A ~~8/6 ~ 1224~~. 8/7 → 0610

Nistor again after a gap of 7 days

My construction

A, B, L algebras $J \subset L$ ideal

$A \underset{\theta'}{\overset{\theta}{\rightrightarrows}} L \otimes B$ cong mod $J \otimes B$

$S = \bigoplus_{n \geq 0} t^n J^n \subset \mathbb{C}[t] \otimes L$

$K =$ ideal $(1-t^2) J^2 S$ in S

$S_\natural = \bigoplus_{n \geq 0} t^n J^n_\#$ $\qquad J^n_\# = J^n / \sum_{i+j=n} [J^i, J^j]$

J-adic trace $\qquad = \begin{cases} J^n / [J, J^{n-1}] & n \geq 1 \\ L_\natural & n = 0 \end{cases}$

$\mu_m : S \longrightarrow J_\#^{2m+1}$ $\qquad P_m(z) = \prod_{k=1}^{m} \left(1 - \frac{z}{2k-1}\right)$

Def: $\mu_m(t^n x) = \underbrace{P_m(n) \frac{1-(-1)^n}{2}}_{\text{vanishes } 0 \leq n \leq 2m} \#_{2m+1}(x) \qquad x \in J^n$

μ_m is the composition

$\mu_m = \#_{2m+1} \frac{1}{2}(\delta_1 - \delta_{-1}) P_m(t\partial_t)$

$P_m(t\partial_t)$ carries K^{m+1} into K

which is killed by δ_1, δ_{-1}

μ_m clearly ~~factors~~ defined on $S_\natural$ so it's a trace on S.

(B) 0647 Anyway the basic construction is

$$p = \tfrac{1}{2}(\theta + \theta') : A \longrightarrow L \otimes B$$
$$q = \tfrac{1}{2}(\theta - \theta') : A \longrightarrow J \otimes B$$

$p + tq : A \longrightarrow S \otimes B$ linear resp 1

curvature $(1-t^2)q^2 : \bar{A}^{\otimes 2} \longrightarrow (1-t^2)J^2 \otimes B \subset K \otimes B$

$p+tq$ induces

$$RA \longrightarrow R(S \otimes B) \longrightarrow S \otimes RB$$

$$X(RA) \xrightarrow{u_*} X(S \otimes RB) \xrightarrow{\alpha} S_\natural \otimes X(RB)$$

Basic map is

$$X(RA) \xrightarrow{u_*} X(S \otimes RB) \xrightarrow{\alpha} S_\natural \otimes X(RB) \xrightarrow{\mu_m} J_\#^{2m+1} \otimes X(RB)$$

$$\cup \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup$$

$$F^p_{IA} \longrightarrow F^p_{K \otimes RB + S \otimes IB} \longrightarrow \sum_{i \geq 0} \natural(K^i) \otimes F^{p-2i}_{IB} \longrightarrow J_\#^{2m+1} \otimes F^{p-2m}_{IB}$$

Thus get a map of towers

$$\mathcal{X}_A \xrightarrow{ch^{2m}(\theta,\theta')} [redacted]\ J_\#^{2m+1} \otimes \mathcal{X}_B[2m]$$

Claim

$$\begin{array}{ccc} \mathcal{X}_A & \longrightarrow & J_\#^{2m+3} \otimes \mathcal{X}_B[2m+2] \\ \downarrow & & \downarrow \iota_\# \\ J_\#^{2m+1} \otimes \mathcal{X}_B[2m] & \xrightarrow{S} & J_\#^{2m+1} \otimes \mathcal{X}_B[2m+2] \end{array}$$

commutes

Ⓒ 0850

Now I want to relate my construction to Nistors.

$$Q = QA = \bigoplus Q_n \qquad Q_n = \Omega^n A$$

RQ, $X(RQ)$ inherit gradings

degree ops D, L_D

canonical ϕ and h_D $\qquad [L_D, h_D] = 0$

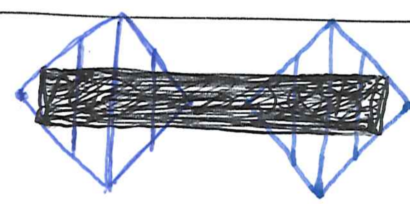
My map

$$\begin{array}{ccccc} X(RA) & \longrightarrow & S_\natural \otimes X(RB) & \xrightarrow{\mu_m} & J_\#^{2m+1} \otimes X(RB) \\ & & \downarrow & & \end{array}$$

0910 $\quad A \xrightarrow{\rho + tq} S \otimes B$

induces $\quad RA \longrightarrow S \otimes RB$

" $\quad X(RA) \longrightarrow S_\natural \otimes X(RB)$



Introduce $\quad Q = QA = \bigoplus_n Q_n$

θ, θ' give rise to a homom.

$$Q \longrightarrow E \otimes B \qquad a_0 da_1 \cdots da_n \longmapsto \rho a_0 \delta a_1 \cdots \delta a_n$$

satisf $\quad Q_n \longrightarrow J^n \otimes B$

whence $\quad Q \xrightarrow[\text{resp!}]{\text{linear map}} S \otimes B \qquad$ compat w. grading $D \longleftrightarrow t\partial_t$

induces $\quad RQ \longrightarrow S \otimes RB \qquad$ "

" $\quad X(RQ) \longrightarrow S_\natural \otimes X(RB). \qquad L_D \longleftrightarrow t\partial_t$

(D) [redacted] Comm. diag.

$$\begin{tikzcd}
X(RA) \arrow[r, "\iota_*"] & X(RQ) \arrow[r] \arrow[d, "P_m(L_D)\gamma_-"] & S_{\natural} \otimes X(RB) \arrow[d, "P_m(t\partial_t)\pi_-"] \\
 & \mathcal{J}X_{\geq 2m+1} \arrow[r] & S_{\natural, \geq 2m+1} \otimes X(RB) \arrow[d, "\delta_1"] \\
 & & J_{\#}^{2m+1} \otimes X(RB)
\end{tikzcd}$$

Way to say things I think is as follows

my map

$$X(RA) \longrightarrow X(S \otimes RB) \longrightarrow S_{\natural} \otimes X(RB) \xrightarrow{\mu_m} J_{\#}^{2m+1} \otimes X(RB)$$

coincides with

$$X(RA) \xrightarrow{\iota_*} X(RQ) \xrightarrow{P_m(L_D)\gamma_-} \mathcal{J}X_{\geq 2m+1} \longrightarrow J_{\#}^{2m+1} \otimes X(RB)$$

In order to write this out I just have to define both maps, then state they ~~are~~ coincide

Steps are:

$$\begin{tikzcd}
RA \arrow[r, "(p+tq)_*"] \arrow[d, "\iota_*"] & R(S \otimes B) \arrow[r] \arrow[d, equal] & S \otimes RB \\
RQ \arrow[r, "(\ )_*"] & R(S \otimes B) &
\end{tikzcd}$$

Yields

$$\begin{tikzcd}
X(RA) \arrow[r] \arrow[d] & S_{\natural} \otimes X(RB) \\
X(RQ) \arrow[ur, "(\ )_*"'] &
\end{tikzcd}$$

(E) Important objects are $\underbrace{p+tq,\ v}$ so far

linear resp. 1.

$$\begin{tikzcd} A \arrow[d, "\iota"'] \arrow[dr, "p+tq"] & \\ Q \arrow[r, "v"'] & S\otimes B \end{tikzcd}$$

11/4 Continue the analysis. So where are we? I have to explain the map

$$X(RA) \xrightarrow{\iota_*} X(RQ) \xrightarrow{\gamma_-} \gamma_- X_{\geq 0} = \gamma_- X_{\geq 1}$$

$$\xrightarrow{S_1} \gamma_- X_{\geq 2} = \gamma_- X_{\geq 3}$$

$$\cdots$$

$$\xrightarrow{S_m} \gamma_- X_{\geq 2m} = \gamma_- X_{\geq 2m+1} \longrightarrow J_{\#}^{2m+1} \otimes X(RB)$$

Only the last part has not been defined.

$$X_{\geq k} = \bigoplus_{n\geq k} \underbrace{X(RQ)_n}_{L_D = n \text{ here}}$$

~~Recall that we have a homom~~

explanation: $\quad Q \xrightarrow{v} S\otimes B \quad$ lin resp 1, comp $D \leftrightarrow t\partial_t$

$$X(RQ) \xrightarrow{v_*} S_\gamma \otimes X(RB) \quad \text{comp } L_D \leftrightarrow t\partial_t$$

then map is

$$X(RQ)_{\geq k} \xrightarrow{(v_*)_{\geq k}} S_{\gamma, \geq k} \otimes X(RB) \xrightarrow{\delta_1} J_{\#}^k \otimes X(RB).$$

But this is not the way to think. Go back to definition of v and factor it

$$Q \xrightarrow{t^D} \bigoplus t^n Q_n \longrightarrow$$

(F) 1535 See if I can concentrate enough so as to finish the Nistor section.

My construction

$$A \overset{\theta}{\underset{\theta'}{\rightrightarrows}} L\otimes B \quad \text{cong. mod } J\otimes B$$

$$p = \tfrac{1}{2}(\theta+\theta') : A \longrightarrow L\otimes B$$

$$q = \tfrac{1}{2}(\theta-\theta') : \bar{A} \longrightarrow J\otimes B$$

$$p+tq : A \longrightarrow (L+tJ)\otimes B \subset S\otimes B$$

linear resp. L

induces

$$RA \longrightarrow R(S\otimes B) \longrightarrow S\otimes RB$$

$$X(RA) \longrightarrow X(S\otimes RB) \longrightarrow S_\natural \otimes X(RB) \xrightarrow{\mu_m} J_\#^{2m+1}\otimes X(RB)$$

$$\cup \qquad\qquad \cup \qquad\qquad \cup \qquad\qquad \cup$$

Claim $$F^p_{IA} \longrightarrow F^p_{K\otimes RB + S\otimes IB} \longrightarrow \sum_{i\geq 0} \natural(K^i)\otimes F^{p-2i}_{IB} \longrightarrow J_\#^{2m+1}\otimes F^{p-2m}_{IB}$$

Get $$\mathcal{X}_A \longrightarrow J_\#^{2m+1}\otimes \mathcal{X}_B[2m] \quad \text{call this } ch^{2m}(\theta,\theta)$$

Given $$\tau : J_\#^{2m+1} \longrightarrow \mathbb{C} \quad \text{get}$$

$$ch^{2m}(\theta,\theta',\tau) \in HC^{2m}(A,B)$$

Joachim's version of Nistor

Introduce $Q = QA = \Omega A$ with $\circ$

graded as vector space $Q = \bigoplus_n Q_n$ where $Q_n = \Omega^n A$

canon. ident. $Q \simeq A * A \Rightarrow$ two canonical embeddings are $\iota(a) = a + da$, $\iota^\gamma(a) = a - da$, γ autom. of order 2 : $\gamma = (-1)^n$ on Q.

θ, θ' induces a homom.

$$Q \xrightarrow{\Theta} L\otimes B \qquad Q_n^* \longrightarrow J^n\otimes B$$

$$a_0 da_1 \dots da_n \longmapsto pa_0 qa_1 \dots qa_n$$

(6)

get factorization

$$A \xrightarrow[\text{hom.}]{i} Q \xrightarrow[\text{linear resp. 1}]{\otimes t^0} S \otimes B \qquad \text{comp. grading}$$

get

$$RQ \longrightarrow R(S \otimes B) \longrightarrow S \otimes RB \qquad \text{comp. grading}$$

$$X(RQ) \xrightarrow{?} S_q \otimes X(RB)$$

my map is therefore

$$X(RA) \xrightarrow{\iota_*} X(RQ) \xrightarrow{P_m(L_D)\gamma_m} X(RQ)_{\geq 2m+1} \longrightarrow S_{q \geq 2m+1} \otimes X(RQ)$$

$$\longrightarrow J_\#^{2m+1} \otimes X(RQ)$$

What points to emphasize?

Q graded as a vector space

RQ and X(RQ) inherit gradings.

D.

There will be a problem linking the grading and the filtration. Maybe I should go over the link.

The alg Q is graded as a vector space:

$$Q = \bigoplus_n Q_n \qquad 1 \in Q_0$$

Although this grading is not compatible with the alg st the ~~[illegible]~~ decreasing filtration

$$Q_{\geq k} = \bigoplus_{n \geq k} Q_n \qquad \text{arising from this grading}$$

is compatible with the alg. structure:

$$Q_{\geq i}\, Q_{\geq j} \subset Q_{\geq i+j} \qquad 1 \in Q_{\geq 0}$$

(H) (Actually this writing project is rather challenging because there is so much to organize. There are too many ideas for me to handle all at once, in practice, too many maps to label and organize.)

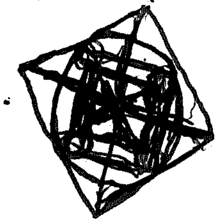
At the moment I am thinking about the end of the argument, the end map. So what do we do?

We have a homomorphism

$$Q \xrightarrow{\nu} L \otimes B$$

arising from the pair $\theta, \theta' : A \longrightarrow L \otimes B$. Specifically

$$a_0 da_1 \cdots da_n \longmapsto p a_0 q a_1 \cdots q a_n$$



One has $\boxed{\nu \iota = \theta, \quad \nu \iota^{\delta} = \theta'}$

Also $\nu(Q_n) \subset \nu(Q_{\geq n}) \subset J^n \otimes B.$

So where next?

$$Q \xrightarrow{t^D} \bigoplus_{k \in \mathbb{Z}} t^k Q_{\geq k} \longrightarrow \bigoplus t^k J^k \otimes B$$

$$Q \xrightarrow[\text{linear resp } 1]{t^D} Q^t \xrightarrow[\text{homom. comp. with } T \longrightarrow L^t.]{\nu^t} L^t \otimes B$$

$$RQ \longrightarrow R_T Q^t \longrightarrow R_{L^t}(L^t \otimes B)$$

At the moment I am trying to manouver things, but the problem is still the assertions.

(I) I already decided that the [redacted] good approach is to set up the maps on the supercomplex level, then claim they have appropriate property with respect to the filtration. Thus you want to identify your map $X(RA) \longrightarrow J_{\#}^{2m+1} \otimes X(RB)$ with a specific composition of

$$X(RA) \xrightarrow{\iota_*} X(RQ) \xrightarrow{\gamma_-} \gamma_- X_{\geq 0} = \gamma_- X_{\geq 1}$$

$$\xrightarrow{S_1} \gamma_- X_{\geq 2} = \gamma_- X_{\geq 3}$$

$$\cdots$$

$$\xrightarrow{S_{2m-1}} \gamma_- X_{\geq 2m} = \gamma_- X_{\geq 2m+1}$$

followed by a map

$$e : X_{\geq 2m+1} \longrightarrow J_{\#}^{2m+1} \otimes X(RB)$$

8/8 – 0636

objects: The map

$$X(RA) \xrightarrow{\iota_*} X(RQ) \xrightarrow{P_m(L_D)\gamma_-} X(RQ)_{\geq 2m+1}$$

The last map $\ell_k : X(RQ)_{\geq k} \longrightarrow J_{\#}^{k} \otimes X(RB)$

relations: 1) $\ell_{2m+1} \circ P_m(L_D)\gamma_- \iota_* : X(RA) \longrightarrow J_{\#}^{2m+1} \otimes X(RB)$ coincides with my map.

2) $P_m(L_D)\gamma_- \iota_*$ carries $F^p X(RA)$ into $F^{p-2m} X(RQ)_{\geq 2m+1}$ for all p, so one has a map of towers

$$\mathcal{X}_A \longrightarrow \mathcal{X}_{\geq 2m+1}[2m]$$

3) This class of this map in

$$HC^{2m}(\mathcal{X}_A, \mathcal{X}_{\geq 2m+1}) = HC^{2m}(A^b, Q^b_{\geq 2m+1})$$

(J) is (essentially) Nistor's bivariant character $ch^{2m}(\iota, \iota^\gamma)$ for the universal quasi-homomorphism.

Explain essentially

overall factor of 2
Nistor's class lies in
$HC^{2m}(A, Q^b_{m+1})$
he uses different L_D, h_D
but his uniqueness arg. shows
difference is killed by S

It still seems that the last map is the awkward point, owing to the fact that the filtration + grading are both involved. It would be nice to get an outline of the essential ideas. I would like to have a list of definitions and assertions whose proofs can be filled in by the reader.

Let us go over the last map carefully.

We begin with $\theta, \theta' : A \longrightarrow L \otimes B$ cong. mod $J \otimes B$. Get homom.

$$Q \xrightarrow{w} L \otimes B$$

properties

$$w\iota = \theta, \quad w\iota^\gamma = \theta'$$

$$w(a_0 da_1 \dots da_n) = p^{a_0} q^{a_1} \dots q^{a_n}$$

$$w(Q_n) \subset w(Q_{\geq n}) \subset J^n \otimes B$$

Get linear map resp ι.

$$Q \xrightarrow{f} S \otimes B \subset L^+ \otimes B$$

$$f(a_0 da_1 \dots da_n) = t^n p^{a_0} q^{a_1} \dots q^{a_n}$$

(K) Get homomorphism of graded algebras

$$Q^t \xrightarrow{w^t} L^t \otimes B$$

comp. w. $T \longrightarrow L^t$

$$f = \text{composition}: \quad Q \xrightarrow{t^D} Q^t \xrightarrow{w^t} L^t \otimes B$$

$$f = w^t t^D$$

f gives rise to

homom. $f_* : RQ \longrightarrow R(S \otimes B) \longrightarrow S \otimes RB$

map of sets $f_{**} : X(RQ) \longrightarrow X(S \otimes RB) \longrightarrow S_{\not\,} \otimes X(RB)$

w^t gives rise to

$$\begin{tikzcd} R_T Q^t \arrow[r] \arrow[dr, "w^t_*"'] & R_{L^t}(L^t \otimes B) \arrow[d, equal] \\ & L^t \otimes RB \end{tikzcd}$$

$$\begin{tikzcd} X_T(R_T Q^t) \arrow[r] \arrow[dr, "w^t_{**}"'] & X_{L^t}(L^t_t \otimes RB) \arrow[d, equal] \\ & L^t_{\not\,} \otimes X(RB) \end{tikzcd}$$

8/8-0841 Let's try introducing notation.

$f = p + tq : A \longrightarrow S \otimes B$

$g = w^t t^D : Q \longrightarrow S \otimes B \quad \subset \quad L^t \otimes B$

$f_* : RA \longrightarrow S \otimes RB \quad$ or $\quad L^t \otimes B$

$g_* : RQ \longrightarrow S \otimes RB \quad\quad L^t \otimes B$

(L)

$$f_{**} : X(RA) \longrightarrow S_b \otimes X(RB)$$
$$g_{**} : X(RQ) \longrightarrow \quad ''$$

$$\iota : A \longrightarrow Q$$
$$\iota_* : RA \longrightarrow RQ$$
$$\iota_{**} : X(RA) \longrightarrow X(RQ).$$

Then $f = g\iota \implies f_* = g_*\iota_* \implies f_{**} = g_{**}\iota_{**}$

~~[illegible]~~

$$t^D : Q \longrightarrow Q^t$$
$$t^D : RQ \longrightarrow RQ^t$$
$$t^{L_D} : X(RQ) \longrightarrow X(RQ)^t$$
$$w^t : Q^t \longrightarrow L^t \otimes B$$
$$w^t_* : (RQ)^t \longrightarrow L^t \otimes RB$$
$$w^t_{**} : X(RQ)^t \longrightarrow L^t_b \otimes X(RB).$$

before I can write this down I need I think

$$R_T(Q^t) \simeq (RQ)^t$$
$$X_T(R_T(Q^t)) \simeq (X(RQ))^t$$
$$R_{L^t}(L^t \otimes B) = L^t \otimes RB$$
$$X_{L^t}(R_{L^t}(L^t \otimes B)) = X_{L^t}(L^t \otimes RB) = L^t_b \otimes X(RB)$$

and some of these need amplification by formulas.

(M) Let us consider then the concrete statements I need and the proofs.

What do I need in order to link my construction with Nistor's?

Nistor objects.

$$A^b = \text{mixed complex } (\Omega A, b, B)$$

$$Q^b_{\geq k} = \text{————————} \quad (\Omega Q_{\geq k}, b, B).$$

"last" maps

$$\left|\begin{array}{l} l_k : \Omega Q_{\geq k} \longrightarrow J_\#^k \otimes \Omega B \\ l_k : Q^b_{\geq k} \longrightarrow J_\#^k \otimes B^b \end{array}\right.$$

He constructs $S_k : Q^b_{\geq k} \longrightarrow Q^b_{\geq k+1}[2]$ S-module level

$$\ni \quad [S_k][\iota_k] = S \in HC^2(Q^b_{\geq k+1}, Q^b_{\geq k+1})$$
$$[\iota_k][S_k] = S \in HC^2(Q^b_{\geq k}, Q^b_{\geq k})$$

and notes that $[S_k]$ with this property is unique up to a class killed by S.

(This is not what I want to use but so what)

He defines the biv. char of the univ. quis to be

$$ch^{2k} = [S_k] \cdots [S_1][\iota - \iota^\gamma] : HC^{2k}(A^b, Q^b_{\geq k+1})$$

1315. I have to find minimum things to say. Let's try to organize the assertions.

The goal is to link my bivariant Chern character $ch^{2m}(\theta, \theta') \in HC^{2m}(A^b, J_\#^{2m+1} \otimes B^b)$

(N) with Nistor's. To show they are essentially the same. Better ~~notation~~

$$ch^{2m}(\theta, \theta', \tau) \in HC^{2m}(A, B)$$

do you mention that one has a quasi-homom. Consisting of two homos.

8/8 - 1524

Start with $\theta, \theta' : A \longrightarrow L \otimes B$ congruent modulo $J \otimes B$ and a J-adic trace on J^{2m+1}. ~~[illegible]~~ Aim to construct

$$ch^{2m}(\theta, \theta', \tau) \in HC^{2m}(A, B).$$

Nistor construction (essentially)

$$Q = QA \quad \cancel{\text{filtering}}$$

$$Q^b_{\geq k} = B(\Omega Q_{\geq k})$$

$$\iota_k : \mathcal{X}_- Q^b_{\geq k+1} \longrightarrow \mathcal{X}_- Q^b_{\geq k}$$

$$\exists \ S_k : \mathcal{X}_- Q^b_{\geq k} \longrightarrow \mathcal{X}_- Q^b_{\geq k+1}[2]$$

$$S_k \iota_k \sim S : \mathcal{X}_- Q^b_{\geq k+1} \longrightarrow \mathcal{X}_- Q^b_{\geq k+1}[2]$$

$$\iota_k S_k \sim S : \mathcal{X}_- Q^b_{\geq k} \longrightarrow \mathcal{X}_- Q^b_{\geq k}[2]$$

0 ~~So what happens~~ 8/9 - 0630

version of Nistor's construction

$Q = QA$, graded as a vector space

$Q_n = \Omega^n$.

$Q = QA$

grading as vector space

assoc. | filtration | comp. with alg. structure
| $\mathbb{Z}/2$ grading

D, $\gamma = (-1)^D$

induced gradings on RQ, $X(RQ)$

D, $L_D = L(1, D)$

canon. $\phi : RQ \longrightarrow \Omega^2(RQ)$, $h_D = h^\phi(1, D)$.

$L_{D*} = [\partial, h_D]$, $[L_D, h_D] = 0$

~~[illegible]~~ Should I write down things.

-OK what comes next???

I am trying to explain Joachim's ~~construction~~ version of Nistor's bivariant Chern character for the universal quasi-homomorphism.

(p) 8/10 – 0551.

points. my construction amounts to

$$X(RA) \longrightarrow X(S_\natural \otimes RB) \xrightarrow{\alpha_\natural} S_\natural \otimes X(RB) \xrightarrow{\text{Mem}} J_\#^{2m+1} \otimes X(RB)$$

together with its filt. behavior

$$F^p_{IA} \qquad\qquad J_\#^{2m+1} \otimes F^{p-2m}_{IB}$$

next – to ~~compare~~ link my ~~construction~~ to Nistor's. to give our version of Nistor's construction.

$Q = QA$ equipped with filtration.

8/10 – 1324. ~~So I am trying to do all~~

1330 Try to recall Nistor construction

filtration $\Omega Q_{\geq k}$ of ΩQ by mixed subcomplexes. $\iota_k : \Omega Q_{\geq k+1} \longrightarrow \Omega Q_{\geq k}$ inclusion

$[\iota_k] \in HC^0(\Omega Q_{\geq k+1}, \Omega Q_{\geq k})$. Nistor ~~shows~~ constructs

$\exists\ [S_k] \in HC^2(\Omega Q_{\geq k}, \Omega Q_{\geq k+1}) \ni$

$$[\iota_k][S_k] = S \in HC^2(\Omega Q_{\geq k+1}, \Omega Q_{\geq k+1})$$

$$[S_k][\iota_k] = S \in HC^2(\Omega Q_{\geq k}, \Omega Q_{\geq k})$$

such a class unique up to a class killed by S.

$$\gamma_- = \frac{1-\gamma}{2}$$

$$\begin{pmatrix} \iota_k^+ & \\ & \iota_k^- \end{pmatrix} \begin{pmatrix} \gamma_+ S_k \gamma_+ & \gamma_+ S_k \gamma_- \\ \gamma_- S_k \gamma_+ & \gamma_- S_k \gamma_- \end{pmatrix} = \begin{pmatrix} S^+ & \\ & S^- \end{pmatrix}$$

Replace S_k by its average wrt γ

$$\frac{1}{2}(S_k + \gamma S_k \gamma)$$

(Q) 8/11 - 0544

new approach
first our construction
then link with Nistor's

new approach is to start ~~by recalling~~ with a suitable version of Nistor's construction.

$(\Omega Q)_{\geq k}$ mixed subcomplex of ΩQ.

$$\iota_k \in HC^0((\Omega Q)_{\geq k+1}, (\Omega Q)_{\geq k})$$

$$\exists\ S_k \in HC^2((\ \ )_{\geq k}, (\Omega Q)_{\geq k+1})$$

$$S_k \iota_k = S \in HC^2((\Omega Q)_{\geq k+1}, (\Omega Q)_{\geq k+1})$$

$$\iota_k S_k = S \in HC^2((\Omega Q)_{\geq k}, (\Omega Q)_{\geq k})$$

S_k unique up to a class killed by S.

γ, $\gamma_- = \frac{1}{2}(1-\gamma)$. $\gamma_-(\Omega Q)_{\geq k} = \gamma_-(\Omega Q)_{\geq k+1}$ k even

rep. S_k by $\frac{1}{2}(S_k + \gamma S_k \gamma)$ can suppose S_k, γ commute

get rest.

$$\iota_k \in HC^0(\gamma_-(\Omega Q)_{\geq k+1}, \gamma_-(\Omega Q)_{\geq k})$$

$$S_k \in \qquad {}_{\geq k} \qquad {}_{\geq k+1}$$

analogous identities.

$$Ch^0(\iota, \iota^\gamma) \in HC^0(\Omega A, \gamma_- \Omega Q_{\geq 1})$$

class of $\Omega A \xrightarrow{\Omega(\iota)} \Omega Q \xrightarrow{\gamma_-} \gamma_-(\Omega Q) = \gamma_-(\Omega Q)_{\geq 1}$

$$Ch^{2m}(\iota, \iota^\gamma) = S_{2m-1} \cdots S_3 \cdot S_1 \cdot Ch^0(\iota, \iota^\gamma)$$

(R) next describe present X version of this. I want to get things clear enough to understand myself. You have to decide what needs explaining.

Start with $F^p\Omega_{\geq k}$.

bifiltration of $\Omega = \Omega Q$

Can transport via $X \simeq \Omega$ to obtain $F^pX_{\geq k}$

Claim the canonical heq $X \sim \Omega$ induces heq $F^pX_{\geq k} \sim F^p\Omega_{\geq k}$ for all p, k.

Cor. $\mathcal{X}_{\geq k} = (X_{\geq k}/F^pX_{\geq k}) \sim \theta(\Omega_{\geq k})$.

so bivariant classes between $\Omega_{\geq k}$ can be constructed from maps of towers.

now bring in D, L_D, h_D.

0836 first thing I want

$\iota_k \in HC^0(\mathcal{X}_{\geq k+1}, \mathcal{X}_{\geq k})$

given by inclusion $X_{\geq k+1} \subset X_{\geq k}$ (which carries $F^pX_{\geq k+1}$ into $F^pX_{\geq k}$

$S_k \in HC^2(\mathcal{X}_{\geq k}, \mathcal{X}_{\geq k+1})$ given by

$$1 - k^{-1}L_D : X_{\geq k+1} \longrightarrow X_{\geq k}$$

which carries $F^pX_{\geq k+1} \longrightarrow F^{p-2}X_{\geq k}$ by ...

(8) 139 Outline so let's use this pen a bit.

What I seem to have evolved is the idea of ~~all~~ starting with a version of Nistor's ~~construction namely: bivers~~ bivariant Chern character for the universal quasi-homom. This means describing classes

$$\iota_k \in HC^0\big((\Omega Q)_{\geq k+1}, (\Omega Q)_{\geq k}\big)$$

let's try out the notation

$$\iota_k \in HC^0\big(\gamma_- Q^b_{\geq k+1}, \gamma_- Q^b_{\geq k}\big)$$

$$S_k \in HC^2\big(\gamma_- Q^b_{\geq k}, \gamma_- Q^b_{\geq k+1}\big) \qquad k \geq 1$$

$$ch^{2m}(\iota, \iota^\gamma) = S_{2m-1} \cdots S_3 \cdot S_1 \, ch^0(\iota, \iota^\gamma)$$

$$\in HC^{2m}(A^b, \gamma_- Q^b_{\geq 2m+1})$$

$ch(\iota, \iota^\gamma)$ = class of

$$\gamma_- \iota^b = \tfrac{1}{2}\big(\iota^b - (\iota^\gamma)^b\big) : \text{[redacted]}$$

$$A^b \xrightarrow{\iota^b} Q^b \xrightarrow{\gamma_-} \gamma_- Q^b = \gamma_- Q^b_{\geq 1}.$$

~~To describe these classes~~

Now discuss, describe X-version.

Have ~~XXX~~ $\underbrace{X(RQ)}_{X} \simeq \underbrace{\Omega Q}_{\Omega}$

Define $FPX_{\geq k}$ etc. ~~S~~ Then you have the lemma about the behavior. Your problem is the fact you haven't written out the details. haven't got the details straight in your own mind.

(T) 1204. The point is to use
1356 Put the pieces together.

Suppose you define $F^pX_{\geq k}$ to corresp to $F^p\Omega_{\geq k}$ under the isom. $X \xrightarrow{\sim} \Omega$. (Check that or note the $F^p\Omega_{\geq k}$ stable under b, d, K etc. so $F^pX_{\geq k}$ is a subcomplex of X. The SDR of X onto PX, Ω onto $P\Omega$, and isom $PX \simeq P\Omega$ have to induce ~~the same thing and~~ SDRs, etc. where Ω replaced by $F^p\Omega_{\geq k}$.)

The problem with this approach is that it can't handle D. L_D, h_D are defined on $X(RQ)$ because RQ depends on Q as vector space. These words are all clear, but ~~the point is~~ I need to make them convincing.

Given Q with grading $Q = \bigoplus_{n\in\mathbb{Z}} Q_n$, $1 \in Q_0$ and filtration $Q_{\geq k} = \bigoplus_{n\geq k} Q_n$ compat with alg str. $Q_{\geq i}\cdot Q_{\geq j} \subset Q_{\geq i+j}$, $1 \in Q_{\geq 0}$. Form graded algebra $Q^t = \bigoplus_k t^k Q_{\geq k} \subset T'\otimes Q$. Q' graded T algebra. Have

$$T'\otimes_T Q^t \xrightarrow{\sim} T'\otimes Q \quad \text{alg. isom.}$$

$$T\otimes Q \xrightarrow{\sim} Q^t \quad \text{graded } T\text{-module.}$$

$$t\partial_t + D \longleftrightarrow t\partial_t$$

(u)

$$Q \xrightarrow{t^D} Q^t$$

$$T\otimes Q \xrightarrow{\sim} Q^t$$

$$f\otimes x \longmapsto f\, t^D x$$

$$t\partial_t + D \longleftrightarrow t\partial_t .$$

$$\therefore \quad R_T(T\otimes Q) \xrightarrow{\sim} R_T(Q^t)$$

$$\begin{array}{c} \| \\ T\otimes RQ \end{array}$$

let's go onto ~~the derivation~~ L_D

$$\begin{array}{ccc} F^p_{I_T(Q^t)} X_T(R_T(Q^t)) & \subset & X_T(R_T(Q^t)) \\ \| & & \| \\ (F^p X)^t & & X(RQ)^t \end{array}$$

Establish ahead of time that

$$T\otimes X(RQ) \xrightarrow{\sim} X_T(R_T(Q^t))$$

$$f\otimes \xi \longmapsto f\, t^{L_D}(\xi)$$

What about L_D on $X_T(R_T(Q^t))$?

You have D on Q extended to Q^t so as to commute with T-module structure, whence have L_D on $X_T(R_T(Q^t))$. Now have

$$X(R(Q)) \longrightarrow X_T(R_T(Q^t))$$

(V)

$$Q \xrightarrow{t^P} Q^t$$

$$D \qquad t\partial_t$$

$$\begin{array}{ccc} X_T(R_T(Q^t)) & & \\ \shortparallel & & \\ X(RQ)^t & \subset & T' \otimes X(RQ) \\ L_D & & L_D \end{array}$$

You have on $X(RQ)^t$ both $t\partial_t$ and L_D

What happens is that on ~~RQ^t~~, RQ^t, ~~$X(RQ)^t$~~ we have $t\partial_t - D$ vanishes on the image of t^{L_D} and ~~on image~~ $[(t\partial_t - D), t^{-1}] = -t^{-1}$. Thus

$$(D - t\partial_t) : (RQ)^t \longrightarrow t^{-1}(RQ)^t$$

so

$$\begin{array}{ccc} (L_D - t\partial_t) : F^P_{I_T(Q^t)} X_T(R_T(Q^t)) & \longrightarrow & t^{-1} F^{P-2}_{I_T(Q^t)} X_T(R_T(Q^t)) \\ \shortparallel & & \shortparallel \\ (F^P X)^t & & (F^{P-2} X)^t \end{array}$$

means that $F^P X_{\geq k} \xrightarrow{L_D - k} F^{P-2}_{\geq k+1} X$

as claimed.

(W) What about $\gamma_{(-1)^D}$

On Q^t we $\gamma =$ extended in obvious way to commute with t and also $(-1)^{t\partial_t}$ which changes t to $-t$. We have

$$(-1)^D - (-1)^{t\partial_t} : R_T(Q^t)$$
$$\| \qquad\qquad$$
$$(RQ)^t \longrightarrow t^{-1}(RQ)^t$$

OK on image of $t^D : RQ \longrightarrow (RQ)^t$

OK on t^{-1} $\qquad (-1)^D(t^{-1}) - (-1)^{t\partial_t}(t^{-1})$

$$= t^{-1} - (-t^{-1}) = 2t^{-1}.$$

So we have two autos. $(-1)^D$ and $(-1)^{t\partial_t}$ of $X_T(R_T(Q^t)) = X(RQ)^t$ preserving

$$F^p_{I_T(Q^t)} X_T(R_T(Q^t)) = (F^p X)^t$$

For any element [redacted] $\rho(x) \in R_T(Q^t)$ have

$$\cancel{(-1)^D - (-1)^{t\partial_t}(\rho(x))}$$

Wait. Think of X^t with the filtr $(F^p_I X)^t$. Two autos of R^t preserving I^t

but $(F^p_I X)^t$ is $F^p_{I^t} X(R^t)$ so have spanning elts.

$$(I^t)^{n+1} + [(I^t)^n, R_t] \qquad b((I^t)^n dR^t)$$

go thru such generators to calculate
$\gamma - (-1)^{t\partial_t} : (F^p X)^t \longrightarrow t^{-1}(F^p X)^t$

(X) which seems to be exactly what I want.

Recall the concrete approach.

$F^p X_{\geq k}$ spanned by

$$(F^p X)^t = F^p_{I^t}\, X_T(R^t)$$

$p = 2n$ ~~ap~~ is

$$(I^t)^{n+1} + [(I^t)^n, R^t] \Longleftrightarrow b\left((I^t)^n d(R^t)\right)$$

$$h_D = h^\phi(1, D) \qquad \phi$$

I think what I want to do is to ~~affixe~~ illustrate the $X(RQ)$ approach from the ΩQ one. So one has ~~$X(RQ)$~~

On one hand we have $F^p\Omega_{\geq k}$

$$(F^p\Omega)^t = F^p(\Omega^t)$$

$$= [(\Omega^p)^t, Q^t] \oplus \bigoplus_{n>p} (\Omega^n)^t$$

~~spanned by~~ So

$F^p\Omega_{\geq k}$ spanned by ~~$[\Omega^p, Q]_{\geq k} \oplus$~~

$$[x_0 dx_1 \cdots dx_p, x_{p+1}], \qquad x_0 dx_1 \cdots dx_n \qquad n > p$$

$$\sum \operatorname{ord}(x_i) \geq k.$$

~~Translates into~~

$F^{2n} X_{\geq k}$ spanned by $(I^{n+1})_{\geq k} + [I^n, R]_{\geq k}$

$$\rho(x_0)\,\omega(x_1, x_2) \cdots \omega(x_{2j-1}, x_{2j}) \qquad j \geq n+1$$

(Y) If you don't have the proof in your mind you can't outline it.

Recall so far

~~Nistor~~ Nistor's character for univ. quasi-hom.

$$\iota_k \in HC^0(Q^b_{\geq k+1}, Q^b_{\geq k})$$

$$\exists \quad S_k \in HC^2(Q^b_{\geq k}, Q^b_{\geq k+1})$$

$$\ni \quad S_k \iota_k = S, \quad i_k S_k = S.$$

S_k unique up to Ker S

rep. S_k by $\frac{1}{2}(S_k + \gamma S_k \gamma)$ etc.

So we get

$$\begin{array}{ccccc} X(RA) & \xrightarrow{\iota_*} & X(RQ) & \xrightarrow{P_m(L_D)\gamma_-} & \gamma_- X(RQ)_{\geq 2m+1} \\ \cup & & \cup & & \cup \\ F^p_{IA} & \longrightarrow & P^p_{IQ} & \longrightarrow & \gamma_- F^{p-2m} X_{\geq 2m+1} \end{array}$$

gives Nistor class

$$ch^{2m}(\iota, \iota^\gamma) \in HC^{2m}(X_A, \gamma_- X_{\geq 2m+1})$$
$$= HC^{2m}(A^b, \gamma_- Q^b_{\geq 2m+1})$$

Next I need last map, point is we have $Q \longrightarrow L \otimes B$

$$Q_{\geq k} \longrightarrow J^k \otimes B$$

$- \quad /$

(2) 8/12 - 0520

go over the whole proof.

my construction

$\theta, \theta' : A \longrightarrow L \otimes B$ cong. mod $J \otimes B$

$p = \frac{1}{2}(\theta + \theta')$, $q = \frac{1}{2}(\theta - \theta') : A \longrightarrow J \otimes B$

$p + tq : A \longrightarrow S \otimes B$ linear resp 1.

$u = (p + \cancel{tq})_* : RA \longrightarrow S \otimes RB$, $IA \longrightarrow K \otimes RB + S \otimes IB$

$$X(RA) \xrightarrow{u_*} X(S \otimes RB) \xrightarrow{\alpha} S_{\natural} \otimes X(RB) \xrightarrow{\mu_m} J^{2m+1}_{\#} \otimes X(RB)$$

$$F^p_{IA} \longrightarrow F^p_{K \otimes RB + S \otimes IB} \longrightarrow \sum_{i \geq 0} \natural(K^i) \otimes F^{p-2i}_{IB} \longrightarrow J^{2m+1}_{\#} \otimes F^{p-2m}_{IB}$$

link my construction to Nistor's.

$Q = QA$ filtration $Q_{\geq k}$ autom γ order 2

induced filt. $\Omega Q_{\geq k}$ mixed subcxs.

$\iota_k \in HC^0(\Omega Q_{\geq k+1}, \Omega Q_{\geq k})$

$\exists$ $S_k \in HC^2(\Omega Q_{\geq k}, \Omega Q_{\geq k+1})$ inverse for ι_k up to S

$S_k \iota_k = S$, $\iota_k S_k = S$. unique up to Ker S

can suppose $[S_k, \gamma] = 0$ whence rest.

$\iota_k \in HC^0(\gamma_- \Omega Q_{\geq k+1}, \gamma_- \Omega Q_{\geq k})$

$S_k \in HC^2(\gamma_- \Omega Q_{\geq k}, \cancel{\gamma_-} \Omega Q_{\geq k+1})$

same props.

$\gamma_- \Omega Q_{\geq k+1} = \gamma_- \Omega Q_{\geq k}$ k even

A) 8/12 - 0531

$$Ch^{2m}(\iota, \iota^\gamma) \overset{def}{=} S_{2m-1} \cdot \cdots \cdot S_3 \cdot S_1 \cdot Ch^0(\iota, \iota^\gamma)$$
$$\in HC^{2m}(\Omega A, \gamma_- \Omega Q_{\geq 2m+1})$$

$$Ch^0(\iota, \iota^\gamma) = \gamma_- \iota_* = \tfrac{1}{2}(\iota_* - \iota^\gamma_*)$$
$$: \Omega A \xrightarrow{\iota_*} \Omega Q \xrightarrow{\gamma_-} \gamma_- \Omega Q = \gamma_- \Omega Q_{\geq 1}.$$

next. θ, θ' induce

$$Q \longrightarrow L \otimes B \quad , \quad Q_{\geq k} \longrightarrow J^k \otimes B$$

trace map $\forall k$
$$\Omega Q_{\geq k} \xrightarrow{l_k} J^k_\# \otimes \Omega B$$

$$Ch^{2m}(\theta, \theta') = l_{2m+1} \cdot Ch^{2m}(\iota, \iota^\gamma)$$
$$\in HC^{2m}(\Omega A, J^{2m+1}_\# \otimes \Omega B).$$

Claim this agrees with my
$$Ch^{2m}(\theta, \theta') = [\mu_m \propto u_*]$$
$$\in HC^{2m}(\chi_A, J^{2m+1}_\# \otimes \chi_B)$$

I want to go over the steps of the proof many times today until it becomes incredibly clear in my mind.

First step is to pass from the mixed complexes $\Omega Q_{\geq k}$ to the corresp. towers.

$\Omega Q_{\geq k}$ spanned by $x_0 dx_1 \cdots dx_n$ $\quad \sum \mathrm{ord}(x_i) \geq k.$
compatible with DG alg structure | grading product d

B) hence compatible with b, κ, etc.

Recall linear isom. $X(RQ) \simeq \Omega Q$

and the description of the structure on $X(RQ)$ in terms ΩQ and its operations.

this structure

$$\left|\begin{array}{l} \text{product on } RQ \\ \text{pairing } \natural(xdy) = -\sum_{i=0}^{n-1} \kappa^{2i} b(x \circ y) + \sum_{0}^{n-1} \kappa^{2i} d(xy) \\ \text{filtr. } F^p_{IQ} \qquad\qquad\qquad\qquad + \kappa^{2n}(x\,dy). \end{array}\right.$$

0735 go over the steps.

~~hom~~ θ, θ' yield hom. $Q \longrightarrow L \otimes B$

~~hom~~ $\ni$ $\qquad Q_{\geq k} \longrightarrow J^k \otimes B \qquad \forall k.$

get $\qquad \Omega Q \longrightarrow \Omega_L(L \otimes B) = L \otimes \Omega B$

DG alg homom. $\ni$

$$\begin{array}{ccc} \Omega Q_{\geq k} & \longrightarrow & J^k \otimes \Omega B \qquad \forall k \\ & \searrow & \| \\ & & \Omega_L (L \otimes B)_{\geq k} \end{array}$$

~~and get~~

Claim get map ~~of mixed cx~~.

$$\Omega Q_{\geq k} \longrightarrow J^k_{\#} \otimes \Omega B \qquad \forall k$$

compatible ~~to~~ d, b, κ, etc. This means we have

$$X(RQ)_{\geq k} \longrightarrow J^k_{\#} \otimes X(RB)$$

comp with differentials, $F^p_{\cancel{IQ}} X_{\geq k} \longrightarrow J^k_{\#} \otimes F^p_{IB}$.

c) Go over facts, see what is obvious, + what requires proof.

θ, θ' induce hom $Q \longrightarrow L \otimes B$

$\ni \quad Q_{\geq k} \longrightarrow J^k \otimes B \qquad \forall k$

get DG hom $\Omega Q \longrightarrow \Omega_L (L \otimes B) = L \otimes \Omega B$

$\ni \quad \Omega Q_{\geq k} \longrightarrow \Omega_L (L \otimes B)_{\geq k} = J^k \otimes \Omega B \qquad \forall k$

get map $\Omega Q_{\geq k} \longrightarrow J^k_{\#} \otimes \Omega B \qquad \forall k$

compat with $d, b, K, P, \ldots$

get hom $RQ \longrightarrow L \otimes RB$

$\ni \quad RQ_{\geq k} \longrightarrow J^k \otimes RB$

get map $X(RQ)_{\geq k} \longrightarrow J^k_{\#} \otimes \cancel{X}(RB)$

comp with $\partial, \ \iota_k$)

$\ni \quad F^p X_{\geq k} \longrightarrow J^k_{\#} \otimes F^p_{IB}$

<u>This is all straightforward.</u> The main point not ~~obviously~~ tautological is why

$\Omega Q_{\geq k} \longrightarrow J^k_{\#} \otimes \Omega B$

commutes with b.

Now the t version.

hom. $Q^t \longrightarrow L^t \otimes B$

$\Omega_T (Q^t) \longrightarrow \Omega_{L^t} (L^t \otimes B)$

$\cancel{\Phi} (\Omega Q)^t \longrightarrow L^t \otimes \Omega B$

D) So review the definition of

$$X_{\geq k} \longrightarrow J^k_\# \otimes X(RB)$$

~~two~~ possibilities:

$$\begin{tikzcd} X_{\geq k} \arrow[r, phantom, "\simeq"] & \Omega Q_{\geq k} \arrow[d, "\text{trace}"] \\ J^k_\# \otimes X(RB) \arrow[r, phantom, "\simeq"] & J^k_\# \otimes \Omega B \end{tikzcd}$$

~~Now $X_{\geq k}$ spanned by $\rho(x_0)\,\omega(x_1, x_2)\cdots\omega(x_{2n-1}, x_{2n})$~~

Why does this commute?

$$\begin{tikzcd} X(RQ)_{\geq k} \arrow[r, phantom, "\simeq"] \arrow[d] & \Omega Q_{\geq k} \arrow[d] \\ X(R(L\otimes B))_{\geq k} \arrow[d] & \Omega(L\otimes B)_{\geq k} \\ X(L\otimes RB)_{\geq k} & \end{tikzcd}$$

~~no good~~.

doesn't work ~~so~~ without difficulty.
So instead what I propose to do is to use

$$\begin{array}{ccc} Q^t & \longrightarrow & L^t \otimes B \\ \shortparallel & & \shortparallel \\ \bigoplus t^k Q_{\geq k} & \longrightarrow & \bigoplus t^k J^k \otimes B \end{array}$$

E) Then apply

$$\begin{tikzcd}
X_T(R_T(Q^t)) \arrow[r] \arrow[d, phantom, "\wr\|"] & X_{L^t}(R_{L^t}(L^t \otimes B)) \arrow[d, phantom, "\wr\|"] \\
\Omega_T(Q^t) \arrow[r] \arrow[d, phantom, "\|"] & \Omega_{L^t}(L^t \otimes B) \otimes_{L^t} \arrow[d, phantom, "\|"] \\
(\Omega Q)^t \arrow[r] & L^t_q \otimes \Omega B
\end{tikzcd}$$

Repeat the logic:

θ, θ' induce a hom. $Q \longrightarrow L \otimes B$ such that $Q_{\geq k} \longrightarrow J^k \otimes B$ for all k.

Thus we have a homom.

$$Q^t \longrightarrow L^t \otimes B$$

of graded T-algs. Comm. diags

$$\cancel{X_T(R_T(Q^t)) \longrightarrow X_{L^t}(R_{L^t}(L^t \otimes B))}$$

$$\begin{array}{ccc}
R_T(Q^t) & \longrightarrow & R_{L^t}(L^t \otimes B) = L^t \otimes RB \\
\cup & & \\
I_T(Q^t) & \longrightarrow & I_{L^t}(L^t \otimes B) = L^t \otimes IB
\end{array}$$

$$\begin{array}{ccccc}
X(RQ)^t & = & X_T(R_T(Q^t)) & \longrightarrow & X_{L^t}(L^t \otimes RB) = L^t_q \otimes X(RB) \\
\cup & & \cup & & \cup \qquad\qquad \cup \\
(F^pX)^t & = & F^p_{I_t(Q^t)} & \longrightarrow & F^p_{L^t \otimes IB} = L^t_q \otimes F^p_{IB}
\end{array}$$

F) Also comm. diag.

$$\begin{tikzcd}
X_T(R_T(Q^t)) \arrow[r] \arrow[d, "\wr"] & X_{L_t}(R_{L_t}(L_t\otimes B)) \arrow[d, "\wr"] \\
\Omega_T(Q^t) \arrow[r] & \Omega_{L_t}(L_t\otimes B)
\end{tikzcd}$$

leading to

$$\begin{tikzcd}
X^t \arrow[r] \arrow[d, "\wr"] & L^t_\flat \otimes X(RB) \arrow[d, "\wr"] \\
(\Omega Q)^t \arrow[r] & L^t_\flat \otimes \Omega B
\end{tikzcd}$$

1545 $\quad 2\frac{1}{4}$ hours.

Review what we did learn, namely

θ, θ' induce $\quad \begin{array}{c} Q \longrightarrow L\otimes B \\ \cup \qquad\quad \cup \\ Q_{\geq k} \longrightarrow J^k\otimes B \end{array}$ $\quad$ hom. of filtered alg.

whence hom. $\quad Q^t \longrightarrow L^t\otimes B$

whence

$$\begin{array}{ccccc}
\Omega_T(Q^t) & \longrightarrow & \Omega_{L^t}(L^t\otimes B) & \longrightarrow & \Omega_{L^t}(L^t\otimes B)\otimes_{L^t} t \\
\| & & \| & & \| \\
(\Omega Q)^t & & L^t\otimes\Omega B & & L^t_\flat\otimes\Omega B
\end{array}$$

also

$$\begin{array}{ccc}
X_T(R_T(Q^t)) & \longrightarrow & X_{L^t}(R_{L^t}(L^t\otimes B)) \\
\| & & \| \\
X^t & & L^t_\flat\otimes X(RB)
\end{array}$$

have comm. diag.

$$\begin{tikzcd}
X^t \arrow[r, equal] \arrow[d, "\wr"] & X_T(R_T(Q^t)) \arrow[r] \arrow[d, "\wr"] & X_{L^t}(R_{L^t}(L^t\otimes B)) \arrow[r, equal] \arrow[d, "\wr"] & L^t_\flat\otimes X(RB) \arrow[d, "\wr"] \\
(\Omega Q)^t \arrow[r, equal] & \Omega_T(Q^t) \arrow[r] & \Omega_{L^t}(L^t\otimes B)\otimes_{L^t} t \arrow[r, equal] & L^t_\flat\otimes\Omega B
\end{tikzcd}$$

6) ~~Identifies~~

Canonical htpy equiv.

$$\begin{array}{ccc} X^t & \xrightarrow[\text{map}]{\text{last}} & L^t_{\mathfrak{h}} \otimes X(RB) \\ \wr & & \wr \\ (\Omega Q)^t & \xrightarrow{tr} & L^t_{\mathfrak{h}} \otimes \Omega B \end{array}$$

Find last map compatible with filtrations, i.e.

$$F^p X^t \longrightarrow L^t_{\mathfrak{h}} \otimes F^p_{IB} X(RB)$$

and that it is canonically htpy equivalent to trace map $(\Omega Q)^t \longrightarrow L^t_{\mathfrak{h}} \otimes \Omega B$ respecting F^p- filtrations.

Next now that the last map is under control we need to focus on the key point, which is how D enters. Here we ~~play off the fact that~~ use the fact RQ, $X(RQ)$ depends only on Q as vector space with 1.

Let's try to find an order.

$$\begin{array}{ccccc} A & \xrightarrow{p+tq} & S \otimes B & \subset & L^t \otimes B \\ \iota \downarrow & & \| & & \\ Q & \xrightarrow{?} & S \otimes B & & \end{array}$$

So far have ~~defined~~ established

$$(F^p X)^t = \sum_k t^k F^p X_{\geq k} = F^p_{I_t(Q^t)} X_T(R_T(Q^t))$$

This comes from the definition

~~$(F^p X)^t$~~ $\quad X^t \simeq (\Omega Q)^t \quad ?$

H) definition of $F^pX_{\geq k}$:

$$\begin{array}{ccccccc}
X(RQ)^t & = & X_T(R_T(Q^t)) & \simeq & \Omega_T(Q^t) & = & (\Omega Q)^t \\
\cup & & \cup & & \cup & & \\
(F^pX)^t & = & F^p_{I_T(Q^t)} X_T(R_T(Q^t)) & \simeq & F^p\Omega_T(Q^t) & = & (F^p\Omega Q)^t
\end{array}$$

$$\underbrace{F^p_{I_T(Q^t)} X_T(R_T(Q^t)) \simeq F^p\Omega_T(Q^t)}_{\text{relative version of first thm.}}$$

defn of $(F^p\Omega Q)_{\geq k}$ as $F^p(\Omega Q_{\geq k})$

What does this mean in concrete terms?

$$F^p_{I_T(Q^t)} X_T(R_T(Q^t)) : \quad I_T(Q^t)^{n+1} + [I_T(Q^t)^n, R_T(Q^t)]$$
$$\Longleftrightarrow \; \natural\left(I_T(Q^t)^n \, d\, R_T(Q^t)\right)$$

what is $I_T(Q^t)$? Can define it as spanned by $\rho(y_0)\,\omega(y_1, y_2) \cdots \omega(y_{2n-1}, y_{2n})$ for $n \geq 1$, can assume y_j homogeneous.

$y_j = t^{k_j} x_j \qquad x_j \in Q_{\geq k_j}$. Then this elt is $t^{\sum k_j} \rho(x_0)\, \omega^n(x_1, \dots, x_{2n})$

1650 It perhaps is not important what $I_T(Q^t)$ is, rather just that it's ~~a~~ graded° ideal. From this one knows ~~that~~ for $I_T(Q^t)^n = \sum t^k (IQ^n)_{\geq k}$ that

$$(IQ^n)_{\geq k} = \sum_{\sum k_j = k} (IQ)_{\geq k_1} \cdots (IQ)_{\geq k_n}$$

I) Return to how D enters. You control $F^pX_{\geq k}$ by means of

$$(F^pX)^t = F^p_{I_T(Q^t)} X_T(R_T(Q^t)) \qquad \text{OKAY}$$

Now you have D on Q, RQ. Get L_D on $X(RQ)$. Extend D to Q^t

$$Q^t \subset T' \otimes Q$$

~~Notes~~ How D enters. Now consider D.

Try this way. Start with $Q = \bigoplus Q_n$, $Q_{\geq k} = \bigoplus_{n \geq k} Q_n$, define D on Q by $D = n$ on Q_n. Extend D as derivation on RQ, L_D on $X(RQ)$. grading on Q as v.s. with 1 induces gradings on alg RQ and on scx $X(RQ)$. Identify degree operators.

$$Q \xrightarrow{t^D} Q^t \qquad \text{extend to isom of graded } T\text{-modules}$$

$$T \otimes Q \xrightarrow{\sim} Q^t$$

induces

$$\begin{array}{ccc} R_T(T \otimes Q) & \xrightarrow{\sim} & R_T(Q^t) \\ \shortparallel & & \shortparallel \\ T \otimes RQ & & (RQ)^t \end{array}$$

This is the extension of $t^D : RQ \longrightarrow (RQ)^t$ to a T module map.

$$T \otimes X(RQ) = X(T \otimes RQ) \xrightarrow{\sim} X_T((RQ)^t) = X(RQ)^t$$

J) In practice this means that $X(RQ)_{\geq k} = \bigoplus_{n\geq k} X(RQ)_n$, where $L_D = n$ on $X(RQ)_n$.

Next point?

I guess you really [redacted] want to understand L_D and h_D on $X(RQ)$. Observe that

$$X(RQ) \xrightarrow{t^{L_D}} X(RQ)^t$$

$$\dot{L}_D \qquad\qquad t\partial_t, L_D$$

$$T \otimes X(RQ) \xrightarrow{\sim} X(RQ)^t \subset T' \otimes X(RQ)$$

$$t\partial_t * L_D \qquad\qquad t\partial_t$$

8/13 – 05 20 ~~Consider $X(RQ)$~~

~~Suppose you understand~~

~~$Q^t = \bigoplus t^k Q_{\geq k} \subset T' \otimes Q$~~

~~graded T-subalgebra~~

last map (trace map).

θ, θ' induce $Q \longrightarrow L \otimes B$ homs of
$Q_{\geq k} \longrightarrow J^k \otimes B$ filt. algs.

~~get~~ i.e. hom. $Q^t \longrightarrow L^t \otimes B$ of gr T-algs

. get

$$\begin{array}{ccccccc}
X(RQ)^t & = & X_T(R_T Q^t) & \longrightarrow & X_{L^t}(R_{L^t}(L^t \otimes B)) & = & L^t_{\natural} \otimes \cancel{X}(RB) \\
\wr| & & \wr| & & \wr| & & \wr| \\
(\Omega Q)^t & = & \Omega_T(Q^t)\otimes_T & \longrightarrow & \Omega_{L^t}(L^t \otimes B)\otimes_{L^t} & = & L^t_{\natural} \otimes \Omega B
\end{array}$$

K)

canon. lin. isom. $\Longrightarrow$ $X \circ R \simeq \Omega$

$$\begin{tikzcd} X(RQ)_{\geq k} \arrow[r, "\text{last map}"] \arrow[d, "\wr"] & J^k_{\#} \otimes X(RB) \arrow[d, "\wr"] \\ \Omega Q_{\geq k} \arrow[r, "\text{trace map of Nistor}"'] & J^k_{\#} \otimes \Omega(B) \end{tikzcd}$$

Also the bottom map in square compat. with Hodge filtration i.e. $F^p(\Omega Q)^t \longrightarrow L^t_{\flat} \otimes F^p \Omega B$

whence $\quad F^p X_{\geq k} \longrightarrow J^k_{\#} \otimes F^p_{IB}$.

So now I understand the last map and trace map. But now I have to ~~understand the role~~ bring in D.

bring in D. Here use $X(RQ)$ depends on Q as vect. sp with 1. Grading on Q induces gradings on RQ, $X(RQ)$. Degree ops D, L_D. First claim is consistency of the induced grading + filtration. Because

$$Q_{\geq k} = \bigoplus_{n \geq k} Q_n$$

it follows that

$$RQ_{\geq k} = \bigoplus RQ_n$$

Why is this true? Matter of defn. How is $X_{\geq k} = X(RQ)_{\geq k}$ defd? Concretely what happens is this:

$X_{\geq k}$ ~~is spann~~ $\simeq$ $\Omega_{\geq k}$

$\therefore$ $X_{\geq k}$ spanned by $\rho(x_0)\,\omega^m(x_1, x_{2m}) \begin{cases} 1 \\ d\rho(x_{2m+1}) \end{cases}$ $\quad \sum \operatorname{ord} x_i \geq k$

4) X_n spanned by

$$\rho(x_1)\cdots\rho(x_m)\begin{cases}1\\ d\rho(x_{m+1})\end{cases} \qquad \sum |x_i| = n$$

question is why $X_{\geq k} = \bigoplus_{n\geq k} X_n$

Enough to consider $R_{\geq k}$ and $R'_{\geq k} = \bigoplus_{n\geq k} R_n$

$$\omega(x_1, x_2) = \rho(x_1 x_2) - \rho(x_1)\rho(x_2)$$

$$\therefore \quad R_{\geq k} \subset R'_{\geq k}$$

But $\rho(Q_n)\, R_{\geq k} \subset R_{\geq k+n}$

$$\Rightarrow \quad R'_{\geq l}\, R_{\geq k} \subset R_{\geq l+k} \qquad \text{etc.}$$

Let's try a more abstract proof.

$$T \otimes Q \xrightarrow{\sim} Q^t$$
$$t^{-k} \otimes x \longmapsto t^{-k}\, t^D x.$$

is om. of graded T modules resp. 1.

induces

$$\begin{tikzcd}
R_T(T\otimes Q) \arrow[r, "\sim"] & R_T(Q^t) \arrow[r] & R_{T'}(T'\otimes Q) \\
T\otimes RQ \arrow[u, "\wr"] \arrow[r] & (RQ)^t \arrow[u] \arrow[r, phantom, "\subset"] & T'\otimes RQ \arrow[u, "\wr"]
\end{tikzcd}$$

$$T\otimes Q \xrightarrow{\sim} Q^t \subset T'\otimes Q$$

$$\begin{tikzcd}
R_T(T\otimes Q) \arrow[r, "\sim"] & R_T(Q^t) \arrow[r] & R_{T'}(T'\otimes Q) \\
T\otimes RQ \arrow[u, "\cong"] & & T'\otimes RQ \arrow[u, "\cong"]
\end{tikzcd}$$

M)

$$\begin{tikzcd} \overset{x}{Q} \arrow[r, "1\otimes"] & T\otimes Q \arrow[r, "\sim"] & \overset{t^D x}{Q^t} \arrow[r, phantom, "\subset"] & T'\otimes Q \end{tikzcd}$$

$$\begin{tikzcd} RQ \arrow[r] \arrow[dr, "1\otimes"'] & R_T(T\otimes Q) \arrow[r, "\sim"] & R_T(Q^t) \arrow[r] & R_{T'}(T'\otimes Q) \\ & T\otimes RQ \arrow[u, "\wr"] \arrow[r, "\sim"] & (RQ)^t \arrow[r, phantom, "\subset"] & T'\otimes RQ \arrow[u, "\wr"] \end{tikzcd}$$

logic? At the moment Q is only a graded vector space with $1 \in Q_0$, ~~This enables you to prove~~ and $Q_{\geq k} \overset{def}{=} \bigoplus_{n \geq k} Q_n$.

$$\text{equiv. to} \quad \begin{tikzcd} T\otimes Q \arrow[r, "\sim"] & Q^t \arrow[r, phantom, "\subset"] & T'\otimes Q \\ 1\otimes x \arrow[rr, mapsto] & & t^D x \end{tikzcd}$$

Then

$$\begin{tikzcd} RQ \arrow[r] & R_T(T\otimes Q) \arrow[r, "\sim"] \arrow[d, phantom, "\shortparallel"] & R_T(Q^t) \arrow[r] & R_{T'}(T'\otimes Q) \arrow[d, phantom, "\shortparallel"] \\ & T\otimes RQ & & T'\otimes RQ \end{tikzcd}$$

The first thing ~~that happens~~ to consider is the grading. This is unclear because you mix the grading & filtration

You have $Q \xrightarrow{t^D} T'\otimes Q$ hom resp 1

induces

$$\begin{tikzcd} RQ \arrow[r] \arrow[dr] & R_{T'}(T'\otimes Q) \\ & T'\otimes RQ \arrow[u, "\wr"] \end{tikzcd}$$

a homom. $\ni \rho(x) \longmapsto \rho(t^D x)$, so we get a grading on RQ ~~as subalg~~. compat. with alg structure.

N) Repeat: You say that RQ depends only on Q as vector space with d, and this means the linear grading on Q induces an alg. grading on RQ. Why?

$$Q \xrightarrow{t^D} T' \otimes Q \qquad \text{lin. resp } 1$$

induces

$$\begin{tikzcd} RQ \arrow[r] \arrow[dr, "X"'] & R_{T'}(T' \otimes Q) \\ & T' \otimes RQ \arrow[u, "s"'] \end{tikzcd}$$

so we get a homom.

$$RQ \xrightarrow{X} T' \otimes RQ$$

$$\rho(x) \longmapsto \rho(t^D x) = t^{|x|}\rho(x) \qquad x \text{ hom.}$$

this homom. inj (set $t=1$)
and the image is a graded subalgebra isomorphic to RQ under ~~f~~ the specialization map $T' \otimes RQ \longrightarrow RQ$. Thus RQ ~~has~~ has the structure of a graded alg. $\ni X = t^D$.

Now that the grading is straight I consider the filtration.

$$T \otimes Q \longrightarrow T' \otimes Q \qquad \text{graded } T\text{-module map } x \mapsto t^D x$$

induces

$$\begin{array}{ccc} R_T(T \otimes Q) & \longrightarrow & R_{T'}(T' \otimes Q) \\ \| & & \| \\ T \otimes RQ & \longrightarrow & T' \otimes RQ \end{array} \qquad \text{graded } T\text{-alg map}$$

0) grading works as follows

$$Q \xrightarrow{t^D} T' \otimes Q \quad \text{induces hom.}$$

$$RQ \longrightarrow R_{T'}(T' \otimes Q) = T' \otimes RQ$$

$$\rho(x) \longmapsto t^{|x|}\rho(x) \qquad \times \text{ hom.}$$

This is a lifting of RQ to a graded subalg of $T' \otimes RQ$, lifting means wrt specialization

$$T' \otimes RQ \longrightarrow RQ \qquad t \mapsto 1.$$

Similarly get map of scxs.

$$X(RQ) \longrightarrow X_{T'}(R_{T'}(T' \otimes Q)) = T' \otimes X(RQ)$$

This is a lifting of $X(RQ)$ to a graded ~~super~~ subcomplex of $T' \otimes X(RQ)$.

But now what about the filtrations assoc to these gradings. By defn we have

$$T \otimes Q \xrightarrow{\sim} Q^t \subset T' \otimes Q$$

$$T \otimes RQ \xrightarrow{\sim} (RQ)^t \subset T' \otimes RQ$$

$$T \otimes X(RQ) \xrightarrow{\sim} X(RQ)^t \subset T' \otimes X(RQ).$$

provided $Q_{\geq k}$, $(RQ)_{\geq k}$, $X(RQ)_{\geq k}$ defined in terms of the grading, e.g. $(RQ)_{\geq k} = \bigoplus_{n \geq k} (RQ)_n \quad \forall k$.

But then it follows that we have

p) It follows that we have

$$Q^t \subset T' \otimes Q$$

inducing isomorphisms.

$$R_T(Q^t) \xrightarrow{\sim} (RQ)^t \qquad X_T(R_T(Q^t)) \xrightarrow{\sim} X(RQ)^t$$

Why. because:

$$T \otimes Q \xrightarrow{\sim} Q^t \subset T' \otimes Q$$

$$\begin{array}{ccccc} R_T(T\otimes Q) & \xrightarrow{\sim} & R_T(Q^t) & \longrightarrow & R_{T'}(T'\otimes Q) \\ \| & & & & \| \\ T \otimes RQ & \xrightarrow{\sim} & (RQ)^t & \subset & T' \otimes RQ \end{array}$$

So where are we now? ~~Yesterday~~

Past stages involve

recall of Nistor ~~[illegible]~~ N.8

end map N.11

grading N.12

So now what next. I have I think understood why

$$T \otimes X(RQ) \xrightarrow{\sim} X_T(R_T(Q^t)) \xrightarrow{\sim} X(RQ)^t \subset T' \otimes X(RQ)$$

So now I can examine the behavior of L_D, h_D.

L_D, h_D operators on $X_T(R_T(Q^t))$

The point is that we have L_D, h_D acting on $X(RQ)$

5) So what do we do next.
Identify my map with

$$X(RA) \xrightarrow{\iota_*} X(RQ) \xrightarrow{P_m(L_D)\gamma_-} \gamma_- X_{\geq 2m+1} \xrightarrow{\ell} J_{\#}^{2m+1} \otimes X(RB)$$

I have to list the identifications.
~~[illegible]~~

$$\begin{tikzcd}
X(RA) \arrow[r, "\iota_*"] & X(RQ) \arrow[r, Rightarrow] \arrow[d, "\gamma_-"] & S_q \otimes X(RB) \arrow[d, "\pi_-"] \\
& \gamma_- X \arrow[d, equal] & \pi_- S_q \otimes X(RB) \arrow[d, equal] \\
& \gamma_- X_{\geq 1} \arrow[d, "1 - L_D"] & \pi_- S_{q,\geq 1} \otimes X(RB) \arrow[dd, "1 - t\partial_t"] \\
& \gamma_- X_{\geq 2} \arrow[d, equal] & \\
& \gamma_- X_{\geq 3} & \pi_- S_{q,\geq 3} \otimes X(RB) \\
& \vdots & \\
& \gamma_- X_{\geq 2m+1} \arrow[r] \arrow[dr, "\ell"'] & \pi_- S_{q,\geq 2m+1} \otimes X(RB) \arrow[d, "S_1"] \\
& & J_{\#}^{2m+1} \otimes X(RB)
\end{tikzcd}$$

Let's be explicit about the maps.

$$\begin{tikzcd}
A \arrow[r, "\iota", "\text{homo.}"'] & Q \arrow[r, "t^D", "\text{lin. map}"'] & Q^t \arrow[r, "\text{homo of graded } T\text{-algs.}"'] & L^t \otimes B \\
& D & t\partial_t & t\partial_t
\end{tikzcd}$$

$$\begin{tikzcd}
X(RA) \arrow[r, "\iota_*"] & X(RQ) \arrow[r, "t^{L_D}"] & X(RQ)^t \arrow[r] & L^t_q \otimes X(RB) \\
& L_D & t\partial_t & t\partial_t
\end{tikzcd}$$

T) Now what??

I think you must go back to the ~~[illegible]~~ trace map. It is defined by ~~taking~~ stapling the homom.

$$Q^t \longrightarrow L^t \otimes B$$

getting

$$\begin{array}{ccc} X_T(R_T(Q^t)) & \longrightarrow & X_{L^t}(R_{L^t}(L^t \otimes B)) \\ \| & & \| \\ X(RQ)^t & & L^t_\flat \otimes X(RB) \end{array}$$

and restricting to degree k to get

$$X(RQ)_{\geq k} \longrightarrow J^k_\# \otimes X(RB).$$

Other point is that we have a T-module map

$$T \otimes X(RQ) \xrightarrow[\sim]{t^{L_D}} X(RQ)^t \longrightarrow L^t_\flat \otimes X(RB)$$

which means that all we need is to take

$$X(RQ)_\bullet \longrightarrow X(RQ)^t \longrightarrow L^t_\flat \otimes X(RB)$$

What I am trying to say is that

$$X(RQ)^t \longrightarrow L^t_\flat \otimes X(RB)$$

is a T-module homom., the former is free over T generated by $t^{L_D} X(RQ) = \bigoplus t^n X(RQ)_n$

Thus $\quad X(RQ)_{\geq k} = \bigoplus_{n \geq k} X_n$

$$t^k X(RQ)_{\geq k} = \bigoplus_n t^{k-n}(t^n X_n)$$

$$\begin{array}{ccc} t^n X_n & \longrightarrow & t^n J^n_\# \otimes X(RB) \\ \cong \downarrow (t^{-1})^{n-k} & & \downarrow (t^{-1})^{n-k} \\ t^k X_n & \longrightarrow & t^k J^k_\# \otimes X(RB) \end{array}$$

u) So what am I ~~[illegible]~~ led to next

Let's set it up. ~~Claim~~

$$A \xrightarrow{\iota} Q \longrightarrow S\otimes B$$

$A \longrightarrow S\otimes B$ is the composition

$$\begin{tikzcd}
A \arrow[r, "\iota"] & Q \arrow[r, "t^D"] & \oplus\, Q^{t,\geq 0} \arrow[r] \arrow[d, phantom, "\cap"] & S\otimes B \arrow[d, phantom, "\cap"] \\
 & & Q^t \arrow[r] & S^t \otimes B
\end{tikzcd}$$

which gives commutativity

$$X(RA) \longrightarrow X(RQ) \longrightarrow S_t \otimes X(RB)$$

$$X(RA) \longrightarrow X(RQ) \longrightarrow \bigoplus_{k\geq 0} t^k X(RQ)_{\geq k} \longrightarrow S\otimes X(RB)$$

So why am I confused at this point?

We [illegible] from

Idea for tomorrow

~~Consider~~ Describe X version of Nistor construction.

grading on Q $\Longrightarrow$ gradings on RQ, X(RQ)

$\Longrightarrow$ filtration on RQ, X(RQ).

V) 8/14 – 0536

X version of what Nistor does

$Q = QA = \Omega A$ with $\circ$, $Q = \oplus Q_n$ graded on v.s. $Q_n = \Omega^n A$

RQ, $X(RQ)$ inherit gradings

$X(RQ)_n$ ~~spanned~~ spanned by $\rho(x_1)\cdots\rho(x_g)$
$\natural(\rho(x_1)\cdots\rho(x_j)\,d\rho(x_{j+1}))$

$\sum |x_i| = n$.

assoc. filtration $X(RQ)_{\geq k} = \bigoplus_{n \geq k} X(RQ)_n$

spanned by above elts with $\sum \mathrm{ord}(x_i) \geq k$

D on RQ, L_D on $X(RQ)$, h_D

Next ~~defn~~ can I define $(IQ)_{\geq k}$?

~~Can I define~~ $IQ = \mathrm{Ker}(RQ \longrightarrow Q)$

$$(IQ)_{\geq k} = IQ \cap (RQ)_{\geq k}$$

define then

$$(IQ^n)_{\geq k} = \sum_{k_1+\cdots+k_n=k} (IQ)_{\geq k_1} \cdots (IQ)_{\geq k_n}$$

$$\left(F^{2m}_{IQ} X(RQ)\right)_{\geq k} = (IQ^{m+1})_{\geq k} + \sum_{k_1+k_2=k} [(IQ)^m_{\geq k_1}, (RQ)_{\geq k_2}]$$

$$\rightleftarrows \sum_{k_1+k_2=k} \natural\left((IQ^m)_{\geq k_1}\, d(RQ)_{\geq k_2}\right)$$

defines a bifiltration $F^p X_{\geq k}$ of $X = X(RQ)$

can state lemma

$L_D, h_D : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k}$ $\qquad \gamma - (-1)^k : F^p X_{\geq k} \to F^p X_{\geq k+1}$.

$L_D - k : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k+1}$

W) Other lemma is under the rain.

$X(RQ) \simeq \Omega Q$

$F^p X_{\geq k} \simeq F^p (\Omega Q)_{\geq k}$.

and the canonical htpy equiv. $X(RQ) \sim \Omega Q$ induces a heq $F^p X_{\geq k} \sim F^p(\Omega Q)_{\geq k}$.

Next the trace map. ~~You have~~

θ, θ' induce $Q \longrightarrow L \otimes B$

$\ni$ $Q_{\geq k} \longrightarrow J^k \otimes B$

$$\left.\begin{array}{l} RQ \longrightarrow L \otimes RB \\ RQ_{\geq k} \longrightarrow J^k \otimes RB \end{array}\right) \text{NO}$$

needs work

~~Let~~ Instead like p+tq you form

$Q \longrightarrow S \otimes B$ lin. resp. 1 resp. grading

$X(RQ) \longrightarrow S_{t} \otimes X(RB)$

third lemma says ~~$F^pX_{\geq k}$~~ $F^p X_{\geq k} \longrightarrow J^k_{\#} \otimes F^p_{IB}$.

8/15 - 0538 Review program

1) my construction $X(RA) \longrightarrow J^{2m+1}_{\#} \otimes X(RB)$

F^p_{IA} $\qquad J^{2m+1}_{\#} \otimes F^{p-2m}_{IB}$

2) version of Nistor construction

$\iota_k \in HC^0(\gamma_-(\Omega Q)_{\geq k+1}, \gamma_-(\Omega Q)_{\geq k})$

$\exists$ $S_k \in HC^2(\gamma_-(\Omega Q)_{\geq k}, \gamma_-(\Omega Q)_{\geq k+1})$

unique up to Ker S $\qquad S_k \iota_k = S$, $\iota_k S_k = S$.

$$\Omega A \xrightarrow{\iota_*} \Omega Q \xrightarrow{\gamma_-} \gamma(\Omega Q)_{\geq 1} \xrightarrow{S_1} \gamma_-(\Omega Q)_{\geq 3} \xrightarrow{S_{2m-1}} \gamma(\Omega Q)_{\geq 2m+1}$$ defines

X) triv. Ch. char of universal quasi-homom.
$$ch^{2m}(\iota, \iota^\gamma) \in HC^{2m}(\Omega A, K(\Omega Q)_{\geq 2m+1})$$

trace ~~map~~ map $\qquad \Omega Q_{\geq k} \longrightarrow J^k_\# \otimes \Omega B$

class $\ell_k \in HC^0(\Omega Q_{\geq k}, J^k_\# \otimes \Omega B)$

~~[illegible]~~ put
$$ch^{2m}(\theta, \theta') = \ell_{2m+1}\, ch^{2m}(\iota, \iota^\gamma).$$
$$\in HC^{2m}(\Omega A, J^{2m+1}_\# \otimes \Omega B).$$

3) X-version of Nistor construction: ~~[illegible]~~
~~Take case~~

X-version of Nistor construction:

Consider $X = X(RQ)$

grading $\quad Q = \oplus Q_n$

inherited gradings on RQ, $X(RQ)$

D, L_D, h_D. $\qquad$ also γ

assoc. filtration $\quad (RQ)_{\geq k}$, $X(RQ)_{\geq k} = X_{\geq k}$.

$$(IQ)_{\geq k} = IQ \cap (RQ)_{\geq k}.$$

$$(IQ^n)_{\geq k} =$$

$$b(IQ^n d(RQ))_{\geq k} =$$

<u>Lemma 1.</u> $\quad L_D, h_D : F^pX_{\geq k} \longrightarrow F^{p-2}X_{\geq k}$

$\qquad L_D - k : F^pX_{\geq k} \longrightarrow F^{p-2}X_{\geq k+1}$

$\qquad \gamma - (-1)^k : \quad \dots \quad \dots \quad F^pX_{\geq k+1}$

$$R^t = \oplus t^k R_{\geq k} \subset T' \otimes R$$

$$I^t = \oplus t^k (I \cap R_{\geq k}) \quad \cancel{T' \otimes R}$$
$$= (T' \otimes I) \cap R^t \subset T' \otimes I$$

y) What's confusing is that I is an arbitrary ideal. We need to know

$$X_T(R^t) \quad \text{is torsion-free over } T$$

because then the local. map

$$X_T(R^t) \longrightarrow X_{T'}(T' \otimes_T R^t)$$
$$\| $$
$$X_{T'}(T' \otimes R)$$
$$\|$$
$$T' \otimes X(R)$$

is injective, identifying

$$X_T(R^t) \simeq X^t$$

But then we have an ideal $I^t \subset R^t$ and

$$(F^p X)^t \underset{\text{def}}{=} F^p_{I^t} X_T(R^t)$$

Then all we need is a relative version of $h_D : F^p_I \subset F^{p-2}_I$.

We do need $\phi : R^t \longrightarrow \Omega^2_T(R^t)$
torsion-free

i.e. want $\phi : R \longrightarrow \Omega^2 R$ to be compatible with filtration.

$$L_D - k : I_{\geq k} \longrightarrow R_{\geq k+1}$$

because $I_{\geq k}/I_{\geq k+1} \subset \underbrace{R_{\geq k}/R_{\geq k+1}}_{\text{killed by } L_D}$

or better: $(L_D - k)(I_{\geq k}) \subset (L_D - k)(R_{\geq k}) \subset R_{\geq k+1}$

Note you don't need ~~$L_D - k$~~ $\quad I_{\geq k} = I \cap R_{\geq k}$

2) only that $R_{\geq k} \cdot I_{\geq \ell} \subset I_{\geq k+\ell}$ etc

so that I^t is an ideal in R^t.

~~Point is that~~ You went over all this before

3) X version of Nistor's construction.

Consider $X = X(RQ)$.

grading on Q as v.s. induces gradings on RQ, X

D, L_D, h_D

assoc. filt $X_{\geq k}$

$I_{\geq k} = IQ \cap RQ_{\geq k}$

$(I^n)_{\geq k} \overset{def}{=}$

$\natural (I^n dR)_{\geq k} \overset{defn}{=}$

$F^p X_{\geq k} =$

Lemma 1: behavior of L_D, h_D, γ wrt $F^p X_{\geq k}$

Recall
$X \cong \Omega$ canonical v.s. isomorphism
$X \sim \Omega$ Canonical homotopy equivalence

Lemma 2. canon htpy equiv $X \sim \Omega$ induces heq
$F^p X_{\geq k} \sim F^p \Omega_{\geq k}$ $\forall p, k$.

---

final step - trace maps 1140

$\theta, \theta' : A \longrightarrow L \otimes B$

induce
$$\begin{array}{ccc} Q & \longrightarrow & L \otimes B \\ \cup & & \cup \\ Q_{\geq k} & & J^k \otimes B \end{array}$$
homo. of filtered algs.

~~An~~ Let's try to understand what is needed.

I have supposedly identified Nistor's $ch^{2m}(\epsilon, \iota^\gamma)$

with $X(RA) \xrightarrow{u_*} X(RQ) \xrightarrow{P_m(L_D)\gamma_-} \gamma_- X(RQ)_{\geq 2m+1}$.

$\boxed{A}$ and its filtration behavior.

i.e. $F^p_{IA} \longrightarrow \gamma_- F^{p-2m} X_{\geq 2m+1}$

(so that we get $X_A \to X_Q \longrightarrow \gamma_- X_{\geq 2m+1}[2m]$)

Anyway I now need the X-version of the trace map. There's are still some problems here.

So let's review. I have to get agreement with my construction:

$$\begin{tikzcd} X(RA) \arrow[r] \arrow[d] & X(S\otimes RB) \arrow[r, "\alpha"] \arrow[d, equal] & S_q \otimes X(RB) \arrow[r] \arrow[d, equal] & J^{2m+1}_\# \otimes X(RB) \\ X(RQ) \arrow[r] & X(S\otimes RB) \arrow[r, "\alpha"] & S_q \otimes X(RB) & \end{tikzcd}$$

So far we have used

$$\begin{tikzcd} A \arrow[r, "p+t\xi"] \arrow[d, "\iota"] & S\otimes B \\ Q \arrow[ur, "w"] & t\partial_t \\ D & \end{tikzcd}$$

w linear resp 1. compat with grading

leads to

$$\begin{tikzcd} X(RA) \arrow[d] \arrow[dr, "(p+t\xi)_*"] & & \\ X(RQ) \arrow[r, "w_*"] \arrow[d, "P_m(L_D)\gamma_-"] & X(S\otimes RB) \arrow[r, "\alpha"] & S_q \otimes X(RB) \arrow[d, "P_m(t\partial_t)\gamma_-^t"] \\ X(RQ)_{\geq 2m+1} \arrow[rr] & & S_{q,\geq 2m+1} \otimes X(RB) \arrow[d, "ev_1"] \\ & & J^{2m+1}_\# \otimes X(RB) \end{tikzcd}$$

B] critical point to understand is why

$$\begin{tikzcd}
X(RQ) \arrow[r, "\omega_*"] & S_q \otimes X(RB) \\
X(RQ)_{\geq k} \arrow[u, phantom, "\cup"] \arrow[r] \arrow[dr, bend right, "tr"'] & S_{q,\geq k} \otimes X(RB) \arrow[u, phantom, "\cup"] \arrow[d, "\omega_1"] \\
 & J^k_{\#} \otimes X(RB)
\end{tikzcd}$$

commutes.

But first I have to define trace map.

$$\begin{aligned} Q &\longrightarrow L \otimes B \\ Q_{\geq k} &\longrightarrow J^k \otimes B \end{aligned} \qquad \text{filt.}$$

$$Q^t \longrightarrow L^t \otimes B \qquad \text{hom. of gr } T\text{-alg.}$$

$$\begin{tikzcd}
R_T Q^t \arrow[r] \arrow[d, equal] & R_{L^t}(L^t \otimes B) \arrow[d, equal] \\
RQ^t \arrow[r] & L^t \otimes RB
\end{tikzcd}$$

$$\begin{tikzcd}
X_T(RQ^t) \arrow[r] \arrow[d, equal] & X_{L^t}(L^t \otimes RB) \arrow[d, equal] \\
X(RQ)^t \arrow[r] \arrow[d, "\int"] & L^t_q \otimes X(RB) \arrow[d, "\int"] \\
(\Omega Q)^t \arrow[r] & L^t_q \otimes \Omega B
\end{tikzcd}$$

~~what next~~

So we know that $Q^t \xrightarrow{\omega^t} L^t \otimes B$ induces the <u>trace map</u> after applying the relative $X \cdot R$ in the two cases.

c) So now I have X version of the trace map of Nistor.

So what should be possible is to ~~state~~ describe following picture

$$\underbrace{X(RA) \longrightarrow X(RQ) \longrightarrow \underline{X}(RQ)}_{ch^{2m}(\iota, \iota^\gamma)}\underbrace{{}_{\geq 2m+1} \longrightarrow J_\#^{2m+1} \otimes X(RB)}_{\text{trace map}}$$

and its filtration behavior.

So I define the trace map as induced by the homom.: $Q^t \longrightarrow L^t \otimes B$

$$RQ^t = R_T(Q^t) \longrightarrow R_{L^t}(L^t \otimes B) = L^t \otimes RB$$
$$IQ^t \qquad\qquad\qquad\qquad L^t \otimes IB$$

$$\begin{array}{ccc} X_T(RQ^t) & \longrightarrow & X_{L^t}(L^t \otimes RB) \\ \| & & \| \\ X(RQ)^t & & L_\natural^t \otimes X(RB) \\ \cup & & \cup \\ F^p_{IQ^t} X(RQ)^t & \longrightarrow & L_\natural^t \otimes F^p_{IB} X(RB). \\ \| & & \\ (F^p X)^t & & \end{array}$$

1409 Start again to clear up things.

~~#~~ X-version of trace map.

hom. of filtered algs $Q \longrightarrow L \otimes B$, $Q_{\geq k} \longrightarrow J^k \otimes B \ \forall k$

hom of graded alg. $Q^t \longrightarrow L^t \otimes B$

compatible with evident hom. $T \longrightarrow L^t$.

So get $X_T(R_T(Q^t)) \longrightarrow X_{L^t}(R_{L^t}(Q^t))$

D] such that

$$F^{p}_{I_T Q^t} X_T(R_T(Q^t)) \longrightarrow F^{p}_{L^t \otimes IB} X_{L^t}(L^t \otimes RB)$$

i.e.

$$\begin{array}{ccc} X(RQ)^t & \longrightarrow & L^t_\natural \otimes X(RB) \\ \cup & & \cup \\ (F^p X)^t & \longrightarrow & L^t_\natural \otimes F^p_{IB} \end{array}$$

I like the following

hom of alg.

$$Q \longrightarrow L \otimes B \quad , \quad Q_{\geq k} \longrightarrow J^k \otimes B \quad \forall k$$

gives hom. of gr alg.

$$Q^t \longrightarrow L^t \otimes B \qquad \text{comp. with } T \to L^t$$

yields

$$\begin{array}{ccc} R_T(Q^t) & \longrightarrow & R_{L^t}(L^t \otimes B) \\ \| & & \| \\ (RQ)^t & \longrightarrow & L^t \otimes RB \end{array}$$

such that $(IQ)^t \longrightarrow L^t \otimes IB$

then yields

$$\begin{array}{ccc} X_T((RQ)^t) & \longrightarrow & X_{L^t}(L^t \otimes RB) \\ \| & & \| \\ X(RQ)^t & & L^t_\natural \otimes X(RB) \end{array}$$

such that $F^p_{IQ^t} X(RQ)^t \longrightarrow L^t_\natural \otimes F^p_{IB} X(RB)$

Thus get $\forall k$

$$X(RQ)_{\geq k} \longrightarrow J^k_\# \otimes X(RB)$$

s.t.

$$(F^p X)_{\geq k} \longrightarrow J^k_\# \otimes F^p_{IB}$$

E] But in fact we have the extra information that $X(RQ)^t \longrightarrow L^t_{\natural} \otimes X(RB)$ is a T-module map and that

$$T \otimes X(RQ) \xrightarrow{\sim} X(RQ)^t$$

In other words $T \otimes Q \xrightarrow[\sim]{t^D} Q^t$

$$T \otimes RQ \xrightarrow[\sim]{t^D} RQ^t$$

$$T \otimes X(RQ) \xrightarrow[\sim]{t^{LD}} X(RQ)^t$$

So how can I describe this sensibly?

~~...~~ I know that I can also describe the trace map

$$X(RQ)^t \longrightarrow L^t_{\natural} \otimes X(RB)$$

as the unique T-module map extending the map as follows

$$Q \xrightarrow{t^D} Q^t \longrightarrow L^t \otimes B$$

linear resp 1

induces $RQ \xrightarrow{t^D} RQ^t \longrightarrow L^t \otimes RB$

induces $X(RQ) \xrightarrow{t^{LD}} X(RQ)^t \longrightarrow L^t_{\natural} \otimes RB$

better

~~$X\mathbb{B}$~~ $Q \longrightarrow L^t \otimes B$

~~$X(RQ)$~~

$RQ \longrightarrow R_{L^t}(L^t \otimes B) = L^t \otimes RB$

$X(RQ) \longrightarrow X_{L^t}(L^t \otimes RB) \xrightarrow{\sim} L^t_{\natural} \otimes X(RB)$.

F In concrete terms we have the sort of map like $p+tq$:

$$Q \longrightarrow Q^t \longrightarrow L^t \otimes B$$

which yields

$$X(RQ) \longrightarrow X(L^t \otimes RB) \longrightarrow L^t_q \otimes X(RB)$$

which yields

$$\begin{tikzcd} T \otimes X(RQ) \arrow[r, "\Phi"] \arrow[d, "S"] & L^t_q \otimes X(RB) \\ X(RQ)^t \arrow[ur, dashed] & \end{tikzcd}$$

can be understood as just

$$X(RQ)_n \subset X(RQ)_{\geq k} \longrightarrow J^k_\# \otimes X(RB) \longrightarrow J^n_\# \otimes X(RB)$$

for $n \geq k$. induced by $J^k \subset J^n$

1500

Let's go back & keep on trying to get to the bottom of things. ~~Start with~~

I have

$$X(RA) \longrightarrow X(RQ) \longrightarrow X(L^t \otimes RB) \longrightarrow L^t_q \otimes X(RB)$$

Try to list the steps to follow.

my map $A \xrightarrow{p+tq} S \otimes B$ induces

$$X(RA) \longrightarrow X_S(R_S(S \otimes B)) = S_q \otimes X(RB)$$

followed by μ_n.

[5] Keep on reviewing

my maps can be described as follows

One has $\quad p+tq : A \longrightarrow S\otimes B \quad$ lin. resp. 1.

This induces

$$X(RA) \longrightarrow X_S(R_S(S\otimes B)) = S_q \otimes X(RB)$$

which we can then follow by

$$\begin{tikzcd} S_q \otimes X(RB) \arrow[d, "\mu_m"] \\ J_\#^{2m+1} \otimes X(RB). \end{tikzcd}$$

I propose to factor $p+tq$ into

$$A \xrightarrow[\text{hom.}]{i} Q \xrightarrow[\text{lin. resp 1 and grading.}]{t^D} \underset{Q^{t,\geq 0}}{\text{[redacted]}} \xrightarrow[\text{hom. graded}]{w} S\otimes B$$

which induces

$$\begin{tikzcd} X(RA) \arrow[r, "\iota_*"] & X(RQ) \arrow[r] \arrow[dr, "t^{L_D}"'] & X(R(Q^{t,\geq 0})) \arrow[r] \arrow[d] & S_q \otimes X(RB) \arrow[d, phantom, "\Vert"] \\ & & X(RQ)^{t,\geq 0} \arrow[r, "w_*"] & S_q^\bullet \otimes X(RB) \end{tikzcd}$$

OK, so there's still this part I don't get,

~~Let us consider~~ namely the use of S versus L^t.

Let's go on. My map

$$X(RA) \longrightarrow X_S(R_S(S\otimes B)) = S_q \otimes X(RB)$$

factor $\quad p+tq : A \longrightarrow S\otimes B$

into $\quad A \xrightarrow{\iota} Q \xrightarrow[\text{lin. resp 1 and grading}]{} S\otimes B$

H] whence my map ω factored

$$\begin{tikzcd}
X(RA) \arrow[r] & X(RQ) \arrow[r] \arrow[d, "P_m(L_D)\gamma_-"] & X_S(R_S(S\otimes B)) = S_\flat \otimes X(RB) \arrow[d, "P_m(t\partial_t)\gamma^t_-"] \\
& \gamma_- X(RQ)_{\geq 2m+1} \arrow[r] & S_{\flat, \geq 2m+1} \otimes X(RB) \arrow[d, "\delta_1"] \\
& & J_\#^{2m+1} \otimes X(RB)
\end{tikzcd}$$

What is easy is the following

$$\begin{tikzcd}
X(RA) \arrow[r, "\iota_*"] & X(RQ) \arrow[r, "\omega"] \arrow[d, "P_m(L_D)\gamma_-"] & S_\flat \otimes X(RB) \arrow[d, "P_m(t\partial_t)\gamma^{\Sigma}_{-}"] \\
& X(RQ)_{\geq 2m+1} \arrow[r, "\omega_{\geq 2m+1}"] \arrow[dr, "\ell_{2m+1}"'] & S_{\flat, \geq 2m+1} \otimes X(RB) \arrow[d, "ev_1"] \\
& & J_\#^{2m+1} \otimes X(RB)
\end{tikzcd}$$

ω induced by

$$\begin{array}{ccc} Q & \longrightarrow & S\otimes B \\ a_0\, da_1 \cdots da_n & & t^n \rho a_0\, \varrho a_1 \cdots \varrho a_n \end{array}$$

lin. resp. 1. ::

I think I want to keep after this because there might be a cleaner version.

I) So what actually happens?

Go back to

$$Q \xrightarrow[\text{linear resp 1 \& grading}]{t^D} Q^t \xrightarrow[\text{homom.}]{\xi} L^t \otimes B$$

$$D \longleftrightarrow t\partial_t$$

$$\begin{array}{ccccc}
RQ & \longrightarrow & R_T Q^t & \xrightarrow{\xi_*} & R_{L^t}(L^t \otimes B) \\
\| & & \| & & \| \\
RQ & \xrightarrow{t^D} & (RQ)^t & \xrightarrow{\xi_*} & L^t \otimes RB
\end{array}$$

$$\begin{array}{ccccc}
X(RQ) & \longrightarrow & X_T((RQ)^t) & \longrightarrow & X_{L^t}(L^t \otimes RB) \\
\| & & \| & & \| \\
X(RQ) & \xrightarrow{t^{L_D}} & (X(RQ))^t & \xrightarrow{\xi_{**}} & L^t_\natural \otimes X(RB)
\end{array}$$

$$X(RA) \longrightarrow X(RQ) \xrightarrow{t^{L_D}} X(RQ)^t \longrightarrow L^t_\natural \otimes X(RB)$$

Concentrate on the fact that

$$Q \xrightarrow[\text{lin. resp. 1 and grading}]{} L^t \otimes B \qquad ?$$

~~$Q \to L^t \otimes B$.~~

$\therefore$ get $\quad X(RQ) \longrightarrow L^t_\natural \otimes X(RB) \qquad$ αv_* like αu_*

Here seems to be a point: Consider

$$Q \longrightarrow S \otimes B$$

linear resp 1 and grading.

J] This induces

$$X(RQ) \longrightarrow X(S \otimes RB) \longrightarrow S_q \otimes X(RB)$$

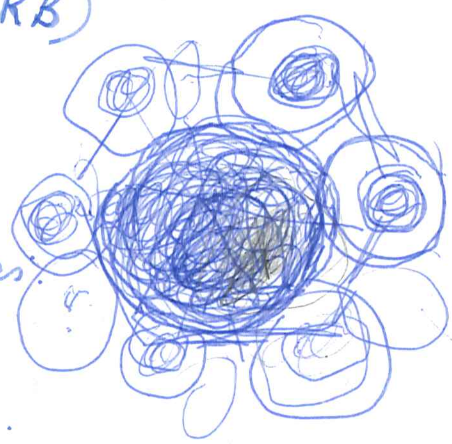
compatible with grading:

$$\begin{tikzcd} X(RQ) \arrow[r] \arrow[d, leftrightarrow, "L_D"] & S_q \otimes X(RB) \arrow[d, leftrightarrow, "t\partial_t"] \\ {} & {} \end{tikzcd}$$

<u>other point</u> is that S_q is a graded T-module, so we have extension

concentrate

$$\begin{tikzcd} T \otimes X(RQ) \arrow[r] \arrow[d, equal] & S_q \otimes X(RB) \\ X(RQ)^t & \end{tikzcd}$$



Somehow think along these lines:
You have $A \xrightarrow{p+tq} S \otimes B$
lin resp 1   graded alg.

$$\begin{tikzcd} RA \arrow[rr, "\text{hom.}"'] \arrow[dr] & & S \otimes RB \\ & \mathcal{D}(RA) \arrow[ur, "\text{graded}"'] & \end{tikzcd}$$

Actually what is $\mathcal{D}(RA)$? The $\mathbb{N}$ graded algebra generated by RA.
So what do we have??

K] 8/16 - 0524

go over steps - what to say

my map

$$X(RA) \to X(S\otimes RB) \longrightarrow S_{L} \otimes X(RB) \longrightarrow J_{\#}^{2m+1} \otimes X(RB)$$

$$A \xrightarrow{p+tq} S\otimes B$$

~~??~~ graded alg

$$RA \longrightarrow S\otimes RB$$

$$RA \longrightarrow RQ \longrightarrow S\otimes RB$$

I am beginning to think that I can use $\mathcal{D}(RA)$ instead of RQ. ~~So how does the~~

If A is a vector space with 1 then there is a corresponding ~~??~~ $\mathbb{N}$-graded vector space with 1 generated by A namely

$$\mathcal{D}'A = A \oplus \bar{A} \oplus \bar{A} \oplus \cdots = \mathbb{C}[t]\otimes A / \cancel{?}(t)\otimes 1_A$$

Then

$$\begin{tikzcd} A \arrow[rr, "p+tq"] \arrow[dr] & & S\otimes B \\ & \mathcal{D}'A \arrow[ur] & \end{tikzcd}$$

you do factor in this way

$$RA \longrightarrow R(\mathcal{D}'A) \longrightarrow RQ \longrightarrow S\otimes RB$$
$$R(\mathcal{D}'A) = \mathcal{D}(RA)$$

wait.

$$\begin{tikzcd} A \arrow[rr, "p+tq"] \arrow[d] & & S\otimes B \\ A\oplus\bar{A} \arrow[r] & \mathcal{D}'A \arrow[r] & Q \arrow[u] \end{tikzcd}$$

vector space level

OKAY

~~??~~ but I don't see F^p connected with $R(A\oplus\bar{A})$

L get back into the spirit, find what to say ...

$$A \xrightarrow{p+tq} S\otimes B$$

$$X(RA) \longrightarrow X_S(R_S(S\otimes B)) = S_\natural \otimes X(RB)$$

Recall what I liked yesterday about the trace map at the end.

homom. of filtered algs.

$$Q \longrightarrow L\otimes B \quad , \quad Q_{\geq k} \longrightarrow J^k \otimes B$$

hom. of gr alg.

$$Q^t \longrightarrow L^t \otimes B$$

$$\begin{array}{ccc} T & \longrightarrow & L^t \\ \downarrow & & \downarrow \\ Q^t & \longrightarrow & L^t\otimes B \end{array}$$

yields

$$\begin{array}{ccc} R_T(Q^t) & \longrightarrow & R_{L^t}(L^t\otimes B) \\ \parallel & & \parallel \\ (RQ)^t & & L^t \otimes RB \end{array}$$

s.t. $\quad (IQ)^t \longrightarrow L^t \otimes IB$

then get

$$\begin{array}{ccc} X_T((RQ)^t) & \longrightarrow & X_{L^t}(L^t\otimes RB) \\ \parallel & & \parallel \\ X(RQ)^t & & L^t_\natural \otimes X(RB) \end{array}$$

s.t. $\quad F^p_{(IQ)^t} X(RQ)^t \longrightarrow L^t_\natural \otimes F^p_{IB} X(RB)$

Thus we get $\quad X(RQ)_{\geq k} \longrightarrow J^k_\# \otimes X(RB) \quad \forall k$

$$\ni \quad F^p X_{\geq k} \longrightarrow J^k_\# \otimes F^p_{IB}$$

Note all done on filtered algebra level.

Next go back to factor of $p+tq$

$$A \xrightarrow{\iota} Q \longrightarrow$$

M]

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[rr, bend right, "p+tq"'] & Q \arrow[r, "r"] & S \otimes B \end{tikzcd}$$

$$\begin{tikzcd} X(RA) \arrow[r, "\iota_*"] & X(RQ) \arrow[r] \arrow[d] & S_{\natural} \otimes X(RB) \arrow[d] \\ & \gamma_- X(RQ)_{\geq 2m+1} \arrow[r] & \gamma_{t,-} S_{\natural, 2m+1} \otimes X(RB) \arrow[d] \\ & & J_{\#}^{2m+1} \otimes X(RB) \end{tikzcd}$$

to get straight

$$Q \xrightarrow{r} S \otimes B \subset L^t \otimes B \quad \text{linear resp } t.$$

$$Q \xrightarrow{t^D} Q^t \xrightarrow{w} L^t \otimes B$$

$$X(RQ) \xrightarrow{t^{hD}} X(RQ)^t \xrightarrow{\ell} L^t_{\natural} \otimes X(RB).$$

Let's go over the steps.

my map + filtration behavior yields

$$ch^{2m}(\theta, \theta') \in HC^{2m}(\mathcal{X}_A, J_{\#}^{2m+1} \otimes \mathcal{X}_B).$$

version of Nistor's construction

$$ch^{2m}(\iota, \iota') \in HC^{2m}(\Omega A, \gamma_-(\Omega Q)_{\geq 2m+1})$$

$$\mathcal{X}(\theta, \theta')_0 \in HC^0((\Omega Q)_{\geq 2m+1}, J_{\#}^{2m+1} \otimes \Omega B)$$

$$ch^{2m}(\theta, \theta') = \mathcal{X}(\theta, \theta') \cdot ch^{2m}(\iota, \iota') \in HC^{2m}(\Omega A, J_{\#}^{2m+1} \otimes \Omega B)$$

$\boxed{N}$ ~~exactly what is at stake~~

next point is to relate these two.

need X version of Nistor's construction:

~~$Q = \bigoplus Q_n$, $Q_n = \Omega^n A$~~

~~$Q_{\geq k} = \bigoplus_{n \geq k} Q_n$ filtration compat with alg st.~~

grading on Q induces grading on RQ, $X(RQ)$.

D, L_D, h_D. (canon. ϕ).

assoc. filtrations.

~~8/16~~ 8/16 - 1620

Outline

my construction $\rightsquigarrow ch^{2m}(\theta, \theta') \in HC^{2m}(\mathcal{X}_A, J_{\#}^{2m+1} \otimes \mathcal{X}_B)$

Nistor's " $\rightsquigarrow \left(\begin{array}{l} ch^{2m}(c, c') \in \\ ch^{2m}(\theta, \theta') \in H^{2m}(\Omega A, J_{\#}^{2m+1} \otimes \Omega B) \end{array} \right)$

X-version of Nistor's construction.

Up to now have considered ΩQ, $\Omega Q_{\geq k}$

Now you want to look at RQ and $X(RQ)$.

To define $X_{\geq k}$, $F^p X_{\geq k}$

$\natural(\rho x_1 \cdots \rho x_m \, d(\rho x_{m+1}))$ $\qquad \sum |x_i| = n.$

0| Grading undefined on ΩQ side.

Go over the points! ~~Pilot~~

At the moment I seem to be ~~more~~ inclined to doing the filtered version $X(RQ) = \Omega Q$ and $x(RQ) \sim \Omega Q$.

~~Define~~ So define $RQ_{\geq k}$

8/17 - 0514

my construction $A \xrightarrow{p+tq} S \otimes B$

$$X(RA) \longrightarrow X_S(R_S(S \otimes B)) = S_\natural \otimes X(RB) \longrightarrow J_\#^{2m+1} \otimes X(RB)$$

$$F^p_{IA} \qquad\qquad J_\#^{2m+1} \otimes F^{p-2m}_{IB}$$

$$\mathcal{X}_A \longrightarrow \mathcal{J}_\#^{2m+1} \otimes \mathcal{X}_B[2m]$$

$$ch^{2m}(\theta, \theta') \in HC^{2m}(\mathcal{X}_A, J_\#^{2m+1} \otimes \mathcal{X}_B).$$

~~So~~ next a version Nistor's construction

next a version of Nistor's construction

Let $Q = QA$ ~~concretely it is~~ be the free product $A * A$ in the cat of unital algebras. Let γ be the canonical autom of order 2 interchanging the two copies of A, let qA be the ~~canonical ideal~~ kernel of the ~~canonical~~ obvious hom. $A * A \longrightarrow A$. Recall Q can be identified with ~~the~~ ΩA equipped with Fed. product

$$x \circ y = xy - (-1)^{|x|} dx\,dy$$

~~Let~~ gu

P 0539

our version of Nistor's construction

$Q = QA,\ qA,\ \gamma$ resp. $A * A$ in the cat of unital algs
the kernel of $A * A \to A$
the canonical autom of order 2

$Q = \Omega A$ equipped with Fed prod ...

$\iota a = a + da$, $\iota^\gamma a = a - da$

~~$Q_n = \Omega^n A$, get grading of Q as vector space~~

~~$Q = \bigoplus_n Q_n$. $1 \in Q_0$~~

~~such that $\gamma = (-1)^n$ on Q_n and such that~~

$Q_n = \Omega^n A$, $Q_{\geq k} = \bigoplus_{n \geq k} Q_n$

8/18 Review again.

X version of Nistor's construction

$Q = \bigoplus Q_n$ $1 \in Q_0$ grading of Q as vector space

induces gradings $RQ = \bigoplus RQ_n$, $X(RQ) = \bigoplus_n X(RQ)_n$

$X(R)_n$ spanned by elts $\rho x_1 \cdots \rho x_m$ $\natural(\rho x_1 \cdots \rho x_m\, d(\rho x_{m+1}))$ $\sum |x_i| = n$
$*$

get filtrations $(RQ)_{\geq k}$ $X(RQ)_{\geq k}$

$X(RQ)_{\geq k}$ spanned by $*$ where $\sum \operatorname{ord}(x_i) \geq k$

---

start again 0617

X version of Nistor's construction

Recall $X(RQ) \simeq \Omega Q$ and $F^p_{IQ} X(RQ) = F^p \Omega Q$

also $X(RQ) \sim \Omega Q$, $F^p_{IQ} X(RQ) \sim F^p \Omega Q$

claim this extends to filtered algebras

$Q_{\geq i} \cdot Q_{\geq j} \subset Q_{\geq i+j}$ $1 \in Q_{\geq 0}$

Q] Define $X(RQ)_{\geq k}$ to be spanned by above elts. with $\sum \mathrm{ord}(x_i) \geq k$.

Define $(IQ)_{\geq k} = IQ \cap (RQ)_{\geq k}$

$$(IQ^m)_{\geq k} = \sum_{k_1+\cdots+k_m=k} (IQ)_{\geq k_1}\cdots(IQ)_{\geq k_m}$$

$$\natural\,(IQ^m\, dRQ)_{\geq k} = \sum_{k_1+k_2=k} \natural\,\big((IQ^m)_{\geq k_1}\, d(RQ)_{\geq k_2}\big)$$

$\natural\,(IQ^m\, dIQ)_{\geq k}$, $[IQ^m, IQ]_{\geq k}$ similarly

Put together to define $(F^p_{IQ}\, X(RQ))_{\geq k}$

abbreviations $X = X(RQ)$ $X_{\geq k} = X(RQ)_{\geq k}$

$R = RQ$, $I = IQ$, $F^pX = F^p_{IQ}\, X(RQ)$,

$F^pX_{\geq k} = (F^p_{IQ}\, X(RQ))_{\geq k}$

Claim then that the canon. identification $X \cong \Omega$ identifies $F^pX_{\geq k}$ with $F^p(\Omega Q_{\geq k})$. and that canon. htpy equiv. $X \sim Q$ induces a htpy eq. $F^pX_{\geq k} \sim F^p\Omega Q_{\geq k}$.

<u>Proof.</u> $T' = \mathbb{C}[t,t^{-1}]$ $T = \mathbb{C}[t^{-1}] \subset T'$

can identify filtration on V with graded T submodule of $T'\otimes V$.

R

$\Omega^n_T(Q^t)$ direct summand of $Q^t \otimes_T \cdots \otimes_T Q^t$

which embeds in $\Omega^n_{T'}(T'\otimes Q) = T'\otimes \Omega^n Q$

$$\therefore \quad \Omega_T(Q^t) \xrightarrow{\sim} (\Omega Q)^t \subset T'\otimes \Omega Q$$

Relative versions over T, T'.

$$\begin{array}{ccc}
X_T(R_T Q^t) & \cong & \Omega_T(Q^t) \\
\downarrow & & \downarrow \\
X_{T'}(R_{T'}(T'\otimes Q)) & = & \Omega_{T'}(T'\otimes Q) \\
\parallel & & \parallel \\
T'\otimes X(RQ) & & T'\otimes \Omega Q
\end{array}$$

gives $\quad X(RQ)^t = (\Omega Q)^t$

i.e. $\quad X(RQ)_{\geq k} = (\Omega Q)_{\geq k}$

Similarly

$$I_T(Q^t) = \operatorname{Ker}(R_T Q^t \longrightarrow Q^t)$$
$$= \operatorname{Ker}((RQ)^t \longrightarrow Q^t)$$
$$= \bigoplus_k t^k \operatorname{Ker}(RQ_{\geq k} \to Q_{\geq k})$$

$$(IQ)^t = \bigoplus_k t^k (IQ \cap (RQ)_{\geq k})$$

$$I_T(Q^t)^m = (IQ^t)^m = \left(\sum t^k (IQ)_{\geq k}\right)^m$$
$$= \sum_k t^k \sum_{k_1+\cdots+k_m} (IQ)_{\geq k_1} \cdots (IQ)_{\geq k_m}$$

so that

$$F^p_{I_T Q} X_T(R_T(Q^t)) = F^p \Omega_T(Q^t) = \bigoplus t^k F^p(\Omega Q)_{\geq k}$$

$$(F^p X)^t \qquad\qquad F^p(\Omega Q)^t =$$

S] 1036 Review earlier stuff.

Q filtered alg $\quad Q_{\geq i} \cdot Q_{\geq j} \subset Q_{\geq i+j}$, $1 \in Q_{\geq 0}$

$T' = \mathbb{C}[t, t^{-1}]$, $\quad T = \mathbb{C}[t^{-1}] \subset T'$.

$$Q^t = \bigoplus t^k Q_{\geq k}$$

Identify decreasing filtrations on V with a graded T-submodule of $T' \otimes V$.

A graded T submodule of $T' \otimes V$ has the form $\bigoplus_{k \in \mathbb{Z}} t^k V_{\geq k}$ where $(V_{\geq k})$ is a decreasing filtration of V.

If $(V_{\geq k})_{k \in \mathbb{Z}}$ is a decreasing filtration

Given a ~~vector space~~ vector space V equipped with decreasing filtn by subspaces $V_{\geq k}$, $k \in \mathbb{Z}$, ~~of~~ put

$$V^t = \bigoplus_k t^k V_{\geq k} \subset T' \otimes V.$$

~~This is a graded T-submodule of $T' \otimes V$.~~

Equivalence between decreasing filtrations on V and graded T-submodules STOP

1046. Q filtered alg, $Q_{\geq k}$

$$Q^t = \bigoplus t^k Q_{\geq k} \subset T' \otimes Q$$

$$\begin{array}{ccc} \Omega^n_T(Q^t) & \longrightarrow & \Omega^n_{T'}(T' \otimes Q) = T' \otimes \Omega^n Q \\ | & & | \text{ dir. sum. of} \\ \text{direct summand of } Q^t \otimes_T \overset{n}{\cdots} \otimes_T Q^t & \longleftrightarrow & T' \otimes Q^{\otimes n} \end{array}$$

$\therefore \quad \Omega^n_T(Q^t) \longrightarrow T' \otimes \Omega^n Q \quad$ inj.

$\boxed{T}$ whence $\Omega_T(Q^t) \xrightarrow{\sim} (\Omega Q)^t \subset T' \otimes \Omega Q$.

Logic: Have $Q^t \subset T' \otimes Q$

this induces

$$\Omega_T(Q^t) \longrightarrow \Omega_{T'}(T' \otimes Q) = T' \otimes \Omega Q$$

the image is $(\Omega Q)^t = \oplus\, t^k (\Omega Q)_{\geq k}$

But $\Omega^n_T(Q^t)$ direct summand of $Q^t \otimes_T \cdots \otimes_T Q^t$

which is T-flat, so $\Omega^n_T(Q^t)$ T-flat

$\therefore$ map injective conclude $\Omega_T(Q^t) \xrightarrow{\sim} (\Omega Q)^t \subset T' \otimes \Omega Q$

Similarly have

$$\begin{array}{ccc} R_T(Q^t) & \xrightarrow{*} & R_{T'}(T' \otimes Q) = T' \otimes RQ \\ \| & & \| \\ \Omega^{ev}_T(Q^t) & \longrightarrow & T' \otimes \Omega^{ev} Q \end{array}$$

so find map* injective

$$R_T(Q^t) \longrightarrow (RQ)^t \subset T' \otimes RQ$$

Then

$$\begin{array}{ccc} X_T(R_T(Q^t)) & \longrightarrow & X_{T'}(R_{T'}(T' \otimes Q)) \\ \| & & \| \\ X_T((RQ)^t) & & X_{T'}(T' \otimes RQ) \\ & \searrow & \| \\ & & T' \otimes X(RQ) \end{array}$$

~~iso~~ isom to $\Omega_T(Q^t) \longrightarrow T' \otimes \Omega Q$, injective,

$\therefore$ $X_T(R_T(Q^t)) \simeq X_T((RQ)^t) \simeq (X(RQ))^t \subset T' \otimes X(RQ)$

11) What assertions have we ~~discussed~~ handled?

1) $\Omega_T(Q^t)$ $\quad T$ flat

2) $\Omega_T(Q^t) \xrightarrow{\sim}$ [redacted] $(\Omega Q)^t \subset T' \otimes \Omega Q$

3) $X_T(R_T(Q^t)) \xrightarrow{\sim} X_T((RQ)^t) \xrightarrow{\sim} (X(RQ))^t \subset T' \otimes X(RQ)$

[redacted] Behind these lies [redacted]

$$(\Omega Q)^t \overset{\text{def}}{=\!=} \bigoplus t^k (\Omega Q)_{\geq k}$$

4) $(\Omega Q)_{\geq k}$ spanned by $x_0\, dx_1 \ldots dx_n$ $\quad \sum \operatorname{ord}(x_i) \geq k$

$$(X(RQ))^t \overset{\text{def}}{=\!=} \bigoplus t^k X(RQ)_{\geq k}$$

5) $X(RQ)_{\geq k}$ spanned by $\begin{array}{l} \rho x_1 \cdots \rho x_m \\ \natural(\rho x_1 \cdots \rho x_m\, d(\rho x_{m+1})) \end{array}$ $\quad \sum \operatorname{ord}(x_i) \geq k$

Next

$$\begin{array}{ccc} I_T(Q^t) = \operatorname{Ker}\left(R_T(Q^t) \longrightarrow Q^t\right) \\ \downarrow \cong \\ (IQ)^t = \operatorname{Ker}\left((RQ)^t \longrightarrow Q^t\right) \end{array}$$

provided $\quad (IQ)_{\geq k} \overset{\text{defn}}{=} IQ \cap (RQ)_{\geq k}$

6) $I_T(Q^t) \xrightarrow{\sim} (IQ)^t$

V

7) $$(I_T Q)^m \xrightarrow{\sim} (IQ^m)^t$$

provided $$(IQ^m)_{\geq k} = \sum_{\sum k_i = k} (IQ)_{\geq k_1} \cdots (IQ)_{\geq k_m}$$

8) $$F^p_{I_T(Q^t)} X_T(R_T(Q^t)) \xrightarrow{\sim} \left(F^p_{IQ} X(RQ)\right)^t$$

provided

$$\natural\left(IQ^m \, d\, RQ\right)_{\geq k} = \sum_{k_1+k_2=k} \natural\left((IQ^m)_{\geq k_1} d(RQ)_{\geq k_2}\right)$$

and similarly for $\natural\left(IQ^m \, d\, IQ\right)_{\geq k}$, $[IQ^m, RQ]_{\geq k}$

9) canonical identification induces

$$X(RQ)^t \simeq (\Omega Q)^t$$

$$\left(F^p_{IQ} X(RQ)\right)^t \simeq F^p(\Omega Q)^t$$

similarly for canonical heq ~

10) $X(RQ) \simeq \Omega Q$ induces

$$X(RQ)_{\geq k} \simeq (\Omega Q)_{\geq k}$$

$$\left(F^p_{IQ} X(RQ)\right)_{\geq k} \simeq F^p(\Omega Q)_{\geq k}$$

~ " ~

~

W] So far I have reviewed the filtered algebra version of $F^p_{IQ} X(RQ) = F^p \Omega Q$ and so.

Next need grading.

$$Q = \bigoplus Q_n \qquad 1 \in Q_0$$

(*) $$Q_{\geq k} = \bigoplus_{n \geq k} Q_n$$

D degree op., $D = n$ on Q_n

Use RQ ~~and~~ hence also $X(RQ)$ depends only on Q as vector space with 1.

RQ, $X(RQ)$ inherits gradings.

$X(RQ)_n$ spanned by $\rho x_1 \cdots \rho x_n$

$$\natural \left(\rho x_1 \cdots \rho x_n \, d(\rho x_{n+1}) \right)$$

D on RQ is unique derivation extending D on Q

deg op on $X(RQ)$ is $L_D = L(1, D)$

canonical ϕ and $h_D = h^\phi(1, D)$.

~~So what do I want?~~

$$\cancel{\boxed{T \otimes Q \xrightarrow{(t^D)} Q^t \subset T' \otimes Q}}$$

(*) means
$$T \otimes Q \xrightarrow{\sim} Q^t$$
$$t^{-m} \otimes x \longmapsto t^{-m}(t^D x)$$

isom. of graded T-modules.

$$\therefore \quad \begin{array}{ccc} R_T(T \otimes Q) & \xrightarrow{\sim} & R_T(Q^t) \\ \| & \nearrow_{\sim} & \\ T \otimes RQ & & \end{array}$$

Also $(t^{L_D})_\bullet : T \otimes X(RQ) \xrightarrow{\sim} X(RQ)^t$.

$\boxed{X}$ Key results concern L_D, h_D, γ relative to $F^pX_{\geq k}$

1) $h_D : F^pX_{\geq k} \longrightarrow F^{p-2}X_{\geq k}$ also L_D

2) $L_D - k : F^pX_{\geq k} \longrightarrow F^{p-2}X_{\geq k+1}$

3) $\gamma - (-1)^k : F^pX_{\geq k} \longrightarrow F^pX_{\geq k+1}$

Translate into

1) $h_D : (F^pX)^t \longrightarrow (F^{p-2}X)^t$

2) $L_D - t\partial_t : (F^pX)^t \longrightarrow t^{-1}(F^{p-2}X)^t$

3) $\gamma - (-1)^{t\partial_t} : (F^pX)^t \longrightarrow t^{-1}(F^pX)^t$

where $(F^pX)^t = \sum t^k \left(F^p_{IQ}X(RQ)\right)_{\geq k} \subset \left(X(RQ)\right)^t$

But because of

$$X_T(R_T(Q^t)) \xrightarrow{\sim} X(RQ)^t$$

$$F^p_{I_T(Q^t)}X_T(R_T(Q^t)) \xrightarrow{\sim} (F^pX)^t$$

1) is a relative form of the calculation

$$h_D : F^p_I X(R) \longrightarrow F^{p-2}_I X(R)$$

2) results from the definition of $F^p_I X(R)$

$L_D - t\partial_t$ on $X_T(R_T(Q^t))$ extends $D - t\partial_t$ derivation on $R_T(Q^t)$ which extends $D - t\partial_t$ on Q^t which maps Q^t into $t^{-1}Q^t$.

Y] So far I have been going over the first two lemmas.

Recall the construction: X version of Nistor

~~Identify~~ $\mathcal{X}_{\geq k} = (X_{\geq k}/F^p X_{\geq k}) \sim \theta((\Omega Q)_{\geq k})$.

$$\begin{array}{ccc} 1 - k^{-1}L_D : X_{\geq k} & \longrightarrow & X_{\geq k+1} \\ \cup & & \cup \\ F^p X_{\geq k} & \longrightarrow & F^{p-2} X_{\geq k+1} \end{array}$$

Defines map $\mathcal{X}_{\geq k} \longrightarrow \mathcal{X}_{\geq k+1}[2]$

i.e. a class in $HC^2(\mathcal{X}_{\geq k}, \mathcal{X}_{\geq k+1})$

Next $L_D = [\partial, h_D]$ together with

$h_D : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k}$

says $S_k \iota_k = S$ and $\iota_k S_k = S$.

Now γ autom of order 2 on X preserving $F^p X_{\geq k}$, so get γ on $\mathcal{X}_{\geq k}$. Also want

$\gamma_- \mathcal{X}_{\geq k+1} \xrightarrow{\sim} \gamma_- \mathcal{X}_{\geq k}$ k even.

$$X_{\geq k+1}/F^p X_{\geq k+1} \longrightarrow X_{\geq k}/F^p X_{\geq k}$$

$\gamma \to (-1)^k \gamma$

γ γ

$$\begin{array}{ccc} V & \longrightarrow & W \\ \| & & \\ V^+ & \longrightarrow & W^+ \\ \oplus & & \oplus \\ V^- & \longrightarrow & W^- \end{array}$$

Z

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \equiv & \\
0 \arrow[r] & F^p X_{\geq k+1} \arrow[r] \arrow[d] & F^p X_{\geq k} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_{\geq k+1} \arrow[r] \arrow[d] & X_{\geq k} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & 0 \\
 & X_{\geq k+1}/F^p X_{\geq k+1} \arrow[r] \arrow[d] & X_{\geq k}/F^p X_{\geq k} \arrow[r] \arrow[d] & {} & \\
 & 0 & 0 & &
\end{tikzcd}$$

$$0 \to \overset{\gamma = (-1)^k}{X_{\geq k+1} \cap F^p X_{\geq k} / F^p X_{\geq k+1}} \to X_{\geq k+1}/F^p X_{\geq k+1}$$

$$\to X_{\geq k}/F^p X_{\geq k} \to \underbrace{X_{\geq k} / X_{\geq k+1} + F^p_{\leq} X_{\geq k}}_{\gamma = (-1)^k} \to 0$$

Point. $\gamma_- X_{\geq k+1} = \gamma_- X_{\geq k}$

and $\gamma_- F^p X_{\geq k+1} = \gamma_- F^p X_{\geq k}$

So the point which I forgot is that
$\gamma = (-1)^k$ on $F^p X_{\geq k} / F^p X_{\geq k+1}$ means $\gamma_- F^p X_{\geq k+1} = \gamma_- F^p X_{\geq k}$
for k even all p.

(A) 8/19 · 0615

$$Q^t = \bigoplus t^k Q_{\geq k} \subset T' \otimes Q$$

injective

$$Q^t \otimes_T Q^t \subset Q^t \otimes_T (T' \otimes Q) \subset (T' \otimes Q) \otimes_T (T' \otimes Q) = T' \otimes Q^{\otimes 2}$$

8/20 – 0549

D section

grading $Q = \bigoplus Q_n$

RQ, $X(RQ)$ depend only

induced gradings RQ_n, $X(RQ)_n$

degree ops D on RQ, L_D on $X(RQ)$

first result is consistency of filtration + grading, clear from element description:

$\rho x_1 \cdots \rho x_m \quad \sum |x_i| = n \quad$ vs. $\quad \sum_i \mathrm{order}(x_i) \geq k.$

alternative viewpoint

$$\begin{array}{ccc} X(RQ) & \xrightarrow{t^{L_D}} & T' \otimes X(RQ) \\ \downarrow{\scriptstyle 1\otimes} & & \\ T \otimes X(RQ) & \xrightarrow{\sim} & X(RQ)^t \end{array}$$

from grading

$$X(RQ) \xrightarrow{1\otimes} T \otimes X(RQ) \xrightarrow{\sim} X(RQ)^t \subset T' \otimes X(RQ)$$

left out $X_T(R_T(Q^t))$

$$T \otimes X(RQ) \xrightarrow{\sim} X_T(R_T(Q^t)) \xrightarrow{\sim} X(RQ)^t \subset T' \otimes X(RQ)$$

(B) Point is that the fact that
Point to make?

On one hand you have $X_{\geq k}$ understood, described, controlled, by X^t which sits

$$X_T \xrightarrow{\sim} X^t \subset T' \otimes X$$

On the other hand ~~the~~ you have

$Q \longrightarrow Q^t$ linear resp 1, ~~[illegible]~~
such that $T \otimes Q \xrightarrow{\sim} Q^t$

hence $T \otimes X \xrightarrow{\sim} X_T$

Do it again:

On one hand you have ~~[illegible]~~ the filtration $Q_{\geq k}$ described by $Q^t \subset T' \otimes Q$

On the other hand you have the grading described by D, (also by the subspace $t^D Q \subset T' \otimes Q$).

~~On the 1 hand~~

Repeat. You have

filtration $Q_{\geq k}$ $\longleftrightarrow$ $Q^t \subset T' \otimes Q$

grading Q_n $\longleftrightarrow$ D

how do you express the assertion that the filtration ~~is ass~~ arises from the ~~filtra~~ grading

$\exists$ $T \otimes t^D Q \xrightarrow{\sim} Q^t$

In more detail: ~~t^D maps Q into~~

(C) In more detail, the map

$$t^D : Q \longrightarrow T' \otimes Q$$

induces an isomorphism of graded T-modules.

$$T \otimes Q \xrightarrow{\sim} Q^t$$

This means that ~~t^D actually maps Q~~ the image of t^D is contained in Q^t and that the extension to a T-module map is an isomorphism.

Repeat: have

$$\text{filtration } Q_{\geq k} \longleftrightarrow Q^t \subset T' \otimes Q$$

$$\text{grading } Q_n \longleftrightarrow D$$

grading defines a $\overset{\text{graded v.s.}}{}$ map $t^D : Q \longrightarrow T' \otimes Q$ which extends to a graded T-module map

$$T \otimes Q \longrightarrow T' \otimes Q$$

and the claim is that

$$Q_{\geq k} = \bigoplus_{n \geq k} Q_n \quad \forall k \iff T \otimes Q \xrightarrow{\sim} Q^t$$

Proof: $\Longrightarrow$

$$Q^t = \bigoplus_k t^k Q_{\geq k}$$

$$= \bigoplus_k \bigoplus_{n \geq k} t^k Q_n$$

$$= \bigoplus_{n \geq k} t^{-(n-k)} t^n Q_n$$

$$= \mathbb{C}[t^{-1}] \otimes \underbrace{\bigoplus_n t^n Q_n}_{t^D Q}$$

(D) $\Leftarrow$ Assume $T\otimes Q \xrightarrow{\sim} Q^t$
look at degree k and you get

$$\begin{tikzcd}
\bigoplus_{p\geq 0} \mathbb{C}t^{-p}\otimes Q_{k+p} \arrow[r, "\sim"] \arrow[dr, "\sim"'] & t^k Q_{\geq k} \arrow[d, equal] \\
& t^k \bigoplus_{p\geq 0} Q_{k+p}
\end{tikzcd}$$

Repeat: have

filtration $Q_{\geq k}$ described by the graded T-submodule $Q^t \subset T'\otimes Q$

grading Q_n —— operator D

the grading defines a map of graded vector spaces $\quad t^D : Q \longrightarrow T'\otimes Q$

$$D \longleftrightarrow t\partial_t$$

which extends to a graded Tmodule map

$$T\otimes Q \longrightarrow T'\otimes Q$$

and we have

$$Q_{\geq k} = \bigoplus_{n\geq k} Q_n \quad \forall k \iff T\otimes Q \xrightarrow{\sim} Q^t$$

Now ~~this~~ $R_S A$ depends only on A as S-bimodule with $1 \implies$

$$\begin{tikzcd}
X_T R_T(T\otimes Q) \arrow[r, "\sim"] \arrow[d, equal] & X_T R_T(Q^t) \arrow[r] \arrow[d, equal] & X_{T'} R_{T'}(T'\otimes Q) \arrow[d, equal] \\
T\otimes XRQ \arrow[r] & X(RQ)^t & T'\otimes XRQ \arrow[l]
\end{tikzcd}$$

conclusion is that $t^D : Q \longrightarrow Q^t$ induces ~~an iso~~

$$T\otimes X(RQ) \xrightarrow{\sim} \cancel{X(RQ)^t}\; X_T(R_T(Q^t)) \xrightarrow{\sim} X(RQ)^t$$

(E) Not clear how much of this I have to say.

So what do we have at the moment?

Just the statement that $t^D : Q \longrightarrow Q^t$ induces $T \otimes X(RQ) \xrightarrow{\sim} X_T(R_T(Q^t)) \simeq X(RQ)^t$

Behavior of L_D, γ, h_D.

extend L_D to $X(RQ)^t \subset$

8/21 - 0507 ~~What is~~ the problem.

$$X(RQ)_{\geq k} = \bigoplus_{n \geq k} X(RQ)_n$$

$$X_T(R_T(Q^t)) \xrightarrow{\sim} X(RQ)^t = \bigoplus t^k X(RQ)_{\geq k} \subset T' \otimes X(RQ)$$

there's a glitch with the definition of D, L_D, h_D on $X(RQ)^t$. I need ~~the following~~ a relative form.

The point is to ~~consider~~ generalize the objects Q

Q has grading as vector space such that the associated $\mathbb{Z}/2$ grading and associated filtration are comp. with alg. structure

(F) 8/21 - 0540

The problem is that I still can't organize the proof in my mind. I want now to sit down and outline the whole thing. The main topics

~~my~~ our construction

Nistor's construction (our version)

X-version of Nistor's construction

$\underbrace{X \text{ analogue of } ((\Omega Q)_{\geq k}, b, B)}$

~~?~~ bifiltration $(F^p X_{\geq k})$

behavior of L_D, h_D, γ wrt $(F^p X_{\geq k})$

last map

link between ~~my~~ our + Nistor's constructions

I can review pieces I understand but I would like to get a hold ~~of~~ the whole picture.

Let's take the ~~?~~ graded approach first.

I would like to ~~start~~ emphasize the structure on Q

| graded as vector space
| $\mathbb{Z}/2$ graded
| filtered as algebra

emphasize also canonical ident

$$X(RQ) = \Omega Q \quad \& \quad F^p_{IQ} X(RQ) \simeq F^p \Omega Q$$

and canonical heq.

$$X(RQ) \sim \Omega Q \quad + \quad F^p_{IQ} X(RQ) \sim F^p \Omega Q$$

[redacted]

6

So far we emphasize

structure on Q | grading as v.s. with 1 | assoc. filt. | assoc $\mathbb{Z}/k$ gr. | ~~an~~ as alg.

canonical ident. $X(RQ) \simeq \Omega Q \ni F^p \sim$

canon. heq $\sim \ni \sim$

first point is the filtration on Q leads to filtrations on $(RQ)_{\geq k}$, $(X(RQ))_{\geq k}$, $(F^p_{IQ} X(RQ))_{\geq k}$

consequence is that we get a tower

$$\mathcal{X}_{\geq k} = \left(X_{\geq k}/F^p X_{\geq k}\right) \sim \Theta\left((\Omega Q)_{\geq k}\right)$$

---

grading. RQ, $X(RQ)$ depend only on Q as vector space with 1.

filtration on Q induces a filtration on RQ, $X(RQ)$. why precisely?

$$Q^t \subset T' \otimes Q$$

$$R_T(Q^t) \longrightarrow R_{T'}(T' \otimes B) = T' \otimes RQ$$

~~compatible w~~ The image is a graded T-submodule of $T' \otimes RQ$, hence of the form $(RQ)^t$ for some filtration.

fight with Erica over bike

(H) Review: Start again:

structure on Q

grading as v.s. with 1

assoc. filt. | resp alg structure

$\mathbb{Z}/2$-grading |

get induced γ and filtration on each of the objects ΩQ, $F^p\Omega Q$, RQ, IQ, $X(RQ)$, $F^p_{IQ} X(RQ)$ compatible with the structure on these objects + relations between them discussed before.

explain $Q^t \subset T' \otimes Q$.

T subalgebra

$$\Omega_T(Q^t) \longrightarrow \Omega_{T'}(T' \otimes Q) = T' \otimes \Omega Q$$

image is evidently $(\Omega Q)^t$ for the filtration.

Take

$$F^p\Omega_T(Q^t) \longrightarrow T' \otimes F^p \Omega Q$$

$$b\,\Omega_T^{p+1}(Q^t)$$

$$[\Omega_T^p(Q^t), Q^t] \longrightarrow T' \otimes [\Omega^p Q, Q]$$

in degree k coefficient

all this is the easy part.

(I) Go on to the grading.

~~grading~~

Point is that RQ depends on Q as vector space with 1, and similarly $X(RQ)$.

How do you get induced grading on RQ?

$t^D : Q \longrightarrow T' \otimes Q$ hn. resp. 1.

$$RQ \longrightarrow R_{T'}(T' \otimes Q) = T' \otimes RQ$$

Obviously, we expand on what we understand without getting the whole picture.

Q graded as vector space with 1

$Q = \bigoplus_n Q_n \qquad 1 \in Q_0$

$D = n$ on Q_n

$$t^D : Q \longrightarrow T' \otimes Q$$

$$D \longleftrightarrow t\partial_t$$

t^D hn. resp 1.

$$RQ \longrightarrow R_{T'}(T' \otimes Q) = T' \otimes RQ$$

lifting of RQ rel. spec. at $t = 1$. to graded subspace. $\therefore$ get grading on RQ

get D on $RQ \longleftrightarrow t\partial_t$ for above map, which is then t^D.

Similarly get grading and t^{-D} is degree operator.

(J) What to prove next?

need something reasonable.

Concentrate. Note that

$$Q \longrightarrow T \otimes Q \xrightarrow{\sim} Q^t \subset T' \otimes Q$$

ind.

$$\begin{tikzcd} RQ \arrow[r] \arrow[dr, "1\otimes"'] & R_T(T\otimes Q) \arrow[r, "\sim"] \arrow[d, equal] & R_T(Q^t) \arrow[r] & R_{T'}(T'\otimes Q) \arrow[d, equal] \\ & T\otimes RQ \arrow[r, "\sim"] & (RQ)^t \subset & T'\otimes RQ \end{tikzcd}$$

1) Q is graded, $\Longrightarrow$ RQ graded

2) Q is filtered $\Longrightarrow$ RQ filtered

to see 1) take $Q \xrightarrow{t^D} T' \otimes Q$

get $RQ \longrightarrow T' \otimes RQ$

~~and D on RQ~~ image is homogeneous

$\therefore$ get $(RQ)_n$

to see 2) take $Q^t \subset T' \otimes Q$

get $R_T Q^t \longrightarrow T' \otimes RQ$

image is $(RQ)^t$

~~When ... fails~~

3) When $Q_{\geq k} = \bigoplus_{n \geq k} Q_k$ have $T \otimes Q \xrightarrow{\sim} Q^t \subset T' \otimes Q$

$T \otimes RQ \xrightarrow{\sim} R_T(Q^t) \longrightarrow T' \otimes RQ$

composition injective

Conclusion $T \otimes RQ \longrightarrow (RQ)^t$

(K) conclude $T \otimes RQ \xrightarrow{\sim} R_T(Q^t) \xrightarrow{\sim} (RQ)^t$
$\subset T' \otimes RQ$

try again

3) When $(Q_{\geq k})$ assoc. grading have

$$\underbrace{Q \subset T \otimes Q \xrightarrow{\sim} Q^t \subset T' \otimes Q}_{t^0}$$

so $RQ \subset T \otimes RQ \simeq R_T(Q^t) \to T' \otimes RQ$

$$\begin{array}{c} R_T(Q^t) \\ \downarrow \\ (RQ)^t \end{array}$$

No you're confusing things.
You want to say that the grading

$$RQ \overset{t^0}{\subset} T' \otimes RQ$$

determines the filt. i.e.

$$T \otimes RQ \twoheadrightarrow (RQ)^t$$

---

grading of Q gives ~~subspace~~ $RQ \longrightarrow T' \otimes RQ$
homog. image $\oplus\, t^n \otimes (RQ)_n$, gives grading RQ.
filt. of Q gives $R_T(Q^t) \twoheadrightarrow T' \otimes RQ$
image is $(RQ)^t$. Now grading $\to$ filt on Q
says $T \otimes Q \xrightarrow{\sim} Q^t \Rightarrow T \otimes RQ \xrightarrow{\sim} R_T(Q^t)$
$\Rightarrow T \otimes RQ \twoheadrightarrow (RQ)^t$ whence grading $\to$ filt on RQ.