January 1, 1994

$\Lambda \tilde{\otimes} B(\mathbb{C})$ has basis u^n, Bu^n for $n \geq 0$ and b, B are given by

$$b(u^n) = -BSu^n = -B\begin{cases} u^{n-1} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b(Bu^n) = 0$$

$$B(u^n) = Bu^n \qquad B(Bu^n) = 0$$

Let us determine all maps of mixed complexes

$$f : M \longrightarrow \Lambda \tilde{\otimes} B(\mathbb{C})[1]$$

f has the form

$$f = \sum_{n \geq 0} \left(u^n f_{2n+1} + Bu^n f_{2n+2} \right) \qquad f_k : M_k \to \mathbb{C}$$

Then $\quad Bf = \sum_{n \geq 0} Bu^n f_{2n+1} = -fB = -\sum_{n \geq 0} \left(u^n f_{2n+1} B + Bu^n f_{2n+2} B \right)$

so

$$\boxed{\begin{aligned} f_{2n+1} &= -f_{2n+2} B \\ 0 &= f_{2n+1} B \end{aligned} \quad \text{for } n \geq 0}$$

Also $\quad +bf = \sum_{n \geq 1} -Bu^{n-1} f_{2n+1} = -fb = -\sum_{n \geq 0} \left(u^n f_{2n+1} b + Bu^n f_{2n+2} b \right)$

so

$$\boxed{\begin{aligned} f_{2n+3} &= f_{2n+2} b \\ 0 &= f_{2n+1} b \end{aligned} \quad \text{for } n \geq 0}$$

So f has the form

$$f = \sum_{n \geq 0} \left(-u^n f_{2n+2} B + Bu^n f_{2n+2} \right)$$

where f_{2n+2} $n \geq 0$ satisfies

$$+f_{2n+4} B = f_{2n+2} b \qquad n \geq 0$$

$$0 = +f_{2n+2} B b$$

The second condition is redundant since $+f_{2n+2} Bb = -f_{2n+2} bB = -f_{2n+4} B^2 = 0$. Thus a map of mixed complexes $f: M \to \Lambda \tilde{\otimes} B(\mathbb{C})[1]$ is equivalent to [redacted] $(\psi_2, \psi_4, \dots)$ where $\psi_{2n}: M_{2n} \to \mathbb{C}$ satisfies $\psi_2 b \mp \psi_4 B = 0$, $\psi_4 b \mp \psi_6 B = 0, \dots$ but not $\psi_2 B = 0$.

Example we have in mind: A unital alg. $\rho: A \to \mathbb{C}$ linear $\ni$ $\rho(1) = 1$. Then ρ induces a homomorphism $RA \to \mathbb{C}$ (which is a trace on RA in particular) with components $\rho\omega^n : \Omega^{2n} A \to \mathbb{C}$.

Then we know $\psi_{2n} = \frac{(-1)^n}{n!} \rho\omega^n$ (OK as is) [redacted] for $n \geq 0$ satisfy $\psi(b + B) = 0$, $\psi \kappa^2 = \psi$. Thus we get a map

$$\sum_{n \geq 0} \left(-u^n \psi_{2n+2} B + Bu^n \psi_{2n+2} \right) : \bar{\Omega} A \longrightarrow \Lambda \tilde{\otimes} B(\mathbb{C})[1]$$

which lifts $\overbrace{\bar{\Omega} A \longrightarrow \mathbb{C}[1]}^{\text{the map}}$ given by

$$-\psi_2 B = \psi_0 b = \rho b : \bar{\Omega}^1 A \longrightarrow \mathbb{C}$$
$$a_0 da_1 \longmapsto \rho([a_0, a_1]).$$

So far we have lifted[redacted] the map

$$\bar{\Omega} A \longrightarrow \mathbb{C}[1]$$

given by ρb to a map

$$f : \bar{\Omega} A \longrightarrow \Lambda \tilde{\otimes} B(\mathbb{C})[1]$$

given by $\psi \in (\bar{\Omega} A)^*$, ψ even support in degrees ≥ 2 such that $-\psi(b+B) = \rho b$. [redacted]

Then we define: E = h-fibre of f.

Claim $\exists$ a lifting

$$\begin{array}{ccccl} & & E & = & \text{h-fibre}\left(\bar{\Omega}A \xrightarrow{f} \Lambda\tilde{\otimes} B(\mathbb{C})[1]\right) \\ & \nearrow & \downarrow & & \\ \bar{\Omega}\tilde{A} & \longrightarrow & \Omega A & = & \text{h-fibre}\left(\bar{\Omega}A \to \mathbb{C}[1]\right) \end{array}$$

suffices to show f pulled back to $\bar{\Omega}\tilde{A}$ is null-homotopic. This should mean ψ is $(\varphi(b+B))_{\geq 2}$ where $\varphi \in (\bar{\Omega}\tilde{A})^*$ is odd.

Suppose we combine ρ and ψ to get $\rho + \psi \in (\Omega A)^*$ such that $(\rho+\psi)(b+B) = 0$. (The specific cochain[redacted] is [redacted] $\rho + \psi = \sum \frac{~~(-1)^n~~}{n!} \rho \omega^n$) OK as is

Then $\rho + \psi$ pulled back to $(\bar{\Omega}\tilde{A})^*$ is a coboundary $\varphi(b+B)$, so we win.

In fact the specific $\rho + \psi$ is the K^2-invariant cocycle corresponding to the trace given by the homom. $\rho_* : RA \longrightarrow \mathbb{C}$. Lifting back to $R(\tilde{A})$ we should be able to deform this homom. to zero, and this should give φ as a sort of Chern-Simons thing.

January 3, 1994

From Karoubi's paper: a defn. of cohomology with arb. coeffs. in terms of noncomm. diff'l forms. We get some interesting contractions for $(\Omega A, d)$, also for $\mathrm{Cone}(\mathbb{C} \to (\Omega A, d))$, $\mathrm{Cone}(\mathbb{C} \to (C(A), D))$, ($D$ the Alexander-Spanier differential). Karoubi uses an augmentation $A \to \mathbb{C}$, but the formulas work for any linear retraction $\rho : A \to \mathbb{C}$. His formulas are based on the b' operator, and I will now explain them from the viewpoint of the standard bimodule resolutions.

Recall the standard normalized resolution $A * \mathbb{C}[\varepsilon]$, $a_0 \varepsilon \cdots \varepsilon a_{n+1} = a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1}$ in degree $n+1$, b' the superderivation $\ni b'(a) = 0$, $b'(\varepsilon) = 1$. If l_ε is left multiplication by ε, then $[b', l_\varepsilon] = 1$. Let $\rho : A \to \mathbb{C}$ be a retraction: $\rho(1) = 1$. Then we have an injection i

$$\begin{array}{ccccccc}
A & \underset{b'}{\overset{l_\varepsilon}{\rightleftarrows}} & A \otimes A & \underset{b'}{\overset{l_\varepsilon}{\rightleftarrows}} & \Omega^1 A \otimes A & \underset{b'}{\overset{l_\varepsilon}{\rightleftarrows}} & \\
\uparrow \downarrow \rho & & \uparrow \downarrow 1\otimes\rho & & \uparrow \downarrow 1\otimes\rho & & \\
\mathbb{C} & \underset{\rho}{\overset{\iota}{\rightleftarrows}} & A & \underset{b'_\rho}{\overset{d}{\rightleftarrows}} & \Omega^1 A & \underset{b'_\rho}{\overset{d}{\rightleftarrows}} &
\end{array}$$

such that $i d = l_\varepsilon i$:

$$\begin{aligned}
a_0 da_1 \cdots da_n &\overset{i}{\longmapsto} a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon \\
&\overset{l_\varepsilon}{\longmapsto} [\varepsilon, a_0] \cdots [\varepsilon, a_n] \varepsilon \\
&\overset{1 \otimes \rho}{\longmapsto} da_0 \cdots da_n
\end{aligned}$$

Define $b'_\rho = (1 \otimes \rho) b' i$:

$$\begin{aligned}
a_0 da_1 \cdots da_n &\overset{i}{\longmapsto} a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon \\
&\overset{b'}{\longmapsto} (-1)^n a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_{n-1}]
\end{aligned}$$

$$= (-1)^{n-1} a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_{n-1}] (a_n \varepsilon - \varepsilon a_n)$$

$$\xrightarrow{1\otimes\rho} (-1)^{n-1} a_0\, da_1 \cdots da_{n-1} (a_n - \rho a_n)$$

I should have also pointed out (before introducing b'_ρ) that $(1\otimes\rho)\, l_\varepsilon = d(1\otimes\rho)$.

$$a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1} \xrightarrow{1\otimes\rho} a_0\, da_1 \cdots da_n\, \rho a_{n+1}$$

$$\xrightarrow{d} da_0\, da_1 \cdots da_n\, \rho a_{n+1}$$

$$(1\otimes\rho)\, l_\varepsilon \left(a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1} \right) = (1\otimes\rho) \left([\varepsilon, a_0] \cdots [\varepsilon, a_n] \varepsilon a_{n+1} \right)$$

$$= da_0 \cdots da_n\, \rho a_{n+1}.$$

(Also $(1\otimes\rho)\, l_\varepsilon(a) = (1\otimes\rho)(\varepsilon a) = \rho a$.)

Then

$$[d, b'_\rho] = d(1\otimes\rho) b' i + (1\otimes\rho) b' \overbrace{id}^{l_\varepsilon i}$$

$$= (1\otimes\rho)(l_\varepsilon b' + b' l_\varepsilon) i = (1\otimes\rho) i = 1$$

Next do similarly for the unnormalized standard resolution $A * \mathbb{C}[h]$. d on ΩA will be replaced by the Alexander-Spanier differential D on $A * \mathbb{C}[h]$. Now we know $[b', D] = 0$; instead of l_ε we use $D' = D + r_h$, where

$$r_h(a_0 h \cdots h a_n) = (-1)^n (a_0 h \cdots h a_n h)$$

$$r_h(a_0, \ldots, a_n) = (-1)^n (a_0, \ldots, a_n, 1)$$

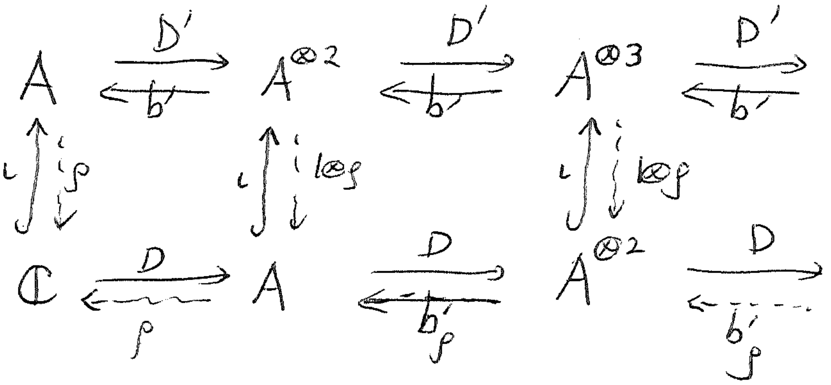
r_h is $-\lambda^{-1} s$ which we know is a contraction for b'. Then $[b', D'] = [b', D + r_h] = [b', r_h] = 1$.

Also recall that

$$D(a_0, \dots, a_n) = \sum_{i=0}^{n+1} (-1)^i (\dots a_{i-1}, 1, a_i, \dots) \quad \text{so}$$

$$(D + r_h)(a_0, \dots, a_n) = \sum_{i=0}^{n} (-1)^i (\text{———})$$

With these changes the corresponding diagram is



$$i(a_0, \dots, a_n) = (a_0, \dots, a_n, 1) \qquad i(a_0 h \dots h a_n) = a_0 h \dots h a_n h$$

$$(1\otimes\rho)(a_0, \dots, a_{n+1}) = (a_0, \dots, a_n)\rho a_{n+1} \qquad (1\otimes\rho)(a_0 h \dots h a_{n+1}) = a_0 h \dots h a_n \rho a_{n+1}$$

$$D' i(a_0 h \dots h a_n) = D'(a_0 h \dots a_n h)$$
$$= D(a_0 h \dots h a_n) h \quad = iD(a_0 h \dots h a_n)$$

$$(1\otimes\rho) D'(a_0 h \dots h a_{n+1}) = (1\otimes\rho)(D(a_0 h \dots a_n) h a_{n+1})$$
$$= D(a_0 h \dots h a_n) \rho a_{n+1}$$
$$= D(1\otimes\rho)(a_0 h \dots h a_{n+1})$$

$$b'_\rho(a_0 h \dots h a_n) = (1\otimes\rho)\, b'\, i\, (a_0 h \dots h a_n)$$
$$= (1\otimes\rho)\, b'(a_0 h \dots h a_n h)$$

$$b'_\rho(a_0, \dots, a_n) = (1\otimes\rho)\, b'(a_0, \dots, a_n, 1)$$

[redacted]

$$= (1\otimes\rho)\left(\sum_{i=0}^{n-1} (-1)^i (\dots, a_i a_{i+1}, \dots, a_n, 1) + (-1)^n (a_0, \dots, a_n) \right)$$

$$= \sum_{i=0}^{n-1} (-1)^i (\dots, a_i a_{i+1}, \dots, a_n) + (-1)^n (a_0, \dots, a_{n-1}) \rho a_n$$

$$= b'(a_0, \dots, a_n) + (-1)^n (a_0, \dots, a_{n-1}) \rho a_n$$

$$= b'(a_0, \dots, a_{n-1}) \cdot (1, a_n) + (-1)^{n-1} (a_0, \dots, a_{n-1}) (a_n - \rho a_n)$$

$$= (-1)^{n-1} (a_0, \dots, a_{n-1}) * (a_n - \rho a_n)$$ see p. 269

Maybe the simplest formula is

$$\boxed{b'_\rho (a_0, \dots, a_n) = b'(a_0, \dots, a_n) + (-1)^n (a_0, \dots, a_{n-1}) \rho a_n}$$

But in comparing with Karoubi we see we ~~[illegible]~~ still can use $[D, b'] = 0$ again to get

$$\boxed{[D, K] = 1}$$

where

$$\boxed{K(a_0, \dots, a_n) = (-1)^n (a_0, \dots, a_{n-1}) \rho a_n}$$

Variation: Instead of a scalar-valued ρ we can consider $\rho : A \to R$ linear $\rho(1) = 1$, then use $R \xrightarrow{i_A \otimes 1} A \otimes R \longrightarrow R$, $a \otimes x \longmapsto \rho(a) x$ to get b'_ρ on $\Omega A \otimes R$ instead of on ΩA. We can also tensor with any R-module M to get b'_ρ on $\Omega A \otimes M$.

January 5, 1994

Consider the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega A \longrightarrow \bar{\Omega} A \longrightarrow 0$$

of mixed complexes. A linear splitting (r, ℓ) compatible with the grading is equivalent to a linear retraction $\rho : A \longrightarrow \mathbb{C}$. Such a splitting is compatible with B. Choose such a splitting. Then we get a map of mixed complexes

(1) $$\bar{\Omega} A \longrightarrow \mathbb{C}[1]$$

given by $r[b, \ell]$ as usual. This map ~~[illegible]~~ lifts is given by the map $\bar{\Omega}^1 A = \Omega^1 A \longrightarrow \mathbb{C}$ which ~~[illegible]~~ $a_0 da_1 \in \bar{\Omega}^1 A$ to itself $a_0 da_1 \in \Omega^1 A$, applies b to get $[a_0, a_1]$, then applies ρ to get $\rho[a_0, a_1] = (\rho b)[a_0, a_1]$. Thus (1) is the map given by ρb.

Furthermore one has

(2) $$\Omega A = \text{h-Fibre}\left(\rho b : \bar{\Omega} A \longrightarrow \mathbb{C}[1]\right)$$

Claim we can lift ρb to a map of mixed complexes

$$f : \bar{\Omega} A \longrightarrow \Lambda \tilde{\otimes} B(\mathbb{C})[1]$$

~~[illegible]~~ A map of this sort which is compatible with B has the form

$$f = \sum_{n \geq 0} -u^n \left(f_{2n+2} B\right) + B u^n f_{2n+2}$$

For it to be compatible with b means the sum of

$$bf = \sum_{n \geq 1} Bu^{n-1}\left(f_{2n+2}B\right) \qquad \begin{aligned} b(u^n) &= -BSu^n = -Bu^{n-1} \\ b(Bu^n) &= -BSBu^n = 0 \end{aligned}$$

$$fb = \sum_{n \geq 0} -u^n\left(f_{2n+2}B\right)b + Bu^n f_{2n+2}b$$

must be zero, which yields

$$f_{2n+4}B + f_{2n+2}b = 0$$

$$f_{2n+2}Bb = 0$$

The second condition follows from the first.

So we conclude that a lifting of ρb to f is the same as $f_2 + f_4 + \cdots \in (\Omega A)^*$ satisfying $f_{2n}b + f_{2n+2}B = 0$ for $n \geq 1$ and $-f_2 B = \rho b$. We know a solution of these equations is given by

$$f_{2n} = \rho \frac{(-\omega)^n}{n!} \qquad n \geq 1.$$

I notice at this point that $(f_{2n})_{n \geq 1}$ fits with $f_0 = \rho$. This suggests putting them together. Let's introduce

$$F = \text{h-Fibre}\left(\Lambda \tilde{\otimes} B(\mathbb{C})[1] \longrightarrow \mathbb{C}[1]\right)$$

A map of mixed complexes $M \longrightarrow F$ is the same as a sequence $(f_{2n})_{n \geq 1}$, $f_{2n} \in (M_{2n})^*$ such that $f_{2n}b + f_{2n+2}B = 0$ for $n \geq 1$, together with $f_0 \in M_0^*$ such that $f_0 b + f_2 B = 0$ and $f_0 B = 0$.

Note that the h-fibre of a map $f: X \to Y$ of mixed complexes is $X_n \oplus Y_{n+1}$ in degree n

with $b = \begin{pmatrix} b & 0 \\ f & -b \end{pmatrix}$, $B = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$

where the sign of f is negociable.

Check:

$$[b,B] = \left[\begin{pmatrix} b & 0 \\ f & -b \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} \right] = \begin{pmatrix} \overset{0}{\overset{\shortparallel}{bB+Bb}} & 0 \\ \underset{\underset{0}{\shortparallel}}{fB-Bf} & \underset{\underset{0}{\shortparallel}}{bB+Bb} \end{pmatrix}$$

Let us compare $F = \text{hofibre}(\Lambda \otimes B(\mathbb{C})[1] \xrightarrow{p} \mathbb{C}[1])$ with the actual fibre $K = \text{Ker}\, p$. Note that there's an obvious injective map from the fibre to the hofibre.

Picture: degree: 0 1 2

$$F: \quad \mathbb{C} \underset{0}{\overset{1}{\leftrightarrows}} \mathbb{C} \underset{1}{\overset{0}{\leftrightarrows}} \mathbb{C} \underset{0}{\overset{1}{\leftrightarrows}} \mathbb{C} \underset{1}{\overset{0}{\leftrightarrows}}$$

$$\begin{array}{cc} \shortparallel & \shortparallel \end{array}$$

$$K: \quad \mathbb{C} \underset{0}{\overset{1}{\leftrightarrows}} \mathbb{C} \underset{1}{\overset{0}{\leftrightarrows}}$$

A map $M \longrightarrow F$ we have identified with an even $b+B$ cocycle supported in $n \geq 0$, i.e. $(f_0, f_2, \cdots)$ such that $f_0 B = 0$, $f_0 b + f_2 B = 0$, $f_2 b + f_4 B = 0, \cdots$

A map $K \longrightarrow F$ should be the same with support in $n \geq 2$. The inclusion $K \hookrightarrow F$ is the clear on the level of cocycles. Also given $M \xrightarrow{f} F$ represented by $(\cdots, 0, f_0, f_2, \cdots)$, we have $f_0 B = 0$, so if we can write $f_0 = gB$, then $(\cdots, 0, g, 0, \cdots)$ which is an odd cocycle has coboundary $(\cdots, 0, f_0, gb, 0, \cdots)$, so subtracting we get a deformation to a map $M \xrightarrow{f'} K$.

Now we can discuss the purpose of introducing F. Actually it turns out that K is more important. The diagram is

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K[-1] & = & K[-1] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Lambda\otimes B(\mathbb{C}) & \longrightarrow & E & \longrightarrow & \bar{\Omega}A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \Omega A & \longrightarrow & \bar{\Omega}A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $E \overset{\text{defn}}{=} \text{hofibre}(\bar{\Omega}A \xrightarrow{f} \Lambda\tilde{\otimes}B(\mathbb{C})[1])$. This means that sitting over ΩA, we have a mixed complex E which has one extra element in each degree >0, and is quis to ΩA, and has Connes' property.

The reason I was led to consider F is that $\bar{\Omega}A \xrightarrow{f} \Lambda\tilde{\otimes}B(\mathbb{C})[1]$ given by $(f_2, f_4, \cdots)$ lifts to $\Omega A \xrightarrow{f} F$ given by $(f_0, f_2, f_4, \cdots)$. One then has

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}[-1] & \longrightarrow & \text{hofib}(\mathbb{C}\xrightarrow{1}\mathbb{C}) & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F[-1] & \longrightarrow & \text{hofib}(\Omega A\to F) & \longrightarrow & \Omega A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \overset{\exists}{\nearrow} & \downarrow & & \\
0 & \longrightarrow & \Lambda\tilde{\otimes}B(\mathbb{C}) & \longrightarrow & \underset{\| \, E}{\text{hofib}(\bar{\Omega}A\to\Lambda\tilde{\otimes}B(\mathbb{C})[1])} & \longrightarrow & \bar{\Omega}A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The middle vertical exact sequence should split because E has Connes' property.

At this point we reach the natural question whether E comes from an A-bimodule resolution of A.

January 6, 1994

Recall we have defined a left action of $A * \mathbb{C}[\varepsilon]$ on $A * \mathbb{C}[h]$ by

$$a \cdot \alpha = a\alpha$$
$$\varepsilon \cdot \alpha = h b'(h\alpha) = h\alpha - h^2 b'(\alpha)$$

and also a right action by

$$\alpha \cdot a = \alpha a$$
$$\alpha \cdot \varepsilon = (-1)^{|\alpha|} b'(\alpha h) h = \alpha h + (-1)^{|\alpha|} b'(\alpha) h^2$$

Check compatibilty with b':

$$b'(a \cdot \alpha) = b'(a\alpha) = a b'(\alpha) = a \cdot b'(\alpha)$$
$$b'(\varepsilon \cdot \alpha) = b'(h\alpha) = \alpha - h b'(\alpha)$$
$$= b'(\varepsilon) \cdot \alpha - \varepsilon \cdot b'(\alpha)$$

$$b'(\alpha \cdot a) = b'(\alpha a) = b'(\alpha) a = b'(\alpha) \cdot a$$
$$b'(\alpha \cdot \varepsilon) = b'(\alpha h) = b'(\alpha) h + (-1)^{|\alpha|} \alpha$$
$$= b'(\alpha) \cdot \varepsilon + (-1)^{|\alpha|} \alpha \cdot b'(\varepsilon)$$

By acting on 1 the left and right actions give rise to liftings of $A * \mathbb{C}[\varepsilon]$ into $A * \mathbb{C}[h]$:

$$a_0 \varepsilon \cdots \varepsilon a_{n+1} = a_0 \varepsilon [a_1, \varepsilon] \cdots [a_n, \varepsilon] a_{n+1} \longmapsto a_0 h [a_1, h] \cdots [a_n, h] a_{n+1}$$
$$\text{———} = a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1} \longmapsto a_0 [h, a_1] \cdots [h, a_n] h a_{n+1}$$

and these coincide with the two liftings given by the simplicial normalization theorem.

Having defined a left and a right action the natural question is whether one has a bimodule structure.

$$(a\cdot\alpha)\cdot a' = a\alpha a' = a\cdot(\alpha\cdot a')$$

$$\varepsilon\cdot(\alpha\cdot a) = h\alpha a - h^2 b'(\alpha a)$$
$$= (h\alpha - h^2 b'(\alpha))a = (\varepsilon\cdot\alpha)\cdot a$$

$$(a\cdot\alpha)\cdot\varepsilon = a\alpha h + (-1)^{|a\alpha|} b'(a\alpha)h^2$$
$$= a(\alpha h + (-1)^{|\alpha|} b'(\alpha) h^2) = a\cdot(\alpha\cdot\varepsilon)$$

$$(\varepsilon\cdot\alpha)\cdot\varepsilon = (h\alpha - h^2 b'(\alpha))\cdot\varepsilon$$
$$= h\alpha h + (-1)^{|h\alpha|} b'(h\alpha)h^2 - h^2 b'(\alpha)h$$
$$= \overset{①}{h\alpha h} - (-1)^{|\alpha|}\alpha h^2 + (-1)^{|\alpha|} h \overset{③}{b'(\alpha)} h^2$$
$$- h^2 \overset{②}{b'(\alpha)} h$$

$$\varepsilon\cdot(\alpha\cdot\varepsilon) = \varepsilon\cdot(\alpha h + (-1)^{|\alpha|} b'(\alpha) h^2)$$
$$= h\alpha h - h^2 b'(\alpha h) + (-1)^{|\alpha|} h b'(\alpha) h^2$$
$$= \overset{①}{h\alpha h} - h^2 \overset{②}{b'(\alpha)} h - (-1)^{|\alpha|} h^2\alpha$$
$$+ (-1)^{|\alpha|} h \overset{③}{b'(\alpha)} h^2$$

$$\boxed{(\varepsilon\cdot\alpha)\cdot\varepsilon - \varepsilon\cdot(\alpha\cdot\varepsilon) = (-1)^{|\alpha|}[h^2, \alpha]}$$

compare p. 93-94

Let $R = A * \mathbb{C}[h]/([h^2, A])$. Then $h^2 \in$ the center of R. We've seen that there is a canonical lifting $\Omega A \longrightarrow A * \mathbb{C}[h]$ compatible with alg. struc.

$$a_0 da_1 \cdots da_n \longmapsto a_0[h, a_1]\cdots[h, a_n]$$

[redacted]

Notice that wrt the embedding
$\Omega A \hookrightarrow R$ d on ΩA is the restriction of ad h on R. This is because $[h,[h,a]] = [h^2,a] = 0$ in R. Thus we get a homomorphism from the cross-product algebra $\Omega A \tilde{\otimes} \mathbb{C}[h]$ [redacted] to R. It should be clear this is an isomorphism as we have a map also [redacted] $A * \mathbb{C}[h] \longrightarrow$ [redacted] $\Omega A \tilde{\otimes} \mathbb{C}[h]$.

The next project is to calculate the commutator quotient space as A-bimodule for $\Omega A \tilde{\otimes} \mathbb{C}[h]$.

Actually there is a point I have omitted, and this is the fact that there is a canonical lifting of $A * \mathbb{C}[\varepsilon] = \Omega A \tilde{\otimes} \mathbb{C}[\varepsilon]$ into R, which can be described as acting on 1 either by the left or the right action [redacted]. Note R is a bimodule over $A * \mathbb{C}[\varepsilon]$ by p. 303. Also the two liftings of $A * \mathbb{C}[\varepsilon]$ into $A * \mathbb{C}[h]$ coincide in R as

$$a_0\, h[a_1,h] \cdots [a_n,h]\, a_{n+1} = a_0 [h,a_1] \cdots [h,a_n]\, h a_{n+1}$$

since [redacted] $h[a,h] + [a,h]h = [a,h^2] = 0$

$$\Longrightarrow \quad h[a,h] = [h,a]h$$

Now onto $R \otimes_A$. No good because $\Omega A \tilde{\otimes} \mathbb{C}[h]$ is not a "projective" bimodule in degrees >0. You have $Ah^2 = h^2A$ in degree 2.

List minor ideas. Maps of bimod-resolutions (really these acyclic DG algs.)

$$e\Omega\tilde{A}e \xleftarrow{\sim} A * \mathbb{C}[h] \longrightarrow A * \mathbb{C}[\varepsilon]$$

When you pass to $-\otimes_A$ you get

$$e\Omega\tilde{A} \xleftarrow{\sim} C(A) \longrightarrow \Omega A$$

We know these are not algebra homomorphisms, but they do become so if one uses the modified product on $C(A)$ (i.e. the one induced from $\widetilde{\Omega}\tilde{A}$).

There's a large supply of contractions for $(C(A), b')$, e.g. l_ξ, r_ξ and each one leads to a ~~quotient~~ mixed complex of $\widetilde{\Omega}\tilde{A}$.

In constructing $0 \to K \to E \to \Omega A \to 0$ you use $\psi = \sum \rho \frac{(-\omega)^n}{n!}$ which is actually the cocycle corresponding to the trace $RA \xrightarrow{\rho_*} \mathbb{C}$, and the map $\widetilde{\Omega}\tilde{A} \to E$ comes from a cochain related to a deformation of ρ to zero (up in $\tilde{A}$). Is there any connection between these cocycles and the corresponding map $\widetilde{R}\tilde{A} \to RA$?

Miscellaneous formulas:

$$\mathbb{C}[h]\,\tilde{\otimes}\,\mathbb{C}[d] \cong \big(\mathbb{C}[u]\otimes\mathbb{C}[\varepsilon]\big)\,\tilde{\otimes}\,\mathbb{C}[\partial]$$

$$\big(A * \mathbb{C}[h]\big)\,\tilde{\otimes}\,\mathbb{C}[\partial] \cong \Big(A * \big(\mathbb{C}[\varepsilon]\otimes\mathbb{C}[u]\big)\Big)\,\tilde{\otimes}\,\mathbb{C}[\partial]$$

$$\Omega A\,\tilde{\otimes}\,\mathbb{C}[d] \cong QA\,\tilde{\otimes}\,\mathbb{C}[d]$$

$$a-(da)d \longleftrightarrow p^a$$
$$da \longleftrightarrow q^a$$

$$\mathbb{C}[F]\,\tilde{\otimes}\,\mathbb{C}[d] = \mathbb{C}[\varepsilon]\,\tilde{\otimes}\,\mathbb{C}[d]$$

$[d,F]=1$ $\qquad$ $[d,\varepsilon]=0$ $\qquad$ both Clifford algebras

$F^2=1$ $\qquad$ $\varepsilon^2=0$ $\qquad$ isom. quadratic spaces

One has besides $\Omega A\,\tilde{\otimes}\,\mathbb{C}[d] = QA\,\tilde{\otimes}\,\mathbb{C}[d]$

also $\Omega A \subset A*F = QA\,\tilde{\otimes}\,\mathbb{C}[F]$

$$a_0da_1\cdots da_n \longmapsto a_0[F,a_1]\cdots[F,a_n]$$

Problem is to construct a map $\Omega A \longrightarrow \Omega\tilde{A}$ corresponding to the nonunital alg homom. $A \longrightarrow \tilde{A}$. Apparently this is the map of Coquereaux–Kastler. It's an algebra homomorphism and it's compatible with b. Formula $a_0da_1\cdots da_n \mapsto a_0\,e\,da_1\,e\,da_2\cdots e\,da_n\,e$. I now feel that this map is too naive as it does not use a choice of retraction $\rho: A \longrightarrow \mathbb{C}$.

January 7, 1994

The problem to focus upon is this: We know that the obvious map

$$(1) \qquad P\bar{\Omega}\tilde{A}/P\bar{\Omega}\tilde{\mathbb{C}} \twoheadrightarrow P\bar{\Omega}A$$

is a surjective quis of B-acyclic mixed complexes. [redacted] Hence it is a heq of mixed cxs. The problem is to construct [redacted] a homotopy inverse explicitly.

Note, if $M' \to M$ is an injection of mixed complex which are B-acyclic, then by the long exact sequence in B-homology, the cokernel M/M' is B-acyclic.

The obvious map (1) is induced by the (nonunital) homomorphism $\bar{\Omega}\tilde{A} \to \Omega A$, recall here that $\bar{\Omega}\tilde{A}$ is the augmentation ideal in $\Omega\tilde{A}$. Thus $\bar{\Omega}\tilde{A}/\bar{\Omega}\tilde{\mathbb{C}}$ maps naturally to $\Omega A/\Omega\mathbb{C} = \bar{\Omega}A$.

Next consider the harmonic decomposition

$$\begin{array}{ccccccl} P\bar{\Omega}\tilde{\mathbb{C}} & \hookrightarrow & P\bar{\Omega}\tilde{A} & \twoheadrightarrow & P\bar{\Omega}\tilde{A}/P\bar{\Omega}\tilde{\mathbb{C}} & & \text{B-acyclic} \\ \oplus & & \oplus & & \oplus & & \\ P^{\perp}\bar{\Omega}\tilde{\mathbb{C}} & \hookrightarrow & P^{\perp}\bar{\Omega}\tilde{A} & \twoheadrightarrow & P^{\perp}\bar{\Omega}\tilde{A}/P^{\perp}\bar{\Omega}\tilde{A} & & B=0 \\ & & & & & & \text{b-acyclic} \end{array}$$

Conclude that $\bar{\Omega}\tilde{A}/\bar{\Omega}\tilde{\mathbb{C}}$ has Connes' property, and therefore the surjective quis

$$\bar{\Omega}\tilde{A}/\bar{\Omega}\tilde{\mathbb{C}} \longrightarrow \bar{\Omega}A$$

is a homotopy equivalence of mixed complexes.

Simpler problem is to construct explicitly a homotopy inverse for the surj. quis

$$(2) \qquad C^\lambda(A)/C^\lambda(\mathbb{C}) \longrightarrow\!\!\!\!\rightarrow \overline{C}^\lambda(A)$$

This follows from an explicit h-inv. for (1) by taking the cokernel of B and using
$P\bar{\Omega}\tilde{A}/B(P\bar{\Omega}\tilde{A}) = C^\lambda(A)$, $P\bar{\Omega}A/B(P\bar{\Omega}A) = \bar{C}^\lambda(A)$.

To get ideas we consider now various arguments that [redacted] (2) is a quis.

1) filtration (written by Jacek). Define increasing filtration of A by [redacted] $F_{-1}A = 0$, $F_0 A = \mathbb{C}$, $F_1 A = A$. Then

$$gr\, A = \mathbb{C} \oplus \bar{A} \qquad \text{with zero mult on } \bar{A}.$$

and
$$gr\, C^\lambda(A) = C^\lambda(gr\, A)$$
$$= C^\lambda(\mathbb{C}) \oplus \left[\bar{A} \overset{!}{\otimes}\right] \oplus \Sigma\left[\bar{A} \overset{!}{\otimes}\right]^{(2)}_{-1} \oplus \cdots$$

by Goodwillie's thm. on semi-direct products. Result follows by considering $\{F_p C^\lambda(A)\} \to \{F_p \bar{C}^\lambda(A)\}$, where F_p on $\bar{C}^\lambda(A)$ is stupid skeletal filtration.

2) [LQ] argument

$$C^\lambda(A) \sim B(\bar{\Omega}\tilde{A})$$
$$\downarrow \text{quis} \quad (\text{column-wise quis})$$
$$B(\Omega A)$$

so
$$C^\lambda(A)/C^\lambda(\mathbb{C}) \sim B(\bar{\Omega}\tilde{A})/B(\bar{\Omega}\mathbb{C}) \xrightarrow{\text{quis}} B(\Omega A)/B\mathbb{C}$$

But $\mathcal{B}(\Omega A)/\mathcal{B}\mathbb{C} = \mathcal{B}(\bar{\Omega} A)$

and $\mathcal{B}(\bar{\Omega} A) \sim \bar{C}^\lambda(A)$.

3) Similar to 2) but using the double complex $C^\lambda(A \oplus \varepsilon A)$, here $A \oplus \varepsilon A$ means $A \otimes \mathbb{C}[\varepsilon]$ where $\mathbb{C}[\varepsilon]$ is the DG algebra with differential $b'(\varepsilon) = 1$.

$$C^\lambda(A \oplus \varepsilon A) \quad \text{similar to} \quad \mathrm{Cone}\left(CC(A) \to C^\lambda(A)\right)$$

$$\bar{C}^\lambda(A \oplus \varepsilon A) \quad \text{———} \quad \mathrm{Cone}\left(CC(A) \to \bar{C}^\lambda(A)\right)$$

$$\sim \mathrm{Cone}\left(\hat{C}(A) \to \bar{C}^\lambda(A)\right)$$

Use fact that the columns $p \geq 1$ for $C^\lambda(A \oplus \varepsilon A)$ and $\bar{C}^\lambda(A \oplus \varepsilon A)$ are related like the standard and standard normalized resolutions.

4) Relative Lie algebra homology, especially for a pair $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{h}$ reductive in $\mathfrak{g}$. We will apply this Lie alg homology theorem (due to Koszul?) in the case $(\mathfrak{gl}(A), \mathfrak{gl}(\mathbb{C}))$, then use invariant theory.

Before getting involved with Lie algebra stuff, I want to record earlier observations. We constructed an extension of mixed complexes

$$0 \longrightarrow \Lambda \tilde{\otimes} \mathcal{B}(\mathbb{C}) \longrightarrow E \longrightarrow \bar{\Omega} A \longrightarrow 0$$

starting from a choice of ρ. A natural question is whether there is a canonical extension. A simple question concerns the extension of complexes

$$0 \longrightarrow \mathcal{B}(\mathbb{C}) \longrightarrow E/\mathrm{Ker}(B) \longrightarrow \bar{C}^\lambda(A) \longrightarrow 0$$

corresponding to the odd reduced cyclic cohomology classes of A.

Look at this question in general: Given an exact sequence of complexes

$$0 \longrightarrow S \xrightarrow{i} E \xrightarrow{j} Q \longrightarrow 0$$

we can choose a splitting (r, l), whence we get a map of complexes $u = r[d,l] : Q \longrightarrow S[1]$. A change in lifting is given by a $v \in \mathrm{Hom}_0(Q,S)$, which alters u by $[d,v]$.

In our ~~situation~~ we choose a $\rho : A \to \mathbb{C}$ and we get $\phi\left(\frac{\omega^n}{n!}\right) : \bar{C}^\lambda_{2n-1}(A) \longrightarrow \mathbb{C}$ for $n \geq 1$, whence a map $\bar{C}^\lambda(A) \xrightarrow{u_\rho} C^\lambda(\mathbb{C})^{[1]} = B(\mathbb{C})[1]$, whence an extension $0 \to C^\lambda(\mathbb{C}) \longrightarrow E_\rho \longrightarrow \bar{C}^\lambda(A) \longrightarrow 0$. A different choice of ρ, say ρ', leads to $u_\rho - u_{\rho'} = [d, v(\rho,\rho')]$, and $v(\rho,\rho')$ can be interpreted as an isom. $v(\rho,\rho') : E_\rho \xrightarrow{\sim} E_{\rho'}$. The question of whether the ~~extn~~ is canonically amounts to transitivity $v(\rho,\rho') + v(\rho',\rho'') \overset{?}{=} v(\rho,\rho'')$ and this is the point that fails, I think. $v(\rho,\rho')$ is obtained by integrating along the line from ρ to ρ'. All we get is

$$v(\rho',\rho'') - v(\rho,\rho'') + v(\rho,\rho') = [d, w(\rho,\rho',\rho'')]$$

where $w(\rho,\rho',\rho'')$ is an integral over the triangle with vertices ρ, ρ', ρ''.

Another point concerns $\bar{\Omega}\tilde{\mathbb{C}} = \bar{\Omega}(\mathbb{C}[e])$

We have

$$\kappa\Big(\big(e-\tfrac{1}{2}\big)de^n\Big) = (-1)^n \big(e-\tfrac{1}{2}\big)de^n$$

$$\kappa(de^n) = (-1)^{n-1} de^n$$

Thus $P\bar{\Omega}\tilde{\mathbb{C}}$ has basis e, de, $(e-\frac{1}{2})de$, de^3, $(e-\frac{1}{2})de^4$;

$$\xleftarrow{0} e \underset{0}{\overset{1}{\rightleftarrows}} de \underset{1/2}{\overset{0}{\rightleftarrows}} \big(e-\tfrac{1}{2}\big)de^2 \underset{0}{\overset{3}{\rightleftarrows}} de^3 \underset{\frac{1}{2}}{\overset{0}{\rightleftarrows}} \big(e-\tfrac{1}{2}\big)de^4$$

$$b\Big(\big(e-\tfrac{1}{2}\big)de^{2n}\Big) = -\big(e-\tfrac{1}{2}\big)de^{2n-1}e + \underbrace{e\big(e-\tfrac{1}{2}\big)}_{\frac{1}{2}e}de^{2n-1}$$

$$= \underbrace{\big(e-\tfrac{1}{2}\big)(e-1)}_{\frac{1}{2}-\frac{1}{2}e} de^{2n-1} + \tfrac{1}{2}e\,de^{2n-1}$$

$b\Big(\big(e-\tfrac{1}{2}\big)de^{2n}\Big) = \frac{1}{2}de^{2n-1}$
$B\,\big(e-\tfrac{1}{2}\big)de^{2n} = (2n+1)de^{2n+1}$

Thus $P\bar{\Omega}\tilde{\mathbb{C}} \cong \Lambda \otimes B(\mathbb{C})$ is clear, more or less.

January 13, 1994

I have been reviewing Lie and BRS cohomology from Jan-Feb 1990, and now I want to record some applications to cyclic homology.

Let $\mathfrak{g} = \mathfrak{gl}_n \mathbb{C}$, $\tilde{\mathfrak{g}} = \mathfrak{gl}_n A$ for n large relative to cohomological degree being considered. I work [redacted] in the cochain setting: $\Lambda \tilde{\mathfrak{g}}^*$ gives the Lie cohomology of $\tilde{\mathfrak{g}}$; this is not rigorous unless [redacted] we assume A finite dimensional [redacted] (which is already an interesting case).

According to LQT the Lie cochain complex for $\tilde{\mathfrak{g}}$ is

$$(\Lambda \tilde{\mathfrak{g}}^*)^{\mathfrak{g}} = S\left(\Sigma C^\lambda(A)^*\right)$$

where $S = \mathrm{Sym}_{\text{super}}$ is [redacted] the symmetric alg in the super sense. Similarly the relative Lie cochain complex for $(\tilde{\mathfrak{g}}, \mathfrak{g})$ is

$$\left(\Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^*\right)^{\mathfrak{g}} = S\left(\Sigma \bar{C}^\lambda(A)^*\right)$$

The relation between Lie and relative Lie cohomology can be described as follows. We have exact seq.

$$0 \longrightarrow \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \tilde{\mathfrak{g}}/\mathfrak{g} \longrightarrow 0$$

of $\mathfrak{g}$ modules which gives rise to a filtration of the DG algebra $\Lambda \tilde{\mathfrak{g}}^*$, namely, the J-adic filtration $\Lambda \tilde{\mathfrak{g}}^* \supset J \supset J^2 \supset \cdots$

where $J = \Lambda \tilde{\mathfrak{g}}^* \cdot (\tilde{\mathfrak{g}}/\mathfrak{g})^*$. [redacted] Then

$$\mathrm{gr}_J \Lambda \tilde{\mathfrak{g}}^* = \Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^* \otimes \Lambda \mathfrak{g}^*$$

is the Lie complex for $\mathfrak{g}$ acting on $\Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^* = (\Lambda\tilde{\mathfrak{g}}^*)_{hor}$, where horizontal refers to the natural action of $\mathfrak{g}[\varepsilon]$ on $\Lambda\tilde{\mathfrak{g}}^*$ (restriction of $\tilde{\mathfrak{g}}[\varepsilon]$ action given by the operators ι_X, L_X).

This filtration gives rise to a spectral sequence

$$E_1^{pq} = H^q(\mathfrak{g}, \Lambda^q(\tilde{\mathfrak{g}}/\mathfrak{g})^*) \Longrightarrow H^n(\tilde{\mathfrak{g}})$$

When $\tilde{\mathfrak{g}}, \mathfrak{g}$ is a reductive pair

$$\| \qquad (\Lambda^p(\tilde{\mathfrak{g}}/\mathfrak{g})^*)^{\mathfrak{g}} \otimes H^q(\mathfrak{g})$$

and we get
$$E_2^{pq} = H^p(\tilde{\mathfrak{g}}, \mathfrak{g}) \otimes H^q(\mathfrak{g}) \Longrightarrow H^n(\tilde{\mathfrak{g}})$$

A connection, i.e. $\mathfrak{g}$-splitting of $0 \to \mathfrak{g} \to \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}/\mathfrak{g} \to 0$, yields an isomorphism of graded algebras

1) $$\Lambda\tilde{\mathfrak{g}}^* = \Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^* \otimes \Lambda\mathfrak{g}^*$$

compatible with the action of $\mathfrak{g}[\varepsilon]$. I recall (see Feb 25, 1990 p.252-254) that d on $\Lambda\tilde{\mathfrak{g}}^*$ can be described in terms of data ∇, L_X, F on $\Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^* = (\Lambda\tilde{\mathfrak{g}}^*)_{hor}$.

In the example $\tilde{\mathfrak{g}} = \mathfrak{gl}_n A$, $\mathfrak{g} = \mathfrak{gl}_n \mathbb{C}$, the connection corresponds to a splitting of

$$0 \longrightarrow \mathbb{C} \longrightarrow A \longrightarrow \bar{A} \longrightarrow 0$$

i.e. to a retraction $\rho : A \longrightarrow \mathbb{C}$. The isomorphism

$$(\Lambda\tilde{\mathfrak{g}}^*)^{\mathfrak{g}} = (\Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^* \otimes \Lambda\mathfrak{g}^*)^{\mathfrak{g}}$$

should yield
$$C_n^\lambda(A) \cong (\bar{A} \oplus \mathbb{C})_\lambda^{\otimes n+1}$$

Better, it should be true that the [redacted] filtration $(J^p)^{\mathfrak{g}}$ on $(\Lambda \tilde{\mathfrak{g}}^*)^{\mathfrak{g}}$ should correspond to the filtration $F_p C^\lambda(A)$ associated to the algebra filtration

$$F_p A = \begin{cases} 0 & p<0 \\ \mathbb{C} & p=0 \\ A & p>0 \end{cases}$$

and the isom. $(gr\, \Lambda \tilde{\mathfrak{g}}^*)^{\mathfrak{g}} = (\Lambda (\tilde{\mathfrak{g}}/\mathfrak{g})^* \otimes \Lambda \mathfrak{g}^*)^{\mathfrak{g}}$ should correspond to

$$gr\, C^\lambda(A) = C^\lambda(\underbrace{\mathbb{C} \oplus \bar{A}}_{gr\, A}).$$

Next use GFT formula for semi-direct products:

$$C^\lambda(\mathbb{C} \oplus \bar{A}) = C^\lambda(\mathbb{C}) \oplus (\bar{A} \otimes \mathcal{B}) \oplus [\bar{A} \otimes \mathcal{B} \otimes]^{(2)}_\lambda \oplus \cdots$$

where $\mathcal{B}$ is the bar construction for $\mathbb{C}$ (with counit), i.e. [redacted] tensor coalgebra $T(\mathbb{C}[1])$ with diffl b':

$$\begin{array}{ccccccccc} & & 3 & & 2 & & 1 & & 0 \\ \mathbb{C} & \xrightarrow{1} & \mathbb{C} & \xrightarrow{0} & \mathbb{C} & \xrightarrow{1} & \mathbb{C} & \xrightarrow{0} & \mathbb{C} \end{array}$$

Notice that $\bar{\mathcal{B}} = \bar{T}(\mathbb{C}[1])$ has two canonical [redacted] contractions. If I use the model $\mathbb{C} * \mathbb{C}[h] = \mathbb{C}[h]$, $b'(h) = 1$ for $\bar{\mathcal{B}}$, then left and right multiplication by h coincide (also left and right multiplication by ε [redacted] coincide).

The thing that I really want to do is to find a [redacted] lifting $\bar{C}^\lambda(A) \longrightarrow C^\lambda(A)$ whose coboundary is ~~the~~ odd map $\bar{C}^\lambda(A) \longrightarrow C^\lambda(\mathbb{C})[1]$

given by the Chern character forms associated to ρ.

So far we have discussed the Lie complex $\Lambda\tilde{\mathfrak{g}}^*$ with $\mathfrak{g}[\varepsilon]$ action, which is a version of (or an analogue of) $\Omega(P)$ where P is a principal G-bundle. For $\Omega(P)$ it ~~is~~ is natural to [redacted] simultaneously consider $\Omega(P)\otimes W(\mathfrak{g})$, the [redacted] BRS algebra, so we consider the analogue

$$\Lambda\tilde{\mathfrak{g}}^* \otimes W(\mathfrak{g}) = \Lambda\tilde{\mathfrak{g}}^* \otimes \Lambda(\mathfrak{g}[\varepsilon])^*$$

It should be true that invariant theory gives [redacted]

$$\left(\Lambda\tilde{\mathfrak{g}}^*\otimes W(\mathfrak{g})\right)^{\mathfrak{g}} = S\left(\sum C^\lambda(A\times\mathbb{C}[\varepsilon])^*\right)$$

~~To make this work consider the algebra~~

~~$A \times \mathbb{C}[\varepsilon] \cong \mathbb{C}\oplus(A\oplus\mathbb{C}[\varepsilon])$~~

Note that

$$A\times\mathbb{C}[\varepsilon] = \tilde{A}\oplus\mathbb{C}\varepsilon$$

$$\begin{aligned}(a, 0) &\longleftarrow a\\ (0, \varepsilon) &\longleftarrow \varepsilon\\ (1, 1) &\longleftarrow 1\end{aligned}$$

Thus $A\times\mathbb{C}[\varepsilon]$ is the semi-direct product algebra $\tilde{A}\oplus\mathbb{C}\varepsilon$, where $A\cdot\varepsilon = \varepsilon\cdot A = 0$, and where the differential is $d(\varepsilon) = e^{\perp}$, e denoting the identity in A. Notice that as algebra $\tilde{A}\oplus\mathbb{C}\varepsilon = \widetilde{A\oplus\mathbb{C}\varepsilon}$, i.e. there's a natural augmentation. However this augmentation is not compatible with d. Actually we want to look at

$$\left(\Lambda\tilde{\mathfrak{g}}^* \otimes W(\mathfrak{g})\right)_{bas} = \left(\Lambda\tilde{\mathfrak{g}}^* \otimes S\mathfrak{g}^*\right)^{\mathfrak{g}}$$

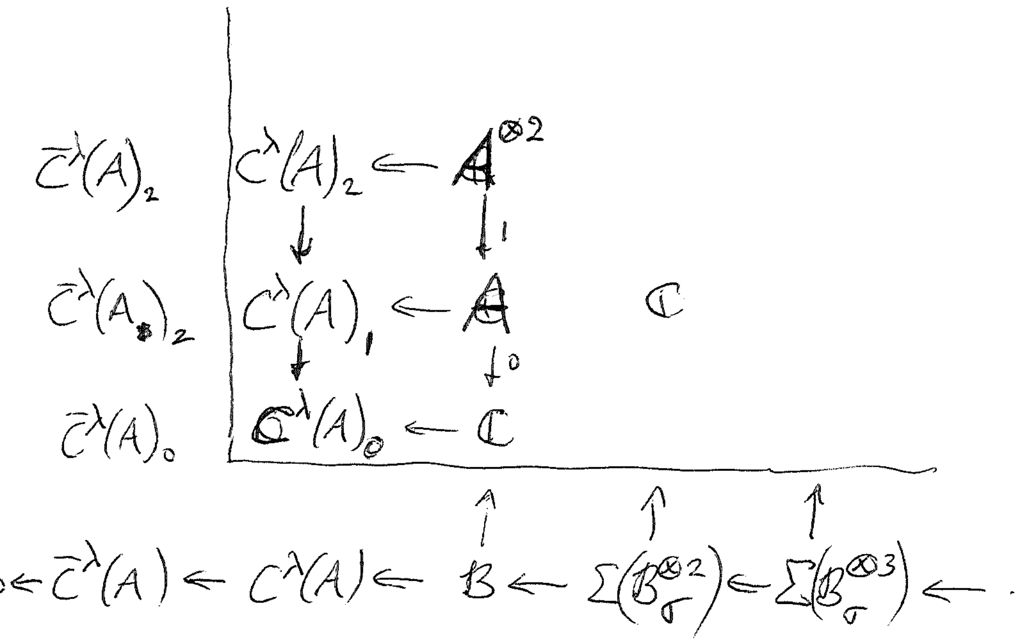
the DG algebra used to calculate equivariant cohomology. This corresponds to reduced cyclic complex

$$\bar{C}^\lambda(\tilde{A} \oplus \mathbb{C}\varepsilon) = \bar{C}^\lambda(\tilde{A}) \oplus \varepsilon B \oplus (\varepsilon B)^{\otimes 2}_\lambda \oplus \cdots$$

Interestingly this is the same as

$$C^\lambda(A \oplus \mathbb{C}\varepsilon) = C^\lambda(A) \oplus \varepsilon B \oplus (\varepsilon B)^{\otimes 2}_\lambda \oplus \cdots$$

except for the horizontal differential d. Here B is the bar construction of A. Picture:



The rows should be exact because we have $(\tilde{A} \oplus \mathbb{C}\varepsilon / \mathbb{C})^{\otimes q+1}_\lambda$ for the q-th row, and $\tilde{A} \oplus \mathbb{C}\varepsilon/\mathbb{C} = A \oplus \mathbb{C}\varepsilon$ with $d(\varepsilon) = -e$.

The obvious map $A \oplus \mathbb{C}\varepsilon \longrightarrow \bar{A}$ should be a heq. In fact we have

$$0 \longrightarrow \mathbb{C} \oplus \mathbb{C}\varepsilon \longrightarrow A \oplus \mathbb{C}\varepsilon \longrightarrow \bar{A} \longrightarrow 0$$

and a splitting given by ρ.

In this double complex the rows are exact, the columns $p > 0$ are exact except for the $\mathbb{C}\varepsilon^{\otimes p}$ along the edges.

Idea: Recall we developed a DG algebra of cochains: $T(A^*)$ roughly, $\mathrm{Hom}(\mathcal{B}, \mathbb{C})$ precisely, where $\mathcal{B}$ = bar construction of A. Note that $T(\bar{A}^*)$ is a graded subalgebra of $T(A^*)$ such that if $\rho \in A^*$ satisfies $\rho(1) = 1$, then

$$T(\bar{A}^*) * \mathbb{C}[\rho] \xrightarrow{\sim} T(A^*)$$

There is a clear analogy with

$$\Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* \xrightarrow{\sim} \Omega(P)$$

which suggests we look for a (super) derivation ∇ of degree $+1$ on $T(\bar{A}^*)$ which is roughly a contraction of d. ∇ is determined by its effect $\nabla : \bar{A}^* \longrightarrow \bar{A}^* \otimes \bar{A}^*$ in degree 1. Dually we are looking for a map $\mu : \bar{A} \otimes \bar{A} \longrightarrow \bar{A}$ obtained in some way from the multiplication $A \otimes A \longrightarrow A$ and ρ. The obvious thing to do is to lift $\bar{A} \otimes \bar{A} \longrightarrow A \otimes A$ by $\bar{a}_1 \otimes \bar{a}_2 \longmapsto (a_1 - \rho a_1) \otimes (a_2 - \rho a_2)$, the multiply to get $a_1 a_2 - a_1 \rho a_2 - \rho a_1 a_2 + \rho a_1 \rho a_2$ then project to get $\overline{a_1 a_2} - \bar{a}_1 \rho a_2 - \rho a_1 \bar{a}_2 \in \bar{A}$. Thus

$$\boxed{\mu(\bar{a}_1, \bar{a}_2) = \overline{a_1 a_2} - \bar{a}_1 \rho a_2 - \rho a_1 \bar{a}_2}$$

Now extend μ to a coderivation of degree

-1 on the tensor coalgebra $T(\bar{A}[1])$. Let's compute the square of μ which is a coderivation of degree -2, hence determined by its effect $\mu^2 : \bar{A}^{\otimes 3} \longrightarrow \bar{A}$. We have in general

$$p_n \mu = p_1^{\otimes n} \Delta^{(n)} \mu = p_1^{\otimes n} \sum_{i=1}^{n} (1^{\otimes i-1} \otimes \mu \otimes 1^{\otimes n-i}) \Delta^{(n)}$$

$$= \sum_{i=1}^{n} (p_{i-1} \otimes p_1\mu \otimes p_{n-i}) \Delta^{(3)}.$$ So in particular

$$p_2 \mu = (p_1\mu \otimes p_1 + p_1 \otimes p_1\mu)\Delta$$ i.e.

$$\mu(\bar{a}_1, \bar{a}_2, \bar{a}_3) = (p_1\mu \otimes p_1 + p_1 \otimes p_1\mu)(\bar{a}_1, \bar{a}_2, \bar{a}_3)$$

$$= \mu(\bar{a}_1, \bar{a}_2) \otimes \bar{a}_3 + \bar{a}_1 \otimes \mu(\bar{a}_2, \bar{a}_3)$$

$$= (\overline{a_1 a_2} - \bar{a}_1 \rho a_2 - \rho a_1 \bar{a}_2) \otimes a_3$$

$$- \bar{a}_1 \otimes (\overline{a_2 a_3} - \bar{a}_2 \rho a_3 - \rho a_2 \bar{a}_3)$$

Actually I should have put

$$\mu(\bar{a}_1, \bar{a}_2, \bar{a}_3) = \mu(\bar{a}_1, \bar{a}_2) \otimes \bar{a}_3 - \bar{a}_1 \otimes \mu(\bar{a}_2, \bar{a}_3)$$

so $$\mu^2(\bar{a}_1, \bar{a}_2, \bar{a}_3) = \mu(\mu(\bar{a}_1, \bar{a}_2), \bar{a}_3) - \mu(\bar{a}_1, \mu(\bar{a}_2, \bar{a}_3))$$

$$= \mu(\overline{a_1 a_2} - \bar{a}_1 \rho a_2 - \rho a_1 \bar{a}_2, \bar{a}_3)$$

$$= \overset{(1)}{\overline{a_1 a_2 a_3}} - \rho(a_1 a_2)\bar{a}_3 - \overset{(3)}{\overline{a_1 a_2}\rho a_3}$$

$$- \underset{(4)}{\overline{a_1 \rho a_2 a_3}} + \overset{(7)}{\bar{a}_1 \rho a_2 \rho a_3} + \rho a_1 \rho a_2 \overset{(2)}{\bar{a}_3}$$

$$- \overset{(5)}{\rho a_1 \overline{a_2 a_3}} + \overset{(6)}{\rho a_1 \bar{a}_2 \rho a_3} + \rho a_1 \rho a_2 \bar{a}_3$$

$$\mu(\bar{a}_1, \mu(\bar{a}_2, \bar{a}_3)) = \mu(\bar{a}_1, \overline{a_2 a_3} - \bar{a}_2 \rho a_3 - \rho a_2 \bar{a}_3)$$

$$= \overset{(1)}{\overline{a_1 a_2 a_3}} - \bar{a}_1 \rho(a_2 a_3) - \overset{(5)}{\rho a_1 \overline{a_2 a_3}}$$

$$- \overset{(3)}{\overline{a_1 a_2}\rho a_3} + \bar{a}_1 \rho a_2 \rho a_3 + \overset{(6)}{\rho a_1 \bar{a}_2 \rho a_3}$$

$$- \overset{(4)}{\overline{a_1 \rho a_2 a_3}} + \overset{(7)}{\bar{a}_1 \rho a_2 \rho a_3} + \overset{(2)}{\rho a_1 \rho a_2 \bar{a}_3}$$

Thus

$$\mu^2(\bar{a}_1,\bar{a}_2,\bar{a}_3) = \bar{a}_1\left(\rho(a_2a_3)-\rho a_2\rho a_3\right) - \left(\rho(a_1a_2)-\rho a_1\rho a_2\right)\bar{a}_3$$

$$= \bar{a}_1\ \omega(a_2,a_3) - \omega(a_1,a_2)\,\bar{a}_3$$

This should mean that for ∇ on $T(\bar{A}^*[1])$ we have ∇^2 is ~~the~~ inner derivation given by the curvature ω. Consequently $\nabla^2=0$ on the commutator quotient space.

January 16, 1994

Let's continue with the reduced analogue of the bar construction.

Let A be a unital algebra, let $\rho: A \to \mathbb{C}$ be a retraction. On the tensor coalgebra $T(\bar{A}[1])$ there is a degree -1 coderivation b'_ρ defined by the formula

$$b'_\rho(\bar{a}_1, \bar{a}_2) = \overline{a_1 a_2 - \rho a_1 a_2 - a_1 \rho a_2}$$

This construction is obviously functorial in the pair (A, ρ), so to understand we can use the surjection $\tilde{A} \longrightarrow A$ to realize $T(\bar{A}[1])$ as a quotient coalgebra of $T(A[1])$. Then b'_ρ on $T(\bar{A}[1])$ comes from $b'_{\tilde{\rho}}$ on $T(A[1])$ where $\tilde{\rho}: \tilde{A} \longrightarrow \mathbb{C}$ is $\mathbb{C} \oplus A \xrightarrow{(id,\rho)} \mathbb{C}$. Thus our reduced situation is obtained by descent in a certain sense from an [redacted] unreduced situation.

In the case of $\tilde{a}$ a retraction $\tilde{a} \to \mathbb{C}$ is equivalent to an arbitrary linear functional $\rho: a \longrightarrow \mathbb{C}$. So from now on consider first (A, ρ), where A is possibly non unital and ρ is arbitrary, then specialize A to be unital and ρ to be such that $\rho(1) = 1$, and check that things descend to the reduced case.

Define b'_ρ on $T(A[1])$ to be the coderivation of degree -1 such that

$$b'_\rho(a_1, a_2) = a_1 a_2 - \rho a_1 a_2 - a_1 \rho a_2$$

Then $b'_\rho = b' - (\rho \otimes 1 - 1 \otimes \rho)\Delta$; note that $(\rho\otimes 1 - 1\otimes\rho)\Delta$ should be the inner coderivation associated to ρ.

The effect of b'_ρ on $T(A^*[1])$, more precisely $\mathbb{C}$-valued cochains should be

$$f \longmapsto (-1)^{|f|+1} f\big(b' - (\rho\otimes 1 - 1\otimes\rho)\Delta\big)$$
$$= \delta f + (\rho\otimes f - (-1)^{|f|} f\otimes\rho)\Delta$$
$$= \delta f + \rho f - (-1)^{|f|} f\rho$$
$$= \big(\delta + \mathrm{ad}(\rho)\big) f$$

Also we have

$$\boxed{\begin{aligned} b'_\rho(a_0,\dots,a_n) &= (a_0a_1, a_2, \dots, a_n) - \dots + (-1)^{n-1}(a_0,\dots,a_{n-1}a_n) \\ &\quad - \rho a_0 (a_1,\dots,a_n) + (-1)^{n}(a_0,\dots,a_{n-1})\rho a_n \end{aligned}}$$

Note that if we pass to reduced chains, i.e. look at the image in $\bar{A}^{\otimes n+1}$, then the preceding vanishes for $a_0 = 1$ and for $a_n = 1$, provided $\rho(1) = 1$.

A consequence of this formula

$$\boxed{b'_\rho = b' - (\rho\otimes 1)(1-\lambda)}$$

We have seen that b'_ρ becomes $\delta + \mathrm{ad}\,\rho$ on bar cochains, i.e. elts of $T(A^*[1])$. What happens for Hochschild cochains, i.e. linear fnls on $A^{[1]}_{\wedge}\otimes T(A[1]) = \Omega^1_{coalg} T(A[1])^{\natural}$?

Let $f, g \in T(A^*[1])$ and consider $\natural(\partial f\, g) \in A^{*}\otimes T(A^*[1])$. We should have

$$\delta_\rho\big(\natural(\partial f\, g)\big) = \natural\Big(\partial(\delta f + [\rho, f])g + (-1)^{|f|}\partial f\,(\delta g + [\rho, g])\Big)$$

$$= \natural\big(\partial(\delta f)g + (-1)^{|f|}\partial f\,\delta g\big)$$

$$+ \natural\Big([\partial\rho, f]g + \underbrace{[\rho, \partial f]g + (-1)^{|f|}\partial f[\rho, g]}_{[\rho,\ \partial f\, g] \text{ killed by } \natural}\Big)$$

$$\natural([\partial\rho, f]g) - (-1)^{|f||g|}\overbrace{\natural([\partial\rho\, g, f])}^{0} = -(-1)^{|f||g|}\natural(\partial\rho[g, f])$$

$$= +\natural(\partial\rho[f, g])$$

Thus
$$\boxed{\delta_\rho\big(\natural(\partial f\, g)\big) = \delta\big(\natural(\partial f\, g)\big) + \natural(\partial\rho[f, g])}$$

The principle (which I neglected to mention) is that the coderivation b'_ρ on $T(A[1])$ should induce (via Lie derivative) an operator b_ρ on $A^{[1]}_{\hat{}}\otimes T(A[1]) = \Omega^1_{\text{coalg}} T(A[1])^\natural$. The above formula gives the effect on Hochschild cochains. To find what b_ρ is it suffices to assume $f \in A^*$, $g \in (A^*)^{\otimes n-1}$. Then

$$\natural(\partial\rho[f, g])(a_0, a_1, \dots, a_n)$$

$$= \rho(a_0) f(a_1) g(a_2, \dots, a_n)\ (-1)^{n+n-1}$$

$$(-1)^n \rho(a_0) g(a_1, \dots, a_{n-1}) f(a_n)\ (-1)^{n+n-1}$$

$$= -\rho(a_0) f(a_1) g(a_2, \dots, a_n) + (-1)^{n-1}\rho(a_0) f(a_n) g(a_1, \dots, a_{n-1})$$

Up to sign this is [redacted] $(fg)(1-\lambda)(\rho\otimes 1)(a_0, \dots, a_n)$.

$$fg\,(1-\lambda)(\rho\otimes 1)(a_0,\dots,a_n)$$
$$= fg\,(1-\lambda)\,\rho(a_0)(a_1,\dots,a_n)$$
$$= fg\,\rho(a_0)\Big[(a_1,\dots,a_n) - (-1)^{n-1}(a_n,a_1,\dots,a_{n-1})\Big]$$
$$= (-1)^{n-1}\rho(a_0)\Big(f(a_1)g(a_2,\dots,a_n) - (-1)^{n-1} f(a_n)g(a_1,\dots,a_{n-1})\Big)$$

(So there are some sign problems still).

Let's look at

$$b(a_0,\dots,a_n) = (a_0a_1,a_2,\dots,a_n) + \sum_{i=1}^{n-1}(-1)^i(\dots,a_ia_{i+1},\dots)$$
$$+(-1)^n(a_na_0,a_1,\dots,a_{n-1})$$

We know that the RHS is degenerate if $a_i=1$ for some $i=1,\dots,n$. If $a_0=1$ the RHS is

$$(a_1,\dots,a_n) + (-1)^n(a_n,a_1,\dots,a_{n-1}) = (1-\lambda)(a_1,\dots,a_n)$$
$$= (1-\lambda)(\rho\otimes 1)(a_0,\dots,a_n)$$

(Better notation maybe would be ι_ρ instead of $\rho\otimes 1$). In any case it's clear we should have

$$\boxed{b_\rho = b - (1-\lambda)(\rho\otimes 1)}$$

Let's use ι_ρ from now on. Then we have

$$b_\rho(a_0,\dots,a_n) = b(a_0,\dots,a_n) - b\,\rho(a_0)(1,a_1,\dots,a_n)$$
$$= b(a_0-\rho a_0, a_1,\dots,a_n)$$

so this is well-defined on $\bar{A}[1]\otimes T(\bar{A}[1])$. Let's now drop the [1].

Finally consider

$$
\begin{array}{ccccccccccc}
0 & \leftarrow & C^\lambda(A) & \leftarrow & A\otimes T(A) & \xleftarrow{1-\lambda} & T(A) & \xleftarrow{N_\lambda} & C^\lambda(A) & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & \bar{C}^\lambda(A) & \leftarrow & \bar{A}\otimes T(\bar{A}) & \xleftarrow{1-\lambda} & T(\bar{A}) & \leftarrow & \bar{C}^\lambda(A) & \leftarrow & 0
\end{array}
$$

We would like to check that

$$(1-\lambda)b'_\rho = b_\rho(1-\lambda) \qquad N_\lambda b_\rho = b'_\rho N_\lambda$$

But this is clear from

$$b'_\rho = b' - \iota_\rho(1-\lambda)$$
$$b_\rho = b - (1-\lambda)\iota_\rho$$

Here are some additional points I want to record from scratch paper. First the calculation on page 322 is motivated by

$$
\begin{array}{ccc}
f\otimes g & \longmapsto & \natural(\partial f g) \\
T(A^*)\otimes T(A^*) & \longrightarrow & A^*\otimes T(A^*) \\
\Big\downarrow \delta_\rho\otimes 1 + 1\otimes\delta_\rho & & \Big\downarrow L(\delta_\rho) \\
T(A^*)\otimes T(A^*) & \longrightarrow & A^*\otimes T(A^*)
\end{array}
$$

Second there was an earlier idea ~~[illegible]~~ ~~[illegible]~~ that I am working with $\bar{\Omega}\tilde{A}$ but using a different splitting of

$$0 \longrightarrow A^{\otimes n} \longrightarrow \Omega^n\tilde{A} \longrightarrow A^{\otimes n+1} \longrightarrow 0$$

The usual splitting leads to $\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}$, but we want to modify this splitting using $\iota_\rho : A^{\otimes n+1} \to A^{\otimes n}$.

This amounts to conjugating via $\begin{pmatrix} 1 & 0 \\ \iota_\rho & 1 \end{pmatrix}$. We have

$$\begin{pmatrix} 1 & 0 \\ \iota_\rho & 1 \end{pmatrix}\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\iota_\rho & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \iota_\rho & 1 \end{pmatrix}\begin{pmatrix} b-(1-\lambda)\iota_\rho & 1-\lambda \\ b'\iota_\rho & -b' \end{pmatrix}$$

$$= \begin{pmatrix} b-(1-\lambda)\iota_\rho & 1-\lambda \\ \iota_\rho b + b'\iota_\rho - \iota_\rho(1-\lambda)\iota_\rho & -(b'-\iota_\rho(1-\lambda)) \end{pmatrix}$$

Let's calculate

$$(\iota_\rho b)(a_0,\dots,a_n) = \rho(a_0a_1)(a_2,\dots,a_n) + \rho(a_0)\sum_{i=1}^{n-1}(-1)^i(a_1,\dots,a_ia_{i+1},\dots,a_n) + (-1)^n\rho(a_na_0)(a_1,\dots,a_{n-1})$$

$$(b'\iota_\rho)(a_0,\dots,a_n) = \rho(a_0)\sum_{i=1}^{n-1}(-1)^{i-1}(a_1,\dots,a_ia_{i+1},\dots,a_n)$$

$$\iota_\rho(1-\lambda)\iota_\rho(a_0,\dots,a_n) = \iota_\rho(1-\lambda)\rho(a_0)(a_1,\dots,a_n)$$
$$= \rho(a_0)\iota_\rho\big((a_1,\dots,a_n) + (-1)^n(a_n,a_1,\dots,a_{n-1})\big)$$
$$= \rho(a_0)\rho(a_1)(a_2,\dots,a_n) + (-1)^n\rho(a_0)\rho(a_n)(a_1,\dots,a_{n-1})$$

$$\big(+(\iota_\rho b + b'\iota_\rho) \mp \iota_\rho(1-\lambda)\iota_\rho\big)(a_0,\dots,a_n)$$
$$= \omega(a_0,a_1)(a_2,\dots,a_n) + (-1)^n\omega(a_n,a_0)(a_1,\dots,a_{n-1})$$

$$\boxed{+(\iota_\rho b + b'\iota_\rho) \mp \iota_\rho(1-\lambda)\iota_\rho = \iota_\omega(1+\lambda)}$$

$$(b'_\rho)^2 = \left(b' - \iota_\rho(1-\lambda)\right)^2$$

$$= \cancel{b'^2} - b'\iota_\rho(1-\lambda) - \iota_\rho \underbrace{(1-\lambda)b'}_{b(1-\lambda)} + \iota_\rho(1-\lambda)\iota_\rho(1-\lambda)$$

$$= \left[-(b'\iota_\rho + \iota_\rho b) + \iota_\rho(1-\lambda)\iota_\rho\right](1-\lambda)$$

$$= -\iota_\omega(1+\lambda)(1-\lambda)$$

$$\therefore \boxed{(b'_\rho)^2 = -\iota_\omega(1-\lambda^2)}$$

which agrees with $(\delta + ad_\rho)^2 = ad(\omega)$

Similarly

$$(b_\rho)^2 = \left(b - (1-\lambda)\iota_\rho\right)^2 = \cancel{b^2} - \overbrace{b(1-\lambda)}^{(1-\lambda)b'}\iota_\rho - (1-\lambda)\iota_\rho b + (1-\lambda)\iota_\rho(1-\lambda)\iota_\rho$$

$$= (1-\lambda)\left[-(b'\iota_\rho + \iota_\rho b) + \iota_\rho(1-\lambda)\iota_\rho\right]$$

$$= -(1-\lambda)\,\iota_\omega(1+\lambda)$$

$$\boxed{(b_\rho)^2 = -(1-\lambda)\iota_\omega(1+\lambda)}$$

Also

$$\boxed{\tilde{b} = \begin{pmatrix} b_\rho & 1-\lambda \\ \iota_\omega(1+\lambda) & -b'_\rho \end{pmatrix}}$$

relative to our splitting of $\Omega\tilde{A}$

From $\tilde{b}^2 = 0$ we get an extra identity

$$\boxed{\iota_\omega(1+\lambda)\, b_\rho - b'_\rho\, \iota_\omega(1+\lambda) = 0}$$

<u>January 20, 1994</u>

V Jones construction.

Let B be a subalgebra of A, let $\rho : A \to B$ be a B-bimodule map such that $\rho(1) = 1$. Define a product on $A \otimes_B A$ by

$$(a_1' \otimes a_1'')(a_2' \otimes a_2'') = a_1' \rho(a_1'' a_2') \otimes a_2''$$

This is associative making $A \otimes_B A$ into a nonunital algebra.

<u>Prop.</u> $\sum x_i \otimes y_i \in A \otimes_B A$ is an identity element $\iff$ $\forall a \in A$, $\sum x_i \rho(y_i a) = a$, $\sum \rho(a x_i) y_i = a$.

Proof. $(\sum x_i \otimes y_i)(a' \otimes a'') = \sum x_i \rho(y_i a') \otimes a''$

$(a' \otimes a'')(\sum x_i \otimes y_i) = \sum a' \rho(a'' x_i) \otimes y_i = \sum a' \otimes \rho(a'' x_i) y_i$

so the direction $\Leftarrow$ is clear. Conversely assuming

$$\sum x_i \rho(y_i a') \otimes a'' = a' \otimes a'' \qquad \forall a', a''$$

apply $1 \otimes \rho : A \otimes_B A \to A \otimes_B B \simeq A$ to get $\sum x_i \rho(y_i a') \rho(a'') = a' \rho(a'')$, and then put $a'' = 1$. $\square$

Remarks: 1) Can separate proof into left and right identity parts.

2) $A \otimes_B A$ is a ~~quotient~~ of the ideal $A \otimes B \otimes A$ in the GNS construction Γ associated to $\rho : A \to B$. When $A \otimes_B A$ is unital, it is a quotient of $\Gamma = A \oplus A \otimes B \otimes A$.

Assume from now on that $A \otimes_B A$ has an identity element $\sum_{i=1}^{n} x_i \otimes y_i$, which is necessarily unique.

Define a left and right action of $A \otimes_B A$ on A by

$$(a' \otimes a'') * a = a' \rho(a''a)$$

$$a * (a' \otimes a'') = \rho(aa') a''$$

Check these define unital left & right $A \otimes_B A$ module structures on A, but not a bimodule structure. Observe that the operator $a \longmapsto (a' \otimes a'') * a = a' \rho(a''a)$ is a kind of rank 1 operator, with linear functional part [redacted] $a \longmapsto \rho(a''a)$. Denote this right B-module map $A \longrightarrow B$ by $\varphi(a'')$. Thus we have a map

1) $\qquad \varphi : A \longrightarrow \mathrm{Hom}_{B^{op}}(A, B) \qquad \varphi(a)(\alpha) = \rho(a\alpha)$

which is compatible with the evident left B, right A module structures. We also have ~~an~~ alg homomorphism

2) $\qquad A \otimes_B A \xrightarrow{1 \otimes \varphi} A \otimes_B \mathrm{Hom}_{B^{op}}(A, B) \longrightarrow \mathrm{Hom}_{B^{op}}(A, A)$

corresponding to the action of $A \otimes_B A$ on A.

Prop. (Assuming $A \otimes_B A$ unital). [redacted] A is a finite projective right B module. The maps 1) and 2) are isomorphisms.

Proof. The identity $\sum_{i=1}^{n} x_i \rho(y_i a) = a$ means that we have right B module maps

$$A \xrightarrow{\begin{pmatrix} \varphi(y_1) \\ \varphi(y_n) \end{pmatrix}} B^n \xrightarrow{(x_1 \quad x_n)} A$$

with composition the identity. Thus A is a summand of the right B-module B^n, hence is finite projective. Moreover the x_i generate A and the $\varphi(y_i)$ generate the left B module $\mathrm{Hom}_{B^{op}}(A,B)$. (If $\lambda \in \mathrm{Hom}_{B^{op}}(A,B)$, then $\lambda(a) = \sum \lambda(x_i)\varphi(y_i a)$, i.e. $\lambda = \sum \lambda(x_i)\varphi(y_i)$.) Thus the map φ is surjective.

On the other hand the identity $\sum \rho(a x_i) y_i = a$ shows that φ is injective. ($\varphi(a) = 0 \Rightarrow \rho(a x_i) = 0 \Rightarrow a = 0$.) Thus φ is an isomorphism and so 2) is an isomorphism as A is finite projective.

Similarly is a finite projective left B-module and we have isomorphisms

3) $$A \xrightarrow{\sim} \mathrm{Hom}_B(A,B) \qquad a' \mapsto (a \mapsto \rho(a a'))$$

4) $$A \otimes_B A \xrightarrow{\sim} \mathrm{Hom}_B(A,B) \otimes_B A \simeq \mathrm{Hom}_B(A,A)$$
$$a' \otimes a'' \longmapsto \qquad (a \mapsto \rho(a a') a'')$$

corresponding to the right action of $A \otimes_B A$ on A.

One can check that $\varphi : A \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(A,B)$ respects the right $A \otimes_B A$ module structures, where on the right one takes the transpose of the left action.

I think we can summarize much of what is happening by saying that one has Morita equivalence data between B and $A \otimes_B A$. Namely $\boxed{{}_{A\otimes_B A}P_B = {}_{A\otimes_B A}A_B \,,\; {}_B Q_{A\otimes_B A} = {}_B A_{A\otimes_B A}}$

and as part of this we know that P and Q are dual and the two algebras are commutants. Sometime I have to write this out.

But for the moment let us concentrate on the iteration aspect of Jones's construction.

First we have (using summation convention)

$$a x_i \otimes y_i = x_j \rho(y_j a x_i) \otimes y_i$$
$$= x_j \otimes \rho(y_j a x_i) y_i = x_j \otimes y_j a$$

$$(a_1 x_i \otimes y_i)(a_2 x_j \otimes y_j) = a_1 x_i \rho(y_i a_2 x_j) \otimes y_j$$
$$= a_1 a_2 x_j \otimes y_j$$

$$(x_i \otimes y_i a_1)(x_j \otimes y_j a_2) = x_i \otimes \rho(y_i a_1 x_j) y_j a_2$$
$$= x_i \otimes y_i a_1 a_2$$

The upshot is that we have a homomorphism, in fact, a canonical homom. $A \longrightarrow A \otimes_B A$, $a \longmapsto a x_i \otimes y_i = x_i \otimes y_i a$. It can be described as associating to a the element of $A \otimes_B A$ corresponding to multiplication by a on ${}_{A \otimes_B A} A_B$:

$$(a x_i \otimes y_i) * a' = a x_i \rho(y_i a') = a a'.$$

Also it sends a to the operator on ${}_B A_{A \otimes_B A}$ given by right mult. by a:

$$a' * (a x_i \otimes y_i) = \rho(a' a x_i) y_i = a' a$$

Denoting this homom. by $u : A \to A \otimes_B A$ we get an A bimodule structure on $A \otimes_B A$ using left and right multiplication by $u(a)$:

$$u(a)(a' \otimes a'') = (a x_i \otimes y_i)(a' \otimes a'')$$
$$= a x_i \rho(y_i a') \otimes a'' = a a' \otimes a''$$
$$(a' \otimes a'') u(a) = (a' \otimes a'')(x_i \otimes y_i a)$$
$$= a' \otimes \rho(a'' x_i) y_i a = a' \otimes a'' a$$

Thus the A-bimodule structure on $A \otimes_B A$ obtained from u coincides with the obvious bimodule structure.

Next we would like a retraction $A \otimes_B A \xrightarrow{r} A$ for u (i.e. $ru = 1$) which is an A-bimodule map. Any bimodule map $A \otimes_B A \to A$ has the form $a' \otimes a'' \mapsto a' \xi a''$ where $\xi \in$ centralizer of B in A. The condition $ru = 1$ means that $\sum x_i \xi y_i = 1$.

At the moment I don't see any obvious choice for ξ. In Jones's situation, where A, B are factors, ξ is probably a multiple of the identity for the following reasons. First since $\sum x_i \otimes y_i \in (A \otimes_B A)^A$ centralizer for the A bimodule structure, one has $\sum x_i y_i \in$ center of A, so $\sum x_i y_i =$ scalar as A is a factor. Then positivity conditions should imply the scalar is > 0.

January 22, 1994

Continue with the V Jones construction.

Recall the situation. One is given two algebras [redacted] B, A a homomorphism $B \to A$ and a B-bimodule map $\rho : A \to B$. We assume $\exists\ x_i \otimes y_i \in A \otimes_B A$ (summation convention) such that [redacted]

$$x_i \rho(y_i a) = a \quad , \quad \rho(a x_i) y = a.$$

We then have isomorphisms

1) $$A \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(A, B) \qquad a \mapsto (a' \mapsto \rho(aa'))$$

2) $$A \otimes_B A \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(A, A) \qquad a_1 \otimes a_2 \mapsto (a \mapsto \underbrace{(a_1 \otimes a_2) * a}_{\text{defn}\ =\ a_1 \rho(a_2 a)})$$

Proof of second: Given $f \in \mathrm{Hom}_{A^{op}}(A, A)$ send it to $f(x_i) \otimes y_i \in A \otimes_B A$. Then

$$f \longmapsto f(x_i) \otimes y_i \longmapsto \big(a \mapsto f(x_i)\rho(y_i a)\big)$$
$$f(x_i)\rho(y_i a) = f(x_i \rho(y_i a)) = f(a)$$

$$a_1 \otimes a_2 \longmapsto (a \mapsto a_1 \rho(a_2 a)) \longmapsto a_1 \rho(a_2 x_i) \otimes y_i$$
$$a_1 \rho(a_2 x_i) \otimes y_i = a_1 \otimes \rho(a_2 x_i) y_i = a_1 \otimes a_2$$

[redacted]

Consequences of 2).

(i) algebra structure on $A \otimes_B A$

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = a_1 \rho(a_2 a_3) \otimes a_4$$

The identity element is $x_i \otimes y_i$.

(ii) homomorphism $A \longrightarrow A \otimes_B A$

$$a \longmapsto a x_i \otimes y_i = x_i \otimes y_i a$$

The A-bimodule structure on $A \otimes_B A$ given by this homomorphism coincides with the obvious bimodule homomorphism.

(iii) A-bimodule map $\mu : A \otimes_B A \longrightarrow A$

$$a_1 \otimes a_2 \longmapsto a_1 a_2 .$$

(Note that in (iii) the possible A-bimodule maps $A \otimes_B A \longrightarrow A$ are described by the image of $1 \otimes 1$ which can be any element η of A centralized by B. Since $x_i \otimes y_i$ lies in the center of the A-bimodule $A \otimes_B A$, its image $x_i \eta y_i$ lies in the center of A.)

At the moment starting from $B \to A \xrightarrow{\rho} B$, $x_i \otimes y_i$ we have constructed

$$\underset{\text{hom alg}}{A \longrightarrow A \otimes_B A} \xrightarrow[\text{A-bimodule map}]{\mu} A$$

and it remains to construct the corresponding identity element in $(A \otimes_B A) \otimes_A (A \otimes_B A)$. Take $(x_i \otimes 1) \otimes (1 \otimes y_i)$ and check that this works

Let's change notation a bit: $R = A \otimes_B A$ and put (a_1, a_2) for $a_1 \otimes a_2 \in R$. Then

$$(x_i, 1)\, \mu((1, y_i)(a_1, a_2)) = (x_i, 1)\, \mu((\rho(y_i a_1), a_2))$$

$$= (x_i, 1)\, \rho(y_i a_1) a_2 = (x_i, \rho(y_i a_1) a_2)$$

$$= (x_i \rho(y_i a_1), a_2) = (a_1, a_2).$$

and the other direction is

$$\mu((a_1, a_2)(x_i, 1))(1, y_i) = \mu((a_1, \rho(a_2 x_i)))(1, y_i)$$
$$= a_1\rho(a_2 x_i)(1, y_i) = (a_1\rho(a_2 x_i), y_i)$$
$$= (a_1, \rho(a_2 x_i) y_i) = (a_1, a_2).$$

Thus we can iterate the construction. The next stage is to consider

$$[redacted] \quad R \otimes_A R \longrightarrow Hom_{A^{op}}(R, R)$$

$$R \otimes_A R = (A \otimes_B A) \otimes_A (A \otimes_B A) \cong A \otimes_B A \otimes_B A$$

Put $S = A \otimes_B A \otimes_B A$ and write (a_1, a_2, a_3) for $a_1 \otimes a_2 \otimes a_3$. Then S acts on R by

$$(a_1, a_2, a_3) * (a', a'') = (a_1, a_2\rho(a_3 a')a'')$$

the product in S is

$$(a_1, a_2, a_3)(a_4, a_5, a_6) = (a_1, a_2\rho(a_3 a_4)a_5, a_6)$$

[redacted]

the identity element is $(x_i, 1, y_i)$

the R-bimodule structure is

$$(a', a'')(a_1, a_2, a_3) = (a'\rho(a'' a_1), a_2, a_3) = ((a', a'') * a_1, a_2, a_3)$$

$$(a_1, a_2, a_3)(a', a'') = (a_1, a_2, \rho(a_3 a')a'') = (a_1, a_2, a_3 * (a', a''))$$

the R-bimodule map $S = R \otimes_A R \longrightarrow R$ is

$$\mu'(a_1, a_2, a_3) = (a_1\rho(a_2), a_3)$$

The next stage is

$$T = S \otimes_R S \xrightarrow{\sim} \text{[redacted]} \; \mathrm{Hom}_{R^{op}}(S, S)$$

$$\big[(a_1, a_2, a_3) \otimes (a_4, a_5, a_6)\big] * (a', a'', a''')$$

$$= (a_1, a_2, a_3)\, \mu'\big((a_4, a_5\rho(a_6 a')a'', a''')\big)$$

$$= (a_1, a_2, a_3)\big(a_4\rho(a_5\rho(a_6 a')a''), a'''\big)$$

$$= \big(a_1, a_2\rho\big(a_3 a_4 \text{[redacted]} \rho(a_5\rho(a_6 a')a'')\big), a'''\big)$$

$$= \big(a_1, a_2\rho(a_3 a_4)\rho(a_5\rho(a_6 a')a''), a'''$$

$$= \big(a_1, a_2\rho\big(\rho(a_3 a_4)a_5\rho(a_6 a')a''\big), a'''\big)$$

Notice this action depends on $(a_1, a_2, \rho(a_3 a_4)a_5, a_6)$

Thus we can identify

$$T = A \otimes_B A \otimes_B A \otimes_B A$$

acting on S by

$$\boxed{(a_1, a_2, a_3, a_4) * (a', a'', a''') = (a_1, a_2\rho(a_3\rho(a_4 a')a''), a''')}$$

Product in T seems to be

$$\boxed{(a_1, a_2, a_3, a_4)(a_5, a_6, a_7, a_8) = (a_1, a_2\rho(a_3\rho(a_4 a_5)a_6), a_7, a_8)}$$

Identity element: [redacted] [redacted]

$$(x_i \otimes 1) \otimes_A (x_i \otimes y_i) \otimes_A (1 \otimes y_i) \in R \otimes_A R \otimes_A R$$

$$\uparrow \qquad\qquad\qquad \uparrow \cong$$

$$\big[(x_i \otimes 1) \otimes (x_j \otimes y_j)\big] \otimes \big[(x_k \otimes y_k) \otimes (1 \otimes y_i)\big] \in (R \otimes_A R) \otimes_R (R \otimes_A R)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$(x_i \otimes x_j \otimes y_j) \otimes (x_k \otimes y_k \otimes y_i) \in S \otimes_R S$$

which becomes

$$(x_i, x_j \rho(y_j x_k) y_k, y_i) = \boxed{(x_i, x_j, y_j, y_i)}$$

Thus the identity in T is 1.

S-bimodule structure on T:

not correct ?

$$(a_1, a_2, a_3, a_4)(a_5, a_6, a_7) = (a_1, a_2, a_3 \rho(a_4 a_5) a_6, a_7)$$
$$(a_1, a_2, a_3)(a_4, a_5, a_6, a_7) = (a_1, a_2 \rho(a_3 a_4) a_5, a_6, a_7)$$

$\mu : T = S \otimes_R S \longrightarrow S$ is

$$(a_1, a_2, a_3, a_4) \longmapsto (a_1, a_2 a_3, a_4)$$

<u>Summary</u>: Assume given algebras B, A a homom. $A \to B$ and a B-bimodule map $\rho : A \to B$ such that $\exists\ x_i \otimes y_i \in A \otimes_B A$ such that

$$x_i \rho(y_i a) = a \qquad \rho(a x_i) y_i = a \qquad \forall a.$$

Then A is a finite projective right B-module

$$A \xrightarrow{\sim} \mathrm{Hom}_{B^o}(A, B) \qquad a \mapsto (\alpha \mapsto \rho(a\alpha))$$

$$A \otimes_B A \xrightarrow{\sim} \mathrm{Hom}_{B^o}(A, A) \qquad a_1 \otimes a_2 \mapsto (\alpha \mapsto (a_1 \otimes a_2) * \alpha)$$

We get a new pairs A, $R = A \otimes_B A$ with homom. $A \to R$ and bimodule map satisfying the same conditions.

$$A \otimes_B A \longrightarrow A, \quad a_1 \otimes a_2 \mapsto a_1 a_2$$

January 24, 1994 (David is 30)

Recall that if we choose a [redacted] retraction $\rho : A \longrightarrow \mathbb{C}$, then we get a modified derivation [redacted] b'_ρ on the tensor coalgebra $T(A)$ $(|A| = 1)$ which then descends to $T(\bar{A})$. $(b'_\rho)^2 = -\iota_\omega(1-\lambda^2)$, where ω is the curvature.

Suppose $A = \tilde{\mathcal{a}}$ where $\mathcal{a}$ is nonunital, and let us take ρ to be the obvious retraction $\ni \rho(\mathcal{a}) = 0$. Recall the general formula

$$b'_\rho(a_1, a_2) = a_1 a_2 - \rho(a_1) a_2 - a_1 \rho(a_2)$$

It's clear then that we have maps

$$(T(\mathcal{a}), b') \hookrightarrow (T(\tilde{\mathcal{a}}), b'_\rho) \longrightarrow\!\!\!\!\!\rightarrow (T(\bar{\tilde{\mathcal{a}}}), b'_{\rho\ \text{descended}})$$

(Note that $b'_\rho : \tilde{\mathcal{a}} \otimes \tilde{\mathcal{a}} \longrightarrow \tilde{\mathcal{a}}$ is the multiplication on $\mathbb{C} \oplus \mathcal{a}$ [redacted] given by $(a_1, a_2) \longmapsto a_1 a_2$ for $a_1, a_2 \in$ [redacted] $\mathcal{a}$, $(x, a) \longmapsto 0$, $(a, x) \longmapsto 0$ for $a \in \mathcal{a}$, $x \in \mathbb{C}$ and $(x, y) \longmapsto -xy$ for $x, y \in \mathbb{C}$. This is associative so $(b'_\rho)^2 = 0$, which checks with the fact that $\rho : \tilde{\mathcal{a}} \longrightarrow \mathbb{C}$, $\rho(x, a) = x$ is a homomorphism.)

Suppose now that $\mathcal{a}$ is unital. Then we have maps

$$T(\mathcal{a}, b') \hookrightarrow T(\tilde{\mathcal{a}}, b') \longrightarrow T(\mathcal{a}, b')$$

The preceding is not very well organized, and there do not seem to be a clear conclusion. I guess the point is that there is only one embedding $\mathcal{a} \hookrightarrow \tilde{\mathcal{a}}$, but many retractions $\tilde{\mathcal{a}} \longrightarrow \mathcal{a}$ described by the image of $1 \in \mathbb{C} \subset \tilde{\mathcal{a}}$. For $\mathcal{a}$ unital one thus has two retractions $1 \longmapsto 0 \in \mathcal{a}, e \in \mathcal{a}$ which can be joined by a path.

Let's now go on to

$$A \times \mathbb{C}[\varepsilon] = (A \times \mathbb{C}) \oplus \mathbb{C}\varepsilon = \tilde{A} \oplus \mathbb{C}\varepsilon$$

and recall we are interested in $\bar{C}^\lambda(A \times \mathbb{C}[\varepsilon])$. The idea is to describe this via the tensor coalgebra $T(\overline{A \times \mathbb{C}[\varepsilon]})$ with a suitable degree -1 derivation. For this derivation we have to choose a retraction of $\tilde{A} \oplus \mathbb{C}\varepsilon$ onto $\mathbb{C}$. There are two choices. First there is the canonical choice such that $\rho(A \oplus \mathbb{C}\varepsilon) = 0$. This is not compatible with $d(\varepsilon) = 1$ so we will have $b_\rho'^2 = d^2 = 0$ but $[b_\rho', d] \neq 0$. Second there is the choice which comes from a retraction of A onto $\mathbb{C}$. This has $\rho(e^\perp) = \rho(1-e) = 0$ and $\rho(\varepsilon) = 0$ so it's compatible with d. Then we should have $b_\rho'^2 \neq 0$, $[d, b_\rho'] = 0$. In both cases we can identify $T(\overline{A \times \mathbb{C}[\varepsilon]})$ with $T(A \oplus \mathbb{C}\varepsilon)$.

Noncommutative version of ∇:

Let T be a graded algebra, let ∇ be a derivation of degree $+1$, let $\omega \in T^2$ satisfying $\nabla(\omega) = 0$, $\nabla^2 = \mathrm{ad}(\omega)$. Let ρ be of degree $+1$ and form $T * \mathbb{C}[\rho]$. Define d on $T * \mathbb{C}[\rho]$ to be the derivation of degree $+1$ such that

$$d\rho = -\rho^2 + \omega$$

$$dx = \nabla x - [\rho, x] \qquad x \in T$$

Then d is well-defined and $d^2 = 0$. Check the latter first:

$$d(d\rho) = -d\rho\,\rho + \rho\, d\rho + d\omega$$

[illegible]

$$= -(-\rho^2+\omega)\rho + \rho(-\rho^2+\omega) + \underset{0}{\nabla\omega} - \rho\omega + \omega\rho$$

$$= 0$$

$$d(dx) = (\nabla - \mathrm{ad}(\rho))\nabla x - [d\rho, x] + [\rho, \nabla x - [\rho, x]]$$

$$= [\omega, x] - [d\rho, x] - [\rho^2, x] = 0.$$

Why d is well-defined. Note: $\tilde{\nabla} = \nabla - \mathrm{ad}\,\rho$ is a derivation $T \longrightarrow T * \mathbb{C}[\rho]$, hence one has a homomorphism

$$T \longrightarrow (T * \mathbb{C}[\rho]) \oplus \varepsilon (T * \mathbb{C}[\rho]) \qquad |\varepsilon| = -1, \; \varepsilon^2 = 0$$

$$x \longmapsto x + \varepsilon \tilde{\nabla} x$$

This extends to a homomorphism

$$T * \mathbb{C}[\rho] \longrightarrow (T * \mathbb{C}[\rho]) \oplus \varepsilon (T * \mathbb{C}[\rho])$$

such that $\rho \longmapsto \rho + \varepsilon(-\rho^2 + \omega)$, which has the form $1+\varepsilon d$, where d is the required derivation of $T * \mathbb{C}[\rho]$

Problem: What sort of "interior product" structure would guarantee that a DG algebra has the form $T * \mathbb{C}[\rho]$, where T is the "horizontal" subalgebra?

Recall that if $R = T * \mathbb{C}[\rho]$, then

$$\Omega^1 R = R \otimes_T \Omega^1 T \otimes_T R \oplus \underbrace{R\, d\rho\, R}_{\cong R \otimes R}$$

Thus there is a derivation $\partial : R \longrightarrow R \otimes R$ such that $\partial\rho = 1$ and $\partial(T) = 0$. The conjecture is that conversely given a derivation ∂ and element ρ of this sort we have $R = T * \mathbb{C}[\rho]$, where $T = \mathrm{Ker}(\partial)$.

January 25, 1994

Yesterday given a graded alg T equipped with degree 1 derivation ∇ and degree 2 element ω such that $\nabla^2 = +\mathrm{ad}(\omega)$, $\nabla\omega = 0$, we constructed a differential d of degree $+1$ on $T * \mathbb{C}[\rho]$ by the formulas

$$d\rho = -\rho^2 + \omega$$
$$dx = \nabla x - [\rho, x] \qquad x \in T.$$

We now point out the similarity with the Alexander-Spanier differential on $A * \mathbb{C}[h]$. This is the special case of the preceding construction where $T = A$, $\nabla = \omega = 0$ and $\rho = -h$.

Return to A unital algebra, $\rho : A \longrightarrow \mathbb{C}$ linear retraction. Say A finite-diml, so we can work dually with the DG algebra $T(A^*)$ with differential δ given by $\delta\theta + \theta^2 = 0$, where $\theta \in T^1(A^*) \otimes A = A^* \otimes A$ is the canonical element. (Notice the similarity with the Lie formalism: If $\{X_a\}$ is a basis for A, then $\theta = \theta^a X_a$ where $\delta\theta^a = -f^a_{bc}\theta^b\theta^c$ with $\{f^a_{bc}\}$ the structural constants: $X_b X_c = f^a_{bc} X_a$.)

Now we have

$$T(A^*) = T(\bar{A}^*) * \mathbb{C}[\rho]$$

January 27, 1994

Suppose A finite dimensional to simplify, let X_a be a basis, let $\theta^a \in A^*$ be the dual basis, let $\{f^a_{bc}\}$ be the structural constants: $X_b X_c = f^a_{bc} X_a$. The dual of the bar construction of A is $T(A^*) = \mathbb{C}\langle\theta^a\rangle$ where the θ^a have degree 1 and the differential δ is defined by

$$\delta\theta + \theta^2 = 0 \qquad\qquad \theta = \theta^a X_a \in A^* \otimes A$$

In [redacted] terms of components

$$\delta\theta^a + f^a_{bc}\theta^b\theta^c = 0$$

[redacted] So far A can be non-unital, but now suppose A is unital (with identity element e) and let $\rho : A \to \mathbb{C}$ [redacted] be a retraction $\rho(e)=1$. Choose the basis X_a such that $X_0 = 1$ and $\rho(X_a) = 0$ for $a > 0$. Use i, j, k for indices > 0. Then A has the basis e, X_i and A^* has the dual basis ρ, θ^i. The θ^i are a basis for $\bar{A}^*$. We have

$$T(A^*) = T(\bar{A}^*) * \mathbb{C}[\rho]$$

and the differential δ on $T(A^*)$ is determined by

$$\delta\theta + \theta^2 = 0 \qquad\qquad \theta = \rho e + \underbrace{\theta^i X_i}_{\bar{\theta}}$$

Thus $\delta(\rho e + \bar{\theta}) + (\rho e + \bar{\theta})^2 = 0$ which splits into

$$\delta\rho + \rho^2 = \omega \qquad\qquad \omega = -f^0_{ij}\theta^i\theta^j$$

$$(\delta + ad(\rho))\theta^i + f^i_{jk}\theta^j\theta^k = 0.$$

Recall that we know already that $\nabla = \delta + \mathrm{ad}(\rho)$ is a degree one derivation on $T(\bar{A}^*)$, that $\omega = \delta\rho + \rho^2 \in T(\bar{A}^*)^2$, and $\nabla(\omega) = 0$, $\nabla^2 = \mathrm{ad}(\omega)$. Also the diffl δ can be reconstructed from ∇, ω by the formulas

$$\delta\rho = -\rho^2 + \omega$$
$$\delta x = \nabla x - [\rho, x] \qquad x \in T(\bar{A}^*).$$

Yesterday I had the idea to pursue the analogy with principal G-bundles: i.e. $\Omega(P) = \Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^*$, the foliation filtration. The exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow A \longrightarrow \bar{A} \longrightarrow 0$$

is analogous to the sequence $0 \to S \to T \to Q \to 0$ for a foliation, because $\mathbb{C}$ is a subalgebra, and hence the ideal J in $T(A^*)$ generated by $\bar{A}^*$ is closed under δ. One has a canonical isom.

$$gr_J\, T(A^*) = T(\bar{A}^*) * \mathbb{C}[\rho']$$

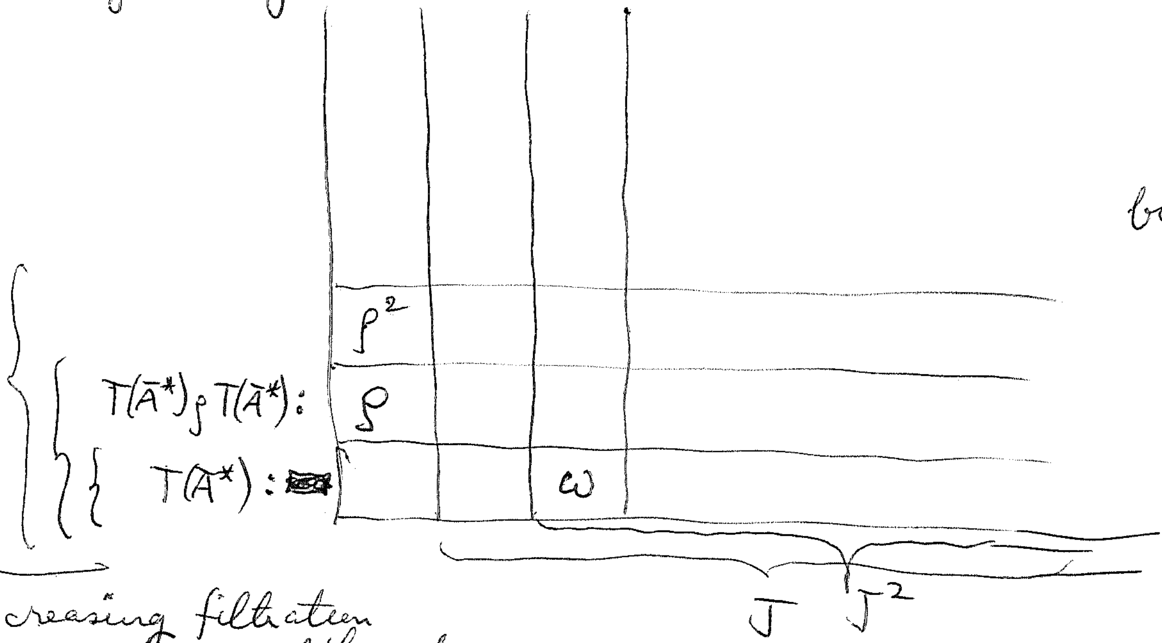
where ρ' is the canonical generator for $A^*/\bar{A}^* \simeq \mathbb{C}$. The differential on $gr_J\, T(A^*)$ is determined by

$$\delta\rho' = -\rho'^2$$
$$\delta x = -[\rho', x] \qquad x \in T(\bar{A}^*)$$

In other words we have the Alexander-Spanier diffl (for $h = -\rho'$).

I now want to check the equivalence between the algebra $T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*)$ and the BRS type algebra $T(A^*) * \mathbb{C}\langle \chi, \varphi \rangle$.

Before doing this recall the pictures from BRS theory (my version Jan-Feb 1990). The bigrading arising from $T(A^*) = T(\bar{A}^*) * \mathbb{C}[\rho]$ has picture

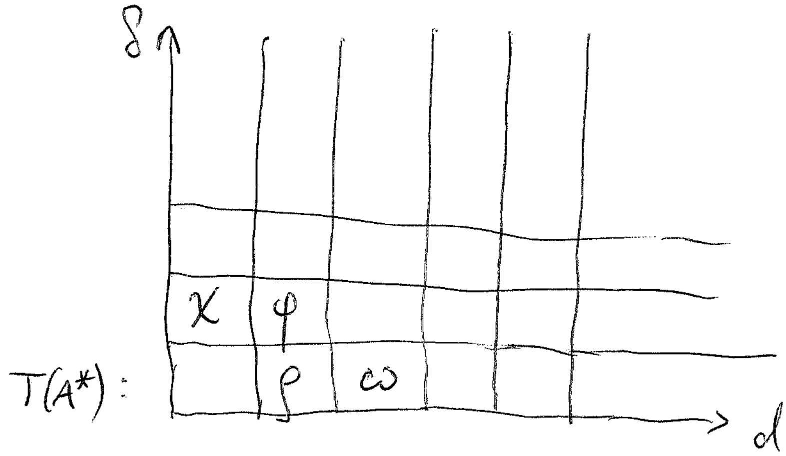


not a bigraded diffl algebra

increasing filtration is canonical although δ has order 1.

The J-adic filtration is what gives rise to Wodzicki's filtration (Leray spectral sequence for $P \to B$ geometrically).

The BRS alg $T(A^*) * \mathbb{C}\langle \chi, \varphi \rangle$ has the picture



This is a bigraded diffl algebra with diffls defined by

$$\delta\chi + \chi^2 = 0 \qquad d\chi = \varphi$$
$$\delta\varphi + [\chi,\varphi] = 0 \qquad d\varphi = 0$$
$$\delta\rho + [\chi,\rho] + \varphi = 0 \qquad d\rho = -\rho^2 + \omega$$
$$\delta\omega + [\chi,\omega] = 0 \qquad d\omega = -[\rho,\omega]$$

Let us now consider the algebra $A \times \mathbb{C}[\varepsilon] = \tilde{A} \oplus \mathbb{C}\varepsilon$ and the basis $1, e, X_i, \varepsilon$. To begin suppose $A = \mathbb{C}$ so that we can ignore the X_i. The BRS algebra is $T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*)$ and it has two diffls d', d'' where d' comes from $d(\varepsilon) = 1 - e$ and d'' comes from the product in the alg $\tilde{A} \oplus \mathbb{C}\varepsilon$.

Let $\chi, \rho, \theta^i, -\varphi$ be the dual basis to $1, e, X_i, \varepsilon$, and let $\Theta = \chi 1 + \rho e + \theta^i X_i - \varphi\varepsilon$ be the canonical element. The diffs d' and d'' are determined by

$$0 = [d', \Theta] = d'\chi 1 + d'\rho e + d'\theta^i X_i - d'\varphi\varepsilon - \varphi(1-e)$$

whence
$$\boxed{d'\chi = \varphi, \quad d'\rho + \varphi = 0, \quad d'\theta^i = d'\varphi = 0}$$

and
$$0 = [d'', \Theta] + \Theta^2 = d''\chi 1 + d''\rho e + d''\theta^i X_i - d''\varphi\varepsilon$$
$$+ (\chi 1 + \rho e - \varphi\varepsilon + \bar{\theta})^2$$

For the moment ignore $\bar{\theta}$. Then

$$(\chi 1 + \rho e - \varphi\varepsilon)^2 = \chi^2 1 + \rho^2 e + [\chi,\rho]e - [\chi,\varphi]\varepsilon$$

so
$$\boxed{d''\chi + \chi^2 = 0, \quad d''\rho + \rho^2 + [\chi,\rho] = 0, \quad d''\varphi + [\chi,\varphi] = 0}$$

The total diffl is $D = d' + d''$:

$$DX + X^2 = \varphi$$

$$D\varphi + [X,\varphi] = 0$$

$$D\rho + \rho^2 + [X,\rho] + \varphi = 0$$

Now use the bigrading where $|X| = (0,1)$, $|\rho| = (0,1)$, $|\varphi| = (1,1)$. Write $D = d + \delta$ where $|d| = (1,0)$, $|\delta| = (0,1)$. Then

$$\delta X + X^2 = 0 \qquad\qquad dX = \varphi$$

$$\delta\varphi + [X,\varphi] = 0 \qquad\qquad d\varphi = 0$$

$$\delta\rho + [X,\rho] + \underset{\substack{\parallel \\ dX}}{\varphi} = 0 \qquad\qquad d\rho + \rho^2 = 0.$$

which are the BRS algebra relations (ignoring ω).

Let's finish the calculation keeping track of everything:

$$\Theta = X1 + \rho e - \varphi\varepsilon + \theta^i X_i$$

$$D\Theta + \Theta^2 = D(\text{———}) + (X1 + \rho e - \varphi\varepsilon + \theta^i X_i)^2$$

$$DX\,1 - \varphi\,1 + X^2 1$$

$$D\rho\, e + \varphi e + [X,\rho]e + \rho^2 e + f^0_{jk}\theta^j\theta^k e$$

$$-D\varphi\,\varepsilon - [X,\varphi]\varepsilon$$

$$D\theta^i X_i + [X,\theta^i]X_i + [\rho,\theta^i]X_i + f^i_{jk}\theta^j\theta^k X_i$$

$$D\overset{0,1}{\chi} + \overset{0,2}{\chi^2} = \overset{1,1}{\varphi}$$
$$D\overset{1,1}{\varphi} + [\overset{0,1}{\chi}, \overset{1,1}{\varphi}] = 0$$

$$D\rho + \overset{2,0}{\rho^2} + [\overset{0,1}{\chi}, \overset{1,0}{\rho}] - \overset{2,0}{\omega} + \overset{1,1}{\varphi} = 0$$
$$D\overset{1,0}{\theta^i} + [\overset{0,1}{\chi}, \overset{1,0}{\theta^i}] + \overset{(2,0)}{[\rho, \theta^i]} + f^i_{jk}\,\overset{(2,0)}{\theta^j\theta^k} = 0$$

$\delta\chi + \chi^2 = 0$	$d\chi = \varphi$
$\delta\varphi + [\chi, \varphi] = 0$	$d\varphi = 0$
$\delta\rho + [\chi, \rho] + \varphi = 0$	$d\rho + \rho^2 = \omega$
$\delta\theta^i + [\chi, \theta^i] = 0$	$d\theta^i + [\rho, \theta^i] + f^i_{jk}\theta^j\theta^k = 0$

These formulas define the differentials δ, d on [redacted] $T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*) = T(A^*) * \mathbb{C}\langle \chi, \varphi \rangle$. It should now be clear that we can identify this with the BRS algebra.

Return to $T(A^*) = T(\bar{A}^*) * \mathbb{C}[\rho]$ with differential δ. Let's examine

$$gr_J\, T(A^*) = T(\bar{A}^*) * \mathbb{C}[\rho]$$

where the differential is the derivation such that

$$\delta\rho = -\rho^2$$
$$\delta x = -[\rho, x] \qquad x \in T(\bar{A}^*)$$

Note that

$$J^p/J^{p+1} = \mathbb{C}[\rho] \otimes \bar{A}^* \otimes \mathbb{C}[\rho] \otimes \cdots \otimes \bar{A}^* \otimes \mathbb{C}[\rho]$$

where there are p copies of $\bar{A}^*$.

Recall we are ultimately interested in the homology of $T(A^*)_\natural$. Since $T(A^*)$ and $gr_J\, T(A^*)$ are the same as graded algebras the induced filtration on [redacted] $T(A^*)_\natural$ should satisfy $gr\{T(A^*)_\natural\} = \{gr\, T(A^*)\}_\natural$

Notice that δ on $gr\, T(A^*)$ is the sum of the derivation δ_1 [redacted] $= -ad(\rho)$ and [redacted] the derivation δ_2 defined by $\delta_2 \rho = \rho^2$, $\delta_2 x = 0$ for $x \in T(\bar{A}^*)$. On $\{gr\, T(A^*)\}_\natural$ we thus have $\delta = \delta_2$. Since [redacted] $H^*(\mathbb{C}[\rho], \delta) = \mathbb{C}[0]$ it follows that

$$H^q\left((J^p/J^{p+1})_\natural, \delta\right) = \begin{cases} H^q(\mathbb{C}[\rho]_\natural, \delta) & p = 0 \\ (\bar{A}^*)_\lambda^{\otimes p} & p \geq 1,\ q = 0 \\ 0 & p \geq 1,\ q \geq 1 \end{cases}$$

In more detail for $p \geq 1$ we have

$$(J^p/J^{p+1})_\natural = \left[\mathbb{C}[\rho] \otimes \bar{A}^* \otimes\right]^{(p)}_\lambda$$

where the λ occurs because $\bar{A}^*$ has degree 1.

Now [redacted] $\mathbb{C}[\rho]_\natural = \mathbb{C}\natural(\rho) \oplus \mathbb{C}\natural(\rho^3) \oplus \mathbb{C}\natural(\rho^5) \oplus \cdots$

Picture of E^1 term is then

ρ^3			
ρ			
	$\bar{A}^*$	$(\bar{A}^*)_\lambda^{\otimes 2}$	$(\bar{A}^*)_\lambda^{\otimes 3}$

(ignore the $\mathbb{C}$ in degree 0)

Rest of entries are zero.

So now we return to the problem of deforming $T(A^*)_{\mathfrak{g}}$ to a subcomplex having the shape of this E^1 term. There actually should be a subcomplex consisting of $T(\bar{A}^*)_{\mathfrak{g}}$ (which is a subcomplex as $\nabla^2 = ad(\omega)$ on the commutator quotient space) plus linear combinations of the Chern-Simons forms. The problem is then to find an explicit deformation retraction to this subcomplex.

Perhaps it's possible to use Laplacean type methods.

January 28, 1994

A has basis e, X_i; A^* has dual basis ρ, θ^i.
$\tilde{A} \oplus \mathbb{C}\varepsilon$ has basis $1, \varepsilon, e, X_i$ and the dual basis for $(\tilde{A} \oplus \mathbb{C}\varepsilon)^*$ is $\chi, \varphi, \alpha, \theta^i$.

$T(A^*)$ generated by ρ, θ^i with diff'l

$$\delta\rho + \rho^2 = \omega \qquad \omega = -f^0_{jk}\theta^j\theta^k$$

$$(\delta + \mathrm{ad}_\rho)\theta^i + f^i_{jk}\theta^j\theta^k = 0$$

$T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*) = \mathbb{C}\langle \chi, \varphi, \alpha, \theta^i \rangle$ with diff'l

$$D\chi + \chi^2 = \varphi$$

$$D\varphi + [\chi, \varphi] = 0 \qquad \text{(Bianchi identity for } \uparrow \text{)}$$

$$D(\chi + \alpha) + (\chi + \alpha)^2 = \omega$$

(i.e. $D\alpha + [\chi, \alpha] + \varphi + \alpha^2 = \omega$, see p 347)

$$(D + \mathrm{ad}(\chi + \alpha))\theta^i + f^i_{jk}\theta^j\theta^k = 0$$

The homomorphism

$$\begin{array}{ccc} T(A^*) & \longrightarrow & T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*) \\ \| & & \| \\ \mathbb{C}\langle \rho, \theta^i \rangle & & \mathbb{C}\langle \chi, \varphi, \alpha, \theta^i \rangle \end{array}$$

induced by the canonical surjection

$$\tilde{A} + \mathbb{C}\varepsilon \longrightarrow\!\!\!\!\!\rightarrow A \qquad \begin{array}{ll} 1 \mapsto e & \varepsilon \mapsto 0 \\ e \mapsto e & X_i \mapsto X_i \end{array}$$

is given by $\rho \longmapsto \chi + \alpha$, $\theta^i \longmapsto \theta^i$

Let us check that $T((\tilde{A} \oplus \mathbb{C}\varepsilon / \mathbb{C})^*) = \mathbb{C}\langle \varphi, \alpha, \theta^i \rangle$ is closed under $\nabla = D + \mathrm{ad}(\chi + \alpha)$.

This is clear for the θ^i.

$$\nabla\alpha = D\alpha + [X,\alpha] + [\alpha,\alpha]$$
$$= \omega - \varphi - \alpha^2 + 2\alpha^2 = \omega + \alpha^2 - \varphi$$

$$\nabla\varphi = \cancel{D\varphi + [X,\varphi]} + [\alpha,\varphi]$$

Thus ∇ on $T((\tilde{A}\oplus\mathbb{C}\varepsilon/\mathbb{C})^*) = \mathbb{C}\langle\varphi,\alpha,\theta^i\rangle$ is given by

$$\boxed{\begin{aligned} \nabla\alpha &= \alpha^2 + \omega - \varphi \\ \nabla\varphi &= [\alpha,\varphi] \\ \nabla\theta^i &+ f^i_{jk}\theta^j\theta^k = 0 \end{aligned}}$$

and the last formula also gives ∇ on $T(\bar{A}^*) = \mathbb{C}\langle\theta^i\rangle$. The homomorphism $T(\bar{A}^*) \hookrightarrow T((\tilde{A}\oplus\mathbb{C}\varepsilon/\mathbb{C})^*)$ is compatible with ∇.

Now there is a retraction homomorphism

$$\begin{array}{ccc} T((\tilde{A}\oplus\mathbb{C}\varepsilon/\mathbb{C})^*) & \longrightarrow & T(\bar{A}^*) \\ \| & & \| \\ \mathbb{C}\langle\varphi,\alpha,\theta^i\rangle & \longrightarrow & \mathbb{C}\langle\theta^i\rangle \end{array}$$

sending $\varphi \mapsto \omega$, $\alpha \mapsto 0$, $\theta^i \to \theta^i$. This is compatible with ∇:

$$\begin{array}{ccc} \varphi & \overset{\nabla}{\longmapsto} & [\alpha,\varphi] \\ \downarrow & & \downarrow \\ \omega & \overset{\nabla}{\longmapsto} & 0 \end{array} \qquad\qquad \begin{array}{ccc} \alpha & \overset{\nabla}{\longmapsto} & \alpha^2 + \omega - \varphi \\ \downarrow & & \downarrow \\ 0 & \overset{\nabla}{\longmapsto} & 0 \end{array}$$

I would like to make this retraction into a deformation retraction, and this task should involve the Chern-Simons forms.

<u>January 29, 1994</u>

Review the situation: I have been led to the DG algebra $\tilde{A} \oplus \mathbb{C}\varepsilon$ [redacted] and [redacted] the DG alg. homom.

1) $$T(A^*) \longrightarrow T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*)$$

by analogy with the homomorphism

2) $$\Omega(P) \longrightarrow \Omega(P) \otimes W(\mathfrak{g})$$

for a principal bundle. A retraction for 2) is given by a connection in P. In analogy with this a retraction for 1) is obtained from a retraction $\rho : A \longrightarrow \mathbb{C}$. I would like next to make the retraction for 1) into a deformation retraction. To this end I will study the analogous situation for 2).

We need a homotopy joining

3) $$\begin{array}{ccccc} \Omega(P) \otimes W(\mathfrak{g}) & \longrightarrow & \Omega(P) & \hookrightarrow & \Omega(P) \otimes W(\mathfrak{g}) \\ x, \ \mathcal{X} & \longmapsto & x, A & \longmapsto & x, A \end{array}$$

to the identity. Here $W(\mathfrak{g}) = \Lambda \mathfrak{g}^*_{\mathcal{X}} \otimes S\mathfrak{g}^*_{\varphi}$ where $\mathcal{X}$ is the universal connection form and $\varphi = d\mathcal{X} + \mathcal{X}^2$ is its curvature. A is the connection form in $(\Omega^1(P) \otimes \mathfrak{g})^{\mathfrak{g}}$, and $F = dA + A^2 \in (\Omega^2(P)_{hor} \otimes \mathfrak{g})^{\mathfrak{g}}$ is its curvature.

We have a 1-parameter family of homomorphisms

$$\begin{array}{cccc} u_t : & \Omega(P) \otimes W(\mathfrak{g}) & \longrightarrow & \Omega(P) \otimes W(\mathfrak{g}) \\ & x, \ \mathcal{X} & \longmapsto & x, \ t\mathcal{X} + (1-t)A \end{array}$$

joining $u_0 =$ the map 3) to $u_1 =$ identity. We want to [redacted] construct [redacted] a suitable homotopy between u_0 and u_1 which shows that the induced maps on basic cohomology are the same.

Again proceed infinitesimally. But first let's simplify a bit and consider

$$u_t : W(\mathfrak{g}) \longrightarrow \Omega \qquad (= \Omega(P) \otimes W(\mathfrak{g}))$$
$$X \longmapsto A_t \qquad (= tX + (1-t)A)$$

Infinitesimally we consider then a pair consisting of a homomorphism u and derivation $\mathring{u}$ relative to u:

$(u, \mathring{u}) : W(\mathfrak{g}) \longrightarrow \Omega$, which is compatible with d and the $\mathfrak{g}[\varepsilon]$ action.

$$u(X) = A \qquad \mathring{u}(X) = \mathring{A}$$

[redacted]

$$\iota_X A = X \qquad \iota_X \mathring{A} = 0$$
$$L_X A + [X, X] = 0 \qquad L_X \mathring{A} + [X, \mathring{A}] = 0$$

This would be clearer if I work out the relations in Ω, then give the universal versions.

We start with $A \in (\Omega^1 \otimes \mathfrak{g}^*)^{\mathfrak{g}}$ satisfying

$$\iota_X A = X \qquad L_X A + [X, A] = 0$$

Then the variation $\mathring{A}$ satisfies

$$\iota_X \mathring{A} = 0 \qquad L_X \mathring{A} + [X, \mathring{A}] = 0$$

so $\mathring{A} \in (\Omega^1(P)_{hor} \otimes \mathfrak{g})^{\mathfrak{g}}$.

F is defined by $F = dA + A^2$ and

we have

$$\iota_X F = 0 \qquad L_X F + [X, F] = 0$$

$$dF + [A, F] = 0$$

[redacted] The variation of [redacted] F [redacted] is $\mathring{F} = d\mathring{A} + [A, \mathring{A}]$ and it satisfies

$$\iota_X \mathring{F} = 0 \qquad L_X \mathring{F} + [X, \mathring{F}] = 0$$

$$d\mathring{F} + [A, \mathring{F}] + [\mathring{A}, F] = 0$$

Check $\left(d + \mathrm{ad}(A)\right)\mathring{F} = \left(d + \mathrm{ad}(A)\right)^2 \mathring{A} = \mathrm{ad}(F)\mathring{A} = [F, \mathring{A}]$.

There's a philosophy here that the variations live in $(\Omega_{hor} \otimes \mathfrak{g})^{\mathfrak{g}}$ and are related by [redacted] $\nabla(\mathring{A}) = \mathring{F}$ where $\nabla = d + \mathrm{ad}(A)$.

Go back to $u, \mathring{u} : W(\mathfrak{g}) \longrightarrow \Omega$. We know $\mathring{u}$ induces a map

$$\Omega^1_{W(\mathfrak{g})} \xrightarrow{\ \mathring{u}_*\ } \Omega$$

of $W(\mathfrak{g})$ [redacted] modules, where Ω is a $W(\mathfrak{g})$ module via u. [redacted] Let us denote the canonical derivation $W(\mathfrak{g}) \longrightarrow \Omega^1_{W(\mathfrak{g})}$ by a dot. Then

$$\Omega^1_{W(\mathfrak{g})} = W(\mathfrak{g}) \otimes \mathfrak{g}^*_{\mathring{\chi}} \oplus W(\mathfrak{g}) \otimes \mathfrak{g}^*_{\mathring{\varphi}}$$

is the free $W(\mathfrak{g})$ module generated by the components of $\mathring{\chi}$ and $\mathring{\varphi}$. Let us describe the induced operators d, ι_X, L_X on $\Omega^1_{W(\mathfrak{g})}$. Its enough to give their effect on the generators.

One has

$$\iota_X \dot{x} = 0 \qquad L_X \dot{x} + [X, \dot{x}] = 0$$

$$\iota_X \dot{\varphi} = 0 \qquad L_X \dot{\varphi} + [X, \dot{\varphi}] = 0$$

$$d\dot{x} + [x, \dot{x}] = \dot{\varphi}$$

$$d\dot{\varphi} + [x, \dot{\varphi}] = [\varphi, \dot{x}]$$

January 30, 1994

First let us discuss [redacted] homotopy formulas occurring for DG algebras. We have encountered two types.

The first occurs in the case of the standard bimodule resolution, where one [redacted] changes the canonical element $h = 1\otimes 1 \in A\otimes A$ to something more like a partition of unity. This idea comes from [redacted] what I learned from Teleman, and it occurs already in Lars Kadison's thesis. It is based on the observation that if $u_0, u_1 : R \to S$ are two homomorphisms, then $u_1 - u_0$ is a derivation from R to either bimodule ${}_{u_1}S_{u_0}$, ${}_{u_0}S_{u_1}$:

$$(u_1-u_0)(xy) = \big(u_1(x) - u_0(x)\big)u_0(y) + u_1(x)\big(u_1(y)-u_0(y)\big)$$

$$= \big(u_1(x)-u_0(x)\big)u_1(y) + u_0(x)\big(u_1(y)-u_0(y)\big)$$

The other occurs in [redacted] using the Cartan homotopy formula. One has a 1-parameter family of homomorphisms $u_t : R \to S$, whence an induced map

$$\iota(u_t, \dot u_t) : \Omega^1 R \longrightarrow {}_{u_t}S_{u_t} \qquad \iota(u_t,\dot u_t)(x\,dy) = u_t(x)\dot u_t(y)$$

Integrating gives

$$u_1 - u_0 = \left(\int_0^1 dt\, \iota(u_t,\dot u_t)\right)\circ d \;:\; R \xrightarrow{d} \Omega^1 R \longrightarrow S$$

In both situations we factor $u_1 - u_0$ through $\Omega^1 R$, which is where one actually constructs the homotopy. On the surface at least these two situations seem unrelated.

Let's consider the example of a derivation D on R corresponding to an $\mathbb{N}$-grading. Take $u_t = t^D$. Then $\dot{u}_t = \frac{1}{t} t^D D$, and

$$\iota(u_t, \dot{u}_t)(x\,dy) = t^D x \, \tfrac{1}{t} t^D Dy = \tfrac{1}{t} t^D (x\,Dy)$$

$$\int_0^1 dt \;\iota(u_t, \dot{u}_t)(x\,dy) = \int_0^1 dt \; t^{D-1}(x\,Dy)$$

$$= \frac{1-P}{D}(x\,Dy)$$

where $P = \lim_{t \to 0} t^D$ is the projection on the degree zero subalgebra.

Let's return to the family of homoms.

$$T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*) \xrightarrow{\;u_t\;} T((\tilde{A} \oplus \mathbb{C}\varepsilon)^*)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\mathbb{C}\langle X, \varphi, \rho, \theta^i \rangle \qquad\qquad \mathbb{C}\langle X, \varphi, \rho, \theta^i \rangle$$

$$X \longmapsto X_t = (1-t)\rho + tX$$

$$\varphi = dX + X^2 \longmapsto dX_t + X_t^2$$

$$\rho, \theta^i \longmapsto \rho, \theta^i$$

Notice that u_1 is the identity and u_0 is essentially the retraction $T((\tilde{A} \oplus \varepsilon\mathbb{C})^*) \longrightarrow T(A^*)$ sending X, φ to ρ, ω.

Notice also that $X_t = \rho + t(X - \rho)$ is the radial retraction of X to ρ, so that u_t should be t^D where D corresponds to a grading on $R = \mathbb{C}\langle X, \varphi, \rho, \theta^i \rangle$.

Let $\alpha = X - \rho$. Then we can also describe R as $\mathbb{C}\langle \alpha, d\alpha, \rho, \theta^i \rangle$. In effect $X = \rho + \alpha$ and

$$\varphi = dX + X^2$$
$$= d(\rho+\alpha) + (\rho+\alpha)^2$$
$$= d\alpha + \omega + [\rho, \alpha] + \alpha^2$$

so that X, φ can be obtained from $\alpha, d\alpha, \rho, \Theta^i$. Conversely

$$\alpha = X - \rho$$
$$d\alpha = dX - d\rho$$
$$= \varphi - X^2 - \omega + \rho^2$$

so that $\alpha, d\alpha$ can be obtained from $X, \varphi, \rho, \Theta^i$.

Clearly $u_t : \alpha \longmapsto t\alpha$, $d\alpha \longmapsto t\,d\alpha$ so that $u_t = t^D$, where $D = 1$ on $\alpha, d\alpha$ and $D = 0$ on ρ, Θ^i.

Summary of formulas for V. Jones construction

Given algebras $B \subset A$, and a B-bimodule map $\rho : A \longrightarrow B$. Assume $\exists\, x_i, y_i \in A$, $1 \le i \le n$ such that

$$x_i \rho(y_i a) = a \qquad \forall a$$
$$\rho(a x_i) y_i = a$$

<u>Put</u> $A_0 = B$, $A_1 = A$.

Define $A_2 = A \otimes_B A$ equipped with product

$$(a_1, a_2)(b_1, b_2) = (a_1 \rho(a_2 b_1), b_2)$$

This makes A_2 an algebra with the identity (x_i, y_i).

Canonical homomorphism

$$A_1 \longrightarrow A_2 \qquad a \longmapsto (a x_i, y_i) = (x_i, y_i a)$$

Left A_2-module structure on A_1

$$(a_1, a_2)^*(b_1) = (a_1 \rho(a_2 b))$$

<u>which</u> extends the left mult action of A_1 on itself

Define $A_3 = A \otimes_B A \otimes_B A$ with product

$$(a_1, a_2, a_3)(b_1, b_2, b_3) = (a_1, a_2 \rho(a_3 b_1) b_2, b_3)$$

This makes A_3 an alg with the identity $(x_i, 1, y_i)$.

Canonical homomorphism

$$A_2 \longrightarrow A_3 \qquad (a_1, a_2) \longmapsto (a_1, 1, a_2)$$

We have a left action of A_3 on A_2 given by [redacted] omitting the last b in the formula for the product in A_3. This action extend the left mult. action of A_2 on itself

Define $A_4 = A \otimes_B A \otimes_B A \otimes_B A$ with product

$$(a_1, a_2, a_3, a_4)(b_1, b_2, b_3, b_4) = (a_1, a_2\rho(a_3\rho(a_4 b_1) b_2), b_3, b_4)$$

identity element: (x_i, x_j, y_j, y_i).

canonical homomorphism

$$A_3 \longrightarrow A_4 \qquad (a_1, a_2, a_3) \longmapsto (a_1, a_2 x_i, y_i, a_3) = (a_1, x_i, y_i a_2, a_3)$$

such that the left regular repn of A_3 is extended by the left action of A_4 on A_3 given by omitting the last b in the formula for the product in A_4.

Define $A_5 = A \otimes_B A \otimes_B A \otimes_B A \otimes_B A$ with prod.

$$(a_1, a_2, a_3, a_4, a_5)(b_1, b_2, b_3, b_4, b_5)$$
$$= (a_1, a_2, a_3\rho(a_4\rho(a_5 b_1) b_2) b_3, b_4, b_5)$$

This makes A_5 an alg with the identity $(x_i, x_j, 1, y_j, y_i)$.
Canonical homomorphism $A_4 \longrightarrow A_5$

$$(a_1, a_2, a_3, a_4) \longmapsto (a_1, a_2, 1, a_3, a_4)$$

The general picture seems pretty clear. Note that A_4 has a natural left action on A_2. This extends the A_3 actions on A_2.