mathcal{a}$ is not a bimodule over $\mathcal{M}(\mathcal{a})$, because the left mult. $a \mapsto \mu \times a$ need not commute with the right mult. $a \mapsto a \times \nu$; this is clear when $\mathcal{a}^2 = 0$, where $\mathcal{M}(\mathcal{a}) =$ [redacted] $\mathrm{Hom}(\mathcal{a},\mathcal{a})^2$.

However if $\mathcal{a}^2 = \mathcal{a}$, then we have

$$(\mu \times a) \times \nu = \mu \times (a \times \nu) \qquad \forall a \in \mathcal{a},\ \mu, \nu \in \mathcal{M}(\mathcal{a})$$

because $(\mu \times a_1 a_2) \times \nu = ((\mu \times a_1) a_2) \times \nu = (\mu \times a_1)(a_2 \times \nu)$

[redacted] $\mu \times (a_1 a_2 \times \nu) = \mu \times (a_1 (a_2 \times \nu)) = (\mu \times a_1)(a_2 \times \nu)$.

Thus $a^2 = a \implies a$ is an $M(a)$ bimodule.

I next want to calculate $M(a)$ in an interesting case. Let

$$a = \begin{pmatrix} \mathbb{C}e & \mathbb{C}y \\ \mathbb{C}x & 0 \end{pmatrix}$$

where the multiplication table is

	e	x	y
e	e	0	y
x	x	0	0
y	0	0	0

This is the ideal AeA in $A = \begin{pmatrix} \mathbb{C}e & \mathbb{C}y \\ \mathbb{C}x & \mathbb{C}e^{\perp} \end{pmatrix}$, where as we have seen $\mathrm{Mod}(A)$ is the category of supercomplexes. We note that the left annihilator $\{a \mid a\,a = 0\}$ is $\mathbb{C}y$ and the right annihilator is $\mathbb{C}x$, so that $a \to M(a)$ is injective.

Let $\mu \in M(a)$. From $\mu \times (a_1 a_2) = (\mu \times a_1) a_2$ we get

$$(\mu \times e) e = \mu \times e \implies \mu \times e = c_1 e + c_2 x$$

$$(\mu \times e) y = \mu \times y \implies \mu \times y = c_1 y$$

$$\left.\begin{aligned} (\mu \times x) e &= \mu \times x \\ (\mu \times x) y &= 0 \end{aligned}\right\} \implies \mu \times x = c_3 x$$

so we have

$$(*) \qquad \mu \times (e \;\; x \;\; y) = (e \;\; x \;\; y) \begin{pmatrix} c_1 & & \\ c_2 & c_3 & \\ & & c_1 \end{pmatrix}$$

with $c_i \in \mathbb{C}$.

From $(a_1 a_2) \times \mu = a_1 (a_2 \times \mu)$ we get

Note that

$$e \longleftrightarrow c_1 = 1, \quad c_2 = c_3 = b_2 = b_3 = 0$$

$$x \longleftrightarrow c_2 = 1, \quad c_1 = c_3 = b_2 = b_3 = 0$$

$$y \longleftrightarrow b_2 = 1, \quad c_1 = c_2 = c_3 = b_3 = 0$$

$$e^{\perp} \longleftrightarrow c_3 = b_3 = 1, \quad c_1 = c_2 = b_3 = 0.$$

It seems that in $\mathcal{M}(a)$, $e^{\perp}$ splits into two idempotents. The one corresponding to $c_3 = 1$, $b_3 = 0$ has left mult $e \mapsto 0$, $x \mapsto x$, $y \mapsto 0$ and right multiplication zero.

$$e(e\times\mu) = e\times\mu \implies e\times\mu = b_1 e + b_2 y$$
$$x(e\times\mu) = x\times\mu \implies x\times\mu = b_1 x$$
$$\left.\begin{array}{l} e(y\times\mu) = y\times\mu \\ x(y\times\mu) = 0 \end{array}\right\} \implies y\times\mu = b_3 y$$

Thus

$$(**)\qquad \begin{pmatrix} e \\ x \\ y \end{pmatrix}\times\mu = \begin{pmatrix} b_1 & & b_2 \\ & b_1 & \\ & & b_3 \end{pmatrix}\begin{pmatrix} e \\ x \\ y \end{pmatrix} \qquad \text{with } b_i\in\mathbb{C}$$

Finally examine the condition $a_1(\mu\times a_2) = (a_1\times\mu)a_2$.

$$\begin{pmatrix} e \\ x \\ y \end{pmatrix}\left(\mu\times\begin{pmatrix} e & x & y \end{pmatrix}\right) = \begin{pmatrix} e \\ x \\ y \end{pmatrix}\begin{pmatrix} e & x & y \end{pmatrix}\begin{pmatrix} c_1 & & \\ c_2 & c_3 & \\ & & c_1 \end{pmatrix}$$
$$= \begin{pmatrix} e & 0 & y \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 & & \\ c_2 & c_3 & \\ & & c_1 \end{pmatrix} = \begin{pmatrix} c_1 e & 0 & c_1 y \\ c_1 x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} e \\ x \\ y \end{pmatrix}\times\mu\right)\begin{pmatrix} e & x & y \end{pmatrix} = \begin{pmatrix} b_1 & & b_2 \\ & b_1 & \\ & & b_3 \end{pmatrix}\begin{pmatrix} e \\ x \\ y \end{pmatrix}\begin{pmatrix} e & x & y \end{pmatrix}$$
$$= \begin{pmatrix} b_1 & & b_2 \\ & b_1 & \\ & & b_3 \end{pmatrix}\begin{pmatrix} e & 0 & y \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1 e & 0 & b_1 y \\ b_1 x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives the condition $b_1 = c_1$. But $(*)$ and $(**)$ give the complete description of operators $a\mapsto\mu\times a$ compatible with right mult. by $\mathcal{A}$, resp. $a\mapsto a\times\mu$ ——— left ———. Thus we see $\mathfrak{M}(\mathcal{A})$ is 5 dimensional.

April 8, 1994

Consider non unital rings A, B, C etc.
Call A *perfect* when $A^2 = A$.
Call an extension of A:

$$0 \longrightarrow K \longrightarrow B \longrightarrow A \longrightarrow 0$$

central when $KB = BK = 0$. In this case the product $b_1 b_2$ for $b_i \in B$ depends only on the images of b_1, b_2 in A.

If C is perfect and $B \xrightarrow{\pi} A$ is a central extension, then a homomorphism $u : C \to B$ is determined by the composition $C \to B \to A$. This is because $u(c_1 c_2) = u(c_1) u(c_2)$ depends only on the images of $u(c_1), u(c_2)$ in A.

Alternative: First note that for any ring A we have another ring $A \otimes_A A$ with product

$$(a_1, a_2)(a_3, a_4) = (a_1 a_2, a_3 a_4)$$

and a homomorphism $A \otimes_A A \xrightarrow{v} A$ given by $v((a_1, a_2)) = a_1 a_2$. $\mathrm{Ker}(v)$ is clearly contained in the left and right annihilator of $A \otimes_A A$.

(Actually a slightly bigger ring is defined, namely $A \otimes_{A^2} A$. Check associativity:

$$\big((a_1, a_2)(a_3, a_4)\big)(a_5, a_6) = (a_1 a_2, a_3 a_4)(a_5, a_6) = (a_1 a_2 \underbrace{a_3 a_4}, a_5 a_6)$$

$$(a_1, a_2)\big((a_3, a_4)(a_5, a_6)\big) = (a_1, a_2)(a_3 a_4, a_5 a_6) = (a_1 a_2, \underbrace{a_3 a_4}_{\in A^2} a_5 a_6).$$

More generally $A^m \otimes_{A^{m+n}} A^n$ has a ring structure defined in the same way.)

Suppose now $C^2 = C$ and $B \twoheadrightarrow A$ is a central extension. Then given two homom. $u, u' : C \to B$ which agree in A we have

$$\begin{tikzcd}
C \otimes_C C \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & C \arrow[d, shift left] \arrow[d, shift right] \\
B \otimes_B B \arrow[r] \arrow[d] & B \arrow[d] \\
A \otimes_A A \arrow[r] \arrow[ur, dotted] & A
\end{tikzcd}$$

(The dotted arrow exists making the triangles commutative as B is a central extension of A.)

showing that the two arrows $C \otimes_C C \rightrightarrows B$ coincide hence $C \rightrightarrows B$ coincide.

Claim that if $A^2 = A$, then $A \otimes_A A \to A$ is the universal central extension of A.* To see this let $B \to A$ be a central extension. Then we have

$$\begin{tikzcd}
B \otimes_B B \arrow[r, "v"] \arrow[d] & B \arrow[d] \\
A \otimes_A A \arrow[r] \arrow[ur, "u"] & A
\end{tikzcd}$$

Since $A \otimes_A A$ is a quotient ring of $B \otimes_B B$ and v is a homom. it follows that u is a homomorphism. Since $A \otimes_A A$ is perfect it is the unique homomorphism $A \otimes_A A \to B$ of extensions of A. (*: You first check that $A \otimes_A A \to A$ is a central extension.)

Note that if B is perfect, then u is surjective. Thus the perfect central extensions of A are quotient algebras of $A \otimes_A A$ which are over A.

Suppose we have central extensions $C \twoheadrightarrow B$, $B \twoheadrightarrow A$, where C (hence B, A) are perfect. I claim that $C \twoheadrightarrow A$ is a central extension. Let $I \subset J \subset C$, $K \subset B$ be the ideals such that $B = C/I$, $A = C/J$ and $B/K = A$. Then $K = J/I$. Since $B \twoheadrightarrow A$ is central we have $BK = KB = 0$, i.e. $CJ, JC \subset I$. Then $C^2 J \subset CI = 0$, $JC^2 \subset IC = 0$ as $C \to B$ is central. Then $C^2 = C \Rightarrow CJ = JC = 0$, whence $C \longrightarrow C/J = A$ is central.

So now let $B \twoheadrightarrow A$ be central with $B^2 = B$.

One has

$$\begin{tikzcd}
B \otimes_B B \arrow[r, twoheadrightarrow, "\text{central}"] \arrow[d] & B \arrow[d, "\text{central}"] \\
A \otimes_A A \arrow[ur] \arrow[r, twoheadrightarrow] & A
\end{tikzcd}$$

so $B \otimes_B B \longrightarrow A$ is a perfect central extension. Thus we get a surjection $A \otimes_A A \longrightarrow B \otimes_B B$ over A. Then the two compositions $A \otimes_A A \rightleftarrows B \otimes_B B$ must be the identity since uniqueness for maps between perfect central extensions. Thus we find $B \otimes_B B = A \otimes_A A$. And applying this to $B = A \otimes_A A$ we find $(A \otimes_A A) \otimes_{(A \otimes_A A)} (A \otimes_A A) \xrightarrow{\sim} (A \otimes_A A)$.

Let $A^2 = A$, let K_A be the left and right annihilator of A. Recall the exact sequence

$$0 \longrightarrow K_A \longrightarrow A \longrightarrow \mathcal{M}(A) \longrightarrow \underset{\substack{\text{outer multiplier}\\ \text{algebra}}}{\mathcal{M}(A)/A} \longrightarrow 0$$

Then A/K_A (denote this $\bar{A}$) is perfect and $A \twoheadrightarrow \bar{A}$ is a central extension. Also $K_{\bar{A}} = 0$, since $A \to \bar{A} \to A/K_{\bar{A}}$ is a central extension, so the kernel is contained in K_A. Note A/K_A is the smallest ~~quotient~~ algebra of A such that $A \to A/K_A$ is central.

On the other hand we have seen that $A \otimes_A A \to A$ is the largest perfect central extension of A. It's clear that we have

$$A \otimes_A A = \bar{A} \otimes_{\bar{A}} \bar{A} \qquad \overline{A \otimes_A A} = \bar{A}.$$

Next I want to discuss multipliers. Start with A such that $A^2 = A$, $K_A = 0$. Let $\mu \in M(A)$. Define

$$\mu^\alpha \cdot (a_1, a_2) = (\mu \cdot a_1, a_2)$$

$$(a_1, a_2) \cdot \mu^\alpha = (a_1, a_2 \cdot \mu)$$

Claim $\mu^\alpha \in M(A \otimes_A A)$. First check μ^α well-defd:

$$(\mu \cdot (a_1 a), a_2) = ((\mu \cdot a_1) a, a_2) = (\mu \cdot a_1, a a_2)$$

so $\mu^\alpha \cdot$ is defd on $A \otimes_A A$. Similarly for $\cdot \mu^\alpha$.

$$\mu^\alpha \cdot ((a_1, a_2)(a_3, a_4)) = \mu^\alpha \cdot (a_1 a_2, a_3 a_4)$$
$$= (\mu \cdot (a_1 a_2), a_3 a_4)$$

$$(\mu^\alpha \cdot (a_1, a_2))(a_3, a_4) = (\mu \cdot a_1, a_2)(a_3, a_4)$$
$$= ((\mu \cdot a_1) a_2, a_3 a_4)$$

these are equal

$$\left((a_1,a_2)\circ\mu^\alpha\right)(a_3,a_4) = (a_1, a_2\circ\mu)(a_3,a_4)$$
$$= \left(a_1(a_2\circ\mu), a_3a_4\right)$$
$$(a_1,a_2)\left(\mu^\alpha\circ(a_3,a_4)\right) = (a_1,a_2)(\mu\cdot a_3, a_4)$$
$$= \left(a_1a_2, (\mu\cdot a_3)a_4\right)$$

But
$$\left(a_1(a_2\circ\mu), a_3a_4\right) = \left(a_1, (a_2\circ\mu)a_3a_4\right)$$
$$= \left(a_1, a_2(\mu\circ a_3)a_4\right)$$
$$= \left(a_1a_2, (\mu\circ a_3)a_4\right).$$

Finally $\left((a_1,a_2)(a_3,a_4)\right)\cdot\mu^\alpha = (a_1,a_2)\left((a_3,a_4)\cdot\mu^\alpha\right)$ is similar to the $\mu^\alpha\circ$ case.

Next given $\mu,\nu\in \mathcal{M}(A)$ we have

$$(\mu\nu)^\alpha\circ(a_1,a_2) = \left((\mu\nu)\circ a_1, a_2\right) = \left((\mu\circ(\nu\cdot a_1)), a_2\right)$$
$$\mu^\alpha\circ\left(\nu^\alpha\circ(a_1,a_2)\right) = \mu^\alpha\circ(\nu\circ a_1, a_2) = \left(\mu\circ(\nu\circ a_1), a_2\right)$$

so $(\mu\nu)^\alpha\circ = (\mu^\alpha\circ)(\nu^\alpha\circ)$ and similarly for right multiplication. Thus we have a homomorphism

$$\alpha: \mathcal{M}(A) \longrightarrow \mathcal{M}(A\otimes_A A)$$

Claim α is <u>injective</u>, for if

$$\mu^\alpha\circ(a_1,a_2) = (\mu\cdot a_1, a_2) = 0 \qquad \forall (a_1,a_2)\in A\otimes_A A$$

then $0 = (\mu\circ a_1)a_2 = \mu\circ(a_1a_2)$ $\forall a_1,a_2\in A$, $\Rightarrow \mu\circ = 0$.
Similarly $\circ\mu^\alpha = 0 \Rightarrow \circ\mu = 0$, and so $\mu^\alpha = 0 \Rightarrow \mu = 0$.

Next consider the exact sequence

$$0 \longrightarrow K_{\tilde{A}} \longrightarrow \tilde{A} \xrightarrow{\ \phi\ } \mathcal{M}(\tilde{A})$$

where $\tilde{A} = A\otimes_A A$. $\quad \begin{matrix}\phi(x)\mu = \phi(x\cdot\mu) \\ \mu\,\phi(x) = \phi(\mu\cdot x)\end{matrix}$. Thus $x\in K_{\tilde{A}}$

$\Longrightarrow \quad x\cdot\mu\,,\ \mu\cdot x \in K_{\tilde{A}}$ for all $\mu \in \mathcal{M}(\tilde{A})$. So any $\mu \in \mathcal{M}(\tilde{A})$ induces a multiplier μ^β on A defined by

$$\mu^\beta\cdot(a_1a_2) = d\big(\mu\cdot(a_1,a_2)\big), \quad (a_1a_2)\cdot\mu^\beta = d\big((a_1,a_2)\cdot\mu\big)$$

Here we use $A\otimes_A A/K_{\tilde{A}} \xrightarrow{\sim} A$, $(a_1,a_2) \longmapsto a_1a_2$

It's clear we get in this way a homom.

$$\beta : \mathcal{M}(\tilde{A}) \longrightarrow \mathcal{M}(A).$$

Let us check that α, β are inverses of each other. Take $\mu \in \mathcal{M}(A)$. Then

$$\begin{aligned}(\mu^\alpha)^\beta\cdot(a_1a_2) &= d\big(\mu^\alpha\cdot(a_1,a_2)\big)\\ &= d\big((\mu\cdot a_1, a_2)\big)\\ &= (\mu\cdot a_1)a_2\\ &= \mu\cdot(a_1a_2)\end{aligned}$$

Similarly $\cdot(\mu^\alpha)^\beta = \cdot\mu$. $\quad\therefore\ (\mu^\alpha)^\beta = \mu$.

Next let $\nu \in \mathcal{M}(\tilde{A})$. Then

$$\begin{aligned}(\nu^\beta)^\alpha\cdot\big((a_1,a_2)(a_3,a_4)\big) &= (\nu^\beta)^\alpha(a_1a_2, a_3a_4)\\ &= \big(\nu^\beta\cdot(a_1a_2), a_3a_4\big)\\ &= \big(d(\nu\cdot(a_1,a_2)), d(a_3,a_4)\big)\end{aligned}$$

$$\begin{aligned}\nu\cdot\big((a_1,a_2)(a_3,a_4)\big) &= \big(\nu\cdot(a_1,a_2)\big)(a_3,a_4)\\ &= \big(d(\nu\cdot(a_1,a_2)), d(a_3,a_4)\big)\end{aligned}$$

Thus $(\nu^\beta)^\alpha\cdot = \nu\cdot$ on $A\otimes_A A$, and similarly $\cdot(\nu^\beta)^\alpha = \cdot\nu$. $\quad\therefore\ (\nu^\beta)^\alpha = \nu$.

Let's now consider Morita equivalence for non unital rings. Suppose given the following data

A, B rings, ${}_A X_B$, ${}_B Y_A$ bimodules

$X \otimes_B Y \xrightarrow{\langle | \rangle} A$ $\quad$ A-bimodule map

$Y \otimes_A X \xrightarrow{\langle | \rangle} B$ $\quad$ B-bimodule map

such that

$$\begin{tikzcd} X \otimes_B Y \otimes_A X \arrow[r] \arrow[d] & X \otimes_B B \arrow[d] \\ A \otimes_A X \arrow[r] & X \end{tikzcd}$$

and

$$\begin{tikzcd} Y \otimes_A X \otimes_B Y \arrow[r] \arrow[d] & Y \otimes_A A \arrow[d] \\ B \otimes_B Y \arrow[r] & Y \end{tikzcd}$$

commute:
$$\langle x_1 | y \rangle x_2 = x_1 \langle y | x_2 \rangle$$
$$\langle y_1 | x \rangle y_2 = y_1 \langle x | y_2 \rangle$$

This data should be equivalent to a ring R having the block matrix form $R = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$, i.e. a bimodule structure over $S = \mathbb{C}[e]$ which is compatible with product (product gives S-bimodule map $R \otimes_S R \to R$).

Suppose now that this data such that the pairings $X \otimes_B Y \to A$, $Y \otimes_A X \to B$ are isomorphisms. Let's call an A module M <u>good</u>

when $A \otimes_A M \xrightarrow{\sim} M$.

Let's show that M good A-module $\Longrightarrow Y \otimes_A M$ is a good B-module

$$\begin{tikzcd}
(\langle y_1 | x\rangle, y_2, m) & & (y_1, x, y_2, m) \arrow[ll, mapsto] \\
B \otimes_B Y \otimes_A M \arrow[d] & & Y \otimes_A X \otimes_B Y \otimes_A M \arrow[ll, "\sim"'] \arrow[d, "\sim"] \\
Y \otimes_A M & & Y \otimes_A A \otimes_A M \arrow[ll]
\end{tikzcd}$$

$$(\langle y_1|x\rangle y_2, m), \quad (y_1, \langle x|y_2\rangle m) \longleftarrow\!\shortmid \; (y_1, \langle x|y_2\rangle, m)$$

$$(\langle y_1|x\rangle y_2, m) = (y_1, \langle x|y_2\rangle, m) = (y_1, \langle x|y_2\rangle m)$$

This should that either M good A-module or Y good $A^{?}$ module $\Longrightarrow$ $Y \otimes_A M$ is a good B-module.

Similarly N good B-module or X good $B^{?}$-module $\Longrightarrow$ $X \otimes_B N$ is a good A-module.

But $X \otimes_B Y \otimes_A M \xrightarrow{\sim} A \otimes_A M \xrightarrow{\sim} M$ for M good and $Y \otimes_A X \otimes_B N \xrightarrow{\sim} B \otimes_B N \xrightarrow{\sim} N$ for N good. The functors $M \longmapsto Y \otimes_A M$, $N \longmapsto X \otimes_B N$ give an equivalence between good A-modules and good B-modules.

Suppose now that A is good as A-module: $A \otimes_A A \xrightarrow{\sim} A$. Then for any $N \in Mod(A)$ one has $A \otimes_A N$ is good. Moreover $N \mapsto A \otimes_A N$ retracts $Mod(A)$ onto the full subcategory of good modules. This is actually a right adjoint functor, namely, if M is good, then

$$\mathrm{Hom}_A(M, \underbrace{A\otimes_A N}_{G(N)}) \xrightarrow{\sim} \mathrm{Hom}_A(\underbrace{M}_{F(M)}, N)$$

F inclusion of good modules

The adjunction maps are

$\alpha : A\otimes_A N \longrightarrow N$ $\qquad (a, n) \longmapsto an$

$\beta : M \xrightarrow{\sim} A\otimes_A M$ $\qquad$ inverse of isom. $(a, m) \longmapsto am$.

Check: $\quad F \xrightarrow{F.\beta} FGF \xrightarrow{\alpha.F} F \quad$ is the identity:

$$M \simeq A\otimes_A M \longrightarrow M$$
$$am \longleftrightarrow (a, m) \longmapsto am$$

$$G \underset{\sim}{\xrightarrow{\beta.G}} GFG \xrightarrow{G.\alpha} G \quad \text{is also the identity}$$

$$A\otimes_A N \simeq A\otimes_A A\otimes_A N \longrightarrow A\otimes_A N$$
$$(a_1, a_2 n) \longleftrightarrow (a_1, a_2, n) \longmapsto (a_1, a_2 n)$$

Now suppose $A\otimes_A A = A$, $B\otimes_B B = B$. (Note that in general

$$Y\otimes_A A = Y\otimes_A X\otimes_B Y = B\otimes_B Y$$

so that Y is a good right A-module iff it is a good left B-module.)

Then we have

$$X\otimes_B (Y\otimes_A A) = A\otimes_A A = A$$

$$(Y\otimes_A A)\otimes_A X = Y\otimes_A (X\otimes_B B) = B\otimes_B B = B$$

or $\quad (Y\otimes_A A)\otimes_A X = B\otimes_B Y\otimes_A X = B\otimes_B B = B$

Better would be to point out that on good A-modules M one has

$$Y \otimes_A M = (Y \otimes_A A) \otimes_A M$$

and that on good B-modules N

$$X \otimes_B N = (X \otimes_B B) \otimes_B N$$

Thus if we stick to good modules we can suppose the bimodules X, Y are good on either side.

Add:

$$(X \otimes_B B) \otimes_B (Y \otimes_A A) = X \otimes_B Y \otimes_A A \otimes_A A = A \otimes_A A \otimes_A A$$

$$(Y \otimes_A A) \otimes_A (X \otimes_B B) = B \otimes_B Y \otimes_A X \otimes_B B = B \otimes_B B \otimes_B B$$

Let's next discuss when an A-module M is good. Recall that A-modules are the same as unital $\tilde{A}$ modules and that $\otimes_A = \otimes_{\tilde{A}}$. From

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{C} \longrightarrow 0$$

we get

$$0 \longrightarrow \operatorname{Tor}_1^{\tilde{A}}(\mathbb{C}, M) \longrightarrow A \otimes_A M \longrightarrow M \longrightarrow \underbrace{\mathbb{C} \otimes_A M}_{M/AM} \longrightarrow 0$$

One sees in particular that if $A^n = 0$, then any good module M, more generally one such that $M = AM$, is zero:

$$M = AM \subset A^2M \subset \cdots \subset A^nM = 0.$$

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Suppose A nu ring such that $A^2 = A$, let $A' = A \otimes_A A$ be its universal central extension. One has the exact sequence of rings

$$0 \longrightarrow K \longrightarrow A' \xrightarrow{(a_1,a_2) \longmapsto a_1 a_2} A \longrightarrow 0$$

This is an exact sequence of A-bimodules since $KA' = A'K = 0$. Then

$$A \otimes_A K \longrightarrow A \otimes_A A' \xrightarrow{(a_1,a_2,a_3) \longmapsto (a_1, a_2 a_3)} A \otimes_A A \longrightarrow 0$$

is exact and $A \otimes_A K = A^2 \otimes_A K = A \otimes_A AK = 0$.
Thus we have an isomorphism

$$\begin{array}{ccc} A \otimes_A A' & \xrightarrow{\sim} & A' \\ \| & & \| \\ A \otimes_A A \otimes_A A & \xrightarrow{\sim} & A \otimes_A A \\ (a_1, a_2, a_3) & \longmapsto & (a_1, a_2 a_3) = (a_1 a_2, a_3) \end{array}$$

showing that A' is a good A-module.

Let M be a good A-module:

$$A \otimes_A M \xrightarrow{\sim} M \qquad (a,m) \longmapsto am$$

Then

$$\begin{array}{ccccc} A \otimes_A A \otimes_A M & \xrightarrow{\sim} & A \otimes_A M & \xrightarrow{\sim} & M \\ (a_1, a_2, m) & \longmapsto & (a_1, a_2 m) & \longmapsto & a_1 a_2 m \end{array}$$

showing $A' \otimes_{A'} M = A' \otimes_A M \xrightarrow{\sim} M$, so M is good A'-module. Conversely if M is a good A'-module: $A' \otimes_A M \xrightarrow{\sim} M$, then $KA' = 0 \implies KM = 0$, and so M is an A-module with A'-action obtained via the homom. $A' \twoheadrightarrow A$.

Moreover $A'\otimes_{A'} M \xrightarrow{\sim} M \Longrightarrow$ $A'M = M$, so $K\otimes_{A'} M = K\otimes_{A'} A'M = KA'\otimes_{A'} M$ $= 0$ whence $A'\otimes_{A'} M = A'/K\otimes_{A'} M = A\otimes_{A'} M = A\otimes_{A} M$ showing that M is a good A-module.

Thus <u>good</u> A <u>modules and good</u> A' <u>modules are the same.</u>

<u>Example:</u> Take $A = \begin{pmatrix} \mathbb{C}e & \mathbb{C}x \\ \mathbb{C}y & 0 \end{pmatrix}$

$$e^2 = e$$
$$ex = x,\ xe = 0$$
$$ey = 0,\ ye = y$$
$$xy = yx = 0$$
$$x^2 = y^2 = 0$$

Then $\tilde{A} = \begin{pmatrix} \mathbb{C}e & \mathbb{C}x \\ \mathbb{C}y & \mathbb{C}e^{\perp} \end{pmatrix}$ is the unital algebra whose unital modules are supercomplexes:

$$eV \underset{x}{\overset{y}{\rightleftarrows}} e^{\perp}V$$

So A-modules are the same as supercomplexes.

I believe good A-modules are those supercomplexes such that $y: eV \to e^{\perp}V$ is an isomorphism, but this needs checking.

Interesting point: If $A_1 = \begin{pmatrix} 0 & \mathbb{C}x \\ \mathbb{C}y & \mathbb{C}e^{\perp} \end{pmatrix}$, then $\tilde{A}_1 = \tilde{A}$, so that A_1-modules are the same as supercomplexes. Good A_1-modules should be supercomplexes such that $x: e^{\perp}V \xrightarrow{\sim} eV$. Thus although the module categories for A and A_1 are the same, the good module subcategories are different.

Let A be a nu ring, let e be an idempotent in A. Claim for any A-module M we have

$$eA \otimes_A M \simeq eM$$

[redacted]

Proof. Consider the maps

$$eM \subset M \longrightarrow eA \otimes_A M \xrightarrow{\psi} eM$$

$$m \longmapsto (e, m)$$

$$(ea, m) \longmapsto eam$$

Then $\quad em \overset{\varphi}{\longmapsto} (e, em) \overset{\psi}{\longmapsto} eem = em$

$$(ea, m) \overset{\psi}{\longmapsto} eam \overset{\varphi}{\longmapsto} (e, eam) = (eea, m) = (ea, m)$$

Alternative: $\tilde{A}$ = ring obtained by adjoining 1. Then $e\tilde{A} \subset A\tilde{A} = A$ so $e\tilde{A} = e^2\tilde{A} \subset eA \subset e\tilde{A} \implies e\tilde{A} = eA$.

Then

$$eA \otimes_A M = e\tilde{A} \otimes_{\tilde{A}} M = eM.$$

by applying e to $\tilde{A} \otimes_{\tilde{A}} M = M$.

$\blacksquare$

A corollary is that eA is a good $A^\wedge$ module. Similarly for any A^{r}-module M we have

$$M \otimes_A Ae \simeq Me$$

so Ae is a good A-module.

I now want to prove that

$$\boxed{Ae \otimes_B eA \xrightarrow{\sim} AeA \otimes_{AeA} AeA \qquad B = eAe}$$

Let us start with $Ae \otimes_B eA$ equipped with the evident AeA bimodule structure and the map

$$Ae \otimes_B eA \xrightarrow{\pi} AeA \qquad (a_1 e, e a_2) \mapsto a_1 e a_2$$

of bimodules over AeA. Define a product on $Ae \otimes_B eA$ by

$$(a_1 e, e a_2)(a_3 e, e a_4) = (a_1 e\, e a_2 a_3 e, e a_4)$$

It's well-defined because its the composition

$$(Ae \otimes_B eA) \otimes (Ae \otimes_B eA) \longrightarrow Ae \otimes_B B \otimes_B eA \to Ae \otimes_B eA$$

Note that

$$(a_1 e, e a_2)(a_3 e, e a_4) = \underbrace{a_1 e a_2}_{\pi(a_1 e, e a_2)} (a_3 e, e a_4) = (a_1 e, e a_2) \underbrace{a_3 e a_4}_{\pi(a_3 e, e a_4)}$$

This implies that $Ae \otimes_B eA$ equipped with this product and π is a central extension of AeA. $Ae \otimes_B eA$ is perfect since $(a_1 e, e)(e, e a_2) = (a_1 e, e a_2)$.

Finally

$$(Ae \otimes_B eA) \otimes_{(Ae \otimes_B eA)} (Ae \otimes_B eA) = (Ae \otimes_B eA) \otimes_{AeA} (Ae \otimes_B eA)$$

$$= Ae \otimes_B B \otimes_B eA = Ae \otimes_B eA$$

Here we have used

$$eA \otimes_{AeA} Ae = e(AeA) \otimes_{AeA} (AeA)e = e(AeA)e = eAe = B$$

and the fact that B is unital and eA is a unital B module to get $B \otimes_B eA \xrightarrow{\sim} eA$.

At this point we have Morita equivalence

$$eA \otimes_{(Ae \otimes_B eA)} Ae \xrightarrow{\sim} eA \otimes_{AeA} Ae = B$$

$$Ae \otimes_B eA = Ae \otimes_B eA$$

and hence an equivalence

$$N \longmapsto Ae \otimes_B N \quad , \quad M \longmapsto eA \otimes_{(Ae \otimes_B eA)} M = eM$$

between B modules and good $Ae \otimes_B eA$ modules.
But the latter are the same as good AeA modules.

April 10, 1994

A nonunital algebra, recall the standard normalized resolution for $\tilde{A}$

$$\xrightarrow{b'} \tilde{A}\otimes A\otimes \tilde{A} \xrightarrow{b'} \tilde{A}\otimes\tilde{A} \xrightarrow{b'} \tilde{A} \to 0$$

can be used to calculate $\mathrm{Tor}_n^{\tilde{A}}(N,M)$ as the homology of

A is better so as not to specify unital modules

$$\longrightarrow N\otimes A^{\otimes 2}\otimes M \xrightarrow{b'} N\otimes A\otimes M \xrightarrow{b} N\otimes M \to 0 \to$$

Here N is an A_r-module and M and A_ℓ-module. In particular when $N = \mathbb{C} = \tilde{A}/A$, the homology of

$$(*) \qquad \longrightarrow A^{\otimes 2}\otimes M \longrightarrow A\otimes M \longrightarrow M \to 0$$

is $\mathrm{Tor}_*^A(\mathbb{C},M)$. Thus M is a good A-module iff $\mathrm{Tor}_n^A(\mathbb{C},M) = 0$ for $n=0,1$.

Acyclicity of $(*)$ is a higher order version of goodness. For example, if e is an idempotent in A, then $\tilde{A}e = Ae$ we have seen is good. But it's excellent, i.e. $(*)$ is acyclic, since $\tilde{A}e$ is a projective hence flat $\tilde{A}$-module (unital module) so $\mathrm{Tor}_n^A(\mathbb{C}, Ae) = 0$ for $n\geq 1$, and also $\mathrm{Tor}_0^A(\mathbb{C},\tilde{A}e) = \tilde{A}e/A\tilde{A}e = \tilde{A}e/Ae = 0$. This argument shows that <u>if</u> M <u>is a</u> A <u>module which is flat</u> (i.e. flat as unital $\tilde{A}$-module, equivalently $-\otimes_A M$ is exact), <u>and if</u> $M = AM$, <u>then</u> M <u>is excellent.</u>

To be more concrete, one knows

$$\xrightarrow{b'} A^{\otimes 2}\otimes\tilde{A} \xrightarrow{b'} A\otimes\tilde{A} \xrightarrow{b'} \tilde{A} \longrightarrow \mathbb{C} \to 0$$

is exact with contraction inserting 1 with the appropriate sign. Thus as M is A-flat and $\tilde{A} \otimes_A M = \tilde{A} \otimes_{\tilde{A}} M = M$, we find

$$\to A^{\otimes 2} \otimes M \longrightarrow A \otimes M \longrightarrow M \longrightarrow M/AM \to 0$$

is zero, so $M/AM = 0$, then gives the exactness of $(*)$.

A further example: Suppose the multiplication map $A \otimes M \to M$ has a section s which is A linear. Then $M \xrightarrow{s} A \otimes M \subset \tilde{A} \otimes M \longrightarrow M$ is the identity showing that M is a projective $\tilde{A}$ module, hence flat, [redacted] Thus M is A-flat and $M = AM$, so M is excellent. In this case the section s gives a specific contraction for $(*)$.

[redacted]

Next return to A nonunital, e idempotent in A, and let $R = Ae \otimes_B eA$, $B = eAe$. R has an evident A-bimodule structure and a compatible product $(a_1 e, e a_2)(a_3 e, e a_4) = (a_1 e a_2 a_3 e, e a_4)$ [redacted] such that the map $R \longrightarrow AeA$, $(a_1 e, e a_2) \longmapsto a_1 e a_2$, is a map of A-bimodules as well as a homomorphism of algs, and moreover is [redacted] surjective. One has

$$(a_1 e, e a_2)(a_3 e, e a_4) = (a_1 e a_2 a_3 e, e a_4) = (a_1 e a_2) \cdot (a_3 e, e a_4)$$
$$= (a_1 e, e a_2 a_3 e a_4) = (a_1 e, e a_2) \cdot (a_3 e a_4)$$

showing the product is given by either bimodule structure. It follows that R is a [redacted] central extension of AeA. We have seen it is the universal central extension.

Now consider the idempotent $(e,e) \in R$.
We have $(e,e)R = eR = eAe \otimes_B eA = eA$
and similarly $R(e,e) = Ae$, $(e,e)R(e,e) = B$.

Thus
$$\begin{tikzcd} R(e,e) \otimes_B (e,e)R \arrow[r] & R \\ Ae \otimes_B eA \arrow[u, equal] \arrow[ur, "\text{identity}"'] & \end{tikzcd}$$

The point of this discussion is that from a general pair (A,e) we obtain a new pair $(R,(e,e))$ such that $R = R(e,e) \otimes_B (e,e)R$. The basic equivalence of interest, namely between B-modules and good R modules does not depend on A, so that the primary case of interest is where $A = Ae \otimes_B eA$.

April 11, 1994

Consider a nonunital alg A, an idempotent $e \in A$, and put $A' = Ae \otimes_B eA$, where $B = eAe$.

We know already that A' is a good module over itself.

Claim that if eA is a flat B module, then A' is an excellent A' module, equivalently A' is H-unital.

Why? Put $e' = (e,e) \in A'$. Then e' is an idempotent such that $A'e' = Ae$, $e'A' = eA$, $e'A'e' = B$, and $A' \xleftarrow{\sim} A'e' \otimes_B e'A'$. We know already that $A'e'$ is an excellent A' module. If $e'A' = eA$ is a flat B-module, then it is a filtered inductive limit of free B-modules, so A' is a filtered inductive limit of finite direct sums of the excellent A'-module $A'e'$, so the claim is clear. Alternatively we go from the acyclic complex

$$\longrightarrow A'^{\otimes 2} \otimes A'e' \longrightarrow A' \otimes A'e' \longrightarrow A'e' \longrightarrow 0 \cdots$$

and tensor $- \otimes_B e'A'$ which is an exact functor to get the bar complex for A'.

Let $A \subset R$ be a right ideal such that $A^2 = A$. Then for all left R-modules M one has $A \otimes_A M \xrightarrow{\sim} A \otimes_R M$.

Why? Check that
$$A \times M \longrightarrow A \otimes_A M$$
$$(a,m) \longmapsto a \otimes_A m$$

is R-bilinear, i.e $ar \otimes_A m = a \otimes_A rm$. Since $A^2 = A$ one can assume $a = a_1 a_2$. Then

$$a_1 \underbrace{a_2 r}_{\in A} \otimes_A m = a_1 \otimes_A a_2 rm = a_1 a_2 \otimes_A rm$$

Thus we have a well-defined map

$$A \otimes_R M \longrightarrow A \otimes_A M \qquad a \otimes_R m \mapsto a \otimes_A m$$

which is inverse to the obvious map the other way.

Interchanging left + right yields:

Let A be a left ideal in a ring R such that $A^2 = A$. Then for all right R-modules M one has

$$M \otimes_A A \xrightarrow{\sim} M \otimes_R A$$

In particular if A is a flat A-module, then A is a flat R module.

By flat A module N we mean an A-module such that $M \longmapsto M \otimes_A N$ is exact for $M \in Mod(A_r)$. This is the same as a flat unital $\tilde{A}$ module. If A is unital and N is a unital A module, then denoting the identity of A by e, one has

$$M \otimes_A N = M \otimes_A eN = Me \otimes_A N$$

so that N is flat as nonunital module iff it is flat as unital module.