Construction of Nistor's bivariant Chern character ①

Lemma 1. Let S, R be algebras and let $K \subset S$, $I \subset R$ be ideals. If

$$\alpha : X(S \otimes R) \longrightarrow S_\natural \otimes X(R)$$

is the canonical map, then for all p

$$\alpha\left(F^p_{K\otimes R + S \otimes I} X(S\otimes R)\right) \subset \sum_{i\geq 0} \natural(K^i) \otimes F_I^{p-2i} X(R).$$

The proof is straightforward checking.

Let L be an algebra, let $J \subset L$ be an ideal. (In the Nistor application $L = \mathcal{L}(H)$, J is a Schatten ideal.)

Let S be the graded algebra

$$S = \bigoplus_{n\geq 0} t^n J^n \subset \mathbb{C}[t] \otimes L$$

Since S is generated by L and tJ, one has

$$S_\natural \simeq \bigoplus_{n\geq 0} t^n J^n_\# \qquad J^n_\# = \begin{cases} L/[L,L] & n=0 \\ J^n/[J,J^{n-1}] & n\geq 1 \end{cases}$$

Let $K \subset S$ be the ideal

$$K = S(1-t^2)J^2 = \sum_{n\geq 0} (1-t^2)t^n J^{n+2}$$

Lemma 2. The obvious maps (for $m \geq 0$)

$$\bigoplus_{n=0}^{2m+1} t^n J^n \longrightarrow S/K^{m+1}$$

$$\bigoplus_{n=0}^{2m+1} t^n J^n_\# \longrightarrow (S/K^{m+1})_\natural$$

are bijections.

This lemma implies there is a unique trace $\tau_m : S \longrightarrow J_{\#}^{2m+1}$, $m \geq 0$ such that

$$\tau_m(K^{m+1}) = 0$$

$$\tau_m(t^n J^n) = 0 \qquad \text{for} \qquad 0 \leq n \leq 2m$$

$$\tau_m(t^{2m+1}x) = \frac{(-1)^m 2^m m!}{1\cdot 3 \cdots (2m-1)} \#(x) \qquad x \in J^{2m+1}$$

where $\#(x)$ is the image of x in $J_{\#}^{2m+1}$.

The numerical factors have been introduced so that these traces are compatible for different m in the sense that one has a commutative square

$$\begin{tikzcd}
S/K^{m+1} \arrow[r, "\tau_m"] \arrow[d, "1-\frac{D}{2m-1}"'] & J_{\#}^{2m+1} \arrow[d, "\iota_{\#}"] \\
S/K^{m} \arrow[r, "\tau_{m-1}"] & J_{\#}^{2m-1}
\end{tikzcd} \qquad m \geq 1$$

Here D is the derivation $t\frac{d}{dt}$ on S, and $i_{\#}$ is the map induced by the inclusion of J^{2m+1} in J^{2m-1}.

Let A, B be algebras and let

$$A \underset{\bar{\theta}}{\overset{\theta}{\rightrightarrows}} L \otimes B$$

be homomorphisms which are congruent modulo $J \otimes B$. Let

$$p = \frac{\theta+\bar\theta}{2} : A \longrightarrow L\otimes B$$
$$q = \frac{\theta-\bar\theta}{2} : \bar A \longrightarrow J\otimes B$$

Then $p+tq : A \longrightarrow S\otimes B$ is a linear map respecting identity elements whose curvature is

$$(1-t^2)q^2 : \bar A^{\otimes 2} \longrightarrow (1-t^2)J^2\otimes B \subset K\otimes B$$

Hence $p+tq : A \longrightarrow (S/K)\otimes B$ is a homomorphism.

Let $u : RA \to S\otimes RB$ be the homomorphism such that

$$\begin{tikzcd} A \arrow[r, "p+tq"] \arrow[d, "\rho_A"'] & S\otimes B \arrow[d, "1\otimes\rho_B"] \\ RA \arrow[r, "u"] & S\otimes RB \end{tikzcd}$$

commutes. Clearly

$$u(IA) \subset K\otimes RB + S\otimes IB.$$

Consider the composite map of supercomplexes

$$X(RA) \xrightarrow{u_*} X(S\otimes RB) \xrightarrow{\alpha} S_\natural \otimes X(RB) \xrightarrow{\tau_m} J_\#^{2m+1} \otimes X(RB)$$

Then

$$F^p_{IA}X(RA) \xrightarrow{u_*} F^p_{K\otimes RB+S\otimes IB}X(S\otimes RB)$$
$$\xrightarrow{\alpha} \sum_{i\geq 0} \natural(K^i)\otimes F^{p-2i}_{IB}X(RB) \qquad \text{(Lemma 1)}$$
$$\xrightarrow{\tau_m} J_\#^{2m+1}\otimes F^{p-2m}_{IB}X(RB) \qquad (\text{as } \tau_m K^{m+1}=0)$$

Consequently if $\tau : J_\#^{2m+1} \to \mathbb{C}$ is a J-adic trace on J^{2m+1}, then one has a map of supercomplexes

$$\tau\tau_m \alpha u_* : X(RA) \longrightarrow X(RB)$$

carrying $F^p_{IA} X(RA)$ into $F^{p-2m}_{IB} X(RB)$ for all p, yielding a class in $HC^{2m}(A,B)$.

It remains to check that if $\tau = \tau' \iota_\#$ with $\tau' : J^{2m-1}_\# \longrightarrow \mathbb{C}$, then the class in $HC^{2m}(A,B)$ represented by $\tau\tau_m \alpha u_*$ is the image under S of the class in $HC^{2m-2}(A,B)$ represented by $\tau' \tau_{m-1} \alpha u_*$. For this we construct a suitable homotopy joining $\iota_\# \tau_m \alpha u_*$ and $\tau_{m-1} \alpha u_* : X(RA) \longrightarrow J^{2m-1}_\# \otimes X(RB)$.

From the square on p.2 one has

$$\tau_{m-1} - \iota_\# \tau_m = \frac{1}{2m-1} \tau_{m-1} D$$

Extend D to $S \otimes RB$ and $S_q \otimes X(RB)$ in the obvious way. Then

$$Du : RA \xrightarrow{u} S \otimes RB \xrightarrow{D} S \otimes RB$$

is a derivation relative to u, so it gives rise to a contracting homotopy

$$h = h(u, Du) : X(RA) \longrightarrow X(S \otimes RB)$$

for the Lie derivative $L(u, Du)$:

$$L(u, Du) = [\partial, h]$$

One has a commutative diagram

$$\begin{tikzcd}
X(RA) \arrow[r, "u_*"] \arrow[dr, "L(u,Du)"'] & X(S \otimes RB) \arrow[r, "\alpha"] \arrow[d, "L(1,D)"] & S_q \otimes X(RB) \arrow[d, "D"] \\
 & X(S \otimes RB) \arrow[r, "\alpha"] & S_q \otimes X(RB)
\end{tikzcd}$$

so $$\tau_{m-1}\dot{\alpha} u_* - \iota_\# \tau_m \alpha u_* = \frac{1}{2m-1}\tau_{m-1} D\alpha u_*$$

where

$$\tau_{m-1} D\alpha u_* = \tau_{m-1}\alpha L(u, Du) = \tau_{m-1}\alpha[\partial, h] = [\partial, \tau_{m-1}\alpha h]$$

Here $\tau_{m-1}\alpha h$ is the composite map

$$X(RA) \xrightarrow{h} X(S\otimes RB) \xrightarrow{\alpha} S_\natural \otimes X(RB) \xrightarrow{\tau_{m-1}} J_\#^{2m-1} \otimes X(RB)$$

One has

$$F_{IA}^{p} X(RA) \xrightarrow{h} F^{p-2}_{K\otimes RB + S\otimes IB} X(S\otimes RB)$$

$$\xrightarrow{\alpha} \sum_{i\geq 0} \natural(K^i)\otimes F_{IB}^{p-2-2i} X(RB)$$

$$\xrightarrow{\tau_{m-1}} J_\#^{2m-1} \otimes F_{IB}^{p-2m} X(RB)$$

Thus $\tau'\tau_{m-1}\alpha u_*$, $\tau' i_\# \tau_m \alpha u_* : X(RA) \longrightarrow X(RB)$, considered as maps of order $\leq 2m$ with respect to the filtrations, are joined by the homotopy $\frac{1}{2m-1}\tau'\tau_{m-1}\alpha h$, which also has order $\leq 2m$, so the classes represented by these maps in $HC^{2m}(A, B)$ coincide.

June 16, 1993

Consider the standard unnormalized and normalized resolutions of the A-bimodule A. The former is the DG algebra

$$R = T_A(A\otimes A) = A * \mathbb{C}[D] \qquad D = 1\otimes 1$$

where the grading and differential are

$$|a| = 0 \qquad |D| = 1$$

$$\partial(a) = 0 \qquad \partial(D) = 1.$$

The standard normalized resolution is

$$R/I = A * \mathbb{C}[d] \qquad \mathbb{C}[d] = \mathbb{C}\oplus\mathbb{C}d,\ d^2=0$$

$$I = \text{ideal } RD^2R \text{ generated by } D^2.$$

(One has $\partial(D^2) = \partial(D)D - D\partial(D) = D - D = 0$ so I is closed under ∂.)

Here are some facts we have established:

I is flat as a left or right R-module

$$gr_I R = \bigoplus_{n\geq 0} I^n/I^{n+1} = T_{R/I}(I/I^2)$$

$$I/I^2 = R/I \otimes_{\mathbb{C}[d]} R/I \quad \text{where}$$

$$D^2 \longleftrightarrow 1\otimes 1$$

Thus

$$I^n/I^{n+1} = (R/I\otimes_{\mathbb{C}[d]} R/I)\otimes_{R/I}\overset{(n)}{\cdots}\otimes_{R/I}(R/I\otimes_{\mathbb{C}[d]}R/I)$$

$$= R/I\otimes_{\mathbb{C}[d]}\overset{(n+1)}{\cdots}\otimes_{\mathbb{C}[d]} R/I$$

Recall also that R/I is the cross product:

$$R/I = A * \mathbb{C}[d] = \Omega A \tilde{\otimes} \mathbb{C}[d] = (\mathbb{C}+\mathbb{C}d)\tilde{\otimes}\Omega A$$

This gives an isomorphism

$$* \qquad \mathbb{C}[d] \otimes (\Omega A)^{\otimes n+1} \xrightarrow{\sim} I^n/I^{n+1}$$

In more detail

$$(\mathbb{C}[d]\otimes\Omega A)\otimes\cdots\otimes\Omega A \longrightarrow I^n/I^{n+1}$$

$$(a+bd)\otimes\omega_0\otimes\omega_1\otimes\cdots\otimes\omega_n \longmapsto (a+bD)\omega_0 D^2\omega_1\cdots\omega_{n-1}D^2\omega_n$$

$a,b\in\mathbb{C}$.

We can lift $*$ into I^n as follows. Observe one has a canonical homomorphism

$$\Omega A \longrightarrow R$$

$$a_0\,da_1\cdots da_n \longmapsto a_0[D,a_1]\cdots[D,a_n]$$

associated to the homomorphism $A\to R$ and the derivation $a\mapsto[D,a]$ relative to this homom. Thus we lift $(*)$ to the map

$$(a+bd)\otimes\omega_0\otimes\cdots\otimes\omega_n \longmapsto (a+bD)\,\omega_0 D^2\omega_1\cdots D^2\omega_n$$

$$(\mathbb{C}+\mathbb{C}d)\otimes(\Omega A)^{\otimes n+1} \hookrightarrow (\mathbb{C}+\mathbb{C}D)(\Omega A)D^2\cdots D^2(\Omega A)$$

Consider the subalgebra of R generated by ΩA and D^2. It seems from the above that this subalgebra is $\Omega A * \mathbb{C}[D^2]$ and that we have

$$** \qquad (\mathbb{C}\oplus\mathbb{C}D)\otimes(\Omega A * \mathbb{C}[D^2]) \xrightarrow{\sim} R$$

Note that $\partial(D^2)=0$ and $\partial([D,a]) = [\partial D, a] = 0$.

It should be possible to prove the isomorphism ** directly as follows. We have the map

$$\Phi : (\mathbb{C}\oplus\mathbb{C}D)\otimes(\Omega A * \mathbb{C}[D^2]) \longrightarrow R$$

in one direction. We make R act on the space on the left by defined left multiplication by a and by D. If $x, y \in \Omega A * \mathbb{C}[D^2]$ define

$$D\cdot(x + Dy) = (D^2 y) + Dx$$

$$a\cdot(x + Dy) = (ax - day) + D(ay)$$

Check that this is compatible with multiplication in A, and check that it is compatible with left multiplication by D and a on R via Φ.

$$\Phi\{a\cdot(x+Dy)\} = \Phi\{(ax - day) + D(ay)\}$$

$$= ax - [D,a]y + Day$$

$$= ax + aDy = a\,\Phi(x+Dy)$$

This left multiplication by A, D on $(\mathbb{C}+\mathbb{C}D)\otimes(\Omega A * \mathbb{C}[D^2])$ extends to a left module structure over R since $R = A * \mathbb{C}[D]$. Acting on $1\otimes 1$ in $(\mathbb{C}+\mathbb{C}D)\otimes(\Omega A * \mathbb{C}[D^2])$ gives a map

$$\Psi : R \longrightarrow (\mathbb{C}+\mathbb{C}D)\otimes(\Omega A * \mathbb{C}[D^2]).$$

One has $\Phi\Psi = id_R$ because $\Phi\Psi$ is an R-module map $R \to R$ sending 1 to 1. Finally check that Ψ is onto: Let's calculate $[D,a]$ acting on $1\otimes(\Omega A * \mathbb{C}[D^2])$. One has

$$[D,a]\cdot x = D(ax) - a\cdot Dx$$

where $a \cdot Dx = -da\,x + D(ax)$

Thus $[D,a]\cdot x = da\,x$. So it's clear that the image of Ψ contains $1\otimes(\Omega A * \mathbb{C}[D^2])$, and then also $D\otimes(\Omega A * \mathbb{C}[D^2])$, whence Ψ is surjective.

One further point is that the lifting $R/I \longrightarrow R$ we obtained before by using left multiplication on R by elts of A and the operator $D\partial D = D - D^2\partial$ agrees with the preceding lift of $R/I = (\mathbb{C}+\mathbb{C}d)\otimes\Omega A$ to the subspace $(\mathbb{C}+\mathbb{C}D)\otimes\Omega A$ of R. In effect for $x,y \in \Omega A * \mathbb{C}[D^2]$

$$(D-D^2\partial)(x+Dy) = Dx + D^2y - \cancel{D^2\partial(x)} - D^2\underbrace{\partial(Dy)}_{y}$$
$$= Dx$$

whence $(\mathbb{C}+\mathbb{C}D)\otimes\Omega A$ is stable under the modified action of R.

Let h be a contraction on a complex (E,∂), i.e. $[\partial,h]=1$. Let $k = h\partial h$, $u = h^2$. Then k is a special contraction: $[\partial,k]=1$, $k^2=0$, and u is a degree 2 endomorphism of E: $[\partial,u]=0$ which commutes with k since [redacted] $u=h^2$ commutes with both h and ∂.

Conversely given operators k, u on E of degrees 1 and 2 resp. satisfying $[\partial,k]=1$, $k^2=0$, $[\partial,u]=0$, $[k,u]=0$ put $h = k + u\partial$. Then $[\partial,h] = [\partial,k] + [\partial,u]\partial + u[\partial,\partial]$ $= 1$, and $h\partial h = (k+\cancel{u\partial})\partial(k+\cancel{u\partial}) = k\partial k = k(\partial k + k\partial) = k$

and $h^2 = \cancel{k^2} + u\partial k + ku\partial + \cancel{u\partial u\partial} = u$.

Thus we have established

Prop. On a complex (E, ∂) one has an equivalence between contractions h on one hand, and pairs (k, u) consisting of a special contraction k and degree 2 endomorphism u which commute, on the other hand.

Cor. The categories of DG modules over the DG algebras $\mathbb{C}[h]$ with $|h| = 1$, $\partial(h) = 1$ and $\mathbb{C}[k,u]/(k^2)$ with $|k| = 1$, $|u| = 2$, $\partial(k) = 1$, $\partial(u) = 0$ are equivalent.

Another way of putting this is that we have an isomorphism of graded algebras

$$\mathbb{C}[h] \tilde{\otimes} \mathbb{C}[\partial] \;\simeq\; \mathbb{C}[k,u]/(k^2) \tilde{\otimes} \mathbb{C}[\partial]$$

$$h \longleftrightarrow k + u\partial$$

$$h^2 \longleftrightarrow u$$

$$h\partial h \longleftrightarrow k$$

Better to write $\left(\mathbb{C}[k]/(k^2) \tilde{\otimes} \mathbb{C}[\partial]\right) \otimes \mathbb{C}[u]$ maybe.

June 17, 1993

Recall R is the DG algebra

$$R = A * \mathbb{C}[D] \qquad |a|=0 \quad \partial(a)=0$$
$$|D|=1 \quad \partial(D)=1$$

Picture of R:

$$A \xleftarrow{\partial} ADA \xleftarrow{\partial} ADADA \xleftarrow{\partial}$$

It is the standard resolution of the bimodule A. If we remove A in degree zero and shift degrees we have a simplicial bimodule where d_i replaces the D in position $i+1$ by 1 and s_i replaces the D in position $i+1$ by D^2.

$I \subset R$ is the ideal RD^2R. One has

$$R/I = A * \mathbb{C}[d] \qquad \mathbb{C}[d] = \mathbb{C} + \mathbb{C}d, \ d^2=0$$
$$= \mathbb{C}[d] \hat{\otimes} \Omega A$$

Picture of R/I:

$$A \longleftarrow AdA \longleftarrow Ad\bar{A}dA \longleftarrow \cdots$$

It is the standard normalized resolution of the bimodule A.

For understanding the I-adic filtration on R we use the following description of R. One has a homom. of algs.

$$\phi : \Omega A \longrightarrow R$$
$$a_0 da_1 \cdots da_n \longmapsto a_0 [D,a_1] \cdots [D,a_n]$$

associated to the inclusion $A \rightarrow R$ and derivation $A \rightarrow R \xrightarrow{D} R$. Thus we have a homom. of algebras

$$\phi : \Omega A * \mathbb{C}[D^2] \longrightarrow R$$

Since $\partial([D,a]) = [1,a] = 0$, $\partial(D^2) = D - D = 0$ the image of $\Omega A * \mathbb{C}[D^2]$ is contained in Ker ∂ on R. One has the map

$$\circledast \qquad (\mathbb{C}+\mathbb{C}D)\otimes(\Omega A * \mathbb{C}[D^2]) \xrightarrow{\phi} R$$

$$1\otimes x \longmapsto \phi(x)$$

$$D\otimes x \longmapsto D\phi(x).$$

This map ϕ is an isomorphism, and $\Omega A * \mathbb{C}[D^2]$ is identified via ϕ with Ker ∂ on R.

We know [redacted] $(\mathbb{C}+\mathbb{C}d)\otimes(\Omega A * \mathbb{C}[D^2])$ is canonically identifiable with $gr_I R$, and that $\circledast$ is interpretable as an isomorphism of $gr_I R$ and R [redacted] as A-bimodule complexes.

In particular we get a lifting

$$l: \; R/I = \mathbb{C}[d]\tilde{\otimes}\Omega A \xrightarrow{\phi} R$$

of the standard normalized resolution into the standard unnormalized resolution. Clearly

$$a_0 da_1 \cdots da_n \overset{l}{\longmapsto} a_0[D,a_1]\cdots[D,a_n]$$

$$d\cdot a_0 da_1 \cdots da_n \overset{l}{\longmapsto} Da_0[D,a_1]\cdots[D,a_n]$$

Now take $a_0\cdot d\cdot a_1 \cdot \cdots \cdot a_n \cdot d\cdot a_{n+1} \in A\underbrace{d\bar{A}\cdots d\bar{A}}_{n\ \bar{A}'s}dA$

[redacted] This element is equal to

[redacted] $a_0 da_1 \cdots da_n d\cdot a_{n+1}$

$$= (-1)^n a_0 d\cdot da_1 \cdots da_n a_{n+1}$$

$$= (-1)^{n+1} da_0 da_1 \cdots da_n a_{n+1} + (-1)^n d\cdot a_0 da_1 \cdots da_n a_{n+1}$$

so one has

$$l(a_0\cdot d\cdot a_1 \cdots d\cdot a_{n+1}) = (-1)^{n+1}\left([D,a_0]-Da_0\right)[D,a_1]\cdots[D,a_n]a_{n+1}$$

$$\boxed{l(a_0\cdot d\cdot a_1 \cdots d\cdot a_{n+1}) = (-1)^n a_0 D[D,a_1]\cdots[D,a_n]a_{n+1}}$$

Note that this element is killed by the face operators $d_1, \ldots, d_n$ hence the lifting l of R/I coincides with the one obtained from the simplicial normalization theorem.

June 17, 1993

$$R = A * \mathbb{C}[D] \qquad \partial(a)=0\ ,\ \partial(D)=1$$

$$gr_I R = A * \mathbb{C}[d,u] \qquad \text{where } \mathbb{C}[d,u]$$

is the commutative DG algebra with $|d|=1, |u|=2$ $\partial(d)=1\ ,\ \partial(u)=0$.

Let δ be the superderivation (of degree +1) on $\mathbb{C}[d,u]$ such that $\delta(d)=0$ and $\delta(u) = ud = d\cdot u$. Then $\delta^2(d)=0\ ,\ \delta^2(u) = \delta(ud) = (ud)d + u\delta(d) = 0$, so $\delta^2=0$. Also $(\partial\delta+\delta\partial)(d) = \delta(1)=0$,

$$(\partial\delta+\delta\partial)(u) = \partial(ud) = u$$

Thus $[\partial,\delta]$ = multiplication by n on u^n and $u^n d$.

Now δ extends to a superderivation (of degree +1) on $gr_I R = A * \mathbb{C}[d,u]$ vanishing on A. On

$$I^n/I^{n+1} = \underbrace{(R/I)\underset{\overset{\shortparallel}{u}}{D^2}\cdots\cdots\underset{\overset{\shortparallel}{u}}{D^2}(R/I)} \qquad n \text{ factors } D^2=u$$

we have $[\partial,\delta]=n$,

so $\frac{1}{n}\delta$ is a special contraction on I^n/I^{n+1} which is compatible with the A-bimodule structure, more generally the R/I bimodule structure

At this point we have constructed a A-bimodule SDR "containing" the canonical map $R \to R/I$. One assumes characteristic zero in the above, however, instead of δ on I^n/I^{n+1} one can use the operator that replaces the first u by ud with the appropriate sign:

$$x_0D^2x_1\cdots D^2x_n \longmapsto (-1)^{|x_0|}x_0(D^2d)x_1\cdots D^2x_n$$

This will be a special contraction.

Let's check this carefully for $n=1$. One has

$$I/I^2 \xleftarrow{\sim} R/I \otimes_{\mathbb{C}[d]} R/I$$

Define k on the right side by

$$k(x\otimes y) = (-1)^{|x|}(xd\otimes y) \qquad x, y\in R/I$$

k is well-defined because

$$k(xd\otimes y) = (-1)^{|x|+1}(xdd\otimes y) = 0$$

$$k(x\otimes dy) = (-1)^{|x|}(xd\otimes dy) = (-1)^{|x|}(xdd\otimes y) = 0.$$

Also

$$x\otimes y \xrightarrow{k} (-1)^{|x|}\, xd\otimes y$$

$$\xrightarrow{\partial} (-1)^{|x|}\Big(\partial(x)d\otimes y + (-1)^{|x|} x\, \overbrace{\partial(d)}^{1}\otimes y + (-1)^{|x|+1}\, xd\otimes\partial(y)\Big)$$

$$x\otimes y \xrightarrow{\partial} \partial(x)\otimes y + (-1)^{|x|} x\otimes\partial(y)$$

$$\xrightarrow{k} (-1)^{|x|+1}\,\partial(x)d\otimes y + xd\otimes\partial(y)$$

$$\therefore\quad (k\partial+\partial k)(x\otimes y) = x\otimes y$$

June 22, 1993 (53 years old)

I need to reconcile my approach to Nistor's which Joachim follows.

Nistor considers $Q = QA$ with its $q = qA$-adic filtration. There is a corresponding filtration on the mixed complex $(\Omega Q, b, B)$, denote it $\mathcal{F}_k \Omega Q$. Nistor constructs bivariant classes $ch_n \in HC^{2n}(A, \mathcal{F}_{n+1}\Omega Q)$ and essentially (up to S) characterizes them.

Joachim uses $X(RQ)$ instead of ΩQ: There are corresponding filtrations $\mathcal{F}_k RQ$ and $\mathcal{F}_k X(RQ)$.

I want now to try to describe features of the constructions. I said that ΩQ is a mixed complex with filtration $\mathcal{F}_k \Omega Q$ by mixed subcomplexes. So the S-module $\mathcal{B}(\Omega Q)$ is filtered. Similarly in Joachim's case there is the filtration $F^P_{IQ} X(RQ)$ giving rise to the tower $\mathcal{X}_Q$. A point to be checked carefully is that one has a corresponding filtration $\mathcal{F}_k \mathcal{X}_Q$.

In the case of a quasi-homomorphism

$$A \rightrightarrows L \otimes B \quad \text{cong mod } J \otimes B$$

we have homomorphisms

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] & Q \arrow[r] & L \otimes B \\ & q^k \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & J^k \otimes B \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

whence maps

$$\begin{tikzcd}
\Omega A \arrow[r, shift left] \arrow[r, shift right] & \Omega Q & \\
& \mathcal{F}_k \Omega Q \arrow[u, phantom, "\cup"] \arrow[r] & J^k_\# \otimes \Omega B
\end{tikzcd}$$

In the $X(R)$ framework one has homomorphism

$$\begin{tikzcd}
RA \arrow[r, shift left] \arrow[r, shift right] & RQ \arrow[r] & L \otimes RB \\
& \mathcal{F}_k RQ \arrow[u, phantom, "\cup"] \arrow[r] & J^k \otimes RB \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

whence maps

$$\begin{tikzcd}
X(RA) \arrow[r, shift left] \arrow[r, shift right] & X(RQ) & \\
& \mathcal{F}_k X(RQ) \arrow[u, phantom, "\cup"] \arrow[r] & J^k_\# \otimes X(RB).
\end{tikzcd}$$

The idea in both frameworks is to deform $\mathcal{F}_1 \left| \begin{matrix} \Omega Q \\ X(RQ) \end{matrix} \right.$ into $\mathcal{F}_k \left| \begin{matrix} \Omega Q \\ X(RQ) \end{matrix} \right.$ so as to get the Chern character.

Let's go over some ideas. First recall that if $J \subset L$ is an ideal, then the J-adic filtration J^p can actually be viewed as an increasing filtration provided [redacted] we change p to $-p$. Thus if

$$F_p L = \begin{cases} J^{-p} & p \leq 0 \\ L & p \geq 0 \end{cases}$$

[redacted] we have $1 \in F_0 L$, $F_p L \cdot F_q L \subset F_{p+q} L$. Then it is natural to consider

$$S = \bigoplus_{p\in\mathbb{Z}} h^p F_p L = \bigoplus_{n\in\mathbb{Z}} t^n J^n \qquad h = t^{-1}$$

as a graded algebra over $\mathbb{C}[h] = \mathbb{C}[t^{-1}]$. When we divide by the ideal $(h) = (t^{-1})$ we obtain the associated graded algebra.

$$\bigoplus_{n\in\mathbb{Z}} t^n J^n \Big/ \bigoplus_{n\in\mathbb{Z}} t^{n-1} J^n = \bigoplus_{n\geq 0} J^n / J^{n+1}$$

Observe that S is generated by t^{-1}, L, tJ so

$$S_\natural = S \Big/ \underbrace{[S,t]}_{0} + [S,L] + [S,tJ]$$

$$= \bigoplus_{n\in\mathbb{Z}} t^n \left(J^n \big/ [J^n, L] + [J^{n-1}, J] \right)$$

$$= \bigoplus_{n\leq 0} t^n L_\natural \oplus \bigoplus_{n>0} t^n J^n_\#$$

Let's consider now the $\mathfrak{q}$-adic filtration on Q. Assign to this the graded algebra

$$Q^t = \bigoplus_{n\in\mathbb{Z}} t^n \mathfrak{q}^n \qquad \text{over } \mathbb{C}[t^{-1}].$$

Suppose we form relative forms:

$$\Omega_{\mathbb{C}[t^{-1}]} Q^t$$

You would like to see that this is $\bigoplus_{n\in\mathbb{Z}} t^n \mathcal{F}_n(\Omega Q)$ for the induced filtration.

June 25, 1993

Consider the category of filtered vector spaces defined as follows. An object in a vector space V together with an increasing filtration F_pV, $p \in \mathbb{Z}$ such that * $\boxed{\bigcup F_p V = V}$ Morphisms are linear map respecting the filtrations.

Let h be an indeterminate and associate to $(V, (F_pV))$ the graded module over $\mathbb{C}[h]$

$$\bigoplus_{p\in\mathbb{Z}} h^p F_p V \quad \subset \quad \mathbb{C}[h] \otimes V$$

In this way we get an equivalence of the category of filtered vector spaces with the full subcategory of graded $\mathbb{C}[h]$-modules which are torsion-free. The inverse functor takes $M = \bigoplus_{p \in \mathbb{Z}} M_p$ into $V = \varinjlim_p M_p$ where $\xrightarrow{h} M_p \xrightarrow{h} M_{p+1} \xrightarrow{h}$ are the arrows, and $F_pV = \operatorname{Im}(M_p \longrightarrow V)$. For M torsion-free $M_p \simeq F_pV$.

Note that when M and V correspond

$$\mathbb{C}[h,h^{-1}] \otimes_{\mathbb{C}[h]} M = \mathbb{C}[h,h^{-1}] \otimes V$$

$$\mathbb{C}[h]/(h-1) \otimes_{\mathbb{C}[h]} M = M/(h-1)M = V$$

$$\mathbb{C}[h]/(h) \otimes_{\mathbb{C}[h]} M = M/hM = \bigoplus_{p \in \mathbb{Z}} h^p (F_pV/F_{p-1}V)$$

$$\simeq gr(V).$$

Tensor product: If $M = \bigoplus h^p F_p V$, $N = \bigoplus h^p F_p W$, then $M \otimes_{\mathbb{C}[h]} N$ is also torsion-free so it corresponds to a filtered vector space, namely

$V\otimes W$ with

$$F_p(V\otimes W) = \sum_i F_iV\otimes F_{p-i}W \subset V\otimes W.$$

The reason is that we have

$$\begin{tikzcd}
M\otimes N \arrow[r, two heads] & M\otimes_{\mathbb{C}[h]} N \arrow[r, two heads, "\text{specializing}", "h\to 1"'] & V\otimes W \\
h^p\bigoplus_i F_iM\otimes F_{p-i}N \arrow[u, phantom, "\cup"] \arrow[r, two heads] & h^p\sum_i F_iM\otimes F_{p-i}N \arrow[u, phantom, "\cup"] \arrow[r, two heads] & F_p(V\otimes W) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

About exact sequences: The torsion-free graded $\mathbb{C}[h]$-modules do not form an abelian category. A map $f: M\longrightarrow N$ has torsion-free kernel and image but the cokernel may have torsion. To simplify suppose f injective and consider the corresponding filtered vector spaces V, W. We have

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & N \arrow[r] & N/fM \arrow[r] & 0
\end{tikzcd}$$

or

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & F_pV \arrow[r] \arrow[d] & F_pW \arrow[r] \arrow[d] & F_p(W/V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & W/V \arrow[r] \arrow[d] & 0 \\
 & V/F_pV \arrow[r] \arrow[d] & W/F_pW \arrow[r] \arrow[d] & W/F_pW+V \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We see that N/fM is torsion-free iff $F_p(W/V)\to W/V$ is injective iff $V/F_pV\to W/F_pW$ injective iff

$$V\cap F_pW = F_pV \qquad \text{all } p.$$

Thus the good case is when the filtration on V is the [redacted] filtration induced from the filtration on W.

Example: Let $I \subset R$ be an ideal in R. Then we can fit the I-adic filtration in the above picture by
$$F_p I = \begin{cases} R & p \geq 0 \\ I^{-p} & p < 0. \end{cases}$$

Note that although one can always choose complements $F_{p-1}V \oplus K_p = F_pV$ for a filtered vector spase V, however it is not true that V then splits as $\bigoplus_{p \in \mathbb{Z}} K_p$ and $\bigcap F_p V$, as one see from the example of an adic filtration.

One [redacted] purpose of the above discussion is the following. Consider a filtered algebra
$$A = \bigcup F_p A \quad , \quad F_pA \cdot F_qA \subset F_{p+q}A \quad , \quad 1 \in F_0 A$$
Either $1 \in F_{-1}A$ or not.

If $1 \in F_{-1}A$, then $F_pA \subset F_pA \cdot F_{-1}A \subset F_{p-1}A$ so $F_pA = A$ for all p. Put another way
$$A^h = \bigoplus h^p F_p A$$
is then an algebra over $\mathbb{C}[h, h^{-1}]$ so $F_pA = A$, $\forall p$.

If $1 \notin F_{-1}A$, then $\mathbb{C} \cap F_pA = \begin{cases} \mathbb{C} & p \geq 0 \\ 0 & p < 0, \end{cases}$

so
$$0 \longrightarrow \mathbb{C} \longrightarrow A \longrightarrow \bar{A} \longrightarrow 0$$
is compatible with filtrations, i.e. (when we define $F_p\mathbb{C} = \begin{cases} \mathbb{C} & p \geq 0 \\ 0 & \end{cases}$)
$$0 \longrightarrow \mathbb{C}[h] \longrightarrow A^h \longrightarrow \bar{A}^h \longrightarrow 0$$

is an exact sequence of torsion-free graded $\mathbb{C}[h]$-modules.

So now we relative differential forms on A^h relative to $\mathbb{C}[h]$, in other word we work with algebras over the commutative ring $\mathbb{C}[h]$ and the corresponding differential forms in this setting. We have

$$\Omega^n_{\mathbb{C}[h]} A^h = A^h \otimes_{\mathbb{C}[h]} \overbrace{(A^h/\mathbb{C}[h]) \otimes_{\mathbb{C}[h]} \cdots\cdots \otimes_{\mathbb{C}[h]} (A^k/\mathbb{C}[h])}^{n}$$

This is a torsion free $\mathbb{C}[h]$ module, [redacted] hence corresponds to a filtered vector space. The vector space is found by specializing $h \longmapsto 1$. Thus we get $\Omega^n A = A \otimes \bar{A}^{\otimes n}$ with the induced filtration:

$$F_p \Omega^n A = \text{[redacted]} \sum_{p_0 + \cdots + p_n = p} F_{p_0} A \; d(F_{p_1} A) \cdots d(F_{p_n} A)$$

Another point is that by specializing to $h = 0$ we find

$$\mathrm{gr}\, \Omega^n A = \Omega^n (\mathrm{gr}\, A)$$

June 26, 1993

The missing point is to relate 1) my construction based on the homomorphism

$$RA \longrightarrow S \otimes RB \qquad S = \bigoplus_{n \geq 0} t^n J^n$$

together with the traces τ_m on S to 2) Joachim's construction using

[redacted]

$$\begin{tikzcd}
RA \arrow[r, "\pi_- \iota_*"] \arrow[dr, "\frac{1}{2}(\iota_* - \iota^\gamma_*)"'] & RQ \arrow[r, "(1-\frac{D}{2m-1}) \cdot\cdot (1-\frac{D}{3})(1-\frac{D}{1})"] & RQ & \\
& \mathcal{F}_1 RQ \arrow[u, phantom, "\cup"] \arrow[r] & \mathcal{F}_{2m+1} RQ \arrow[u, phantom, "\cup"] \arrow[r] & J^{2m+1} \otimes RB.
\end{tikzcd}$$

[redacted] Joachim's construction can be described in better notation as follows. Recall that the grading $Q = \bigoplus_{n \geq 0} \Omega^n$ of Q as vector space with $\mathcal{I}$ induces a grading on RQ and D is the degree operator. Put $R = RQ$ and write $R = \bigoplus R_n$ for the grading so that $D = n$ on R_n. Recall that we have a homomorphism $RQ \longrightarrow L \otimes RB$ such that $R_n \longrightarrow J^n \otimes RB$. Joachim's construction consists in using

$$RQ \xrightarrow{(1-\frac{D}{2m-1}) \cdots\cdot (1-D)\pi_-} RQ$$

to map R into $R_{\geq 2m+1} = \bigoplus_{n \geq 2m+1} R_n$ and then the maps $R_n \longrightarrow J^n \otimes RB \subset J^{2m+1} \otimes RB$ for $n \geq 2m+1$.

But all we have really is the map

$$RQ = \bigoplus R_n \longrightarrow \bigoplus t^n J^n \otimes RB$$

[redacted] where D on RQ corresponds to $D = t\frac{d}{dt}$ on $S \otimes RB$. So as I suspected before, it should ~~suffices~~ suffice to consider D on S.

Let us define the trace τ_m on S as follows. Consider the distribution

$$\mu_m : \frac{1}{2}(\delta_1 - \delta_{-1})\left(1 - \frac{D}{2m-1}\right)\cdot\cdot\left(1-\frac{D}{3}\right)(1-D) : \mathbb{C}[t] \to \mathbb{C}$$

Thus $\mu_m(t^n) = 0$ n even

$$\mu_m(t^n) = \left(1 - \frac{n}{2m-1}\right)\cdots\left(1-\frac{n}{3}\right)(1-n) \qquad n \text{ odd}$$

we note that μ_m kills t^n for $0 \leq n \leq 2m$ and that

$$\mu_m(t^{2m+1}) = \frac{(-2)(-4)\cdots(-2m)}{1\cdot 3 \cdots 2m-1}$$

Also μ_m kills the ideal $(1-t^2)^{m+1}\mathbb{C}[t]$, since polynomials in this ideal vanish to order $m+1$ at ± 1 and μ_m is of order m.

Consider now

$$\begin{tikzcd}
S \arrow[r, phantom, "\subset"] \arrow[d] & \mathbb{C}[t]\otimes L \arrow[d, "\mu_m\otimes 1"'] \\
J^{2m+1} \arrow[r, phantom, "\subset"] \arrow[d, "\#_{2m+1}"'] & L \\
J^{2m+1}_{\#} &
\end{tikzcd}$$

June 30, 1993

More on Nistor. I would like reconcile my construction of the bivariant Chern character with Nistor's construction.

He works with the mixed complex $(\Omega Q, b, B)$ where $Q = QA$. This inherits a filtration $\Omega Q_{\geq n}$ from the $\mathfrak{q} = \mathfrak{q}A$ adic filtration of Q: $Q_{\geq n} = \mathfrak{q}^n$. One can organize this filtration by introducing the graded algebra

$$Q^t = \bigoplus_{n \in \mathbb{Z}} t^n \mathfrak{q}^n \quad \text{over} \quad \mathbb{C}[t^{-1}]$$

Then one should have

$$(\Omega Q)^t = \bigoplus_{n \in \mathbb{Z}} t^n \Omega Q_{\geq n} = \Omega(Q^t; \mathbb{C}[t^{-1}]).$$

More precisely, $\Omega(Q^t; \mathbb{C}[t^{-1}])$ is a graded algebra over $\mathbb{C}[t^{-1}]$ which is torsion free and $t^n \Omega Q_{\geq n}$ is the deg n subspace.

Continuing with this idea of working with $\mathbb{C}[t^{-1}]$ as ground ring, we note that all the operators b, d, κ, B, P etc. make sense on $\Omega(Q^t; \mathbb{C}[t^{-1}]) = \bigoplus t^n \Omega Q_{\geq n}$. So everything we did like $X(RQ) \simeq \Omega Q$, $F^p_{IQ} \simeq F^p \Omega Q$ should generalize. In particular ~~all these~~

$$R(Q^t; \mathbb{C}[t^{-1}]) \overset{\text{def}}{=} \Omega^{ev}(Q^t; \mathbb{C}[t^{-1}]) \quad \text{with } \circ$$

should have the universal mapping property expected. Since it's torsion-free over $\mathbb{C}[t^{-1}]$ we should have

$$R(Q^t; \mathbb{C}[t^{-1}]) = \bigoplus_n t^n RQ_{\geq n}$$

where $\{RQ_{\geqslant n}\}$ is the filtration on RQ inherited from the $\mathfrak{q}$-adic filtration on Q.

[redacted]

Write $RQ^t = R(Q^t; \mathbb{C}[t^{-1}])$.

We have $IQ^t = \bigoplus_{q\geqslant 1}(\Omega^{2q}Q)^t = \bigoplus_n \bigoplus_{q\geqslant 1} t^n\Omega^{2q}Q_{\geqslant n}$

So it should be possible to continue this to the X-complex as follows:

$$\underset{\bigoplus t^n X(RQ)_{\geqslant n}}{\underset{\|}{X(RQ)^t}} = X(RQ^t, \mathbb{C}[t^{-1}]) \simeq \Omega Q^t$$

Also $$F^p_{IQ^t}X(RQ)^t \simeq \underset{\text{Hodge filtration.}}{F^p\Omega Q^t}$$

Notice that when this involves

$$b(\Omega^{p+1}Q)^t = [\Omega^pQ^t, Q^t]$$
$$= \bigoplus t^n \underbrace{\sum_{i+j=n}[\Omega^pQ_{\geqslant i}, Q_{\geqslant j}]}_{b(\Omega^{p+1}Q_{\geqslant n})}$$

So the moral seems to be that we can identify (up to contractible complexes) what Nistor uses, namely, the mixed complex $\Omega Q_{\geqslant n}$ or rather the [redacted] supercomplex $(\Omega Q_{\geqslant n}, b+B)$ with Hodge filtration, with what Joachim uses, namely the supercomplex $X(RQ)_{\geqslant n}$ together with the appropriate version of $F^p_{IQ}X(RQ)_{\geqslant n}$ filtration. What I mean by appropriate is that

$$F^{2n}_{IQ} X(RQ)^+_{\geq k} = (IQ)^{n+1}_{\geq k} + \underbrace{[(IQ)^n, RQ]_{\geq k}}_{\sum_{i+j=k}[(IQ)^n_{\geq i}, RQ_{\geq j}]}$$

$$F^{2n}_{IQ} X(RQ)^-_{\geq k} = \underbrace{\natural((IQ)^n\, d(RQ))_{\geq k}}_{\sum_{i+j=k} \natural((IQ)^n_{\geq i},\, d(RQ)_{\geq j})}$$

At this point I would like to claim that it should be possible to do Nistor's argument using $S = \bigoplus t^n J^n$ and the traces μ_m.

July 1, 1993

Review progress so far on Joachim's version of Nistor's bivariant Chern character. Let's begin with a description of Nistor's construction. He constructs universal classes Ch_0^{2n} lying in the bivariant $HC^{2n}(A, F_{-n-1})$. What is F_{-n-1}? Let $Q = QA$, $\mathfrak{q} = \mathfrak{q}A$ and consider the mixed complex $(\Omega Q, b, B)$. The $\mathfrak{q}$-adic filtration $Q_{\geq n} = \mathfrak{q}^n$ of Q induces a filtration $\Omega Q_{\geq n}$ on the ~~mixed~~ complex ΩQ:

$$\Omega^{\delta} Q_{\geq n} \quad \text{spanned by } x_0\, dx_1 \ldots dx_{\delta} \\ \text{where } x_i \in \mathfrak{q}^{n_i} \text{ and } \sum n_i \geq n.$$

Nistor's F_{-n-1} is just $\Omega Q_{\geq n+1}$.

Digression: There is an analogous filtration on the cyclic module $C(Q)$: $C(Q)_n = Q^{\otimes n+1}$ of unnormalized chains.

Q is a superalgebra such that $\mathfrak{q}$ is invariant under the $\mathbb{Z}/2$ grading involution γ. Note that this involution is $(-1)^n$ on $\mathfrak{q}^n/\mathfrak{q}^{n+1}$. Thus Q has the two structures: filtered algebra and superalgebra, linked by the property that on $\operatorname{gr} Q$ the degree mod 2 coincides with the $\mathbb{Z}/2$ degree. (Note: this is similar to "special" as in special tower.)

Now when we form $\operatorname{gr} \Omega Q$ for the filtration $\Omega Q_{\geq n}$ we get $\Omega(\operatorname{gr} Q)$. Since the involution γ on Q induces the degree mod 2 involution on $\operatorname{gr} Q$, we see the same holds for $\operatorname{gr} \Omega Q = \Omega(\operatorname{gr} Q)$. In other words γ on $\Omega Q_{\geq n}/\Omega Q_{\geq n+1}$ is $(-1)^n$.

Return now to Nistor's construction.

A linear map $D: Q \longrightarrow Q$ such that $D(\mathbb{C}) \subset \mathbb{C}$ induces an operation L_D on ΩQ, which is the obvious "derivation" type extension of D to the tensor products $Q \otimes \bar{Q}^{\otimes n}$. (suppose to be on the safe side that $D(1) = 0$.) Nistor shows that L_D is homotopic to zero with respect to the differential $b + B$; this is a generalization of Rinehart's calculation.

We have another way to understand this L_D, namely using the canonical identification

$$X(RQ) \simeq \Omega Q$$

But [redacted] I see that I have made a mistake. The L_D that I define on $X(RQ)$ does not correspond to Nistor's.

Recall the definition. The point is that by the universal mapping property of RQ, D extends to a derivation on RQ (here we use $D(1) = 0$), and this gives rise to a Lie derivative operator L_D on $X(RQ)$, and also to a homotopy operator h_D satisfying $L_D = [\partial, h_D]$.

Let's continue with Joachim's rather than Nistor's construction. This means we will work with $X(RQ)$ rather than ΩQ. We take D to be the grading operator on Q given by the linear isomorphism $Q \simeq \Omega A$.

First recall the Nistor approach with his L_D.

He starts with

$$\Omega A \xrightarrow{\iota_* - \iota_*^\gamma} \Omega Q$$

i.e. up to a factor of 2 with

$$\Omega A \xrightarrow{\iota_*} \Omega Q \xrightarrow{\pi_-} \pi_- \Omega Q$$

where π_- projects on the negative subspace for the $\mathbb{Z}/2$-grading arising from γ. We have

$$\pi_- \Omega Q = \pi_- \Omega Q_{\geq 1}$$

since we have seen that $\Omega Q_{\geq n} / \Omega Q_{\geq n+1}$ is of parity $n + 2\mathbb{Z}$. ~~Then he considers~~

~~$$\Omega Q_{\geq n} \xrightarrow{(1-L_D)(1-\frac{1}{2}L_D)\cdots(1-\frac{1}{n}L_D)} \Omega Q_{\geq n+1}$$~~

~~Its homotopy formula is used to show that~~

Nistor's homotopy formula says

$$S(L_D) = [\partial, j_D] + \text{error term } f$$

Actually I find it hard to describe Nistor's approach since I tend to replace mixed complexes by supercomplexes with filtration. My goal at this point should be to find exactly what to say to convince the reader that my construction and Nistor's yield the same bivariant Chern character.

So I should first do things in the way Joachim does. This means instead of $(\Omega Q, b, B)$ and the filtration $\Omega Q_{\geq n}$ by mixed subcomplexes, we will use the filtered supercomplex

$X(RQ)$ with $F^p_{IQ}X(RQ)$ and the corresponding filtration given for each n by $X(RQ)_{\geq n}$ equipped with $\underbrace{F^p_{IQ}X(RQ)_{\geq n}}$ filtration. Here we have to define carefully as I discussed yesterday. Recall that $F^p_{IQ}X(RQ)_{\geq n}$ corresponds under the canonical identification $X(RQ) = \Omega Q$ to $F^p_{Hodge}(\Omega Q_{\geq n}) = b(\Omega^{p+1}Q_{\geq n}) \oplus (\Omega^{>p}Q)_{\geq n}$.

What happens then is that Nistor's mixed complex $\Omega Q_{\geq n}$ is replaced by the filtered supercomplex $X(RQ)_{\geq n}, \{F^p_{IQ}X(RQ)_{\geq n}\}$. The lemma we need then to push things through is that

$$L_D\left(F^p_{IQ}X(RQ)_{\geq n}\right) \subset F^{p-2}_{IQ}X(RQ)_{\geq n} \qquad \forall p$$

Assuming this Joachim's construction goes as follows

Start with

$$\begin{tikzcd} X(RA) \arrow[r, "\iota_*"] & X(RQ) \arrow[r, "\pi_-"] & \pi_-X(RQ) \arrow[d, equal] \\ & & \pi_-X(RQ)_{\geq 1} \end{tikzcd}$$

Then follow with

$$\begin{tikzcd} \pi_-X(RQ)_{\geq 1} \arrow[r, "1-L_D"] & \pi_-X(RQ)_{\geq 3} \arrow[r, "1-\frac{1}{3}L_D"] & \pi_-X(RQ)_{\geq 5} \end{tikzcd}$$

So far we work with supercomplexes. But now examine what happens to $F^p_{IA}X(RA)$.

$$\begin{tikzcd} F^p_{IA}X(RA) \arrow[r, "\iota_*"] & F^p_{IQ}X(RQ) \end{tikzcd}$$

$$\xrightarrow{\pi_-} \pi_-F^p_{IQ}X(RQ)_{\geq 1}$$

$$\xrightarrow{1-D} \pi_- F^{p-2}_{IQ} X(RQ)_{\geqslant 3}$$

$$\xrightarrow{1-\frac{1}{3}D} \pi_- F^{p-4}_{IQ} X(RQ)_{\geqslant 5}$$

Let's try to put this differently. Let's consider [illegible] the identification $X(RQ) = \Omega Q$, $F^p_{IQ} X(RQ) = b\Omega^{p+1}Q \oplus \Omega^{>p}Q$. [illegible] Let D be the derivation on RQ arising from [illegible] the grading on $Q = \oplus \Omega^n$. Consider L_D on $X(RQ)$. ?

July 3, 1993

The problem is still to compare my construction of the biv. Chern character associated to a quasi-homomorphism

(1) $$A \rightrightarrows L\otimes B \quad \text{cong mod } J\otimes B$$

with Nistor's construction. From (1) we obtain homs.

$$\begin{tikzcd}
A \arrow[r, shift left] \arrow[r, shift right] & Q \arrow[r] & L\otimes B \\
& \bigoplus_n t^n Q_{\geq n} \arrow[r] \arrow[d, equal] & \bigoplus_n t^n J^n \otimes B \arrow[d, equal] \\
& Q_t \arrow[r] & S\otimes B
\end{tikzcd}$$

$$(RQ)_t \longrightarrow S\otimes RB$$

$$(IQ)_t \longrightarrow S\otimes IB$$

$$X(RQ)_t \longrightarrow S_4 \otimes X(B)$$

The next ingredient is the based linear map

$$Q = \bigoplus_n Q_n \xhookrightarrow{t^D} \bigoplus_n t^n Q_{\geq n} = Q_t$$

Calculate curvature

[redacted]

$$t^D(x\circ y) = t^{|x|+|y|}xy - t^{|x|+|y|+2}dx\,dy$$

$$t^D x \circ t^D y = t^{|x|+|y|}(xy - dx\,dy)$$

∴ The curvature is

$$t^{|x|}x \otimes t^{|y|}y \longmapsto (1-t^2)\,t^{|x|+|y|}\,dx\,dy$$

and this maps $\bar{Q}^{\otimes 2}$ to

$$\sum (1-t^2)t^n Q_{\geq n} = (1-t^2)g^2 Q_t$$

Note that t^D becomes a homomorphism when t is specialized to ± 1.

Instead of $p + t_g : A \longrightarrow S \otimes B$ we now have $Q \xrightarrow{t^D} Q_t \longrightarrow S \otimes B$, which also becomes a homomorphism ~~[illegible]~~ [redacted] modulo $K \otimes B$.

Proceed as for A:

$$\begin{tikzcd} Q \arrow[r, "t^D"] & Q_t \end{tikzcd}$$

linear
assume values
in $(1-t^2)g^2 Q_t$.

$$\begin{tikzcd} RQ \arrow[r, "u'"] & (RQ)_t \\ IQ \arrow[u, phantom, "\cup"] \arrow[r] & K' \arrow[u, phantom, "\cup"] \end{tikzcd}$$

K' = inverse image of $(1-t^2)g^2 Q_t$

Note that

$$\begin{tikzcd} RQ \arrow[r] & (RQ)_t \arrow[r] & S \otimes RB \\ & K' \arrow[r] & K \otimes RB + S \otimes IB \\ & (IQ)_t \arrow[r] & S \otimes IB \end{tikzcd}$$

$$\begin{tikzcd} X(RQ) \arrow[r] & X(RQ)_t \arrow[r] & S_\natural \otimes X(RB) \\ F^p_{IQ} \arrow[r] & F^p_{K'} \arrow[r] & \sum_{i \geq 0} \natural(K^i) \otimes F^{p-2i}_{IB} X(RB) \end{tikzcd}$$

Now the question is whether there is any relation between the filtration $F^p_{K'}$ of $X(RQ)_t$ and the double filtration $\left(F^p_{IQ} X(RQ)\right)_{\geq n}$ I discussed before.

In general it should be interesting to study RQ where Q is a filtered algebra equipped with linear splitting of the filtration. In this case we have a linear map

$$Q \longrightarrow \bigoplus_n t^n Q_{\geq n} = Q_t$$

which becomes a homomorphism at $t = 1$. Deformation of Q to $\text{gr}\, Q$.

Points worth remembering:

1) Puzzle about $\bigoplus_{n \geq 0} t^n L^n$ versus $\bigoplus_{n \in \mathbb{Z}} t^n L^n$ arising in connection with the canonical traces μ_m. Trace μ_m defined on $\bigoplus_{n \in \mathbb{Z}} t^n L^n$ but what happens to the ideal K?

2) One can define $(RQ)_t = \bigoplus_n t^n RQ_{\geq n}$ as the image of the unique graded algebra homom.

$$R(Q_t) \longrightarrow \bigoplus_n \mathbb{C} t^n \otimes RQ$$

corresponding to the homomorphism $R(Q_t) \to RQ$ where $t \longmapsto 1$. Similarly for $(\Omega Q)_t$, $(X(RQ))_t$, $\left(F^p_{IQ} X(RQ)\right)_t$

3) If one uses $\bigoplus_{n \in \mathbb{Z}} t^n F_n$ then one can handle both universal enveloping alg. increasing filtrations and adic filtrations at the same time.

July 8, 1993

Recall $S = \bigoplus_{n \geq 0} t^n J^n$. I have various comments to record.

First I claim

$$\circledast \qquad S/K \xrightarrow{\sim} L \times_{L/J} L .$$

Here $K = (1-t^2)J^2S = \sum_n (1-t^2)t^n J^{n+2}$ is the ideal generated by $(1-t^2)J^2$ in S. The two maps $S/K \rightrightarrows L$ are given by specializing tx to $\pm x$, $x \in J$. As we know $L \oplus tJ \xrightarrow{\sim} S/K$ it is clear $\circledast$ is an isomorphism.

Thus the trace

$$\mu_m : S/K^{m+1} \longrightarrow J^{2m+1}_\#$$

I construct is a trace on a nilpotent extension of $L \times_{L/J} L$ of order m. So when we have $A \overset{\theta}{\underset{\theta'}{\rightrightarrows}} L$ congruent modulo J, we get from

$$\begin{tikzcd} & S/K^{m+1} \arrow[r, "\mu_m"] \arrow[d] & J^{2m+1}_\# \\ A \arrow[r, "(\theta, \theta')"] & L \times_{L/J} L & \end{tikzcd}$$

a map $HC_{2m}A \longrightarrow J^{2m+1}_\#$.

Using the lifting $p + tq : A \longrightarrow S$ we get a trace on RA

$$RA \longrightarrow S/K^{2m+1} \longrightarrow J^{2m+1}_\#$$

$$a_0 da_1 \cdots da_{2n} \longmapsto (p+tq)a_0 (1-t^2)^n qa_1 \cdots qa_{2n} \longrightarrow \mu_m(t(1-t^2)^n qa_0 \cdots qa_{2n})$$

Now

$$\mu_m\left(t(1-t^2)^n qa_0 \cdots qa_{2n}\right) = \begin{cases} 0 & n \neq m \\ \frac{2^m m!}{1\cdot 3 \cdots 2m-1} \#_{2m+1}(qa_0 \cdots qa_{2n}) & \text{if } n = m \end{cases}$$

Using the rescaling

$$f(a_0, \ldots, a_{2n}) = \frac{(-1)^n}{n!} \tau\left(\rho a_0\, \omega(a_1,a_2) \cdots \omega(a_{2n-1}, a_{2n})\right)$$

between K^2-invariant $b+B$ cocycle and trace on RA we see that the $b+B$ cocycle associated to μ_m has only one component, namely the reduced cyclic $2m$ cocycle

$$\frac{(-2)^m}{1\cdot 3 \cdots 2m-1} \#_{2m+1}\left(qa_0 \cdots qa_{2m}\right)$$

in degree $2m$.

(Observe the numerical factors check since

$$\mathrm{tr}(pq^{2n+1})B = (2n+1)\,\mathrm{tr}(q^{2n+1})$$

$$\mathrm{tr}(pq^{2n+1})b = 2\,\mathrm{tr}(q^{2n+3})$$

This is true for any trace on RA.)

Second, consider the comparison: $L_t = \bigoplus_{n\in\mathbb{Z}} t^n J^n$, $S = \bigoplus_{n\geq 0} t^n J^n$. Instead of the ideal generated in S by $(1-t^2)J^2$ look at the ideal generated by $(1-t)J$. Put $K = (1-t)JS$.

Now we have the derivations $t\frac{d}{dt}$ and $\frac{d}{dt}$ acting on L_t and S. We have at least formally

$$e^{x\frac{d}{dt}}(t) = t + x$$

We can use $e^{x\frac{d}{dt}}$ on S because $\frac{d}{dt}$ is locally nilpotent. This automorphism for $x=1$ relates $(1-t)JS$ to $tJS = S_{\geqslant 1}$

We can also use $e^{x\frac{d}{dt}}$ on $\Big(\prod_{n<0} t^n$ [redacted] $L\Big)\times S$ which is a completion of L_t. Subalgebra of $L[[t^{-1}]][t]$ such that the coeff of t^n lies in J^n. $\bigoplus_{n\in\mathbb{Z}} t^n J^n_\#$ is the space of continuous traces on this completion.

An interesting point is that

$$S/t^nJ^nS = \bigoplus_{k=0}^{n-1} t^kJ^k$$

$$L_t/t^nJ^nL_t = \bigoplus_k t^k(J^k/J^n)$$

so the sort of traces of interest cohomologically, i.e. J-adic traces on J^k which don't [redacted] vanish on J^n for n large, do not appear as traces on $L_t/t^nJ^nL_t$.

July 15, 1993:

Some related calculations

$L^t = \bigoplus_{k\in\mathbb{Z}} t^kJ^k$ Note that

$(t^{-1}-1)L^t \supset (1-t)JL^t$

[redacted] and $L^t/(t^{-1}-1)L^t \xrightarrow{\sim} L$

We have

$$L^t/(1-t)JL^t \xrightarrow{\sim} (\mathbb{C}[t^{-1}]\otimes L/J)\times_{L/J} L$$

Also it seems that

$$L^t/(1-t^2)J^2L^t \xrightarrow{\sim} \Big(\bigoplus_{k\leq 0} t^k(L/J^2) \oplus t(J/J^2)\Big) \times_{L/J^2\times L/J^2} (L\times L)$$

Note that $L^t/(t^{-2}-1)L^t \xrightarrow{\sim} L\times L$ and that

$$(t^{-2}-1)L^t \supset (t^{-1}-t)JL^t \supset (1-t^2)J^2L^t$$

In any case before doing these calculations I should have asked whether the trace μ_0 of interest on $S/(1-t^2)J^2S \simeq S\oplus tJ$ is defined on $L^t/(1-t^2)J^2L^t$, because the answer seems to be negative.

July 17, 1993

I seem to have progressed a bit in understanding the difference between working with $L^t = \bigoplus_{k\in\mathbb{Z}} t^k J^k$ and $S = L^{t,\geq 0} = \bigoplus_{k\geq 0} t^k J^k$. Recall that the point behind L^t is to enable one to handle filtrations by means of graded $\mathbb{C}^t = \mathbb{C}[t^{-1}]$-modules. Yesterday I noticed the following. [redacted]

Consider the homomorphism

$$\begin{tikzcd} Q \arrow[r] & L\otimes B \\ \arrow[u, phantom, "\cup"] Q_{\geq k} \arrow[r] & J^k\otimes B \arrow[u, phantom, "\cup"] \end{tikzcd}$$

in the Nistor situation. This gives rise to a homomorphism of graded $\mathbb{C}^t$-algebras

$$Q^t \longrightarrow L^t \otimes B$$

[redacted]hence to a homom. of graded $\mathbb{C}^t$-algebras

$$R(Q^t;\mathbb{C}^t) \longrightarrow R(L^t\otimes B;L^t) = L^t\otimes RB$$

hence to a map of supercomplexes of graded $\mathbb{C}^t$-modules

$$X(R(Q^t;\mathbb{C}^t),\mathbb{C}^t) \longrightarrow X(L^t\otimes RB;L^t) = L^t_\natural \otimes X(RB)$$

Now localizing [redacted] by inverting t on the left yields a homom

$$R(Q^t;\mathbb{C}^t) \longrightarrow \mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}^t} R(Q^t;\mathbb{C}^t) = \mathbb{C}[t,t^{-1}]\otimes RQ$$

and map

$$X(R(Q^t;\mathbb{C}^t);\mathbb{C}^t) \longrightarrow \mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}^t} X(R(Q^t;\mathbb{C}^t);\mathbb{C}^t) = \mathbb{C}[t,t^{-1}]\otimes X(RQ)$$

Now in general neither of these canonical maps into the localization is injective, [redacted] unless one makes the flatness assumption that t^{-1} is injective on $R(Q^t;\mathbb{C}^t)$ and $X(R(Q^t;\mathbb{C}^t);\mathbb{C}^t)$ respectively. But I think we know that as $\mathbb{C}^t$ modules

$$X(R(Q^t;\mathbb{C}^t);\mathbb{C}^t) \cong \Omega(Q^t;\mathbb{C}^t)$$

$$\Omega^n(Q^t;\mathbb{C}^t) = Q^t\otimes_{\mathbb{C}^t}(Q^t/\mathbb{C}^t)\otimes_{\mathbb{C}^t}\cdots\otimes_{\mathbb{C}^t}(Q^t/\mathbb{C}^t)$$

The question arises as to when multiplication by t^{-1} is injective. Recall there are two cases:

$1\notin Q_{\geq 1}$. In this case we have

$$\mathbb{C}\cap Q_{\geq k} = \begin{cases}\mathbb{C} & k\leq 0\\ 0 & k>0\end{cases}$$

so that multiplication by t^{-1} on $Q^t/\mathbb{C}^t$ is injective.

$1\in Q_{\geq 1}$. In this case from $Q_{\geq i}\cdot Q_{\geq j}\subset Q_{\geq i+j}$ and $Q_{\geq j}\supset Q_{\geq j+1}$ we conclude that $Q_{\geq j}=Q_{\geq j+1}$ for all j. Thus $Q^t = \mathbb{C}[t,t^{-1}]\otimes Q_{\geq 0}$ is a $\mathbb{C}^t$ module on which t^{-1} is invertible. It follows that

$$\Omega^n(Q^t;\mathbb{C}^t) = Q^t\otimes_{\mathbb{C}[t,t^{-1}]}(Q^t/\mathbb{C}[t,t^{-1}])\otimes_{\mathbb{C}[t,t^{-1}]}$$

$$= \Omega^n(Q^t;\mathbb{C}[t,t^{-1}])$$

so that multiplication by t^{-1} is invertible

Thus things work (just barely). The problem I thought occurred was that in

$$\begin{tikzcd} X(R(Q^t;\mathbb{C}^t);\mathbb{C}^t) \arrow[r] \arrow[d, "\varphi"'] & L^t_q\otimes X(RB) \\ X(RQ)^t & \end{tikzcd}$$

if φ were not [redacted] an isomorphism then I would not obtain the desired maps $X(RQ)_{\geq k} \longrightarrow J^k_{\#} \otimes X(RB)$. [redacted] The hope was that this might indicate perhaps that working with L^t involving negative powers of t had defects.

But observe that it might be possible to turn the non-injectivity of t^{-1} on $L^t_\natural$ into an argument for using the relative R and X for Q^t relative $\mathbb{C}^t$. The point is how [redacted] can we actually construct the map

$$X(RQ)_{\geq k} \longrightarrow J^k_{\#} \otimes X(RB) \ ?$$

Joachim's method. Start with $RQ \longrightarrow L \otimes RB$ carrying $(RQ)_{\geq k}$ to $J^k \otimes RB$ for all k. Form $\Omega(RQ) \longrightarrow \Omega(L \otimes RB; L) = L \otimes \Omega(RB)$ and check that $\Omega(RQ)_{\geq k}$ goes to $J^k \otimes \Omega(RB)$. Calculate that if we replace J^k by $J^k_{\#}$ that the map is compatible with b, hence one has a map $\Omega(RQ)_{\geq k} \longrightarrow J^k_{\#} \otimes \Omega(RB)$. Then take induced map on the quotients: $X(RQ)_{\geq k} \longrightarrow J^k_{\#} \otimes X(RB)$.

July 20, 1993

Digression: What is the relation between $\Omega A = T_A(\Omega^1 A)$ and $T_A(A\otimes A)$? We have a bimodule embedding $\Omega^1 A \hookrightarrow A\otimes A$, $da \longmapsto 1\otimes a - a\otimes 1 = [D,a]$ where $D = 1\otimes 1$. This extends to a homomorphism of graded algebras

$$\Omega A \longrightarrow T_A(A\otimes A) = A * \mathbb{C}[D]$$

$$a_0 da_1 \dots da_n \longmapsto a_0[D,a_1]\cdots[D,a_n]$$

This is part of the lifting (p 151) of $A*\mathbb{C}[d]$ into $A*\mathbb{C}[D]$.

Can the differential d on ΩA be extended to a differential δ on $A*\mathbb{C}[D]$? I should be able to assign $\delta(D)$ arbitrarily to obtain a superderivation on $A*\mathbb{C}[D]$ [illegible] such that $\delta a = [D,a]$ for all $a\in A$. Let's calculate $\delta^2 a = \delta(\delta a)$

$$\delta^2 a = \delta[D,a] = [\delta(D),a] - [D,\delta a]$$

$$[D,\delta a] = [D,[D,a]] = [D^2,a]$$

Thus if $\delta(D) = D^2$ we have $\delta^2(a) = 0$. But **also** $\delta(D^2) = \delta(D)D - D\delta(D) = D^2D - DD^2 = 0$, whence $\delta^2 = 0$.

Calculation gives

$$\delta(a_1) = 1\otimes a_1 - a_1\otimes 1$$

$$\delta(a_1\otimes a_2) = 1\otimes a_1\otimes a_2 - a_1\otimes 1\otimes a_2 + a_1\otimes a_2\otimes 1$$

$$\delta(a_1\otimes a_2\otimes a_3) = 1\otimes a_1\otimes a_2\otimes a_3 - a_1\otimes 1\otimes a_2\otimes a_3 + a_1\otimes a_2\otimes 1\otimes a_3$$
$$- a_1\otimes a_2\otimes a_3\otimes 1$$

Thus it seems the Alexander-Spanier differential is meaningful in the noncommutative setting.

On $T_A(A \otimes A)$ we have superderivations ∂ and δ of degrees -1 and $+1$ resp. We have

$$(\delta\partial + \partial\delta)(D) = \delta(1) + \partial(D^2) = 0$$

$$(\delta\partial + \partial\delta)(a) = \partial\delta(a) = \partial[D,a] = [1,a] = 0$$

Thus $[\partial, \delta] = 0$.

Note 10/15/93 You should have posed the question of describing the possible superderivations of $T_A(A \otimes A) = A * \mathbb{C}[D]$. A lifting homomorphism

$$A * \mathbb{C}[D] \longrightarrow \mathbb{C}[\varepsilon] \otimes (A * \mathbb{C}[D])$$

where $|\varepsilon| = -r$, $\varepsilon^2 = 0$ has the form $1 + \varepsilon \otimes \delta$ where δ is a (super)derivation of degree $+r$. Thus δ is equivalent to a degree $+r$ derivation $\delta : A \longrightarrow A * \mathbb{C}[D]$ and the element $\delta(D) \in A * \mathbb{C}[D]$ of degree $r+1$.

Only choice of a canonical sort for $\delta(a)$ is $[D,a]$ or zero, also $[D^k, a]$. Similarly $\delta(D)$ must be a scalar times D^k.

If one wants δ to be of degree $+1$, then there are two independent possibilities: $\delta_1(a) = [D,a]$, $\delta_1(D) = [D,D] = 2D^2$ (i.e. $\delta_1 = \mathrm{ad}(D)$), and $\delta_2(a) = 0$ $\delta_2(D) = D^2$. Then the Alexander-Spanier diffl is $\delta_1 - \delta_2$, while $\delta_1 = \mathrm{ad}(D)$ is the difference of the left and right contractions. These supercommute with ∂. Squaring gives a quadratic function from the space spanned by δ_1, δ_2 to degree 2 derivations, the image is multiples of $\mathrm{ad}(D^2)$. Both $(\delta_1 - \delta_2)^2$ and δ_2^2 are zero, so the quadratic function is *hyperbolic*. Better to take $\delta_1 - \delta_2$ and δ_2 as basis.

August 22, 1993

This is about the puzzle S vs. L^t, an observation which seems worth recording. In the Nistor situation we have the algebras

$$\begin{tikzcd} \mathbb{C} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & S \arrow[d, phantom, "\cap"] \\ T \arrow[r, phantom, "\subset"] & L^t \end{tikzcd} \qquad (1)$$

and the square [redacted]

$$\begin{tikzcd} Q \arrow[r, "\xi"] \arrow[d, "t^D"'] & S\otimes B \arrow[d, phantom, "\cap"] \\ Q^t \arrow[r] & L^t\otimes B \end{tikzcd} \qquad (2)$$

$$\xi(a_0da_1 \ldots da_n) = t^n p a_0 g a_1 \cdots g a_n$$

where t^D, ξ [redacted] are linear maps resp. 1 and the others are homomorphisms. Apply the relative $X \circ R$ functor to the algebras in (2) relative to the corresponding algebras in (1), and we get

$$\begin{tikzcd} X(RQ) \arrow[r] \arrow[d] & S_\natural \otimes X(RB) \arrow[d] \\ X_T(R_T(Q^t)) \arrow[r] & L^t_\natural \otimes X(RB) \end{tikzcd}$$

which can be identified with

$$\begin{tikzcd} X(RQ) \arrow[r, "\alpha v_*"] \arrow[d, "t^{L_D}"'] & S_\natural \otimes X(RB) \arrow[d, phantom, "\cap"] \\ X(RQ)^t \arrow[r, "\ell^t"] & L^t_\natural \otimes X(RB) \end{tikzcd}$$

ℓ^t "trace" map

$v : RQ \to S\otimes RB$ homom induced by ξ

Recall that t^{L0} extends to an isomorphism of graded T-modules

$$T \otimes X(RQ) \xrightarrow{\sim} X(RQ)^t$$

and that ℓ^t is a T-module map. Thus ℓ^t is the T-module extension of αv_*.

Question: Is there any significance to ~~the~~ fact that

$$\ddot{S}_\natural \xrightarrow{\sim} L^t_\natural$$

in the case $[L,L] = L$? There's still the fact that the trace μ_m on S, when extended to $L^t_\natural$, does not vanish on an interesting ideal of L^t.

Sept 21, 1993

HPT. Recall

Prop. Let (E,d) be a complex, $|d|=-1$. One has an equivalence

$$\{ h \in End(E)_{+1} \mid [d,h]=1 \} \overset{\sim}{\longleftrightarrow} \left\{ (k,u) \,\middle|\, \begin{array}{ll} |k|=1, & |u|=2 \\ [d,k]=1, & k^2=0 \\ [d,u]=0 & [k,u]=0 \end{array} \right\}$$

$$h \longmapsto (hdh, h^2)$$

$$k+ud \longleftarrow\!\shortmid (k,u)$$

Analogy: contraction ~ connection

special contraction ~ flat connection

Next dilating a contraction to a special contraction: Consider

$E \oplus E[1]$: $E_n \oplus E_{n-1}$ in degree n

$$d' = \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix} \qquad \text{Put } h' = \begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}$$

$$\begin{matrix} E_{n+1} \\ \oplus \\ E_n \end{matrix} \xleftarrow{\begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}} \begin{matrix} E_n \\ \oplus \\ E_{n-1} \end{matrix} \qquad \text{so } |h'|=1.$$

Also:

$$[d',h'] = \begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}\begin{pmatrix} d & \\ & -d \end{pmatrix} + \begin{pmatrix} d & \\ & -d \end{pmatrix}\begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}$$

$$= \begin{pmatrix} hd & h^2d \\ d & hd \end{pmatrix} + \begin{pmatrix} dh & -dh^2 \\ -d & dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since $[d,h^2] = [d,h]h - h[d,h] = h - h = 0$.

$$h'^2 = \begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}\begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix} = \begin{pmatrix} h^2-h^2 & -h^3+h^3 \\ 0 & -h^2+h^2 \end{pmatrix} = 0$$

Thus h' is a special contraction on $E \oplus E[1]$. Then $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad i^* = (1 \quad 0)$

yield $\quad i^* h' i = (1 \quad 0)\begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = h$

and as expected for $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad j^* = (1 \quad 0)$.

$$h^2 = i^* h' i i^* h' i = i^* h'(1 - jj^*)h' i = -(i^* h' j)(j^* h' i)$$

$$i^* h' j = -h^2 \qquad j^* h' i = 1.$$

Recall

Prop. Given sdr (h, e): $hdh = h$, $h^2=0$, $[d,h] = 1-e$ and perturbation of differential d: $(d-\theta)^2 = 0$ we have a perturbed sdr $(\tilde{h}, \tilde{e})$ given by

$$\tilde{h} = h\frac{1}{1-\theta h} = \frac{1}{1-h\theta}h \qquad \tilde{e} = \frac{1}{1-h\theta}e\frac{1}{1-\theta h}$$

Slightly new proof: Following shows we have an sdr

$$\begin{aligned}\tilde{h}(d-\theta)\tilde{h} &= \frac{1}{1-h\theta}h(d-\theta)h\frac{1}{1-\theta h} \\ &= \frac{1}{1-h\theta}\underbrace{(\underbrace{hdh}_{h} - h\theta h)}_{h(1-\theta h)}\frac{1}{1-\theta h} = \tilde{h}\end{aligned}$$

Formula for $\tilde{e}$ is verified as in part 4:

$$(1-h\theta)[d-\theta, \tilde{h}](1-\theta h) = (1-h\theta)(d-\theta)h + h(d-\theta)(1-\theta h)$$

$$\begin{aligned} = \ & dh - h\theta dh - \theta h + h\theta^2 h \\ & hd - \underbrace{hd\theta h}_{h\theta^2 h} - h\theta + h\theta^2 h \end{aligned} \qquad \begin{aligned} &= 1 - e - \theta h - h\theta + h\theta^2 h \\ &= (1-h\theta)(1-\theta h) - e \end{aligned}$$

- Recall $f : X \to Y$ map of complexes is a h.eq. $\iff$ $C(f) : X[1] \oplus Y$, $\tilde{d} = \begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}$ is contractible.

Proof of $\Rightarrow$. Let $g : Y \to X$ map of cxs be a homotopy inverse for f so that $\exists\, h_X, h_Y$ such that $1 - gf = [d, h_X]$, $1 - fg = [d, h_Y]$. Let

$$\tilde{h} = \begin{pmatrix} -h_X & g \\ 0 & h_Y \end{pmatrix}$$

Then

$$[\tilde{d}, \tilde{h}] = \begin{pmatrix} dh + hd + gf & -dg + gd \\ -fh + hf & fg + dh + hd \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ [h,f] & 1 \end{pmatrix}$$

is invertible and homotopic to zero, so $C(f)$ is contractible.

Note $\tilde{h}$ is a special contraction iff

$$h_X^2 = h_Y^2 = [h, g] = [h, f] = 0$$

Define a <u>special homotopy equivalence</u> between two complexes X, Y to be a quadruple (f, g, h_X, h_Y),

$f : X \to Y$, $g : Y \to X$ degree zero

$h_X : X \to X$, $h_Y : Y \to Y$ degree one

satisfying $[d, f] = [d, g] = 0$, $[d_X, h_X] = 1 - gf$, $[d_Y, h_Y] = 1 - fg$

$$[h, f] = [h, g] = 0, \quad h^2 = 0.$$

Examples : $\begin{tikzcd} X \arrow[loop left, "h_X"] \arrow[r, shift left, "p"] & Y \arrow[l, shift left, "i"] \end{tikzcd}$ s.d.r. $\quad pi = 1$, $1 - ip = [d, h_X]$

$h_X^2 = h_X i = p h_X = 0.$

also $\begin{tikzcd} Y \arrow[r, hook, shift left, "i"] & X \arrow[l, two heads, shift left, "p"] \arrow[loop right, "h_X"] \end{tikzcd}$ s.d.r.

Question: Is this a useful notion? It seems that special h.eqs don't compose.

Sept. 22, 1993

$f : X \longrightarrow Y$ map of complexes. One has the mapping cylinder construction

$$\begin{tikzcd} X \arrow[r, hook, "j"] & \mathrm{Cyl}(f) \arrow[loop above] \arrow[r, two heads, shift right] & Y \arrow[l, hook', shift right] \\ & \text{sdr} & \end{tikzcd}$$

and the Serre construction

$$\begin{tikzcd} X \arrow[r, hook, shift right] & \mathrm{Ser}(f) \arrow[loop above] \arrow[l, two heads, shift right] \arrow[r, two heads, "q"] & Y \\ \text{sdr} & & \end{tikzcd}$$

A retraction wrt $j : X \longrightarrow \mathrm{Cyl}(f)$ is equivalent to $g : Y \to X$, $[d,g]=0$ and h_X such that $[d,h_X] = 1-gf$.

A section wrt $q : \mathrm{Ser}(f) \twoheadrightarrow Y$ is equivalent to $g : Y \to X$, $[d,g]=0$ together with $h_Y \ni [d,h_Y] = 1-fg$.

Formulas:

$$\mathrm{Cyl}(f)_n = X_n \oplus X_{n-1} \oplus Y_n \qquad d = \begin{pmatrix} d_X & -1 & \\ & -d_X & \\ & f & d_Y \end{pmatrix}$$

$$j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad \text{The sdr is}$$

$$\text{inj.} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{proj} = \begin{pmatrix} f & 0 & 1 \end{pmatrix}, \quad \text{htpy} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} d & -1 & \\ & -d & \\ & f & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} +1 & 0 & 0 \\ +d & 0 & 0 \\ -f & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -d & +1 & \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -f & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} f & 0 & 1 \end{pmatrix}$$

retraction wrt $j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is $r = \begin{pmatrix} 1 & -h_x & g \end{pmatrix}$

$$\begin{pmatrix} 1 & -h_x & g \end{pmatrix} \begin{pmatrix} d & -1 & \\ & -d & \\ & f & d \end{pmatrix} = \begin{pmatrix} d_x & -1+gf+h_x d & g d_y \end{pmatrix} = d_x \begin{pmatrix} 1 & -h_x & g \end{pmatrix}$$

using $[d, h_x] = 1 - gf$, $[d, g] = 0$

$$Ser(f)_n = X_n \oplus Y_{n+1} \oplus Y_n \qquad d = \begin{pmatrix} d_x & & \\ f & -d_y & -1 \\ & & d_y \end{pmatrix}$$

The sdr is

$$proj = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad inj = \begin{pmatrix} 1 \\ 0 \\ f \end{pmatrix}, \quad htpy = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} d & & \\ f & -d & -1 \\ & & d \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & -d & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f & +d & +1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -f & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ f \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

The section wrt $g = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} g \\ -h_y \\ 1 \end{pmatrix}$

$$\begin{pmatrix} d & & \\ f & -d & -1 \\ & & d \end{pmatrix} \begin{pmatrix} g \\ -h_y \\ 1 \end{pmatrix} = \begin{pmatrix} d_x g \\ fg - 1 + d_y h_y \\ d_y \end{pmatrix} = \begin{pmatrix} g \\ -h_y \\ 1 \end{pmatrix} d_y$$

using $[d, g] = 0$ and $[d_y, h_y] = 1 - fg$.

September 23, 1993

Recall that if a map f has both a left and a right inverse: $lf = 1$, $fr = 1$, then these coincide and f is invertible:

$$r = (lf)r = l(fr) = l$$

Let $f : X \to Y$ be a map of complexes. Call left h-inverse for f a pair $l : Y \to X$, $h : X \xrightarrow{+1} X$ such that [redacted]

$$[d, l] = 0 \qquad [d, h] = 1 - lf.$$

Similar a right h-inverse is a pair $r : Y \to X$, $k : Y \xrightarrow{+1} Y$ $\ni$ $[d, r] = 0$ $[d, k] = 1 - fr$. Then

$$\begin{aligned} l - r &= l - lfr + lfr - r \\ &= l(1 - fr) - (1 - lf)r \\ &= l\,[d, k] - [d, h]\,r \\ &= [d, lk - hr] \end{aligned}$$

showing that l, r are homotopic.

Next if (l, h) is a left h-inverse and if $u : Y \xrightarrow{+1} X$, then $(l + [d, u],\ h - uf)$ is also a left h-inverse:

$$[d, h - uf] = 1 - lf - [d, u]f = 1 - (l + [d, u])f$$

Thus if (r, k) is a right h-inverse, we have $r = l + [d, -lk + hr]$ by above, so $(r, \underbrace{h + (lk - hr)f}_{lkf + h(1 - rf)})$ is also a left-h-inverse.

Consider

$$\begin{tikzcd} & \mathrm{Cyl}(f) \arrow[dl, bend right, "\rho"'] \arrow[d, shift left, "p"] \\ X \arrow[ur, "j"'] \arrow[r, "f"] & Y \arrow[u, shift left, "\iota"] \end{tikzcd}$$

$\rho = (1 \ -h \ \ l)$ $\quad p = (f \ 0 \ 1)$ $\quad j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \iota = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$1 - \iota p = [d, h]$ $\quad h = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Observe that given maps $\xrightarrow{a}\xrightarrow{b}$ and left h-inverses (l_a, h_a), (l_b, h_b), then ba has left h-inverse $(l_a l_b, \ h_a + l_a h_b a)$:

$$[d, l_a l_b] = [d, l_a] l_b + l_a [d, l_b] = 0$$

$$[d, h_a + l_a h_b a] = 1 - l_a a + l_a (1 - l_b b) a$$
$$= 1 - l_a l_b b a$$

Apply this composition result to the composition $X \xrightarrow{j} \mathrm{Cyl}(f) \xrightarrow{p} Y$ with left h-inverses $(\rho, 0)$ for j and (ι, h) for p. Then we get $pj = f$ with homotopy

$$0 + \rho h j = (1 \ -h \ \ l)\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = h$$

I would like to conclude that the [redacted] cylinder factorization of [redacted] allows us to replace f and the left h-inverse (l, h) [redacted] $j : X \longrightarrow \mathrm{Cyl}(f)$ and the left h-inverse $(\rho, 0)$.

Let's next analyze an embedding of a complex Y into a contractible complex C, better: suppose Y is a subcomplex of a contractible complex C. Let $\Sigma^{-1}(C/Y)=X$, so that one has an exact sequence of complexes

$$0 \longrightarrow Y \longrightarrow C \longrightarrow \Sigma X \longrightarrow 0$$

Splitting this sequence not necessarily respecting the differentials determines a map of complexes $f: X \longrightarrow Y$ such that $C = C(f)$. Changing the splitting by $u: X \xrightarrow{+1} Y$ changes f by $[d,u]$. Thus we would like to choose the splitting so that f has the nicest form.

One has the following diagram of subcomplexes

$$\begin{tikzcd}
X \arrow[d, dash] \\
ZX \arrow[d, dash] \\
BX \arrow[d, dash] \\
0
\end{tikzcd}$$

this layer $\simeq \Sigma X$

$$\begin{tikzcd}
 & C \arrow[d, dash, "\textcircled{A}"] & \\
 & d^{-1}Y \arrow[d, dash, "\textcircled{H}"] & \\
 & Y+dC \arrow[dl, dash] \arrow[dr, dash] & \\
Y \arrow[dr, dash, "\textcircled{B}"'] & & dC \arrow[dl, dash, "\textcircled{dA}"] \\
 & Y\cap dC \arrow[d, dash, "\textcircled{dH}"] & \\
 & dY \arrow[d, dash, "\textcircled{dB}"] & \\
 & 0 &
\end{tikzcd}$$

Note that d induces

$$\begin{tikzcd}
C/dC \arrow[r, "\sim"] & dC \\
\cup & \cup \\
d^{-1}Y/dC \arrow[r, "\sim"] & Y\cap dC \\
\cup & \cup \\
Y+dC/dC \arrow[r, "\sim"] & dY
\end{tikzcd}$$

To get into a standard form, we choose complements

$$A \oplus d^{-1}Y = C$$
$$H \oplus Y+dC = d^{-1}Y$$
$$B \oplus Y\cap dC = Y$$

Then dA, dH, dB give the complements as indicated. Then

$$(A \oplus H \oplus B) \oplus dC = C$$

$$(A \oplus H \oplus dA) \oplus Y = C$$

The second splitting determines the map $f : X \longrightarrow Y$ as the deviation of $A \oplus H \oplus dA$ from being closed under d. Thus f is zero on A, dA and is the isomorphism $H_*(X) \xrightarrow{\sim} H_*(Y)$ from H to dH in the diagram. In other words what happens is that we have chosen $X = X_m \oplus X_c$, $Y = Y_m \oplus Y_c$ and f to be an isomorphism $X_m \xrightarrow{\sim} Y_m$, other components zero.

September 24, 1993

Recall that in the case of a [redacted] general contraction there are three candidates for the perturbed contraction, namely

$$h\frac{1}{1-\theta h}\ , \quad h\frac{1}{1-[\theta,h]}\ , \quad \frac{1}{1-[\theta,h]}h$$

What do we get if we dilate h to

$$H = \begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}$$

on $X \oplus \Sigma X$? The perturbation is $\Theta = \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}$. [redacted] In the case of a special contraction such as H the three candidates above coincide. Let's calculate

$$\Theta H = \begin{pmatrix} \theta h & -\theta h^2 \\ -\theta & \theta h \end{pmatrix} \qquad H\Theta = \begin{pmatrix} h\theta & h^2\theta \\ \theta & h\theta \end{pmatrix}$$

$$[\Theta, H] = \begin{pmatrix} [\theta,h] & -[\theta,h^2] \\ 0 & [\theta,h] \end{pmatrix}$$

$$1-[\Theta,H] = \begin{pmatrix} 1-[\theta,h] & [\theta,h^2] \\ & 1-[\theta,h] \end{pmatrix}$$

Put $G = (1-[\theta,h])^{-1}$. Then

$$1-[\Theta,H] = \begin{pmatrix} 1-[\theta,h] & \\ & 1-[\theta,h \end{pmatrix}\begin{pmatrix} 1 & G[\theta,h^2] \\ 0 & 1 \end{pmatrix}$$

$$(1-[\Theta,H])^{-1} = \begin{pmatrix} 1 & -G[\theta,h^2] \\ & 1 \end{pmatrix}\begin{pmatrix} G & \\ & G \end{pmatrix}$$

$$= \begin{pmatrix} G & -G[\theta,h^2]G \\ & G \end{pmatrix}$$

so the perturbed contraction is

$$H\frac{1}{1-[\Theta,H]} = \begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}\begin{pmatrix} G & -G[\theta,h^2]G \\ & G \end{pmatrix}$$

The compression of this to X is the upper left hand corner $hG = h\frac{1}{1-[\theta,h]}$

The compression to ΣX is the lower right corner

$$-hG - G[\theta,h^2]G$$

Note $\quad Gh = hG - \underbrace{[h,G]}$

$$[h,\frac{1}{1-[\theta,h]}] = -\frac{1}{1-[\theta,h]}[h,1-[\theta,h]]\frac{1}{1-[\theta,h]}$$

$$= G[h^2,\theta]G = -G[\theta,h^2]G$$

Thus $\quad -hG - G[\theta,h^2G] = -Gh = -\frac{1}{1-[\theta,h]}h$

The sign is due to the suspension. The conclusion is that the two compressions give the two candidates $h\frac{1}{1-[\theta,h]}$ and $\frac{1}{1-[\theta,h]}h$.

September 25, 1993

The question I should have asked a long time ago: What are all possible contractions on $F(f) = X \oplus Y[-1]$ with $d = \begin{pmatrix} d & 0 \\ f & -d \end{pmatrix}$?

An operator on $F(f)$ of degree $+1$ has the form $h = \begin{pmatrix} h_X & g \\ v & -h_Y \end{pmatrix}$ where $g : Y \to X$ has degree 0, $h_X : X \to X$, $h_Y : Y \to Y$ have degree 1 and $v : X \to Y$ has degree 2. One has

$$\left[\begin{pmatrix} d_X & 0 \\ f & -d_Y \end{pmatrix}, \begin{pmatrix} h_X & g \\ v & -h_Y \end{pmatrix} \right] = \begin{pmatrix} d_X h_X + h_X d + gf & d_X g - g d_Y \\ f h_X - d_Y v + v d_X - h_Y f & fg + d_Y h_Y + h_Y d_Y \end{pmatrix}$$

This is the identity iff $[d,g] = 0$, $[d_X, h_X] = 1 - gf$, $[d_Y, h_Y] = 1 - fg$, and $[d,v] = [f,h]$. Thus g is a homotopy inverse for f, and $[d,v] = [f,h]$ says that h_X, h_Y are compatible.

Suppose given f, g, h_X, h_Y where everything but the compatibility condition ~~[illegible]~~ holds. Then

$$\left[\begin{pmatrix} d_X & 0 \\ f & -d_Y \end{pmatrix}, \begin{pmatrix} h_X & g \\ 0 & -h_Y \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ [f,h] & 1 \end{pmatrix}$$

is invertible and commutes with d, so we ~~[illegible]~~ get two contractions

$$\begin{pmatrix} 1 & 0 \\ -[f,h] & 1 \end{pmatrix} \begin{pmatrix} h_X & g \\ & -h_Y \end{pmatrix} = \begin{pmatrix} h_X & g \\ -[f,h] h_X & -[f,h] g - h_Y \end{pmatrix}$$

$$\begin{pmatrix} h_X & g \\ & -h_Y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -[f,h] & 1 \end{pmatrix} = \begin{pmatrix} h_X - g[f,h] & g \\ h_Y [f,h] & -h_Y \end{pmatrix}$$

These respectively amount to keeping h_X and changing h_Y to make it compatible, respectively the other order.

Consider HPT. Suppose given $f : X \longrightarrow Y$ a map of complexes compatible with θ, such that f is a heq, where the h-inverse etc. need not respect θ. Form $F(f) = X_n \oplus Y_{n+1}$ in degree n with diffl $\begin{pmatrix} d & \\ f & -d \end{pmatrix}$ and perturbation $\textcircled{\Theta} = \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}$. h.eq. data for f given a contraction H on $F(f)$, then HPT yields three [redacted] possible contractions on $F(f)$ with differential d [redacted] $\textcircled{\Theta}$, namely:

$$H \frac{1}{1 - \textcircled{\Theta} H} , \qquad H \frac{1}{1 - [\textcircled{\Theta}, H]} , \quad \frac{1}{1 - [\textcircled{\Theta}, H]} H$$

Thus we expect different explicit homotopy equivalences on the perturbed complex.

Consider the basic HPT situation

$$\begin{tikzcd} E \arrow[loop right, "h"] \arrow[d, shift right, "p"'] \\ E' \arrow[u, shift right, "i"'] \end{tikzcd} \qquad pi = 1 \qquad 1 - ip = [d, h]$$

[redacted]

Assume perturbations θ of d given on both E, E'.

There are two cases: $[p, \theta] = 0$ and $[i, \theta] = 0$. Consider the former, form the fibre [redacted] $F(p)$.

$$\left[\begin{pmatrix} d & \\ p & -d \end{pmatrix} , \begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} [d,h] + ip & [d,i] \\ ph & pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

provided we assume $ph = 0$.

Better to say there are two cases:

a) $[p,\theta]=0$, $ph=0$

b) $[i,\theta]=0$, $hi=0$.

We already know that the perturbed operators

$$\theta' = p\underbrace{\theta\frac{1}{1-h\theta}i}_{\tilde{i}} = \underbrace{p\frac{1}{1-h\theta}}_{\tilde{p}}\theta i$$

$$\tilde{h} = h\frac{1}{1-\theta h} = \frac{1}{1-h\theta}h$$

in these cases are

a) $\theta' = \theta$, $\tilde{p} = p$

b) $\theta' = \theta$, $\tilde{i} = i$.

Consider case a) from the viewpoint of $F(p)$. We have the contraction and perturbation

$$H = \begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta & \\ & -\theta \end{pmatrix}$$

$$\Theta H = \begin{pmatrix} \theta h & \theta i \\ 0 & 0 \end{pmatrix} \qquad H\Theta = \begin{pmatrix} h\theta & -i\theta \\ 0 & 0 \end{pmatrix}$$

$$1-\Theta H = \begin{pmatrix} 1-\theta h & -\theta i \\ 0 & 1 \end{pmatrix} \qquad 1-H\Theta = \begin{pmatrix} 1-h\theta & +i\theta \\ & 1 \end{pmatrix}$$

Now $\begin{pmatrix} a & b \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b \\ & 1 \end{pmatrix}$ so

$$(1-\Theta H)^{-1} = \begin{pmatrix} \frac{1}{1-\theta h} & \frac{1}{1-\theta h}\theta i \\ 0 & 1 \end{pmatrix} \qquad (1-H\Theta)^{-1} = \begin{pmatrix} \frac{1}{1-h\theta} & -\frac{1}{1-h\theta}i\theta \\ & 1 \end{pmatrix}$$

$$H\frac{1}{1-\Theta H} = \begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{1-\theta h} & \frac{1}{1-\theta h}\theta i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} h\frac{1}{1-\theta h} & h\frac{1}{1-\theta h}\theta i + i \\ 0 & 0 \end{pmatrix}$$

$$\frac{1}{1-H\textcircled{\Theta}}H = \begin{pmatrix} \frac{1}{1-h\theta} & -\frac{1}{1-h\theta}i\theta \\ & 1 \end{pmatrix}\begin{pmatrix} h & i \\ & \end{pmatrix} = \begin{pmatrix} \frac{1}{1-h\theta}h & \frac{1}{1-h\theta}i \\ 0 & 0 \end{pmatrix}$$

so from $H\frac{1}{1-\textcircled{\Theta}H} = \frac{1}{1-H\textcircled{\Theta}}H = \overbrace{\qquad\wedge\qquad}$

we get exactly $\tilde{h}$ $\tilde{i}$ as above.

On the other hand

$$[\textcircled{\Theta}, H] = \begin{pmatrix} \theta h + h\theta & \theta i - i\theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [h,\theta] & [\theta,i] \\ 0 & 0 \end{pmatrix}$$

so $$H(1-[\textcircled{\Theta},H])^{-1} = \begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{1-[h,\theta]} & \frac{1}{1-[h,\theta]}[i,\theta] \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} h\frac{1}{1-[h,\theta]} & h\frac{1}{1-[h,\theta]}[i,\theta] + i \\ 0 & 0 \end{pmatrix}$$

which is not very inspiring, but also

$$(1-[H,\textcircled{\Theta}])^{-1}H = \begin{pmatrix} \frac{1}{1-[h,\theta]} & \frac{1}{1-[h,\theta]}[i,\theta] \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{1-[h,\theta]}h & \frac{1}{1-[h,\theta]}i \\ 0 & 0 \end{pmatrix}$$

which gives a different $\tilde{h}$, $\tilde{i}$.

Let's check this last claim.

$$[d-\theta, h] = 1 - [h,\theta] - ip$$

Recall we are assuming $\theta p = p\theta$ and $ph = 0$.

Thus $p[h,\theta] = 0$

Now [redacted]

$$\left[d-\theta, \frac{1}{1-[h,\theta]} h\right] =$$

$$\underbrace{\left[d-\theta, \frac{1}{1-[h,\theta]}\right] h}_{} + \underbrace{\frac{1}{1-[h,\theta]}\left(1-[h,\theta]-ip\right)}_{1-\left(\frac{1}{1-[h,\theta]}i\right)p}$$

$$= \frac{1}{1-[h,\theta]}\left[d-\theta,[h,\theta]\right]\frac{1}{1-[h,\theta]}h$$

Apply now $0 = [d-\theta,[d-\theta,h]] = [d-\theta, 1-[h,\theta]-ip]$
$= -[d-\theta,[h,\theta]] - [d-\theta,i]p$

Thus $[d-\theta,[h,\theta]] = [\theta,i]p$, and since p kills $\frac{1}{1-[h,\theta]}h$ we win.

Thus we have proved:

Claim: Given
$$\begin{tikzcd} E \arrow[loop right, "h"] \arrow[d, shift right, "p"'] \\ E' \arrow[u, shift right, "i"'] \end{tikzcd}$$
$[d,p] = [d,i] = 0$ $pi = 1$
$[d,h] = 1-ip$ $ph = 0$
θ on E, E', $(d-\theta)^2 = 0$, $[p,\theta] = 0$

Then we have $[d-\theta,p] = [d-\theta,\tilde{i}] = 0$, $p\tilde{i} = 1$, $[d-\theta,$ [redacted] $,\tilde{h}] = 1-\tilde{i}p$ where

$\tilde{h}$ [redacted] $=$ [redacted] $\frac{1}{1-\text{[redacted]}h\theta}h$

$\tilde{i} = \frac{1}{1-h\theta}i$

or

$\tilde{h} = \frac{1}{1-[h,\theta]}h$

$\tilde{i} = \frac{1}{1-[h,\theta]}i$

Claim: Given

$$\begin{tikzcd} E \arrow[loop above, "h"] \arrow[d, shift right, "p"'] \\ E' \arrow[u, shift right, "i"'] \end{tikzcd}$$

$[d,p] = [d,i] = 0$

$[d,h] = 1 - ip$, $pi = 1$

θ on E, E' $\quad [d,\theta] = \theta^2$

Assume $[i,\theta] = 0$ and $hi = 0$. Then we have

$$[d-\theta, \tilde{p}] = [d-\theta, i] = 0$$

$$[d-\theta, \tilde{h}] = 1 - i\tilde{p} \qquad p\tilde{i} = 1$$

where

$$\tilde{h} = h\frac{1}{1-\theta h} \qquad \tilde{h} = h\frac{1}{1-[h,\theta]}$$

$$\tilde{p} = p\frac{1}{1-\theta h} \quad \text{or} \quad \tilde{p} = p\frac{1}{1-[h,\theta]}$$

Same check for the second case: $[d-\theta, h] = 1 - [h,\theta] - ip$

$$\left[d-\theta, \frac{1}{1-[h,\theta]}\right] = \frac{-1}{1-[h,\theta]} \underbrace{[d-\theta, 1-[h,\theta]]}_{[d-\theta,\, ip]} \frac{1}{1-[h,\theta]}$$

$$= \frac{1}{1-[h,\theta]} i[\theta,p]\frac{1}{1-[h,\theta]} = i[\theta,p]\frac{1}{1-[h,\theta]}$$

then

$$\left[d-\theta, h\frac{1}{1-[h,\theta]}\right] = (1-[h,\theta]-ip)\frac{1}{1-[h,\theta]} - h\cancel{\overbrace{\left[d-\theta, \frac{1}{1-[h,\theta]}\right]}^{''}}$$

$$= 1 - i\left(p\frac{1}{1-[h,\theta]}\right) \quad \text{as desired.}$$

The first proof uses

$$\left[\begin{pmatrix} d & \\ i & -d \end{pmatrix}, \overbrace{\begin{pmatrix} 0 & p \\ 0 & -h \end{pmatrix}}^{H}\right] = \begin{pmatrix} pi & dp - pd \\ -hi & ip + dh + hd \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Next I want to copy some formulas related to composition.

$$\begin{tikzcd} X \arrow[r, shift left, "f'"] & Y \arrow[l, shift left, "g'"] \arrow[r, shift left, "f''"] & Z \arrow[l, shift left, "g''"] \end{tikzcd}$$

$h'_X \quad u' \quad h'_Y, h''_Y \quad u'' \quad h''_Z$

$[d,f'] = [d,g'] = 0$
$[d_X, h'_X] = 1 - g'f'$
$[d_Y, h'_Y] = 1 - f'g'$
$[d,u'] = [f',h']$
similarly with ''.

Set
$$\boxed{\begin{aligned} h_X &= h'_X + g' h''_Y f' \\ h_Z &= h''_Z + f'' h'_Y g'' \\ u &= f''u' + u''f' - f''h'_Y h''_Y f' \end{aligned}}$$

$$\begin{aligned} [d_X, h_X] &= [d_X, h'_X] + g'[d_Y, h''_Y]f' \\ &= 1 - g'f' + g'(1 - g''f'')f' \\ &= 1 - (g'g'')(f''f') = 1 - gf \end{aligned}$$

$$\begin{aligned} [d,u] = {} & f''(f'h'_X - \cancel{h'_Y f'}) + (\cancel{f''h''_Y} - h''_Z f'')f' \\ & -f''(\underbrace{[d_Y, h'_Y]}_{\cancel{1} - f'g'} h''_Y - h'_Y \underbrace{[d_Y, h''_Y]}_{\cancel{1} - g''f''})f' \end{aligned}$$

$$\begin{aligned} &= f''f'(h'_X + g'h''_Y f') - (h''_Z + f''h'_Y g'')f''f' \\ &= f\,h_X - h_Z f \end{aligned}$$

Question: How do fibres behave for a composition $f = f''f'$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f'"'] & S(f') \arrow[r, hook] \arrow[d] & * \arrow[d, "F(f')"] \\ & Y \arrow[r, hook] \arrow[dr, "f''"'] & S(f'') \arrow[d, "F(f'')"] \\ & & Z \end{tikzcd}$$

$X_n + Y_{n+1} + Y_n + Z_{n+1} + Z_n$

suggests an exact sequence

$$0 \longrightarrow F(f') \longrightarrow \Phi \longrightarrow F(f'') \longrightarrow 0$$

where $\Phi_n = X_n \oplus Y_{n+1} \oplus Y_n \oplus Z_{n+1}$ and Φ should deform to $F(f''f')$. Formulas:

$$d_\Phi = \begin{pmatrix} d & & & \\ f' & -d & -1 & \\ & & d & \\ & & f'' & -d \end{pmatrix}$$

$$\begin{tikzcd} \Phi \arrow[loop left, "h"] \arrow[r, shift left, "\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f'' & 0 & 1 \end{pmatrix}"] & F(f) \arrow[l, shift left, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ f' & 0 \\ 0 & 1 \end{pmatrix}"] \end{tikzcd} \qquad h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is a special deformation retraction.

Next given $\begin{pmatrix} h' & g' \\ u' & -h' \end{pmatrix}$ for $F(f')$ and similarly for $F(f'')$ we want a contraction for Φ which compresses to the contraction for $F(f)$. Look for a suitable operator on Φ of the form such that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & f'' & 0 & 1 \end{pmatrix} \begin{pmatrix} h' & g' & a & b \\ u' & -h' & c & d \\ & & h'' & g'' \\ & & u'' & -h'' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ f' & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} h'+g'h''f' & g'g'' \\ f''u'+u''f' & -h''-f''h'g' \\ -f''h'h''f' & \end{pmatrix}$$

The answer is:
and it seems that
this is indeed a
contraction of Φ.

$$\begin{pmatrix} h' & g' & g'h'' & g'g'' \\ u' & -h' & -h'h'' & -h'g'' \\ & & h'' & g'' \\ & & u'' & -h'' \end{pmatrix}$$

September 26, 1993

I found yesterday that the formulas of HPT, in which the propagators $\frac{1}{1-h\theta}, \frac{1}{1-\theta h}$ appear, can sometimes be replaced by formulas involving the single propagator $\frac{1}{1-[\theta,h]}$. [redacted] Moreover the proofs simplify in this way. For[redacted] example consider the contractible case $[d,h]=1$. Then

$$\left[d-\theta, h\frac{1}{1-\theta h}\right] = [d-\theta,h]\frac{1}{1-\theta h} + h\frac{1}{1-\theta h}[d-\theta,-\theta h]\frac{1}{1-\theta h}$$

$$= \Big\{1-\theta h-h\theta + \underbrace{h\frac{1}{1-\theta h}\left(\cancel{-\theta^2 h}+\theta+\cancel{\theta^2 h}-\theta h\theta\right)}_{h\theta}\Big\}\frac{1}{1-\theta h} = 1$$

Even with [redacted] the simplified notation of the proof in part 4 we have:

$$(1-h\theta)[d-\theta,\tilde{h}](1-\theta h) = (1-h\theta)(d-\theta)h + h(d-\theta)(1-\theta h)$$

$$= dh-\theta h-h\theta dh+h\theta^2 h$$

$$\quad hd-h\theta-hd\theta h+h\theta^2 h$$

$$= 1-\theta h-h\theta-h\theta^2 h+2h\theta^2 h = (1-h\theta)(1-\theta h)$$

Contrast this with the following

$$[d-\theta,h] = 1-[\theta,h] \Rightarrow [d-\theta,[\theta,h]] = 0$$

so
$$\left[d-\theta, h\frac{1}{1-[\theta,h]}\right] = (1-[\theta,h])\frac{1}{1-[\theta,h]} - h\cancel{\left[d-\theta,\frac{1}{1-[\theta,h]}\right]} = 1$$

Here's the 'new' version of the sdr situation: suppose given $h^2=0$, $hdh=h$, $[d,h]=1-e$. Note that h commutes with $[\theta,h]$ as $h^2=0$. Thus $hG=Gh$ where $G=(1-[h,\theta])^{-1}$. [redacted] Putting $\tilde{h}=hG=Gh$ we have $\tilde{h}^2=0$. Also

$$\begin{aligned} h(d-\theta)h &= hdh - h\theta h \\ &= h - h\theta h \\ &= \left(1-[\theta, h]\right)h \end{aligned}$$

so $$\tilde h(d-\theta)\tilde h = Gh(d-\theta)hG = G\left(1-[\theta,h]\right)hG$$
$$= hG = \tilde h.$$

Next

$$\left[d-\theta,\ h\frac{1}{1-[\theta,h]}\right] = \left(1-[\theta,h]-e\right)\frac{1}{1-[\theta,h]}$$
$$+\,h\frac{1}{1-[\theta,h]}\underbrace{[d-\theta,\ 1-[\theta,h]]}_{[\cancel{d}-\theta,\, e]}\frac{1}{1-[\theta,h]}$$

(note $h(-\theta e + e\theta)$
$= -h\theta e$
$= -[\theta,h]e$
as $he=0$)

$$= 1-\left(1+\frac{1}{1-[\theta,h]}[\theta,h]\right)e\,\frac{1}{1-[\theta,h]}$$

$$= 1-\tilde e \qquad \text{where} \quad \tilde e = \frac{1}{1-[\theta,h]}\,e\,\frac{1}{1-[\theta,h]}$$

OK, it isn't that much simpler a calculation, but the moral is that in the sdr situation I can use the formulas

$$\tilde h = hG = Gh \quad , \qquad \tilde e = GeG$$

where $G = \left(1-[h,\theta]\right)^{-1}$.

Alternative version:

$Gh(\theta e - e\theta) = G[\theta,h]e$

$$[d-\theta, h] = [d-\theta,\ \tilde h\,(1-[\theta,h])]$$

$$1-[\theta,h]-e = [d-\theta,\tilde h]\left(1-[\theta,h]\right) - \underbrace{\tilde h}_{Gh}[\cancel{d}-\theta,\ 1-\overset{e}{\cancel{[\theta,h]}}]$$

$$\begin{aligned}[d-\theta,\tilde h]\left(1-[\theta,h]\right) &= 1-[\theta,h]-e + G[\theta,h]e \\ &= 1-[\theta,h]-\underbrace{\left(1+G[\theta,h]\right)e}_{Ge}\end{aligned}$$

so $$[d-\tilde\theta, \tilde h] = 1-GeG$$

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Next consider GNS framework for dilating an operator h to an operator ε having square zero. Put $A = \mathbb{C} \oplus \mathbb{C}\varepsilon$, $B = \mathbb{C}[h]$, $\rho : A \to B$ given by $\rho 1 = 1$, $\rho\varepsilon = h$. Given a B-module N, there are two canonical dilations to an A module, namely $A \otimes N$ and $\mathrm{Hom}(A, N)$, which can both be identified with N^2. There are also canonical maps:

$$\begin{tikzcd}
N \arrow[r, "\text{universal } i"] \arrow[d, equal] & A \otimes N \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A,N) \arrow[r, "\text{universal } \iota^*"] \arrow[d, equal] & N \arrow[d, equal] \\
N \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & N^2 \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & -h^2 \end{pmatrix}"] & N^2 \arrow[r, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & N
\end{tikzcd}$$

$$\varepsilon = \begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix} \qquad\qquad \varepsilon = \begin{pmatrix} h & 1 \\ -h^2 & -h \end{pmatrix}$$

Another variation: suppose we use $\Theta = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}$ on the big complex $X \oplus \Sigma X$.

Actually consider any dilation $H = \begin{pmatrix} h & u \\ v & w \end{pmatrix}$ of h on X to a complex $X \oplus Y$. I suppose H is a special contraction: $[d, H] = 1$, $d = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$, and $H^2 = 0$. Consider the perturbation $\Theta = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}$. Then the perturbed contraction one calculates:

$$(1 - \Theta H)^{-1} = \left(1 - \begin{pmatrix} \theta h & \theta u \\ 0 & 0 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 1 - \theta h & -\theta u \\ 0 & 1 \end{pmatrix}^{-1}$$

$$H(1 - \Theta H)^{-1} = \begin{pmatrix} h & u \\ v & w \end{pmatrix} \begin{pmatrix} \frac{1}{1-\theta h} & \frac{1}{1-\theta h}\theta u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} h\frac{1}{1-\theta h} & * \\ * & * \end{pmatrix}$$

which yields $h\frac{1}{1-\theta h}$.

This puts a whole new slant on the situation, because the idea of using $\theta=0$ on the complement is rather attractive. It fits with Connes' dilation examples.

Let $f: X \longrightarrow Y$ be a map compatible with θ which is a h eq when θ is ignored. Consider the Serre factorization

$$\begin{tikzcd} X \arrow[r, hook, shift right, "\begin{pmatrix}1\\0\\f\end{pmatrix}"'] & S(f) \arrow[l, two heads, shift right, "(1\ 0\ 0)"'] \arrow[r, shift right, "p=(0\ 0\ 1)"'] & Y \arrow[l, dashed, shift right, "\iota=\begin{pmatrix}g\\-h\\1\end{pmatrix}"'] \end{tikzcd}$$

$S(f)_n = X_n \oplus Y_{n+1} \oplus Y_n$

$d = \begin{pmatrix} d & & \\ f & -d & -1 \\ & & d \end{pmatrix}$

$$\left[\begin{pmatrix} d & & \\ f & -d & -1 \\ & & d \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -f & 0 & 1 \end{pmatrix} = I - \begin{pmatrix} 1 \\ 0 \\ f \end{pmatrix}(1\ 0\ 0)$$

Let be given a contraction for $F(f)$: $\left[\begin{pmatrix} d & 0 \\ f & -d \end{pmatrix}, \begin{pmatrix} h & g \\ u & -h \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then

$$\begin{pmatrix} d & & \\ f & -d & -1 \\ & & d \end{pmatrix}\begin{pmatrix} g \\ -h \\ 1 \end{pmatrix} = \begin{pmatrix} dg \\ fg+dh-1 \\ d \end{pmatrix} = \begin{pmatrix} gd \\ -hd \\ d \end{pmatrix} = \begin{pmatrix} g \\ -h \\ 1 \end{pmatrix} d$$

so $i = \begin{pmatrix} g \\ -h \\ 1 \end{pmatrix}$ is a section of $p = (0\ 0\ 1)$. Also

$$\left[\begin{pmatrix} d & & \\ f & -d & -1 \\ & & d \end{pmatrix}, \begin{pmatrix} h & g & 0 \\ u & -h & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} dh\,{}^{+hd}_{+gf} & dg-gd & -g \\ fh-du \\ +ud-hf & fg+dh \\ +hd & h \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & -g \\ 0 & 1 & h \\ & & 1 \end{pmatrix}$$

$$= I - \begin{pmatrix} -g \\ h \\ 1 \end{pmatrix}(0\ \ 0\ \ 1)$$

thus $h = \begin{pmatrix} h & g & 0 \\ u & -h & 0 \\ 0 & 0 & 0 \end{pmatrix}$ satisfies $[d,h] = 1 - ip$.

Notice that

$$ph = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} h & g & 0 \\ u & -h & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

But $$hi = \begin{pmatrix} h & g & 0 \\ u & -h & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g \\ -h \\ 1 \end{pmatrix} = \begin{pmatrix} hg - gh \\ ug + h^2 \\ 0 \end{pmatrix}$$

$$h^2 = \begin{pmatrix} h^2 + ug & hg - gh & 0 \\ uh - hu & ug + h^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

need not be zero. In fact $h^2 = 0$ iff the contraction for $F(f)$ is special, in which case $hi = 0$.

We learn then that the Serre factorization replaces f by $p : S(f) \to Y$ $\begin{pmatrix} h & g \\ u & -h \end{pmatrix}$ by i, h such that $pi = 1$, $[d, h] = 1 - ip$, and $ph = 0$.

In other words we go from

$$\left[\begin{pmatrix} d & \\ f & -d \end{pmatrix}, \begin{pmatrix} h & g \\ u & -h \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

to $$\left[\begin{pmatrix} d & \\ i & -d \end{pmatrix}, \begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

September 27, 1993

First discuss e, h version of the basic results without assuming h special.

Suppose only that $[d,h] = 1-e$, $[d,\theta]=\theta^2$.

Then
$$[d-\theta, h\frac{1}{1-\theta h}] = (1-e-\check{\theta}h-\check{h}\theta)\frac{1}{1-\theta h}$$

$$+h\frac{1}{1-\theta h}\underbrace{[d-\theta,-\theta h]}_{\substack{(-\cancel{\theta^2 h}+\theta-\theta e \\ +\cancel{\theta^2 h}-\theta h\theta)}}\frac{1}{1-\theta h}$$

$$= \left(1-\theta h-h\theta+h\frac{1}{\cancel{1-\theta h}}(\overset{\theta}{\cancel{\theta-\theta h\theta}})\right)\frac{1}{1-\theta h}$$

$$+\left(-e-h\frac{1}{1-\theta h}\theta e\right)\frac{1}{1-\theta h}$$

$$= 1-\frac{1}{1-h\theta}e\frac{1}{1-\theta h}$$

Thus we have the identity

$$\boxed{[d-\theta, h\frac{1}{1-\theta h}] = 1-\frac{1}{1-h\theta}e\frac{1}{1-\theta h}}$$

Side point: If $1-\theta h$ is invertible, then $1-h\theta$ is also invertible and

$$\frac{1}{1-h\theta} = 1+h\frac{1}{1-\theta h}\theta$$

since $$(1-h\theta)(1+h\frac{1}{1-\theta h}\theta) = 1-h\theta+\underbrace{h\frac{1}{1-\theta h}\theta-h\theta h\frac{1}{1-\theta h}\theta}_{h(1-\theta h)\frac{1}{1-\theta h}\theta = h\theta}$$

and similarly the product the other way round.

So far $\overset{we}{\wedge}$ haven't assumed $e^2=e$.

Now take the case corresponding to $[p,\theta]=0$, $ph=0$. This is: $\boxed{eh = 0 \quad \text{and} \quad e\theta(1-e)=0}$

Note that $0=[d,eh]=e[d,h]=e(1-e)$, where $[d,e]=[d,1-e]=[d,[d,h]]=0$ has been used. Thus $eh=0 \Longrightarrow e$ idempotent. The condition $e\theta(1-e)=0$ means θ carries $\mathrm{Ker}(e)$ into itself, so θ descends to $\mathrm{Im}(e)$. In this case we have

$$e\frac{1}{1-\theta h}=e$$

since $e(1-\theta h)=e-e\theta h=e-e\theta eh=e$. Thus we have $[d-\theta,\tilde{h}]=1-\tilde{e}$ where $\tilde{h}=\frac{1}{1-h\theta}h$, $\tilde{e}=\frac{1}{1-h\theta}e$ and $\tilde{e}\tilde{h}=\frac{1}{1-h\theta}e\frac{1}{1-h\theta}h=\frac{1}{1-h\theta}eh=0$, so that $\tilde{e}$ is also idempotent.

A similar argument should hold in the case corresponding to $[\iota,\theta]=0$, $h\iota=0$, namely where $he=0$ and $e\theta e=\theta e$, i.e. θ carries $\mathrm{Im}(e)$ into itself.

Question: Consider the $(p,\iota,h=h_X,h_Y=0,u=0)$ situation: $X \underset{i}{\overset{p}{\rightleftarrows}} Y$, $[d,p]=[d,i]=0$, $[d,h]=1-ip$, $ph=0$
$pi=1$

Can we dilate to a special situation, i.e. where in addition $h^2=0$, $h\iota=0$. The obvious thing to do, namely dilating h_F leads to $\tilde{X}_n=X_n\oplus X_{n-1}$
$\tilde{Y}_n=Y_n\oplus Y_{n-1}$, $d_{\tilde{X}}=\begin{pmatrix} d & \\ & -d\end{pmatrix}$, $d_{\tilde{Y}}=\begin{pmatrix} d & \\ & -d\end{pmatrix}$, $\tilde{p}=\begin{pmatrix} p & \\ & -p\end{pmatrix}$
$\tilde{g}=\begin{pmatrix} \iota & -hi \\ & -i\end{pmatrix}$, $h_{\tilde{X}}=\begin{pmatrix} h & -h^2 \\ 1 & -h\end{pmatrix}$, $h_{\tilde{Y}}=\begin{pmatrix} 0 & 0 \\ -1 & 0\end{pmatrix}$, $\tilde{u}=\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$

The reason this $h_{\tilde{Y}}$ occurs is because

$$\tilde{p}\tilde{h} = \begin{pmatrix} p & \\ & -p \end{pmatrix}\begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -p & 0 \end{pmatrix}$$

is nonzero.

The best one can do it seems is to first replace h by $h - hip$ so as to make $hi = 0$, whence h is a contraction on $\mathrm{Ker}(p)$ extended by zero on $\mathrm{Im}(i)$. Then you can dilate the h on $\mathrm{Ker}(p)$, which means I guess adding a suspended copy of $\mathrm{Ker}(p)$ to X and using the dilation $\begin{pmatrix} h & -h^2 \\ 1-ip & -h \end{pmatrix}$.

September 28, 1993

Consider a complex

$$\begin{tikzcd} {} \arrow[r, "\partial"] & M_1 \arrow[r, "\partial"] & M_0 \arrow[l, bend right, "h"'] \arrow[r, "\partial"] & M_{-1} \arrow[l, bend right, "h"'] \arrow[r] & \cdots \end{tikzcd}$$

of A modules which is acyclic, let h be a contracting homotopy not respected by A. Define a DG module structure on M over $A * \mathbb{C}[d]/(d^2) = \Omega A \otimes (\mathbb{C} \oplus \mathbb{C}d)$ as follows. We have $\mathrm{End}(M)$ a DG algebra, a homomorphism $A \longrightarrow \mathrm{End}^0(M)$ given by left multiplication such that $[\partial, a] = 0$ for all $a \in A$, and a degree $+1$ operator $d = h\partial h = h - h^2\partial$. We get a DG alg. homom.

$$A * \mathbb{C}[d]/(d^2) \longrightarrow \mathrm{End}(M)$$

$$\partial(a) = 0$$

$$\partial(d) = 1$$

[redacted] whence M becomes a DG module over the former. Now $A * \mathbb{C}[d]/(d^2)$ in degree $n+1$ is spanned by elements

$$a_0 \cdot d \cdot a_1 \cdot d \cdots a_n \cdot d \cdot a_{n+1} = a_0 d [a_1, d] \cdots [a_n, d] a_{n+1}$$

$$= (-1)^n a_0 d [d, a_1] \cdots [d, a_n] a_{n+1} .$$

Suppose that $M_{-2} = M_{-3} = \cdots = 0$ so the $M_{\geq 0}$ is a resolution of M_{-1}, and $\partial(M_{-1}) = 0$. Then for $m \in M_{-1}$ we have $\partial(a_{n+1} m) = 0$ so $d(a_{n+1} m) = h(a_{n+1} m)$ and [redacted]

$$[d, a_n] a_{n+1} m = h a_n a_{n+1} m - a_n h a_{n+1} m$$

$$= [h, a_n] a_{n+1} m .$$

This is killed by ∂ so we can continue and find

$$\left(a_0 d[d, a_1] \cdots\cdots [d, a_n] a_{n+1}\right) m = a_0 h[h, a_1] \cdots [h, a_n](a_{n+1} m)$$

We have

$$\begin{tikzcd}
{} \arrow[r] & M_n \arrow[r] \arrow[d] & M_{n-1} \arrow[r] \arrow[l, bend right, "h"'] & {} \arrow[l, bend right, "{[h,a_1]}"'] \cdots & M_0 \arrow[r] & M_{-1} \arrow[r] \arrow[l, bend right, "{[h,a_n]}"'] & 0 \\
& M_n/\partial M_{n+1} \arrow[ur, bend right] & & & & &
\end{tikzcd}$$

so (up to [redacted] a sign?) the cocycle

$$h[h, a_1] \cdots [h, a_n] \in \mathrm{Hom}\left(M_{-1}, M_n/\partial M_{n+1}\right)$$

represented the element $\in \mathrm{Ext}^n_A\left(M_{-1}, M_n/\partial M_{n+1}\right)$ given by the [redacted] resolution $M_{\geq 0}$ of M_{-1}.

September 30, 1993

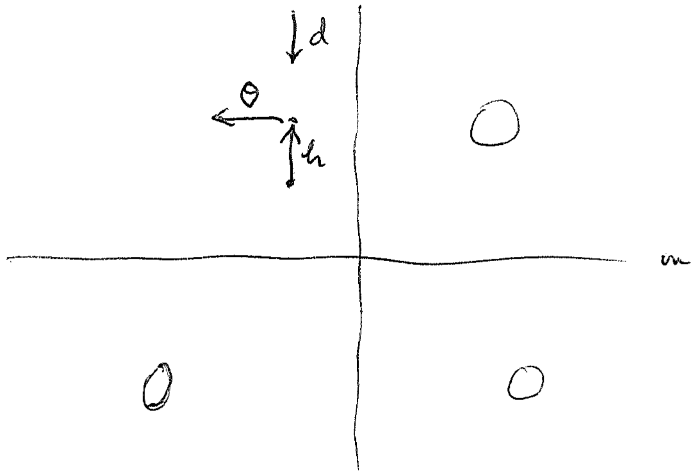
Consider an algebra A, [redacted] C_A the category of DG A-modules (bounded below), $Ho(C_A)$ the corresponding homotopy category.

First construction. Let P be a right DG A-modules, let $X \in C_A$ be acyclic. We want a contraction on $P \otimes_A X$ assuming P is a (bdd below) complex of projective right A-modules. Filter P by the subcomplexes $P_{\leq m}$; we get a corresponding filtration $P_{\leq m} \otimes_A X$ of $P \otimes_A X$ with layers $P_m[m] \otimes_A X$. Moreover there's an obvious splitting of the filtration. This is because $P \otimes_A X$ has a double complex structure and we are looking at the increasing column filtration. So we have a perturbation situation where the differential is $d - \theta$, d vertical differential $(-1)^m \otimes d_X$ on $P_m[m] \otimes_A X$, and $-\theta$ is the horizontal differential. Finally we need a contraction wrt d.

As X is acyclic there is a contraction h on X, but it need not respect the A module structure. Since P_m is projective, the surjection $P_m \otimes A \xrightarrow{\mu} P_m$ given by multiplication has a A-module section [redacted] s. Then $P_m \otimes_A X$ is a direct summand of $(P_m \otimes A) \otimes_A X = P_m \otimes X$ on which one has the contraction $1 \otimes h$, hence one has an induced contraction on $P_m \otimes_A X$.

Here's a variant of this construction. [redacted]
suppose P is a complex of left projective A-modules and we wish to contract~~s~~ $Hom_A(P, X)$. One has a double complex $Hom_A(P_m, X_n)$ of which this mapping complex is a suitable completed total complex. Again P_m a direct summand of the A module $A \otimes P_m$ implies

$\mathrm{Hom}_A(P_m, X)$ is a direct summand of $\mathrm{Hom}_A(A \otimes P_m, X) = \mathrm{Hom}(P_m, X)$, on which we have a contraction. Picture assuming P, X are chain complexes



so it's clear the geometric series $\frac{1}{1-h\theta}$ converges.

Second construction. Given a complex of A-modules X. Let $T = T_A(ADA)$ be the standard resolution where $|D| = 1$ and $\partial(D) = 1$. Then T is the cone on the map $\bar{T} \longrightarrow A$, $\bar{T} = \Omega(T/A)$, which makes $\bar{T}$ a free bimodule resolution of A.

Given X, then $T \otimes_A X$ is the cone on $\bar{T} \otimes_A X \to X$. Now $T \otimes_A X$ is acyclic, left mult by D is a contraction not respecting A. So $\bar{T} \otimes_A X \to X$ is a quis and $\bar{T} \otimes_A X$ is a free A-module resolution of A.

We want to know that if X is a complex of projective modules then $\bar{T} \otimes_A X \longrightarrow X$ is a heq respecting A, equivalently, $T \otimes_A X \underset{A}{\sim} 0$. This follows from $\mathrm{Hom}_A(P, X)$ contractible, when P is projective & X acyclic, but we can give a direct construction as follows. We have double complex $T_m \otimes_A X_n$, so it suffices to give a A-module contraction on $T \otimes_A X_n$ for each n. But we have X_n is a direct summand of the A-module $A \otimes X_n$, [redacted] hense $T\otimes_A X_n$ is a A-module direct summand of $T\otimes_A (A\otimes X_n) = T\otimes \hat{X}_n$. Finally right multiplication by D on T (with sign) gives a contraction of T as A-module.

Next I would like to handle the case of Λ the DG algebra $\mathbb{C}\oplus\mathbb{C}\varepsilon$, $|\varepsilon| =$ [redacted] 1, $\partial(\varepsilon)=0$. The above construction should work I think if [redacted] we are careful about the meaning of projective. Also should work more generally for a connected DG (chain) algebra. Projective should be the same as free, which means there should be a skeletal filtration where the ~~quotients~~ are of form $A\otimes V$, V vector space with differential zero. On the other hand there are angles to be explored involving the bar construction and the adjoint functors.

First construction: Let $\mathcal{B}$ be the bar construction of Λ. Then [redacted] X acyclic Λ-module $\Longrightarrow$ $\mathcal{B}\otimes_\tau X$ is heq to 0 as $\mathcal{B}$-comodules. This holds because the differential in $\mathcal{B}\otimes_\tau X$ is a perturbation of the differential in $\mathcal{B}\otimes X$, and everything respects the left $\mathcal{B}$-comodule ~~structure~~.

[redacted] Next I want to know that the canonical map $\Lambda\otimes_{\pm\tau}\mathcal{B}\otimes_\tau X \longrightarrow X$ is a heq resp. Λ when X is free. There's no obvious way to do this except by a skeletal filtration on X. Actually

$$\mathrm{Cone}\left(\Lambda\otimes_{\pm\tau}\mathcal{B}\otimes_\tau\Lambda \longrightarrow A\right) = T_\Lambda(\Lambda D\Lambda)/(D^2)$$

is the standard normalized resolution of Λ:

$$\Lambda \leftarrow \Lambda\underbrace{D}_{\deg 1}\Lambda \leftarrow \Lambda\underbrace{D\bar{\Lambda}D}_{\deg 3}\Lambda \longleftarrow \Lambda\underbrace{D\bar{\Lambda}D\bar{\Lambda}D}_{\deg 5}\Lambda \leftarrow$$

If we take X of the form $\Lambda \otimes_\tau Q$ for some S module Q, then $\mathrm{Cone}(\Lambda \otimes_{\pm\tau} B \otimes X \to X) = R \otimes_\tau Q$ where $R = T_\Lambda(\Lambda D \Lambda)/(D^2)$. Again perturbing the diffl of $R \otimes Q$ and using right mult. by D on R, we find this is contractible respecting Λ module structure.

Consider now the construction of the derived category $D(\mathcal{C}_\Lambda)$ from $Ho(\mathcal{C}_\Lambda)$.
Let's try to proceed by defining $Ho(\mathcal{C}_\Lambda^f)$ as the full subcategory consisting of $\Lambda \otimes_\tau Q$ with $Q \in \mathcal{C}_S$. We want the inclusion $\iota : Ho(\mathcal{C}_\Lambda^f) \subset Ho(\mathcal{C}_\Lambda)$ to have a right adjoint:

$$[F, X] = [F, \iota^* X]$$

More precisely, for each X in $Ho(\mathcal{C}_\Lambda)$ there is $\iota^* X \in Ho(\mathcal{C}_\Lambda^f)$ and a map $\iota^* X = \iota\iota^* X \to X$ such that

$$[F, \iota^* X] = [\iota F, \iota\iota^* X] \xrightarrow{\sim} [\iota F, X]$$

Try $\iota^* X = \Lambda \otimes_{\pm\tau} B \otimes_\tau X$ and the canonical map to X. If $F = \Lambda \otimes_{\pm\tau} Q$, then $[F, X] = [\Lambda \otimes_{\pm\tau} F, X] = [F, B \otimes_\tau X]$ for any $X \in \mathcal{C}_\Lambda$. Thus it suffices to know that the canonical arrow $\Lambda \otimes_{\pm\tau} B_\tau \otimes X \longrightarrow X$ induces an isomorphism in $Ho(\mathcal{C}_S)$:

$$B \otimes_\tau \Lambda \otimes_{\pm\tau} B \otimes_\tau X \xrightarrow{\sim} B \otimes_\tau X$$

There is an adjunction type arrow going the other way. I feel there is a canonical homotopy equivalence here but this needs checking.

Put $\mathcal{H}_\Lambda, \mathcal{H}_B$ for the homotopy categories of DG Λ-modules and B-comodules resp. Then $F = \Lambda \otimes_\tau -$, $G = B \otimes_\tau -$ are adjoint functors

$$\begin{tikzcd} \mathcal{H}_\Lambda \arrow[r, shift right, "G"'] & \mathcal{H}_B \arrow[l, shift right, "F"'] \end{tikzcd} \qquad \begin{array}{l} FG \xrightarrow{\alpha} id \\ id \xrightarrow{\beta} GF \end{array}$$

α, β are the adjunction maps, and they have the basic property that the compositions

$$F(Q) \xrightarrow{F(\beta_Q)} FGF(Q) \xrightarrow{\alpha_{F(Q)}} F(Q)$$

$$G(M) \xrightarrow{\beta_{G(M)}} GFG(M) \xrightarrow{G(\alpha_M)} G(M)$$

are the identity. Now it seems that in the present case these maps are isomorphisms in the homotopy categories, i.e. we have $FGF \simeq F$ $GFG \simeq G$ canonically.

Let us now define a map in $\mathcal{H}_\Lambda, \mathcal{H}_B$ to be a quis if G, F resp. carries it into an isomorphism; this is equivalent to the cone being killed by G, F resp. Introduce the free, cofree full subcategories $\mathcal{H}'_\Lambda, \mathcal{H}'_B$ as the essential images of F, G resp. Then we have "free" resolutions $FG(M) \xrightarrow{\alpha} M$ since $GFG(M) \xrightarrow{\sim} G(M)$. Also if $X \to Y$ is quis, i.e. $GX \xrightarrow{\sim} GY$, then $[F(Q), X] = [Q, GX] \xrightarrow{\sim} [Q, GY] = [F(Q), Y]$

October 3, 1993

Let Λ be a augmented DG algebra (say supported in degrees ≥ 0), let $\mathcal{B}$ be its bar construction:

$\mathcal{B}$ = tensor coalgebra $T(\Sigma\bar\Lambda)$ equipped with differential $d = d' + d''$, where d' arises from the differential on Λ and d'' arises from the product in Λ.

Notation + recall: tensor coalgebra of V is $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$, $p_n : T(V) \longrightarrow V^{\otimes n}$ is the projection, $\Delta : T(V) \longrightarrow T(V) \otimes T(V)$ given by $(p_i \otimes p_j)\Delta = p_{i+j}$, universal property (assuming C "connected"):

$$\begin{matrix} \operatorname{Hom}_{\text{coalg}}(C, \text{[redacted]}\, T(V)) & = & \operatorname{Hom}_{\substack{\text{vector}\\ \text{spaces}}}(C, V) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \phi & & u \end{matrix}$$

$\phi \longmapsto p_1\phi$, given u the corresp ϕ [redacted] given by $p_n\phi = u^{\otimes n}\Delta^{(n)} : C \xrightarrow{\Delta^{(n)}} C^{\otimes n} \xrightarrow{u^{\otimes n}} V^{\otimes n}$. This uses $p_n = p_1^{\otimes n}\Delta^{(n)}$.

variant universal property: a linear map $u : T(V) \longrightarrow V$ extends uniquely to a coderivation $D : T(V) \longrightarrow T(V)$ given by

$$p_n D = \sum_{i=1}^{n} \left(p_1^{\otimes i-1} \otimes u \otimes p_1^{\otimes n-i}\right)\Delta^{(n)}$$

Return to $T(\Sigma\bar\Lambda)$. We have

$$\begin{tikzcd} \bar\Lambda \arrow[r, "d_{\bar\Lambda}"] \arrow[d, "\cong"', "\sigma"] & \bar\Lambda \arrow[d, "\cong"', "\sigma"] \\ \Sigma\bar\Lambda \arrow[r, "-d_{\Sigma\bar\Lambda}"'] & \Sigma\bar\Lambda \end{tikzcd}$$

d' is by defn. the unique coderivation of degree -1 [redacted] such that $p_1 d' = d_{\Sigma\bar\Lambda} p_1$, i.e. $\boxed{p_1 d' = -\sigma d_{\bar\Lambda} \sigma^{-1} p_1}$

[redacted] Let $\tilde{\mu}$ be defined so that

$$\begin{tikzcd}
\bar{\Lambda} \otimes \bar{\Lambda} \arrow[r, "\mu"] \arrow[d, "\cong"', "\sigma\otimes\sigma"] & \bar{\Lambda} \arrow[d, "\cong"', "\sigma"] \\
\Sigma\bar{\Lambda} \otimes \Sigma\bar{\Lambda} \arrow[r, "\tilde{\mu}"] & \Sigma\bar{\Lambda}
\end{tikzcd}$$

$\mu =$ product in $\bar{\Lambda}$

commutes. Thus

$$\tilde{\mu} = \sigma\,\mu(\sigma\otimes\sigma)^{-1} = \overset{\text{surprise!}}{-}\sigma\,\mu(\sigma^{-1}\otimes\sigma^{-1})$$

d'' is defined to be the unique coderivation of $T(\Sigma\bar{\Lambda})$ of degree -1 such that

$$p_1 d'' = \tilde{\mu}\,p_2\,, \quad \text{i.e.} \quad \boxed{p_1 d'' = -\sigma\,\mu(\sigma^{-1}\otimes\sigma^{-1})p_2}$$

Let $\tau = \sigma^{-1}p_1 : \mathcal{B} = T(\Sigma\bar{\Lambda}) \xrightarrow{p_1} \Sigma\bar{\Lambda} \xrightarrow{\sigma^{-1}} \bar{\Lambda} < \Lambda$

Then
$$\begin{aligned} \tau d' &= -d_{\bar{\Lambda}}\tau \\ \tau d'' &= -\mu(\sigma^{-1}\otimes\sigma^{-1})(p_1\otimes p_1)\Delta = -\mu(\tau\otimes\tau)\Delta \end{aligned}$$

whence
$$\boxed{\tau d_{\mathcal{B}} + d_{\Lambda}\tau + \mu(\tau\otimes\tau)\Delta = 0}$$

so τ is a twisting cochain.

Example: $\Lambda = A$, where A [redacted] is an augmented algebra. Put $(a_1,\dots,a_n)$ for $\sigma a_1\otimes\cdots\otimes\sigma a_n \in (\Sigma\bar{A})^{\otimes n}$.

Then $p_1 d_{\mathcal{B}}(\sigma a_1\otimes\sigma a_2) = \tilde{\mu}(\sigma\otimes\sigma)(a_1\otimes a_2) = \sigma\,\mu(a_1\otimes a_2) = \sigma(a_1 a_2)$

i.e. $p_1 d_{\mathcal{B}} : (a_1, a_2) \longmapsto (a_1 a_2)$. $d_{\mathcal{B}} = b'$ and

$$\Delta(a_1,\dots,a_n) = \sum_{i=0}^{n} (a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_n).$$

So far we have reviewed the construction of the bar construction $\mathcal{B}$ for Λ. Next the adjoint functors between Λ modules, $\mathcal{B}$ comodules.

~~Recall~~ This works more generally. Suppose now that $\mathcal{B}$ is a DG coalgebra related to Λ by a twisting cochain τ.

Recall $R = \mathrm{Hom}(\mathcal{B}, \Lambda)$ is a DG algebra with product $fg = \mu(f \otimes g)A : \mathcal{B} \to \mathcal{B}\otimes\mathcal{B} \to \Lambda\otimes\Lambda \to \Lambda$.

Then $\tau \in \mathrm{Hom}^1(\mathcal{B},\Lambda) = \mathrm{Hom}(\mathcal{B},\Lambda)_{-1}$ satisfies

$$d\tau + \tau^2 = 0 \quad \text{in } R.$$

Example: $\Lambda = A = \tilde{a}$, $\mathcal{B} =$ bar const., τ is $\tau(\sigma a) = a$ for $a \in a$, zero in other degrees. The product in R is

$$(fg)(a_1,\dots,a_n) = \sum_{i=0}^{n} (-1)^{i|g|} f(a_1,\dots,a_i)\, g(a_{i+1},\dots,a_n).$$

Thus $(\tau^2)(a_1,a_2) = -\tau a_1 \tau a_2 = -a_1 a_2$ and

$$(d\tau)(a_1,a_2) = (d\circ\tau - (-1)^{|\tau|}\tau\circ d)(a_1,a_2) = \tau(a_1a_2) = a_1a_2, \quad \text{so} \quad d\tau + \tau^2 = 0.$$

Now $R = \mathrm{Hom}(\mathcal{B},\Lambda)$ acts on $Q \otimes M$, where Q is a right $\mathcal{B}$ comodule$^{(*)}$, M a left Λ module$^{(*)}$:

$$Q\otimes M \xrightarrow{\Delta\otimes 1} Q\otimes B\otimes M \xrightarrow{1\otimes r\otimes 1} Q\otimes\Lambda\otimes M \xrightarrow{1\otimes\mu} Q\otimes M$$

for $r \in R$. This should make $Q\otimes M$ a left R-module, so one obtains a twisted complex $Q \otimes_\tau M$ with differential $d_{Q\otimes M} + \tau$.

Similarly if N is a right Λ-module and P is a left $\mathcal{B}$-comodule, then $N \otimes P$ should be a right R-module, and we get a twisted complex $N \otimes_{-\tau} P$ with differential $d_{N\otimes P} - \tau$.

Universal cases are $\mathcal{B}\otimes_\tau\Lambda$ and $\Lambda\otimes_{-\tau}\mathcal{B}$ in the sense that $Q\otimes_\tau M = Q\otimes^{\mathcal{B}}(\mathcal{B}\otimes_\tau\Lambda)\otimes_\Lambda M$, etc.

---

* means **DG** Λ module, DG $\mathcal{B}$ comodule unless specified otherwise

Consider next the functors

$$F(P) = \Lambda \otimes_{-\tau} P = (\Lambda \otimes_{-\tau} \mathcal{B}) \otimes^{\mathcal{B}} P$$

$$G(M) = \mathcal{B} \otimes_{\tau} M = (\mathcal{B} \otimes_{\tau} \Lambda) \otimes_{\Lambda} M$$

from (left) $\mathcal{B}$ comodules to Λ modules and back.

[redacted] Then FG and GF are respectively given by the Λ bimodules and $\mathcal{B}$ bicomodules at the top:

$$\begin{tikzcd}
\Lambda \otimes_{-\tau} \mathcal{B} \otimes_{\tau} \Lambda \arrow[d, "1 \otimes \eta_{\mathcal{B}} \otimes 1"] \arrow[dd, bend right] & , & \mathcal{B} \otimes_{\tau} \Lambda \otimes_{-\tau} \mathcal{B} \\
A \otimes \Lambda \arrow[d, "\mu"] & & \mathcal{B} \otimes \mathcal{B} \arrow[u, "1 \otimes \varepsilon_{\Lambda} \otimes 1"] \\
A & & \mathcal{B} \arrow[u, "\Delta"] \arrow[uu, bend left]
\end{tikzcd}$$

Define α: (left diagram) Define β: (right diagram)

ε_{Λ} unit for Λ

$\eta_{\mathcal{B}}$ counit for $\mathcal{B}$

α, β gives rise to maps of functors $FG \xrightarrow{\alpha} 1$, $1 \xrightarrow{\beta} GF$. [redacted] Claim these are adjoint functors, which means the compositions

$$F \xrightarrow{F(\beta)} FGF \xrightarrow{\alpha_F} F$$

$$G \xrightarrow{\beta_G} GFG \xrightarrow{G(\alpha)} G$$

are the identity maps. [redacted]

I've forgotten to check that α, β are compatible with twisted differentials. The twisting ~~of the~~ in $\Lambda \otimes \mathcal{B} \otimes \Lambda$ is $ad(\tau)$ relative to the R bimodule structure on $\overset{}{\Lambda} \underset{\substack{\uparrow \\ \text{right}}}{\otimes} \mathcal{B} \underset{\substack{\uparrow \\ \text{left}}}{\otimes} \Lambda$

Let's calculate how the left multiplication relates to α. We will forget the first factor Λ. Then the left multiplication by τ on $\mathcal{B} \otimes \Lambda$ is the top now of

$$\begin{tikzcd}
B\otimes\Lambda\ [redacted] \arrow[r, "\Delta\otimes 1"] \arrow[dr, equal] & B\otimes B\otimes\Lambda \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "\eta_B\otimes 1\otimes 1"] & B\otimes\Lambda\otimes\Lambda \arrow[r, "1\otimes\mu"] \arrow[d, "\eta_B\otimes 1\otimes 1"] & B\otimes\Lambda \arrow[d, "\eta_B\otimes 1"] \\
 & B\otimes\Lambda \arrow[r, "\tau\otimes 1"] & \Lambda\otimes\Lambda \arrow[r, "\mu"] & \Lambda
\end{tikzcd}$$

Thus left and right multiplication by τ on $\Lambda\otimes B\otimes\Lambda$ [redacted] followed by $1\otimes\eta_B\otimes 1$ should be

$$\begin{tikzcd}
\Lambda\otimes B\otimes\Lambda \arrow[r, "1\otimes\tau\otimes 1"] \arrow[r, shift right=2, swap, "\text{[redacted]}"] & \Lambda\otimes\Lambda\otimes\Lambda \arrow[r, shift left, "1\otimes\mu"] \arrow[r, shift right, swap, "\mu\otimes 1"] & \Lambda\otimes\Lambda
\end{tikzcd}$$

and these are equalized by μ, so that $\alpha\circ\mathrm{ad}(\tau) = 0$.

Now compute the composition $F\xrightarrow{F(\beta)} FGF \xrightarrow{\alpha_F} F$.

$$\begin{tikzcd}
\Lambda\otimes_{-\tau}B \arrow[d, "1\otimes\Delta"] \arrow[dr, equal] \arrow[dd, bend right=60, swap, "F(\beta)"] & \text{[redacted]} & \Lambda\otimes_{-\tau}B \\
\Lambda\otimes B\otimes B \arrow[r, "1\otimes\eta_B\otimes 1"] \arrow[d, "1\otimes 1\otimes\varepsilon_\Lambda\otimes 1"] & \Lambda\otimes B \arrow[r, "1\otimes\varepsilon_\Lambda\otimes 1"] \arrow[ur, equal] & \Lambda\otimes\Lambda\otimes B \arrow[u, swap, "\mu\otimes 1"] \\
\Lambda\otimes B\otimes\Lambda\otimes B \arrow[rr, phantom, "="] & (\text{commutes}) & \Lambda\otimes B\otimes\Lambda\otimes B \arrow[u, swap, "1\otimes\eta_B\otimes 1\otimes 1"] \arrow[uu, bend right=60, swap, "\alpha_F"]
\end{tikzcd}$$

Thus the composition is the identity, and should be similarly for $G\xrightarrow{\beta_G} GFG \xrightarrow{G(\alpha)} G$, so we have adjoint functors. This result should be in [HMS].

---

miscellaneous points from scratch work. Recall important example of twisting cochain from $B(A)$ to $R\oplus I[1]$ given by $\rho: A\to R$, such that $\omega: A^{\otimes 2}\to I$.

Summarize preceding: Given DG alg Λ, DG coalg B, $\tau : B \to \Lambda$ a twisting cochain, we have adjoint functors

$$F(P) = \Lambda \otimes_{-\tau} P \quad , \quad G(M) = B \otimes_{\tau} M$$

between (DG) B comodules and Λ modules.

In particular we have [illegible] Λ bimodule $E = \Lambda \otimes_{-\tau} B \otimes_{\tau} \Lambda$ corresponding to FG and a bimodule map $\Lambda \otimes_{-\tau} B \otimes_{\tau} \Lambda \longrightarrow \Lambda$ corresponding to $\alpha : FG \to 1$. So we are able to form a DG algebra $T = T_{\Lambda}(\Sigma E)$ which gives a projective bimodule resolution of Λ. Notice also that E is a "Λ-coalgebra" in the sense that one has besides $E \longrightarrow \Lambda$, the counit, a coproduct $E \longrightarrow E \otimes_{\Lambda} E$. This I think implies a simplicial [illegible] structure on $\bar{T}$. In the present case this amounts to the standard "triple" simplicial structure

$$\begin{tikzcd} {} \arrow[r, shift right=2] \arrow[r, shift right=4] \arrow[r, shift right=6] & (FG)^2 \arrow[l, shift right=2] \arrow[l, shift right=4] \arrow[r, shift right=1] \arrow[r, shift right=3] & FG \arrow[l, shift right=1] \arrow[r] & 1 \end{tikzcd}$$

associated to a pair of adjoint functors. Thus we have a simplicial functor $\{(FG)^{n+1}\}$, faces given by $\alpha : FG \to 1$, degeneracies by $\beta : 1 \to GF$.

On the coalgebra side we have GF given by the B-bicomodule $B \otimes_{\tau} \Lambda \otimes_{-\tau} B$. We have a cosimplicial functor $\{(GF)^{n+1}\}$ with cofaces obtained from $\beta : 1 \to GF$ and codegeneracies from α.

When we form the appropriate trace:
commutator quotient space for bimodules
cocommutator subspace for bicomodules we get

cyclic ~~[illegible]~~ modules

$$((FG)^{n+1})_\natural \qquad ((GF)^{n+1})^\natural$$

which are isomorphic, e.g.

$$(\Lambda \otimes_{-\tau} B \otimes_{\tau} \Lambda)_\natural = \Lambda \otimes_{-\tau} B \otimes_{\tau}$$

$$(B \otimes_{\tau} \Lambda \otimes_{-\tau} B)^\natural = B \otimes_{\tau} \Lambda \otimes_{-\tau}$$

Question: Is this isomorphism related to self-duality of Connes Λ category?

Notice that the above discussion holds without assuming acyclicity of $\Lambda \otimes_{-\tau} B$. Consider $B = \mathbb{C}$. Then $F(P) = \Lambda \otimes P$, $G(M) = M$.

FG corresponds to the Λ-bimodule $\Lambda \otimes \Lambda$ which has Λ coalg structure

GF corresponds to the vector space Λ which has $\mathbb{C}$ alg structure.

The simplicial functor $(FG)^{n+1}$ corresp. to the bimodule resolution

$$\begin{tikzcd} {} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \Lambda \otimes \Lambda \otimes \Lambda \arrow[l, shift right=4] \arrow[l, shift right=6] \arrow[r, shift left=1] \arrow[r, shift right=1] & \Lambda \otimes \Lambda \arrow[l, bend right] \arrow[r] & \Lambda \end{tikzcd}$$

The cosimplicial functor $(GF)^{n+1}$ corresp. to the Amitsur complex

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \Lambda \arrow[r, shift left=1] \arrow[r, shift right=1] & \Lambda \otimes \Lambda \arrow[l, bend right] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \Lambda \otimes \Lambda \otimes \Lambda \arrow[l, bend right] \arrow[l, bend right=50] \end{tikzcd} .$$

October 4, 1993

Review: Λ (DG) algebra, B coalgebra $\tau : B \longrightarrow \Lambda$ twisting cochain, functors $F(P) = \Lambda \otimes_{-\tau} P$, $G(M) = B \otimes_\tau M$. Drop subscript τ

FG given by the Λ bimodule $E = \Lambda \otimes B \otimes \Lambda$

GF ——— B bicomodule $E' = B \otimes \Lambda \otimes B$

simplicial functor $\{(FG)^{n+1}\}$, faces from $\alpha : FG \to 1$, degeneracies from $\beta : 1 \longrightarrow GF$. This functor given by the simplicial Λ bimodule.

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] & E \otimes_\Lambda E \arrow[l, shift right=5] \arrow[l, shift right=6] \arrow[r, shift left=1] \arrow[r, shift right=1] & E \arrow[l, bend right] \arrow[r, dashed] & \Lambda \end{tikzcd}$$

Since E is a free Λ bimodule and $E \xrightarrow{\alpha} \Lambda$ is surjective, we know this simplicial bimodule is a resolution of Λ. Thus it's a free bimodule resolution, so its homotopy type depends only on Λ. We know then that

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] & E \otimes_\Lambda E \otimes_\Lambda \arrow[l, shift right=5] \arrow[l, shift right=6] \arrow[r, shift left=1] \arrow[r, shift right=1] & E \otimes_\Lambda \arrow[l, bend right] \end{tikzcd}$$

gives the Hochschild homology of Λ. In fact the inclusion $\mathbb{C} \subset B$ of coalgebras induces $\Lambda \otimes \Lambda \longrightarrow E$ which extends to a map of simplicial bimodules $\{\Lambda^{\otimes\, n+2}\} \longrightarrow \{T_\Lambda^{n+1}(E)\}$, then to a map of simplicial modules $\{\Lambda^{\otimes n+1}\} \longrightarrow \{[E \otimes_\Lambda]^{(n+1)}\}$, which is a quis.

Consider next the cosimplicial functor $\{(GF)^{n+1}\}$, cofaces from $\beta : 1 \longrightarrow GF$, faces from $\alpha : FG \longrightarrow 1$. This is given by the cosimplicial B-comodule

$$\begin{tikzcd} B \arrow[r, dashed] & E' \arrow[r, shift left=1] \arrow[r, shift right=1] & E' \otimes^B E' \arrow[l, shift right=4] & \cdots \end{tikzcd}$$

which gives a cosimplicial module

$$\begin{tikzcd} E' \otimes^B \arrow[r, shift left=1] \arrow[r, shift right=1] & E' \otimes^B E' \otimes^B & \cdots \end{tikzcd}$$

~~[illegible]~~ This should give the coHochschild homology of B.

In fact the ~~[illegible]~~ augmentation $\Lambda \longrightarrow \mathbb{C}$ induces a map ~~[illegible]~~ of simplicial modules $\{[E\otimes^{B}]^{(n+1)}\} \longrightarrow \{B^{\otimes n+1}\}$ which should be a quis.

Now we have

$$E\otimes_{\Lambda} = (\Lambda\otimes_{-\tau} B\otimes_{\tau}\Lambda)\otimes_{\Lambda} = \Lambda\otimes_{-\tau} B\otimes_{\tau}$$

$$E'\otimes^{B} = (B\otimes_{\tau}\Lambda\otimes_{-\tau}B)\otimes^{B} = \Lambda\otimes_{-\tau}B\otimes_{\tau}$$

and more generally, we have canonical isomorphisms

$$[E\otimes_{\Lambda}]^{(n+1)} = [E'\otimes^{B}]^{(n+1)}$$

for all $n \geq 0$.

We have the following situation. $\{[E\otimes_{\Lambda}]^{(n+1)}\}$ and $\{[E'\otimes^{B}]^{(n+1)}\}$ are both cyclic modules in Connes sense. They are not isomorphic, rather one is obtained from the other by composing with the self duality isomorphism $\Lambda \simeq \Lambda^{op}$. Thus they need not have the same ~~Hochschild~~ ~~homology~~ Hochschild or cyclic homology in general. However if the face + degeneracy maps are quis, then the two cyclic modules have the same cyclic (hence also Hochschild) homology.

This I should be able eventually to build into Tsygan's proof for universal enveloping algs.

October 5, 1993

Λ DG algebra, $E \xrightarrow{\eta} \Lambda$ map of (DG) Λ-bimodules, $C = \text{Cone}(E \to \Lambda) \simeq \Sigma E \oplus \Lambda$ with differential $\begin{pmatrix} -d & \\ \eta & d \end{pmatrix}$. One has a bimodule map $\Lambda \hookrightarrow C$ so we can form

$$R_\Lambda(C) = T_\Lambda(\Sigma E)$$

If C has already an algebra structure, then there is a retraction $R_\Lambda(C) \twoheadrightarrow C$. In fact we know $R_\Lambda(C) = \Omega^{ev}_\Lambda(C)$ equipped with Fedosov product.

Let us try to define a retraction $R_\Lambda(C) \to C$ by making $R_\Lambda(C)$ act on C. Start with the DG algebra $\text{Hom}_{\Lambda^{op}}(C, C)$ of operators compatible with the right Λ module structure. We have a homom. $\Lambda \longrightarrow \text{Hom}_{\Lambda^{op}}(C,C)$ given by left multiplications. To extend this to a bimodule map $\phi : C \longrightarrow \text{Hom}_{\Lambda^{op}}(C,C)$ and then use the universal property of $R_\Lambda(C)$. Such a bimodule map ϕ is equivalent to a bimodule map $\psi : C \otimes_\Lambda C \longrightarrow C$, $x \otimes y \longmapsto \psi(x,y)$. The condition that ϕ extends the left multiplication map means $\psi(1,y) = y$, and the condition that acting on 1 gives a retraction of $R_\Lambda(C)$ onto C means $\psi(x, 1) = x$.

Conclude that if $\exists$ bimodule map $\psi : C \otimes_\Lambda C \to C$ such that $\psi(1,x) = x$, $\psi(x,1) = x$, then we get an action of $R_\Lambda(C)$ on C such that acting on 1 retracts $R_\Lambda(C)$ onto C.

October 7, 1993

Review how the B operator can be understood. If $P \xrightarrow{\eta} A$ is a projective bimodule resolution then

$$P \otimes_A P \underset{\eta\otimes 1}{\overset{1\otimes\eta}{\rightrightarrows}} P$$

are two maps from a projective complex to a complex quis A which become equal in A, so we know there is a homotopy operator $h: P\otimes_A P \xrightarrow{+1} P$ such that

$$[d,h] = 1\otimes\eta - \eta\otimes 1$$

Consider now $h: P\otimes_A P\otimes_A \longrightarrow P\otimes_A$. We have the permutation σ on the former, and

$$[d, h\sigma] = (1\otimes\eta - \eta\otimes 1)\sigma = \eta\otimes 1 - 1\otimes\eta$$

whence $[d, h+h\sigma] = 0$. ~~Thus $B = (h+h\sigma)\Delta$ is a degree +1 endomorphism of $P\otimes_A$ respecting the differential. Here $\Delta : P \to P\otimes_A P$.~~

~~Let's work~~

Thus we have a degree +1 map of complexes

$$P\otimes_A P\otimes_A \xrightarrow{h+h\sigma} P\otimes_A$$

On the other hand the former complex is homotopy equivalent to $P\otimes_A$, so we obtain a degree +1 map on $P\otimes_A$ respecting the differential.

Let's now take P to be the standard resolution (unnormalized): $P_n = A\otimes A^{\otimes n}\otimes A$. In this case we have a "A-coalgebra" structure given by $\eta: P\to A$ and $\Delta: P \longrightarrow P\otimes_A P$ defined by

$$\Delta(1,a_1,\dots,a_n,1)=\sum_{i=0}^{n}(1,a_1,\dots,a_i,1)\otimes_A(1,a_{i+1},\dots,a_n,1)$$

In this case the desired homotopy $h: P\otimes_A P \xrightarrow{+1} P$ is given by

$$h\big((a_0,\dots,a_{i+1})\otimes_A(a'_0,\dots,a'_{j+1})\big)$$
$$=(-1)^i(a_0,\dots,a_i,a_{i+1}a'_0,a'_1,\dots,a'_j,a'_{j+1})$$

This is essentially the product in $\mathrm{Cone}(P\to A)=T_A(ADA)$, the sign due to: $\Sigma P\otimes_A\Sigma P\simeq\Sigma^2(P\otimes_A P)$.

[redacted] One has for $h\Delta: P\longrightarrow P$ the formula

1) $$\boxed{h\Delta(a_0,\dots,a_{n+1})=\sum_{i=0}^{n}(-1)^i(a_0,a_1,\dots,a_i,1,a_{i+1},\dots,a_{n+1})}$$

so $h\Delta: P\otimes_A\longrightarrow P\otimes_A$ is

$$h\Delta\big((a_0,\dots,a_n,1)\otimes_A\big)=\sum_{i=0}^{n}(-1)^i(a_0,\dots,a_i,1,a_{i+1},\dots,a_n,1)\otimes_A$$

or simply

2) $$\boxed{h\Delta(a_0,\dots,a_n)=\sum_{i=0}^{n}(-1)^i(a_0,\dots,a_i,1,a_{i+1},\dots,a_n)}$$

in the Hochschild (b) complex.

Next [redacted]

$$\sigma\Delta\big((a_0,\dots,a_n,1)\otimes_A\big)=\sigma\sum_{i=0}^{n}(a_0,\dots,a_i,1)\otimes_A(1,a_{i+1},\dots,a_n,1)\otimes_A$$
$$=\sum_{i=0}^{n}(-1)^{i(n-1)}(1,a_{i+1},\dots,a_n,1)\otimes_A(a_0,\dots,a_i,1)\otimes_A$$
$$\xrightarrow{h}\sum_{i=0}^{n}\underbrace{(-1)^{i(n-1)}(-1)^{n-i}}_{(-1)^{in-\cancel{i}+n-\cancel{i}}\;=\;(-1)^{(i+1)n}}(1,a_{i+1},\dots,a_n,a_0,\dots,a_i,1)\otimes_A$$

so on the (b) complex

3) $$\boxed{h\sigma\Delta(a_0,\dots,a_n) = \sum_{i=0}^{n}(-1)^{(i+1)n}(1, a_{i+1},\dots,a_n,a_0,\dots,a_i)}$$

Thus $(h+h\sigma)\Delta$ is not Connes' B operator $(1-\lambda)(-\lambda^{-1}s)N_\lambda = (1-\lambda^{-1})sN_\lambda$, nor $(1-\lambda)sN_\lambda$. These two B operators have $B^2=0$.

Consider $h\Delta$ on P. From 1) we see that this is essentially the degree $+1$ coderivation δ of $T_A(ADA)$ such that $\delta(a)=0$, $\delta(D)=D^2$. We have $[d, h\Delta] = (1\otimes\eta-\eta\otimes 1)\Delta = 0$, which agrees with earlier result that δ anticommutes with the differential on $T_A(ADA)$.

One has for $h\sigma\Delta$ on $P\otimes_\Lambda$ that $[d, h\sigma\Delta] = (1\otimes\eta-\eta\otimes 1)\sigma\Delta = (\eta\otimes 1 - 1\otimes\eta)\Delta = 0$. Thus $h\sigma\Delta = sN_\lambda$ anti-commutes with b it seems. Check:

$$bs+sb' = b's+sb'+cs = 1-\lambda$$

$$b(sN_\lambda)+(sN_\lambda)b = bsN_\lambda + sb'N_\lambda = (1-\lambda)N_\lambda = 0.$$

October 8, 1993

A algebra, E chain complex of bimodules equipped with a bimodule map $E \xrightarrow{\eta} A$, $C = \mathrm{Cone}(E \xrightarrow{\eta} A) = \Sigma E \oplus A$ with diffl $\begin{pmatrix} -d & \\ \eta & d \end{pmatrix}$.

The first point [redacted] is to describe products $\psi : C \otimes_A C \longrightarrow C$ not necessarily associative but such that $1 \in A \subset C$ is a unit, i.e. $\psi(1, x) = \psi(x, 1) = x$. Now

$$C \otimes_A C = \Sigma E \otimes_A \Sigma E \oplus \Sigma E \otimes_A A \oplus A \otimes_A \Sigma E \oplus A \otimes_A A$$

$$= \Sigma E \otimes_A \Sigma E \oplus \underbrace{\Sigma E \otimes 1 \oplus 1 \otimes \Sigma E \oplus A}$$

The unit conditions determine ψ on , [redacted] and the rest of ψ consists of operators [redacted] $\Sigma E \otimes_A \Sigma E \longrightarrow \Sigma E$ and $\Sigma E \otimes_A \Sigma E \longrightarrow A$. The latter is zero for degree reasons. [redacted] Let $h : E \otimes_A E \longrightarrow E$ be the map such that [redacted].

$$\psi(\sigma \otimes \sigma) = \sigma h$$

where $\sigma \otimes \sigma$ on the left is

$$E \otimes_A E \xrightarrow[\sigma\otimes\sigma]{\sim} \Sigma E \otimes_A \Sigma E \subset C \otimes_A C$$

and similarly σ on the right is $E \xrightarrow[\sigma]{\sim} \Sigma E \subset C$. One thus has

$$\begin{tikzcd} \text{[redacted]}E \otimes_A \text{[redacted]}E \arrow[r, "h"] \arrow[d, "\sigma\otimes\sigma"] & \text{[redacted]}E \arrow[d, "\sigma"] \\ C \otimes_A C \arrow[r, "\psi"] & C \end{tikzcd}$$

Now $\sigma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $d_C \sigma = \begin{pmatrix} -d & \\ \eta & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -d_E \\ \eta \end{pmatrix}$

$\sigma d_E = \begin{pmatrix} 1 \\ 0 \end{pmatrix} d_E = \begin{pmatrix} d_E \\ 0 \end{pmatrix}$

so $[d, \sigma] = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$. Then

$$[d, \psi(\sigma\otimes\sigma)] = \psi\left([d,\sigma]\otimes\sigma - \sigma\otimes[d,\sigma]\right)$$
$$= \psi\left(\binom{0}{\eta}\otimes\sigma - \sigma\otimes\binom{0}{\eta}\right)$$

[redacted]

$$= \sigma(\eta\otimes 1 - 1\otimes\eta) : E\otimes_A E \longrightarrow E \xrightarrow{\sim} \underset{\subset C}{\Sigma E}$$

Now $[d, \sigma h] = \underbrace{\binom{0}{\eta}}h - \sigma[d,h]$

0 for degree reasons: $h(E\otimes_A E)$ supported [redacted] in degree ≥ 1.

Thus we have $\boxed{[d,h] = 1\otimes\eta - \eta\otimes 1}$

Conclusion is that $\psi : C\otimes_A C \longrightarrow C$ such that $1\in A$ is left & right unit are equivalent to operators $h: E\otimes_A E \xrightarrow{+1} E$ such that $[d,h] = 1\otimes\eta - \eta\otimes 1$. I've assumed A is concentrated in degree 0, E supported in degrees ≥ 1, but the direction $h \mapsto \psi$ should work in general.

Note that once we have ψ and we know $\exists\, \xi\in E_0$ such that $\eta(\xi) = 1$, then it follows that C is acyclic, contracting homotopy $\psi(\xi, -)$; [redacted] one has $d(\xi) = 0$ for degree reasons. Then we deduce that $\eta: E \longrightarrow A$ is a homotopy equivalence (as left or right A mod).

Here's a direct proof (reminiscent of Atiyah's periodicity proof trick). Have maps $E \xrightarrow{\eta} A$ and $A \xrightarrow{\phi} E$, $\phi(a) = a\xi$, such that $\eta\phi = 1$. To show $\phi\eta$ is homotopic to the identity.

One has commutative square

$$\begin{tikzcd} E \arrow[r, "\phi_1"] \arrow[d, "\eta"'] & E\otimes_A E \arrow[d, "1\otimes\eta"] \\ A \arrow[r, "\phi"'] & E \end{tikzcd}$$

$\phi_1(x) = \xi\otimes x$

Better $(1\otimes\eta)\phi_1(x) = (1\otimes\eta)(\xi\otimes x) = \xi\cdot\eta x = \phi\eta(x)$

$(\eta\otimes 1)\phi_1(x) = (\eta\otimes 1)(\xi\otimes x) = x$

Then $h\phi_1 : E \longrightarrow E\otimes_A E \longrightarrow E$ is such that

$$[d, h\phi_1] = [d,h]\phi_1 \qquad ([d,\phi_1]=0 \text{ as } d(\xi)=0)$$
$$= (1\otimes\eta - \eta\otimes 1)\phi_1$$
$$= \xi\eta - 1$$

So far we have C acyclic, more generally contractible as left or right DG module, equivalently $E \longrightarrow A$ is a heq ignoring either the left or right A mod structure. This has been obtained from $h : E\otimes_A E \xrightarrow{+1} E \ni [d,h] = 1\otimes\eta - \eta\otimes 1$.

We next want $E\otimes_A E$ to be heq to E as DG A-bimodule. So assume given a coproduct

$$\Delta : E \longrightarrow E\otimes_A E$$

for which η is both left & right counit. If Δ is coassociative, then we have degeneracies on the presimplicial bimodule of $E\otimes_A \overset{n+1}{\cdots} \otimes_A E$ and we want to use the following simplicial argument that E and $E\otimes_A E$ are heq. One has commutative square

$$\begin{tikzcd} X_1 \arrow[r, "s_1"] \arrow[d, "d_0"'] & X_2 \arrow[d, "d_0"] \\ X_0 \arrow[r, "s_0"'] & X_1 \end{tikzcd}$$

(Recall $d_i s_j = \begin{cases} s_{j-1}d_i & i<j \\ 1 & i=j,j+1 \\ s_j d_{i-1} & i>j+1 \end{cases}$

and also $d_0 \sim d_1$ so $s_0 d_0 = d_0 s_1 \sim d_1 s_1 = 1$.

This idea works as follows. Let

$$k = (h\otimes 1)(1\otimes\Delta) : E\otimes_A E \longrightarrow E\otimes_A E\otimes_A E \longrightarrow E\otimes_A E$$

Then
$$\begin{aligned}[d,k] &= [d, h\otimes 1](1\otimes\Delta)\\ &= (1\otimes\eta\otimes 1 - \eta\otimes 1\otimes 1)(1\otimes\Delta)\\ &= 1\otimes 1 - \Delta(\eta\otimes 1)\end{aligned}$$

and of course $(\eta\otimes 1)\Delta = 1$ on E, ~~so~~ E and $E\otimes_A E$ [redacted] are heq DG bimodules.

October 10, 1993

Continue to analyze what it means for $E \xrightarrow{\eta} A$ to be idempotent up to heq. I want to be able to handle the situation where $A = U(\mathfrak{g})$ and $E = A \otimes \Lambda\mathfrak{g} \otimes A$ is the Koszul bimodule resolution.

We work with $h : E \otimes_A E \longrightarrow E$, bimod map of degree 1 such that $[d,h] = 1\otimes\eta - \eta\otimes 1$, also with $\Delta : E \otimes_A E \longrightarrow E$ a bimodule map of degree 0 compatible with d such that [redacted] $(\eta\otimes 1)\Delta = (1\otimes\eta)\Delta = 1$.

Previously we saw how from this data to show Δ is a homotopy equivalence:

$$\begin{tikzcd} E\otimes_A E \arrow[r, shift left, "1\otimes\Delta"] \arrow[r, shift right, "\Delta\otimes 1"'] & E\otimes_A E\otimes_A E \arrow[r, shift left, "h\otimes 1"] \arrow[r, shift right, "1\otimes h"'] & E\otimes_A E \end{tikzcd}$$

$$[d,(h\otimes 1)(1\otimes\Delta)] = (1\otimes\eta\otimes 1 - \eta\otimes 1\otimes 1)(1\otimes\Delta) = 1 - \Delta(\eta\otimes 1)$$

$$[d,(1\otimes h)(\Delta\otimes 1)] = (1\otimes 1\otimes\eta - 1\otimes\eta\otimes 1)(\Delta\otimes 1) = \Delta(1\otimes\eta) - 1$$

Now conversely we show [redacted], assuming Δ is a homotopy equivalence, [redacted] how to obtain h. Suppose given

$$\begin{tikzcd} E\otimes_A E \arrow[loop left, "k"] \arrow[d, shift left, "\eta\otimes 1"] \\ E \arrow[u, shift left, "\Delta"] \end{tikzcd} \qquad [d,k] = 1 - \Delta(\eta\otimes 1)$$

Then put $h = (1\otimes\eta)k : E\otimes_A E \xrightarrow{k} E\otimes_A E \xrightarrow{1\otimes\eta} E$. Then

$$[d,h] = (1\otimes\eta)(1-\Delta(\eta\otimes 1)) = 1\otimes\eta - \eta\otimes 1.$$

Notice that if we start with h and let $k = (h\otimes 1)(1\otimes\Delta)$, then we get h back again

$$\begin{tikzcd}
E\otimes_A E \arrow[r, "1\otimes\Delta"] \arrow[dr, equal, "1"'] & E\otimes_A E\otimes_A E \arrow[r, "h\otimes 1"] \arrow[d, "1\otimes 1\otimes\eta"] & E\otimes_A E \arrow[d, "1\otimes\eta"] \\
 & E\otimes_A E \arrow[r, "h"] & E
\end{tikzcd}$$

Also $k\Delta = (h\otimes 1)(1\otimes\Delta)\Delta = (h\otimes 1)(\Delta\otimes 1)\Delta$

$$= (h\Delta\otimes 1)\Delta$$

vanishes when $h\Delta = 0$, and

$$(\eta\otimes 1)k = (\eta\otimes 1)(h\otimes 1)(1\otimes\Delta) = (\eta h\otimes 1)(1\otimes\Delta)$$

vanishes when $\eta h = 0$, which is a natural property to require of h.

The reason one might prefer to work with k instead of h is because k might arise conveniently from HPT.

October 11, 1993

Example. Let $A = T(V)$ and let E be the bimodule complex

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[r] & \Omega^1 A \arrow[r] \arrow[d, equal] & \overset{(0)}{A \otimes A} \arrow[r] & 0 \arrow[r] & \cdots \\
 & & A \otimes V \otimes A \arrow[ur, "\partial"'] & & &
\end{tikzcd}$$

$$\partial(1 \otimes v \otimes 1) = 1 \otimes v - v \otimes 1.$$

Then $C = \mathrm{Cone}(E \xrightarrow{\eta} A)$ is the complex

$$\longrightarrow 0 \longrightarrow A \otimes V \otimes A \xrightarrow{-\partial} A \otimes A \xrightarrow{\mu} \overset{0}{A} \longrightarrow 0 \longrightarrow \cdots$$

I want to explicitly show that $\eta \otimes 1 : E \otimes_A E \to E$ is a homotopy equivalence. It's equivalent to construct a contraction on $\mathrm{Cone}(\eta \otimes 1) = C \otimes_A E$. One has

$$C \otimes_A E = \left(A \otimes V \otimes A \xrightarrow{-\partial} A \otimes A \xrightarrow{\mu} A \right) \otimes_A \left(\begin{tikzcd} A \otimes V \otimes A \arrow[d, "\partial"] \\ A \otimes A \end{tikzcd} \right)$$

$$= \left(\begin{tikzcd}
A \otimes V \otimes A \otimes V \otimes A \arrow[r, "-\partial \otimes 1"] \arrow[d, "1 \otimes \partial"] & A \otimes A \otimes V \otimes A \arrow[r, "\mu \otimes 1"] \arrow[d, "-1 \otimes \partial"] & A \otimes V \otimes A \arrow[d, "\partial"] \\
A \otimes V \otimes A \otimes A \arrow[r, "-\partial \otimes 1"] & A \otimes A \otimes A \arrow[r, "\mu \otimes 1"] & A \otimes A
\end{tikzcd} \right)$$

This is really the short exact sequence of complexes

$$0 \longrightarrow A \otimes V \otimes E \longrightarrow A \otimes E \longrightarrow E \longrightarrow 0$$

To get a contraction for C we can choose contractions on each row then perturb. If γ is the row-wise contraction: $[d', \gamma] = 1$, $d' =$ ~~[illegible]~~ horizontal differential, then the desired contraction of $C \otimes_A E$ is $\gamma - \gamma d'' \gamma$, where

d'' is the vertical differential.

Since the rows are short exact sequences [redacted] γ is obtained by choosing a lifting for $A\otimes E \longrightarrow E$, [redacted] equivalently a left connection on E. There's an obvious choice in the present case:

$$A\otimes V\otimes A \xrightarrow{1\otimes\varepsilon_A\otimes 1\otimes 1} A\otimes A\otimes V\otimes A$$

ε_A = unit for A.

$$A\otimes A \xrightarrow{1\otimes\varepsilon_A\otimes 1} A\otimes A\otimes A$$

Recall, given $f: X\to Y$, $C(f)_n = X_{n-1}\oplus Y_n$, $d_C = \begin{pmatrix}-d & \\ f & d\end{pmatrix}$ that a contraction on $C(f)$ has the form $\begin{pmatrix}-h & g\\ u & h\end{pmatrix}$, where $\left[\begin{pmatrix}-d & \\ f & d\end{pmatrix}, \begin{pmatrix}-h & g\\ u & h\end{pmatrix}\right] = \begin{pmatrix} dh+hd+gf & -dg+gd \\ \begin{matrix}-fh+du\\ -ud+fh\end{matrix} & fg+dh+dh\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Observe that the contraction on $C\otimes_A E = \mathrm{Cone}(\eta\otimes 1)$:

$$\boxed{E\otimes_A E} \xrightarrow{\eta\otimes 1} \boxed{E}$$

we have constructed has the form $\begin{pmatrix}-k & \Delta\\ 0 & 0\end{pmatrix}$, where Δ is a lifting of E into $E\otimes_A E$ wrt $\eta\otimes 1$.

Goal is now to find k, Δ. Let κ denote the horizontal contraction (essentially):

$$\begin{tikzcd} A\otimes A\otimes A\otimes V\otimes A \arrow[r, shift right, "\partial\otimes 1\otimes 1"'] & A\otimes A\otimes V\otimes A \arrow[l, shift right, "\kappa\otimes 1\otimes 1"'] \end{tikzcd}$$

$$\begin{tikzcd} A\otimes V\otimes A\otimes A \arrow[r, shift right, "\partial\otimes 1"'] & A\otimes A\otimes A \arrow[l, shift right, "\kappa"'] \end{tikzcd}$$

where $\kappa(a_1\otimes a_2) = a_1\otimes a_2 - a_1 a_2\otimes 1 = a_1(1\otimes a_1 - a_1\otimes 1)$
$= a_1 da_2$

One has the following picture of $E \otimes_A E \xrightarrow{\eta \otimes 1} E$.

$$\begin{tikzcd}
A\otimes V\otimes A\otimes V\otimes A \arrow[d, "-1\otimes\partial"] \arrow[r, shift right, "\partial\otimes 1\otimes 1"'] & A\otimes A\otimes V\otimes A \arrow[l, shift right, "K\otimes 1\otimes 1"'] \arrow[d, "1\otimes\partial"] \arrow[r, shift right, "\mu\otimes 1\otimes 1"'] & A\otimes V\otimes A \arrow[l, shift right, "1\otimes\varepsilon_A\otimes 1\otimes 1"'] \arrow[d, "\partial"] \\
A\otimes V\otimes A\otimes A \arrow[r, shift right, "\partial\otimes 1"'] & A\otimes A\otimes A \arrow[l, shift right, "K\otimes 1"'] \arrow[r, shift right, "\mu\otimes 1"'] & A\otimes A \arrow[l, shift right, "1\otimes\varepsilon_A\otimes 1"']
\end{tikzcd}$$

k is the pair $K\otimes 1\otimes 1$, $K\otimes 1$ above. Let's compute $[d,k]$. This vanishes on the left column. On ~~$A\otimes A\otimes V\otimes A$~~ $1\otimes a\otimes v\otimes 1 \in A\otimes A\otimes V\otimes A$ we get

~~$(\partial\otimes 1\otimes 1)(K\otimes 1\otimes 1)(1\otimes a\otimes v\otimes 1) + (K\otimes 1)(1\otimes\partial)(1\otimes a\otimes v\otimes 1)$~~

~~$= (\partial\otimes 1\otimes 1)(da\otimes v\otimes 1) + (K\otimes 1)(1\otimes a \otimes v - 1\otimes \overset{av}{[illegible]}\otimes 1)$~~

~~$= 1\otimes a\otimes v\otimes 1 - a\otimes 1\otimes v\otimes 1 + da\otimes v - d(av)\otimes 1$~~

~~$= 1\otimes a\otimes v\otimes 1 - a\otimes 1\otimes v\otimes 1$~~

three terms:

$$(\partial\otimes 1\otimes 1)(K\otimes 1\otimes 1)(1\otimes a\otimes v\otimes 1) = (\partial\otimes 1\otimes 1)(da\otimes v\otimes 1)$$
$$= (1\otimes a - a\otimes 1)\otimes v\otimes 1$$

$$(-1\otimes\partial)(K\otimes 1\otimes 1)(1\otimes a\otimes v\otimes 1) = (-1\otimes\partial)(da\otimes v\otimes 1)$$
$$= -da(1\overset{\checkmark}{\otimes} v - v\overset{\checkmark}{\otimes} 1)$$

$$(K\otimes 1)(1\otimes\partial)(1\otimes a\otimes v\otimes 1) = (K\otimes 1)(1\otimes a\otimes v - 1\otimes av\otimes 1)$$
$$= da\otimes v - d(av)\otimes 1$$
$$= da\overset{\checkmark}{\otimes} v - da(v\overset{\checkmark}{\otimes} 1) - \boxed{a\,dv\otimes 1}$$

$= a\otimes v\otimes 1\otimes 1$ in $A\otimes V\otimes A\otimes A$

so

$$\boxed{[d,k](1\otimes a\otimes v\otimes 1) = 1\otimes a\otimes v\otimes 1 - a\otimes 1\otimes v\otimes 1 - a\otimes v\otimes 1\otimes 1}$$

On $1\otimes a\otimes 1 \in A\otimes A\otimes A$ we get

[redacted] $[d,k](1\otimes a\otimes 1)$

[redacted] $= (\partial\otimes 1)(k\otimes 1)(1\otimes a\otimes 1)$

$= (\partial\otimes 1)(da\otimes 1)$

$= 1\otimes a\otimes 1 - a\otimes 1\otimes 1$

Thus if we define $\Delta : E \longrightarrow E\otimes_A E$ [redacted] to be the bimodule map given by

$$\Delta(1\otimes v\otimes 1) = 1\otimes 1\otimes v\otimes 1 + 1\otimes v\otimes 1\otimes 1$$

$$\Delta(1\otimes 1) = 1\otimes 1\otimes 1.$$

then we have

$$\boxed{\begin{array}{l}(\eta\otimes 1)\Delta = 1\\ 1-\Delta(\eta\otimes 1) = [d,k]\end{array}}$$

[redacted] $\Delta(\eta\otimes 1)(1\otimes a\otimes v\otimes 1) = \Delta(a\otimes v\otimes 1)$

$= a(1\otimes 1\otimes v\otimes 1 + 1\otimes v\otimes 1\otimes 1)$

$= (1-[d,k])(1\otimes a\otimes v\otimes 1)$

$\Delta(\eta\otimes 1)(1\otimes a\otimes 1) = \Delta(a\otimes 1) = a(1\otimes 1\otimes 1)$

$= (1-[d,k])(1\otimes a\otimes 1)$

Finally note that

$$(1\otimes\eta)\Delta(1\otimes v\otimes 1) = (1\otimes\mu)(1\otimes v\otimes 1\otimes 1) = 1\otimes v\otimes 1.$$

so that also $\boxed{(1\otimes\eta)\Delta = 1}$.

[redacted] Also $k^2 = 0$, $(\eta\otimes 1)k = 0$

$$(k\Delta)(1\otimes v\otimes 1) = (k\otimes 1\otimes 1)(1\otimes 1\otimes v\otimes 1) = 0$$

so we have an SDR $(\eta\otimes 1, \Delta, k)$.

Let's next compute the $h : E\otimes_A E \longrightarrow E$ arising from k, namely $h = (1\otimes\eta)k$:

Notice that since h has degree $+1$ it has one nonzero component going from $A\otimes A\otimes A$ to $A\otimes V\otimes A$. One has

$$h(1\otimes a\otimes 1) = (1\otimes\mu)(\kappa\otimes 1)(1\otimes a\otimes 1)$$

$$= (1\otimes\mu)(da\otimes 1) = da$$

Thus

$$h : A\otimes A\otimes A \longrightarrow A\otimes V\otimes A$$

$$a_1\otimes a\otimes a_2 \longmapsto a_1\, da\, a_2$$

October 15, 1993

Today I will finish up work on the problem of making an S module out of $B \otimes (E \otimes_A)$, where E is an arbitrary projective bimodule resolution of A. I need to summarize the ideas + formulas for later reference.

[redacted] Example to understand after completing the case of a free algebra is $A = S(V)$, $B = S(\Sigma V) = \Lambda V$ with $|V| = 1$. Use Tsygan notation $V\varepsilon$ for $\Sigma V = V[1]$.

One has the bimodule resolution

$$\begin{tikzcd} A \otimes B \otimes A \arrow[r, "\eta"] \arrow[d, phantom, "\parallel"] & A \arrow[d, phantom, "\parallel"] \\ S(V \oplus V\varepsilon \oplus V) & S(V) \end{tikzcd}$$

$$\eta = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} : V \oplus V\varepsilon \oplus V \longrightarrow V$$

Let's find the left + right contractions. The two liftings for η are

$$i' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \iota'' \colon \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : V \longrightarrow V \oplus V\varepsilon \oplus V$$

The differential in $V \oplus V\varepsilon \oplus V$ is $d = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Let

$$k' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad k'' = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$\boxed{\begin{array}{ll} [d, k'] = 1 - \iota'\eta & k'^2 = k'\iota' = \eta k' = 0 \\ [d, k''] = 1 - \iota''\eta & k''^2 = k''\iota''' = \eta k'' = 0 \end{array}}$$

Apply the functor S to η, ι', ι'' to obtain homomorphisms, and to the odd operators d, k', k'' to obtain derivations on $S(-)$. Actually these are biderivations, i.e. also compatible with coalgebra structure.

On $S(V \oplus V\varepsilon \oplus V)$, $[d,k']$ is the derivation of degree zero [illegible] extending the projection $1-\iota'\eta$, which is 0 on $\mathrm{Im}(\iota')$ and 1 on $\mathrm{Ker}(\eta)$. Thus relative to the splitting

$$S(V \oplus V\varepsilon \oplus V) = S(\iota' V) \otimes S(\mathrm{Ker}\,\eta)$$

$[d,k']$ is the degree operator $1 \otimes N$, so $k'(1\otimes N)^{-1}$ is a SDR of $S(V \oplus V\varepsilon \oplus V)$ onto $S(\iota' V)$.

I should have mentioned earlier the simpler homotopy on $S(V\varepsilon \oplus V)$. Here $d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on V are extended to biderivations on $S(V\varepsilon \oplus V)$. As $[d,k] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on V the derivation $[d,k]$ on $S(V\varepsilon \oplus V)$ is the degree operator N, and then $[d, kN^{-1}] = 1 -$ projection onto $S^0(V\varepsilon \oplus V) = \mathbb{C}$.

In a similar way we can handle

$$\begin{tikzcd} A\otimes B\otimes A\otimes B\otimes A \arrow[r, "\eta\otimes 1"] \arrow[d, equal] & A\otimes B\otimes A \arrow[d, equal] \\ S(V\oplus V\varepsilon\oplus V\oplus V\varepsilon\oplus V) & S(V\oplus V\varepsilon\oplus V) \end{tikzcd}$$

$$[d,k] = \left[\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right] =$$

$$= \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 1 - \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Delta} \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\eta \otimes 1}$$

Then $(1 \otimes \eta)\Delta = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & 0 & 1 \end{pmatrix} \Delta = 1$

and $h = (1 \otimes \eta) k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Comments: In the case $A = U(\mathfrak{g})$, $B = S(\mathfrak{g}\varepsilon)$ the same formulas should hold, where d, k' etc. are extended as coderivations. I think that $A \otimes B \otimes A$, $A \otimes B \otimes A \otimes B \otimes A$ all are naturally coalgebras. An interesting point is the fact that $\mathfrak{g} \oplus \mathfrak{g}\varepsilon = \mathfrak{g} \otimes \mathbb{C}[\varepsilon]/\varepsilon^2$ is a DG Lie algebra (in general $\mathfrak{g} \otimes K$ is a Lie alg when $\mathfrak{g}$ is a Lie alg and K is a comm. alg.), hence $U(\mathfrak{g} \oplus \mathfrak{g}\varepsilon) = A \otimes B$ is naturally a Hopf algebra. The same doesn't work for ~~$\mathfrak{g} \oplus \mathfrak{g}\varepsilon \oplus \mathfrak{g}$~~ $A \otimes B \otimes A$ as $\mathfrak{g} \oplus \mathfrak{g}\varepsilon \oplus \mathfrak{g}$ is not a Lie algebra. In fact $S(\mathfrak{g} \oplus \mathfrak{g}\varepsilon \oplus \mathfrak{g})$ is some sort of model for the homogeneous space $G \times^G G$.

Formulas for $\Lambda \otimes_{-\tau} P$, $B \otimes_\tau M$ in the case of mixed complexes and S-modules. Recall $B = \bigoplus_{n \geq 0} \mathbb{C} u_n$ where $\Delta u_n = \sum_{i=0}^{n} u_i \otimes u_{n-i}$ and $|u_n| = 2n$; also $\eta(u_n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$

If P is a B-comodule, let the coproduct $\Delta : P \longrightarrow B \otimes P$ be denoted $\xi \longmapsto \sum u_n \otimes \phi_n(\xi)$. Then $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ says $\sum u_n \otimes u_m \otimes \phi_m \phi_n \xi = \sum u_i \otimes u_j \otimes \phi_{i+j} \xi$, whence $\phi_m \phi_n = \phi_{m+n}$ and so $\phi_m = S^m$ where $S = \phi_1$ has degree -2.

$\Lambda \otimes P = 1 \otimes P + \varepsilon \otimes P$. Compute action of τ on $\Lambda \otimes P$:

$$\begin{tikzcd}
\Lambda \otimes P \arrow[r, "1 \otimes \Delta"] & \Lambda \otimes B \otimes P \arrow[r, "1 \otimes \tau \otimes 1"] & \Lambda \otimes \Lambda \otimes P \arrow[r, "\mu \otimes 1"] & \Lambda \otimes P \\
\lambda \otimes \xi \arrow[r, mapsto] & \lambda \otimes \sum u_n \otimes S^n \xi \arrow[r, mapsto] & (-1)^{|\lambda|} \lambda \otimes \varepsilon \otimes S \xi \arrow[r] & \varepsilon \lambda \otimes S \xi
\end{tikzcd}$$

Thus $\Lambda \otimes_{-\tau} P$ is $\Lambda \otimes P$ with $\boxed{d = 1 \otimes d_P - \varepsilon \otimes S}$

Similarly
$$\begin{tikzcd}
B \otimes M \arrow[r, "\Delta \otimes 1"] & B \otimes B \otimes M \arrow[r, "1 \otimes \tau \otimes 1"] & B \otimes \Lambda \otimes M \arrow[r, "1 \otimes \mu"] & B \otimes M \\
u_n \otimes \xi \arrow[r, mapsto] & \sum_i u_i \otimes u_{n-i} \otimes \xi \arrow[r, mapsto] & u_{n-1} \otimes \varepsilon \otimes \xi \arrow[r, mapsto] & u_{n-1} \otimes \varepsilon \xi
\end{tikzcd}$$

Thus $B \otimes_\tau M$ is $B \otimes M$ with $\boxed{d = 1 \otimes d_M + S \otimes \varepsilon}$

October 17, 1993 (Cindy is 13)

Recall the general result about the GNS construction

$$\Gamma(A \xrightarrow{in,\rho} RA * C) = A * \tilde{C}$$

Proof: A homom. $\Gamma \to R$ equiv. to a triple (u, e, v), $u: A \to R$ homom., $e \in R$ idempotent, $v: RA * C \to eRe$ homom., such that $eu(a)e = v(\rho a)$. This in turn equiv. to (u, e, v'') with u, e as above and $v'': C \to eRe$ a homom. As (e, v'') equiv to a homom $\tilde{C} \xrightarrow{w} R$, we see $\Gamma \to R$ equiv to pairs $A \to R$, $\tilde{C} \to R$, so we win.

Apply this in the case $A \mapsto \mathbb{C}[\varepsilon] = \mathbb{C}\oplus\mathbb{C}\varepsilon,\ \varepsilon^2 = 0$ and $C \mapsto A$. Note $R(\mathbb{C}[\varepsilon]) = \mathbb{C}[h]$ no relations. Thus we find

$$\boxed{\Gamma(\mathbb{C}[\varepsilon] \xrightarrow{in,\rho} \mathbb{C}[h] * A) = \mathbb{C}[\varepsilon] * \tilde{A}}$$

Here's the way this arises: $\mathbb{C}[h] * A$ is the standard A-bimodule resolution of A; it arises when we consider complexes M of A-modules equipped with a contraction h not respecting the A-action. $\mathbb{C}[\varepsilon] * \tilde{A}$ it is the standard normalized $\tilde{A}$-bimodule resolution of $\tilde{A}$; it arises when we consider complexes of $\tilde{A}$ modules (i.e. non unital modules) equipped with special contraction not respecting the $\tilde{A}$-action.

The above GNS algebra handles the process of dilating a complex of A-modules with contraction h to a complex of $\tilde{A}$ modules with special contraction $\begin{pmatrix} h & -h^2 \\ 1 & -h \end{pmatrix}$

October 27, 1993

Criterion of Cartan-Eilenberg for flatness using linear equations: [redacted]

An R-module M is flat iff given any linear relation $xm = 0$ $\left(\sum_j x_{ij} m_j = 0\right)$, x a matrix ~~over~~ R, m a vector ~~over~~ M, there exists a matrix y over R and vector m' over M such that $xy = 0$ and $m = ym'$.

Proof. Assume M flat and the relation $xm = 0$ given as above. Define the right R-module K by

$$0 \longrightarrow K \longrightarrow R^{\{j\}} \xrightarrow{x} R^{\{i\}} \qquad (r_j) \mapsto (x_{ij} r_j)$$

Then one has an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K \otimes_R M \arrow[r] & M^{\{j\}} \arrow[r, "x"] & M^{\{i\}} \\ & & (m_j) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (x_{ij} m_j) = 0. \end{tikzcd}$$

Thus $(m_j) \in K \otimes_R M$, i.e. $m_j = \sum_k y_{jk}$ [redacted] m'_k where $y_k = (y_{jk}) \in K$ and $m'_k \in M$. Thus $m = ym'$ where $xy = 0$.

Conversely assume this linear equations criterion. To show M flat it suffices to show $J \otimes_R M \to M$ is injective for any right ideal J. [redacted] Let $\sum x_j \otimes_R m_j \in J \otimes_R M$ be in the kernel: $\sum x_j m_j = 0$. Then there exists y matrix over R, m' vector over M such that $\sum_i x_j y_{jk} = 0$ and $m_j = \sum_k y_{jk} m'_k$. Then

$$\sum_j x_j \otimes_R m_j = \sum_{j,k} x_j \otimes_R y_{jk} m'_k$$

$$= \sum_{j,k} x_j y_{jk} \otimes_R m'_k = 0 .$$

Alternative viewpoint: This criterion is equivalent to the key step in Lazard's thm. (M flat iff M filtered inductive limit of f.g. [illegible] projective modules), namely it establishes the fact that the category of f.g. free R-modules equipped with map to M: $R^n \longrightarrow M$ is filtering:

$$\begin{tikzcd}
R^p \arrow[r, "x"] & R^q \arrow[r, dashed, "y"] \arrow[d, "m"] & R^n \arrow[dl, dashed, "m'"] \\
& M &
\end{tikzcd}$$

$xm = 0$

(Note that because we use left modules, composition of maps given by matrices corresponds to multiplication in the opposite order.)

~~Consider now Wodzicki's statement that~~ ~~if I is a non-unital algebra such that $I^2 = I$ [illegible] and I occurs as a~~ ~~if $\mathfrak{a}$ is a left ideal in an algebra R such that $\mathfrak{a}^2 = \mathfrak{a}$, then $\mathfrak{a}$ is flat over R iff $\mathfrak{a}$ is flat over $\tilde{\mathfrak{a}}$.~~

~~Proof. Here's one direction~~

~~Lemma: For any R-module M one has~~

~~$$\mathfrak{a} \otimes_{\tilde{\mathfrak{a}}} M \xrightarrow{\sim} \mathfrak{a} \otimes_R M$$~~

~~It suffices to see that for $a \in \mathfrak{a}$, $m \in M$, we have $a \otimes$~~

Suppose $\mathfrak{a}$ is a left ideal in the algebra R such that $\mathfrak{a}^2 = \mathfrak{a}$. We ask when $\mathfrak{a}$ is flat as R-module, i.e. when the Cartan-Eilenberg criterion is satisfied. Suppose $x \cdot m = 0$ with (x_{ij}) a matrix over R and (m_j) a vector over $\mathfrak{a}$. Since $\mathfrak{a}^2 = \mathfrak{a}$ we can write $(m_j) = (y_{jk})(m'_k)$ with $y_{jk}, m'_k \in \mathfrak{a}$.

Check: $m_1 = (a_1 \ldots a_p)\begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}$ $\quad m_2 = (b_1 \ldots b_q)\begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix}$

Then
$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} a_1 \cdots a_p & 0 \\ 0 & b_1 \cdots b_q \end{pmatrix}\begin{pmatrix} z_1 \\ \vdots \\ z_p \\ y_1 \\ \vdots \\ y_q \end{pmatrix}.$$

Then we have the linear relation in $\mathfrak{a}$
$$(xy)\, m' = 0$$
with xy a matrix over $\mathfrak{a}$, m' a vector over $\mathfrak{a}$. Assume that any such relation in $\mathfrak{a}$ can be factored: $m' = z\, m''$ with z a matrix over $\mathfrak{a}$ and m'' a vector over $\mathfrak{a}$, and $(xy)z = 0$. Then the original relation $x \cdot m = 0$ factors $x(yz) = 0$, $m = (yz)m''$ where yz is a matrix over $\mathfrak{a}$ and m'' is a vector over $\mathfrak{a}$. Thus $\mathfrak{a}$ is flat over R.

Next assume $\mathfrak{a}$ flat over R and suppose given $xm = 0$ with x, m resp matrix, vector over $\mathfrak{a}$. Then by flatness of $\mathfrak{a}$ over R we know $m = ym'$, where y is a matrix over R such that $xy = 0$, and m' is a vector over $\mathfrak{a}$. But $m' = zm''$ where z, m'' are resp. matrix, vector over $\mathfrak{a}$. Thus we have $m = (yz)m''$, where yz is a matrix over $\mathfrak{a}$ such that $x(yz) = 0$, and m'' is a vector over $\mathfrak{a}$. This proves

Prop. Let $\mathfrak{a}$ be a left ideal in an algebra R such that $\mathfrak{a}^2 = \mathfrak{a}$. Then $\mathfrak{a}$ is flat as R-module iff given a linear relation $xm = 0$ where x, m are resp. matrix and vector over $\mathfrak{a}$, ~~there~~ one has $m = ym'$, where y is a matrix over $\mathfrak{a}$ such that $xy = 0$, and m' is a vector over $\mathfrak{a}$.

Notice that this flatness criterion depends only on the algebra $\mathfrak{a}$, not on R, so we find

Cor. Let $\mathfrak{a}$ be a left ideal in R such that $\mathfrak{a}^2 = \mathfrak{a}$. Then $\mathfrak{a}$ is R-flat iff $\mathfrak{a}$ is $\tilde{\mathfrak{a}}$-flat.

October 28, 1993

Consider excision now. Let $I \subset R$ be an ideal, form the DG algebra $R \oplus_i \Sigma I$ with differential given by the inclusion $I \xrightarrow{i} R$. Then we have the bicomplex $C^\lambda(R \oplus_i \Sigma I)$:

$$\begin{array}{|llll}
 & & & I_\sigma^{\otimes 3} \\
R_\lambda^{\otimes 2} & I \otimes R & I_\sigma^{\otimes 2} & \\
R & I & & \\
\hline
\end{array}$$

The total complex is quis to $C^\lambda(\overset{=R/I}{A})$, the p-th column is $C^\lambda(R)$ for $p=0$, and $\Sigma^{p-1}\left[I \overset{!}{\otimes}_R\right]_\sigma^{(p)}$, where $\overset{!}{\otimes}_R$ in this case is concretely $\otimes \mathcal{B}(R) \otimes$. One has

$$\begin{tikzcd}
0 \arrow[r] & C^\lambda(R) \arrow[r] & C^\lambda(R \oplus_i \Sigma I) \arrow[r] \arrow[d, phantom, "\wr"] & \bigoplus_{p \geq 1} \Sigma^{2p-1}\left[I \overset{!}{\otimes}_R\right]_\sigma^{(p)} \arrow[r] & 0 \\
 & & C^\lambda(A) & &
\end{tikzcd}$$

If $C^\lambda(R,I) = \text{Fibre}\left\{C^\lambda(R) \longrightarrow C^\lambda(A)\right\}$ is the relative cyclic complex, then

$$C^\lambda(R,I) \sim \bigoplus_{p \geq 0} \Sigma^{2p}\left[I \overset{!}{\otimes}_R\right]_\sigma^{(p+1)}$$

Here there's a horizontal differential between the columns which has been omitted from the notation. The accurate way to say things is that $C^\lambda(R,I)$ quis $C^\lambda(R\oplus_{i}\Sigma I)/C^\lambda(R)$ which has an increasing filtration with the quotients we have described.

Excision holds when $C^\lambda(\tilde I, I)\to C^\lambda(R,I)$ is a quis. Let's consider the situation where $I^2=I$ and I is a flat $\tilde I$-module. Then we know I is a flat R-module, hence we have a quis

$$[I\overset{!}{\otimes}_R]^{(p)} \longrightarrow I^p\overset{!}{\otimes}_R = I\overset{!}{\otimes}_R$$

so excision follows provided $I\overset{!}{\otimes}_{\tilde I}\longrightarrow I\overset{!}{\otimes}_R$ is a quis.

Consider the bar construction $\mathcal{B}=\mathcal{B}(\tilde I)$:

$$\longrightarrow I\otimes I\otimes I\xrightarrow{b'} I\otimes I\xrightarrow{b'} I\xrightarrow{0} k\longrightarrow 0$$

and recall that $\tilde I\otimes_{\tilde I}\mathcal{B}\otimes_{I}\tilde I$ is the standard $\tilde I$-bimodule resolution of $\tilde I$. Thus $H_n(\mathcal{B}) = \mathrm{Tor}_n^{\tilde I}(k,k)$. H-unital means $H_n(\mathcal{B})=0$, $n\neq 0$. When I is flat as $\tilde I$ module

$$0\longrightarrow I\longrightarrow \tilde I\longrightarrow k\longrightarrow 0$$

is a flat resolution of k, so we have

$$\mathrm{Tor}_n^{\tilde I}(k,k) = 0 \qquad n\geq 2$$

$$0\to \mathrm{Tor}_1^{\tilde I}(k,k)\longrightarrow \underset{\substack{\| \\ (\tilde I/I)\otimes_{\tilde I} I = I/I^2}}{k\otimes_{\tilde I} I}\longrightarrow \underset{\substack{\| \\ k}}{k\otimes_{\tilde I}\tilde I}\longrightarrow k\longrightarrow 0$$

Thus $I=I^2$ and I flat $\tilde I$-module $\Longrightarrow$ I H-unital.

When I is H unital we have the following

$\tilde{I}$-bimodule resolution of I:

$$\xrightarrow{b'} I\otimes I\otimes I \xrightarrow{b'} I\otimes I \xrightarrow{b'} I \to 0$$

Notice that $B\otimes_{\tau} I$, $I\otimes_{-\tau} B$, $\bar{B}$ are the same up to some signs and shifting. In the H-unital case the above $\tilde{I}$-bimodule resolution of I is $I\otimes_{-\tau} B\otimes_{\tau} I$.

Return to our $I=I^2$, I flat over $\tilde{I}$ situation. In this case because of flatness one has that any bimodule of the form $I\otimes V\otimes I$ is acyclic for $H_*(R,-)$. More generally for left (resp. right) ~~modules~~ M (resp. N) we have

$$H_n(R, M\otimes N) = H_n\left(P_{\cdot} \otimes_{R\otimes R^{\circ}} (M\otimes N)\right)$$

$$= H_n\left(N\otimes_R P_{\cdot}\otimes_R M\right) = \mathrm{Tor}_n^R(N,M)$$

and these vanish for $n\neq 0$ when either N or M is flat.

Thus we can use the resolution $I\otimes_{-\tau} B\otimes_{\tau} I$ to compute $H_n(R,I) = H_n(I\overset{!}{\otimes}_R)$. We get the homology of

$$(I\otimes_{-\tau} B\otimes_{\tau} I)\otimes_R = I\otimes_R I\otimes_{-\tau} B\otimes_{\tau}$$

$$= I\otimes_{-\tau} B\otimes_{\tau}$$

This is the same as for $(I\otimes_{-\tau} B\otimes_{\tau} I)\otimes_{\tilde{I}}$, so we find $I\overset{!}{\otimes}_{\tilde{I}} \longrightarrow I\overset{!}{\otimes}_R$ is a qis as desired.

The next thing to do would be to see if this analysis works under the hypothesis of H unitality.

Excision assuming H-unitality

The idea as before is to calculate $I \overset{!}{\otimes}_R$ using the fact that $I \otimes_{-\tau} B \otimes_{\tau} I$ is a R-bimodule resolution of I. For this we need to know that the R-bimodule $I \otimes I$ (more generally $I \otimes V \otimes I$) is acyclic for $H_*(R, -)$, i.e. $H_n(R, I \otimes I) = \mathrm{Tor}_n^R(I, I) = 0, n \neq 0$.

The point here is that $R \otimes_{-\tau} B \otimes_{\tau} I$ is a free R-module resolution of I because

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_{-\tau} B \otimes_{\tau} I \arrow[r] \arrow[d, phantom, "\wr"] & R \otimes_{-\tau} B \otimes_{\tau} I \arrow[r, Rightarrow] & R/I \otimes \underbrace{(B \otimes_{\tau} I)}_{\substack{\wr \\ 0}} \arrow[r] & 0 \\ & I & & & \end{tikzcd}$$

since $(R/I) \cdot I = 0$

Thus $I \overset{!}{\otimes}_R I \sim I \overset{\text{[redacted]}}{\otimes}_R (R \otimes_{-\tau} B \otimes I) = I \otimes_{-\tau} B \otimes_{\tau} I \sim I$.

Next project should be to understand better how the precyclic object $[\Sigma I \overset{!}{\otimes}_R]^{(n+1)}$ represents the relative cyclic homology type. The [redacted] aim should be to generalize from C^λ the cyclic complex to the cyclic bicomplex [redacted] CC, or more generally to a divisible S module, so that one might be able to handle the periodic cyclic case.

Other ideas: 1) link to the pre-cyclic modules $[\Sigma E \overset{\text{[redacted]}}{\otimes}_A]^{(n+1)}$ studied already. In fact this is clear: Look at DG alg $R_R(R \oplus_i \Sigma I) = T_R(\Sigma I)$, except you have to [redacted] somehow get in the $\overset{!}{\otimes}_R$.

2) bring in $\bigoplus_n I^n = T_R(I)$ in the flat case.

October 29, 1993

If M is a flat R-module and N is an R-module of finite presentation, then one has an isomorphism

$$\mathrm{Hom}_R(N,R)\otimes_R M \xrightarrow{\sim} \mathrm{Hom}_R(N,M)$$

In effect both sides are left exact in N and they agree for $N=R$, (and there's a natural map of course).

Hence, given a surjection $L \twoheadrightarrow M$, although there need not be a lifting, there exists a lifting of any map $N \longrightarrow M$ when N is finitely presented:

$$\begin{tikzcd}
\mathrm{Hom}_R(N,R)\otimes_R L \arrow[r] \arrow[d] & \mathrm{Hom}_R(N,L) \arrow[d] \\
\mathrm{Hom}_R(N,R)\otimes_R M \arrow[r, "\sim"] & \mathrm{Hom}_R(N,M)
\end{tikzcd}$$

Recall that if $\mathfrak{a}$ is a left ideal in R then $R/\mathfrak{a}$ is projective $\Longleftrightarrow$ $\mathfrak{a}$ has a right identity. Indeed if $e \in \mathfrak{a}$ is a right identity, i.e. $xe = x$, $\forall x \in \mathfrak{a}$, then $e^2=e$ and $\mathfrak{a} = \mathfrak{a}e \subset Re \subset \mathfrak{a}$, so $\mathfrak{a} = Re$, and $\mathfrak{a}$ is a direct summand of R considered as left R-module, hence $R/\mathfrak{a}$ is projective. Conversely a R-module retraction $R \longrightarrow \mathfrak{a}$ has the form $r \mapsto re$, where e is an idempotent generating $\mathfrak{a}$, so $\mathfrak{a}$ has right identity e.

Further $R/\mathfrak{a}$ is flat $\Longleftrightarrow$ $\mathfrak{a}$ has approximate right identity, i.e. given $x_1, \dots, x_n \in \mathfrak{a}$ $\exists\, e \in \mathfrak{a}$ such that $x_i e = x_i$.

Proof: ($\Longrightarrow$) Have $R/\sum Rx_i \longrightarrow R/\mathfrak{a} \twoheadleftarrow R$

finite pres flat

so we obtain a lifting $R/\sum Rx_i \longrightarrow R$, which gives an element $y \in R$ such that $y \equiv 1 \mod \mathfrak{a}$ and $x_i y = 0$. Thus if $e = 1 - y$ we have $e \in \mathfrak{a}$ and $x_i e = x_i$, showing $\mathfrak{a}$ has approximate right identities.

($\Longleftarrow$) Let S be the monoid $1+\mathfrak{a}$ of elements of R congruent to 1 mod $\mathfrak{a}$ under multiplication. To show $R/\mathfrak{a}$ is the inductive limit of the functor from S to free R-modules sending the only object to R and $1+y \in S$ to right mult. by $1+y$ on R. In other words we are considering the full subcategory of f.g. free modules over $R/\mathfrak{a}$ consisting of the single object given by the canonical surjection $R \longrightarrow R/\mathfrak{a}$. This category is filtering - only have to check equalizer condition: given $1+y_1, 1+y_2 \in S$ want $1+z \in S$ such that $(1+y_1)(1+z) = (1+y_2)(1+z)$, i.e. $(y_1 - y_2)(1+z) = 0$, so it suffices to take $z = -e$ where $y_i e = y_i$. The inductive limit of this functor is $R/\mathfrak{a}$.

Note that the existence of approximate right identities implies $\mathfrak{a}^2 = \mathfrak{a}$.

Q: Wodzicki's triple factorization property implying excision, is it equivalent to $\mathfrak{a}^2 = \mathfrak{a}$, + $\mathfrak{a}$ flat over $\tilde{\mathfrak{a}}$?