January 2, 1992

Yesterday I got tied up with the problem of relating bivariant groups defined as the cohomology groups of a mapping complex

$$H^k(C,C') = H^k(\mathrm{Hom}(C,C'))$$

with cup product induced by composing operators, to bivariant groups defined from H^0 with cup product using the suspension automorphism. The latter construction is discussed in Verdier's paper on derived categories SGA $4\frac{1}{2}$ (at the beginning of the paper). He start with an additive category $\mathcal{A}$ equipped with 'translation' automorphism Σ & defines

$$\mathrm{Hom}^k(C,C') = \mathrm{Hom}(C, \Sigma^k C')$$

with composition

$$(g : C' \longrightarrow \Sigma^\ell C'')(f : C \longrightarrow \Sigma^k C') = ((\Sigma^k g)f : C \longrightarrow \Sigma^{\ell+k} C'')$$

I have the following thoughts about this. What's at issue here is a category whose Hom sets are $\mathbb{Z}$ graded. We can do this with additive categories or discretely, i.e. $\mathrm{Hom}^*(x,x') =$ disjoint union of $\mathrm{Hom}^k(x,x')$, $k \in \mathbb{Z}$. In the latter case this structure amounts to a functor from the category, call it $\mathcal{C}$, to the groupoid $\mathbb{Z}$. [redacted] Restricting to Hom^0 is the same as taking the fibre $\mathcal{C}^0$ over the unique object of the groupoid $\mathbb{Z}$. To say $\mathcal{C}$ arises from $\mathcal{C}^\circ$ and an automorphism means that we have a <u>scinded fibred category</u>.

It's tempting to think that systematic use of fibred category language might allow one to work with the category of spectra easily. I also feel one might profitably look at replacing $\mathbb{Z}$ by its group ring in some way.

Return to problem. Supposing

$$H^k(C, C') = H^k(\mathrm{Hom}(C, C'))$$

is used as defn, how do we identify

$$H^\circ(C, \Sigma^k C') \quad , \quad H^k(C, C')$$

and show that the cup product on the right is $g, f \longmapsto (\Sigma^{|f|} g) f$ on the left.

Regard $H^k(-,-)$ as a bifunctor on the category with morphisms defined via H°. A natural transf.

$$H^\circ(C, \Sigma^k C') \longrightarrow H^k(C, C')$$

is equivalent to one

$$H^\circ(\Sigma^{-k} C, C') \longrightarrow H^k(C, C')$$

is equivalent to one

$$H^\circ(C, C') \longrightarrow H^k(\Sigma^k C, C')$$

which by Yoneda is the same as giving element $\sigma^k_C \in H^k(\Sigma^k C, C)$

compatible with morphisms, i.e.
$\forall f : C \longrightarrow C'$ we have

$$\boxed{f\,\sigma^k_C = \sigma^k_{C'}(\Sigma^k f)} \quad \text{in } H^k(\Sigma^k C, C')$$

We need the following compatibility

$$\boxed{\sigma^{j+k}_C = \sigma^j_C \; \sigma^k_{\Sigma^j C}}$$

$$H^j(\Sigma^j C, C) \qquad H^{j+k}(\Sigma^{k+j} C, \Sigma^j C)$$

Now define the map

$$H^0(C, \Sigma^k C') \longrightarrow H^k(C, C')$$

by
$$f \longmapsto \sigma^k_{C'} f$$

Then
$$H^0(C', \Sigma^j C'') \longrightarrow H^j(C', C'')$$

$$g \longmapsto \sigma^j_{C''}\, g$$

Calculate product on the right:

$$\left(\sigma^j_{C''} g\right)\left(\sigma^k_{C'} f\right) = \sigma^j_{C''}\, \sigma^k_{\Sigma^j C''}\left(\Sigma^k g\right) f$$

$$= \sigma^{j+k}_{C''}\left(\left(\Sigma^k g\right) f\right)$$

Because $\sigma^0_C = id_C$ + $\mathbb{Z}$ is a group these maps $f \longmapsto \sigma^k_{C'} f$ are bijections.

January 14, 1993

I want to summarize some observations made while trying to understand how bivariant cyclic cohomology $HC^*(M, M')$ together with cup product reduces to degree zero and the suspension. Simple situation: the bivariant cohomology with cup product

$$H^*_\Lambda(M, M') = H^*(\mathrm{Hom}_\Lambda(M, M'))$$
$$H^*_S(P, P') = H^*(\mathrm{Hom}_S(P, P'))$$

for mixed complexes and S-modules.

The first point (included in paper) is that these [redacted] are equivalent to graded homotopy categories $Ho^*\mathcal{C}_\Lambda$, $Ho^*\mathcal{C}_S$, that there are canonical elements

$$\sigma_{k,M} \in H^{-k}_\Lambda(M, M[k]) \quad , \quad \sigma_{k,P} \in H^{-k}_S(P, P[k])$$

which allow one to [redacted]

[redacted]

a) define Σ as an automorphism of $Ho\,\mathcal{C}_\Lambda$, $Ho\,\mathcal{C}_S$

b) identify the graded homotopy categories with the graded cats constructed from the ordinary homotopy categories and Σ

This is a mouthful, but the upshot is that [redacted] it is better to start [redacted] with the graded homotopy categories than to obtain them [redacted] by constructing Σ as a functor. This view is also supported by the idea that the graded categories are bifibred categories over the group $\mathbb{Z}$ in some sense.

Next consider the functors $\mathcal{B}$, $\Lambda\otimes -$. The second point is that one should think of adjoint functors between two categories as a bifunctor which is representable in either variable, and then it is natural to form the union of the two categories glued together with new maps given by the bifunctor. In the example being considered the bifunctor is

$$H^*_\tau(P,M) = H^*_\Lambda(\Lambda\otimes P, M) = H^*_S(P, \mathcal{B}M)$$

so we have $(Ho^*\mathcal{C}_S)\cup(Ho^*\mathcal{C}_\Lambda)$ with three kinds of maps: $H^*_\Lambda(M,M')$, $H^*_S(P,P')$, $H^*_\tau(P,M)$. In this category we still have Σ on objects and the σ classes for objects, so we conclude that $(Ho^*\mathcal{C}_S)\cup(Ho^*\mathcal{C}_\Lambda)$ <u>reduces to degree 0 and</u> Σ. It's not necessary to specify the canonical isomorphisms $\mathcal{B}(M[k]) \simeq (\mathcal{B}M)[k]$.

A moral here is that we should probably start thinking in terms of categories whose Hom sets are complexes like

$$\mathrm{Hom}_\Lambda(M,M'), \quad \mathrm{Hom}_\tau(P,M), \quad \mathrm{Hom}_S(P,P').$$

January 18, 1993

List ideas which might be relevant for the program (future program) of constructing homotopy equivalences with the aid of connections.

It seems that the example $\mathcal{C}_\Lambda \rightleftarrows \mathcal{C}_S$ should be studied very carefully, actually the homotopy categories $Ho\,\mathcal{C}_\Lambda \rightleftarrows Ho\,\mathcal{C}_S$ are better.

We have adjoint functors $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ in the homotopy category situation such that the following holds. The adjunction map $\alpha_X : FGX \to X$ is an isomorphism when $X = FY$, and $\beta_Y : Y \to GFY$ is an isomorphism when $Y = GX$. Thus from

$$FY \xrightarrow{F(\beta_Y)} FGFY \underset{\sim}{\xrightarrow{\alpha_{FY}}} FY$$

$$GX \underset{\sim}{\xrightarrow{\beta_{GX}}} GFGX \xrightarrow{G(\alpha_X)} GX$$

(recall these compositions are the identity)

these conditions are equivalent to

$$F(\beta_Y) \text{ isom } \quad \forall Y$$

$$G(\alpha_X) \text{ isom } \quad \forall X.$$

Is it interesting to allow isomorphisms in the graded homotopy category which are inhomogeneous? Units in a bivariant cohomology ring?

February 9, 1993

Suppose T is an operator on V satisfying a polynomial relation $(fg)(T)=0$ where $f,g \in \mathbb{C}[x]$ are relatively prime.

Claim

(i) $V = \mathrm{Ker}_V f(T) \oplus f(T)V$

where $f(T)$ is invertible on $f(T)V$

(ii) Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 0 & 0 \\ 0 & f(T)^{-1} \end{pmatrix}$ relative to this splitting. Then P, R are polynomials in T.

(iii) $\mathrm{Ker}_V f(T) = g(T)V$, $\mathrm{Ker}_V g(T) = f(T)V$.

This is standard starting from a choice of $a,b \in \mathbb{C}[x]$ such that

$$af + bg = 1.$$

One puts $P = (bg)(T)$. Then $f(T)P = 0$

$$1-P = (af)(T) = a(T)f(T) = f(T)a(T)$$

so

$$\mathrm{Im}\,P \subset \mathrm{Ker}\,f(T) \subset \mathrm{Ker}(1-P) \subset \mathrm{Im}\,P$$

$$\mathrm{Im}(1-P) \subset \mathrm{Im}\,f(T) \subset \mathrm{Ker}\,P \subset \mathrm{Im}(1-P)$$

etc. But what is worth mentioning is the link with special contracting homotopies. Write f for $f(T)$ etc. Let

$$d = \begin{pmatrix} & g \\ f & \end{pmatrix} \qquad h = \begin{pmatrix} & a \\ b & \end{pmatrix}$$

Then

$$dh + hd = \begin{pmatrix} gb & \\ & fa \end{pmatrix} + \begin{pmatrix} af & \\ & bg \end{pmatrix}$$

$$= \begin{pmatrix} gb+af & 0 \\ 0 & fa+bg \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Making this contracting homotopy special;

$$hdh = \begin{pmatrix} & a \\ b & \end{pmatrix}\begin{pmatrix} gb & 0 \\ 0 & fa \end{pmatrix} = \begin{pmatrix} 0 & afa \\ bgb & 0 \end{pmatrix}$$

then gives the operator $R = afa$ which is 0 on PV and the inverse of f on $P^\perp V$:

$$(afa)P = (afa)(gb) = 0$$

$$f(afa) = fa(1-gb) = fa = 1-P.$$

March 18, 1993

In connection with the structure on $X(R)$ the following seems to be worth recording. We have two pairings

$$\begin{tikzcd} & R\otimes R \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ X_+ \arrow[rr, shift right, "d"'] & & X_- \arrow[ll, shift right, "b"'] \end{tikzcd}$$

$$\mu(x\otimes y) = xy$$
$$\nu(x\otimes y) = \natural(x\,dy)$$

such that

$$b\nu(x\otimes y) = \mu(x\otimes y) - \mu(y\otimes x)$$
$$d\mu(x\otimes y) = \nu(x\otimes y) + \nu(y\otimes x)$$

The universal situation is

$$\begin{tikzcd} & R\otimes R \arrow[dl] \arrow[dr] & \\ R\otimes R \arrow[rr, shift right, "1+\sigma"'] & & R\otimes R \arrow[ll, shift right, "1-\sigma"'] \end{tikzcd}$$

This is the degree 2 piece of the X-complex of $T(V)$ in the case $V = R$. The other pieces are related to associativity properties which I haven't yet mentioned, namely

$$\mu(xy, z) = \mu(x, yz)$$
$$\nu(x, yz) = \nu(xy, z) + \nu(zx, y)$$

April 9, 1993

Consider $C(A)_n = A^{\otimes n+1}$, $n \geq 0$
with differential b, let $s(a_0,\dots,a_n) = (1,a_0,\dots,a_n)$.
We have

$$[b,s] = \underbrace{[b',s]}_{1} + [c,s]$$

$$cs(a_0,\dots,a_n) = c(1,a_0,\dots,a_n) = (-1)^{n+1}(a_n,a_0,\dots,a_{n-1})$$

$$\therefore \quad \boxed{cs = -\lambda}$$

$$sc(a_0,\dots,a_n) = (-1)^n(1,a_na_0,a_1,\dots,a_{n-1})$$

Thus

$$\boxed{\begin{aligned} [b,s] &= 1 - K \quad \text{where} \\ K &= \lambda - sc \\ K(a_0,\dots,a_n) &= (-1)^n(a_n,a_0,\dots,a_{n-1}) \\ &\quad + (-1)^{n-1}(1,a_na_0,a_1,\dots,a_{n-1}) \end{aligned}}$$

so we have defined the Karoubi operator on unnormalized chains.

Does K on $A^{\otimes n+1}$ satisfy a polynomial identity? Let

$$W_i = A^{\otimes i-1} \otimes 1 \otimes A^{\otimes n-i} \subset A^{\otimes n+1}, \qquad 0 \leq i \leq n$$

and recall that $W_1 + W_2 + \cdots + W_n$ is the degenerate subspace. Also recall that

$$A^{\otimes n+1}/W_1 + \cdots + W_n = A \otimes \bar{A}^{\otimes n} = \Omega^n A$$

Take $\xi = (a_0,\dots,a_{i-1},1,a_{i+1},\dots,a_n) \in W_i$ with $1 \leq i \leq n-1$.

Then
$$K\xi = (-1)^n(a_n,a_0,\dots,a_{i-1},1,a_{i+1},\dots,a_{n-1}) + (-1)^{n-1}(1,a_na_0,\dots,a_{i-1},1,a_{i+1},\dots,a_{n-1}) \in W_{i+1}$$

If $i = n$: $\xi = (a_0,\dots,a_{n-1},1)$, then

$$K\xi = (-1)^n(1, a_0, \ldots, a_{n-1}) + (-1)^{n-1}(1, a_0, \ldots, a_{n-1}) = 0$$

so we have

$$\boxed{\begin{array}{ll} K W_i \subset W_{i+1} & 1 \le i \le n-1 \\ K W_n = 0 & \end{array}}$$

It follows that

$$\begin{aligned} K^n(W_1 + W_2 + \cdots + W_n) &\subset K^{n-1}(W_2 + W_3 + \cdots + W_n) \\ &\subset K^{n-2}(W_3 + \cdots + W_n) \\ &\cdots \\ &\subset K W_n = 0. \end{aligned}$$

But we know already that $(K^n-1)(K^{n+1}-1) = 0$ on $A^{\otimes n+1}/W_1 + \cdots + W_n = \Omega^n A$, so we conclude

$$\boxed{K^n(K^n-1)(K^{n+1}-1) = 0 \quad \text{on } A^{\otimes n+1}}$$

In fact we have the filtration

$$\begin{array}{ccccccccccc} A^{\otimes n+1} & \supset & W_0 + \cdots + W_n & \supset & W_1 + \cdots + W_n & \supset & W_2 + \cdots + W_n & \supset & \cdots \supset W_n & \supset & 0 \\ & | & & | & & | & & | & & | & \\ & K^{n+1}-1 & & K^n - 1 & & K & & K & & K & \end{array}$$

where the layers are killed by the polynomials indicated.

Now go back to $[b, s] = 1 - K$. Since $b^2 = 0$ we have $[b, K] = 0$:
$$\left| \begin{array}{l} b(1-K) = b(bs+sb) = bsb \\ (1-K)b = (bs+sb)b = bsb \end{array}\right.$$

but K does not commute with s.

So K is an endomorphism of the complex $C = C(A)$ satisfying a polynomial identity in each degree, hence we spectral decomposition into [redacted] generalized eigenspaces

$$C = \bigoplus_{z \in \mathbb{C}} \underbrace{\bigcup_n \mathrm{Ker}\left((\kappa - z)^n ; C\right)}_{C_z}$$

which are subcomplexes. From the polynomial relation we know that

$$C_z = \begin{cases} \mathrm{Ker}\left((\kappa-1)^2 ; C\right) & z = 1 \\ \mathrm{Ker}\left((\kappa - \zeta) ; C\right) & z = \zeta \text{ root of unity} \neq 1. \\ \bigcup_n \mathrm{Ker}\left(\kappa^n ; C\right) & z = 0 \\ 0 & z \text{ otherwise} \end{cases}$$

Consider next the homology of C_z. Now in general we know that the spectral decomposition restricts to that of any κ-invariant subspace or quotient space. Thus

$$H_* C = \bigoplus_z H_* C_z$$

is the spectral decomposition of κ acting on $H_* C$. But κ is the identity on $H_* C$ since it is homotopic to the identity on C. Thus

$$\boxed{C_z \text{ is acyclic} \quad z \neq 1}$$

Let's now use $[b, s] = 1 - \kappa$ to construct a contracting homotopy for C_z, $z \neq 1$. One has to proceed differently than in the case of Ω since s doesn't commute with κ. Let P_z be the spectral projection onto C_z.

$$[b, P_z s P_z] = P_z [b, s] P_z = P_z (1 - \kappa) P_z = (1 - \kappa) P_z$$

Thus $(1-K)$ on C_z is homotopic to zero with homotopy operator $P_z s P_z$. If $z \neq 1$, then $1-K$ is invertible on C_z and we get a contracting homotopy using the Green's operator G for $1-K$ as follows

$$[b, GP_z s P_z] = G[b, P_z s P_z] = G(1-K)P_z = P_z .$$

Lumping all $z \neq 1$ together, we proceed as follows. Let $P = P_1$, $P^\perp = 1-P$. Then

$$\begin{aligned} [b, GP^\perp s P^\perp] &= GP^\perp [b,s] P^\perp \\ &= GP^\perp (1-K) P^\perp \\ &= P^\perp \end{aligned}$$

so that $GP^\perp s P^\perp = GsP^\perp$ is a contracting homotopy for $P^\perp C$.

Notice now that we have a proof of the normalization theorem for the Hochschild complex. Namely, C_0 is the degenerate subcomplex (here $C_0 = \bigcup_n \mathrm{Ker}(K^n; C)$) and we have shown that it is acyclic.

Let's go over this carefully. Put $D = C_0 =$ $W_1 + W_2 + \cdots$ for the degenerate subspace. Then K on D is locally nilpotent, and it is invertible on C/D. We are using the spectral projection P_0 to split C into $P_0^\perp C \xrightarrow{\sim} C/D$ and $P_0 C = D$. Also we need the inverse of $1-K$ on D which is given by the geometric series. We want to contract D. We use

$$[b, s] = 1-K \qquad \cancel{[b, P_0 s P_0]}$$

$$[b, P_0 s P_0] = P_0(1-K)P_0 = (1-K)P_0$$

$$[b, \sum_{n \geq 0} K^n P_0 s P_0] = P_0 .$$

There is something noncanonical going on which I would like to understand. I have noticed that $(1-\lambda)sN_\lambda$ is not Connes's B operator on $C(A)$. However Kassel has nicely observed that one should consider s to be any contracting homotopy for b', and in this ~~[illegible]~~ perspective it makes sense for an H-unital algebra. We also have besides the s above, which puts a 1 in front, the operator which puts 1 on the right with the appropriate sign.

Let's work this out using the description $C(A) = T_A(A \otimes A)$ with b' the degree -1 derivation extending the product $A \otimes A \longrightarrow A$. Let $\xi = 1 \otimes 1$, so that $b'(\xi) = 1$. Then

$$s(x) = \xi x \qquad b'(\xi x) = x - \xi b' x$$

Put $h(x) = (-1)^{|x|} x \xi$. Then

$$b' h x = b'(-1)^{|x|} x \xi = \underbrace{(-1)^{|x|}(b'x)\xi}_{-hb'x} + x$$

So $h(a_0, \ldots, a_n) = (-1)^n (a_0, \ldots, a_n, 1)$

$$\lambda h(a_0, \ldots, a_n) = -(1, a_0, \ldots, a_n) = -s(a_0, \ldots, a_n)$$

and $\boxed{h = -\lambda^{-1} s}$. If we use this contracting homotopy for $C^{b'}$, then the corresponding

B operator is

$$B = (1-\lambda)(-\lambda^{-1}s)N_\lambda$$
$$= (1-\lambda^{-1})sN_\lambda$$

and $(1-\lambda^{-1})s(a_0,\dots,a_n)$

$$= (1-\lambda^{-1})(1,a_0,\dots,a_n)$$
$$= (1,a_0,\dots,a_n) - (-1)^{n+1}(a_0,\dots,a_n,1)$$

This is Connes B_0 operator (transposed to chains).

Suppose we use h in place of s as a homotopy operator. Observe

$$(ch)(a_0,\dots,a_n) = c(-1)^n(a_0,\dots,a_n,1)$$
$$= -(a_0,\dots,a_n)$$

$\therefore$ $\boxed{ch = -1}$ (Recall $\boxed{cs = -\lambda \mid h = -\lambda^{-1}s}$)

Then $[b,h] = [b',h] + [c,h]$
$$= \underline{1} + ch + hc = hc$$

and $(hc)^2 = h(-1)c = -hc$. So $-hc$ is an idempotent. In fact

$$(-hc)h = h$$

so that $\mathrm{Im}(-hc) = \mathrm{Im}(h) = A^{\otimes n}\otimes 1 = W_n$ in degree n. Thus all we learn is that W_n is acyclic.

Any further progress in this direction seems to require an understanding of the possible contracting homotopies h such that $[b',h]=1$.

We recall the map

$$\bar{\Omega}\tilde{A} \xrightarrow[(1\ \ (1-\lambda)h)]{} C(A)$$

Check:

$$(1\ \ (1-\lambda)h)\begin{pmatrix} b & (1-\lambda) \\ & -b' \end{pmatrix} = (b\ \ (1-\lambda)(1-hb'))$$

$$= (b\ \ (1-\lambda)b'h) = b\,(1\ \ (1-\lambda)h)$$

$$(1\ \ (1-\lambda)h)\begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix} = ((1-\lambda)hN_\lambda\ \ 0)$$

$$= \underbrace{((1-\lambda)hN_\lambda)}\,(1\ \ (1-\lambda)h)$$

the B you must use in order that $(1\ \ (1-\lambda)h)$ is a map of mixed complexes.

Observe that

$$(1\ \ (1-\lambda)h)\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{d \text{ on } \bar{\Omega}\tilde{A}} = ((1-\lambda)h\ \ 0)$$

so if we could find h such that

$$[b', h] = 1 \qquad ((1-\lambda)h)^2 = 0.$$

Then d on $\bar{\Omega}\tilde{A}$ would descend to $(1-\lambda)h$ and K would descend to give a K on $C(A)$.

Compute κ^{-1} on Ω and lift

$$\kappa^{-1}(a_0\, da_1 \cdots da_n)$$
$$= \kappa^{-1}\left(d(a_0a_1)\, da_2 \cdots da_n - da_0\, a_1\, da_2 \cdots da_n\right)$$
$$= (-1)^{n-1} da_2 \cdots da_n\, d(a_0a_1) + (-1)^n a_1\, da_2 \cdots da_n\, da_0$$

Define $\kappa^*(a_0, \ldots, a_n)$

$$= (-1)^n(a_1, a_2, \ldots, a_n, a_0) + (-1)^{n-1}(1, a_2, \ldots, a_n, a_0a_1)$$
$$= \lambda^{-1}(a_0, \ldots, a_n) + (-1)^{n-1} s(a_2, \ldots, a_n, a_0a_1)$$
$$= \text{———} + s\lambda^{-1}(a_0a_1, a_2, \ldots, a_n)$$
$$= \text{———} + s\lambda^{-1}c(-1)^n(a_1, a_2, \ldots, a_n, a_0)$$
$$= \text{———} + s\lambda^{-1}c\lambda^{-1}(a_0, \ldots, a_n).$$

Thus we have $\boxed{\kappa = \lambda - sc}$ $\boxed{\kappa^* = \lambda^{-1} + s\lambda^{-1}c\lambda^{-1}}$

Recall the identities $\boxed{cs = -\lambda}$ $\boxed{c\lambda^{-1}s = 1}$

$$\kappa\kappa^* = (\lambda - sc)(\lambda^{-1} + s\lambda^{-1}c\lambda^{-1})$$
$$= 1 + \lambda s\lambda^{-1}c\lambda^{-1} - sc\lambda^{-1} \underbrace{- s\underbrace{(cs)}_{-\lambda}\lambda^{-1}c\lambda^{-1}}_{+sc\lambda^{-1}}$$

$$\kappa\kappa^* = 1 + \underbrace{\lambda s\lambda^{-1}c\lambda^{-1}}$$

$-$a projection ~~with~~ with image $= \mathrm{Im}(\lambda s)$

since $(-\lambda s\lambda^{-1}c\lambda^{-1})(\lambda s) = -\lambda s\lambda^{-1}\underbrace{cs}_{-\lambda} = \lambda s$

$$\kappa^*\kappa = (\lambda^{-1} + s\lambda^{-1}c\lambda^{-1})(\lambda - sc)$$
$$= 1 - \lambda^{-1}sc + \cancel{s\lambda^{-1}c} - \cancel{s\lambda^{-1}c}\underbrace{\lambda^{-1}s}_{1}c$$
$$= 1 - \underbrace{\lambda^{-1}sc}$$

a projection with image $= \mathrm{Im}(\lambda^{-1}s)$

since $(\lambda^{-1}sc)\lambda^{-1}s = \lambda^{-1}s.$

The preceding page is probably misguided as there's no reason for κ^* to commute with b.

A better approach is to start with a homotopy operator joining κ^{-1} to the identity on Ω and lift this homotopy operator. Thus on Ω we have

$$[b, -\kappa^{-1}d] = -\kappa^{-1}(1-\kappa) = 1-\kappa^{-1}$$

and $-\kappa^{-1}d = d(-\kappa^{-1})$. So what should I lift $-\kappa^{-1}d$ to?

$$\begin{aligned}(\kappa^{-1}d)(a_0\, da_1 \cdots da_n) &= \kappa^{-1}\, da_0 \cdots da_n \\ &= (-1)^n da_1 \cdots da_n\, da_0 \\ &\Leftrightarrow (-1)^n (1, a_1, \ldots, a_n, a_0) \\ &= s\lambda^{-1}(a_0, \ldots, a_n)\end{aligned}$$

This yields $-s\lambda^{-1}$ as lifting

The other possibility is $-\lambda^{-1}s$ which we know is the other contracting ~~homotopy~~ operator for b'. We have

$$[b, -\lambda^{-1}s] = \underbrace{[b', -\lambda^{-1}s]}_{1} - \underbrace{c\lambda^{-1}s}_{1} - \lambda^{-1}sc$$

$$\therefore\ [b, -\lambda^{-1}s] = -\lambda^{-1}sc = 1-(1+\lambda^{-1}sc)$$

Note $\kappa^* s = (\lambda^{-1} + s\lambda^{-1}c\lambda^{-1})s = \lambda^{-1}s + s\lambda^{-1}$

so maybe you want to combine $-s\lambda^{-1}$ and $-\lambda^{-1}s$

Note that if $h^2=0$, then h commutes with $[b,h]$.

Consider $[b,s] = 1-\kappa$, $\kappa = \lambda - sc$ on $C(A)$. We saw that $\kappa^n(\kappa^n-1)(\kappa^{n+1}-1)=0$ on $C_n = A^{\otimes n+1}$, with $\kappa^n=0$ on $D_n = W_1+\dots+W_n$ and $(\kappa^n-1)(\kappa^{n+1}-1)=0$ on C_n/D_n, hence κ invertible on C_n/D_n. We know then that C_n splits into $C_n = K_n \oplus D_n$ where κ is invertible on K_n. It seems that this splitting is what one obtains from the normalization theorem. To check this carefully.

Recall that C is a simplicial module where the faces are

$$d_i(a_0,\dots,a_n) = (\dots, a_ia_{i+1}, \dots) \qquad 0 \le i \le n-1$$
$$d_n(a_0,\dots,a_n) = (a_na_0, a_1, \dots, a_{n-1})$$

and the degeneracies are

$$s_i(a_0,\dots,a_n) = (\dots, a_i, 1, a_{i+1}, \dots) \qquad i = 0,\dots,n.$$

I think this is OKAY. If so then

$$D_n = \sum_{i=0}^{n-1} s_iC_{n-1} \qquad s_iC_{n-1} = s_iA^{\otimes n} = A^{\otimes i+1}\otimes 1 \otimes A^{\otimes n-i-1}$$
$$= W_{i+1} \text{ in } A^{\otimes n+1}$$
$$= \sum_{i=1}^{n} W_i$$

so a complement to D_n in C_n is $\bigcap_{i=0}^{n-1} \operatorname{Ker} d_i$ I think.

Calculate the behavior of the d_i to κ

Recall $\kappa(a_0,\dots,a_n) = (-1)^n(a_n,a_0,\dots,a_{n-1}) + (-1)^{n-1}(1,a_na_0,a_1,\dots,a_{n-1})$

Then $\boxed{d_0\kappa = 0}$

Suppose next $1 \leq i \leq n-1$. Then

$$(-1)^i d_i K(a_0, \dots, a_n)$$

$$= (-1)^i d_i (-1)^n (a_n, a_0, \dots, a_{n-1}) + (-1)^i d_i (-1)^{n-1} (1, a_n a_0, a_1, \dots, a_{n-1})$$

$$= (-1)^{i+n} (a_n, a_0, \dots, a_{i-1}a_i, \dots, a_{n-1}) + (-1)^{i+n-1} (1, a_n a_0, a_1, \dots, a_{i-1}a_i, \dots, a_{n-1})$$

$$= (-1)^{i+1} K(a_0, \dots, a_{i-1}a_i, \dots, a_n)$$

$$= K(-1)^{i-1} d_{i-1}(a_0, \dots, a_n). \qquad \boxed{d_0 K = 0}$$

Thus we have $\boxed{(-1)^i d_i K = K(-1)^{i-1} d_{i-1} \qquad 1 \leq i \leq n-1}$

So if $\xi \in \bigcap_{i=0}^{n-1} \operatorname{Ker} d_i$, then we have

$$d_i K \xi = 0 \quad \text{for} \quad i = 0, 1, \dots, n-1$$

so it's clear that $K\left(\bigcap_{i=0}^{n-1} \operatorname{Ker} d_i\right) \subset \left(\bigcap_{i=0}^{n-1} \operatorname{Ker} d_i\right)$. Thus it seems so that the complement to D_n given by the simplicial normalization theorem is stable under K, so this must be [redacted] the generalized eigenspace corresponding to the set of nonzero eigenvalues.

April 15, 1993

About Connes's B_0.

Let's write out what might go in the revision of part 2.

I think one should consider cochains f ($f \in (\Omega A)^*$) satisfying $fbd = 0$ in preference to cocycles $fb + fB = 0$. The first reason is that $f(b+B) = 0 \implies fbd = -fBd = 0$, so $fbd = 0$ is a less restrictive condition. Secondly $fbd = 0$ is a homogeneous condition, i.e. it holds iff it holds for all the homogeneous components.

Next we have

Lemma: Assume $fbd = 0$. Then TFAE

1) f harmonic: $fP = f$
2) f K-invariant: $f(1-K) = 0$
3) fd K-invariant: $fd(1-K) = 0$.

It's perhaps better to consider the three conditions on their own, and to discuss implications between them. Obviously

$$f \text{ } K\text{-invariant} \implies fd \text{ } K\text{-invariant} \qquad (\text{as } [d,K]=0)$$

$$\underline{\qquad\qquad} \implies f \text{ harmonic} \qquad \left(\begin{array}{l}\text{as } \operatorname{Im}(1-K) \supset \operatorname{Im} P^\perp \\ \text{or } P^\perp = (1-K)G\end{array}\right)$$

Claim:

ⓐ If $fbB = 0$, then (f harm. $\implies$ f K-invariant)

ⓑ If $fbd = 0$, then (fd K-inv $\implies$ f K-invariant)

Why? Can suppose f homogeneous: $f \in (\Omega^n A)^*$. Then the identity $K^{n(n+1)} = 1 + bB$ on $\Omega^n A$ shows K has

~~[illegible]~~ finite order on $\Omega^n / bB\Omega^n$, hence P is given by averaging wrt κ. Thus when $fbB=0$

$$fP = \frac{1}{n(n+1)} \sum_{0}^{n(n+1)-1} f\kappa^j$$

so $fP = f \iff f(1-\kappa) = 0$.

(b) Can suppose $f \in (\Omega^n A)^*$. Since

$$f(1-\kappa) = f(bd+db) = fdb$$

we have to show $fdb = 0$. But if fd is κ-invariant, then

$$fdb = \frac{1}{n+1} fBb = -\frac{1}{n+1} fbB = -\frac{1}{n+1} fbd \sum_{j=0}^{n-1} \kappa^j = 0.$$

~~A longer argument is that $fbd \Rightarrow f\kappa^n = f$ (as $\kappa^n = 1 - bd\kappa^{-1}$ on Ω^n), so $Pf = \frac{1}{n}\sum_0^{n-1} f\kappa^j$~~

A longer argument uses

$$f - fP = fdGb + \cancel{fbdG}$$

$$= fd(1-\kappa)G^2 b$$

so that fd κ-invariant $\Longrightarrow$ f harmonic. Then note that $fbd=0 \Longrightarrow fbB = \sum fbd\kappa^j = 0$ so part (a) yields f κ-invariant.

An additional point related to ~~[illegible]~~ the longer argument is that

$$fbd = 0 \Longrightarrow f\kappa^n = f \qquad \text{as } \kappa^n = 1 + bd\kappa^{-1} \text{ on } \Omega^n$$

$$\Longrightarrow fP = \frac{1}{n}\sum_{j=0}^{n-1} f\kappa^j$$

(This point is suggested by the desire to link P to Connes's normalizing process.)

Next consider $A = \tilde{a}$ where a is unital, and identify $\bar{\Omega}\tilde{a}$ with
$C(a) \oplus \Sigma C(a)$, more precisely

$$(\bar{\Omega}\tilde{a})_n = a^{\otimes n+1} \oplus a^{\otimes n}$$

where $a^{\otimes n} = 0$ for $n \leq 0$. We have Kassel's map of mixed complexes

$$J = (1 \;\; (1-\lambda)h) : \bar{\Omega}\tilde{a} \twoheadrightarrow C(a)$$

where h is a contracting homotopy for b':

$$[b', h] = 1$$

and $C(a)$ has the differentials b and $B = (1-\lambda)hN_\lambda$.
We obtain Connes's B by taking $h = -\lambda^{-1}s$.
We have $B = B_0 N_\lambda$ where $B_0 = (1-\lambda)(-\lambda^{-1}s) = (1-\lambda^{-1})s$.
Let's record Connes's identity

$$\circledast \qquad bB_0 + B_0 b' = 1 - \lambda$$

i.e.
$$bB_0 + B_0 b' = b(1-\lambda)h + (1-\lambda)hb'$$
$$= (1-\lambda)(b'h + hb') = 1 - \lambda .$$

<u>Claim:</u> Pulling back via $J = (1 \;\; B_0)$ gives a bijection between

1) $\psi \in C(a)^*$ such that $\left| \begin{array}{l} \psi b B_0 = 0 \\ \psi B_0 (1-\lambda) = 0 \end{array} \right.$

2) $f \in (\bar{\Omega}\tilde{a})^*$ such that $\left| \begin{array}{l} fbd = 0 \\ fd(1-\kappa) = 0 \end{array} \right.$

Proof. Let $f = (\psi \;\; \varphi)$ and note that

$$fbd = (\psi \;\; \varphi)\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \psi b & \psi(1-\lambda)-\varphi b' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \psi(1-\lambda)-\varphi b' & 0 \end{pmatrix}$$

$$fd = (\varphi \;\; 0) \qquad fd(1-\kappa) = (\varphi(1-\lambda) \;\; 0)$$

Suppose ψ given satisfying $\psi b B_0 = \psi B_0(1-\lambda) = 0$. Then $J\psi = (\psi \;\; \underbrace{\psi B_0}_{\varphi})$. Clearly we have $fd(1-\kappa) = 0$. To get $fbd = 0$ we must show $\psi(1-\lambda) = \psi B_0 b'$. But by ⊛

$$\psi(1-\lambda) = \psi B_0 b' + \cancelto{0}{\psi b B_0}.$$

Conversely suppose given $f = (\psi \;\; \varphi)$ satisfying $fbd = 0$, $fd(1-\kappa) = 0$, i.e. $\psi(1-\lambda) = \varphi b'$, $\varphi(1-\lambda) = 0$. We will show $\varphi b = 0$. Two proofs: We know (see above) that $fbd = 0$, $fd(1-\kappa) = 0 \Longrightarrow fdb = 0$. But

$$fdb = (\psi \;\; \varphi)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix} = (\varphi \;\; 0)\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix} = (\varphi b \;\; \varphi(1-\lambda)).$$

so $\varphi b = 0$. Directly, can assume $\varphi \in (C(a)_n)^*$. Then $\varphi N_\lambda = (n+1)\varphi$, and

$$(n+1)\varphi b = \varphi N_\lambda b = \varphi b' N_\lambda = \psi(1-\lambda) N_\lambda = 0.$$

so $\varphi b = 0$. Then we have using $b = b' + c$, $c\lambda^{-1}s = 1$

$$\varphi = \varphi c\lambda^{-1}s = -\varphi b'\lambda^{-1}s = \psi(1-\lambda)(-\lambda^{-1}s) = \psi B_0$$

so $f = \psi J$. Clearly we have $\psi B_0 = \psi$ is λ-invariant, so it remains to check that $\psi b B_0 = 0$. But again by $*$ we have

$$\psi(1-\lambda) = \underbrace{\psi B_0}_{b} b' + \psi b B_0$$

so we win.

We see that B_0 is analogous to d in some sense. First of all B_0 on $C(\mathcal{A})$ lifts d on $\Omega\mathcal{A}$. Secondly although $d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ on $\bar{\Omega}\tilde{\mathcal{A}}$ does not descend to $C(\mathcal{A})$, the operator $\begin{pmatrix} 0 & 0 \\ 1 & B_0 \end{pmatrix}$ does descend to B_0:

$$J\begin{pmatrix} 0 & 0 \\ 1 & B_0 \end{pmatrix} = \begin{pmatrix} 1 & B_0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & B_0 \end{pmatrix}$$

$$= \begin{pmatrix} B_0 & B_0^2 \end{pmatrix}$$

$$= B_0 J$$

The big problem is that $B_0^2 \neq 0$.

Question: We have analyzed K on $C(\mathcal{A})$ defined by $[b, s] = 1 - K$. Recall $K = \lambda - sc$. What about $[b, B_0]$?

We have

$$[b, B_0] = bB_0 + B_0 b' + B_0 c$$

$$= 1 - \lambda + (1-\lambda^{-1})sc = 1 - (\lambda - sc) - \lambda^{-1} sc$$

(The following might be useful:
$$[b, B_0] = 1 - \lambda - \lambda^{-1}sc + \lambda(\lambda^{-1}sc)$$
$$= (1-\lambda)(1-\lambda^{-1}sc).)$$

Let's analyze the "new" Karoubi type operator
$$\lambda - sc + \lambda^{-1}sc \quad = K + \lambda^{-1}sc$$
Note that the identity $c\lambda^{-1}s = 1$ implies $\lambda^{-1}sc$ is a projector with image
$$\mathrm{Im}\, \lambda^{-1}c = \mathcal{A}^{\otimes n} \otimes 1 = W_n \subset \mathcal{A}^{\otimes n+1}$$
and $\mathrm{Ker}(\lambda^{-1}sc) = \mathrm{Ker}(c)$.

Recall that $KW_i \subset W_{i+1}$ for $1 \leq i \leq n-1$ and that $KW_n = 0$. Now $\lambda^{-1}sc$ carries $\mathcal{A}^{\otimes n+1}$ into $W_n = a^{\otimes} \otimes 1$. Thus $K + \lambda^{-1}sc$ carries $W_i + \cdots + W_n$ into $W_{i+1} + \cdots + W_n$ for $1 \leq i \leq n-1$, and it is the identity on W_n. Since $K + \lambda^{-1}sc \equiv K$ modulo $W_1 + \cdots + W_n$ we thus see that $K' = K + \lambda^{-1}sc$ satisfies the polynomial relation
$$\boxed{(K'-1)\, K'^{\,n-1}(K'^{\,n}-1)(K'^{\,n+1}-1) = 0 \quad \text{on} \quad C(\mathcal{A})_n = a^{\otimes n+1}}$$

<u>Discussion</u>: My feeling at this point is that further calculations with $C(a)$ are unlikely to lead anywhere on the simplicial normalization problem for entire cyclic cohomology, i.e. whether the Getzler-Szenes and Connes definitions are equivalent.

Most promising approach: We know (I think) that there are heqs for the entire topology

$$\begin{array}{ccc} \bar{\Omega}\tilde{A} & \qquad & \Omega A \\ \wr & & \wr \\ P\bar{\Omega}\tilde{A} & & P\Omega A \end{array}$$

On the other hand the harmonic complexes perhaps can be understood in terms of $X(R\tilde{A})$ and $X(RA)$ respectively. I think the key step will be to construct a lifting

$$\hat{\Omega}A \longrightarrow \hat{\Omega}\tilde{A}$$

corresponding to the nonunital homomorphism $A \to \tilde{A}$. This should be related to lifting $\hat{R}A$ into $\hat{R}(\tilde{A})$, a problem I started looking at while at MIT. (I recall that there were some choices (non canonical stuff) which were confusing.)

~~It seems that the specific choice of~~ $\cancel{B_0}$ ~~is mostly irrelevant. Thus given~~ $\cancel{f \in (\bar{\Omega}\tilde{a})^*}$ ~~satisfying~~ $\cancel{fbd = 0,\ fd(1-\kappa) = 0}$ ~~we know~~ $\cancel{f = (\psi\ \ \varphi)}$ ~~where~~ $\cancel{\psi(1-\lambda) = \varphi b',\ \varphi b = \varphi(1-\lambda) = 0}$. ~~Now we know that~~ $\cancel{\varphi = \psi B_0 = \psi(1-\lambda)(-\lambda^{-1}s)}$

~~Suppose $h \ni (b, h) = 1$.~~
~~We would like to know that $\varphi = \psi(1-\lambda)h$.~~

I have noticed that the homotopy operator s works as well as $-\lambda^{-1}s$. First let's recall that we are studying f such that $fbd = fdb = 0$, which means the components $(\psi\ \varphi)$ satisfy

$$\psi(1-\lambda) = \varphi b' \qquad \varphi b = 0 \qquad \varphi(1-\lambda) = 0.$$

Suppose $f = f_n$ i.e. we have $f = (\psi_n, \varphi_{n-1})$. Then we can think of f as a link between the cyclic cocycles $\psi_n b$ and φ_{n-1}.

$$\begin{tikzcd}
\psi_n b & \\
\psi_n \arrow[u] \arrow[r, "1-\lambda"] & {} \\
& \varphi_{n-1} \arrow[u, "-b'"] \arrow[r] & n\varphi_{n-1}
\end{tikzcd}$$

There seems to be a concept of a good h, one such that any $(\psi\ \varphi)$ as above is such that $\varphi = \psi(1-\lambda)h$. Notice that we have either

$$\varphi = -\varphi c s = \varphi b' s = \psi(1-\lambda)s$$

$$\varphi = \varphi c \lambda^{-1} s = -\varphi b' \lambda^{-1} s = \psi(1-\lambda)(-\lambda^{-1}s) = \psi B_0$$

April 17, 1993

Recall that we have an equivalence between

a) $f \in (\bar{Q}\tilde{a})^*$ such that $fbd = 0$, $fd(1-R) = 0$

b) $\psi \in C(a)^*$ such that $\psi b B_0 = 0$, $\psi B_0(1-\lambda) = 0$

given by $\psi \longmapsto f = \psi J = (\psi \;\; \psi B_0)$. Here $B_0 = (1-\lambda)h$ where h is a suitable contraction: $b'h + hb' = 1$, for example $h = s$ or $(-\lambda^{-1})s$.

Review the proof:

The conditions in a) mean, for $f = (\psi \;\; \varphi)$, that $\psi(1-\lambda) = \varphi b'$, $\varphi(1-\lambda) = 0$ and they imply $\varphi b = 0$. Thus if $\varphi = \psi B_0$ we have $\varphi(1-\lambda) = 0 \iff \psi B_0(1-\lambda) = 0$. Next from $(1-\lambda) = B_0 b' + b B_0$ we have

$$\psi(1-\lambda) = (\psi B_0)b' + \psi b B_0$$

so that if $\varphi = \psi B_0$, then $\psi(1-\lambda) = \varphi b' \iff \psi b B_0 = 0$. The only thing to show is that given f as in a) one has $\varphi = \psi B_0$. But from $\varphi(1-\lambda) = \varphi b = 0$ we have

$$\varphi = -\varphi c s = \varphi b' s = \psi(1-\lambda)s$$

$$\varphi = \varphi c \lambda^{-1} s = -\varphi b' \lambda^{-1} s = \psi(1-\lambda)(-\lambda^{-1}s) = \psi(1-\lambda^{-1})s$$

in the two cases $h = s, (-\lambda^{-1})s$.

Notice that we are trying to prove

$$\varphi = \varphi b' h \quad \text{or} \quad \varphi h b' = 0$$

for our contraction h. Thus if we have an h with this property, then also $hb'h$ has this property, so there are lots of possibilities

It might be worthwhile to try to understand for a unital algebra $\mathcal{A}$ the possible contractions h for b' such that

$$\varphi b = \varphi(1 \otimes 1) = 0 \implies \varphi h b' = 0$$

This might be susceptible to the filtration of $C^\lambda(\mathcal{A})$ Jacek studies.

The above equivalence seems too complicated to include in the revision of part 2. It seems better to consider J going from $b+B$ cocycles in $C(\mathcal{A})^*$ to those in $(\Omega\tilde{\mathcal{A}})^*$.

May 7, 1993

I have noticed a link between simplicial normalization and special contractions in the sense of HPT.

Recall that the standard bimodule resolution (unnormalized) of A:

(1) $$\longrightarrow A\otimes A\otimes A \xrightarrow{b'} A\otimes A \xrightarrow{b'} \overset{\circ}{A} \longrightarrow 0 \longrightarrow 0$$

is the DG algebra given by $T_A(A\otimes A)$, where the differential is the superderivation of degree -1 extension the multiplication $m: A\otimes A \to A$. (This is also a DG coalgebra provided one shifts degrees by 1, namely the bar construction of A.)

If $\eta = 1\otimes 1$, then $b'(\eta^2) = 0$, hence the two-sided ideal generated by η^2 is a DG ideal. The quotient is the normalized standard resolution:

(2) $$\longrightarrow A\otimes \bar{A}^{\otimes 2}\otimes A \longrightarrow A\otimes\bar{A}\otimes A \longrightarrow A\otimes A \longrightarrow \overset{\circ}{A} \longrightarrow 0$$

and it's a DG algebra.

Another description of (1) + (2) is

$$T_A(A\otimes A) = A * \mathbb{C}[\eta]$$

$$T_A(A\otimes A)/(\eta^2) = A * \left(\mathbb{C}[\eta]/(\eta^2)\right)$$

where $\mathbb{C}[\eta]$ is the DG algebra generated by η of degree 1 with $d(\eta) = 1$.

Observe that

$$A * \left(\mathbb{C}[\eta]/(\eta^2)\right) \simeq A * \mathbb{C}[d]/(d^2)$$

$$\simeq \Omega A \,\tilde{\otimes}\, \mathbb{C}[d]/(d^2)$$

May 8, 1993

Let us consider the problem of representing a class in
$$\mathrm{Ext}^n_A(M,N) = H^n(A, \mathrm{Hom}(M,N))$$
explicitly by Hochschild cocycles. (Note that one has both non-normalized and normalized cocycles, which correspond to the two standard resolutions.)

Suppose the class is represented by

$$\begin{tikzcd} \arrow[r] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\ & N & & & & \end{tikzcd}$$

where P is a projective resolution of M. Here M, N etc. are (left) A-modules. Then we have a diagram of bimodules

$$\begin{tikzcd} & & & & A \arrow[d] & \\ \arrow[r] & \mathrm{Hom}(M,P_n) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{Hom}(M,P_0) \arrow[r] & \mathrm{Hom}(M,M) \arrow[r] & 0 \\ & \mathrm{Hom}(M,N) & & & & \end{tikzcd}$$

where the sequence in the middle is exact.

Because P is a projective resolution of M, the complex $\to P_1 \to P_0 \to M \to 0 \to$ is acyclic, so as we work over a field, there exists a contraction h. This gives a right A-module contraction for the bimodule complex

$$(*) \qquad \to \mathrm{Hom}(M,P_1) \to \mathrm{Hom}(M,P_0) \to \mathrm{Hom}(M,M) \to 0 \to \cdots$$

Let us now shift to the bimodule context. Consider a complex Q of A bimodules which is contractible as a complex of right A-modules, e.g. $(*)$.

Look at $\operatorname{Hom}_{A^{op}}(Q,Q)$.
This is a DG algebra, one has a homomorphism $A \longrightarrow Z^0 \operatorname{Hom}_{A^{op}}(Q,Q)$, and an element $h \in \operatorname{Hom}^{-1}_{A^{op}}(Q,Q)$ with boundary 1. Here h is a contraction for Q as right A-module complex. Thus we get

$$A * \mathbb{C}[h] \longrightarrow \operatorname{Hom}_{A^{op}}(Q,Q)$$

a homomorphism of DG algebras, where A has zero diff and $\mathbb{C}[h]$ has differential such that $d(h)=1$. If h is a special contraction, then we get

$$A * \mathbb{C}[h]/(h^2) \longrightarrow \operatorname{Hom}_{A^{op}}(Q,Q)$$

But we have seen that $A * \mathbb{C}[h]$ and $A * \mathbb{C}[h]/(h^2)$ are the standard non-normalized and normalized bimodule resolutions of A respectively.

In the example ⊛ we have a bimodule map $A \longrightarrow Z_0 Q$, hence a map

$$A * \mathbb{C}[h] \to \operatorname{Hom}_{A^{op}}(Q,Q) \longrightarrow \operatorname{Hom}_{A^{op}}(A,Q) = Q$$

compatible with left $A * \mathbb{C}[h]$ multiplication right A-multiplication, and d.

Let's reformulate. $A * \mathbb{C}[h]$ is a DG algebra with grading $|a| = 0$, $|h| = 1$ (lower indexing) and $\partial a = 0$ $\partial h = 1$. A DG module over $A * \mathbb{C}[h]$ is a complex of A-modules together with a contraction. Thus if Q is a complex of bimodules with contraction respecting the right module structure, then Q becomes a left $A * \mathbb{C}[h]$, right A DG bimodule.

Consider now an A-module complex
P which is a resolution of P_0:

$$\longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

and let h be a contraction of the underlying complex of vector spaces. Then P becomes a DG $A * \mathbb{C}[h]$ module. Since the inclusion $P_0 \hookrightarrow P$ is compatible with ∂ and A, we get a map of DG $A * \mathbb{C}[h]$-modules

$$(A * \mathbb{C}[h]) \otimes_A P_0 \longrightarrow P \, .$$

Next I want to discuss special contractions. Consider the simplest example of an acyclic complex of A-modules which is nontrivial, namely, an exact sequence

$$0 \longrightarrow P_2 \underset{\partial_2}{\overset{h_1}{\rightleftarrows}} P_1 \underset{\partial_1}{\overset{h_0}{\rightleftarrows}} P_0 \longrightarrow 0$$

A contraction $h = (h_1, h_0)$ is a pair satisfying

$$\partial_1 h_0 = 1$$
$$\partial_2 h_1 + h_0 \partial_1 = 1$$
$$h_1 \partial_2 = 1$$

and it is special iff $h_1 h_0 = 0$. This is necessarily true: $\partial_2(h_1 h_0) = (1 - h_0 \partial_1) h_0 = h_0 - h_0 = 0$
so $h_1 h_0 = (h_1 \partial_2)(h_1 h_0) = 0$

In this case the $A * \mathbb{C}[h]$ module structure gives

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 A \otimes_A P_0 \arrow[r] \arrow[d] & A \otimes P_0 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0
\end{tikzcd}$$

The maps are

$$\begin{tikzcd} \Omega^1 A\otimes_A P_0 \arrow[r] \arrow[d] & A\otimes P_0 \arrow[d] \\ P_2 \arrow[r, "{[a,h_0]\xi}"] & P_1 \end{tikzcd}$$

$$da\otimes\xi \longmapsto a\otimes\xi - 1\otimes a\xi \qquad a\otimes\xi$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$h_1[a,h_0]\xi \qquad [a,h_0]\xi \qquad a h_0 \xi$$

So the ultimate map $\Omega^1 A\otimes_A P_0 \longrightarrow P_2$ is

$$a_0\, da_1 \otimes \xi \longmapsto \text{[redacted]}\; a_0\, h_1[a_1, h_0]\,\xi$$

The basic derivation is $a \longmapsto h_1[a,h_0] = h_1 a h_0$.

Next consider a longer complex

$$\begin{tikzcd} \cdots \arrow[r, shift left] & P_2 \arrow[l, shift left] \arrow[r, shift left, "\partial_2"'] \arrow[dr, "p_2"'] & P_1 \arrow[l, shift left, "h_1"'] \arrow[r, shift left, "\partial_1"'] \arrow[dr, "p_1"'] & P_0 \arrow[l, shift left, "h_0"'] \arrow[r] & 0 \\ & & N_2 \arrow[u, hook, "\iota_1"'] & N_1 \arrow[u, "\iota_2"'] & \end{tikzcd}$$

Note that from $dh + hd = 1$ we get

$$dhd = d$$

[redacted] Examine:

$$\begin{tikzcd} P_k \arrow[rr, shift left, "\partial_k"'] \arrow[dr, "p_k"'] & & P_{k-1} \arrow[ll, shift left, "h_{k-1}"'] \\ & N_k \arrow[ur, hook, "\iota_{k-1}"'] & \end{tikzcd}$$

$$\iota_{k-1} p_k h_{k-1} \iota_{k-1} p_k = \iota_{k-1} p_k$$

Since ι_{k-1} injective, p_k surjective we have $\boxed{p_k h_{k-1} \iota_{k-1} = 1}$. Thus $r_{k-1} = p_k h_{k-1}$

is a retraction for ι_{k-1} and

$s_k = h_{k-1}\iota_{k-1}$ is a section of p_k.

Can ask whether $h_{k-1} = s_k r_{k-1} = h_{k-1}\iota_{k-1} p_k h_{k-1}$
$= h_{k-1}\partial_k h_{k-1}$. This is $h = h\partial h$, which holds <u>exactly when h is special.</u>

Let's repeat this starting with a complex with contraction

$$\begin{tikzcd}
\cdots \arrow[r, shift right] & P_{k+1} \arrow[l, shift right] \arrow[r, shift right, "\partial_{k+1}"'] \arrow[dr, "p_{k+1}"'] & P_k \arrow[l, shift right, "h_k"'] \arrow[r, shift right, "d_k"'] \arrow[dr, "p_k"'] & P_{k-1} \arrow[l, shift right, "h_{k-1}"'] \arrow[r, shift right] \arrow[dr] & \cdots \arrow[l, shift right] \\
\arrow[ur, hook] & & N_{k+1} \arrow[ul, phantom] \arrow[u, hook, "\iota_k"] & N_k \arrow[u, hook, "\iota_{k-1}"] & {}
\end{tikzcd}$$

We have $dh + hd = 1$ whence $dhd = d$.
Then

$$d_k h_{k-1} d_k = d_k$$

$$\iota_{k-1} p_k h_{k-1} \iota_{k-1} p_k = \iota_{k-1} p_k$$

$$\Longrightarrow \boxed{p_k h_{k-1} \iota_{k-1} = 1} \qquad \text{as } p_k \text{ surj}, \ \iota_{k-1} \text{ inj}$$

$\Longrightarrow$ $r_{k-1} = p_k h_{k-1}$ is a retraction for ι_{k-1}

$s_k = h_{k-1}\iota_{k-1}$ is a section of p_k

so $\boxed{r_k \iota_k = 1} \ \boxed{p_k s_k = 0}$ $\forall k$

also $\boxed{\iota_k r_k + s_k p_k = 1}$

as $\iota_k r_k + s_k p_k = \iota_k p_{k+1} h_k + h_{k-1}\iota_{k-1} p_k$
$= d_{k+1} h_k + h_{k-1} d_k = 1$

Now put $h'_k = s_{k+1} r_k = h_k \iota_k p_{k+1} h_k$

$= h_k d_{k+1} h_k$. Thus $h' = hdh$ is a special contraction. Then

$$r'_{k-1} = p_k h'_{k-1} = \underbrace{p_k h_{k-1} \iota_{k-1}}_{1} p_k h_{k-1} = p_k h_{k-1} = r_{k-1}$$

$$s'_k = h'_{k-1} \iota_{k-1} = h_{k-1}\iota_{k-1} \underbrace{p_k h_{k-1} \iota_{k-1}}_{1} = h_{k-1}\iota_{k-1} = s_k$$

This calculation shows that the process of taking a complex with contraction, [redacted] writing the complex [redacted] as a splicing of short exact sequences, using the contraction to split the short exact sequences, then splicing these splittings to obtain a new contraction, is the same as the process of replacing a contraction h by the special contraction hdh.

Next note the analogy with HPT. We have a complex with A-module structure and contraction independent of each other. In HPT we have a complex with perturbed differential and contraction. (The two might become more similar in an appropriate cochain picture, since $b'\theta = \theta^2$ in your old cochain theory.)

Consider for simplicity a double complex where the rows come with contractions. d = horizontal differential, h = contraction, θ vertical differential:

$$\begin{tikzcd} 0 & P_0 \arrow[l, "d"'] \arrow[r, shift left, "h"] & P_1 \arrow[l, shift left, "d"] \arrow[r, shift left, "h"] & P_2 \arrow[l, shift left, "d"] \arrow[r, shift left, "h"] & \cdots \arrow[l, shift left, "d"] \end{tikzcd}$$

Here the P_n are the columns of the double complex. The total complex has differential $d - \theta$. We then know [redacted] from HPT that the total complex is contractible.

Recall the formulas. There are many, at least three, namely

$$\tilde{h} = h\frac{1}{1-\theta h}\ ,\quad h\frac{1}{1-[\theta,h]}\ ,\quad \frac{1}{1-[h,\theta]}h$$

The last two are easy:

$$[d-\theta, h] = 1-[\theta,h] = 1-\theta h - h\theta = 1-[h,\theta]$$

so one has $[d-\theta,[\theta,h]] = 0$. Thus

$$\left[d-\theta, h\frac{1}{1-[\theta,h]}\right] = [d-\theta,h]\frac{1}{1-[\theta,h]} = 1.$$

and similarly for $\frac{1}{1-[\theta,h]}h$.

The first is harder:

$$\left[d-\theta, h\frac{1}{1-\theta h}\right] = (1-\theta h - h\theta)\frac{1}{1-\theta h} + h\frac{1}{1-\theta h}\underbrace{[d-\theta,-\theta h]}_{-\cancel{\theta^2 h}+\theta+\cancel{\theta^2 h}-\theta h\theta}\frac{1}{1-\theta h}$$

$$= 1 - h\theta\frac{1}{1-\theta h} + h\cancel{\frac{1}{1-\theta h}}\cancel{(1-\theta h)}\theta\frac{1}{1-\theta h}$$

$$= 1$$

These three choices for $\tilde{h}$ coincide when $h^2=0$, e.g.

$$h\frac{1}{1-[\theta,h]} = h\frac{1}{1-h\theta-\theta h+\theta h^2\theta} = h\frac{1}{(1-\theta h)(1-h\theta)}$$

$$= h\frac{1}{1-h\theta}\frac{1}{1-\theta h} = h\frac{1}{1-\theta h}$$

Idea: Study carefully a filtered complex with contractions given on the layers, e.g. a bicomplex with horizontal contraction. The A-module structure might become a perturbation θ of the differential in a suitable cochain calculus.

Cuntz's application to Nistor's bivariant Chern characters needs checking.

$Q = QA$. As a vector space this is $\mathbb{N}$-graded; let N be the corresponding degree operator: $Nx = nx$ for $x \in \Omega^n A$. Note $N1 = 0$. Write $Q = Q_0 \oplus Q_1 \oplus \cdots$ with $Q_n = \Omega^n A$.

Consider RQ. This depends only on Q as a vector space with 1. Claim the $\mathbb{N}$ grading of Q inducing a corresponding $\mathbb{N}$-grading of the algebra RQ. Why? One has an induced grading of TQ and the ideal generated by $1_{TQ} - 1_Q$ is homogeneous since this element has degree zero. Note that

$$RQ = RQ_0 * TQ_1 * TQ_2 * \cdots$$

Put another way the linear operator N on Q extends to a derivation on TQ carrying the ideal generated by $1_{TQ} - 1_Q$ into itself, so it induces a derivation N of RQ. The eigenspaces of N then yield the $\mathbb{N}$-grading of the algebra RQ.

One next has an induced $\mathbb{N}$ grading of $X(RQ)$ as supercomplex. The corresponding degree operator is the Lie derivative $\underbrace{\text{associated}}_{L = L(1,N)}$ to $u = id$, $\hat{u} = N$ on RQ.

Now Cuntz uses a different grading, namely the one obtained from the linear isomorphism

$$X(RQ) = \Omega Q = \bigoplus_n Q \otimes \bar{Q}^{\otimes n} \quad \text{and}$$

the grading on the latter coming from the grading on Q. This isomorphism does the following:

$$\rho(x_0)\,\omega(x_1,x_2)\cdots\omega(x_{2s-1},x_{2s}) \longleftrightarrow x_0\,dx_1 \cdots dx_{2n} \Leftrightarrow (x_0,\dots,x_{2n})$$

$$\natural\big(\rho(x_0)\,\omega(x_1,x_2)\cdots\omega(x_{2s-1},x_{2s})\,d\rho(x)\big) \longleftrightarrow x_0\,dx_1\cdots dx_{2n}\,dx$$

$$\Leftrightarrow (x_0,\dots,x_{2n},x).$$

Let $X(RQ) = \bigoplus E_n$ be Cuntz's grading. He claims that

$$(L - n)E_n \subset E_{n+2} + E_{n+4}$$

Go over the steps. Better, try to understand the filtration $\mathcal{F}^p = \bigoplus_{n \geq p} E_n$ and why $(L-p)\mathcal{F}^p \subset \mathcal{F}^{p+2}$. The idea is that Q has the canonical J-adic filtration and that this induces a filtration on RQ and on $X(RQ)$.

Look at RQ. You have the J-adic filtration on Q. Does this induce a filtration on RQ, a decreasing filtration F^p compatible with product?

Consider a decreasing filtration F^pQ such that $F^0Q = Q$ and $F^pQ \cdot F^qQ \subset F^{p+q}Q$.
Better: $Q = F^0Q \supset F^1Q \supset \cdots$ such that .
There there should be an induced filtration on RQ. Now in fact since RQ depends only on Q as vector space with 1 the filtration of RQ should depend only on the vector space filtration of Q. We might define it by

$$F^pRQ = \sum_{\substack{p_1 + \cdots + p_n \geq p \\ \text{any } n}} \rho(F^{p_1}Q)\cdots\rho(F^{p_n}Q)$$

What we have done is to take the induced filtration in TQ:

$$F^p(Q^{\otimes n}) = \sum_{p_1 + \cdots + p_n \geq p} F^{p_1}Q \otimes \cdots \otimes F^{p_n}Q$$

and take the image filtration in RQ.

Another way to describe what's going on

is to split the filtration:

$$F^p Q = \bigoplus_{n \geq p} Q_n$$

assuming that this exists, i.e. $\exists\ Q = \bigoplus_{n\geq 0} Q_n$ the above holds. Then we get a degree operator N on Q such that $N1 = 0$, so we get an extension of N to a derivation of Q such that $N\rho(x) = \rho(Nx) \quad \forall x \in Q$. Better, the grading of Q induces a grading of the algebra RQ, and the degree operator for the latter is the derivation N of RQ extending N on Q.

So far we only use the linear structure of Q. Suppose now that we use the alg structure and the condition $F^pQ \cdot F^q Q \subset F^{p+q}Q$. Then we have a linear isomorphism

$$RQ = \Omega^{ev} Q = \bigoplus_{n \geq 0} Q \otimes \bar{Q}^{\otimes 2n}$$

and there is a filtration on the right side induced by the filtation on Q, which gives rise to the filtration

$$\mathbb{F}^p = \sum_{\substack{p_0 + \cdots + p_{2s} \geq p \\ \text{any } s}} \rho(F^{p_0}Q)\, \omega(F^{p_1}Q, F^{p_2}Q) \cdots \omega(F^{p_{2s-1}}Q, F^{p_{2s}}Q)$$

The question is whether this agrees with the filtration F^pRQ. Let's try to prove some inclusions. First

$$\omega(x,y) = \rho(xy) - \rho(x)\rho(y)$$

so

$$\begin{aligned} \omega(F^{p_1}Q, F^{p_2}Q) &\subset \rho(F^{p_1}Q\, F^{p_2}Q) + \rho(F^{p_1}Q)\rho(F^{p_2}Q) \\ &\subset \rho(F^{p_1+p_2}Q) + \rho(F^{p_1}Q)\rho(F^{p_2}Q) \end{aligned}$$

$$\subset \quad F^{p_1+p_2}RQ + F^{p_1}RQ \cdot F^{p_2}RQ$$

Now $\{F^pRQ\}$ is an algebra filtration so that we conclude

$$\omega(F^{p_1}Q, F^{p_2}Q) \subset F^{p_1+p_2}RQ$$

hence
$$\mathcal{F}^p = \sum_{\substack{p_0+\cdots+p_{2s} \geq p \\ \text{any } s}} \underbrace{\rho(F^{p_0}Q)}_{F^{p_0}RQ} \underbrace{\omega(F^{p_1}Q, F^{p_2}Q)}_{F^{p_1+p_2}RQ} \cdots \omega(F^{p_{2s-1}}Q, F^{p_{2s}}Q)$$

$$\boxed{\mathcal{F}^p \subset F^pRQ.}$$

Suppose we show that $\rho(F^pQ)\mathcal{F}^q \subset \mathcal{F}^{p+q}$

Then
$$\rho(F^{p_1}Q)\cdots\rho(F^{p_n}Q)\mathcal{F}^q \subset \mathcal{F}^{p_1+\cdots+p_n+q}$$

so
$$F^pRQ \cdot \mathcal{F}^q \subset \mathcal{F}^{p+q}$$

and then applying this to $1 \in \mathcal{F}^0$ we find

$$\boxed{F^pRQ \subset \mathcal{F}^p}$$

To prove $\rho(F^pQ)\mathcal{F}^q \subset \mathcal{F}^{p+q}$ consider

$$\rho(F^pQ)\cdot\rho(F^{p_0}Q)\,\omega(F^{p_1}Q, F^{p_2}Q)\cdots\omega(F^{p_{2s-1}}Q, F^{p_{2s}}Q)$$

where $p_0+\cdots+p_{2s} \geq q$. Suffices to show

$$\rho(F^pQ)\cdot\rho(F^{p_0}Q) \subset \rho(F^{p+p_0}Q) + \omega(F^pQ, F^{p_0}Q)$$

clear from
$$\rho(\underset{F^pQ}{x})\,\rho(\underset{F^{p_0}Q}{y}) = \rho(\underset{F^{p+p_0}Q}{xy}) - \omega(x,y)$$

Thus we have proved $\boxed{F^pRQ = \mathcal{F}^p}$

The next step is to bring in parity. Cuntz claims that $\mathcal{F}^p = E_p \oplus \mathcal{F}^{p+1}$ where $(L-p)E_p \subset E_{p+2} + E_{p+4}$. It seems that for the overall $\mathbb{Z}/2$ grading L has the eigenvalues $0, 2, 4, \cdots$ on $(RQ)_+$ and $1, 3, 5 \cdots$ on $(RQ)_-$. Consider

$$\rho(x_1) \cdots \rho(x_n)$$

where $x_i \in Q$ is homogeneous. This is an eigenvector for L with eigenvalue $\sum_{i=1}^{n} |x_i|$, so the assertion seems clear.

In more detail consider $\mathcal{F}^p / \mathcal{F}^{p+1}$ and the $\mathbb{Z}/2$ grading of this. I claim this quotient is of parity $p + 2\mathbb{Z}$. Take a typical generator for $\mathcal{F}^p$ namely $\rho(x_1) \cdots \rho(x_n)$ where $x_i \in F^{p_i} Q$ and $\sum p_i \geqq p$. Modulo $\mathcal{F}^{p+1}$ we can assume $\sum p_i = p$ and that $x_i \in Q_{p_i}$, whence the parity of the element is $p + 2\mathbb{Z}$.

I want to try to make Joachim's method work in the case $B = \mathbb{C}$. Thus given $A \overset{\alpha}{\underset{\bar\alpha}{\rightrightarrows}} L$ congruent mod I and a trace on I^p, I want to produce classes in $HC^{2n}(A)$ for n large enough.

Now I think Connes cyclic cocycles

$$\mathrm{tr}\left(\varepsilon F[F,\theta]^{2n+1}\right) = 2\,\mathrm{tr}\left(\varepsilon\theta[F,\theta]^{2n}\right)$$

do this for $2n+1 \geq p$.

My idea: We have two liftings

$$\begin{tikzcd} RA \arrow[r, shift left, "\alpha\pi"] \arrow[r, shift right, "\bar\alpha\pi"'] \arrow[d, "\pi"] & L \arrow[d] \\ A \arrow[r] & L/I \end{tikzcd}$$

which are homotopic, specifically by $u_t : RA \to L$, $u_t(\rho a) = (1-t)\alpha a + t\bar\alpha a$. This homotopy gives rise to an odd operator

$$H : X(RA) \longrightarrow X(L)$$

such that
$$\begin{cases} [\partial, H] = (\alpha\pi)_* - (\bar\alpha\pi)_* \\ H\left(F^q_{IA}X(RA)\right) \subset F^q_I X(L) \qquad \forall q \end{cases}$$

It follows that $H : F^q_{IA}X(RA) \longrightarrow F^q_I X(L)$ is an odd map of supercomplexes for $q \geq 1$ (recall that the kernel of $\pi_* : X(RA) \to X(A)$ is $F^1_{IA}X(RA)$).

Now recall also that

$$\overline{HC}_{2n}(A) = H_+\left(\bar X(RA)/F^{2n}_{IA}\bar X(RA)\right)$$
$$\xrightarrow{\sim} H_-\left(F^{2n}_{IA}\bar X(RA)\right) = H_-\left(F^{2n}_{IA}X(RA)\right)$$

Thus we get a map

$$\widetilde{HC}_{2n}(A) \cong H_-(F^{2n}_{IA}X(RA)) \longrightarrow H_+(F^{2n}_{I}X(L))$$
$$= H_+\left(I^{n+1}+[I^n,R] \rightleftarrows \natural(I^n dR)\right)$$
$$\in I^{n+1}+[I^n,R]/[I^n,R]$$
$$= I^{n+1}/I^{n+1}\cap[I^n,R]$$

Notice that the last space is a quotient of $I^{n+1}/[I^n,I]$. There are two things to be desired: first we don't quite get Nistor's range, namely, a class in HC^{2n} for $n+1 \geqslant p$, and we don't get Connes's range: $2n+1 \geqslant p$.

Here seems to be the nature of Joachim's argument at least in the case $B=\mathbb{C}$.

We have $A \overset{\iota}{\underset{\iota^\gamma}{\rightrightarrows}} Q$ congruent modulo J (here $Q=QA$, $J=qA$) and we have a trace τ on J^m. Then we have

$$\begin{tikzcd} X(RA) \arrow[r, "\iota_* - \iota^\gamma_*"', "\text{comm.}"] \arrow[dr] & X(RQ) \arrow[r] & X(Q) \\ & \mathcal{F}^1 \arrow[u, phantom, "\cup"] & F^1 X(Q) \arrow[u, phantom, "\cup"] \\ & \vdots \arrow[u, phantom, "\cup"] & \vdots \arrow[u, phantom, "\cup"] \\ & \mathcal{F}^k \arrow[u, phantom, "\cup"] & F^k X(Q) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where $F^pX(Q)$ is the filtration on $X(Q)$ induced by the filtration $F^pQ = J^p$ on Q. I think this means $F^kX(Q) = F^{2k}_{J}X(Q)$

$$F^p X(Q)_+ = F^p Q = J^p$$

$$F^p \Omega^1(Q)_\natural = [redacted] \sum_{k=0}^{p} \natural\left(J^{p-k} d(J^k)\right)$$

Now for $k \geq 1$ we have

$$\natural\left(J^{p-k} d(J^k)\right) \subset \natural\left(J^{p-k} \sum_{i=0}^{k-1} J^i \, dJ \, J^{k-1-i}\right)$$

$$\subset \sum_{i=0}^{k-1} \natural\left(J^{k-1-i} J^{p-k} J^i \, dJ\right) = \natural\left(J^{p-1} dJ\right)$$

Thus $$F^p \Omega^1(Q)_\natural = \natural\left(J^p dQ + J^{p-1} dJ\right)$$

which proves the claim.

Structure of Joachim's argument. He considers a derivation D on RQ and the corresponding Lie derivative L_D on $X(RQ)$. One has [redacted] the Cartan homotopy formula $L_D = [\partial, h_D]$, where

$$h_D\left(F^p_{IQ} X(RQ)\right) \subset F^{p-2}_{IQ} X(RQ)$$

He defines S_n [redacted] be a polynomial of degree n in D with constant term 1. Thus S_n is homotopic to the identity and of order $\leq$ [redacted] $2n$ with respect to the IQ [redacted] adic filtration. [redacted]

Next he has $j, \bar{j} : X(RA) \longrightarrow X(RQ)$ which are of order zero. Therefore he has

$$X(RA) \xrightarrow{j \cdot \bar{j}} X(RQ) \xrightarrow{S_n} X(RQ)$$

of order $\leq$ [redacted] $2n$.

Next there is the filtration $\{\mathcal{F}^p\}$ on $X(RQ)$ which is associated to eigenvalue decomposition for D.

The point is that we have this homomorphism $Q \longrightarrow \mathcal{L}(H) \otimes B$, hence $RQ \longrightarrow \mathcal{L}(H) \otimes RB$. [redacted]

Now
$$\begin{tikzcd} RQ \arrow[r] \arrow[d] & \mathcal{L}(H) \otimes RB \arrow[d] \\ Q \arrow[r] & \mathcal{L}(H) \otimes B \end{tikzcd}$$
commutes so $IQ \longrightarrow \mathcal{L}(H) \otimes IB$. What about the [redacted] other filtration? Assumption is that the ideal $J = qA \subset Q$ maps to K[redacted]$\otimes B$, $K = \ell_m$.

Notation $\varphi : Q \longrightarrow \mathcal{L}(H) \otimes B$. Then we have
$$\begin{tikzcd} Q \arrow[r, "\varphi"] \arrow[d, "\rho"] & \mathcal{L}(H) \otimes B \arrow[d, "1 \otimes \rho"] \\ RQ \arrow[r, "\varphi_*"] & \mathcal{L}(H) \otimes RB \end{tikzcd}$$
φ_* unique homomorphism such that
$$\varphi_*(\rho(x)) = (1 \otimes \rho)\varphi(x).$$
By assumption $\varphi(\Omega^n) \subset K$[redacted]$^n \otimes B$, K^n[redacted]$= (\ell_p)^n$.
Thus it would appear that $\varphi_*(\mathcal{F}^n) \subset K^n$[redacted]$\otimes RB$.

Let's go over the steps.

We have [redacted] $A \rightrightarrows \mathcal{L} \otimes B$ congruent modulo $J \otimes B$, whence we have
$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\rho"] & Q \arrow[r, "\varphi"] \arrow[d, "\rho"] & \mathcal{L} \otimes B \arrow[d, "1 \otimes \rho"] \\ RA \arrow[r, shift left] \arrow[r, shift right] & RQ \arrow[r, "\varphi_*"] & \mathcal{L} \otimes RB \\ IA \arrow[u, phantom, "\cup"] & IQ \arrow[u, phantom, "\cup"] & \mathcal{L} \otimes IB \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\begin{array}{ccccc} X(RA) & \rightrightarrows & X(RQ) & \longrightarrow & X(\mathcal{L} \otimes RB) \\ \cup & & \cup & & \cup \\ F^p_{IA}(RA) & \rightrightarrows & F^p_{IQ}(RQ) & \longrightarrow & \text{[redacted]} F^p_{\mathcal{L} \otimes IB} X \text{[redacted]} (\mathcal{L} \otimes RB) \end{array}$$

In the end we seem to be heading toward two ideals

$$J \otimes RB \; , \; \mathcal{L} \otimes IB \quad \subset \mathcal{L} \otimes RB$$

We have $\quad \varphi_* : RQ \longrightarrow \mathcal{L} \otimes RB$

$$\varphi_*(\rho(x)) = (1 \otimes \rho)\,\varphi(x)$$

Now what do we know about $\varphi : Q \longrightarrow \mathcal{L} \otimes B$

answer $\quad \varphi(qA) \subset J \otimes B.$

May 18, 1993

Let $R = A * \mathbb{C}[h] \simeq T_A(A\otimes A)$,
let $I = Rh^2R$. Then

$$R/I = A*\left(\mathbb{C}[h]/(h^2)\right) \simeq \Omega A \,\tilde{\otimes}\, \mathbb{C}[d]/(d^2)$$

I would like to calculate $gr_I R = \bigoplus_p I^p/I^{p+1}$.
The first thing we would like to show is that I is flat as a (left) R-module. This implies that

$$I\otimes_R \overset{p}{\cdots} \otimes_R I \xrightarrow{\sim} I^p$$

$$(I/I^2)\otimes_R \overset{p}{\cdots} \otimes_R (I/I^2) \xrightarrow{\sim} I^p/I^{p+1}.$$

and reduces us to understanding I/I^2 as an R/I-module.

Consider

$$0 \longrightarrow \Omega^1_A R \longrightarrow R\otimes_A R \longrightarrow R \longrightarrow 0$$

and the result
$$\Omega^1_A\, T_A(M) = T_A(M)\otimes_A M \otimes_A T_A(M).$$

This tells us that $\Omega^1_A R = R\otimes_A \underbrace{(A\otimes A)}_{AhA} \otimes_A R = \underset{\overset{\|}{R\,dh\,R}}{R\otimes R}$.
We thus have an exact sequence

$$(*) \qquad 0 \longrightarrow R\otimes R \longrightarrow R\otimes_A R \longrightarrow R \longrightarrow 0$$
$$1\otimes 1 \longmapsto h\otimes 1 - 1\otimes h$$

This sequence splits as a sequence of right R-modules. So if M is a R-module we have an exact sequence

$$0 \longrightarrow R\otimes M \longrightarrow R\otimes_A M \longrightarrow M \longrightarrow 0.$$

This implies that if M is A-projective, then M is of projective dimension ≤ 1 over R.

Apply this to $M = R/I \simeq \Omega A \tilde{\otimes} \mathbb{C}[d]/(d^2)$ which we know is free as A module. Thus R/I is of projective-dimension ≤ 1 as R module, implying that I is a projective R-module.

Next take the sequence

$$0 \longrightarrow R \otimes R/I \longrightarrow R \otimes_A R/I \longrightarrow R/I \longrightarrow 0$$

and tensor with R/I on the left to get

$$\underset{\overset{\parallel}{0}}{\mathrm{Tor}_1^R(R/I, R\otimes_A R/I)} \to \underset{\overset{\parallel}{I/I^2}}{\mathrm{Tor}_1^R(R/I, R/I)} \to R/I \otimes R/I \to R/I \otimes_A R/I \to R/I \to 0$$

i.e.

~~[illegible]~~

$$1\otimes 1 \longmapsto h\otimes 1 - 1 \otimes h$$

$$0 \longrightarrow I/I^2 \longrightarrow R/I \otimes R/I \longrightarrow R/I \otimes_A R/I \longrightarrow R/I \longrightarrow 0$$

$$h^2 \longmapsto h\otimes 1 + 1\otimes h \qquad (\text{because } dh^2 = dh\,h + h\,dh)$$

(Check: $h\otimes 1 + 1 \otimes h \longmapsto h(h\otimes 1 - 1\otimes h) + (h \otimes 1 - 1\otimes h)h$
$= h^2 \otimes 1 - 1 \otimes h^2 = 0$
as $h^2 = 0$ in R/I.)

Conclusion: I/I^2 is the image of the R/I bimodule map

$$R/I \otimes R/I \longrightarrow R/I \otimes R/I$$

$$1 \otimes 1 \longmapsto h\otimes 1 + 1 \otimes h$$

~~Why is this well-defined, i.e.~~

$\cancel{a(h\otimes 1 + 1\otimes h) \overset{?}{=} (h\otimes 1 + 1 \otimes h)a}$

May 19, 1993

Review $\quad R = A\langle D\rangle = A * \mathbb{C}[D]$

$$R = T_A(A\otimes A) \qquad D = 1\otimes 1.$$

$$I = RD^2R \qquad R/I = A\langle d\rangle = A*\mathbb{C}[d], \quad d^2=0.$$

$$\Omega^1_A R = R\otimes_A (A\otimes A)\otimes_A R$$

$$\Omega^1_A R = R\otimes R$$

$$d(D) \longleftrightarrow 1\otimes 1$$

$$\begin{array}{ccccccccc} 0 & \to & R\otimes R & \longrightarrow & R\otimes_A R & \longrightarrow & R & \longrightarrow & 0 \\ & & 1\otimes 1 & \longmapsto & D\otimes 1 + 1\otimes D & & & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \to & I/I^2 & \xrightarrow{d} & R/I\otimes R/I & \longrightarrow & R/I\otimes_A R/I & \longrightarrow & R/I \to 0 \\ & & & & \| & & & & \\ & & & & (R/I)\otimes_R \Omega^1_A R\otimes_A (R/I) & & & & \end{array}$$

$$D^2 \overset{d}{\longmapsto} d\otimes 1 + 1\otimes d$$

Conclude I/I^2 is the image of

$$R/I\otimes R/I \longrightarrow R/I\otimes R/I$$

$$x\otimes y \longmapsto x(d\otimes 1 + 1\otimes d)y$$

But $R/I = \mathbb{C}[d]\otimes \Omega A = \Omega A\otimes \mathbb{C}[d]$, so the above map is

$$\begin{array}{ccc} \mathbb{C}[d]\otimes\mathbb{C}[d] & \longrightarrow & \mathbb{C}[d]\otimes\mathbb{C}[d] \\ 1\otimes 1 & \longmapsto & d\otimes 1 + 1\otimes d \\ d\otimes 1 & \longmapsto & d\otimes d \\ 1\otimes d & \longmapsto & d\otimes d \\ d\otimes d & \longmapsto & 0 \end{array}$$

tensored on the left + right by ΩA. Observe that

$d \in R/I$ centralizes $d \otimes 1 + 1 \otimes d$
so the image is a quotient of
$$\mathbb{C}[d] \otimes_{\mathbb{C}[d]} \mathbb{C}[d] = \mathbb{C}[d].$$
In fact this is the image so we find
$$\begin{array}{ccc} R/I \otimes_{\mathbb{C}[d]} R/I & \xrightarrow{\sim} & I/I^2 \\ 1 \otimes 1 & \longmapsto & D^2 \end{array}$$

Our conclusion therefore is that
$$gr_I R = T_{R/I}\left(R/I \otimes_{\mathbb{C}[d]} R/I\right)$$
$$= A\langle d\rangle\langle D^2\rangle \Big/ \left([d, D^2] = 0\right)$$
This is reasonable for the following reason. We know $R = A\langle D\rangle = A * \mathbb{C}[D]$, and $\mathbb{C}[D]$ is additively the same as $\mathbb{C}[d]\langle D^2\rangle / ([d, D^2] = 0)$.

At this point I would like to understand the meaning of this result better. It would be nice to explicitly identify $R = A\langle D\rangle$ with $gr_I R = A\langle d\rangle\langle D^2\rangle \Big/ \left([d, D^2] = 0\right)$.

Note that
$$R/I \otimes_{\mathbb{C}[d]} R/I \otimes_{\mathbb{C}[d]} R/I = \left(\Omega A \otimes \mathbb{C}[d]\right) \otimes_{\mathbb{C}[d]} \cdots$$
$$= \Omega A \otimes \Omega A \otimes \Omega A \otimes \mathbb{C}[d]$$
lifts into R by lifting $a_0\, da_1 \dots da_n$ into $a_0 [D, a_1] \cdots [D, a_n]$ and then sending
$$\omega \otimes \omega' \otimes \omega'' \otimes \left\{\begin{matrix} 1 \\ d \end{matrix}\right. \quad \text{to} \quad \omega D^2 \omega' D^2 \omega'' \left\{\begin{matrix} 1 \\ D \end{matrix}\right.$$

This means that if I fix a basis for ΩA, then I get a basis for $A\langle D\rangle$ consisting of products

$$\omega_0 D^2 \omega_1 D^2 \cdots D^2 \omega_n \begin{cases} 1 \\ D \end{cases} \qquad n \geq 0.$$

where $\omega_0, \omega_1, \ldots, \omega_n$ run over the given basis of ΩA.

Formulas for left multiplication by a, D on R relative to the decomposition

$$R = \bigoplus_{n\geq 0} (\Omega A \cdot D^2)^n \, \Omega A \otimes (\mathbb{C} \oplus \mathbb{C}D)$$

$$D \cdot a = da + aD$$

$$\begin{aligned} D \cdot da &= D[D,a] \\ &= D[D,a] + [D,a]D - [D,a]D \\ &= \big(D(Da - aD) + (Da - aD)D\big) - da\,D \\ &= D^2 a - a D^2 - da\,D \end{aligned}$$

This tells how to move D to the right in the product $D \cdot a_0 da_1 \cdots da_n$

Recall that we have a lifting

$$A\langle d\rangle \longrightarrow A\langle D\rangle$$

obtained by first using the homomorphism

$$\begin{aligned} \Phi : A\langle d\rangle &\longrightarrow A\langle D\rangle \,\tilde{\otimes}\, \mathbb{C}[\partial] \\ a &\longmapsto a \\ d &\longmapsto D\partial D = D - D^2\partial \end{aligned}$$

$\partial^2 = 0$
$\partial D + D\partial = 1$
$\partial a = a\partial$

followed by the action of $A\langle D\rangle \tilde{\otimes} \mathbb{C}[\partial]$ on $A\langle D\rangle$ (where ∂ is the superderivation on $A\langle D\rangle$ such that $\partial(D)=1$) applied to 1.

Let's calculate the lifting

$$\Phi(da) = \Phi(da - ad)$$
$$= [D - D^2\partial, a] = [D,a] - [D^2,a]\partial$$

Thus

$$\Phi(da_1\cdots da_n) = \big([D,a_1]-[D^2,a_1]\partial\big)\big([D,a_2]-[D^2,a_2]\partial\big)\cdots$$
$$= [D,a_1]\cdots[D,a_n]$$
$$- [D^2,a_1]\partial[D,a_2]\cdots[D,a_n]$$
$$- [D,a_1][D^2,a_2]\partial[D,a_3]\cdots$$

Now $\partial[D,a] + [D,a]\partial = [\partial,[D,a]] = [[\partial,D],a] = [1,a] = 0.$

better: $\partial(Da-aD) + (Da-aD)\partial = \partial Da - a\partial D + D\partial a - aD\partial$
$= a - a = 0$

Similarly $[\partial,[D^2,a]] = [[\partial,D^2],a] = 0$, so there are no second order & higher terms in ∂. We have

$$\Phi(da_1\cdots da_n) = [D,a_1]\cdots[D,a_n] + \left\{\begin{array}{l}(-1)^n[D^2,a_1][D,a_2]\cdots \\ +(-1)^{n-1}[D,a_1][D^2,a_2]\cdots \\ \cdots\end{array}\right\}\partial$$

$$= [D,a_1]\cdots[D,a_n] + (-1)^n\left(\sum_{j=1}^{n}(-1)^{j-1}[D,a_1]\cdots[D^2,a_j]\cdots[D,a_n]\right)\partial$$

$$= [D,a_1]\cdots[D,a_n] + (-1)^n[D,[D,a_1]\cdots[D,a_n]]\partial$$

So

$$\Phi(da_1 \cdots da_n\, d) =$$

$$\Big([D,a_1]\cdots[D,a_n] + (-1)^n [D,[D,a_1]\cdots[D,a_n]]\,\partial\Big)(D - D^2\partial)$$

Let's apply these operators to 1.

$$\Phi(da_1 \cdots da_n)(1) = [D,a_1]\cdots[D,a_n]$$

$$\Phi(da_1 \cdots da_n\, d)(1) = [D,a_1]\cdots[D,a_n]D + (-1)^n \underbrace{[D,[D,a_1]\cdots[D,a_n]]}_{D[D,a_1]\cdots[D,a_n] - (-1)^n [D,a_1]\cdots[D,a_n]D}\ \underbrace{\partial(D)}_{1}$$

Thus we get the formulas for the lifting

$$\boxed{\begin{array}{lcl} a_0\, da_1 \cdots da_n & \longmapsto & a_0[D,a_1]\cdots[D,a_n] \\ a_0\, da_1 \cdots da_n\, d & \longmapsto & (-1)^n a_0 D[D,a_1]\cdots[D,a_n] \end{array}}$$

May 20, 1993

Again consider $R = A\langle D\rangle$ and $R/I = A\langle d\rangle$. We have defined a left R/I-module structure on R using left multiplication by $a \in A$ and letting $d \longmapsto D\partial D = D - D^2\partial$. We ask whether there is a similar right R/I-module structure making R into a bimodule over R/I.

Let's shift to a more neutral notation. Let $R = T_A(A\otimes A) = A\langle \xi\rangle$ where $\xi = 1\otimes 1$. ∂ is the degree -1 derivation such that $\partial(\xi) = 1$, $\partial(a) = 0$. Let h be the operator of left multiplication by ξ: $h(x) = \xi x$. Then $[\partial, h](x) = \partial(\xi x) + \xi\partial(x) = \partial(\xi)x - \xi\partial(x) + \xi\partial(x) = x$, so $\boxed{[\partial, h] = 1}$

Let k be right multiplication by ξ with sign: $k(x) = (-1)^{|x|}x\xi$. Then $[\partial, k](x) = \partial\left((-1)^{|x|}x\xi\right) + k\partial(x)$ $= (-1)^{|x|}\partial(x)\xi + x\partial(\xi) + (-1)^{|\partial(x)|}\partial(x)\xi = x$, so $\boxed{[\partial, k] = 1}$

Next $[h,k](x) = h((-1)^{|x|}x\xi) + k(\xi x) =$ $(-1)^{|x|}\xi x\xi + (-1)^{|\xi x|}\xi x\xi = 0$. Thus $\boxed{[h,k] = 0}$

Note the h commutes with right multiplication by a and k commutes with left multiplication by a.

The idea is to use left mult. by $a \in A$ and $h\partial h$ to define left mult. by $A\langle d\rangle$ on R, and to use right mult by $a \in A$ and $k\partial k$ to define right mult. by $A\langle d\rangle$ on R. This doesn't seem to work, since $h\partial h$, $k\partial k$ don't commute.

Let's compute

$$h\partial h\; k\partial k = (h - h^2\partial)(k - k^2\partial)$$

$$= hk - h^2\underbrace{\partial k}_{1-k\partial} - hk^2\partial + h^2\underbrace{\cancel{\partial k^2\partial}}_{k^2\partial}$$

$$= hk - h^2 + h^2k\partial - hk^2\partial$$

$$k\partial k\; h\partial h = \underbrace{kh}_{-hk} - k^2 + \underbrace{k^2h}_{hk^2}\partial - \underbrace{kh^2}_{h^2k}\partial$$

$$\therefore\quad [h\partial h, k\partial k] = -h^2 - k^2$$

May 21, 1993:

Digression: Consider M a mixed complex such that $H^b H^B M = 0$, i.e. the conclusion of Connes's lemma. Then we have an exact sequence of complexes

$$0 \longrightarrow BM \hookrightarrow \mathrm{Ker}(B;M) \longrightarrow \underbrace{\frac{\mathrm{Ker}(B;M)}{BM}}_{\text{acyclic}} \longrightarrow 0$$

where the quotient is acyclic, so this sequence splits. This means there exists a decomposition

$$\mathrm{Ker}(B;M) = BM \oplus L$$

compatible with b where L is contractible. Then L is a sub mixed complex of M, and we have an exact sequence of mixed complexes

$$0 \longrightarrow L \longrightarrow M \longrightarrow M/L \longrightarrow 0$$

Next note that $\mathrm{Ker}(B;M) \longrightarrow \mathrm{Ker}(B;M/L)$ is surjective, i.e. if $x \in M$ is such that $Bx \in L$, then $Bx \in BM \cap L = 0$, so $x \in \mathrm{Ker}(B;M)$. Thus we have

$$\mathrm{Ker}(B;M/L) = \mathrm{Ker}(B;M)/L$$

$$B(M/L) = BM \oplus L/L$$

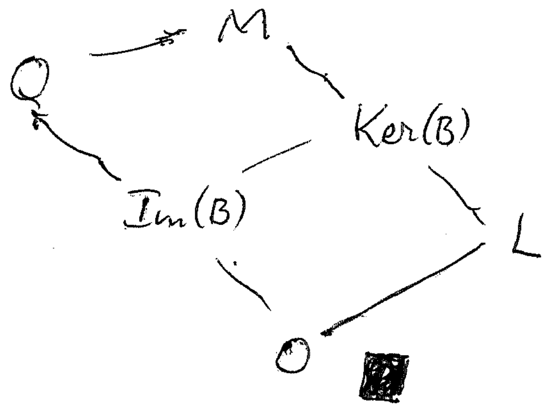
whence B is exact on M/L.

This means that M/L is a free mixed complex, so as $M \twoheadrightarrow M/L$ is a quis we have a section of this map:

$$M = L \oplus M/L$$

Thus we have shown that any mixed complex satisfying the conclusion of Connes lemma is the direct sum of a mixed complex with B exact (i.e. a free mixed complex) and an acyclic complex with $B=0$.

Summarize: If M is such that $H^b H^B M = 0$, then we can choose a complement L to $\mathrm{Im}\, B \subset \mathrm{Ker}\, B$ such that $b(L) \subset L$. Next we can choose a complement Q to $L \subset M$ such that Q is closed under b, B. Picture



$h = s$ or $-\lambda^{-1}s$.

In the example of $C = C(a)$ we know $\mathrm{Ker}\, B = (1-\lambda)C$, so that

$$\mathrm{Im}(B) \simeq M/\mathrm{Ker}(B) = C/(1-\lambda)C = C^\lambda,$$

so it appears that Q looks exactly the same as $P\Omega\tilde{a}$. This leads me to expect that there should be some rather close connection between the two.

Let's review the preceding.

Suppose M is a mixed complex such that $\operatorname{Ker} B/\operatorname{Im} B$ is acyclic wrt b. We claim that M splits into mixed subcomplexes

$$(*) \qquad M = P \oplus L$$

where B is exact on P and $B = 0$ on L. Note then that for such a splitting we have $\operatorname{Ker}(B;M) = \operatorname{Ker}(P;M) \oplus L$, $BM = BP$, and $BP = \operatorname{Ker}(P;M)$, whence

$$L = H(M,B).$$

We have the following picture

$$\begin{tikzcd}[column sep=small]
 & & M \arrow[dl, dash] \arrow[dr, dash] & & \\
 & P \arrow[dr, dash] & & \operatorname{Ker}(B) \arrow[dl, dash] \arrow[dr, dash] & \\
 & & \operatorname{Im}(B) \arrow[dr, dash] & & L \arrow[dl, dash] \\
 & & & 0 &
\end{tikzcd}$$

Note that any subcomplex of M (wrt b) contained in $\operatorname{Ker} B$ is a mixed subcomplex of M. Similarly any subcomplex of M containing $\operatorname{Im}(B)$ is a mixed subcomplex of M. So the existence of the decomposition $(*)$ depends on whether there are complementary subcomplexes for $\operatorname{Im}(B) \subset \operatorname{Ker}(B)$ and $\operatorname{Ker}(B)/\operatorname{Im}(B) \subset M/\operatorname{Im}(B)$. In both cases this follows from the fact that $\operatorname{Ker}(B)/\operatorname{Im}(B)$ is acyclic.

Another point is that if K is an acyclic subcomplex of $\operatorname{Ker}(B)$ such that

$\mathrm{Im}(B) \cap K = 0$, then K can be extended to a complement L for $\mathrm{Im}(B)$ in $\mathrm{Ker}(B)$. In effect we want ~~to~~ choose L so that we have

$$\begin{tikzcd}
 & & \mathrm{Ker}(B) \arrow[dl, dash] \arrow[ddr, dash] & \\
 & \mathrm{Im}(B)\oplus K \arrow[dl, dash] \arrow[ddr, dash] & & \\
\mathrm{Im}(B) \arrow[ddr, dash] & & & L \arrow[dl, dash] \\
 & & K \arrow[dl, dash] & \\
 & 0 & &
\end{tikzcd}$$

and this is possible because [redacted] K, $\mathrm{Ker}(B)/\mathrm{Im}(B)$ acyclic $\Rightarrow$ $\mathrm{Ker}(B)/\mathrm{Im}(B)\oplus K$ acyclic.

Now apply this to $\bar{\Omega}\tilde{a}$. It seems that there is a splitting of the exact sequences

$$0 \longrightarrow \Sigma C' \xrightarrow{\binom{-B_0}{1}} \bar{\Omega}\tilde{a} \xrightarrow{(1\ B_0)} C \longrightarrow 0$$

compatible with both b, B in the case where

$$\binom{-B_0}{1}\Sigma C' \cap B(\bar{\Omega}\tilde{a}) = 0.$$

This holds for $h = s$ and $-\lambda^{-1}s$ as we have seen. Details:

suppose

$$\begin{pmatrix} -B_0 \\ 1 \end{pmatrix} x = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} \qquad x \in \Sigma C', \quad \begin{pmatrix} y \\ z \end{pmatrix} \in \bar{\Omega}\tilde{a}$$

$$\left.\begin{aligned} -B_0 x &= 0 \\ x &= N_\lambda y \end{aligned}\right\} \Longrightarrow -B_0 N_\lambda y = 0$$

But, if $h = -\lambda^{-1}s$,

$$c(-B_0) = c(\text{[redacted]}-1+\lambda)\text{[redacted]}(-\lambda^{-1}s) = 1+\lambda$$

[redacted] and if $h = s$, then

$$c\lambda^{-1}(-B_0) = c\lambda^{-1}(1-\lambda)s = 1-(-\lambda) = 1+\lambda$$

and $(1+\lambda)N_\lambda = 2N_\lambda$. Thus $-B_0 N_\lambda y \Rightarrow N_\lambda y = 0 \Rightarrow x = 0$.

Recall that if M is a mixed complex such that $\operatorname{Ker} B/\operatorname{Im} B$ is acyclic, then we have a splitting into mixed subcomplexes

$$M = Q \oplus L$$

where B is exact on Q and zero on L. Q and L are arbitrary subcomplexes such that the squares in

$$\begin{tikzcd}[column sep=small]
& M \arrow[dl, dash] \arrow[dr, dash] & & \\
Q \arrow[dr, dash] & & \operatorname{Ker} B \arrow[dl, dash] \arrow[dr, dash] & \\
& \operatorname{Im} B \arrow[dr, dash] & & L \arrow[dl, dash] \\
& & 0 &
\end{tikzcd}$$

are bicartesian. Another point is that if K is a subcomplex of $\operatorname{Ker} B$ which is acyclic and such that $\operatorname{Im} B \cap K = 0$, then K can be extended to such an L.

Now we want to apply this to $M = \bar\Omega \tilde{a}$ and $K = \operatorname{Im}\begin{pmatrix} -B_0 \\ 1 \end{pmatrix}$. Notation

$$C_n = a^{\otimes n+1}$$
$$C'_n = a^{\otimes n}$$

$$M = \bar\Omega\tilde{a} = C \oplus C'$$

$$\operatorname{Ker} B = (1-\lambda)C \oplus C'$$

$$\operatorname{Im} B = 0 \oplus N_\lambda C'$$

Here we have canonical choices for Q, L namely

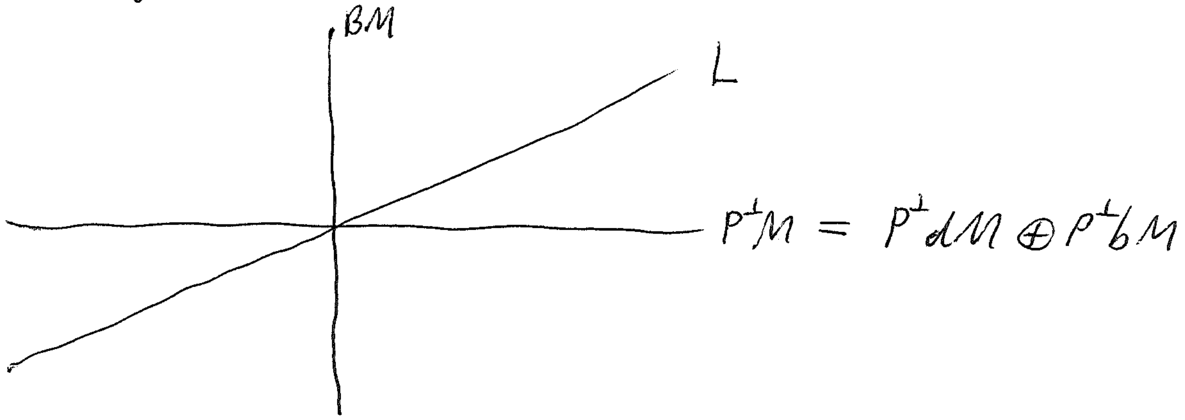
$$Q = P\bar\Omega\tilde{a} \qquad L = P^\perp\bar\Omega\tilde{a}$$

This holds for a general nonunital algebra, however we want to find an L containing K. Recall

$$P^\perp = Gdb + bGd$$

where Gdb is a projection with image $P^\perp dM$
and bGd ———————————————— $P^\perp bM$

Picture of $\operatorname{Ker} B = BM \oplus P^\perp M$



L is supposed to be a subcomplex of $\operatorname{Ker} B$ which is complementary to BM. It is therefore the graph of a map $\alpha = \alpha' + \alpha'' : P^\perp dM \oplus P^\perp bM \longrightarrow BM$ which commutes with b. Now

$$\begin{tikzcd}
P^\perp dM \arrow[r, "\alpha'"] \arrow[d, "b \cong"] & BM \arrow[d, "b"] \\
P^\perp bM \arrow[r, "\alpha''"] & BM
\end{tikzcd}$$

must commute, and b on the left is an isomorphism with inverse Gd. Thus

$$\alpha'' = b\,\alpha'\,Gd$$

must hold for $[b,\alpha] = 0$ and the converse is also true.

At this point we know that the possible L

are described by linear maps
$\alpha' : P^\perp dM \longrightarrow BM = PdM$. We have

$$\alpha' = \begin{pmatrix} 0 & 0 \\ \theta & \theta \end{pmatrix} \qquad \theta : (1-\lambda)C' \longrightarrow N_\lambda C'$$

Now we want to choose θ so that the corresponding graph of α contains K.
We calculate the components of $\begin{pmatrix} -B_0 \\ 1 \end{pmatrix} x$ relative to the decomposition $\operatorname{Ker} B = PdM \oplus P^\perp dM \oplus P^\perp bM$

$$Gdb\begin{pmatrix} -B_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ G_\lambda b & P_\lambda^\perp \end{pmatrix}\begin{pmatrix} -(1-\lambda)h \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ P_\lambda^\perp - G_\lambda(1-\lambda)b'h \end{pmatrix} = \begin{pmatrix} 0 \\ (P_\lambda^\perp h)b' \end{pmatrix}$$

$$bGd\begin{pmatrix} -B_0 \\ 1 \end{pmatrix} = \begin{pmatrix} P_\lambda^\perp & 0 \\ -b'G_\lambda & 0 \end{pmatrix}\begin{pmatrix} -(1-\lambda)h \\ 1 \end{pmatrix} = \begin{pmatrix} -(1-\lambda)h \\ b'(P_\lambda^\perp h) \end{pmatrix}$$

$$P\begin{pmatrix} -B_0 \\ 1 \end{pmatrix} = \begin{pmatrix} -B_0 \\ 1 \end{pmatrix} - \begin{pmatrix} -B_0 \\ (P_\lambda^\perp h)b' + b'(P_\lambda^\perp h) \end{pmatrix} = \begin{pmatrix} 0 \\ (P_\lambda h)b' + b'(P_\lambda h) \end{pmatrix}$$

Here we have used $1 - [P_\lambda^\perp h, b'] = [h - P_\lambda^\perp h, b'] = [P_\lambda h, b']$.
The condition that $K \subset L$ means that

$$P\begin{pmatrix} -B_0 \\ 1 \end{pmatrix} = \alpha' Gdb\begin{pmatrix} -B_0 \\ 1 \end{pmatrix} + \underbrace{\alpha'' bGd}\begin{pmatrix} -B_0 \\ 1 \end{pmatrix}$$

$$b\alpha' Gd\, bGd = b\alpha' Gd$$

$$\begin{pmatrix} 0 \\ [P_\lambda h, b'] \end{pmatrix} = \begin{pmatrix} 0 \\ \theta(P_\lambda^\perp h)b' \end{pmatrix} + \underbrace{\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \theta G_\lambda & 0 \end{pmatrix}\begin{pmatrix} -(1-\lambda)h \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}\begin{pmatrix} 0 \\ -\theta P_\lambda^\perp h \end{pmatrix} = \begin{pmatrix} 0 \\ b'\theta(P_\lambda^\perp h) \end{pmatrix}$$

where $(1-\lambda)\theta = 0$ as θ has image in $N_\lambda C'$.

so θ must satisfy

$$(P_\lambda h)\, b' + b'(P_\lambda h) = \theta(P_\lambda^\perp h)\, b' + b'\,\theta(P_\lambda^\perp h)$$

or $$[P_\lambda h - \theta P_\lambda^\perp h,\ b'] = 0$$

Suppose we put $\theta = P_\lambda\, \varphi(1-\lambda)$, then

$$[P_\lambda (1 - \varphi(1-\lambda))\, h,\ b'] = 0$$

The question is whether we can solve this when φ is completely arbitrary of degree 0.

May 23, 1993 (Alice is 31)

The problem has to do with the following data. One has a complex (C, b') with a contraction h. One also has a subcomplex of C namely $N_\lambda C$. The basic assumption is that the map

$$* \quad N_\lambda C \overset{i}{\subset} C \xrightarrow{h} C \xrightarrow{1-\lambda} (1-\lambda)C$$

measuring the extent to which h fails to preserve the subcomplex is injective. Note that $*$ is a map of complexes.

Let's try to study this situation by itself. Notation

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, shift right, "i"'] & C \arrow[l, dashed, shift right, "r"'] \arrow[r, shift right, "j"'] \arrow[loop above, "h"] & C'' \arrow[l, dashed, shift right, "l"'] \arrow[r] & 0 \end{tikzcd}$$

Denote the differential by d. r, l are a splitting of C. Relative to this splitting we have

$$d = \begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix} \qquad h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

$$\begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} + \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix} = \begin{pmatrix} d_{11}h_{11} + d_{12}h_{21} + h_{11}d_{11} & d_{11}h_{12} + d_{12}h_{22} + h_{12}d_{22} + h_{11}d_{12} \\ d_{22}h_{21} + h_{21}d_{11} & d_{22}h_{22} + h_{22}d_{22} + h_{21}d_{12} \end{pmatrix}$$

It would be better to use the following notation

$$d = \begin{pmatrix} d' & f \\ 0 & d'' \end{pmatrix} \qquad h = \begin{pmatrix} h' & h_{12} \\ g & h'' \end{pmatrix}$$

Then $dh + hd$ [redacted] is

$$\begin{pmatrix} [d',h'] + \text{[redacted]}\, fg & \text{[redacted]}\; fh'' + h'f + d'h_{12} + h_{12}d'' \\ d''g + gd' & [d'',h''] + \text{[redacted]}\, gf \end{pmatrix}$$

so $dh + hd = 1$ means

$$1 = \text{[redacted]}\, fg + [d',h']$$

$$d''g + gd' = 0$$ (means $g : C' \longrightarrow C''$ is a map of complexes)

$$1 = gf + [d'',h'']$$

$$fh'' + h'f + d'h_{12} + h_{12}d'' = 0$$

i.e. g, f are homotopy inverses, and [redacted] h', h'' are the homotopies joining fg and gf to the identity. The last condition says that the homotopies h', h'' are compatible with f up to homotopy.

Let us consider a module M with two submodules K, L such that $K \cap L = 0$. Then we have a nine diagram

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & L \arrow[r, equal] \arrow[d, hook] & L \arrow[d, hook] \\
K \arrow[r, hook] \arrow[d, equal] & M \arrow[r, two heads] \arrow[d] & M/K \arrow[d] \\
K \arrow[r, hook] & M/L \arrow[r, two heads] & M/K+L
\end{tikzcd}$$

(Actually I first should have considered the general case and pointed out the equivalence of

1) $K \longrightarrow M/L$ injective
2) $L \longrightarrow M/K$ injective
3) $K \cap L = 0$.)

Suppose we give a retraction for $K \hookrightarrow M/L$. Then the bottom sequence splits. If we also give a retraction of M onto L, then we can lift $M/K+L$ into M/L and then ~~[illegible]~~ compose with the lifting of M/L into M to get a lifting of $M/K+L$ into M. In particular we obtain a lifting of $M/K+L$ into M/K so the right vertical sequence splits.

Let us now apply these ideas to the following situation:

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$$\begin{tikzcd}
 & BM \arrow[d, hook] & & P_\lambda C' \arrow[ll, "\binom{0}{1}"'] \arrow[d, "B_0"'] \\
C' \arrow[r, hook, "i = \binom{-B_0}{1}"'] & \mathrm{Ker}\, B \arrow[rr, "j = (1\ B_0)"'] \arrow[d] & & (1-\lambda)C \arrow[ll, dashed, bend right, "\binom{1}{0}"'] \arrow[u, dashed, bend right, "-\frac{1}{2}P_\lambda c"'] \arrow[d] \\
 & BM \arrow[rr, dotted, no head] & & \frac{(1-\lambda)C}{B_0 P_\lambda C'}
\end{tikzcd}$$

$$\| \qquad \mathrm{Ker}\, B / \mathrm{Im}\, B \text{ for } (C, b, B).$$

Note that

$$\left(-\tfrac{1}{2}P_\lambda c\right) B_0 = -\tfrac{1}{2} P_\lambda\, c\,(1-\lambda^{-1}) s = -\tfrac{1}{2} P_\lambda (-\lambda - 1) = P_\lambda$$

so that $\boxed{\left(-\tfrac{1}{2}P_\lambda c\right) B_0 = 1 \quad \text{on} \quad P_\lambda C'}$

Thus $B_0\left(-\frac{1}{2}P_\lambda c\right)$ projects onto $B_0 P_\lambda C'$ so its kernel is a complex for $BC \subset (1-\lambda)C = \mathrm{Ker}(B \text{ on } C)$

This kernel is the image of

$$1 - B_0\left(-\tfrac{1}{2}P_\lambda c\right) \quad \text{on} \quad (1-\lambda)C$$

and it is the image of

$$\left(1 + \tfrac{1}{2} B_0 P_\lambda c\right)(1-\lambda) : C \longrightarrow (1-\lambda)C$$

I want a subcomplex of $(1-\lambda)C$ complementary to BC, so I would like to find a retraction of $(1-\lambda)C$ onto BC which commutes with the differential.

Consider the retraction at hand $-\frac{1}{2}B_0 P_\lambda c : (1-\lambda)C \to BC$

Note that

$$P_\lambda b(1-\lambda) = P_\lambda(1-\lambda)b' = 0 \; -$$

so that $\quad P_\lambda c = -P_\lambda b' \quad$ on $(1-\lambda)C$.

Thus our retraction is

$$-\tfrac{1}{2}B_0 P_\lambda c = \tfrac{1}{2}B_0 P_\lambda b' \; : \; (1-\lambda)C \to BC \subset (1-\lambda)C.$$

The differential on $(1-\lambda)c$ is b.

Logical structure of what we want to do

$$\begin{tikzcd}
& L \arrow[d, hook, "\varepsilon"'] & \\
K \arrow[r, hook] & M \arrow[r, two heads, "j"] \arrow[d] & M/K \arrow[ul, "\alpha"'] \arrow[d, "p"] \\
& M/L & M/K+L \arrow[ul, "\beta"]
\end{tikzcd}
\qquad \boxed{pj\varepsilon = 0}$$

Assume α such that $\boxed{\alpha j \varepsilon = 1_L}$.

Then $j\varepsilon\alpha$ is a projection on M/K with image $j\varepsilon L$, so $1_{M/K} - j\varepsilon\alpha$ is a projection whose image is a lift of $M/K+L$. If we further lift this into M we get the map β satisfying

$$\boxed{j\beta p + j\varepsilon\alpha = 1_{M/K}}$$

Then $\quad \alpha j\beta p = \alpha - \alpha j\varepsilon\alpha = \alpha - \alpha = 0 \quad$ so

$$\boxed{\alpha j \beta = 0}$$

Then we get two projections on M

$\varepsilon \alpha j$, $\beta p j$ which annihilate each other. Note

$$\beta p j \underline{\beta p} j = \beta p (1 - j\varepsilon\alpha) j = \beta p j$$

so $\beta p j$ is a projection. So we know that

$$1_M - \varepsilon \alpha j - \beta p j$$

is a projection. Since

$$j(1_M - \varepsilon\alpha j - \beta p j)$$
$$= j - (j\varepsilon\alpha + j\beta p) j = j - j = 0$$

The image of this projection is contained in $\operatorname{Ker}(j) = K$, and in fact the image is clearly K. Now

$$(\varepsilon\alpha j + \beta p j)\varepsilon = \varepsilon\alpha j\varepsilon = \varepsilon$$
$$(\varepsilon\alpha j + \beta p j)\beta = \beta p j \beta = \beta$$
$$\uparrow$$
$$\beta p j \beta p = \beta p (1 - j\varepsilon\alpha) = \beta p \quad + \ p \text{ surjective}$$

Thus $1_M - \varepsilon\alpha j - \beta p j$ is the projection onto K with kernel $\varepsilon L + \beta(M/K+L)$.

Now let's apply this

Recall $\quad cB_0 = c(1-\lambda^{-1})s = -(1+\lambda)$

so $\quad -\frac{1}{2}P_\lambda c B_0 = P_\lambda \frac{1+\lambda}{2} = P_\lambda$

$$\begin{tikzcd}
 & P_\lambda C \arrow[d, "\varepsilon = \begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & \\
C' \arrow[r, "i = \begin{pmatrix} -B_0 \\ 1 \end{pmatrix}"'] & \operatorname{Ker} B \arrow[r, two heads, "j = (1 \;\; B_0)"'] & (1-\lambda)C \arrow[ul, bend right, "\alpha = -\frac{1}{2} P_\lambda c"'] \arrow[d, "p"] \\
 & & (1-\lambda)C/BC \arrow[ul, "\beta"]
\end{tikzcd}$$

Then

$$1_{(1-\lambda)C} - j\varepsilon\alpha = 1 - (1 \;\; B_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\left(-\tfrac{1}{2} P_\lambda c\right)$$

$$= 1 + \tfrac{1}{2} B_0 P_\lambda c$$

is a projection on $(1-\lambda)C$ whose image is a lifting of $(1-\lambda)C/BC$. Use the section $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of j and define β by

$$\beta p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\left(1 + \tfrac{1}{2} B_0 P_\lambda c\right)$$

Then

$$\varepsilon\alpha j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\left(-\tfrac{1}{2} P_\lambda c\right)(1 \;\; B_0) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} P_\lambda c & P_\lambda \end{pmatrix}$$

$$\beta p j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\left(1 + \tfrac{1}{2} B_0 P_\lambda c\right)(1 \;\; B_0)$$

$$= \begin{pmatrix} 1 + \frac{1}{2} B_0 P_\lambda c & B_0 P_\lambda^\perp \\ 0 & 0 \end{pmatrix}$$

$$1_{\operatorname{Ker} B} - \varepsilon\alpha j - \beta p j = \begin{pmatrix} -\frac{1}{2} B_0 P_\lambda c & -B_0 P_\lambda^\perp \\ \frac{1}{2} P_\lambda c & P_\lambda^\perp \end{pmatrix} = \begin{pmatrix} -B_0 \\ 1 \end{pmatrix}\left(\tfrac{1}{2} P_\lambda c \;\; P_\lambda^\perp\right)$$

Note that

$$\begin{pmatrix} \frac{1}{2} P_\lambda c & P_\lambda^\perp \end{pmatrix} \begin{pmatrix} -B_0 \\ 1 \end{pmatrix} = \left(-\tfrac{1}{2} P_\lambda c\right) B_0 + P_\lambda^\perp$$
$$= P_\lambda + P_\lambda^\perp = 1.$$

so that we have a projection onto $K = iC'$.

At this point we have explicitly decomposed Ker B into

$$K = iC' \quad , \quad \varepsilon P_\lambda C = \mathrm{Im}\, B, \quad \text{[redacted]} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + \tfrac{1}{2} B_0 P_\lambda c)(1-\lambda) C$$

The problem is that the latter is not a subcomplex.

May 26, 1993

Review: Consider M a mixed complex satisfying the conclusion of Connes's lemma. Let $K \subset {}_BM$ satisfy $bK \subset K$, K is acyclic wrt b, $K \cap BM = 0$. Then we know the quotient mixed complex $C = M/K$ satisfies the conclusion of Connes's lemma (better terminology: has Connes's property). Why?

The key point is to show ${}_BM \to {}_B(M/K)$ is surjective. Clear, because if $x \in M$ satisfies $Bx \in K$, then $Bx \in K \cap BM = 0$, so $x \in {}_BM$. We have exact sequences

$$\begin{tikzcd}
 & BM \arrow[r, "\sim"] \arrow[d, hook] & BC \arrow[d, hook] \\
K \arrow[r, hook, "i"'] \arrow[d, equal] & {}_BM \arrow[r, two heads, "j"'] \arrow[d] & {}_BC \arrow[d] \\
K \arrow[r, hook] & {}_BM/BM \arrow[r, two heads] & {}_BC/BC
\end{tikzcd}$$

Note $BM \xrightarrow{\sim} BC$ because it is surjective and the kernel is $BM \cap K = 0$. The bottom exact sequence where K, ${}_BM/BM$ are acyclic by assumption, shows ${}_BC/BC$ is acyclic.

Consider the problem of extending K is a subcomplex of ${}_BM$ which is complementary to BM. This can be done as follows. Because K is contractible the bottom exact sequence splits, so we can lift ${}_BC/BM$ into ${}_BM/BM$. Because ${}_BM/BM$ is contractible we can lift ${}_BM/BM$ into ${}_BM$. Thus by composing these liftings we find a subcomplex Q of ${}_BM$ which is complementary to $K \oplus BM$. Thus $K \oplus Q$ is a complement to BM inside ${}_BM$.

Next let's carry this out concretely in the Connes situation where $M = \Omega\tilde{a}$ and $K = \begin{pmatrix} -B_0 \\ 1 \end{pmatrix} C'$. Recall the picture

$$\begin{tikzcd}
 & P_\lambda C \arrow[d, hook, "\varepsilon = \binom{0}{1}"] & \\
C' \arrow[r, hook, "\iota = \binom{-B_0}{1}"'] & \operatorname{Ker} B \arrow[r, two heads, "j = (1\ \ B_0)"'] \arrow[d] & (1-\lambda)C \arrow[ul, dashed, "\alpha = t"'] \arrow[d, "p"] \\
 & \operatorname{Ker} B / \operatorname{Im} B \arrow[r] & (1-\lambda)C/BC \arrow[ul, dashed, "\beta"]
\end{tikzcd}$$

Put $t = -\frac{1}{2} P_\lambda c$.

Then $tB_0 = P_\lambda(-\frac{1}{2})c(1-\lambda^{-1})s$
$= P_\lambda \frac{\lambda+1}{2} = P_\lambda$

$$\boxed{tB_0 = P_\lambda}$$

$(B_0t)^2 = B_0P_\lambda t = B_0 t$

Thus $B_0 t$ is a projection on $(1-\lambda)C = {}_BC$ with image BC. Also $1 - B_0 t$ is a projection onto a lift of $(1-\lambda)C/BC$.

Now define

$$\beta p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 - B_0 t) = \begin{pmatrix} 1 - B_0 t \\ 0 \end{pmatrix}$$

Then

$$\beta p j = \begin{pmatrix} 1 - B_0 t \\ 0 \end{pmatrix}(1 \quad B_0) = \begin{pmatrix} 1 - B_0 t & \overbrace{B_0 - B_0 P_\lambda}^{B_0 P_\lambda^\perp} \\ 0 & 0 \end{pmatrix}$$

$$\varepsilon \alpha j = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t (1 \quad B_0) = \begin{pmatrix} 0 & 0 \\ t & P_\lambda \end{pmatrix}$$

$$1 - \beta p j - \varepsilon \alpha j = \begin{pmatrix} B_0 t & -B_0 P_\lambda^\perp \\ -t & P_\lambda^\perp \end{pmatrix} = \begin{pmatrix} -B_0 \\ 1 \end{pmatrix}(-t \quad P_\lambda^\perp)$$

Note

$$(-t \quad P_\lambda^\perp)\begin{pmatrix} -B_0 \\ 1 \end{pmatrix} = P_\lambda + P_\lambda^\perp = 1$$

so $(-t \quad P_\lambda^\perp)$ is a retraction onto $iC' = K$.

Also

$$(-t \quad P_\lambda^\perp)\varepsilon = (-t \quad P_\lambda^\perp)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = P_\lambda^\perp = 0 \text{ on } P_\lambda C$$

so $(-t \quad P_\lambda^\perp)$ kills the image of ε.

Now the defect with α, β is that they are not compatible with the differential. The image of β is not a subcomplex of $\operatorname{Ker} B$. The idea is to use the contractibility of C' so as to modify β modulo $\operatorname{Im} B$ to become compatible with the differential. [redacted]

Let's compute the discrepancy of βp wrt the differentials.

$$\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}\begin{pmatrix} 1-B_0 t \\ 0 \end{pmatrix} - \begin{pmatrix} 1-B_0 t \\ 0 \end{pmatrix} b = \begin{pmatrix} -[b, B_0 t] \\ 0 \end{pmatrix}$$

We know the image of this is contained in $K \oplus BM$, in fact this is a map of complexes

$$(1-\lambda)C \xrightarrow{p} (1-\lambda)C/BC \xrightarrow{[b,\beta]} iC' \oplus \varepsilon P_\lambda C$$

We know the projection of $iC' \oplus \varepsilon P_\lambda C$ onto iC' is given by $(-t \quad P_\lambda^\perp)$, so we get

$$(-t \quad P_\lambda^\perp)\begin{pmatrix} -[b, B_0 t] \\ 0 \end{pmatrix} = t[b, B_0 t]$$

from $(1-\lambda)C$ to C'. But

$$\begin{aligned} t[b, B_0 t] &= t b B_0 t - tB_0 t b \\ &= t(-B_0 b')t - tb \qquad\qquad (1-\lambda)t = 0 \\ &= -(P_\lambda b' t + tb) \\ &= -(b't + tb) \end{aligned}$$

Recall that $b' N_\lambda = N_\lambda b$ says $b'(P_\lambda C) \subset (P_\lambda C)$ $\therefore$ $P_\lambda b' P_\lambda = b' P_\lambda$ and $P_\lambda b' t = b't$.

Check $\qquad (b't + tb) b + (-b')(b't + tb) = 0$.

(Note that $b't + tb$ [redacted]: $(1-\lambda)C \longrightarrow C'$ is of degree -1).

Thus $b't + tb$ is compatible with the differentials

Also

$$
\begin{aligned}
(b't + tb)B &= (b't+tb)B_0 N_\lambda \\
&= b' P_\lambda N_\lambda + t(-B_0 b')N_\lambda \\
&= b' N_\lambda - \underbrace{P_\lambda b' N_\lambda}_{P_\lambda N_\lambda b \,=\, N_\lambda b \,=\, b' N_\lambda} \\
&= 0
\end{aligned}
$$

so that $b't+tb : (1-\lambda)C/BC \longrightarrow C'$ is well-defined.

Next because C' is contractible we can write $b't+tb$ as a coboundary:

$$(-b')(-h)(b't+tb) - (-h)(b't+tb)b$$

$$= \underbrace{b'hb'}_{b'}t + b'_\lambda \overset{h}{t} b + hb'tb = b't+tb.$$

So now we modify β to $\tilde{\beta}$ defined by

$$\tilde{\beta}p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1-B_0 t) + \begin{pmatrix} -B_0 \\ 1 \end{pmatrix}(-h)(b't+tb)$$

Then

$$\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}\tilde{\beta}p = \begin{pmatrix} b(1-B_0 t) \\ 0 \end{pmatrix} + \begin{pmatrix} -B_0 \\ 1 \end{pmatrix}(-b')(-h)(b't+tb)$$

$$(\tilde{\beta}p)b = \begin{pmatrix} (1-B_0 t)b \\ 0 \end{pmatrix} + \begin{pmatrix} -B_0 \\ 1 \end{pmatrix}(-h)(b't+tb)b$$

so

$$\begin{pmatrix} b & 1-\lambda \\ & -b' \end{pmatrix}(\tilde{\beta}p) - (\tilde{\beta}p)b = \begin{pmatrix} 0 \\ b't+tb \end{pmatrix} : (1-\lambda)C/BC \longrightarrow BM.$$

$$\tilde{\beta}p = \begin{pmatrix} 1 - B_0 t + B_0 h b' t + B_0 h t b \\ -h(b't+tb) \end{pmatrix}$$

[redacted] Summarizing we find that

$\cdot\ \tilde\beta p : (1-\lambda)C \longrightarrow \operatorname{Ker} B$ kills BC

and satisfies $[b, \tilde\beta p] : (1-\lambda)C \longrightarrow \operatorname{Im} B$. Also it lifts $(1-\lambda)C/BC$ into $\operatorname{Ker} B$. Now apply $P^\perp = 1 - P$ to $\tilde\beta p$. This does not change $\tilde\beta p$ modulo $P \operatorname{Ker} B = \operatorname{Im} B$, so $P^\perp \tilde\beta p$ is also a lifting of $(1-\lambda)C/BC$ into $\operatorname{Ker} B$. But now we have

$$[b, P^\perp \tilde\beta p] = P^\perp [b, \tilde\beta p] = 0$$

so that the image of $P^\perp \tilde\beta p$ is a subcomplex of $\operatorname{Ker} B$ which is complementary to $K \oplus BM$.

One has $\quad P\tilde\beta p = \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \qquad \gamma : (1-\lambda)C/BC \to P_\lambda C$

so $$j P^\perp \tilde\beta p = j\tilde\beta p - jP\tilde\beta p$$
$$= 1 - B_0 t + B_0 \gamma = 1 - B_0(t-\gamma)$$

Thus $\tilde t = t - \gamma : (1-\lambda)C/BC \longrightarrow P_\lambda C$ gives a better retraction of $(1-\lambda)C$ onto BC. ~~Now one can repeat the above~~ [redacted] ~~splitting of Ker B into three parts using $\tilde t$ instead of t and obtain a splitting Ker B = K ⊕ M ⊕ Im $P^\perp$~~ It seems in fact that we have actually split $(1-\lambda)C = {}_BC$ into BC and a complementary subcomplex Q. Then [redacted] we can lift this Q into $\operatorname{Ker} B$.

Nistor application. Suppose given

$$A \rightrightarrows L\otimes B \longrightarrow L\otimes B/J\otimes B \qquad \tau \text{ given on } J^p.$$

the usual quasi-homomorphism situation. The first case to understand is when $B = \mathbb{C}$. In this case we can suppose $L = QA$, $J = qA$. Joachim's idea is to use

$$X(RA) \rightrightarrows X(RQ)$$

and the Lie derivative + Cartan homotopy associated to the derivation of RQ arising from the standard grading of $Q = \Omega A$. The idea I would like to pursue is to make use of the graded algebra

$$\bigoplus_{n\geq 0} J^n u^n \subset L[u]$$

to handle traces on the powers J^n. Joachim uses the fact that (in the general case where B is present) the obvious homom.

$$Q \xrightarrow{\ \nu\ } L\otimes B$$

carries $q^n A \longmapsto J^n\otimes B$, so that when we consider the homomorphism

$$RQ \xrightarrow{\ \nu'\ } L\otimes RB$$

such that [redacted] $\nu'\rho = (1\otimes\rho)\nu$, then ν' carries the degree n subspace R_n of RQ into $J^n\otimes RB$. [redacted] So we have homomorphisms

$$RA \rightrightarrows RQ = \bigoplus_n R_n \longrightarrow \Big(\bigoplus_n J^n\Big)\otimes RB$$

and corresponding maps on X-complexes

$$X(RA) \rightrightarrows X(RQ) \longrightarrow \left(\bigoplus_n J^n\right)_\natural \otimes X(RB)$$

Recall ~~that~~ that $\left(\bigoplus_n J^n\right)_\natural = L_\natural \oplus \bigoplus_{n\geq 1} J^n/[J, J^{n-1}]$

Let's now take $B = \mathbb{C}$. In this case $\left(\bigoplus_n J^n\right)_\natural \otimes X(RB)$ is even, so we are working essentially with traces on RA and RQ up to homotopy. Let's calculate the homomorphisms

$$RA \overset{\theta_*}{\underset{\theta^\gamma_*}{\rightrightarrows}} RQ \longrightarrow \bigoplus_n J^n u^n \qquad \begin{matrix} L = Q \\ J = q \end{matrix}$$

$$\rho(a) \longmapsto \rho(\theta a) = \rho(a) + \rho(da) \longmapsto a + (da)u$$
$$ \longmapsto \rho(\theta^\gamma a) = \rho(a) - \rho(da) \longmapsto a - (da)u$$

$$\omega(a_1, a_2) \longmapsto a_1a_2 + d(a_1a_2)u - (a_1 + da_1)u \circ (a_2 + da_2)u$$
$$= a_1a_2 + d(a_1a_2)u - a_1a_2 + da_1da_2 - (a_1 da_2 + da_1 a_2)u + da_1da_2u^2$$
$$= (da_1da_2)(1-u^2)$$

Thus we have the homomorphisms

$$RA \ni a_0 da_1 \cdots da_{2n} \longmapsto a_0 da_1 \cdots da_{2n}(1-u^2)^n \pm da_0 \cdots da_{2n}\, u(1-u^2)^n \in \bigoplus_{n\geq 0} q^n u^n$$

The next step is to consider traces on the algebra $\bigoplus q^n u^n$. The commutator quotient space is

$$\left(\bigoplus q^n u^n\right)_\natural = Q_\natural \oplus \bigoplus_{n\geq 1} \left(q^n/[q, q^{n-1}]\right) u^n$$

However the traces we are interested in when pulled back via $a_0 da_1 \ldots da_{2n} \mapsto da_0 \cdots da_{2n}\, u(1-u^2)^n$

have to vanish on IA^N for some N.

Analytic motivation says we want to start with a linear functional on $\mathfrak{g}^p/[\mathfrak{g},\mathfrak{g}^{p-1}]$ and use the same linear functional on $\mathfrak{g}^n$ for $n \geq p$.

So suppose we are given τ on $\mathfrak{g}^p/[\mathfrak{g},\mathfrak{g}^{p-1}]$. Then we can construct the following sort of trace T on $\bigoplus_{n\geq 0} \mathfrak{g}^n u^n$. For $n<p$ we define $T|\mathfrak{g}^n u^n = 0$, and for $n \geq p$ we take $T(xu^n) = \tau(x)\cdot c_n$. In other words T is a sort of tensor product of τ and a linear functional f on $\mathbb{C}[u]$ such that $f(u^n)=0$ for $n<p$.

We want T when pulled back via the $\frac{1}{2}\times$ the difference of the two homomorphisms $RA \rightrightarrows \bigoplus \mathfrak{g}^n u^n$ to vanish on some power of IA. Thus

$$\tau(da_0 \cdots da_{2n})\, f\big(u(1-u^2)^n\big) = 0$$

for $n \geq q$. Notice that we can suppose f is odd.

Next note that
$$(1-u^2)^{n+1} = (1-u^2)^n - u^2(1-u^2)^n$$
$$(1-u^2)^{n+2} = (1-u^2)^n - 2u^2(1-u^2)^n + u^4(1-u^2)^n$$

i.e. $\sum_{n\geq q} \mathbb{C}\, u(1-u^2)^n = \mathbb{C}[u^2]\, u(1-u^2)^q$. This means that f is a distribution supported at $u=\pm 1$, which implies it has the form

$$f(u^n) = \tfrac{1}{2}(\delta_1 - \delta_{-1})(\text{poly in } D)(u^n) \qquad D = u\frac{d}{du}$$

Now let us look at Joachim's choice:

$$\mu^{(k)} = \tfrac{1}{2}(\delta_1 - \delta_{-1})(1-D)\left(1-\frac{D}{3}\right)\cdots\left(1-\frac{D}{2k-1}\right)$$

Then $\mu^{(k)}$ kills even powers of u and u[redacted]$, u^3, \ldots, u^{2k-1}$ as well as $u\,\mathbb{C}[u^2]$[redacted]$(1-u^2)^n$ for $n > k$. The latter is true because differentiation lower the order of vanishing by 1. Thus we have

$$\mu^{(k)}\left(u(1-u^2)^n\right) = \begin{cases} 0 & \text{for } k \neq n \\ \left[\dfrac{(-1)^k}{1\cdot 3 \cdots (2k-1)}\, u(1+u)^k \dfrac{D^k(1-u)^k}{(-1)^k k!}\right]_{u=1} & \\ \quad = \dfrac{2^k k!}{1\cdot 3\cdots(2k-1)} & \text{for } k = n. \end{cases}$$

Apply [redacted] $\mu^{(k)}$ "$\otimes$" τ to $RA \longrightarrow \bigoplus q^n u^n$ sending $a_0 da_1 \ldots da_{2n} \longmapsto \begin{pmatrix} pa_0\, qa_1 \cdots qa_{2n} \\ + qa_0 \cdots qa_{2n} u \end{pmatrix}(1-u^2)^n$

and we get a cochain with only the component

$$\tau(qa_0 \cdots qa_{2k}) \cdot \frac{2^k k!}{1\cdot 3 \cdots (2k-1)}$$

This is a reduced cyclic $2k$-cocycle. (This is not quite the $b+B$ cocycle associated to the trace on RA [redacted] because we still should divide by $\frac{1}{k!}$ as part of the rescaling.)

--

Recall the situation with a quasi-homom.

$$A \underset{\bar\theta}{\overset{\theta}{\rightrightarrows}} L\otimes B \longrightarrow (L/J)\otimes B \qquad \tau \text{ on } J^p$$

We get maps linear sending 1 to 1

(1) $$A \rightrightarrows \frac{\theta+\bar\theta}{2}a \pm \left(\frac{\theta-\bar\theta}{2}a\right)u \in L\otimes B \oplus (J\otimes B)u$$

which extend to homoms.

(2) $$RA \rightrightarrows \left(\oplus J^n u^n\right)\otimes RB.$$

and maps of complexes

$$X(RA) \rightrightarrows \left(\oplus J^n u^n\right)_{\natural} \otimes X(RB)$$

[redacted] We then use the trace τ on J^p to get a trace $\tilde\tau$ on $\oplus J^n u^n$, whence we have a map of complexes $X(RA) \longrightarrow X(RB)$.

The problem is to control [redacted] IA and the filtrations on the X-complexes. Notice that unless we specialize u to ± 1, the linear maps (1) are not homomorphisms, hence the homoms. (2) do not carry IA to IB. The solution of the problem should arise by working infinitesimally around $u=\pm 1$.

The idea is to look at quotients of $\oplus J^n u^n$ on which the [redacted] trace $\tilde\tau$ makes sense. [redacted] R. We have

$$\begin{array}{ccccc} \bigoplus\limits_{n\geq 0} J^n u^n & \subset & L[u] & & \\ \downarrow & & \downarrow & & \\ L\oplus J\bar u & \subset & L[u]/(1-u^2) & \xrightarrow[\sim]{u\mapsto(1,-1)} & L\times L \end{array}$$

Also

$$\begin{tikzcd}
\bigoplus_{n\geq 0} J^n u^n \arrow[r, phantom, "\subset"] \arrow[d] & L[u] \arrow[d] & \\
\bigoplus_{n\leq 2k+1} J^n \bar{u}^n \arrow[r, phantom, "\subset"] & L[u]/(1-u^2)^{k+1} \arrow[r, "\sim"] & L[u]/(u-1)^{k+1} \times L[u]/(u+1)^{k+1}
\end{tikzcd}$$

Note that

$$\tfrac{1}{2}(\delta_1 - \delta_{-1})(1-D)\left(1-\frac{D}{3}\right)\cdots\left(1-\frac{D}{2k-1}\right) : L[u] \longrightarrow L$$

kills Lu^{2n} for all n and $Lu, Lu^3, \ldots, Lu^{2k-1}$. Also it kills the ideal $(u^2-1)^{k+1}L[u]$, and we know it's nontrivial on $Lu(u^2-1)^k$. We can understand this map (really it amounts to a linear functional on $\mathbb{C}[u]$) as the odd linear functional on $\mathbb{C}[u]$ obtained from the one on $L[u]/(u-1)^{k+1}$ and the corresponding one on $L[u]/(u+1)^{k+1}$ (via $u \mapsto -u$) which vanishes on $u, u^3, \ldots, u^{2k-1}$ and is non-trivial. The dimensions check since $L[u]/(u-1)^{k+1}$ has dim. $k+1$.

What I want to look at concerns the behavior of the canonical filtrations on X complexes associated to the homom.

$$\begin{tikzcd}
RA \arrow[r] & \left(\bigoplus_{n\geq 0} J^n u^n\right) \otimes RB \arrow[d] \\
& L[u]/(u-1)^{k+1} \otimes RB
\end{tikzcd}$$

What's going on is that we have a variation of a homomorphism $RA \longrightarrow L \otimes RB$

Consider (R, I) and (S, J), and a homomorphism $R \xrightarrow{p^t} S[[t]]$. To begin we consider the weakest assumption: $p^t(I) \subset J + S[[t]]t$, i.e. $p_0(I) \subset J$. Then

$$I \longrightarrow J + St + St^2 + \cdots$$
$$I^2 \longrightarrow J^2 + Jt + St^2 + \cdots$$
$$I^3 \longrightarrow J^3 + J^2t + Jt^2 + St^3 + \cdots$$

so we have $\boxed{p_k(I^n) \subset J^{n-k}}$

Consider the map of supercomplexes

$$p^t_* : X(R) \longrightarrow X(S)[[t]]$$

I would like to understand the behavior of this map with respect to the adic filtrations on the X complexes. Specifically to show that

$$(p^t_*)_n : X(R) \longrightarrow X(S)$$

(the coefficient of t^n) carries $F^p_I X(R)$ into $F^{p-2n}_J X(S)$. The idea is to reduce to the case of a derivation, more precisely, to the case where $(S, J) = (R, I)$ and $p^t = e^{tD} : R \longrightarrow R[[t]]$ with D a derivation of R. In this case $p^t_* = e^{tL_D}$, where L_D is the Lie derivative on $X(R)$ belonging to D, and $(p^t_*)_n = L^n_D$, so the desired result follows from

$$L_D F^p_I X(R) \subset F^{p-2}_I X(R).$$

Consider the canonical factorization

$$R \hookrightarrow \mathcal{D}R \xrightarrow{e^{tD}} \mathcal{D}R[[t]] \longrightarrow S[[t]]$$

of p^t. [redacted] ~~Recall $\mathcal{D}R$ is the universal algebra with derivation generated by R. Also $R \mapsto \mathcal{D}R$ is left adjoint to $S \mapsto S[[t]]$.~~

Recall $R \mapsto \mathcal{D}R$ is left adjoint to $S \mapsto S[[t]]$:

$$\mathrm{Hom}_{algs}(\mathcal{D}R, S) = \mathrm{Hom}_{alg}(R, S[[t]])$$

$\mathcal{D}R$ is generated by canonical linear maps $p_k : R \longrightarrow \mathcal{D}R$, $k \geq 0$ subject to the relations saying that $p^t = \sum p_k t^k : R \longrightarrow \mathcal{D}R[[t]]$ is a homomorphism. $\mathcal{D}R$ has a canonical grading wrt $\mathbb{N}$ where $|p_k(x)| = k$ and a canonical derivation D of degree $+1$ such that $p_k = \frac{D^k}{k!} p_0$. $\mathcal{D}R$ is the universal algebra with derivation generated by R.

Starting with $I \subset R$ we obtain an ideal J in $\mathcal{D}R$ defined as the ideal generated by $p_0(I)$. [redacted] Since the homom. $\mathcal{D}R \xrightarrow{\varphi} S$ carrying the universal $p^t = e^{tD} p_0 : R \longrightarrow \mathcal{D}R[[t]]$ to $p^t : R \longrightarrow S[[t]]$ carries $p_0 : R \hookrightarrow \mathcal{D}R$ to $p_0 : R \longrightarrow S$, we have $\varphi(J \text{ in } \mathcal{D}R) = \varphi(\text{ideal gen by } p_0 I)$ $\subset$ ideal gen by $\varphi p_0(I) = p_0(I)$ in S $\subset J$.

Thus [redacted] $\varphi : \mathcal{D}R \longrightarrow S$ carries the situation $R \xrightarrow{p^t} \mathcal{D}R[[t]]$, $I \longrightarrow J + \mathcal{D}R[[t]]t$ (i.e. $p_0 : I \longrightarrow J$) in the given $R \xrightarrow{p^t} S[[t]]$, $p_0 : I \longrightarrow J$.

Consider two homs. $A \underset{\theta^\gamma}{\overset{\theta}{\rightrightarrows}} L$ congruent modulo the ideal $J \subset L$. Form the poly family of homs

$$u^t : RA \longrightarrow L$$

$$u^t \rho(a) = pa + tqa \qquad p = \frac{\theta+\theta^\gamma}{2} \qquad q = \frac{\theta-\theta^\gamma}{2}$$

The curvature of $p+tq$ is

$$\begin{aligned}(p+tq)b' - (p+tq)^2 &= p^2+q^2+t(pq+qp) - (p^2+t(pq+qp)+t^2q^2) \\ &= (1-t^2)\,q^2\end{aligned}$$

Thus $p+tq$ is a homomorphism for $t=\pm 1$ and a hom modulo J^2. $u_t : (RA, IA) \longrightarrow (L, J^2)$ is not restricted, although to (L,J) it is.

Let $H: X(RA) \longrightarrow X(L)$ be the odd operator such that $[\partial, H] = u^1_* - u^{-1}_*$ given by the integrated Cartan homotopy formula for u^t, $t \in [-1,1]$. Then $H\left(F^p_{IA} X(RA)\right) \subset F^{p-1}_{J^2} X(L)$ and H restricted to $F^p_{IA} X(RA)$ commutes with ∂ for $p \geq 1$, because $u^{\pm 1}_*$ factor through the quotient $X(RA)/F^1_{IA}X(RA) = X(A)$. Thus we get induced maps for $m \geq 1$

$$\begin{aligned}\widehat{HC}_{2m} A &= H_-\left(F^{2m}_{IA} X(RA)\right) \longrightarrow H_+\left(F^{2m-1}_{J^2} X(L)\right) \\ &= H_+\left(J^{2m} \leftrightarrows \natural\left(J^{2m}dL + J^{2m-2}d(J^2)\right)\right) \\ &\subset J^{2m}/[J^{2m-2}, J^2]\end{aligned}$$

Remarks: 1) The above construction is motivated by the idea that given

$$\begin{tikzcd} & L \arrow[d] \\ A \arrow[ur, shift left, "\theta"] \arrow[ur, shift right, "\theta^\tau"'] \arrow[r] & L/J \end{tikzcd} \qquad \tau \text{ on } J^p$$

τ defines an odd class on L/J which becomes trivial on A for two reasons and the difference of these reasons is an even class on A.

2) The above construction seems to break down when τ is defined on J and one wants to obtain the element of $\overline{HC}^0 A = \bar{A}^*$ given by τg. My feeling is that this difficulty arises because we use odd cocycles on the subcomplexes $F^p_{IA} X(RA)$, and we perhaps want to trangress them to even cocycles on $\chi^p(RA, IA)$. Notice that extending cocycle on $F^p_{IA}X(RA) = b\Omega^{p+1}A \oplus \Omega^{\geq p}A$ by zero on $\Omega^{<p}A$ and then applying the boundary yields a cocycle supported in a single degree. More precisely, let f be an odd cocycle on $F^{2m}_{IA}X(RA) = b\Omega^{2m+1} + \Omega^{2m+1} + \cdots$ Then $f = (f_{2m+1}, f_{2m+3}, \cdots)$, where $f_{2m+1}b = f_{2m+3}B$, $f_{2m+3}b = f_{2m+5}B$, etc. Then extending f by $f_{2m-1} = f_{2m-3} = \cdots 0$ and applying $B-b$ gives $f_{2m+1}B$ in degree $2m$ and all other components 0. This is a reduced cyclic cocycle: $(f_{2m+1}B)b = -f_{2m+1}bB = -f_{2m+3}B^2 = 0$.

June 5, 1993

Lemma: $I \subset R$ ideal. Then the canonical map $X(S\otimes R) \longrightarrow S_\natural \otimes X(R)$ carries $F^p_{S\otimes I} X(S\otimes R)$ into $S_\natural \otimes F^p_I X(R)$.

Proof: $(S\otimes I)^p = S\otimes I^p \subset S\otimes R$.

$$[(S\otimes I)^n, (S\otimes R)] \subset [S\otimes I^n, S\otimes R] \subset S\otimes [I^n, R]$$

Thus

$$(S\otimes I)^{n+1} \longrightarrow S_\natural \otimes I^{n+1}$$

$$[(S\otimes I)^n, (S\otimes R)] \longrightarrow S_\natural \otimes [I^n, R]$$

Also in $\Omega^1_S(S\otimes R) = S\otimes \Omega^1 R$ we have

$$(S\otimes I)^{n+1} d(S\otimes R) \subset S\otimes I^{n+1} \cdot S\otimes dR \subset S\otimes I^{n+1} dR$$

$$(S\otimes I)^n d(S\otimes I) \subset S\otimes I^n \cdot S\otimes dI \subset S\otimes I^n dI.$$

Suppose now that we are given algebras R, R', L an ideal $J \subset L$ and homomorphisms

$$R \overset{\theta}{\underset{\bar\theta}{\rightrightarrows}} L\otimes R' \qquad \text{congruent modulo } J\otimes R'.$$

Consider then $R \xrightarrow{(\theta, g)} (L\oplus J)\otimes R'$ where $L\oplus J$ is the semi-direct product algebra in which $J^2 = 0$ and $g = \bar\theta - \theta$. This is a homomorphism, so we get an induced map

NO

$$(\theta, g)_* : X(R) \longrightarrow (L\oplus J)_\natural \otimes X(R')$$

Now $(L \oplus J)_\natural = L_\natural \oplus J/[L,J]$

In particular we get a map

$$X(R) \longrightarrow J/[L,J] \otimes X(R')$$

which we might think of as a Lie derivative $L(\theta, \mathring{\theta})$, where $\theta : R \longrightarrow L \otimes R'$ and $\mathring{\theta} = \bar{\theta} - \theta : R \longrightarrow J \otimes R'$ is a derivation relative to θ. Thus NO

$$L(\theta,\mathring{\theta})(x) = \mathring{\theta}(x) \in J \otimes R' \longrightarrow J/[L,J] \otimes R'$$

$$L(\theta,\mathring{\theta})\natural(x\,dy) = \natural(\mathring{\theta}x\, d\theta y + \theta x\, d\mathring{\theta} y)$$

$$\in \natural(J \otimes R' \cdot L \otimes \Omega' R' + L \otimes R' \cdot J \otimes \Omega' R')$$

$$\longrightarrow J/[J,L] \otimes \Omega' R'_\natural$$

The next point is that if $I \subset R$, $I' \subset R'$ are ideals compatible with $\theta, \bar{\theta}$:

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & L \otimes R' \arrow[d] \\ R/I \arrow[r, shift left] \arrow[r, shift right] & L \otimes R'/I' \end{tikzcd}$$

Then we have

$$\begin{tikzcd} R \arrow[r, "{(\theta,q)}"] \arrow[d] & (L \oplus J) \otimes R' \arrow[d] \\ R/I \arrow[r] & (L \oplus J) \otimes R'/I' \end{tikzcd}$$

i.e. (θ, q) carries I into $(L \oplus J) \otimes I'$, so by the lemma above

$$(\theta, \bar{q})_* : X(R) \longrightarrow \left(L_\natural \oplus J/[J,L]\right) \otimes X(R')$$

carries $F^p_I X(R)$ into $\left(L_\natural \oplus J/[J,L]\right) \otimes F^p_{I'} X(R')$.

At this point then we have the bivariant Chern character of degree 0, i.e. given $A \rightrightarrows L \otimes B$ congruent mod $J \otimes B$ we have corresponding homomorphisms

$$RA \rightrightarrows L \otimes RB \quad \text{cong mod } J \otimes RB$$

carrying IA into $L \otimes IB$. From the above discussion we get

$$X(RA) \longrightarrow J/[J,L] \otimes X(RB)$$

carrying $F^p_{IA} X(RA)$ to $J/[J,L] \otimes F^p_{IB} X(RB)$ for all p. Thus a trace on J as L-bimodule determines a map of towers $\mathcal{X}_A \longrightarrow \mathcal{X}_B$ of order ≤ 0.

Here's a correct version of ~~variation on~~ the preceding, ~~where we keep things symmetrical~~. Consider

$$A \longrightarrow (L + tJ) \otimes B$$

$$a \longmapsto pa + tqa$$

$$(p+tq)b' - (p+tq)^2$$
$$= p^2+q^2+t(pq+qp)$$
$$-(p^2+tpq+tqp+t^2q^2)$$
$$= (1-t^2)q^2$$

where $t^2 = 1$. This is a homomorphism A) hence we get an induced map

$$X(A) \longrightarrow (L+tJ)_\natural \otimes X(B)$$

But $[L+tJ, L+tJ] = [L,L] + t[J,L] + [J,J] = [L,L] + t[J,L]$ so that $(L+tJ)_\natural = L_\natural \oplus tJ/[L,J]$

Thus we get

$$X(A) \longrightarrow \left(L_\natural \oplus tJ/[J,L]\right) \otimes X(B)$$

Moreover if we do this for

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\rho"] & (L+tJ)\otimes B \arrow[d, "1\otimes\rho"] \\
RA \arrow[r, "\psi"] \arrow[d] & (L+tJ)\otimes RB \arrow[d] \\
A \arrow[r] & (L+tJ)\otimes B
\end{tikzcd}$$

then it is clear that ψ carries IA into $(L+tJ)\otimes IB$ and thus

$$\psi_* : X(RA) \longrightarrow \left(L_\natural + t\, J/[L,J]\right) \otimes X(RB)$$

carries $F^p_{IA} X(RA)$ into $\text{————} \otimes F^p_{IB} X(RB)$

situation: $A \underset{\theta^\gamma}{\overset{\theta}{\rightrightarrows}} L\otimes B$ cong. mod $J\otimes B$

$S = \bigoplus_{n\geq 0} t^n J^n \subset L[t]$. One has a [redacted] based linear map

$$(*) \qquad A \longrightarrow S\otimes B \qquad\qquad p = \frac{\theta+\theta^\gamma}{2} : A \to L\otimes B$$

$$a \longmapsto pa + tqa \qquad\qquad q = \frac{\theta-\theta^\gamma}{2} : \bar{A} \to J\otimes B$$

with curvature $(1-t^2)q^2 : \bar{A}^{\otimes 2} \longrightarrow (1-t^2)J^2\otimes B$.

Let K be the ideal $(1-t^2)J^n S = \sum_{n\geq 0}(1-t^2)t^n J^{n+2}$ in S. [redacted] If we replace S by S/K^m in $(*)$ we obtain a based linear map with nilpotent curvature. Assuming A quasi-free [redacted] there is a systematic way to straighten such maps to homomorphisms. Thus we get a [redacted] compatible family of homs.

$$A \longrightarrow S/K^m \otimes B$$

whence a compatible family of maps of supercomplexes

$$X(A) \longrightarrow (S/K^m)_\natural \otimes X(B)$$

Let us now study S/K^m and $(S/K^m)_\natural$. The following should be true

Prop: [redacted] The obvious maps

$$\bigoplus_{n<2m} t^n J^n \longrightarrow S/K^m$$

$$L_\natural \oplus \bigoplus_{0<n<2m} t^n J^n/[J^{n-1},J] \longrightarrow (S/K^m)_\natural$$

are isomorphisms.

Proof. Let's check for $m = 1$ carefully.
We have for $m=1$ a comm. square

$$\begin{tikzcd} L \oplus tJ \arrow[r, "\alpha"] \arrow[d, hook] & S/K \arrow[d] \\ L \oplus tL \arrow[r, "\sim"] & L[t]/(1-t^2)L[t] \end{tikzcd}$$

The top arrow α is surjective because for $t^n x \in t^n J^n$ $\{n \geq 2\}$ we have $t^n x = t^{n-2}x - (1-t^2)t^{n-2}x$ $\equiv t^{n-2}x$ mod K. The square then shows that α is an isomorphism. Next we have

$$\begin{aligned} (S/K)_\natural &= L+tJ/[L+tJ, L+tJ] \\ &= L+tJ/[L,L]+[J,J]+t[L,J]+t[L,J] \\ &= L_\natural + t\,J/[J,L] \end{aligned}$$

For general m we have

$$\begin{tikzcd} L \oplus tJ \oplus \cdots \oplus t^{2m-1}J^{2m-1} \arrow[r, "\alpha"] \arrow[d, hook] & S/K^m \arrow[d] \\ L \oplus tL \oplus \cdots \oplus t^{2m-1}L \arrow[r, "\sim"] & L[t]/(1-t^2)^m L[t] \end{tikzcd}$$

α surjective because given $t^n x \in t^n J^n$ for $n \geq 2m$ we have

$$t^n x = \underbrace{t^{n-2m}(t^2-1)^m x}_{\in K^m} + \underbrace{t^{n-2m}\left(t^{2m} - (t^2-1)^m\right)x}_{\in t^{n-2}J^n + t^{n-4}J^n + \cdots}$$

So by induction $t^n J^n \subset \mathrm{Im}(\alpha)$ for $n \geq 2m$.
Since S/K^m is generated by L, tJ
the comm. quotient space is obtained by dividing by

brackets with L and tJ. Now

$$[L, L\oplus tJ\oplus\cdots\oplus t^{2m-1}J^{2m-1}]$$
$$\subset [L,L]\oplus t[L,J]\oplus\cdots\oplus t^{2m-1}[L,J^{2m-1}]$$

$$[tJ, L\oplus tJ\oplus\cdots\oplus t^{2m-1}J^{2m-1}]$$
$$\subset t[J,L]\oplus t^2[J,J]\oplus\cdots\oplus t^{2m-1}[J,J^{2m-2}]\oplus t^{2m}[J,J^{2m-1}]$$

However modulo $K^m = (1-t^2)^m J^{2m}$ we have

$$t^{2m}x \equiv \left(t^{2m} - (t^2-1)^m\right)x$$

$= \sum_{j=0}^{2m-2} t^{2j}$ [redacted] $\binom{*}{*}x$

so $t^{2m}[J,J^{2m-1}] \subset [J,J^{2m-1}] + t^2[J,J^{2m-1}] + \cdots + t^{2m-2}[J,J^{2m-1}]$

$$\subset [L,L] + t^2[J,J] + \cdots + t^{2m-2}[J,J^{2m-3}]$$

Thus $(S/K^m)_\natural = L/[L,L]\oplus tJ/[L,J]\oplus t^2J^2/[J,J]\oplus\cdots$

$$\oplus\ t^{2m-1}J^{2m-1}/[J,J^{2m-2}].$$

At this point we have lots of maps from $X(A)$ to $X(B)$, [redacted] namely, associated to continuous traces on S for the K-adic topology. The point is to [redacted] understand when they are equivalent

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Review: Assuming A quasi-free we have constructed compatible maps

$$X(A) \longrightarrow (S/K^{m+1})_\natural \otimes X(B)$$

Moreover we have

$$L_\natural \oplus tJ_\# \oplus\cdots\oplus t^{2m+1}J^{2m+1}_\# \xrightarrow{\sim} (S/K^{m+1})_\natural$$

hence we have a canonical trace

$$S/K^{m+1} \longrightarrow J_{\#}^{2m+1}$$

vanishing on the image of $t^n J^n$ for $0 \le n < 2m+1$ and sending $t^{2m+1}x$, $x \in J^{2m+1}$ to the image of x in $J_{\#}^{2m+1}$.

I claim we can compare these canonical traces for different m as follows, namely, one has a commutative square

$$\begin{tikzcd} S/K^{m+1} \arrow[r, "\text{canon}"] \arrow[d, "1-\frac{D}{2m-1}"'] & J_{\#}^{2m+1} \arrow[d, "\frac{-2m}{2m-1}\cdot\text{ obvious map}"] \\ S/K^{m} \arrow[r, "\text{canon}"] & J_{\#}^{2m-1} \end{tikzcd}$$

$D = t\frac{d}{dt}$

Suffices to check this on $t^n J^n$ for $0 \le n \le 2m+1$. For $n < 2m-1$, $1-\frac{D}{2m-1}$ maps $t^n J^n$ into the corresponding subspace of S/K^m which is killed by the canonical trace. For $n = 2m-1$, $1-\frac{D}{2m-1}$ kills $t^{2m-1}J^{2m-1}$ obviously. For $n = 2m$, $1-\frac{D}{2m-1}$ map $t^{2m}J^{2m}$ into the even subspace of S/K^m which is killed by the canonical trace. For $n = 2m+1$ and $t^{2m+1}x \in t^{2m+1}J^{2m+1}$

$$\left(1-\frac{D}{2m-1}\right)t^{2m+1}x = \left(1-\frac{2m+1}{2m-1}\right)t^{2m+1}x = \frac{-2}{2m-1}t^{2m+1}x$$

and

$$t^{2m+1}x - \underbrace{t(t^2-1)^m x}_{\in K^m} = \left(m t^{2m-1} + \text{lower powers of } t\right)x$$

is mapped by the canonical trace to mx in $J_{\#}^{2m-1}$.

Clear.

Proposition: Let S, T be algebras, let

$$\alpha : X(S\otimes T) \longrightarrow S_\natural \otimes X(T)$$

be the canonical map

$$\alpha(s\otimes t) = \natural(s)\otimes t$$

$$\alpha\big((s_1\otimes t_1)\,d(s_2\otimes t_2)\big) = \natural(s_1 s_2)\otimes \natural(t_1\,dt_2)$$

(Alternatively is the map from $X(S\otimes T)$ to the relative complex $X_S(S\otimes T) = S_\natural \otimes X(T)$.)

Let $I\subset S$, $J \subset T$ be ideals. Then for all p one has

$$\boxed{\alpha\left(F^p_{S\otimes J + I\otimes T}X(S\otimes T)\right) \subset \sum_{i\geq 0} \natural(I^i)\otimes F_J^{p-2i}X(T)}$$

Proof: Identify $s\otimes 1$, $1\otimes t$ with s, t and suppress $\otimes$ signs, so that $s\otimes t = st = ts$. One can suppose $p\geq 0$ since $F^p_J X(T) = X(T)$ for $p<0$.

Then

$$F^{2n+1}_{SJ+IT}X(ST)_+ = (SJ+IT)^{n+1}$$

$$\subset \sum_{i=0}^{n+1} I^i J^{n+1-i}$$

$$\xrightarrow{\alpha} \sum_{i=0}^{n+1} \natural(I^i)\, J^{n+1-i}$$

$$= \sum_{i=0}^{n+1} \natural(I^i)\, F_J^{2n-2i+1}X(T)_+$$

for $n\geq 0$. Also

$$\subset \sum_{i\geq 0}\natural(I^i)\,F_J^{2n+1-2i}X(T)_+$$

$$F^{2n}_{SJ+IT}X(ST)_+ = (SJ+IT)^{n+1} + [(SJ+IT)^n, ST]$$

$$\subset \sum_0^{n+1} I^i J^{n+1-i} + \sum_0^n \underbrace{[I^iJ^{n-i}, ST]}$$

$$\subset [I^i, ST]J^{n-i} + I^i[J^{n-i}, ST]$$

$$\subset [I^i,S]TJ^{n-i} + I^iS[J^{n-i},T] \subset [I^i,S]J^{n-i} + I^i[J^{n-i},T]$$

Thus

$$F^{2n}_{SJ+IJ} X(ST)_+ \subset \sum_0^{n+1} I^i J^{n+1-i} + \sum_0^n \left([I^i, S] J^{n-i} + I^i [J^{n-i}, T]\right)$$

$$\xrightarrow{\alpha} \sum_0^{n+1} \natural(I^i)\, J^{n-i+1} + \sum_0^n \natural(I^i)\,[J^{n-i}, T]$$

$$\subset \sum_0^n \natural(I^i)\, F_J^{2n-2i} X(T)_+ + \natural(I^{n+1})\,T$$

$$\subset \sum_{i \geq 0} \natural(I^i)\, F_J^{2n-2i} X(T)_+$$

$$F^{2n}_{SJ+IT} X(ST)_- = \natural\left((SJ+IT)^n d(ST)\right)$$

$$\subset \sum_{i=0}^n \natural\left(I^i J^{n-i} (dST + SdT)\right)$$

$$\subset \sum_{i=0}^n \natural\left(I^i J^{n-i} (dS + dT)\right)$$

$$\xrightarrow{\alpha} \sum_{i=0}^n \natural(I^i)\, \natural(J^{n-i} dT)$$

$$\subset \sum_{i \geq 0} \natural(I^i)\, F_J^{2n-2i} X(T)_-$$

$$F^{2n+1}_{SJ+IT} X(ST)_- = \natural\left((SJ+IT)^{n+1} d(ST) + (SJ+IT)^n d(SJ+IT)\right)$$

$$\subset \sum_{i=0}^{n+1} \natural\left(I^i J^{n+1-i} (dS + dT)\right) +$$

$$\sum_{i=0}^n \natural\left(I^i J^{n-i} (dS\,J + S\,dJ + dI\,T + I\,dT)\right)$$

$$\subset \sum_{i=0}^{n+1} \natural\left(I^i J^{n+1-i} (dS + dT)\right) + \sum_{i=0}^n \natural\left(I^i J^{n+1-i} \cancel{dS} + I^i J^{n-i} \cancel{dI}\right)$$

$$+ \sum_{i=0}^n \natural\left(I^i J^{n-i} dJ + I^{i+1} J^{n-i} dT\right)$$

$$\xrightarrow{\alpha} \sum_0^{n+1} \natural(I^i)\,\natural(J^{n+1-i}dT)$$

$$+ \sum_0^{n} \natural(I^i)\,\natural(J^{n-i}dJ) \quad + \underbrace{\sum_0^{n} \natural(I^{i+1})\,\natural(J^{n-i}dT)}_{\sum_1^{n+1} \natural(I^i)\,\natural(J^{n+1-i}dT)}$$

$$\subset \sum_0^{n} \natural(I^i)\,\natural(J^{n-i+1}dT + J^{n-i}dJ)$$

$$+ \natural(I^{n+1})\,\natural(T\,dT)$$

$$\subset \sum_{i\geq 0} \natural(I^i)\, F_J^{2n+1-2i}X(T).$$

Application: Consider a homomorphism

$$u : R \longrightarrow S \otimes T$$

where $S = S_0 + S_1 + \cdots$ is a graded algebra. Thus $u = \sum u_n$ where $u_n : R \longrightarrow S_n \otimes T$ are linear maps such that $u_n b' = \sum u_i u_{n-i}$ in the evident sense. Note that

$$\mathring{u} = \sum n u_n : R \longrightarrow S \otimes T$$

is a derivation relative to u.

Let $I \subset R$, $J \subset T$ be ideals such that $u_0(I) \subset S_0 \otimes J$. Consider the ideals $S^{>0} = \bigoplus_{n>0} S_n \subset S$ and $J \subset T$. Then

$$S \otimes J + S^{>0} \otimes T = S_0 \otimes J + S^{>0} \otimes T$$

and we have

$$u : I \longrightarrow S \otimes J + S^{>0} \otimes T$$

$$\mathring{u} : R \longrightarrow S^{>0} \otimes T \subset S \otimes J + S^{>0} \otimes T$$

Thus we have a restricted situation and assuming R quasi-free and ϕ given for R we know the odd map $h^\phi(u,\dot{u}) : X(R) \longrightarrow X(S\otimes T)$ carries $F^p_I X(R)$ into $F^p_{S\otimes J + S^{>0}\otimes T} X(S\otimes T)$.

Composing with α shows that

$$\alpha h^\phi(u,\dot{u}) : X(R) \longrightarrow S_\natural \otimes X(T)$$

carries $F^p_I X(R)$ into $\sum_{i\geq 0} \natural\left((S^{>0})^i\right) \otimes F^{p-2i}_J X(T)$

$$\subset \sum_{i\geq 0} \natural\left(S_i + S_{i+1} + \cdots\right) \otimes F^{p-2i}_J X(T)$$

$$\subset \natural(S_0) \otimes F^p_J X(T) \oplus \natural(S_1)\otimes F^{p-2}_J X(T) \oplus \cdots$$

$$= \sum_{n\geq 0} \natural(S_n) \otimes F^{p-2n}_J X(T).$$

(The term for $n=0$ in $\alpha h^\phi(u,\dot{u})$ should be zero as $\dot{u}$ has image in $S^{>0}\otimes T$.)

Special case I had in mind

$$R \xrightarrow{u_0 + \varepsilon u_1} T + \varepsilon T = (\mathbb{C} \oplus \mathbb{C}\varepsilon)\otimes T$$

$$I \longrightarrow J + \varepsilon T$$

i.e. $u_0(I) \subset J$

Then I find that

$$h^\phi(u_0, u_1) : X(R) \longrightarrow X(T)$$

which should be the coefficient of ε in $h^\phi(u_0 + \varepsilon u_1, \varepsilon u_1)$, carries $F^p_I X(R)$ to $F^{p-2}_J X(R)$.

Review

$$A \overset{\theta}{\underset{\bar\theta}{\rightrightarrows}} L\otimes B \quad \text{cong mod } J\otimes B$$

$$\mathcal{S} = \bigoplus_{n\geq 0} t^n J^n \subset \mathbb{C}[t]\otimes L$$

$$\rho = \frac{\theta+\bar\theta}{2} : A \longrightarrow L\otimes B$$

$$q = \frac{\theta-\bar\theta}{2} : \bar{A} \longrightarrow J\otimes B$$

Consider based linear map

(1) $$A \longrightarrow S\otimes B$$
$$a \longmapsto \rho a + t q a$$

The curvature is $(1-t^2)q^2 : \bar{A}^{\otimes 2} \longrightarrow (1-t^2)J^2\otimes B$

$$K = (1-t^2)J^2\cdot S \quad \text{ideal} \subset S.$$

Thus $A \longrightarrow (S/K)\otimes B$ is a homomorphism.

From (1) we obtain

(2)
$$\begin{tikzcd}
RA \arrow[r] & S\otimes RB \\
A \arrow[u, "\rho"] \arrow[r, "\rho+tq"] & S\otimes B \arrow[u, "1\otimes\rho"']
\end{tikzcd}$$

such that this square commutes.

Then $RA \longrightarrow S\otimes RB$

sends $IA \longrightarrow K\otimes RB + S\otimes IB$

so we get

(3) $$X(RA) \longrightarrow S_\natural \otimes X(RB)$$

$$F^p_{IA}X(RA) \longrightarrow \sum_{i\geq 0} \natural(K^i)\otimes F^{p-2i}_{IB} X(RB)$$

$$\bigoplus_{0\le n\le 2m+1} t^n J^n \xrightarrow{\sim} S/K^{m+1}$$

$$\bigoplus_{0\le n\le 2m+1} t^n J^n_\# \xrightarrow{\sim} \left(S/K^{m+1}\right)_\natural$$

Hence there is a canonical trace

$$\left(S/K^{m+1}\right)_\natural \longrightarrow J^{2m+1}_\#$$

which is specified by the properties that it kills t^nJ^n for $0\le n\le 2m$ and sends $\natural(t^{2m+1}x)$, $x\in J^{2m+1}$ to the image of x in $J^{2m+1}_\#$. Compatibility:

$$D = t\frac{d}{dt} \qquad \begin{tikzcd} S/K^{m+1} \arrow[r, "\text{can.}"] \arrow[d, "1-\frac{D}{2m-1}"] & J^{2m+1}_\# \arrow[d, "\frac{-2m}{2m-1}\cdot\text{ obvious map}"] \\ S/K^m \arrow[r, "\text{can}"] & J^{2m-1}_\# \end{tikzcd}$$

Better to define $S/K^{m+1} \xrightarrow{\tau_m} J^{2m+1}_\#$ to be the trace vanishing on t^nJ^n for $0\le n\le 2m$ and such that $\tau_m(t^{2m+1}x) = \dfrac{(-1)^m 2^m m!}{1\cdot 3\cdots(2m-1)}\#(x)$ for $x\in J^{2m+1}$. Then we have for $m\ge 1$

$$\begin{tikzcd} S/K^{m+1} \arrow[r, "\tau_m"] \arrow[d, "1-\frac{D}{2m-1}"] & J^{2m+1}_\# \arrow[d, "i_\#"] \\ S/K^m \arrow[r, "\tau_{m-1}"] & J^{2m-1}_\# \end{tikzcd}$$

$i : J^{2m+1} \to J^{2m-1}$ the inclusion

1993 In addition to

$$\psi_* : X(RA) \longrightarrow X(S\otimes RB) \longrightarrow S_\natural \otimes X(RB)$$

$$F^p_{IA} X(RA) \longrightarrow F^p_{K\otimes RB + S\otimes IB} X(S\otimes RB) \longrightarrow \sum_{i\geq 0} \natural(K^i)\otimes F^{p-2i}_{IB} X(RB)$$

I need to consider $D_*\psi_*$ where $D =$ the derivation $t\frac{d}{dt}$ on S, and D_* denoted the induced map (in this case on $S_\natural \otimes X(RB)$).

One has $\psi : RA \longrightarrow S\otimes RB$ and $\dot\psi = (D\otimes 1)\psi : RA \longrightarrow S\otimes RB$. We then get from the Cartan homotopy formula an odd map $h = h(\psi,\dot\psi) : X(RA) \longrightarrow X(S\otimes RB)$ such that

$$\begin{tikzcd} X(RA) \arrow[r, "\psi_*"] \arrow[rr, bend right, "{[\partial, h]}"'] & X(S\otimes RB) \arrow[r, "D_*"] & X(S\otimes RB) \end{tikzcd}$$

Moreover $h : F^p_{IA} X(RA) \longrightarrow F^{p-2}_{K\otimes RB + S\otimes IB} X(S\otimes RB)$.

This gives

$$\begin{tikzcd} X(RA) \arrow[r, "\psi_*"] \arrow[rr, bend right, "{[\partial, h]}"'] & S_\natural \otimes X(RB) \arrow[r, "D_*"] & S_\natural \otimes X(RB) \end{tikzcd}$$

where $h : F^p_{IA} X(RA) \longrightarrow \sum_{i\geq 0} \natural(K^i)\otimes F^{p-2-2i}_{IB} X(RB)$