November 8, 1993

Here is an improvement to the discussion of the Karoubi operator κ on $C(A)$, which arises from the Alexander-Spanier differential.

Notation: $C(A) = \bigoplus_{n\geq 0} A^{\otimes n+1}$. This can be identified with the graded algebra

$$T_A(A\otimes A) = A * \mathbb{C}[h] \qquad h = 1\otimes 1.$$

where $|h| = 1$. b' is the superderivation of degree -1 such that $b'(a)=0$, $b'(h)=1$. Define d to be the superderivation of degree $+1$ given by

$$d(a) = [h,a] \qquad d(h) = h^2$$

Then we have $\boxed{b'^2 = [b',d] = d^2 = 0}$.

Proof.
$$[b',d](a) = b'[h,a] = [1,a] - [h,0] = 0$$
$$[b',d](h) = b'(h^2) + d(1) = 1\cdot h - h\cdot 1 = 0.$$
$$d^2(a) = d[h,a] = [h^2,a] - [h,[h,a]] = 0.$$
$$d^2(h) = d(h^2) = (h^2)\cdot h - h(h^2) = 0.$$

Note: $C(A)$ equipped with b' is the standard normalized resolution of the A-bimodule A. d is the "Alexander-Spanier" differential:

$$\boxed{\begin{aligned} d(a_0,\dots,a_n) = (1,a_0,\dots,a_n) + \sum_{i=1}^{n}(-1)^i(\dots,a_{i-1},1,a_i,\dots) \\ + (-1)^{n+1}(a_0,\dots,a_n,1). \end{aligned}}$$

Why?
$$(a_0,\dots,a_n) = (a_0h)(a_1h)\cdots(a_{n-1}h)a_n$$
$$d(ah) = [h,a]h + ah^2 = h(ah)$$
$$\begin{aligned} d\{(a_0h)\cdots(a_{n-1}h)a_n\} = {} & h(a_0h)(a_1h)\cdots(a_{n-1}h)a_n \\ & - (a_0h)h(a_1h)\cdots\cdots a_n \\ & \cdots + (-1)^n(a_0h)\cdots(a_{n-1}h)(ha_n - a_nh) \end{aligned}$$

$$\begin{aligned} = \; & 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n \\ & - a_0 \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n \\ & \quad \cdots \\ & + (-1)^n \, a_0 \otimes \cdots \otimes a_{n-1} \otimes 1 \otimes a_n \\ & + (-1)^n \, a_0 \otimes \cdots \cdots \otimes a_n \otimes 1 . \end{aligned}$$

Recall that we defined K on $C(A)$ by

$$K = 1 - [b, s] = \lambda - sc$$

I claim that $\boxed{K = 1 - [b, d]}$

Proof. Since $[b', d] = 0$ we only need to calculate that $[c, d] = 1 - \lambda + sc$. We will use the simplicial structure on $C(A)$:

$$d_i(a_0, \dots, a_n) = \begin{cases} (\cdots, a_i a_{i+1}, \cdots) & 0 \leq i < n \\ (a_n a_0, a_1, \cdots, a_{n-1}) & i = n \end{cases}$$

$$s_i(a_0, \dots, a_n) = (\cdots, a_i, 1, a_{i+1}, \cdots) \qquad 0 \leq i \leq n$$

Note $c = (-1)^n d_n$ on $C(A)_n = A^{\otimes n+1}$ and also $d = s - s_0 + \cdots + (-1)^{n-1} s_n$. Thus

$$cd = cs - (-1)^{n+1} d_{n+1} \cdot \sum_{i=0}^{n} (-1)^i s_i$$

$$= -\lambda + (-1)^n \sum_{i=0}^{n} (-1)^i d_{n+1} s_i \quad\quad d_{n+1} s_i = \begin{cases} s_i d_n & 0 \leq i < n \\ 1 & i = n \end{cases}$$

$$dc = sc - \left(\sum_{i=0}^{n-1} (-1)^i s_i \right) (-1)^n d_n$$

Thus $cd + dc = -\lambda + 1 + sc$. $\square$.

We have a canonical algebra homoms.

$$\Omega A = T_A(\Omega^1 A) \xrightarrow{j} T_A(A\otimes A) = C(A)$$

which is compatible with d. Thus

$$j(a_0 da_1 \cdots da_n) = a_0[h, a_1]\cdots[h, a_n]$$

Note that $\boxed{b'j = 0}$, so $bj = cj$.

What is the cross-over term on $C(A)$ in terms of the description $T_A(A\otimes A)$. One has

$$A^{\otimes n+1} = T^n_A(A\otimes A) = T^{n-1}_A(A\otimes A)\otimes_{\cancel{A}}(\cancel{A}\otimes A)$$

so $c : T^n_A(A\otimes A) \longrightarrow T^{n-1}_A(A\otimes A)$ is given by

$$\boxed{c(\xi\otimes a) = (-1)^n a\xi \qquad \xi\in T^{n-1}_A(A\otimes A)}$$

Thus for $\omega\in\Omega^{n-1}A$ we have

$$\begin{aligned}
cj(\omega da) &= c\{j(\omega)\cdot(1\otimes a - a\otimes 1)\}\\
&= c\{j(\omega)\otimes a - j(\omega)a\otimes 1\}\\
&= (-1)^n\{aj(\omega) - j(\omega)a\}\\
&= (-1)^{n-1} j(\omega a - a\omega)\\
&= jb(\omega da)
\end{aligned}$$

Thus $bj = cj = jb$. Summarizing we have

$$\boxed{[j, d] = [j, b] = [j, \kappa] = 0}$$

Recall that we have a canonical exact sequence

$$\begin{tikzcd} 0 \arrow[r] & D(A) \arrow[r] & C(A) \arrow[r, "p"] & \Omega A \arrow[r] \arrow[l, bend right, dashed, "j"'] & 0 \end{tikzcd}$$

where $D(A)$ in degree n is the degenerate subcomplex $\sum_{i=0}^{n-1} s_i C(A)_{n-1}$. p is not an algebra homomorphism, but we have $\boxed{[p,b]=[p,d]=[p,K]=0}$. The formula $[p,d]=0$ follows from $d = s - \sum(-1)^i s_i$.

j gives a splitting of this sequence compatible with b, d, K. It's clear from the formula for j that its image is the standard complement for $D(A)$ given by the intersection of the kernels of all face operators by the last. Note that these face operators are the n different ~~maps~~

$$(A\otimes A)\otimes_A \cdots^{n} \cdots \otimes_A (A\otimes A) \longrightarrow (A\otimes A)\otimes_A \overset{n-1}{\cdots} \otimes_A (A\otimes A)$$

obtained by collapsing $A\otimes A$ to A via multiplication.

The preceding seems not very useful for cyclic homology purposes, since when we pass to the generalized eigenspace for K and the eigenvalue 1 we obtain $P\Omega A$ on which B is not exact.

Note that the fact that the d homology of $C(A)$ occurs at the eigenvalue 1 gives

$$\boxed{H(C(A), d) = H(P\Omega A, d) = \mathbb{C}.}$$

November 27, 1993

Note the reference Coquereaux - Kastler Pacific J Math 137, in which is constructed a lifting of ΩA into $\Omega\tilde{A}$ compatible with multiplication but not d. I think this can be understood by observing that

$$\Omega^1\tilde{A} = e\Omega^1\tilde{A}e \oplus e\Omega^1\tilde{A}e^{\perp} \oplus e^{\perp}\Omega^1\tilde{A} \oplus e^{\perp}\Omega^1\tilde{A}e^{\perp}$$

$$\simeq \Omega^1 A \oplus A_{(0)} \oplus {}_{(0)}A \oplus 0$$

Here $A_{(0)}$ denotes the $\tilde{A}$-bimodule on which the left (resp. right) multiplication by $a \in A$ is usual (resp. zero); similarly for ${}_{(0)}A$. One has $A_{(0)} \simeq e\Omega^1\tilde{A}e^{\perp}$, $a \longmapsto a\,de$, etc.

Since $\Omega^1 A$ is a direct summand of $\Omega^1\tilde{A}$ as $\tilde{A}$ bimodule the algebra

$$\Omega\tilde{A} = T_{\tilde{A}}(\Omega^1\tilde{A})$$

should have $\Omega A = T_{A}(\Omega^1 A) = T_{\tilde{A}}(\Omega^1 A)$ as subalgebra and quotient algebra, ~~[illegible]~~ better: $\Omega\tilde{A}$ has ΩA as retract.

Now consider $\mathfrak{a}$ nonunital and study $\Omega^1\tilde{\mathfrak{a}}$. Recall $\Omega^1\tilde{\mathfrak{a}} \xleftarrow{\sim} \mathfrak{a}\otimes\mathfrak{a} \oplus \mathfrak{a}$

$$a_0da_1,\ da_1 \longleftarrow\!\shortmid\ a_0\otimes a_1,\ a_1$$

Define $u : \Omega^1\tilde{\mathfrak{a}} \longrightarrow \mathfrak{a}_{(0)}$ by $\boxed{\begin{aligned} u(a_0da_1) &= a_0a_1 \\ u(da_1) &= a_1 \end{aligned}}$

Then u is a bimodule map over $\tilde{\mathfrak{a}}$. Check:

$$u(a a_0 da_1) = a a_0 a_1 = a\, u(a_0 da_1)$$
$$u(a\, da_1) = a a_1 = a\, u(da_1)$$

$$u(a_0 da_1 a) = u(a_0 d(a_1 a) - a_0 a_1 da)$$
$$= a_0 a_1 a - a_0 a_1 a$$
$$= 0 = u(a_0 da_1) a$$
$$u(da_1 a) = u(d(a_1 a) - a_1 da)$$
$$= a_1 a - a_1 a = 0 = u(da_1) a$$

Similarly there is a bimodule map

$$v : \Omega^1 \tilde{\mathcal{A}} \longrightarrow {}_{(0)}\mathcal{A} \qquad \boxed{\begin{aligned} v(a_0 da_1) &= 0 \\ v(da_1) &= a_1 \end{aligned}}$$

Check:
$$v(a a_0 da_1) = 0 = a\, v(a_0 da_1)$$
$$v(a \cdot da_1) = 0 = a\, v(da_1)$$
$$v(a_0 da_1 a) = v(a_0 d(a_1 a) - a_0 a_1 da)$$
$$= 0 = v(a_0 da_1) a$$
$$v(da_1 a) = v(d(a_1 a) - a_1 da) = a_1 a = v(da_1) a$$

We thus have exact sequences of $\tilde{\mathcal{A}}$ bimodules

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{A} d\mathcal{A} \arrow[d] & & \\
0 \arrow[r] & d\mathcal{A}\mathcal{A} \arrow[r] & \Omega^1 \tilde{\mathcal{A}} \arrow[r, "u"] \arrow[d, "v"] & \mathcal{A}_{(0)} \arrow[r] & 0 \\
 & & {}_{(0)}\mathcal{A} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Note that $u = v$ when $\mathcal{A}$ has 0 multiplication, so $d\mathcal{A}\mathcal{A} = \mathcal{A}d\mathcal{A}$: $da_0 a_1 = -a_0 da_1$

However <u>when</u> $\mathcal{A}$ <u>is unital</u>, <u>these</u> exact sequences split and are transverse.

Why? Let e be the identity of A

Then $l_u : a \longmapsto a\,de = a\,e\,de\,e^{\perp}$ is a bimodule lifting $A_{(0)} \longrightarrow \Omega^1\tilde{A}$ for u, and similarly $l_v : a \longmapsto de\,a = e^{\perp}de\,e\,a$ is a bimodule lifting ${}_{(0)}A \longrightarrow \Omega^1\tilde{A}$ for v. Also $v\,l_u = 0$ and $u\,l_v(a) = u(de\,a) = u(da - e\,da) = a - ea = 0.$

November 30, 1993

Remarks about K on $C(A)$. First the relation $[c, d] = 1 - K$ (p 260) can be proved using the formula

$$c(\xi h a) = (-1)^{|\xi|+1} a \xi$$

There's no need to use the s_i.

Next for

$$\begin{tikzcd} 0 \arrow[r] & D(A) \arrow[r] & C(A) \arrow[r, "p"] & \Omega A \arrow[r] & 0 \end{tikzcd}$$

One has $p(a_0 h a_1 \cdots a_{n-1} h a_n) = (a_0 \cdot d \cdot a_1 \cdots a_{n-1} \cdot d \cdot a_n) 1$

$= a_0 da_1 \cdots da_n$

p arises from the left action of $C(A)$ on ΩA where $a \in A$ acts by left multiplication by a on ΩA and h acts by the operator d on ΩA. Thus this sequence is a sequence of $C(A)$ modules and $D(A)$ is a left ideal in $C(A)$. It seems to be the sum of the ideal $C(A) h^2 C(A)$ and the left ideal $C(A) h$.

In future: the Coquereaux-Kastler paper suggests looking also at ΩA.

December 1, 1993

Let M be a mixed complex which is homotopy equivalent to zero, i.e. $\exists$ h of degree 1 such that $[b,h]=1$, $[B,h]=0$. Then $[b, hbh] = 1$, $[B, hbh] = 0$ so we can assume h special. Then

$$M = hM \oplus bM \quad \text{where} \quad hM \underset{b}{\overset{h}{\rightleftarrows}} bM$$

are inverse, so we have

$$M = (\mathbb{C} \oplus \mathbb{C}b) \otimes hM$$

where hM is closed under B. This shows a contractible mixed complex has the form $M = (\mathbb{C} \oplus \mathbb{C}b) \otimes N$ where N is a complex with differential B. Splitting (N, B) into a contractible (i.e. B-acyclic) subcomplex and a minimal subcomplex (i.e. $B=0$), we see M has Connes's property. Thus M <u>heq</u> $0 \implies M$ <u>has Connes's property</u>.

Another way to see this is to note that if a map $f: M \to M'$ of mixed complexes is ~ 0, then the induced map $H(M,B) \longrightarrow H(M',B)$ of the complexes with diffl b is also heq 0. Thus $M \mapsto H(H(M,B), b)$ is a functor on the homotopy category $\mathrm{Ho}\,\mathcal{C}_\Lambda$ of mixed complexes. Thus a mixed complex which is heq to one satisfying Connes's property also has Connes's property.

Mixed complexes with Connes's property probably do not form a thick subcategory of $\mathrm{Ho}\,\mathcal{C}_\Lambda$. Example: condition $Q' \cap \mathrm{Im}\, B = 0$ in the proof of Connes's lemma.

Note also that the homology $H_n(M/\mathrm{Ker}B, b)$ and $H_\nu(M, b+B)$ are functors on $\mathrm{Ho}\,\mathcal{C}_\Lambda$, but not $H_n(M/\mathrm{Ker}\,b, B)$. Also $H_n(M/\mathrm{Ker}\,B, b)$ and $H_\nu(M, b+B)$ are not functors in the derived category $D\mathcal{C}_\Lambda$.

A better result is that mixed complexes having Connes's property are precisely those which are homotopy equivalent to a B-acyclic mixed complex.

Consequences is that a quasi isomorphism between mixed complexes with Connes property is a homotopy equivalence (because this is true for B-acyclic mixed complexes)

Concerning K *on* C(A). If you use the Alexander-Spanier differential d instead of S, then K commutes with d, and there is a *canonical special contraction* $(1-K)^{-1}d = d(1-K)^{-1}$ on the degenerate complex.

December 2, 1993

Coquereaux-Kastler: $\mathcal{A}$ unital algebra with identity element e. Consider the right ideal $e\Omega\tilde{\mathcal{A}} \subset \Omega\tilde{\mathcal{A}}$. This is spanned by $a_0 da_1 \cdots da_n$ for all n and $a_i \in \mathcal{A}$. Thus $e\Omega\tilde{\mathcal{A}} = C(\mathcal{A})$ as graded vector spaces. The multiplocation on $e\Omega\tilde{\mathcal{A}}$ when transfered to $C(\mathcal{A})$ is

$$(a_0,\dots,a_n) \underset{\wedge}{\overset{*}{}} (a_{n+1},\dots,a_k) = \sum_{j=0}^{n} (-1)^{n-j} (a_0,\dots,a_j a_{j+1} \dots a_k)$$

(1)
$$= (a_0,\dots,a_n)(a_{n+1},\dots,a_k)$$
$$+ (-1)^n b'(a_0,\dots,a_n)(e,a_{n+1},\dots,a_k)$$

Recall the image of the homomorphism

$$\psi : \Omega A \longrightarrow C(A)$$
$$a_0 da_1 \cdots da_n \longmapsto a_0 [h,a_1] \cdots [h,a_n]$$

coincides with the complement $\bigcap_{i<\text{last}} \operatorname{Ker} d_i$ for $D(A)$ in $C(A)$. Thus $b'\psi = 0$. Therefore one has the CK observation that the products on $\psi(\Omega A)$ coming from $\Omega\tilde{\mathcal{A}}$ ~~and~~ from $C(A)$ coincide.

But actually for this coincidence we only need b' ~~to~~ vanish not the individual faces. Write (1) as $\xi * \eta = \xi . \eta + (-1)^{|\xi|} b'\xi . s\eta$. So $\xi * \eta = \xi\eta$ for all η if $b'\xi = 0$. Conversely taking $\eta = e$ we have $\xi * e = \xi + (-1)^{|\xi|} b'\xi . (e,e)$ and this implies $b'\xi = 0$; sticking a 1 at the right is injective as $c\lambda' s = 1$.

So it seems that under the identification

$e\Omega\tilde{A} = C(A)$ the subspace $e\Omega\tilde{A}e$ corresponds to the kernel of b', and the two products coincide on this subspace.

In future: describe $e\Omega\tilde{A}e$ better using $\Omega^1\tilde{A} = \Omega^1 A \oplus {}_0A \oplus A_0$ and describe the homomorphism. Go back to problem of a lifting for $\hat{R}\tilde{A} \longrightarrow \mathbb{C} \times \hat{R}A$.

December 14, 1993

Motivation: We know that $P\Omega A$ [redacted] is almost B-acyclic; it fails only because of $\mathbb{C} = P\Omega\mathbb{C}$. We have an [redacted] exact sequence of mixed complexes

$$0 \longrightarrow \mathbb{C} \longrightarrow P\Omega A \longrightarrow P\bar{\Omega} A \longrightarrow 0$$

where $P\bar{\Omega}A$ is B-acyclic (i.e. free). This sequence splits compatibly with B, yielding a map $P\bar{\Omega}A \longrightarrow \mathbb{C}[1]$ of mixed complexes such that $P\Omega A$ is the h-fibre. We can lift this map [redacted] as follows:

$$\circledast \qquad \begin{tikzcd} & \Lambda \tilde{\otimes} B(\mathbb{C})[1] \arrow[d] \\ P\bar{\Omega}A \arrow[r] \arrow[ur, dashed] & \mathbb{C}[1] \end{tikzcd}$$

since $P\bar{\Omega}A$ is free and the vertical map is a quis. Taking the h-fibre F of the lift we get

$$\begin{tikzcd} 0 \arrow[r] & \Lambda \otimes B(\mathbb{C}) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & P\bar{\Omega}A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & P\Omega A \arrow[r] & P\bar{\Omega}A \arrow[r] & 0 \end{tikzcd}$$

where the [redacted] vertical arrows are quis. Then F is a minimal free cover of $P\Omega A$, and perhaps we might locate it inside $C(A)$ or $\bar{\Omega}\tilde{A}$ or $P\bar{\Omega}\tilde{A}$.

Now we know that if we split the exact sequence of complexes

$$0 \longrightarrow \bar{C}^\lambda[1] \longrightarrow P\bar{\Omega} \longrightarrow \bar{C}^\lambda \longrightarrow 0$$

then we get an S operator on $\bar{C}^\lambda$, and further that $P\bar{\Omega} = \Lambda \tilde{\otimes} \bar{C}^\lambda$. The adjunction property for $\Lambda\tilde{\otimes}-$ and $B(-)$ reduce to lifting in the diagram of S-modules

$$\begin{tikzcd}
 & \mathcal{B}(\Lambda\tilde{\otimes}\mathcal{B}(\mathbb{C}))[1] \arrow[d] \\
\bar{C}^\lambda \arrow[r] \arrow[ru, dashed] & \mathcal{B}(\mathbb{C})[1]
\end{tikzcd}$$

where its possible because of general properties of adjoint functors: $\begin{tikzcd} GFG \arrow[r, shift right, twoheadrightarrow] & G \arrow[l, shift right, dashed] \end{tikzcd}$

So there's a canonical way to obtain a lifting ⊛ once we have an "explicit S-operator" on $\bar{C}^\lambda$.

Let's examine this abstractly. Suppose M is a free mixed complex. Then $\exists$ h of degree -1 on M such that $[B,h] = 1$. One has

$$[b,[b,h]] = [b^2,h] = 0$$

$$[B,[b,h]] = -[b,[B,h]] = -[b,1] = 0$$

Thus $[b,h]$ is a map of mixed complexes: $M \longrightarrow M[2]$. In particular it gives a map of complexes $M/BM \longrightarrow M/BM[2]$.

Example. $M = \Lambda\tilde{\otimes}Q = 1\otimes Q + B\otimes Q$ where (Q,d,S) is an S-module. Recall

$$b(1\otimes x + B\otimes y) = 1\otimes dx - B\otimes Sx - B\otimes dy$$

$$B(\text{———————}) = B\otimes x.$$

Thus

$$b = \begin{pmatrix} d & 0 \\ -S & -d \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and if} \quad h = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then $[B,h] = 1$, and

$$[b,h] = \left[\begin{pmatrix} d & \\ -S & -d \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} -S & \cancel{d-d} \\ & -S \end{pmatrix}$$

This h is special, but let's return to a general contraction h for B. Exact sequence of cxs.

$$\begin{tikzcd} 0 \arrow[r] & M/BM[1] \arrow[r, "i"] & M \arrow[r, "j"] \arrow[l, dashed, bend right, "r"'] & M/BM \arrow[r] \arrow[l, dashed, bend right, "l"'] & 0 \end{tikzcd}$$

Now h determines a splitting of this sequence (ignoring b). Set $\left(\begin{array}{l} l = hi \\ r = jh \end{array} \right.$

Then $\left. \begin{array}{l} lj = hij = hB \\ ir = ijh = Bh \end{array} \right) \Rightarrow ir + lj = hB + Bh = 1$

Also $jlj = jhB = j(1-Bh) = j \Rightarrow jl = 1$ as j is surjective. Similarly $iri = Bhi = (1-hB)i = i \Rightarrow ri = 1$ as i is injective. Also $irlj = Bh^2B = h^2B^2 = 0$ $\Rightarrow rl = 0$. Note $[B,h^2] = [B,h]h - h[B,h] = 1\cdot h - h\cdot 1 = 0$.

Thus we have the splitting determined by h.

Also we have ~~$lr = hBh$~~

$$lr = hijh = hBh$$

is the special contraction associated to h.

One has $l' = (lr)i = l$ so we get

$$r' = j(lr) = r$$

the same splitting from $h' = hBh$.

Recall that the [illegible] splitting l, r determines S on M/BM by

$$-S = r[b,l] = [b,r]l$$

(These are equal as $0 = [b, rl] = [b,r]l - r[b,l]$ as r has degree -1.)

$$[b,h]i = [b,hi] = [b,l] = -iS$$

$$j[b,h] = [b,jh] = [b,r] \underset{\substack{\uparrow \\ \text{why?}}}{=} -Sj$$

$$[b,r] = [b,r](ir + lj) = [b,r]lj$$
$$= -Sj$$

$j[b,h]$ [redacted] $= -Sj$ means $[b,h]$ induces $-S$ on M/BM. Note that

$$r[b,h]l = jh(bh+hb)hi = j(hbh^2 + h^2bh)i$$

vanishes [redacted] when h is special.

To summarize we find that when $[B,h] = 1$, then $[b,h]$ induces $-S$ on M/BM.

December 15, 1993

Small observation about motivating B. Starting with d, b on Ω we get K defined by $1 - K = [b, d]$ and we derive the formulas $K^n = 1 + K^n bd$, $K^{n+1} = 1 - db$ on Ω^n. Then we get the polynomial relation, the spectral projection P with $P\Omega = \mathrm{Ker}(1-K)^2$. Then we have on $P\Omega$

$$K^n = (1 + K - 1)^n = 1 + n(K-1) = 1 + bd$$

$$K^{n+1} = \left(\qquad \right)^{n+1} = 1 + (n+1)(K-1) = 1 - db$$

so $\quad \frac{1}{n} bd = -\frac{1}{n+1} db \quad$ or $\quad b((n+1)d) + (nd)b = 0.$

Thus if we define $B = NPd$ we have $bB + Bb = 0$ on both $P\Omega$ and $P^{\perp}\Omega$, so on Ω.

∎

December 23, 1993

I propose to analyze standard bimodule resolutions of A. The standard normalized resolution is the DG algebra given by the graded algebra

$$A * \mathbb{C}[\varepsilon] \qquad \begin{array}{l} |a| = 0 \\ |\varepsilon| = 1 \end{array} , \quad \varepsilon^2 = 0$$

equipped with diffl b', where

$$b'(a) = 0 \qquad b'(\varepsilon) = 1$$

One has the 1-1 correspondence

$$(a_0, \dots, a_{n+1}) \longleftrightarrow a_0 \varepsilon a_1 \varepsilon \cdots a_n \varepsilon a_{n+1}$$
$$= a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1}$$

between $A \otimes \bar{A}^{\otimes n} \otimes A$ and $(A * \mathbb{C}[\varepsilon])_{n+1}$ since

$$b'(a_0[\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1}) = (-1)^n a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] a_{n+1}$$
$$= (-1)^n a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_{n-1}] (\varepsilon a_n a_{n+1} - a_n \varepsilon a_{n+1})$$

we have the correspondence

$$\Omega^n A \otimes A \quad \simeq \quad (A * \mathbb{C}[\varepsilon])_{n+1} \qquad n \geq 0$$
$$a_0 da_1 \cdots da_n \otimes a_{n+1} \longleftrightarrow a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1}$$

such that b' on the left is my formula

$$b'(\omega da \otimes a') = (-1)^{|\omega|+1} (\omega \otimes aa' - \omega a \otimes a')$$
$$= (-1)^{|\omega|} (\omega a \otimes a' - \omega \otimes aa')$$

On $A * \mathbb{C}[\varepsilon]$ we have the $\overbrace{\text{superderivation}}^{ad(\varepsilon)}$ of degree +1 and square zero such that

$$[b', ad(\varepsilon)] = 0$$

Thus $\mathrm{Ker}(b')$ is a graded DG subalgebra of $A * \mathbb{C}[\varepsilon]$ equipped with diffl $ad(\varepsilon)$. It can be canonically identified with ΩA. Thus we have subalgebras $\mathbb{C}[\varepsilon]$, ΩA of $A * \mathbb{C}[\varepsilon]$.

Claim one has linear isomorphisms

$$\mathbb{C}[\varepsilon] \otimes \Omega A \xrightarrow{\sim} A * \mathbb{C}[\varepsilon].$$

$$\Omega A \otimes \mathbb{C}[\varepsilon] \xrightarrow{\sim} A * \mathbb{C}[\varepsilon]$$

given by multiplication in $A * \mathbb{C}[\varepsilon]$. The former follows from the fact that b' and l_ε, left mult. by ε, satisfy the CAR $(b')^2 = (l_\varepsilon)^2 = 0$ $[b', l_\varepsilon] = 1$. The latter is similar with

$$r_\varepsilon(\alpha) = (-1)^{|\alpha|} \alpha \varepsilon \qquad \alpha \in A * \mathbb{C}[\varepsilon]$$

So $\Omega A = \mathrm{Ker}(b')$ is a canonical subspace of $A * \mathbb{C}[\varepsilon]$, but we have two choices at least of complements $\varepsilon \Omega A$ and $\Omega A \varepsilon$.

We can also average, namely

$$\left[b', \frac{l_\varepsilon + r_\varepsilon}{2}\right] = 1 \quad , \quad \left(\frac{l_\varepsilon + r_\varepsilon}{2}\right)^2 = 0$$

Check: $$l_\varepsilon r_\varepsilon(\alpha) = (-1)^{|\alpha|} l_\varepsilon(\alpha\varepsilon) = (-1)^{|\alpha|} \varepsilon \alpha \varepsilon$$
$$r_\varepsilon l_\varepsilon(\alpha) = r_\varepsilon(\varepsilon\alpha) = (-1)^{|\alpha|+1} \varepsilon \alpha \varepsilon.$$

$$b' r_\varepsilon \alpha = (-1)^{|\alpha|} b'(\alpha\varepsilon) = (-1)^{|\alpha|}(b'\alpha)\varepsilon + \alpha$$
$$r_\varepsilon b' \alpha = (-1)^{|\alpha|+1}(b'\alpha)\varepsilon. \qquad \therefore [b', r_\varepsilon] = 1$$

The significance of this is not clear.

Next consider applying $M \longmapsto M \otimes_A = M_\natural$ to the resolution $A * \mathbb{C}[\varepsilon]$. Begin with the standard isomorphism

$$(\Omega^n A \otimes A)_\natural \simeq \Omega^n A$$

$$\natural(a_0 da_1 .. da_n \otimes a_{n+1}) \longleftrightarrow a_{n+1} a_0 da_1 .. da_n$$

This gives the isomorphism

$$\left((A * \mathbb{C}[\varepsilon])_{n+1}\right)_\natural \simeq \Omega^n A \qquad n \geq 0$$

$$\natural(a_0[\varepsilon,a_1] \cdots [\varepsilon,a_n]\varepsilon\, a_{n+1}) \longleftrightarrow a_{n+1} a_0\, da_1 .. da_n$$

Let's compute the composition

$$\Omega A \simeq \mathrm{Ker}(b') \subset A * \mathbb{C}[\varepsilon] \xrightarrow{\natural} \Omega A[1] \oplus A_\natural$$

$$a_0 da_1 . da_n \longmapsto a_0[\varepsilon,a_1] \cdots [\varepsilon,a_n] =$$

$$a_0[\varepsilon,a_1] .. [\varepsilon,a_{n-1}]\varepsilon a_n - a_0[\varepsilon,a_1] .. [\varepsilon,a_{n-1}] a_n \varepsilon$$

$$\overset{\natural}{\longmapsto} a_n a_0 da_1 .. da_{n-1} - a_0 da_1 .. da_{n-1} a_n$$

$$= (-1)^n b(a_0 da_1 ... da_n)$$

What's clear is that

$$\Omega A \xrightarrow{\sim} (\Omega A)\varepsilon \subset A * \mathbb{C}[\varepsilon] \xrightarrow{\natural} \Omega A[1] \oplus A_\natural$$

$$a_0 da_1 . da_n \longmapsto a_0[\varepsilon,a_1] .. [\varepsilon,a_n]\varepsilon \longmapsto a_0 da_1 .. da_n$$

is the identity essentially. On the other hand

$$\Omega A \xrightarrow{\sim} \varepsilon(\Omega A) \subset A * \mathbb{C}[\varepsilon] \xrightarrow{\natural} \Omega A[1] \oplus A_\natural$$

$$a_0 da_1 . da_n \longmapsto \varepsilon a_0[\varepsilon,a_1] .. [\varepsilon,a_n] =$$

$$[\varepsilon,a_0][\varepsilon,a_1] .. [\varepsilon,a_n] + \overbrace{a_0 \varepsilon[\varepsilon,a_1] .. [\varepsilon,a_n]}^{(-1)^n}$$

$$\overset{\natural}{\longmapsto} (-1)^{n+1} b(da_0 .. da_n) + (-1)^n a_0 da_1 .. da_n$$

This map is $1-bd = \kappa + db$ up to sign.

December 24, 1993

Using $A * \mathbb{C}[\varepsilon] = \Omega A \oplus \Omega A \otimes \varepsilon$, let's calculate $(A*\mathbb{C}[\varepsilon])\otimes_A$ and $(A*\mathbb{C}[\varepsilon])/[-,-]$. As usual suppress $\otimes$ signs. We have seen

$$(A*\mathbb{C}[\varepsilon])_n \otimes_A \simeq \begin{cases} A_\natural & n=0 \\ (\Omega^{n-1}A)\varepsilon & n>0 \end{cases}$$

but let's check this directly. One has

$$[a, \omega + \eta\varepsilon] = [a,\omega] + [a,\eta]\varepsilon - \eta da$$

so

$$[A, A*\mathbb{C}[\varepsilon]] = [A,\Omega A] + \{-\eta da + [a,\eta]\varepsilon \mid \eta \in \Omega A\}$$

This should be complementary to $\Omega A\varepsilon$ in degrees >0. Suppose

$$[a,\omega] - \eta da + [a,\eta]\varepsilon = \xi\varepsilon.$$

Actually things look nicer if we put a on the right:

$$[\omega_i + \eta_i\varepsilon, a_i] = [\omega_i, a_i] + \eta_i da_i + [\eta_i, a_i]\varepsilon$$

(use summation convention)

Suppose this $= \xi\varepsilon$

Then $\xi = [\eta_i, a_i]$ and $[\omega_i, a_i] + \eta_i da_i = 0$. Apply b to the latter & use $b[\omega_i, a_i] = 0$ as $b^2=0$. Get $[\eta_i, a_i] = 0$, so $\xi = 0$. Thus

$$[A*\mathbb{C}[\varepsilon], A] \cap (\Omega A)\varepsilon = 0$$

On the other hand the sum of these subspaces contains $(\Omega A)\varepsilon$ and all

$[\omega, a] + \eta\, da$ with $a \in A$, $\omega, \eta \in \Omega A$.

Taking $\omega = 0$ get $\Omega^{>0}A$, so

$$[A * \mathbb{C}[\varepsilon], A] + (\Omega A)\varepsilon = \left([A,A] + \Omega^{>0}A\right) \oplus (\Omega A)\varepsilon$$

Next $[\varepsilon, \omega + \eta\varepsilon] = d\omega + d\eta\,\varepsilon$. We want to calculate $(A * \mathbb{C}[\varepsilon])\otimes_A$ modulo the image of $d\Omega A + (d\Omega A)\varepsilon$. Under the isom

$$(A * \mathbb{C}[\varepsilon])\otimes_A \simeq \Omega A[1] \oplus A_\natural$$

$(d\Omega A)\varepsilon$ goes into $d\Omega A$. Let's calculate what happens to $da_0 da_1 \cdots da_n = da_0 \cdots da_{n-1}(\varepsilon a_n - a_n\varepsilon)$. The image in $(A * \mathbb{C}[\varepsilon])\otimes_A$ is

$$\natural\{(a_n da_0 \cdots da_{n-1} - da_0 \cdots da_{n-1} a_n)\varepsilon\}$$

which corresponds to the following elt of ΩA

$$-[da_0 \cdots da_{n-1}, a_n] = (-1)^{n+1}\, bd(a_0 da_1 \cdots da_n)$$

Thus

$$A * \mathbb{C}[\varepsilon] / [-,A] + [-,\varepsilon] = \left(\Omega A / d\Omega A + bd\Omega A\right)[1] \oplus A_\natural$$

But $\Omega A / d\Omega A + bd\Omega A = \Omega A / d\Omega A + (1-\kappa)\Omega A$ is the reduced cyclic complex except in degree zero where it is A instead of $\bar{A}$.

Let us next consider standard bimodule resolutions for A and $\tilde{A}$. There are four

$$\begin{tikzcd} \tilde{A} * \mathbb{C}[h] \arrow[d] & A * \mathbb{C}[h] \arrow[d] \\ \tilde{A} * \mathbb{C}[\varepsilon] & A * \mathbb{C}[\varepsilon] \end{tikzcd}$$

The point is that $\tilde{A} * \mathbb{C}[\varepsilon]$ and $A * \mathbb{C}[h]$ are very close.

We are assuming that is unital so that $\tilde{A} \xrightarrow{\sim} \mathbb{C} \times A$; let e denote the identity of A.

Recall the GNS algebra and the result

$$\Gamma(A' \longrightarrow RA' * C) = A' * \tilde{C}$$

If we apply this when A', C are respectively $\mathbb{C}[\varepsilon]$, A we get

$$* \qquad \boxed{\Gamma(\mathbb{C}[\varepsilon] \longrightarrow \mathbb{C}[h] * A) = \mathbb{C}[\varepsilon] * \tilde{A}}$$

Here $R(\mathbb{C}[\varepsilon]) = \mathbb{C}[h]$ with $h = \rho(\varepsilon)$. This means that standard normalized resolution DG algebra for $\tilde{A}$ is the GNS algebra for dilating h in the standard unnormalized resolution of A to a special contraction. In terms of DG modules:

A DG module over $\tilde{A} * \mathbb{C}[\varepsilon]$ is a complex E of $\tilde{A}$ modules equipped with special contraction indepedent of the $\tilde{A}$ module structure. E then splits $eE \oplus e^{\perp}E$, where eE is a complex of A modules and A acts on $e^{\perp}E$ by zero. Then ε is a dilation of the contraction $e\varepsilon e$ on eE.

Recall that in the GNS algebra

$$\Gamma = \Gamma(A' \longrightarrow B) = A \oplus A\otimes B\otimes A$$

the element $e = 1\otimes 1\otimes 1$ is idempotent and $e\Gamma e = B$. Thus from $*$ we get

$$\boxed{e(\tilde{A} * \mathbb{C}[\varepsilon])e = A * \mathbb{C}[h]}$$

and this is easy to understand because in degree n we have

$$\begin{tikzcd}[row sep=tiny] e(\tilde{A} \underset{\cup}{*} \mathbb{C}[\varepsilon])_n e = e(\tilde{A}\otimes \tilde{A}^{\otimes n-1}\otimes \tilde{A})e = A^{\otimes n+1} \\ a_0\varepsilon a_1 \cdots \varepsilon a_n \arrow[r, leftrightarrow] & \overset{\cup}{a_0 h a_1 \cdots h a_n} \end{tikzcd}$$

Thus we have homomorphisms

$$\begin{tikzcd} \tilde{A} * \mathbb{C}[\varepsilon] \arrow[r, phantom, "\underset{\text{nonunital}}{\supset}"] & e(\tilde{A}*\mathbb{C}[\varepsilon])e \arrow[r, "\sim"] & A * \mathbb{C}[h] \arrow[d] \\ & & A * \mathbb{C}[\varepsilon] \end{tikzcd}$$

Let's now explore Kadison's viewpoint. The idea is to work relative to the separable subalgebra $S = \tilde{\mathbb{C}} \subset \tilde{A}$.

$$\begin{tikzcd} T_{\tilde{A}}(\tilde{A}\otimes\tilde{A}) \arrow[r, two heads, "\pi"] \arrow[d, equal] & T_{\tilde{A}}(\tilde{A}\otimes_{\tilde{\mathbb{C}}}\tilde{A}) \arrow[d, equal] \\ \underbrace{\tilde{A} * \mathbb{C}[h]}_{\text{standard resolution}} \arrow[r, two heads, "\pi"] & \underbrace{\tilde{A}*\mathbb{C}[h]/([e,h])}_{\text{relative std. resolution}} \end{tikzcd}$$

Check. $\tilde{A}\otimes_{\tilde{\mathbb{C}}}\tilde{A}$ is the quotient of the bimodule $\tilde{A}\otimes\tilde{A}$ given by the relations $s(1\otimes 1) = (1\otimes 1)s$ for $s\in\tilde{\mathbb{C}}$, i.e. $eh = he$.

We get a lifting for this surjection π by sending $h \in \tilde{A} * \mathbb{C}[h]/([e,h])$ to the element $ehe + e^\perp h e^\perp \in \tilde{A} * \mathbb{C}[h]$ which commutes with e, and satisfies

$$b'(ehe + e^\perp h e^\perp) = e^2 + (e^\perp)^2 = 1 .$$

When we try to do something similar for the normalized resolutions

$$(*) \qquad \tilde{A} * \mathbb{C}[\varepsilon] \xrightarrow{\pi} \overbrace{\tilde{A} * \mathbb{C}[\varepsilon] / ([e,\varepsilon])}^{\mathbb{C}[\varepsilon] \times (A * \mathbb{C}[\varepsilon])}$$

it doesn't work because $(e\varepsilon e + e^\perp \varepsilon e^\perp)^2 \neq 0$.

~~[illegible]~~

In fact

$$(ehe)^2 = ehehe \quad \cancel{eh^2e + eh[e,h]e} \quad \cancel{eh(1-e)he}$$

$$= eh([e,h] + he)e$$

$$= \underbrace{eh[e,h]e} + eh^2e$$

[illegible] $eh(1-e)[e,h] = e[h,1-e][e,h] = e[h,e]^2$

so

$$\boxed{(ehe)^2 = eh^2e + e[h,e]^2}$$

$$\boxed{(e\varepsilon e)^2 = e[\varepsilon,e]^2}$$

Yet π should be a homotopy equivalence of $\tilde{A}$-bimodule resolutions of $\tilde{A}$.

Critical case to examine: $A = \mathbb{C}$, whence $(*)$ is

$$\mathbb{C}[e] * \mathbb{C}[\varepsilon] \longrightarrow \mathbb{C}[e] \otimes \mathbb{C}[\varepsilon]$$

December 25, 1993

The problem is to construct a section of the surjection of normalized resolutions

$$\tilde{A} * \mathbb{C}[\varepsilon] \twoheadrightarrow \tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon, e])$$

We know how to do this on the level of the unnormalized resolutions, so we are led to examine liftings from the normalized to the unnormalized resolution. We look at this for a general algebra A, not just $\tilde{A}$.

We want a bimodule section of

$$A * \mathbb{C}[h] \twoheadrightarrow A * \mathbb{C}[\varepsilon]$$

The simplicial normalization theorem gives us two canonical sections, namely

$$\begin{tikzcd}
a_0 \varepsilon a_1 \cdots \varepsilon a_n \varepsilon a_{n+1} \arrow[d, equal] & \\
a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1} \arrow[r, mapsto] \arrow[dd, equal] & a_0 [h, a_1] \cdots [h, a_n] h a_{n+1} \\
 & \text{killed by } d_0, \ldots, d_{n-1} \\
a_0 \varepsilon [a_1, \varepsilon] \cdots [a_n, \varepsilon] a_{n+1} \arrow[r, mapsto] & a_0 h [a_1, h] \cdots [a_n, h] a_{n+1} \\
 & \text{killed by } d_1, \ldots, d_n
\end{tikzcd}$$

The Alexander-Spanier differential d on $A * \mathbb{C}[h]$ (the super derivation of degree $+1$ defined by $d(a) = [h, a]$, $d(h) = h^2$) gives rise to a DG algebra lifting $\Omega A \longrightarrow A * \mathbb{C}[h]$

$$a_0 da_1 \cdots da_n \longmapsto a_0 [h, a_1] \cdots [h, a_n]$$

which makes $A * \mathbb{C}[h]$ into a bimodule over ΩA. I have to check that the two liftings given by the simp. norm. th. coincide with this lifting on ΩA.

To do this, and even more, define a left action of $A * \mathbb{C}[\varepsilon]$ on $A * \mathbb{C}[h]$ by

$$a \circ \alpha = a\alpha$$
$$\varepsilon \circ \alpha = h\, b'(h\alpha) = h\alpha - h^2 b'(\alpha)$$

Check that $\varepsilon \cdot (\varepsilon \cdot \alpha) = 0$. ~~Then for $\omega \in \Omega A$~~

For $\beta \in \mathrm{Ker}(b') \subset A * \mathbb{C}[h]$, we have

$$\varepsilon \cdot (a \cdot \beta) - a \cdot (\varepsilon \cdot \beta) = ha\beta - ah\beta = [h, a]\beta$$

Thus $(a_0 \varepsilon [a_1, \varepsilon] \cdots [a_n, \varepsilon] a_{n+1}) \cdot \beta = a_0 h [a_1, h] \cdots [a_n, h] a_{n+1} \beta$

which means that acting on 1 yields the second lifting to $\bigcap_{i>0} \mathrm{Ker}\, d_i$.

On the other hand, let us define a right action of $A * \mathbb{C}[\varepsilon]$ on $A * \mathbb{C}[h]$ by

$$\alpha \cdot a = \alpha a$$
$$\alpha \cdot \varepsilon = (-1)^{|\alpha|} b'(\alpha h) h = \alpha h + (-1)^{|\alpha|} b'(\alpha) h^2$$

Check: $(\alpha \cdot \varepsilon) \cdot \varepsilon = (-1)^{|\alpha|+1} b'(\underbrace{(\alpha \cdot \varepsilon) h}) h = 0$.

$$(-1)^{|\alpha|} \underbrace{b'(\alpha h)}\, \underbrace{h^2}$$

both killed by b'

Again if $b'(\beta) = 0$ then

$$(\beta \cdot \varepsilon) \cdot a - (\beta \cdot a) \cdot \varepsilon = \beta h a - \beta a h = \beta [h, a]$$

so that

$$\beta \cdot \left(a_0[\varepsilon,a_1]\cdots[\varepsilon,a_n]\varepsilon a_{n+1}\right) = \beta a_0[h,a_1]\cdots[h,a_n]h a_{n+1}$$

which means that acting on 1 gives the first lifting to $\bigcap_{i<\text{last}} \operatorname{Ker} d_i$.

Now let's return to

$$\begin{tikzcd}
\tilde{A} * \mathbb{C}[h] \arrow[r, two heads] \arrow[d] & \tilde{A} * \mathbb{C}[h]/([h,e]) \arrow[l, dashed, bend right, "u"'] \arrow[d] \\
\tilde{A} * \mathbb{C}[\varepsilon] \arrow[r, two heads] & \tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon,e])
\end{tikzcd}$$

u is the DG alg homom. such that $u(a)=a$ $u(h)= h^{\natural} = ehe + e^{-1}he^{-1}$. Note $[e,h^{\natural}]=0$ and $h^{\natural} \longmapsto h$ under the surjection $\twoheadrightarrow$. u commutes with b' since $b'(h^{\natural}) = e^2+(e^{-1})^2 = 1$. Recall we have defined $\alpha\cdot\varepsilon$ on $\tilde{A} * \mathbb{C}[h]/([h,e])$ by

$$\alpha\cdot\varepsilon = (-1)^{|\alpha|}\, b'(\alpha h)h$$

Thus $u(\alpha\cdot\varepsilon) = (-1)^{|\alpha|}\, b'(u(\alpha)h^{\natural})h^{\natural}$, so we get a right action of $\tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon,e])$ on $\tilde{A} * \mathbb{C}[h]$ by defining

$$\alpha\bullet\varepsilon = (-1)^{|\alpha|}\, b'(\alpha h^{\natural})h^{\natural} \qquad \alpha\in\tilde{A} * \mathbb{C}[h]$$

Finally we define an action of $\tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon,e])$ on $\tilde{A} * \mathbb{C}[\varepsilon]$ by

$$\alpha\circ\varepsilon = (-1)^{|\alpha|}\, b'(\alpha \varepsilon^{\natural})\varepsilon^{\natural} \qquad \varepsilon^{\natural} = e\varepsilon e + e^{-1}\varepsilon e^{-1}$$

Check well-defined

$$(\alpha e)\circ\varepsilon = (-1)^{|\alpha|} b'(\alpha \underbrace{e\varepsilon^{\natural})\varepsilon^{\natural}} = (-1)^{|\alpha|} b'(\alpha\varepsilon^{\natural})\varepsilon^{\natural} e$$

$$= (\alpha\circ\varepsilon)e$$

$$(\alpha\circ\varepsilon)\circ\varepsilon = (-1)^{|\alpha|+1} b'((\alpha\circ\varepsilon)\varepsilon^{\natural})\varepsilon^{\natural}$$

$$= (-1)\ b'(\underbrace{b'(\alpha\varepsilon^{\natural})}\underbrace{(\varepsilon^{\natural})^2})\varepsilon^{\natural} = 0$$

both killed by b'.

Thus we have the section

$$\begin{tikzcd} \tilde{A} * \mathbb{C}[\varepsilon] \arrow[r, two heads] & \tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon, e]) \arrow[l, dashed, bend right] \end{tikzcd}$$

$$a_0[\varepsilon^{\natural}, a_1]\cdots[\varepsilon^{\natural}, a_n]\varepsilon^{\natural} a_{n+1} \longleftarrow\!\!\shortmid\ a_0[\varepsilon, a_1]\cdots[\varepsilon, a_n]\varepsilon a_{n+1}$$

This is a bimodule map compatible with b'.

December 26, 1993

Recall that yesterday we described the two canonical bimodule liftings (given by the simp. norm. thm) for the map of std resolutions

$$A * \mathbb{C}[h] \longrightarrow A * \mathbb{C}[\varepsilon]$$

using a left + right action of $A*\mathbb{C}[\varepsilon]$ on $A*\mathbb{C}[h]$. The right action is given by

$$\alpha \cdot a = \alpha a$$

$$\alpha \cdot \varepsilon = (-1)^{|\alpha|} b'(\alpha h) h = \alpha h + (-1)^{|\alpha|} b'(\alpha) h^2$$

and yields the lifting

$$l(a_0 \varepsilon a_1 \varepsilon \cdots a_n \varepsilon a_{n+1}) = a_0 [h,a_1] \cdot\cdot [h,a_n] h a_{n+1}$$

We used this to get a lifting for the Kadison map for normalized resolutions ⊛

$$\begin{tikzcd} \tilde{A} * \mathbb{C}[h] \arrow[r] \arrow[d] & \tilde{A} * \mathbb{C}[h]/([h,e]) \\ \tilde{A} * \mathbb{C}[\varepsilon] \arrow[r] & \tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon,e]) \end{tikzcd}$$

$$\begin{tikzcd} \tilde{A} * \mathbb{C}[h] \arrow[r, "h^\natural \leftarrow\!\shortmid\, h"] \arrow[d] & \tilde{A} * \mathbb{C}[h]/([h,e]) \arrow[r, phantom, "="] \arrow[d] \arrow[dl, "u"', "\varepsilon^\natural \leftarrow\!\shortmid\, h"] & \mathbb{C}[h] \times (A * \mathbb{C}[h]) \arrow[d] \\ \tilde{A} * \mathbb{C}[\varepsilon] \arrow[r, two heads, "⊛"'] & \tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon,e]) \arrow[u, dashed, bend right, "l"'] \arrow[r, phantom, "="] & \mathbb{C}[\varepsilon] \times (A * \mathbb{C}[\varepsilon]) \end{tikzcd}$$

Here $\varepsilon^\natural = e\varepsilon e + e^\perp \varepsilon e^\perp$, and the lifting for ⊛ is ul i.e.

$$
\begin{aligned}
& ul\left(a_0[\varepsilon,a_1]\cdots[\varepsilon,a_n]\varepsilon a_{n+1}\right) \\
&= u\left(a_0[h,a_1]\cdots[h,a_n]h a_{n+1}\right) \\
&= a_0[\varepsilon^\natural,a_1]\cdots[\varepsilon^\natural,a_n]\varepsilon^\natural a_{n+1}
\end{aligned}
$$

Now restrict to subbimodules on which e is the identity

$$\begin{tikzcd}
 & A * \mathbb{C}[h] \arrow[dl, "u"'] \arrow[d, shift left] \\
e(\tilde{A} * \mathbb{C}[\varepsilon])e \arrow[r] & A * \mathbb{C}[\varepsilon] \arrow[u, dashed, shift left, "l"']
\end{tikzcd}$$

$h \mapsto e\varepsilon e$

Here $u(a_0 h \cdots h a_{n+1}) = a_0 \varepsilon \cdots \varepsilon a_{n+1}$ is an isomorphism

We get the lifting

$$
\begin{aligned}
& ul\left(a_0[\varepsilon,a_1]\cdots[\varepsilon,a_n]\varepsilon a_{n+1}\right) \\
&= u\left(a_0[h,a_1]\cdots[h,a_n]h a_{n+1}\right) \\
&= a_0[e\varepsilon e,a_1]\cdots[e\varepsilon e,a_n]e\varepsilon e\, a_{n+1} \\
&= a_0[\varepsilon,a_1]e[\varepsilon,a_2]e\cdots e[\varepsilon,a_n]e\,\varepsilon\, a_{n+1}
\end{aligned}
$$

$[e\varepsilon e,a] = e[\varepsilon,a]e$

Next consider what happens on commutator quotient spaces.

$$\begin{tikzcd}
(\tilde{A} * \mathbb{C}[\varepsilon])\otimes_{\tilde{A}} \arrow[r, shift left] & (\tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon,e]))\otimes_{\tilde{A}} \arrow[l, dashed, shift left, "ul"'] \\
\Omega\tilde{A} \arrow[u, "\cong"'] \arrow[r] & \Omega\mathbb{C} \times \Omega A \arrow[u, "\cong"']
\end{tikzcd}$$

Start with $a_0 da_1 \cdots da_n \in \Omega^n A$, $n>0$ It goes to

$$\natural\left(a_0[\varepsilon,a_1]\cdots[\varepsilon,a_n]\varepsilon\right) \in \left(\tilde{A} * \mathbb{C}[\varepsilon]/([\varepsilon,e])\right)\otimes_{\tilde{A}}$$

Apply ul to get

$$\begin{tikzcd}
\natural(\overline{a_0[\varepsilon,a_1]e\cdots e[\varepsilon,a_n]e\varepsilon e}) \arrow[d] & \in & (\tilde{A}*\mathbb{C}[\varepsilon])\otimes_{\tilde{A}} \arrow[d, phantom, "\wr"] \\
a_0da_1e\cdots eda_ne & \in & \Omega\tilde{A}
\end{tikzcd}$$

Put another way we have

$$\begin{tikzcd}
 & \natural(a_0[h,a_1]\cdots[h,a_n]h) & & a_0[h,a_1]\cdots[h,a_n] \\
 & (A*\mathbb{C}[h])\otimes_A \arrow[dl, "u"', "\cong"] \arrow[d] & = & C(A) \arrow[d, shift right] \\
e(\tilde{A}*\mathbb{C}[\varepsilon])e\otimes_A \arrow[r, two heads] \arrow[d, phantom, "\parallel"] & (A*\mathbb{C}[\varepsilon])\otimes_A & = & \Omega A \arrow[u, dashed, shift right, "l"'] \\
e(\Omega\tilde{A})e & & & a_0da_1\cdots da_n
\end{tikzcd}$$

$$u\natural(a_0[h,a_1]\cdots[h,a_n]h) = \natural(a_0[\varepsilon,a_1]e\cdots e[\varepsilon,a_n]e\varepsilon e)$$

$$\longmapsto a_0da_1e\cdots eda_ne \in e(\Omega\tilde{A})e.$$

I guess I know that

$$\begin{tikzcd}
 & C(A) \\
e(\Omega\tilde{A})e \arrow[ur, "u"', "\sim"] & \Omega A \arrow[u, "l"']
\end{tikzcd}$$

are algebra homomorphisms, so that it should be possible to transport the Alexander-Spanier differential to $e(\Omega\tilde{A})e$. NO, u isn't, but

$$e(\Omega\tilde{A})e \xrightarrow{\sim} {}_{b'}C(A)$$

should be alg isom.

December 27, 1993

Key diagram of A-bimodule resolutions

$$\begin{tikzcd}
& A * \mathbb{C}[h] \arrow[dl, "u"', "\simeq"] \arrow[d, "p"', shift right] \\
e(\tilde{A} * \mathbb{C}[\varepsilon])e \arrow[r, two heads] & A * \mathbb{C}[\varepsilon] \arrow[u, bend right, "\ell"']
\end{tikzcd}$$

$u(h) = e\varepsilon e$

u and the two surjections are DG algebra maps.

$$\boxed{\begin{aligned} u(a_0 h a_1 \cdots a_n h a_{n+1}) &= a_0 \varepsilon a_1 \varepsilon \cdots a_n \varepsilon a_{n+1} \\ &= a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1} \end{aligned}}$$

Beware, in using this formula, not to write 1 for the identity of A. Thus

$$\begin{aligned} u(h^2) = u(ehehe) &= e\varepsilon e \varepsilon e = e[\varepsilon, e]\varepsilon e \\ &= [\varepsilon, e](1-e)\varepsilon e = [\varepsilon, e](1-e)[\varepsilon, e] \\ &= e[\varepsilon, e]^2 \end{aligned}$$

$$\boxed{\ell(a_0[\varepsilon, a_1] \cdots [\varepsilon, a_n]\varepsilon a_{n+1}) = a_0[h, a_1] \cdots [h, a_n] h a_{n+1}}$$

$$\boxed{\begin{aligned} u\ell(a_0[\varepsilon, a_1] \cdots [\varepsilon, a_n]\varepsilon a_{n+1}) &= a_0[e\varepsilon e, a_1] \cdots [e\varepsilon e, a_n] e\varepsilon e a_{n+1} \\ &= a_0[\varepsilon, a_1]e \cdots [\varepsilon, a_n] e \varepsilon a_{n+1} \end{aligned}}$$

ℓ is not compatible with multiplication, but arises from the right action of $A * \mathbb{C}[\varepsilon]$ on $A * \mathbb{C}[h]$ defd by

$$\begin{aligned} \alpha \cdot a &= \alpha a \\ \alpha \cdot \varepsilon &= (-1)^{|\alpha|} b'(\alpha h) h = \alpha h + (-1)^{|\alpha|} b'(\alpha) h^2 \end{aligned}$$

When we [redacted] apply $\otimes_A$ and make standard identification we get the diagram

$$\begin{tikzcd}
& A * \mathbb{C}[h] \arrow[dl, "u"', "\cong"] \arrow[d, "p"', shift right] \\
e\Omega\tilde{A} \arrow[r, two heads] & \Omega A \arrow[u, bend right, "l"', shift right]
\end{tikzcd}$$

$$u(a_0 h \cdots h a_n) = a_0 da_1 \cdots da_n \in e\Omega\tilde{A}$$
$$p(\text{———}) = \text{———} \in \Omega A$$
$$l(a_0 da_1 \cdots da_n) = a_0 [h, a_1] \cdots [h, a_n]$$
$$ul(a_0 da_1 \cdots da_n) = a_0 da_1 e \cdots da_n e$$

u is not a homomorphism, but [redacted] restricts to give an isomorphism of algebras $b'(A * \mathbb{C}[h]) \xrightarrow{\sim} e(\Omega\tilde{A})e$. (cf. p269) In particular ul is a multiplicative lifting of ΩA into $e\Omega\tilde{A} \subset \tilde{\Omega}\tilde{A}$; this is a result of Coquereaux-Kastler.

Further work: Analyze

$$a_0 da_1 \cdots da_n = a_0 da_1 \begin{pmatrix} e \\ e^\perp \end{pmatrix} \cdots da_n \begin{pmatrix} e \\ e^\perp \end{pmatrix} \quad \text{using} \quad \begin{array}{l} e^\perp da e^\perp = 0 \\ da_1 e^\perp da_2 = a_1 de^2 a_2 \end{array}$$

This is related to $\Omega^1\tilde{A} = \underset{\Omega^1 A}{\underset{\cong}{e\Omega^1\tilde{A}e}} \oplus \underset{deA}{\underset{\parallel}{e^\perp\Omega^1\tilde{A}e}} \oplus \underset{Ade}{\underset{\parallel}{e\Omega^1\tilde{A}e^\perp}}$

and $\Omega\tilde{A} = T_{\tilde{A}}(\Omega^1\tilde{A})$.

There's also the GNS result

$$\boxed{\Gamma\left(\mathbb{C}[\varepsilon] \longrightarrow A*\mathbb{C}[h]\right) = \tilde{A} * \mathbb{C}[\varepsilon]}$$

and its interpretation using complexes of A modules equipped with contraction independent of the A module structure.

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M a B-acyclic mixed complex, h an operator of degree -1 such that $[B,h]=1$. Then $[b,h]$ has degree -2 and it commutes with both b and B, hence we obtain an S-module structure on $(M/BM, b)$ with $S = -[b,h]$.

Define

$$u : \Lambda \tilde{\otimes} (M/BM) \xrightarrow{\sim} M$$

by

$$u(1 \otimes \bar{m}) = hBm$$

$$u(B \otimes \bar{m}) = Bm$$

This is well-defined, compatible with B as $BhB = B(1-Bh) = B$, compatible with $d = 1 \otimes b - S \otimes B$ on the left and b on the right:

$$bu(1\otimes\bar{m}) = bhBm = [b,h]Bm - hbBm$$

$$= hB(bm) - B(-[b,h])m$$

$$= u(1\otimes b\bar{m} - B\otimes S\bar{m}) = ud(1\otimes\bar{m})$$

$$bu(B\otimes\bar{m}) = bBm = -Bbm = u(-B\otimes b\bar{m})$$

$$= ud(B\otimes\bar{m}).$$

and bijective, the inverse being

$$v(m) = 1\otimes\bar{m} + B\otimes\overline{hm}$$

Check $\quad uv(m) = u(1\otimes\bar{m} + B\otimes\overline{hm}) = hBm + Bhm = m$

$$vu(1\otimes\bar{m}) = v(hBm) = 1\otimes\overline{hBm} + B\otimes\overline{h^2Bm}$$

$$= 1\otimes\overline{m - \cancel{Bhm}} + B\otimes\overline{\cancel{Bh^2m}} = 1\otimes\bar{m}$$

$$vu(B\otimes\bar{m}) = v(Bm) = 1\otimes\overline{\cancel{Bm}} + B\otimes\overline{hBm} = B\otimes\overline{m - Bhm} = B\otimes\bar{m}$$