

[h] Can't use ink pen with this paper so instead maybe make a summary. What I need is to organize exactly what is needed.

My construction

$$A \xrightarrow{\quad} L \otimes B \quad \text{cong mod } J \otimes B$$

$$S = \bigoplus_{n \geq 0} t^n J^n \quad K = (1-t^2)J^2 S$$

$$A \xrightarrow{p+tg} S \otimes B \quad \text{hom. mod } K \otimes B$$

$$RA \xrightarrow{u} S \otimes RB \quad IA \rightarrow K \otimes RB + S \otimes IB$$

$$X(RA) \xrightarrow{u_*} X(S \otimes RB) \xrightarrow{\chi} S_h \otimes X(RB) \xrightarrow{\mu_m} J_{\#}^{2m+1} \otimes X(RB)$$

$$F_{IA}^p \xrightarrow{\quad} F_{K \otimes RB + S \otimes IB}^p \rightarrow \sum \mathbb{Z}(k_i) \otimes F_{IB}^{p-2i} \rightarrow J_{\#}^{2m+1} \otimes F_{IB}^{p-2m}$$

μ_m construction

$$S_h \xrightarrow{\pi} \pi S \xrightarrow{P_m(t \partial_t)} J_{\#}^{2m+1}$$

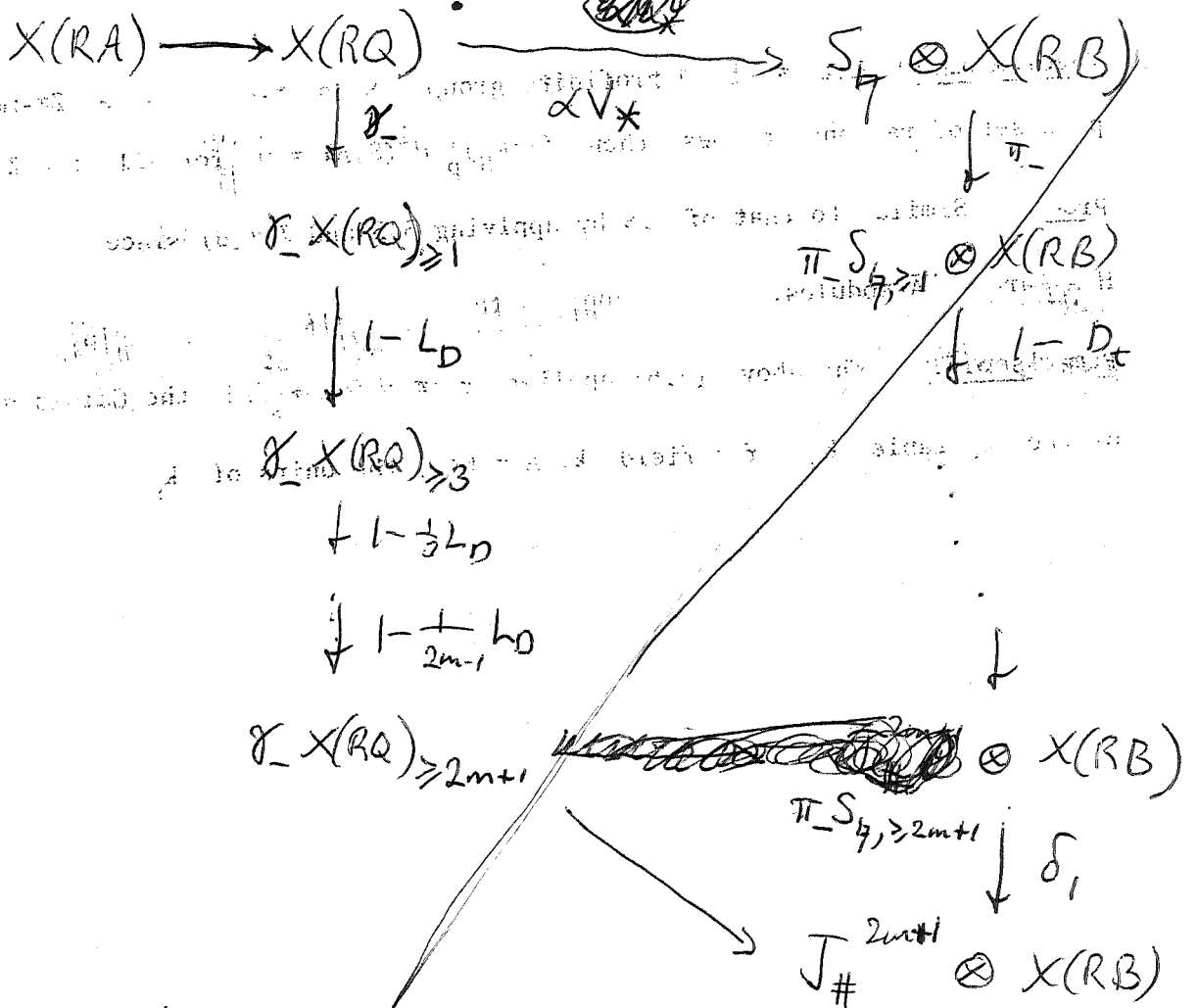
$$S \xrightarrow{\pi} \pi S \xrightarrow{1-D_t} \pi S \xrightarrow{1-\frac{1}{3}D_t} \dots \xrightarrow{1-\frac{1}{2m-1}D_t} \pi S \xrightarrow{\delta_1} J_{\#}^{2m+1}$$

Joachim's version of Nistor construction

First structure on $X(RQ)$. Describe what is needed. The output will be a map

$$X(RA) \rightarrow J_{\#}^{2m+1} \otimes X(RB) \quad , \quad F_{IA}^p \rightarrow J_{\#}^{2m+1} \otimes F_{IB}^{p-2m}$$

i) It is obtained



Notice that we need to know something about

$$\begin{array}{ccc}
 X(RQ) & \xrightarrow{\quad} & S \otimes X(RB) \\
 \cup & & \cup \\
 X(RQ)_{\geq k} & \xrightarrow{\quad} & S_{\geq k} \otimes X(RB)
 \end{array}$$

This is something new. I've been working on

$$X(RQ)_{\geq k} \xrightarrow{\quad} J_{\#}^k \otimes X(RB)$$

Logic: You take my map $X(RA) \rightarrow S_{\mathbb{Z}} \otimes X(RB)$
and interpolate $X(RQ)$

$$\begin{array}{ccccccc}
 X(RA) & \longrightarrow & X(RQ) & \longrightarrow & S_{\mathbb{Z}} \otimes X(RB) \\
 RA & \xrightarrow{t} & RQ & \xrightarrow{t, \geq 0} & S \otimes RB
 \end{array}$$

So I have to work on understanding

$$X(RQ) \longrightarrow S_{\mathbb{Z}} \otimes X(RB)$$

and to understand why it is compatible with the grading, i.e. why L_D on the left is compatible with $D_t = tD_t$ on the right.

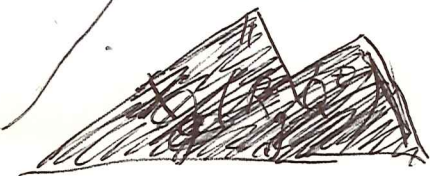
Notice that, when you've done this you get automatically

$$\begin{array}{ccc}
 X(RQ)_{\geq k} & \longrightarrow & S_{\mathbb{Z}, \geq k} \otimes X(RB) \\
 & & \downarrow \\
 & & J^k_{\#} \otimes X(RB)
 \end{array}$$

which I struggled over yesterday.

Yesterday the point was ~~fixed~~

$$X(RQ)^t = \bigoplus t^k X(RQ)_{\geq k}$$



$$X_T(R_T Q^t) \longrightarrow X(RQ)^t$$



[k]

Anyway yesterday I did this:

Q alg w filt $Q_{\geq k}$, hom of filt algs.

$$Q \longrightarrow L \otimes B, \quad Q_{\geq k} \longrightarrow J^k \otimes B$$

$$Q^t \longrightarrow L^t \otimes B$$

$$X_T(R_T Q^t) \longrightarrow L_{\mathbb{Z}}^t \otimes X(RB), \quad F_{I_T Q^t}^P \longrightarrow L_{\mathbb{Z}}^t \otimes F_{IB}^P$$

||

$$X(RQ) \xrightarrow{t} \left(F_{IQ}^P X(RQ) \right)^t \longrightarrow L_{\mathbb{Z}}^t \otimes F_{IB}^P$$

In addition $D: \bar{Q} \longrightarrow Q$ linear resp $Q_{\geq k}$

$$h_D: \left(F_{I_T Q^t}^P X_T(R_T Q^t) \right) \subset F_{I_{-Q^t}}^{P-2} X_T(R_T Q^t)$$

||

$$h_D: \left(F_{IQ}^P X(RQ) \right)^t \longrightarrow \left(F_{IQ}^{P-2} X(RQ) \right)^t$$

~~How does this commute?~~

What does the above gives?

Given $Q, Q_{\geq k}, Q \longrightarrow L \otimes B, Q_{\geq k} \longrightarrow J^k \otimes B, D, D(Q_{\geq k}) \subset Q_{\geq k}$ we get

$$X(RQ)_{\geq k} \longrightarrow J_{\#}^k \otimes X(RB)$$

$$\left(F_{IQ}^P X(RQ) \right)_{\geq k} \longrightarrow J_{\#}^k \otimes F_{IB}^P X(RB)$$

and $h_D \left(F_{IQ}^P \right)_{\geq k} \subset \left(F_{IQ}^{P-2} \right)_{\geq k}$

[2] So the next thing is to link these two. Notice that so far D has not been related to the maps $Q \rightarrow L \otimes B$

In fact suppose we ~~assume known the result on D~~ ~~choose~~ assume D arises from a splitting: $Q_{\geq k} = \bigoplus_{n \geq k} Q_n$ of the filtration. Then

$$Q \xrightarrow{t^D} Q^t \longrightarrow L^t \otimes B$$

Start again. Yesterday ~~proved~~ understood two results

$Q, Q_{\geq k}$ filtered alg,

1) If $D: \bar{Q} \rightarrow Q$ ^{mean} resp. filtration, then

$$h_D: \left(\frac{FP}{IQ} X(RQ) \right)_{\geq k} \longrightarrow \left(\frac{FP^{-2}}{IQ} X(RQ) \right)_{\geq k}$$

2) If $Q \rightarrow L \otimes B, Q_{\geq k} \rightarrow J^k \otimes B$ homom. of filtered algs, then one has induced maps

$$X(RQ)_{\geq k} \rightarrow J^k_{\#} \otimes X(RB), \left(\frac{FP}{IQ} X(RQ) \right)_{\geq k} \rightarrow J^k_{\#} \otimes \frac{FP}{IB} X(RB)$$

Now ~~take~~ suppose given $Q_n \subset Q$ & $Q_{\geq k} = \bigoplus_{n \geq k} Q_n$ ^{$1 \leq Q_0$}

Need the rest of the picture, so return to the link. ~~Have~~ my map

$$\boxed{m} \quad X(RA) \xrightarrow{\alpha u_*} S \otimes X(RB)$$

and want to factor

$$X(RA) \xrightarrow{\alpha x} X(RQ) \xrightarrow{\alpha v_*} S \otimes X(RB)$$

need $RQ \longrightarrow S \otimes RB$

which like α arises from a linear map in this case

$$Q \longrightarrow S \otimes B$$

and really is the map

$$Q \xrightarrow{t^D} Q_{t, \geq 0} \longrightarrow S \otimes B$$

$$X(RQ) \xrightarrow{t^{L_D}} \bigoplus_{k \geq 0} t^k X(RQ)_{\geq k} \longrightarrow S \otimes X(RB)$$

Claim that L_D on the left ~~is~~ is compat with D_t on the right.

Note that



you have analyzed the 2nd map; you know that the ~~the~~ degree k component

$$X(RQ)_{\geq k} \longrightarrow J_{\#}^k \otimes X(RB) \text{ carries } \begin{pmatrix} F^P \\ I^Q \end{pmatrix}_{\geq k} \text{ into } J_{\#}^k \otimes F_{IB}^P.$$

Now you must understand the ~~the~~ first map.

It is perfectly transparent from the viewpoint

[n] of the grading, namely $X(RQ)_n$ goes by mult. by t^n into $t^n X(RQ)_{\geq n}$.

Another point $Q \xrightarrow{t^D} S \otimes B$ is ~~a graded~~ compatible with grading so where are we? means ~~are~~ D and D_t are compatible, so this extends ~~immediately~~ to X -R-ads.

Consider

$$\begin{array}{ccc}
 Q & \xrightarrow{t^D} & Q^t \\
 \parallel & & \parallel \\
 \bigoplus_k Q_k & \xrightarrow{\quad} & \bigoplus_k t^k Q_{\geq k} \\
 \downarrow \chi_R & & \downarrow t^k \chi_R
 \end{array}$$

induces

$$\begin{array}{ccc}
 RQ & \longrightarrow & R_{\mathbb{C}^t}(Q^t) = (RQ)^t \\
 X(RQ) & \longrightarrow & X_{\mathbb{C}^t}(R_{\mathbb{C}^t} Q^t) = X(RQ)^t \\
 & & \searrow t^{L_0}
 \end{array}$$

Now I don't know very much about t^{L_0} behavior wrt. $F_{IQ}^P X(RQ)$. Could I find out more

□ Back to earlier struggle. I'm trying to produce an account of Joachim's approach to Nistor. This is based on $X(RQ)$

Q filtered algebra $Q = Q_{\geq 0} \supseteq Q_{\geq 1} \supseteq \dots$ etc. with linear splitting: $Q_n \subset Q_{\geq n}$

$Q_{\geq k} = \bigoplus_{n \geq k} Q_n$, $1 \in Q_0$. Get $D = n$ on Q_n

extend to D on RQ & grading $RQ = \bigoplus_n (RQ)_n$

~~$D = n$ on $(RQ)_n$~~ and to $X(RQ) = \bigoplus_n X(RQ)_n$

$L_D = n$ on $X(RQ)_n$.

Question: You've defined $X(RQ)_{\geq k}$

via $X(RQ)^t = X_{\mathbb{C}^t}(R_{\mathbb{C}^t} Q^t)$. And defined $X(RQ)_n$ via D . How do you see \blacktriangle that $X(RQ)_{\geq k} = \bigoplus_{n \geq k} X(RQ)_n$?

There should be some mechanism for handling gradings. A grading on Q is a map $Q \xrightarrow{t^D} \mathbb{C}[t, t^{-1}] \otimes Q$ of some sort. We start with such a thing and ~~obtain~~ obtain Q^t .

What I want is meaning of splitting a filtration. Filtration on Q is graded \mathbb{C}^t algebra module

[P]

Lifting of Q into Q^t
Lifting of Q into Q^t .

$$Q = \bigoplus_n Q_n$$

Idea: A grading on Q yields a

Can. map $Q \xrightarrow{t^D} \bigoplus_{t \in \mathbb{Z}} Q_t$

Consider the graded \mathbb{C}^t submodule generated by $t^D Q = \bigoplus_{t \leq n} Q_n \subseteq \bigoplus_{t \leq n} Q$. When we take ~~the~~ the \mathbb{C}^t submodule we have

$$\bigoplus_{n \leq k} \bigoplus_{k \leq n} t^k Q_n = \bigoplus_k t^k \bigoplus_{n \geq k} Q_n$$

Conversely ~~the~~ given $Q \geq k$ and ~~the~~ Q^t we have surjective $Q^t \rightarrow Q$ sending $t \mapsto 1$ and a lifting should be the same as a grading. Not quite.

$t^D \geq 0$
 $Q \geq 0$
graded complement
for $t^D Q \subseteq Q^t$
 $t^2 Q \geq 2$

[9] when we've split Q we have an isom

$$\mathbb{C}[t^{-1}] \otimes Q \xrightarrow{\sim} Q^t$$

of graded \mathbb{C}^t -modules, then an isom.

$$\mathbb{C}[t^{-1}] \otimes RQ \xrightarrow{\sim} RQ^t$$

etc.

~~Consider the following~~

Return now to Joachim's version of Moster construction. Work with $X(RQ)$. On one hand Q has grading D , and this induces similar structure on RQ , $X(RQ)$.

$$Q \xrightarrow{t^D} \mathbb{C}[t, t^{-1}] \otimes Q \implies Q^t \subset \mathbb{C}[t, t^{-1}] \otimes Q$$

yields

$$RQ \xrightarrow{\quad} \mathbb{C}[t, t^{-1}] \otimes RQ$$

yields

$$X(RQ) \xrightarrow{\quad} \mathbb{C}[t, t^{-1}] \otimes X(RQ)$$

OKAY at this point we have the grading and filtration organized on $X(RQ)$. From the filt side we get

$$h_D: X(RQ) \longrightarrow X(RQ)$$

$$\left(F_{IQ}^P X(RQ) \right)_{\geq k} \longrightarrow \left(F_{I^2 Q}^{P-2} \dots \right)$$

r 7/21 - 0530

Start with a review of Joachim's version of the Nistor construction.

Let $Q = QA$, $Q_{\geq k} = \mathcal{O}_B^k$ filtration

$Q_n = \Omega^n$ grading

From the filtration get ~~filtration~~ filtration

~~$X(RQ)_{\geq k}$ defined by~~
 ~~$X(RQ)^t = \text{Image}$~~

~~$$X_{\mathbb{C}^t}(R_{\mathbb{C}^t} Q^t) \rightarrow \mathbb{C}[t, t^{-1}] \otimes X(RQ)$$~~

Also for $(\frac{FP}{IQ} X(RQ))_{\geq k}$

From the grading we get a grading on $X(RQ)$ ~~etc. maps~~ defined how?

The grading gives an isom of \mathbb{C}^t modules

~~$$\mathbb{C}[t^{-1}] \otimes Q \xrightarrow{\sim} Q^t$$~~

~~$$X_{\mathbb{C}^t}(R_{\mathbb{C}^t}(\mathbb{C}^t \otimes Q)) \xrightarrow{\sim} X_{\mathbb{C}^t}(R_{\mathbb{C}^t} Q^t)$$~~

because $X(RQ)$ depends on Q and $\mathbb{C} \rightarrow \mathbb{C}^t$

~~$$\mathbb{C}^t \otimes X(RQ) \xrightarrow{\parallel} X(RQ)^t$$~~

~~Define $\mathcal{E}(Q) = \mathcal{E}^2 \mathcal{R} \mathcal{E}(Q) = \mathcal{O}$~~
~~then~~

5

$Q = QA$, $Q_{\geq k} = \sigma_f^k$, $Q_n = \Omega^n$
filtration yields filtrations on $X(RQ)$, $F_{IQ}^P X(RQ)$

$$\begin{array}{ccc} (F_{IQ}^P X(RQ))^t & = & F_{I_{\mathbb{C}^t Q^t} Q^t}^P X(R_{\mathbb{C}^t} Q^t) \\ \text{SI} & & \text{SI} \\ (F_{\Omega Q}^P)^t & & F_{\Omega_{\mathbb{C}^t Q^t} Q^t}^P \end{array}$$

grading of Q as vector space with 1 ~~side~~ yields a grading of $X(RQ)$

$$Q \xrightarrow{t^D} \mathbb{C}[t, t^{-1}] \otimes Q$$

map of graded vector spaces with 1 sides

$$X(RQ) \longrightarrow \mathbb{C}[t, t^{-1}] \otimes X(RQ)$$

corresp to a grading on $X(RQ)$.

D on Q gives rise to $L_D^{h_D}$ on $X(RQ)$
focus, maybe list objects

$\text{filtration } X(RQ)_{\geq k}$, $(F_{IQ}^P X(RQ))_{\geq k}$
 $\text{grading } X(RQ)_n$
 h_D, h_D

(u) Repeat

Let $Q, (Q \geq k)$ be a filtered algebra

Then $X(RQ), F_{IQ}^p X(RQ)$ inherit filtrations

Let $D: Q \rightarrow Q$ be linear resp. filt

Then L_D, h_D resp $X(RQ)_{\geq n}, \text{map } F_{\geq k}^p \rightarrow F_{\geq k}^{p-2}$

Let $Q = \bigoplus Q_n, 1 \in Q_0$ be a grading of Q . Then $X(RQ)$ inherit grading

$$Q_{\geq k} = \bigoplus_{n \geq k} Q_n \implies X(RQ)_{\geq k} = \bigoplus_{n \geq k} X(RQ)_n$$

$$D = n \text{ on } Q_n \implies L_D = n \text{ on } X(RQ)_n$$

~~Now put this together~~ Now put this together

For each k have mixed complex $\Omega Q_{\geq k}$

~~The cyclic homology type of~~ $\Omega Q_{\geq k}$ is rep by the tower $X(RQ)_{\geq k} / (F_{IQ}^p X(RQ))_{\geq k}$

Note that

$$X(RQ)^t / (F_{IQ}^p X(RQ))^t$$

Tower of graded \mathbb{Q}^t -modules, clearly special.

Notation needed,

Claim is that in \mathbb{Q} derived cat of mixed complexes we have certain maps and properties.

USERS/1QV1LEN/O/NN.22

V ~~W~~ 7/21 - 1342 run into a problem
 namely ~~_____~~

$$S_n = 1 - \frac{1}{n} L_D : FPX_{\geq n} \longrightarrow FX_{\geq n+1}^{p-2} ?$$

\Rightarrow I know $L_D : FPX_{\geq n} \longrightarrow FP^{-2}X_{\geq n} / FP^{-2}X_{\geq n+1}$

I know that ~~_____~~ $L_D : FPX_{\geq n} \longrightarrow FP^{-2}X_{\geq n}$
 \cup \cup
 $L_D : FPX_{\geq n+1} \longrightarrow FP^{-2}X_{\geq n+1}$

but I don't ~~necessarily~~ yet have $L_D = n$ on this quotient.

$$D : (\mathbb{I}Q)_{\geq n} \longrightarrow (RQ)_{\geq n}$$

See if it works using ^{the} T-formalism. Consider

~~_____~~ So what do I use at this point?

~~_____~~ Go back to L_D on $X(RQ)^t$. We

$$D = n \text{ on } Q_{\geq n} / Q_{\geq n+1}$$

What you would like to do is to

~~Consider~~ Grading

$$\mathbb{C}[t^{-1}] \otimes Q \xrightarrow{\sim} Q^t$$

isomorphism
of graded v.s.

$$t\partial_t \otimes 1 + 1 \otimes D \longleftrightarrow t\partial_t$$

$$L_D : FP_{\mathbb{I}TQ^t} X_T(R_T Q^t) \longrightarrow FP_{\mathbb{I}TQ^t}^{-2} X_T(R_T Q^t)$$

~~_____~~

W

So what to do.

$$t^D : Q \longrightarrow \mathbb{C}[t, t^{-1}] \otimes Q$$

So what happens. You have

$$\mathbb{C}[t^{-1}] \otimes Q \xrightarrow{\sim} Q^t = \bigoplus_k t^k Q_{\geq k}$$

$$Q \xrightarrow{t^D} \mathbb{C}[t, t^{-1}] \otimes Q$$

So the fact is that

$$\mathbb{C}[t^{-1}] \otimes_{t\partial_t + L_D} X(RQ) \xrightarrow{\sim} X(RQ)^t \quad t\partial_t$$

$$\begin{array}{ccc}
 \downarrow & & \downarrow \\
 X(RQ) & \xrightarrow{\quad} & X(RQ)^t / t^{-1} X(RQ)^t \\
 L_D & & t\partial_t
 \end{array}$$

But now consider

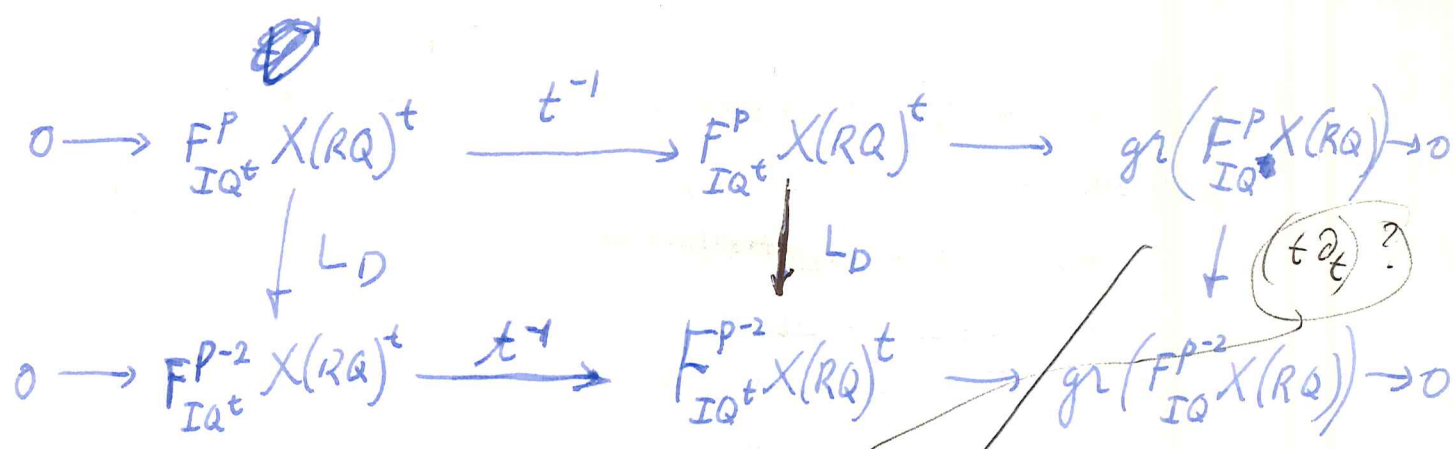
$$F_{IQ^t}^P X(RQ)^t$$

Take L_D on $F_{IQ^t}^P X(RQ)^t$

~~L_D commutes~~ L_D commutes with t^{-1} .

$$F_{IQ^t}^P X(RQ)^t \subset X(RQ)^t \quad L_D$$

x



$t \partial_t$ canonical map induced by the inclusion



Point is that

$$F_{IQ^t}^P X(RQ)^t \longrightarrow gr(F_{IQ^t}^P X(RQ)^t)$$

should be a proof.

$$\left(F_{IQ}^P X(RQ) \right)_{\geq k}$$

What I want is to take $F_{IQ}^P X(RQ)_{\geq k}$ and to define

7/22-0522 Yesterday I encountered another problem. I need to know that $S_n = 1 - \frac{1}{n} L_D$ going from $X_{\geq n}$ to $X_{\geq n+1}$ actually carries $F_{IQ}^P X_{\geq n}$ into $F_{IQ}^{P-2} X_{\geq n+1}$. So far I only know that $S_n(F_{IQ}^P X_{\geq n}) \subset F_{IQ}^{P-2} X_{\geq n} \cap X_{\geq n+1}$

~~The problem is thus~~ First do it elementwise

Consider $(IQ_{\geq k})$

[y] Consider $(\mathbb{I}\mathbb{Q}^n)_{\geq k}$ spanned by

$$\xi = p(x_0) \omega(x_1, x_2) \cdots \omega(x_{2n-1}, x_{2n}) \quad \sum \text{ord}(x_i) \geq k$$

modulo $(\mathbb{I}\mathbb{Q}^n)_{\geq k+1}$ I can assume

x_i ~~homogeneous~~ homogeneous: $x_i \in \mathbb{Q}_{k_i}$

$$\sum k_i = k.$$

Then $D p(x_0) = p(Dx_0) = k_0 p(x_0)$

$$D \omega(x_1, x_2) = D(p(x_1, x_2) - p(x_1)p(x_2))$$

$$= p(D(x_1, x_2)) - p(Dx_1)p(x_2) - p(x_1)p(Dx_2)$$

$$= \underbrace{p(D(x_1, x_2) - Dx_1 x_2 - x_1 Dx_2)}_{\in p(\mathbb{Q}_{>k_1+k_2})} + \underbrace{\omega(Dx_1, x_2) + \omega(x_1, Dx_2)}_{(k_1+k_2) \omega(x_1, x_2)}.$$

$$\in p(\mathbb{Q}_{>k_1+k_2})$$

$$(k_1+k_2) \omega(x_1, x_2).$$

Thus

$$\underline{D(\mathbb{I}\mathbb{Q}^n)_{\geq k} \subset (\mathbb{I}\mathbb{Q}^{n-1})_{\geq k+1}}$$

$$D\xi - k\xi \in (\mathbb{I}\mathbb{Q}^{n-1})_{\geq k+1}.$$

This should work. Let's find an abstract argument.

~~What?~~ What is needed is to amplify

$$h_D: \mathbb{F}^p X_{\geq k} \rightarrow \mathbb{F}^{p-2} X_{\geq k} \quad \text{to include}$$

$$L_D - k: \mathbb{F}^p X_{\geq k} \rightarrow \mathbb{F}^{p-2} X_{\geq k+1}.$$

Thus you ~~also~~ want $h_D - t\partial_t$ to map $(\mathbb{F}^p X)^t$ into $t^{-1}(\mathbb{F}^{p-2} X)^t$

[2] What is the point? ~~class~~

~~Byzantine~~

$$X = X(RQ)$$

$$F^P = F_{IQ}^P$$

$$X^t = X_T(R_T Q^t)$$

$$(F^P X)^t = F^P X_T(R_T Q^t)$$

This time you want: $T/t^{-1}T \cong C$.

$$X^t/t^{-1}X^t = X(R(\text{gr}Q))$$

How are

$$\boxed{(F^P X)^t / t^{-1}(F^P X)^t}$$

$$F^P(X^t)/t^{-1}F^P(X^t)$$

and

$$F^P X(R(\text{gr}Q))$$

related. I want a

First look at the algebras. I have R^t with ideal I^t over $T = C^t$. ~~Pass~~

I am ~~not~~ interested in R^t/mR^t $m = t^{-1}$

actually is $(I^t)^n / m(I^t)^n = (I^t/mI^t)^n$

↑
NO ?

$$I^t = \bigoplus t^k I_{\geq k}$$

$$(I^t)^n = \bigoplus t^k \left(\sum_{\sum k_i = k} I_{\geq k_1} \cdots I_{\geq k_n} \right)$$

Meaning of the calculation that

$$D\omega(x_1, x_2) = \rho(D(x_1, x_2) - Dx_1 x_2 - x_1 Dx_2) + \omega(Dx_1, x_2) + \omega(x_1, Dx_2)$$

Statement that D is a derivation on the associated graded algebra $\text{gr}Q$.

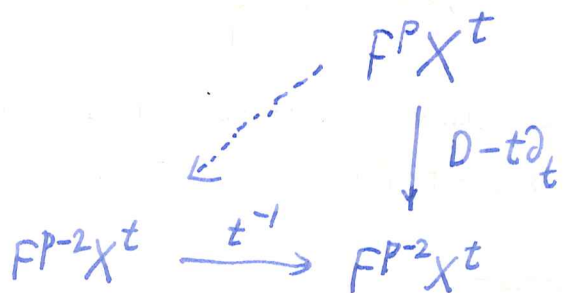
$$D(I_{\geq k}) \subset I_{\geq k} + R_{\geq k+1}$$

A) $D(I^t) \subset I^t + t^{-1}R^t$

Thus $D((I^n)^t) = D(I^t)^n$

$\subset (I^t)^n + t^{-1}(I^t)^{n-1}$

Want $D - t\partial_t : I^t \rightarrow t^{-1}R^t$



So see if we can make this work.

Details and review. We have

Q alg w. filtration $Q_{\geq k}$ comp alg st. $\bigcup Q_{\geq k} = Q$

form $Q^t = \bigoplus_{k \in \mathbb{Z}} t^k Q_{\geq k} \subset \mathbb{C}[t, t^{-1}] \otimes Q$

graded $T = \mathbb{C}^t$ alg

induced filtrations $(RQ)_{\geq k}, X(RQ)_{\geq k}$

also $(IQ^n)_{\geq k}, (F_{IQ}^{FP} X(RQ))_{\geq k}$

formulas. $X(RQ)^t = X_T(R_T Q^t) \simeq \Omega_T(Q^t)$

$(F_{IQ}^{FP} X(RQ))^t = F_{I_T Q^t}^{FP} X_T(R_T Q^t) \simeq FP \Omega_T(Q^t)$

grading $Q_{\geq k} = \bigoplus_n Q_n$ comp $Q_{\geq k} = \bigoplus_{n \geq k} Q_n, 1 \in Q_0$

$D: \bar{Q} \rightarrow Q$ $D = n$ on Q_n . D on Q^t comm. with t^{-1}

get D_n^{deriv} on RQ , on RQ^t
 $\hookrightarrow D$ on $X(RQ)$, on $X(RQ)^t$

B) ~~Concentrate~~ Concentrate on the properties of L_D, h_D on $X = X(RQ)$ relative to the bifiltration $F^p X_{\geq k}$:

$$h_D : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k}$$

$$L_D - h : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k+1}$$

Proof of former. Extend D to Q^t to commute with T , then to $\text{Dom } R_T Q^t = (RQ)^t$, then to L_D, h_D on ~~relative version of~~ $X_T(R_T Q^t) = X(RQ)^t$. We know by relative version of L_D, h_D :

$$L_D, h_D : F_{I_T Q^t}^p X_T(R_T Q^t) \longrightarrow F_{I_T Q^t}^{p-2} X_T(R_T Q^t)$$

$$\parallel \quad \parallel$$

$$(FPX)^t \quad (F^{p-2}X)^t$$

Proof of latter. Consider $L_D - D_t = L_D - t\partial_t$ on $X_T(R_T Q^t) = X(RQ)^t$. Then what happens to ~~$(FPX)^t$~~ $(FPX)^t$ ~~?~~? What happens to I^t . Claim $(L_D - D_t)(I^t) \subset t^{-1}R^t$

suffices to check for generator of I^t .

$$0 \longrightarrow I^t \longrightarrow R^t \longrightarrow Q^t \longrightarrow 0$$

$$\downarrow$$

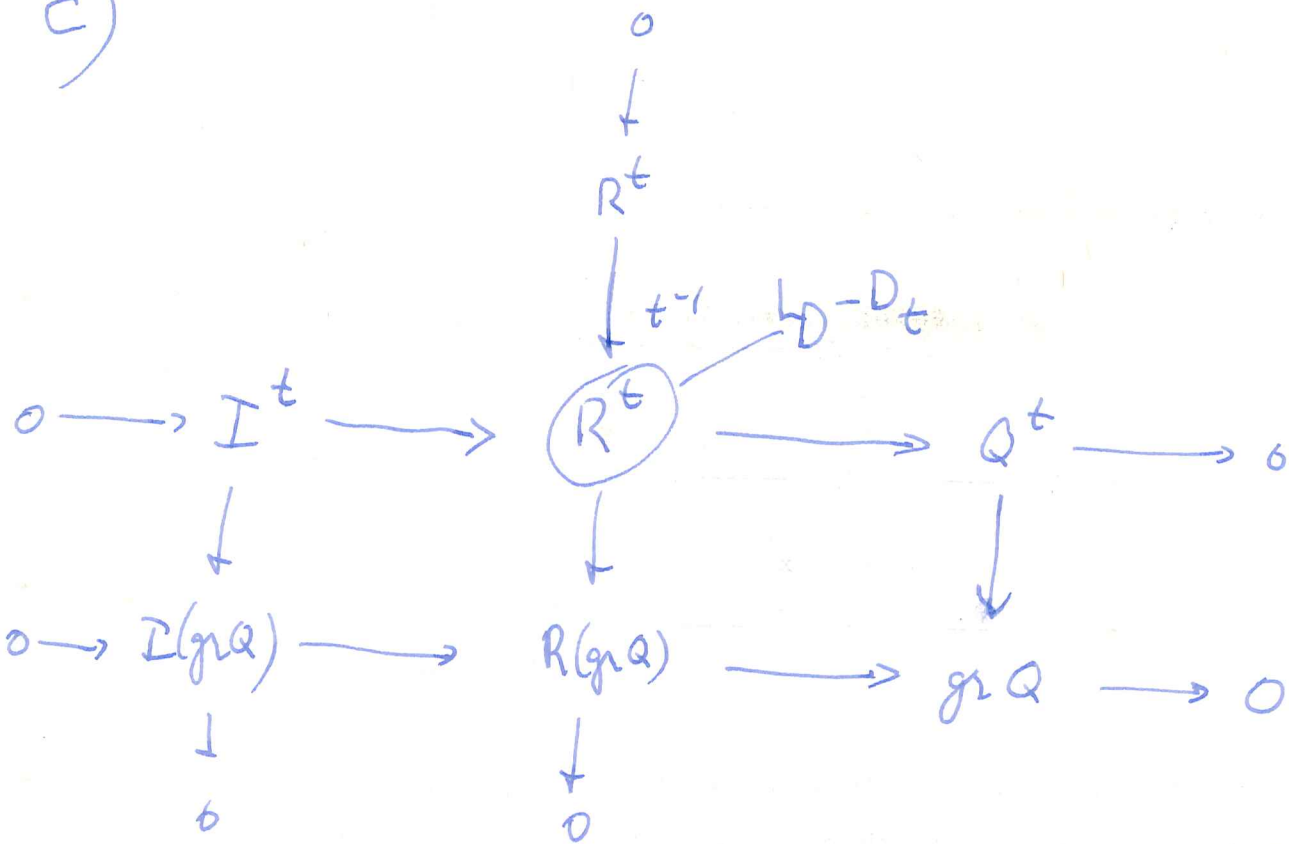
$$R^t / t^{-1}R^t$$

$$\parallel$$

$$0 \longrightarrow I(\text{gr} Q) \longrightarrow R(\text{gr} Q) \longrightarrow \text{gr} Q \longrightarrow 0$$

here $L_D = D_t$

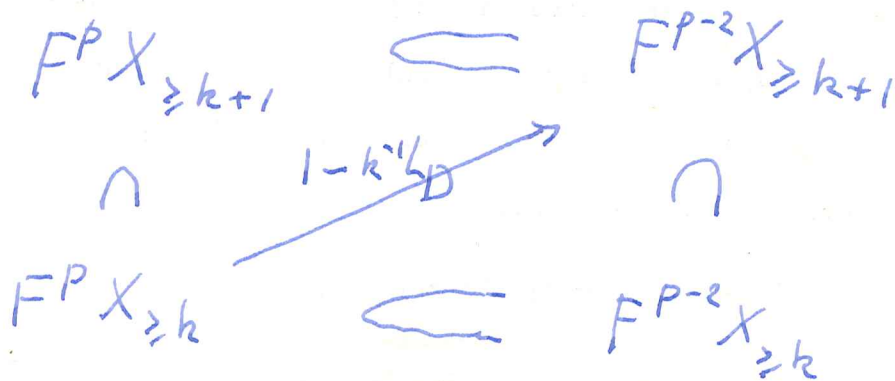
c)



$L_D - D_t : R^t \longrightarrow t^{-1}R^t$ So what next
 you idiot??

$$L_D^{-k} : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k+1}$$

$$h_D : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k}$$



The other point is ~~how to get from $F^p X_{\geq k}$~~

$$\begin{aligned}
 F^p X_{\geq k} &\sim \Theta(F^p(\Omega Q_{\geq k})) \\
 (X_{\geq k}^p = X_{\geq k}/F^p X_{\geq k}) &\sim \Theta(\Omega Q_{\geq k})
 \end{aligned}$$

$$D) \quad (IQ^n)_{\geq k} = \sum_{\sum k_i = k} (IQ)_{\geq k_1} \cdots (IQ)_{\geq k_n}$$

$$F^{2n} X_{\geq k} = IQ_{\geq k}^{n+1} + \sum_{i+j=n} [IQ_{\geq i}^n, RQ_{\geq j}] \iff \sum_{i+j=n} h(IQ_{\geq i}^n \cdot d(RQ_{\geq j}))$$

I want the minimum to say about Joachim's version of Nistor's construct.

~~Q filtered alg~~ Q filtered alg, w. filt $Q_{\geq n}$, linear splitting of filt.

$$X = X(RQ)$$

$$FPX = F_{IQ}^P X(RQ)$$

$$X_{\geq k} = (X(RQ))_{\geq k}$$

$$FPX_{\geq k} = (\text{---})_{\geq k}$$

$D =$ deriv. on RQ extending degree op. on Q
 L_D, h_D on X .

Claims ① $(X_{\geq n}^P = X_{\geq n} / FPX_{\geq n}) \sim \mathbb{P}(RQ_{\geq n})$

②

$$\begin{array}{ccc} FPX_{\geq n+1} & \xrightarrow{h_D} & F_{\geq n+1}^{P-2} \\ \cap & \nearrow S_n = 1 - \frac{1}{n} L_D & \cap \\ FPX_{\geq n} & \xrightarrow{h_D} & F_{\geq n}^{P-2} \end{array} \quad n > 0$$

gives

E)

$$\begin{array}{ccc}
 \mathcal{X}_{\geq n+1} & \xrightarrow{S} & \mathcal{X}_{\geq n+1} [2] \\
 \downarrow L_n & \nearrow S_n & \downarrow L_n \\
 \mathcal{X}_{\geq n} & \xrightarrow{S} & \mathcal{X}_{\geq n} [2]
 \end{array}$$

i.e.

$$[L_n] \in HC^0(\mathcal{X}_{\geq n+1}, \mathcal{X}_{\geq n}) = HC^0(\Omega_{\geq n+1}, \Omega_{\geq n})$$

$$[S_n] \in HC^2(\mathcal{X}_{\geq n}, \mathcal{X}_{\geq n+1}) = HC^2(\Omega_{\geq n}, \Omega_{\geq n+1})$$

satisfy

$$[S_n][L_n] = S \in HC^2(\Omega_{\geq n}, \Omega_{\geq n})$$

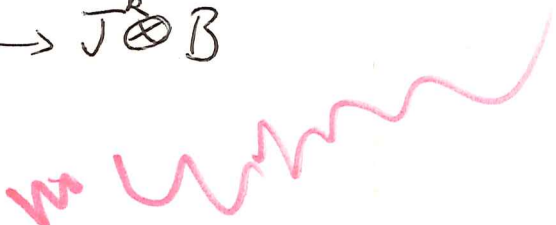
$$[L_n][S_n] = S \in HC^2(\Omega_{\geq n}, \Omega_{\geq n})$$

What's the next point?

Go back over Joachim's version of the Nistor construction

$$\begin{array}{ccccc}
 A & \begin{array}{c} \xrightarrow{L} \\ \xrightarrow{f} \end{array} & Q & \longrightarrow & L \otimes B \\
 & & \downarrow f^k & \longmapsto & J^k \otimes B
 \end{array}$$

$$A \xrightarrow{i} Q \longrightarrow \text{wavy line}$$



My map

$$X(RA) \longrightarrow X(S \otimes RB) \longrightarrow S_{\#} \otimes X(RB) \longrightarrow J_{\#}^{2n+1} \otimes X(RB)$$

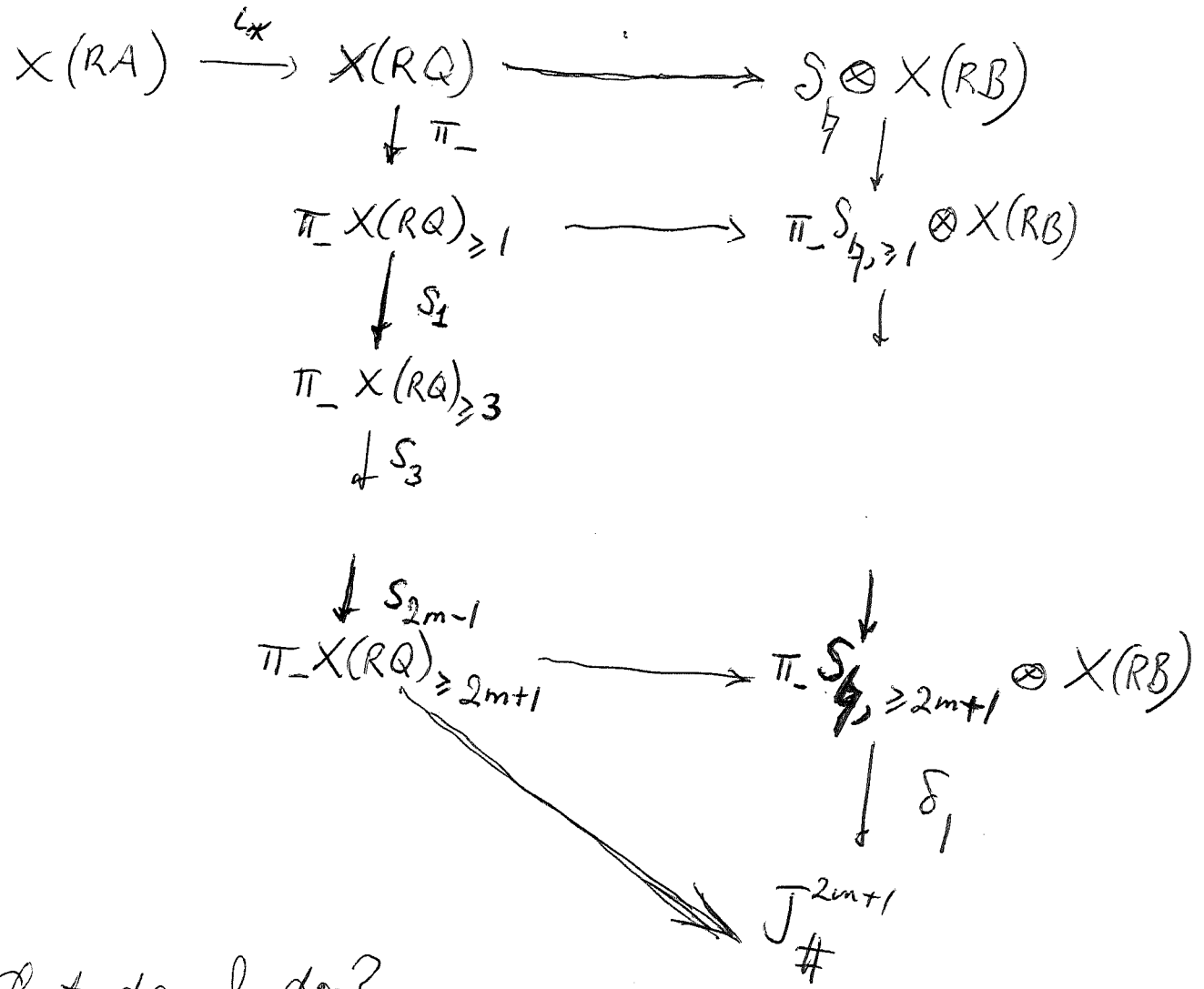
Joachim's map

$$\begin{array}{ccc}
 X(RA) & \longrightarrow & X(RQ) \longrightarrow \\
 & & \cup \\
 & & X(RQ)_{\geq k} \longrightarrow J_{\#}^k \otimes X(RB)
 \end{array}$$

F)

$$\begin{array}{ccc}
 X(RA) & \longrightarrow & S_{\frac{1}{2}} \otimes X(RB) \\
 \downarrow & & \downarrow \\
 X(RQ) & &
 \end{array}$$

I don't know what the next step should be, I guess I can take Toechim's construction



What do I do?

I need to know

$$\begin{array}{ccc}
 X(RQ) & \longrightarrow & S_{\frac{1}{2}} \otimes X(RB) \\
 \cup & & \cup \\
 X(RQ)_{\geq n} & \longrightarrow & S_{\frac{1}{2}, \geq n} \otimes X(RB) \\
 & & \downarrow
 \end{array}$$

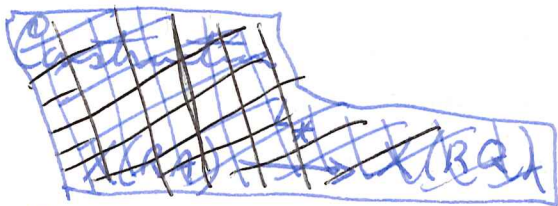
6) 7/23 -

Joachim's construction

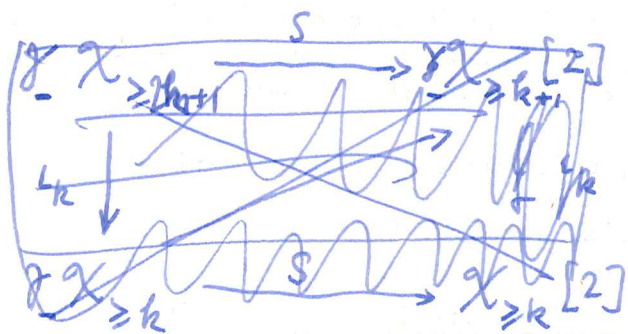
$Q = QA$ filt $Q_{\geq k}$, grading Q_n
 structure on $X = X(RQ)$, $FPX = F_{IQ}^P X(RQ)$

$X_{\geq k}$, $F^P X_{\geq k}$

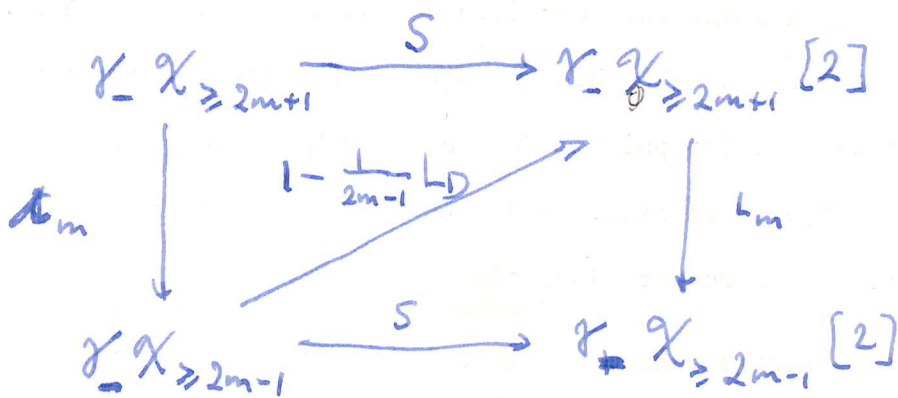
- Prop: 1) $L_D - k : F^P X_{\geq k} \longrightarrow F^{P-2} X_{\geq k+1}$
 2) $h_D : F^P X_{\geq k} \longrightarrow F^{P-2} X_{\geq k}$
 3) $\gamma - (-1)^k : F^P X_{\geq k} \longrightarrow F^P X_{\geq k+1}$



Consequence Put $\mathcal{X}_{\geq k} = (X_{\geq k}^P)$, $\mathcal{X}_{\geq k}^P = X_{\geq k} / F^P X_{\geq k}$
 Then

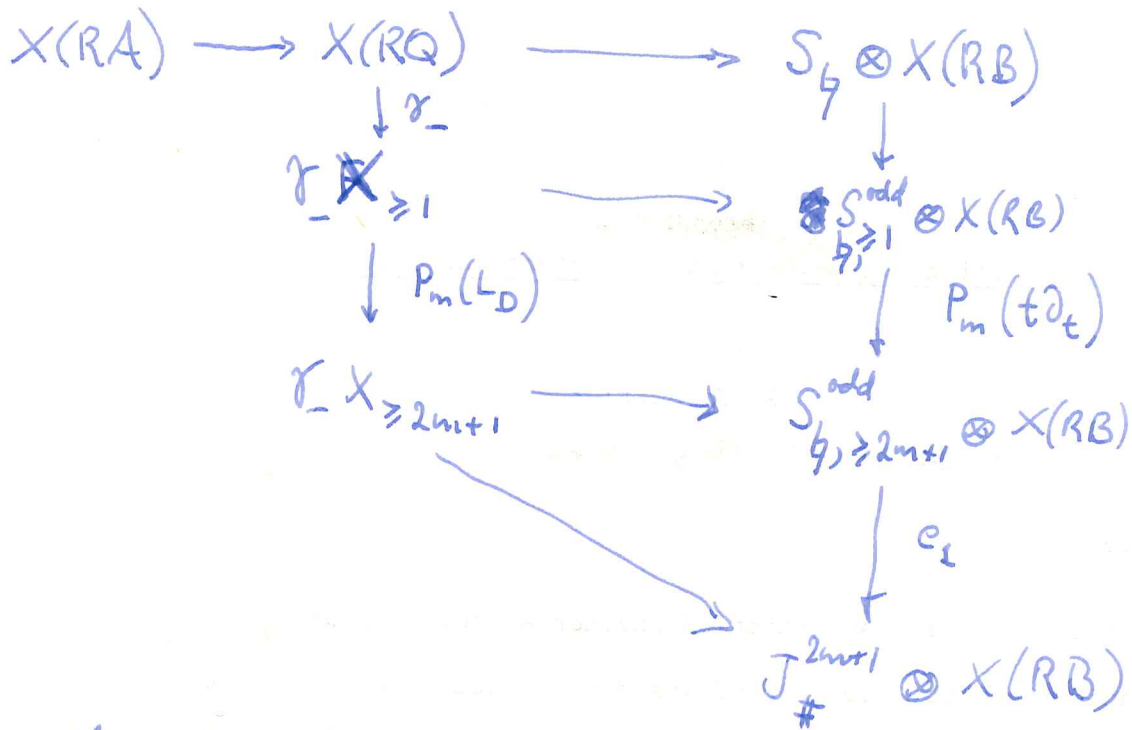


$1 - \frac{1}{2m-1} L_D : \gamma F^P X_{\geq 2m-1} \longrightarrow \gamma F^P X_{\geq 2m} = \gamma F^P X_{\geq 2m+1}$



$[S_m] \circ \dots \circ [S_1] \in HC^{2m}(\gamma X, \gamma X_{\geq 2m+1})$
 $[S_m] \circ \dots \circ [S_1] \circ \gamma \circ L \in HC^{2m}(X_A, \gamma X_{\geq 2m+1})$

H) Joachim-Nistor const.



Before this I should ~~give~~ give Joachim's map

$$X(RA) \xrightarrow{\gamma_i} \gamma_- X(RQ) \xrightarrow{P_m(L_D)} \gamma_- X(RQ)_{\geq 2m+1} \xrightarrow{(*)} J_{\#}^{2m+1} \otimes X(RB)$$

$$FP_{IA} X(RA) \rightarrow \gamma_- FP X_{\geq 1} \longrightarrow \gamma_- FP^{-2m} X_{\geq 2m+1} \longrightarrow J_{\#}^{2m+1} \otimes FP_{IB} X(RB)$$

What remains is the last map $(*)$, its properties, relation to $X(RQ) \longrightarrow S_{\mathbb{Z}} \otimes X(RB)$, the map induced by $X \circ R_0$ applied to

$$Q \xrightarrow[t^D]{\text{linear}} \bigoplus t^n Q_{\geq n} \xrightarrow{\text{hom.}} S \otimes B$$

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 I) $\mathcal{X}_{\geq k} = (X_{\geq k} / F^p X_{\geq k}) \sim Q_{\geq k}^b$

1) $L_D, h_D : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k}$

2) $L_D - k : F^p X_{\geq k} \longrightarrow F^{p-2} X_{\geq k+1}$

3) $\sigma - (-1)^k : F^p X_{\geq k} \longrightarrow F^p X_{\geq k+1}$

~~need~~ need $L_D = [\partial, h_D]$, $[L_D, h_D] = 0$

last needed ~~to~~ to say $\sigma = (-1)^{L_D}$

commute

$$\begin{array}{ccc}
 \sigma_- F^p X_{\geq k+1} & \longrightarrow & \sigma_- F^{p-2} X_{\geq k+1} \\
 \cap & \nearrow_{1 - \frac{1}{k} L_D} & \cap \\
 \sigma_- F^p X_{\geq k} & \longrightarrow & \sigma_- F^p X_{\geq k}
 \end{array}$$

$[D_D, \sigma_- h_D] = \sigma_- L_D$ Anyway

$$\begin{array}{ccc}
 \mathcal{X}_{\geq k+1} & \xrightarrow{s} & \mathcal{X}_{\geq k+1} [2] \\
 \downarrow \iota_k & \nearrow s_k & \downarrow \iota_k \\
 \mathcal{X}_{\geq k} & \xrightarrow{s} & \mathcal{X}_{\geq k} [2]
 \end{array}$$

so $[s_k]$ etc. etc. etc. etc.

Let's ~~try~~ try to define the character

$ch^0 \in HC^0(A, \sigma_- Q_{\geq 1}^b)$

$ch^0 : X(RA) \longrightarrow X(RQ) \xrightarrow{\sigma_-} \sigma_- X(RQ)$

J) Define Chern character - ~~#~~

$$X(RA) \xrightarrow{L_*} X(RQ) \xrightarrow{\gamma_-} \gamma_- X(RQ)_{\geq 1}$$

$$\underline{\gamma_- L_*} = \frac{1}{2} (L_* - L_*^{\sigma})$$

$$S_{2m-1} \cdot S_{2m-3} \cdots S_3 \cdot S_1 \cdot \gamma_- L_*$$

define $Ch^{2m} \in HC^{2m}(A, \underbrace{\gamma Q^b}_{\geq 2m+1})$

$$\gamma_-(\Omega Q)_{\geq 2m+1}$$

$$\gamma_-(\mathcal{X} Q)_{\geq 2m+1}$$

Mistor notation $Q^b =$ mixed ex assoc. to

So how to say all this. We have maps of supercomplexes

$$X(RA) \xrightarrow{L_*} X(RQ) \xrightarrow{\gamma_-} \gamma_- X(RQ)_{\geq 1}$$

$$\xrightarrow{S_1} \gamma_- X(RQ)_{\geq 3} \longrightarrow \dots$$

$$\xrightarrow{S_{2m-1}} \gamma_- X(RQ)_{\geq 2m+1}$$

$$\begin{matrix} FP \\ IA \end{matrix} X(RA) \longrightarrow \begin{matrix} FP \\ IQ \end{matrix} X(RQ) \longrightarrow \gamma_- FPX_{\geq 1}$$

$$\xrightarrow{S_1} \gamma_- FP^{-2} X_{\geq 3} \longrightarrow \dots$$

$$\xrightarrow{S_{2m-1}} \gamma_- FP^{-2m} X_{\geq 2m+1}$$

$$\therefore [S_{2m-1} S_{2m-3} \cdots S_3 S_1 \gamma_- L_*] \in HC^{2m}(A, \gamma_- \mathcal{X} Q_{\geq 2m+1})$$

K) Digress to have fun

Recall $T = \mathbb{C}[t^{-1}] = \mathbb{C}^t$

V filtered v.s. $V_{\geq k}$ ~~$V_{\geq k}$~~

$$V^t = \bigoplus t^k V_{\geq k} \subset \mathbb{C}[t, t^{-1}] \otimes V$$

Can identify a filtration on V with a graded \mathbb{C}^t submodule of $\mathbb{C}[t, t^{-1}] \otimes V$ in this way.

I would like to understand abstractly why $L_D = k : F^p X_{\geq k} \rightarrow F^{p-2} X_{\geq k+1}$ and $\gamma = (-1)^k : F^p X_{\geq k} \rightarrow F^p X_{\geq k+1}$.

This means forming $F^p X^t = \bigoplus t^k F^p X_{\geq k}$ which I know is $F^p_{I_T Q^t} X_T(R_T Q^t)$

To start with $D: \bar{Q} \rightarrow Q$ ~~deg of~~ only linear ~~map~~ but such that $\gamma = (-1)^D$ is comp. with alg. structure. $\therefore \gamma(IQ) \subset IQ$

You want

to consider L_D on $X^t = X(RQ)^t$. commutes with t^{-1} , T linear. Consider $F^p X^t = F^p_{I^t} X_T(R^t)$

Old proof gives $L_D, h_D : F^p X^t \rightarrow F^{p-2} X^t$

but we know $D: R^t \rightarrow R^t$

~~D carries~~ $D = D_t$ on Q^t no.

$$Q^t = \bigoplus t^k Q_{\geq k}$$

$$D - D_t : Q^t \rightarrow t^{-1} Q^t$$

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Joachim's version of Nistor's construction

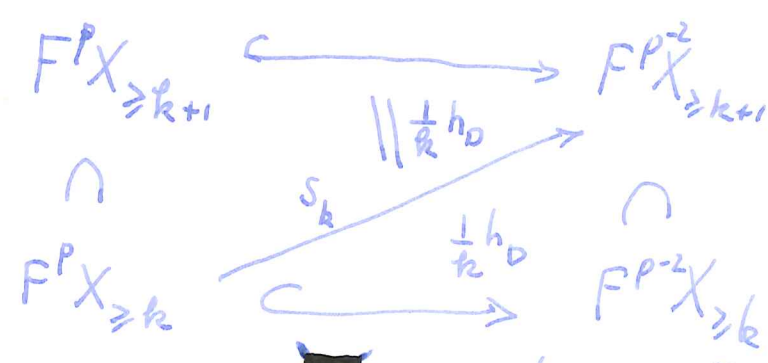
$Q = QA$ filt. $Q_{\geq k}$ comp. w. alg st.
 v.s. grading $Q = \bigoplus Q_n$ ———> filt.
 D degree op.

superalg $\gamma = (-1)^D$ alg. autom order 2
 resp. filt. + grading.

$X = X(RQ), \quad FPX = F_{IQ}^P X(RQ).$

$X_{\geq k}, F^P X_{\geq k}$ induced filt., $X_{\geq k} = (X_{\geq k} / F^P X_{\geq k})$

Claim: $X_{\geq k} \sim$ Hodge tower of $F_{-k_0}^b(QA, gA)^b$
 $\Omega Q_{\geq k}$ in Nistor's notation.



And I want to make ~~to myself~~ that

Claim: $L_D - k : FPX_{\ge k} \rightarrow FPX_{\ge k+1}^{(2)}$

Put $S_k = 1 - \frac{1}{k} L_D$ $k \neq 0$. Then

$X(RA) \xrightarrow{\iota_*} X(RQ) \xrightarrow{\gamma_-} \gamma_- X(RQ) = \gamma_- X(RQ)_{\geq 0}$

$\xrightarrow{S_1} \gamma_- X(RQ)_{\geq 2} = \gamma_- X(RQ)_{\geq 3}$

$\xrightarrow{S_{2m-1}} \gamma_- X(RQ)_{\geq 2m} = \gamma_- X(RQ)_{\geq 2m+1}$

M)

curves

$$F_A^p X(RA) \longrightarrow F_A^p X_{\geq 1} \xrightarrow{\gamma} \gamma F^p X = \gamma F^p X_{\geq 1}$$

$$\xrightarrow{S_1} \gamma F^{p-2} X_{\geq 2} = \gamma F^{p-2} X_{\geq 3}$$

$$\xrightarrow{S_{2m-1}} \gamma F^{p-2m} X_{\geq 2m} = \gamma F^{p-2m} X_{\geq 2m+1}$$

for all p whence we have a map of special towers

$$\mathcal{X}_A \longrightarrow \mathcal{X}_{Q, \geq 2m+1} [2m]$$

i.e. $ch^{2m} \in HC^{2m}(\Omega A, \Omega Q_{\geq 2m+1})$

Nistor uses the map

$$S_n \cdot S_{n-1} \cdots S_1 \gamma_{\bullet}$$

The point is that he constructs

$$[S_k] \in HC^2(\Omega Q_{\geq k}, \Omega Q_{\geq k+1}) \quad k \geq 1.$$

Then defines

$$ch^{2n} = [S_n] \cdots [S_1] [\gamma_{\bullet}] \in HC^{2n}(\Omega A, \Omega Q_{\geq n+1})$$

Point is that on the image γ_{\bullet} ~~S_k~~ for k even unnecessary.

N)

Next step - end map

$$Q \xrightarrow{t^D} Q^t \xrightarrow{\text{hom.}} L^t \otimes B$$

$$X(RQ) \xrightarrow{t^D} X(RQ)^t \xrightarrow{\text{hom.}} L^t \otimes X(RB)$$

The claim is that ~~we have~~ ^{they} homom.

$$Q^t \longrightarrow L^t \otimes B$$

which induces

$$X^t = X(RQ)^t \longrightarrow L^t \otimes X(RB)$$

$$FP X^t \longrightarrow L^t \otimes FP_{IB}$$

This amounts to a ^{compatible} map ~~map~~

$$FP X_{\geq k} \longrightarrow J_{\#}^k \otimes FP_{IB}$$

$$\text{or } X_{\geq k} \longrightarrow J_{\#}^k \otimes X(RB)$$

$$\cup \quad FP X_{\geq k} \longrightarrow J_{\#}^k \otimes FP_{IB}$$

Can define it.

hom $Q^t \longrightarrow L^t \otimes B$ of graded T-alg.

$$\begin{array}{ccc} \text{get } X_T(R_T Q^t) & \longrightarrow & X_{L^t}(R_{L^t}(L^t \otimes B)) \\ \cong \downarrow & & \downarrow \cong \\ X(RQ)^t & & L^t \otimes X(RB) \end{array} \quad , \quad \begin{array}{c} FP_{I_T Q^t} X_T(R_T Q^t) \\ \longrightarrow \\ FP_{I_{L^t}(L^t \otimes B)} X_{L^t}(R_{L^t}(L^t \otimes B)) \end{array}$$

o) Concretely have $Q \longrightarrow L \otimes B$
 $Q_{\geq k} \longrightarrow J^k \otimes B$

$$\begin{array}{ccc} \Omega Q & \longrightarrow & L \otimes \Omega B \\ \cup & & \cup \\ \Omega Q_{\geq k} & \longrightarrow & J^k \otimes \Omega B \end{array}$$

$$\begin{array}{ccc} & & \downarrow \\ & \nearrow & J^k_{\#} \otimes \Omega B \end{array}$$

compatible with b on ΩQ and ΩB .

$$X_{\geq k} \longrightarrow J^k_{\#} \otimes X(RB)$$

$$FPX_{\geq k} \longrightarrow J^k_{\#} \otimes FP_{IB}(RB)$$

~~yes give this for some practice~~

What is next? I need

~~yes give this for some practice~~

$$Q \xrightarrow{t^0} Q^t$$

Not really to do the Nistor const. all you need is the maps

$$X_{\geq k} \longrightarrow J^k_{\#} \otimes X(RB)$$

$$\begin{array}{ccc} \cup & & \\ FPX_{\geq k} & \longrightarrow & J^k_{\#} \otimes FP_{IB} X(RB). \end{array}$$

~~scribble~~
 anyway

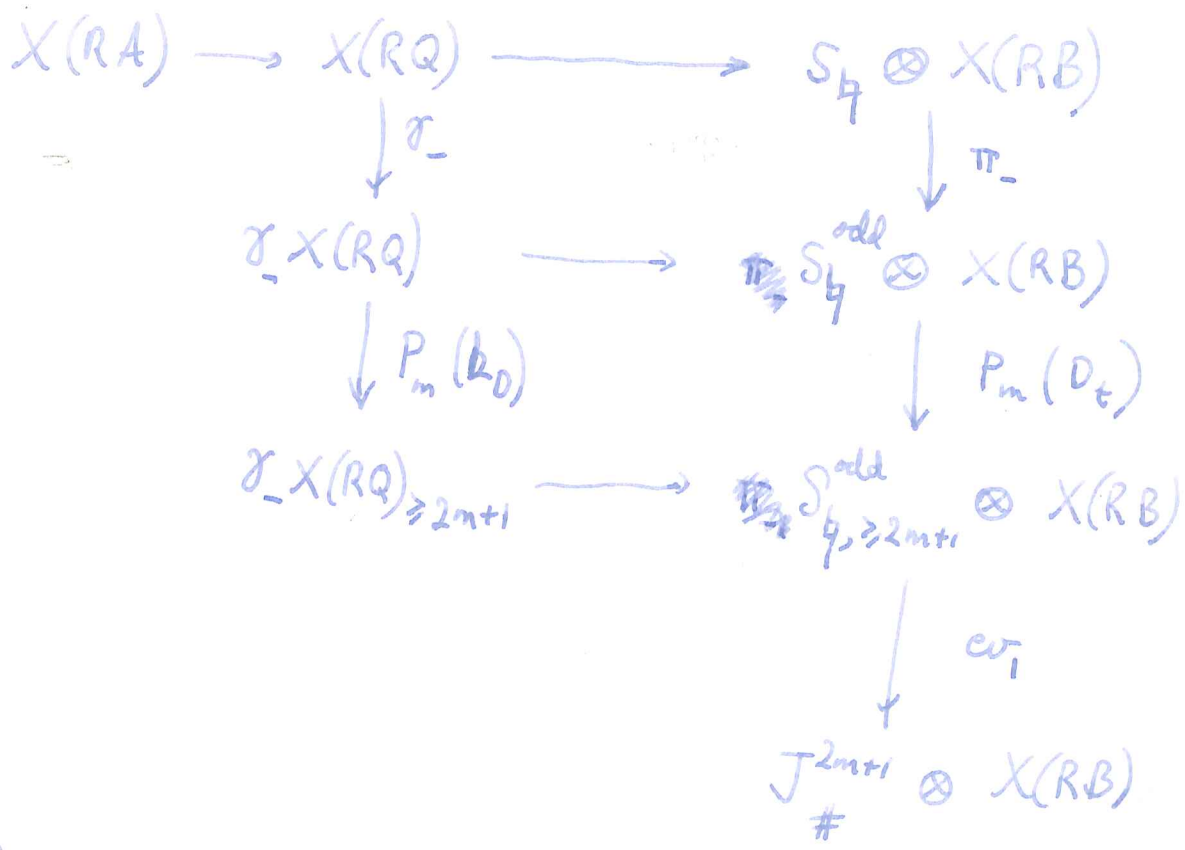
$$HC^0((\Omega Q)_{\geq k}, J^k_{\#} \otimes \Omega B).$$

So combine with

$$Ch^{2m} = [S_{2m-1}] [S_{2m-3}] \cdots [S_1] [\gamma - L_*] \in HC^{2m}(\Omega A, (\Omega Q)_{\geq 2m-1})$$

to get a class in $HC^{2m}(\Omega A, J^k_{\#} \otimes \Omega B)$.

P) Now I want to correlate with my construction



The basic point I am missing?



Your map



induced by



what I need is