April 23, 1992

Recall that a tower of $(\mathbb{Z}/2)$-graded complexes M/F^nM such that $M = F^{-1}M$ and $F^nM/F^{n+1}M$ has zero homology of parity n [redacted] has an associated cyclic 'theory'. So we have three different models for a cyclic 'theory': ⓐ mixed complexes (DG modules over $\mathbb{C}[B]$) ⓑ DG comodules over $\mathbb{C}[u]$, $\deg(u) = 2$ ⓒ towers as above. It would be nice to understand precisely the relations between these models. Recall that ⓐ, ⓑ are linked because $\mathbb{C}[u]$ is the bar construction on $\mathbb{C}[B]$; see Kassel-Husemoller notes.

[redacted]

We can study towers as above via the idea of Postnikov systems. Suppose

$$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow 0$$

is an exact sequence of $(\mathbb{Z}/2)$-graded complexes. Choose a section l of p, whence we obtain an odd map $f : Z \longrightarrow X$ given by $if = [d, l]$. We can view f as the map $Z \longrightarrow \Sigma X$ in the Puppe sequence and identify Y with the mapping fibre of f. This ought to mean that Y is determined by Z, X and the homotopy class of $f \in H^1(\mathrm{Hom}(Z, X))$.

Notice that over a field any $(\mathbb{Z}/2)$ graded complex is quis to its homology equipped with zero differential. I think this means that given the tower M/F^nM as above, then we can construct a quasi-isomorphic tower which is <u>minimal</u> in the sense that $F^nM/F^{n+1}M$ is concentrated in parity $n+1$. ~~What this means is that~~ Thus

$$\hat{M}^{ev} \simeq \prod HH_{2n} \quad , \quad \hat{M}^{odd} \simeq \prod HH_{2n+1} .$$

Let's discuss the first steps of the tower. It starts with M/F^0M which has the form

$$\begin{tikzcd} HH_0 \arrow[r, shift right] & 0 \arrow[l, shift right] \end{tikzcd}$$

The next stage adds HH_1 in odd degree:

$$\begin{tikzcd} HH_0 \arrow[dr] & 0 \\ 0 & HH_1 \end{tikzcd}$$

the differential being the map $HH_0 = HC_0 \xrightarrow{B} HH_1$. The next step is to add HH_2 in even degree, the attaching map being $HC_1 \xrightarrow{B} HH_2$. This gives for the next two steps.

$$\begin{tikzcd} HH_0 \arrow[dr] \arrow[dddr, leftrightarrow] & \\ & HH_1 \arrow[dl] \\ HH_2 \arrow[dr] & \\ & HH_3 \end{tikzcd}$$

One gets the impression in working with this that the complex obtained up to isomorphism depends only on the Connes exact sequence. Indeed the k-invariant for

$$0 \to F^n/F^{n+1} \to M/F^{n+1} \to M/F^n \to 0$$

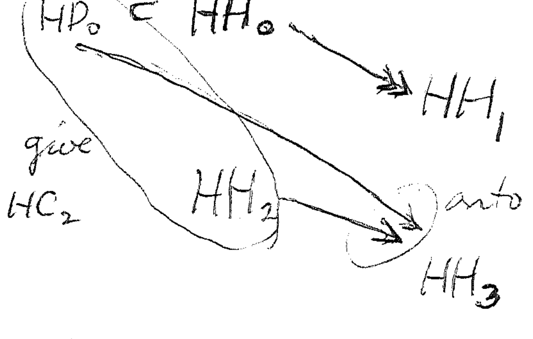
is the map

$$\begin{tikzcd} H_{n+2\mathbb{Z}}(M/F^n) \arrow[r, "B"] \arrow[d, equal] & H_{n+1+2\mathbb{Z}}(F^n/F^{n+1}) \arrow[d, equal] \\ HC_n & HH_{n+1} \end{tikzcd}$$

Next we note that the Connes exact sequence really depends only on the inverse systems HC_{2n} and HC_{2n+1}, i.e. on $\bigoplus HC_n$ as $\mathbb{C}[S]$-modules. In effect we have

$$HC_{n+2} \xrightarrow{S} HC_n \xrightarrow{B} HH_{n+1} \longrightarrow HC_{n+1} \xrightarrow{S} HC_{n-1}$$

so HH_{n+1} (up to isomorphism) is the sum of $HC_n/\mathrm{Im}\, S$ and $\mathrm{Ker}\, S$ on HC_{n+1}. The Connes exact sequence can be split into even and odd parts (the even part having all HC_{2n}). It seems that the minimal total corresponding to an even system $\{HC_{2n}\}$ (i.e. $HC_{odd} = 0$) has surjective differential from even to odd. This is clear inductively



Let's next discuss some examples. To begin recall that $\bigoplus HC_n$ is a graded $\mathbb{C}[S]$-module, and if we restrict to systems such that $HH_n = 0$ for large n, then we can decompose this

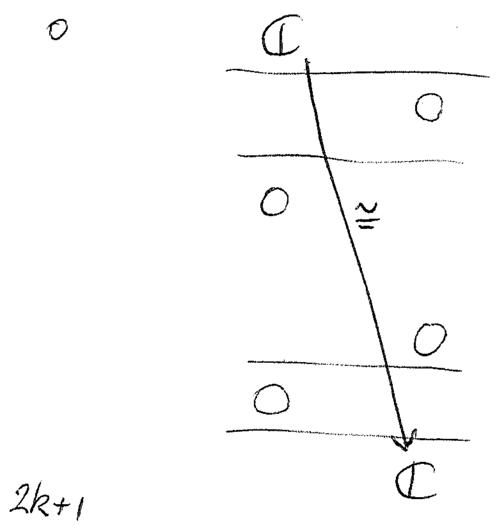
$\oplus HC_n$ into indecomposables which are of the form

$$\begin{tikzcd}
0 \arrow[r] & HC_n \arrow[r, "\sim"] \arrow[d, phantom, "\shortparallel" rotate=90] & HC_{n-2} \arrow[r, "\sim"] & \cdots \arrow[r, "\sim"] & HC_{n-2k} \arrow[r] \arrow[d, phantom, "\shortparallel" rotate=90] & 0 \\
 & \mathbb{C} & & & \mathbb{C} &
\end{tikzcd}$$

The corresponding Hochschild homology is

$HH_{n-2k} = \mathbb{C}$ $\qquad$ $HH_{n+1} = \mathbb{C}$.

The minimal tower in the case $2k = n$ is



In this example the smallest DG$\mathbb{C}[S]$-module for this type is

$$N = \overset{2k}{\mathbb{C}} \oplus 0 \oplus \overset{2k-2}{\mathbb{C}} \oplus \cdots\cdots \oplus \overset{2}{\mathbb{C}} \oplus 0 \oplus \mathbb{C}$$

and the corresponding mixed complex is $N \oplus BN$; with $b = \overset{\overset{0}{\shortparallel}}{d} + \overset{+1\ -2}{BS}$

$$N+BN: \quad \begin{tikzcd}
\overset{2k+1}{\mathbb{C}} \arrow[r, shift right, "0"'] & \overset{2k}{\mathbb{C}} \arrow[l, shift right, "1"'] \arrow[r, shift right, "1"'] & \overset{2k-1}{\mathbb{C}} \arrow[l, shift right, "0"'] \arrow[r, shift right] & \cdots \arrow[l, shift right] \arrow[r, shift right] & \overset{2}{\mathbb{C}} \arrow[l, shift right] \arrow[r, shift right, "1"'] & \overset{1}{\mathbb{C}} \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & \overset{0}{\mathbb{C}} \arrow[l, shift right, "1"']
\end{tikzcd}$$

As a check note that the b homology is $\mathbb{C}$ in degree 0 and $\mathbb{C}$ in degree $2k+1$.

April 24, 1992

Let's bring in the idea of minimal free things. Example: A minimal free commutative DG algebra à la Sullivan has indecomposables isomorphic to the homotopy. I would like the consider the analogue for chain DG $\mathbb{C}[B]$-modules (i.e. mixed complexes) and cochain DG $\mathbb{C}[S]$-modules. (Work dually with the latter instead of chain DG $\mathbb{C}[S^*]$-comodules). What should be true is that if we realize a mixed complex quis-type by a free DG $\mathbb{C}[B]$-module M which is minimal, then the indecomposable space M/BM should be isomorphic to the cyclic homology. In fact for M to be free over $\mathbb{C}[B]$ is equivalent to B being exact.

This seems to be an interesting point, namely given a mixed complex we can always replace it by a mixed complex such that a) B is exact, b) $b=0$ on M/BM. Notice that S is determined by the exact sequence

$$0 \longrightarrow \Sigma(M/BM) \xrightarrow{B\Sigma^{-1}} M \longrightarrow M/BM \longrightarrow 0$$

Indeed there is a well-defined map up to homotopy

$$M/BM \xrightarrow{S} \Sigma^2(M/BM)$$

which is unique since M/BM has zero differential.
Consequence: Any chain $\mathbb{C}[S^*]$-comodule is quis to one with zero differential.

Recall that if X is a chain $\mathbb{C}[S^*]$-comodule, then we have a mixed complex $X+BX$ with differential $b = d + BS$. It seems clear that if we have a mixed complex M as above satisfying a), b), then by choosing a lifting $M/BM \to M$ we obtain an isomorphism

$$\begin{array}{c} (M/BM) \oplus B(M/BM) \xrightarrow{\sim} M \\ \| \\ \mathbb{C}[B] \otimes (M/BM) \end{array}$$

In fact this probably doesn't use the hypothesis b); it should be possible to identify mixed complex M for which B is exact as those of the form $X + BX$ where X is a chain $\mathbb{C}[S^*]$-comodule.

On the comodule side we want to consider X on which S is surjective. I am willing to make finiteness assumptions in order to work with cochain $\mathbb{C}[S]$-modules, i.e. cochain complexes with endo S of degree $+2$. The freeness condition is that S be injective.

Let us now consider a minimal free cochain $\mathbb{C}[S]$-module, say finitely generated. Call it Y. Then Y/SY should be a complex with zero differentials which is the Hochschild homology (dualized). Example: take

$$\mathbb{C}[S] + \mathbb{C}[S]u$$

where $du = S^{n+1}$, so that $\deg(u) = 2n+1$.

The problem is how to link up with my towers of $\mathbb{Z}/2$-graded complexes. The above example shows that the normal methods based on Y/SY do not see the higher order structure.

Let's consider a cochain complex Y which is a $\mathbb{C}[S]$-module. We might write $S = u$ so as to exhibit the connection with equivariant cohomology. Assume Y is free and finitely generated as $\mathbb{C}[S]$-module. Y/SY gives the sort of generators required; it is like the indecomposable space. I want to assume minimality, i.e. that Y/SY has differential zero. Now suppose I localize $Y \otimes_{\mathbb{C}[S]} \mathbb{C}[S, S^{-1}]$. This is a graded $\mathbb{C}[S,S^{-1}]$-module, in fact, complex. Structurally it is [redacted] in each degree an inductive system $\hookrightarrow Y^n$ [redacted] $\xhookrightarrow{S} Y^{n+2}$ [redacted] $\xhookrightarrow{S} Y^{n+4}$ [redacted] ......

[redacted]

April 26, 1992

Let us consider a cochain complex with endomorphism S of degree 2. Assume S injective. Picture:

$$\begin{tikzcd}
X^0 \arrow[rr, hook] \arrow[dr, "d"] & & X^2 \arrow[rr, hook] \arrow[dr, "d"] & & X^4 \arrow[rr, hook] \arrow[dr] & & {} \\
& X^1 \arrow[rr, hook] \arrow[ur, "d"] & & X^3 \arrow[rr, hook] \arrow[ur, "d"] & & {} &
\end{tikzcd}$$

(*)

Taking the inductive limit under S gives a $\mathbb{Z}/2$ graded complex $L^{ev} = \varinjlim X^{2n}$, $L^{od} = \varinjlim X^{2n+1}$. We consider the following subcomplexes of L:

F_0 :	$\text{Ker}\{X^0 \xrightarrow{d} X^1\}$	0
F_0' :	X^0	dX^0
F_1 :	X^0	$\text{Ker}\{X^1 \xrightarrow{d} X^2/X^0\}$
F_1' :	$X^0 + dX^1$	X^1
F_2 :	$\text{Ker}\{X^2 \xrightarrow{d} X^3/X^1\}$	X^1

Thus

F_n :	$\text{Ker}\{X^n \xrightarrow{d} X^{n+1}/X^{n-1}\}$	X^{n-1}
F_n' :	$\underbrace{X^n}_{\text{parity } n}$	$\underbrace{X^{n-1} + dX^n}_{\text{parity } n-1}$

Observe that a) these are increasing subcomplexes

$F_0 \subset F_0' \subset F_1 \subset F_1' \subset \cdots$

b) $F_n \subset F_n'$ is a quis because

$$F_n'/F_n : X^n / \mathrm{Ker}(X^n \xrightarrow{d} X^{n+1}/X^{n-1}) \underset{\sim}{\xrightarrow{d}} \underbrace{X^{n-1} + dX^n / X^{n-1}}_{\subset X^{n+1}/X^{n-1}}$$

has diffl from parity n to $n+1$ an isomorphism.

c)
$$F_n / F_{n-1}' : \frac{\mathrm{Ker}\{X^n \xrightarrow{d} X^{n+1}/X^{n-1}\}}{X^{n-2} + dX^{n-1}} \qquad \bigcirc$$
$$\|$$
$$\frac{\mathrm{Ker}\{X^n/X^{n-2} \xrightarrow{d} X^{n+1}/X^{n-1}\}}{\mathrm{Im}\{X^{n-1}/X^{n-3} \xrightarrow{d} X^n/X^{n-2}\}}$$

is concentrated in parity n.

Let's recall that X as in ⊛ gives rise to a Connes exact sequence (cohomological type) with

$$HC^n = H^n(X, d)$$
$$HH^n = H^n(X/SX, d)$$

Thus $F_n / F_{n-1}' = HH^n[n + 2\mathbb{Z}]$.

Either of the filtrations $\{F_n\}$ or $\{F_n'\}$ leads to an injective inductive system of $\mathbb{Z}/2$-graded complexes

$$F_0 \subset F_1 \subset F_2 \subset \cdots\cdots$$

such that F_n/F_{n-1} has homology concentrated in parity n. The difference between the two filtrations is that

$$F_n/F_{n-1} : \frac{\mathrm{Ker}\{X^n \xrightarrow{d} X^{n+1}/X^{n-1}\}}{X^{n-2}} \overset{d}{\underset{0}{\rightleftarrows}} X^{n-1}/\mathrm{Ker}\{X^{n-1} \xrightarrow{d} X^n/X^{n-2}\}$$

$$F_n'/F_{n-1}' : X^n / X^{n-2} + dX^{n-1} \overset{0}{\underset{}{\leftrightarrows}} X^{n-1} + dX^n / X^{n-1}$$

So F_n/F_{n-1} has an injective differential and F_n'/F_{n-1}' has a surjective differential.

Next we reverse the construction. Suppose given a $\mathbb{Z}/2$ graded complex L equipped with an increasing filtration

$$0 \subset F_0 \subset F_0' \subset F_1 \subset F_1' \subset \cdots$$

with the following properties:

i) exhaustive $L = \cup F_n$

ii) F_n'/F_n has bijective diffl from $n+2\mathbb{Z}$ to $n+1+2\mathbb{Z}$

iii) F_n/F_{n-1}' concentrated in degree $n+2\mathbb{Z}$

~~[illegible]~~

~~Suppose one is given first the filtration~~

Consider the exact sequence

$$0 \longrightarrow F_{n-1}'/F_{n-1} \longrightarrow F_n/F_{n-1} \longrightarrow F_n/F_{n-1}' \longrightarrow 0$$

$$\begin{tikzcd}
n+2\mathbb{Z} & 0 \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & HH^n \arrow[r] & 0 \\
n-1+2\mathbb{Z} & 0 \arrow[r] & \cdot \arrow[u, "s"] \arrow[r, "\sim"] & \cdot \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

This shows that in F_n/F_{n-1} the differential from $n-1+2\mathbb{Z}$ to $n+2\mathbb{Z}$ is injective. Also F_{n-1}' is determined by F_{n-1}, F_n and the properties i)–iii)

Similarly from

$$0 \longrightarrow F_n/F_{n-1}' \longrightarrow F_n'/F_{n-1}' \longrightarrow F_n'/F_n \longrightarrow 0$$

$$\begin{tikzcd}
HH^n & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, "s"] \\
0 & \cdot \arrow[r, phantom, "\sim"] & \cdot
\end{tikzcd}$$

~~[illegible]~~ we see that the differential in F_n'/F_{n-1}' from $n+2\mathbb{Z}$ to $n-1+2\mathbb{Z}$ is surjective.

So we conclude that it suffices to give an exhaustive filtration $\{F_n\}$ of L such that F_n/F_{n-1} has injective differential from $n-1+2\mathbb{Z}$ to $n+2\mathbb{Z}$ for all n.

Now we reconstruct X from $(L, \{F_n\})$ by setting

$$X^n = F_{n+1} \text{ in degree } n + 2\mathbb{Z}$$
$$= F_n' \text{ in degree } n + 2\mathbb{Z}$$

More precisely suppose we set

$$F_n' : \quad \underbrace{X_n}_{\text{degree } n+2\mathbb{Z}} \qquad Y_n$$

Since F_n'/F_{n-1}' has surjective diffl from degree $n+2\mathbb{Z}$ we get

$$X_n/Y_{n-1} \xrightarrow{d} Y_n/X_{n-1}$$

and so $Y_{n-1} = X_{n-1} + dX_n$. Similarly if we ~~started~~ with $\{F_n\}$, and put

$$F_n : \quad Z^n \qquad X^{n-1}$$

then from the injectivity of the differential from degree $n-1+2\mathbb{Z}$ in F_n/F_{n-1} we get

$$Z^n/X^{n-2} \xleftarrow{d} X^{n-1}/Z^{n-1}$$

showing that $Z^{n-1} = \text{Ker}\{X^{n-1} \xrightarrow{d} X^n/X^{n-2}\}$.

I guess the only thing remaining to be checked is that $d(X^n) \subset X^{n+1}$, but this is clear since ~~$d(X_n) \subset Y_n \subset X_{n+1}$~~ X_n = parity n part of F_n', X_{n+1} = parity $n+1$ part of F_{n+1}' and

$$d(F_n') \subset F_n' \subset F_{n+1}'.$$

I think the way to see that this reconstruction works is to draw the picture

$$\begin{array}{lcll} F_0' & : & X^0 & dX^0 \\ F_1' & : & X^0+dX^1 & X^1 \\ F_2' & : & X^2 & X^1+dX^2 \end{array}$$

and note that X^n is defined to be the parity n part of F_n', while the other part is determined so that d be surjective in F_n'/F_{n-1}' from degree $n+2\mathbb{Z}$ to $n-1+2\mathbb{Z}$.

Special cases. ① The minimal complex case is where the differential on X/SX is zero. This is the situation where $d(X^n) \subset X^{n-1}$, i.e. $F_n = F_n'$

② When X comes from a mixed complex M, i.e. $X^n = M^0 \oplus \cdots \oplus M^n$:

$$\begin{tikzcd}
{} & {} \\
M^2 \arrow[u] \arrow[r] & {} \\
M^1 \arrow[u] \arrow[r] & M^0 \arrow[u] \\
M^0 \arrow[u] &
\end{tikzcd}$$

Then $L = M$ with the differential $b+B$. We have

$$\begin{array}{llll}
F_0 : & {}_bM^0 & 0 & \xrightarrow{\sim} \\
F_0' : & M^0 & bM^0 & \\
 & \parallel & & \\
F_1 : & M^0 & {}_bM^1 & \\
 & & & \xleftarrow{\sim} \\
F_1' : & M^0+bM^1 & M^1 \text{[redacted]} & \\
 & & \parallel & \\
F_2 : & M^0+{}_bM^2 & M^1 & \\
 & & & \xrightarrow{\sim} \\
F_2' : & M^0+M^2 & M^1+bM^2 &
\end{array}$$

Observe that $F_0 \subset F_1 \subset F_2 \subset \cdots$ is dual to the tower

$$\begin{array}{llll}
M/F^0 : & M_0^{\blacksquare}/bM_1^{\blacksquare} & & 0 \\
M/F^{0'} : & & M_0 \text{ ——— } M_1/{}_bM_1 & \\
M/F^1 : & M_0^{\blacksquare} & & M_1^{\blacksquare}/bM_2^{\blacksquare} \\
M/F^{1'} : & & M_0+M_2/{}_bM_2 \text{ ——— } M_1 & \\
M/F^2 : & M_0^{\blacksquare}+M_2^{\blacksquare}/bM_3^{\blacksquare} & & M_1^{\blacksquare} \\
M/F^{2'} : & & M_0+M_2 \text{ ——— } M_1+M_3/{}_bM_3 & \\
M/F^3 : & M_0^{\blacksquare}+M_2^{\blacksquare} & & M_1^{\blacksquare}+M_3^{\blacksquare}/bM_4^{\blacksquare}
\end{array}$$

we considered earlier, where

Observe that the dual of

$$\begin{array}{lll}
X^0+dX^1 & \text{is} & X_2/\operatorname{Ker}(S:X_2\to X_0)\cap\operatorname{Ker}(d:X_2\to X_1) \\
\cap & & \uparrow \\
\operatorname{Ker}\{X^2\xrightarrow{d}X^3/X^1\} & \text{is} & X_2/d\operatorname{Ker}(S:X_3\to X_1)
\end{array}$$

Summarize the main ideas.

Minimal mixed complexes [illegible] and minimal $\mathbb{C}[S^*]$-comodules. The former are [illegible] chain [illegible] $\mathbb{C}[B]$-modules M which are free (equivalently B is exact) such that M/BM has zero differential, whence $M/BM = HC$. The latter are chain $\mathbb{C}[S]$-modules X such that S is surjective and $\operatorname{Ker} S$ has zero differential, whence $\operatorname{Ker} S = HH$.

There is an equivalence between chain $\mathbb{C}[S]$-modules X such that S is surjective and towers M/F^n of $\mathbb{Z}/2$-graded complexes such that F^n/F^{n+1} has surjective differential from degree $n+1+2\mathbb{Z}$ to degree $n+2\mathbb{Z}$, (and [illegible] M/F^0 is concentrated in degree zero). Minimal X correspond to minimal towers (i.e. F^n/F^{n+1} concentrated in degree $n+1+2\mathbb{Z}$).

Because $\mathbb{C}[S]$ is a PID, [illegible] chain $\mathbb{C}[S]$-modules are analogous to complexes of modules over a PID, i.e. determined up to noncanonical isomorphism by their homology. (The quasi-isomorphism type of a mixed complex (or chain $\mathbb{C}[S]$-module, or tower..) is completely determined by HC. with the S operator.) Every chain $\mathbb{C}[S]$-module (say bounded in the sense that $HH_n = 0$ for $n \gg 0$) ~~[illegible]~~ is a direct sum of indecomposable ones: $\Sigma^k(\mathbb{C}[S]/S^{n+1})$. Universal coefficient theorem should give bivariant groups noncanonically.

May 8, 1992

Problem – to summarize and organize recent work. The starting point is the problem of properly understanding the known calculation of cyclic theory for universal enveloping algebras and group algebras. Included also should be the algebra of differential operators on a manifold. In these calculations one starts with a nice projective bimodule resolution of A, often given by a Koszul type complex. This resolution gives a nice complex computing the Hochschild homology, and one then exhibits a B operator making it into a mixed complex. The problem is then the shows this mixed complex gives the same cyclic theory as the standard mixed complex of chains on A.

My idea for attacking this problem is to look for a way to construct the cyclic theory of A starting from any projective bimodule resolution E of A. Ideally I would like to use $E\otimes_A$ to calculate the Hochschild homology and then find a way to construct the extra structure on $E\otimes_A$ yielding the cyclic theory. The extra structure should consist of a perturbed differential on $E\otimes_A$. More precisely the Postnikov tower of $E\otimes_A$ when considered as a tower of $\mathbb{Z}/2$-graded complexes should have on it a perturbed differential.

Given $E \xrightarrow{\varepsilon} A$ a projective bimodule chain complex with augmentation ε assumed surjective we construct

a bigraded DG algebra

$$1) \qquad A \xleftarrow{\varepsilon} E \xleftarrow{\varepsilon\otimes 1 - 1\otimes \varepsilon} E\otimes_A E \longleftarrow \cdots$$

The total DG algebra is $T_A(\Sigma E)$ with differential the sum of the differentials coming from E and from ε. Because E is projective and $\varepsilon : E_0 \to A$ is surjective one obtains an acyclic DG algebra with A in degree 0 where the degree n components for $n > 0$ is a projective bimodule. It follows that

$$2) \qquad E \longleftarrow E\otimes_A E \longleftarrow E\otimes_A E\otimes_A E \longleftarrow \cdots$$

is a projective bimodule resolution of A, hence

$$3) \qquad E\otimes_A \longleftarrow [E\otimes_A]^{(2)} \longleftarrow [E\otimes_A]^{(3)} \longleftarrow \cdots$$

computes the Hochschild homology.

One checks that 3) is the b-complex associated to the cyclic object $[E\otimes_A]^{(n+1)}$ in the category of complexes. This immediately gives a mixed complex structure which means we get a cyclic theory refining the Hochschild homology. More precisely we can ~~When E is a resolution of A all the complexes $[E\otimes_A]^{(n+1)}$ are all quis~~ form the double complex (with differentials $b, b', 1-\lambda, N_\lambda$) associated to this cyclic object. This is a double complex of complexes and we take total complexes for the columns. The columns turn out to be contractible so we obtain a B operator as Kassel does in the case of a nonunital algebra.

When E is a <u>resolution</u> of A then all the complexes $[E\otimes_A]^{(n+1)}$ are quis. In this case

3) is quis $E\otimes_A$. If an explicit SDR could be constructed, then we should get via HPT a perturbed differential on $E\otimes_A$.

Recall that cyclic object theory gives a complex

$$E\otimes_A \leftarrow [E\otimes_A]^{(2)}_\lambda \leftarrow [E\otimes_A]^{(3)}_\lambda \leftarrow \cdots$$

calculating the cyclic homology. There is an analogy with the series $\sum_{n\geq 1} \frac{1}{n} tr(K^n)$ for $\log\det(1-K)$. This analogy suggests the importance of the "propagator" K, and indicates that it should be worthwhile to concentrate on chain complexes of bimodules E with augmentation $\varepsilon : E \to A$. We have a procedure for refining such an E with ε surjective into a projective bimodule resolution 2).

There is an analogy with the join operation. Let us identify a pair $(E, \varepsilon : E \to A)$ with a chain complex of bimodules K such that $K_0 = A$ and K_n is projective for $n>0$: One has $K = \text{Cone}(\varepsilon : E \to A)$ and an exact sequence

$$0 \longrightarrow A \longrightarrow K \longrightarrow \Sigma E \longrightarrow 0$$

of bimodule complexes. Observe that the (total) DG algebra

$$A \leftarrow E \leftarrow E\otimes_A E \leftarrow \cdots$$

is $T_A(\Sigma E)$ with extra differential given by ε. This should just be $R_A(K)$ = result of gluing together the complexes $K\otimes_A \overset{n}{\cdots} \otimes_A K$ in a suitable fashion.

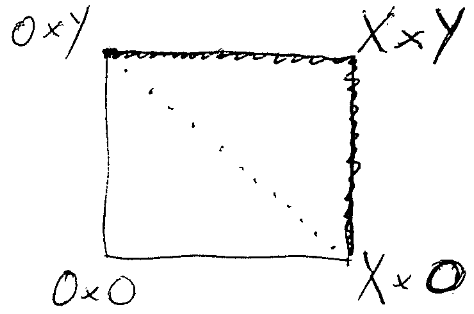
The join comes in when we take $K = \mathrm{Cone}(E \to A)$ and $K' = \mathrm{Cone}(E' \to A)$ and tensor. Then I should be getting something like

$$(E \to A) \otimes_A (E' \longrightarrow A) = (E \otimes_A E' \longrightarrow E \oplus E' \longrightarrow A).$$

Recall that for spaces

$$X * Y = \underrightarrow{\mathrm{holim}} \left(\begin{tikzcd}[row sep=small] & X \\ X \times Y \arrow[ur] \arrow[dr] & \\ & Y \end{tikzcd} \right)$$

and that $C(X * Y) = C(X) \times C(Y)$ where C stands for cone:



The analogue of $R_A(K)$ is to take $C(X)$ and form the free monoid it generates where the vertex of $C(X)$ is identified with the identity. It is therefore a suitable union of the product $C(X)^n$ with certain identifications.

To fix the ideas assume X is a nonempty set. The product $C(X)^n$ consists of all sequences $((t_1, x_1), \dots, (t_n, x_n))$ with $0 \leq t_i \leq 1$ and $x_i \in X$, where (t_i, x_i) is identified with the vertex 0 of $C(X)$ for $t_i = 0$. The free monoid generated by $C(X)$ should consist of all words

$$(t_1 x_1) \cdots (t_n x_n)$$

with $(t_i, x_i) = 1$ (i.e. omitted) if $t_i = 0$.

I guess we get the "join [redacted] monoid" generated by X by taking those words with $\sum t_i = 1$. It seems like we have the geometric realization of a semi-simplicial set, whose non degenerate p-simplices are all sequences $(x_0, \dots, x_p)$ in X.

Some other ideas: If G is a group, then instead of A-bimodule, we can consider G-bisets, i.e. $G\times G$ sets. Consider $G \times G$ sets over G acted on by left + right translation. This category is equivalent to ΔG-sets and the operation $E \times^G E'$ corresponds to the direct product operation on the corresponding ΔG-sets. In effect

$$\begin{tikzcd}
E \times^G E' \arrow[r] \arrow[d] & G \times^G E' \arrow[d] \\
E \times^G G \arrow[r] & G \times^G G
\end{tikzcd}$$

is cartesian ([redacted] this is true for $E = X \times G$ a right G set over G and $E' = G \times X'$ a left G-set over G), so $E \times^G E' \xrightarrow{\sim} E \times_G E'$.

So the construction of a free $G \times G$ space E over G such that [redacted] $E \to G$ is a heq should be the same as constructing $P(\Delta G)$. [redacted]

May 12, 1992

On Burghelea's theorem. Let G be a discrete group. Claim: The groupoid of G-torsors over S^1 is equivalent to the groupoid given by G acting on itself by conjugation.

More precisely, if P is a torsor then we have a monodromy automorphism $T: P \to P$. Pick a basepoint $0 \in S^1$. Associate to P the G torsor P_0, fibre over the basepoint, together with the automorphism T on P_0. This gives us an equivalence of the groupoid of G-torsors over S^1 with the groupoid of G-torsors over a point equipped with automorphism. The inverse equivalence is obtained by glueing, namely using T to identify the ends of $[0,1] \times P_0$. Next given a G-torsor P_0 over a pt with auto T, choose a point $p \in P_0$, and let g be defined by $T(p) = pg$. If $p' = ph$ is another point of P_0, and $T(p') = p'g'$, then $p'g' = T(ph) = T(p)h = pgh = p'(hgh^{-1})$, so $g' = hgh^{-1}$.

The above is awkward probably because I am trying to define the functors in the wrong direction. Instead let us proceed as follows.

Consider G as a G-torsor over a pt. An automorphism of this torsor is given by left multiplication: $T(g) = \mu g$, where $\mu \in G$. Let ${}_\mu G$ denote the G-torsor over a point equipped with autom defined this way. Thus for each $\mu \in G$ we have a G-torsor G_μ over a point with automorphism, and hence a G-torsor over S^1 with a distinguished point over $0 \in S^1$.

Let us next calculate the maps ${}_{\mu}G \xrightarrow{f} {}_{\nu}G$. Such an f is of the form $f(g) = xg$ for a unique $x \in G$. For it to commute with T means

$$\begin{array}{ccc} (Tf)(g) & = & f(Tg) \\ \| & & \| \\ T(xg) & & f(\mu g) \\ \| & & \| \\ \nu x g & & x\mu g \end{array}$$

i.e. $\nu = x\mu x^{-1}$. Thus the full subcategory of G-torsors with automorphisms over a point consisting of the $\{{}_{\mu}G,\ \mu \in G\}$ is isomorphic ~~equivalent~~ to the groupoid given by G acting on itself by inner automorphisms. Since any G-torsor with autom. is isomorphic to ${}_{\mu}G$, we obtain the required equivalence of categories.

Let us consider G-torsors over S^1 whose monodromy is a given conjugacy class $\gamma \subset G$. Let's pick $z \in \gamma$ and let G_z be the centralizer of z. *Claim: The groupoid of* [redacted] *whose G-torsors over S^1 monodromy lies in γ is equivalent to the groupoid of G_z-torsors with monodromy* $\{z\}$.

This is clear from the equivalence of categories established between G-torsors over S^1 and G acting on itself by conjugation. Specifically if we replace G-torsors over S^1 by G-torsors over a pt equipped with automorphisms and then by the equivalent category of $\{{}_{\mu}G,\ \mu \in G\}$, then we want the component of ${}_{\mu}G$ with $\mu \in \gamma$. This component is equivalent to the full subgroupoid consisting of the object ${}_{z}G$, whose group of automorphisms is G_z.

In terms of torsors P over S^1, having monodromy γ we can consider those points $p \in P_0$ such that $T(p) = pz$ and this should be a G_z-torsor over 0. Actually this doesn't depend on the basepoint. More precisely we can consider $\{p \in P \mid T(p) = pz\}$. This will be a G_z-torsor over the circle. In this way the category of G-torsors over S^1 with monodromy γ is equivalent to the category of G_z-torsors over S^1 with monodromy $\{z\}$.

Review. Given an oriented circle S we have the groupoid of G torsors on S. If we pick a basepoint of S, then we get an equivalence of the groupoid given by G acting on itself by conjugation with this groupoid.

A torsor has a monodromy which is a conjugacy class in G. The groupoid of torsors on S with monodromy the conjugacy class of z is equivalent to the category of torsors on S for G_z with monodromy z. This last statement doesn't depend on the basepoint, because given P with monodromy the conjugacy class of z, then $\{p \mid T(p) = p^z\}$ is a G_z-torsor with monodromy z.

What is the role of the basepoint? It gives a fibre functor from torsors ~~to~~ on S to G-torsors with automorphism. Given different basepoints, a path joining them gives an ~~isomorphism~~ isomorphism.

May 13, 1992

Given $z \in G$ let

$$P_z = \mathbb{R} \times^{\mathbb{Z}} G$$

where $1 \in \mathbb{Z}$ acts as left mult. by z on G. Then P_z is a G torsor over $\mathbb{R}/\mathbb{Z} = S^1$, the monodromy at the basepoint is

$$T_z : (0,g) \longrightarrow (1,g) \sim (0,zg)$$

$$\mathrm{Hom}(P_z, P_{z_1}) = \left\{ x \in G \;\middle|\; \begin{tikzcd} g \arrow[r, mapsto] \arrow[d, mapsto] & T_z(g) = zg \arrow[d, mapsto] \\ xg \arrow[r, mapsto] & z_1xg \quad xzg \end{tikzcd} \right\}$$

$$= \{ x \in G \mid z_1 = xzx^{-1} \}.$$

[redacted] Then $z \longmapsto P_z$ is an equivalence of the groupoid $Ad(G)$ given by G acting on itself by conjugation with the groupoid of G-torsors over $S^1 = \mathbb{R}/\mathbb{Z}$.

Let S be an oriented circle, and suppose given a universal covering $R \longrightarrow S$. [redacted]

[redacted] Then R is a principal $\mathbb{Z}$-bundle and the above construction of an equivalence from $Ad(G)$ to G-torsors [redacted] generalizes. [redacted] Observe that a choice of basepoint on S gives rise to a universal covering. In some sense a choice of basepoint and a choice of universal covering should be equivalent. (I guess the key idea is that of a fibre functor. Both a basepoint and a universal covering give rise to a fibre functor from G torsors on S to G torsors over a point equipped with automorphism. Two choices of basepoint give isomorphic fibre functors, but the isomorphism depends on a choice of path joining the basepoints)

The reduction of G-torsors on S with monodromy = the conjugacy class of z to G_z-torsors on S with monodromy $\{z\}$ doesn't use the basepoint.

The next stage, after understanding torsors over S, is to let the circle S vary. We have seen how the category of finite, nonempty cyclic ordered sets and inclusions gives a model for the way circles vary. However a good way to proceed is to return to my old idea of considering circle bundles.

Let us review some of this. Suppose we are interested in the equivariant aspects of the S^1 action on the free loop space BG^{S^1}. The homotopy quotient for the action is The space

1)
$$PS^1 \times^{S^1} (BG)^{S^1}.$$

A map $X \longrightarrow PS^1 \times^{S^1} (BG)^{S^1}$ should be equivalent to a principal S^1 bundle $P \longrightarrow X$, together with an equivariant map

$$P \longrightarrow (BG)^{S^1}$$

which is equivalent to an S^1-equivariant map

$$P \times S^1 \longrightarrow BG$$

i.e. a map $P = P \times^{S^1} S^1 \longrightarrow BG$. Thus the space 1) should be a classifying space for pairs consisting of a principal S^1-bundle $P \longrightarrow G$ together with a principal G-bundle over P. Similarly $(BG)^{S^1}$ is a classifying space for principal G-bundles over

$X \times S^1$, i.e. a circle bundle over X with section.

Let us now apply this in the case of G discrete. Suppose we restrict the monodromy to be a central element $z \in G$. Given P over $X \times S^1$ we can restrict to $X \times \{0\}$ to get a G-torsor P_0 on X. We can reconstruct P as $P_0 \times [0,1]$ using right multiplication by z as clutching data. It is thus clear that the category of G-torsors on $X \times S^1$ with monodromy $\{z\}$ is equivalent to the category of G torsors on X.

Now let us consider a circle bundle Q over X and a G-torsor P on Q with monodromy $\{z\}$. The first thing to do is to take the quotient $P/\langle z \rangle$ by the central subgroup generated by z. This gives a $G/\langle z \rangle$-torsor over Q with trivial monodromy, and we know this is equivalent to a $G/\langle z \rangle$ torsor over X. Let us lift back to this $G/\langle z \rangle$-torsor so as to suppose it is trivial. This means P reduces to a $\langle z \rangle$-torsor.

Let us now consider the case of torsors with group $\langle z \rangle$ over a circle S with monodromy $\{z\}$. If $\langle z \rangle$ is of infinite order, such a torsor is an universal covering of S. So if I have a "family" $Q \longrightarrow X$ of circles and a $\langle z \rangle$-torsor with monodromy z, then I have $Q = R/\langle z \rangle$ where R is a bundle of lines over X and z acts by translation on the lines. Because the fibres of R over X are trivial, we get a section of R over X, and hence a section of P over X. Thus we find in the $\langle z \rangle =$ infinite cyclic case that a $\langle z \rangle$-torsor with monodromy

$\{z\}$ over a circle bundle amounts to a trivialization of the circle bundle.

It seems therefore that (in the infinite cyclic case) G-torsors with monodromy $\{z\}$ over circle bundles are equivalent to $G/\langle z\rangle$ torsors.

May 14, 1992

Let S be an oriented circle, let P be a principal G bundle over S, where G is a discrete group. Choose a metric ~~[illegible]~~ on ~~[illegible]~~ S of unit length. Then ~~[illegible]~~ we obtain an action of $\mathbb{R}$ on S given by translation, and S becomes a torsor under $\mathbb{R}/\mathbb{Z}$. The $\mathbb{R}$-action on S lifts to an action of $\mathbb{R}$ on P commuting with G. Thus we have an action of the product $\mathbb{R} \times G$ on P, and this action is clearly transitive. The stabilizer of a point $p \in P$ is ~~[illegible]~~ infinite cyclic with generator of the form $(+1, z)$ where $T(p) = pz$ and T is the monodromy transformation. (Recall G acts to the right on P. The monodromy transf. is the action of $(1,0)$, so $T(p) = pz$ means ~~[illegible]~~

$$(1,0) \cdot p = pz = (0, z^{-1}) \cdot p \Rightarrow (1,z)p = p.)$$

Let us now restrict to the case where z is central in G. Then the stabilizer $\langle (1,z) \rangle$ of p is the same for all p, so P is a torsor under

$$G' = \mathbb{R} \times G / \langle (1,z) \rangle . \qquad \mathbb{Z} \simeq \langle (1,z) \rangle$$

Let's examine G'. ~~[illegible]~~ We have

$$\begin{tikzcd}
 & & {} \arrow[d] & \\
 & & \mathbb{Z} \arrow[r, phantom, "="] \arrow[d] & \mathbb{Z} \arrow[d] & \\
1 \arrow[r] & G \arrow[r] \arrow[d, equal] & \mathbb{R} \times G \arrow[r] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G \arrow[r] & G' \arrow[r] \arrow[d] & S^1 \arrow[r] \arrow[d] & 1 \\
 & & {} & {} &
\end{tikzcd}$$

Thus G' is an extension of the circle group S^1 by G.

We note that a G'-torsor (over a point) P is equivalent to a G-torsor over a circle (metric of unit length) with monodromy z, the circle being P/G.

So next consider an (oriented) circle bundle $Q \xrightarrow{S^1} X$ together with a G torsor $P \to Q$ having monodromy z on the fibres of $Q \to X$. As above we define an $\mathbb{R}$ action on P over X, and this makes P into a principal G'-bundle over X. The conclusion is that the pair $(Q \xrightarrow{S^1} X,\ P \xrightarrow{G} Q)$ with monodromy z is equivalent to a principal G'-bundle P over X, and Q can be recovered as P/G. This solves the problem of determining the homotopy quotient for the circle action on the free loop space of BG, G discrete. We find a disjoint union (of spaces) indexed by conjugacy classes in G, where the space corresponding to the conjugacy class of $z \in G$, is the classifying space of the extension G'_z of S^1 by G_z defined by the central element z of G_z.

Consider

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
1 \arrow[r] & n\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \langle z \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \mathbb{R} \times G \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{R}/n\mathbb{Z} \arrow[r] & G' \arrow[r] \arrow[d] & G/\langle z \rangle \arrow[r] & 1 \\
 & & 0 & &
\end{tikzcd}$$

where $\langle z\rangle \simeq \mathbb{Z}/n\mathbb{Z}$.

Two simple cases: ① If $n=1$ i.e. $z=$ identity in G, then $G' = S^1 \times G$ so

$$BG' = BS^1 \times BG$$

② If $n=0$, then we have an extension

$$1 \longrightarrow \mathbb{R} \longrightarrow G' \longrightarrow G/\langle z\rangle \longrightarrow 1$$

so that $BG' \sim B(G/\langle z\rangle)$.

Summarize: I guess at this point that I have obtained all of Burghelea's answers by considering G torsors over circles.

May 16, 1992

Recall the diagram learned from Goodwillie

$$\begin{tikzcd}
HC^- \arrow[r] \arrow[d] & HP \arrow[d] \\
HH \arrow[r] & HC
\end{tikzcd}$$

Let's consider this in the case of a group algebra $\mathbb{C}[G]$ and the part of the cyclic theory concerned with the identity conjugacy class. In this case $HH = H(G)$ and the above square is canonically tensoring $H(G)$ with

$$\begin{tikzcd}
\mathbb{C}[u^{-1}] \arrow[r, hook] \arrow[d, two heads] & \mathbb{C}[u, u^{-1}] \arrow[d, two heads] \\
\mathbb{C} \arrow[r, hook] & \mathbb{C}[u].
\end{tikzcd}$$

This implies: A group homology class $\alpha \in H(G)$ has a canonical lifting to a negative cyclic hom class. The associated cyclic class to α lifts to a periodic cyclic homology class. Actually the latter lifting determines the former by the injectivity of $HC^- \hookrightarrow HP$.

In cohomology we have

$$\begin{tikzcd}
(HC^-)^\circ & HP^\circ \arrow[l, two heads] \\
H^\circ(G) \simeq HH^\circ \arrow[u, hook] & HC^\circ \arrow[l, two heads] \arrow[u, hook]
\end{tikzcd}$$

so the assertion is that a group cohomology class α when identified with a Hochschild coh. class lifts canonically to a cyclic cohomology class.

Connes does this directly by interpreting a group cocycle as a cyclic cocycle in some way.

May 17, 1992

Problem: [redacted] the homotopy quotient of the S^1-action on the free loop space of BG is [redacted] a classifying space for G torsors on circle bundles.

May 23, 1992 (Alice is 30)

I would like to discuss G torsors over a cyclic graph S where G is a topological group. The idea is that a G-torsor on S is given by G-torsors over each vertex and each edge together with glueing data identifying the torsor at a vertex with the restriction of the torsor over each edge with that vertex. Another idea is to replace the category of torsors with classifying spaces. Thus the space corresponding to G torsors on S is obtained as follows. We take a copy of BG at each vertex. At each edge we need a space to describe the torsors on the edge together with restriction maps to the ends, for example

1) $$BG^I \longrightarrow BG \times BG \qquad I = [0,1]$$

In this way we obtain a functor from the category of vertices and edges of S to spaces. The space correspond to G torsors on S is the inverse limit. Using BG^I for edges we of course obtain the space

$$\mathrm{Map}(|S|, BG)$$

which is the free loop spaces as $|S| = S^1$.

However it appears that instead of 1) we can use any factorization of the diagonal map

$$BG \xrightarrow[heq]{} \; ? \xrightarrow[fibration]{} BG \times BG$$

so for example we can take

2) $$PG \times^G PG \longrightarrow BG \times BG$$

What this means is that we are forming the homotopy inverse limit of the constant functor on the category of vertices + edges with value BG. If we use 2) for the edges then a point of the inverse image consists of points of BG indexed by vertices together with isomorphisms for each edge of the corresponding fibres of PG at the ends. Thus the inverse limit is a fibre bundle over BG^V, $V = \{\text{vertices}\}$, where the [redacted] fibre is G^E, $E = \{\text{edges}\}$, and G^V acts on G^E as in forming the [redacted] circular product $G \times^G \cdots G \times^G$.

[redacted]

Let Ad(G) denote the category given by G acting on itself by conjugation. I claim there is a canonical map

$$S^1 \times \underbrace{B(Ad(G))}_{N^{cy}(G)} \longrightarrow B(Ad(G))$$

which can be easily understood. It suffices to give a natural transformation from the identify functor on Ad(G) to itself. This means for each object $g \in G$ given an arrow $\varphi(g)$ from g to g in Ad(G). Let us take

$$\varphi(g) = g : g \longrightarrow g$$ (check $ggg^{-1} = g$)

and check this is a natural transformation:

$$\begin{tikzcd} X \arrow[d, "f"] & F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ Y & F(Y) \arrow[r, "\theta_Y"'] & G(Y) \end{tikzcd}$$

Given $h : g_1 \to g_2$ i.e. $hg_1h^{-1} = g_2$

we want

$$\begin{tikzcd} g_1 \arrow[r, "g_1"] \arrow[d, "h"] & g_1 \arrow[d, "h"] \\ g_2 \arrow[r, "g_2"] & g_2 \end{tikzcd}$$

to commute. Clear.

Geometrically given a torsor over S^1 we should get a torsor plus automorphism. This should associate to P the torsor P together with its monodromy automorphism.

Here's Graeme's method for understanding the fact that the category Λ of cyclic graphs has homotopy type BS^1. [redacted] Given a cyclic graph S, i.e. a finite nonempty cyclically ordered set, we associate to S the space $I(S)$ of embeddings $S \hookrightarrow S^1$ compatible with the cyclic order. Fixing a basepoint of S gives a map $I(S) \to S^1$ which is a fibre bundle with fibre an open simplex (= pt in dim 0). Thus $\underline{I}(S) \sim S^1$. $I(S)$ is contravariant in S and a map $S' \subset S$ induces a heq $I(S) \to I(S')$. The fibred category

Λ/I should have [redacted] classifying space which is a qsifibration over $B\Lambda$ with fibre $\sim S^1$; so it remains to see that $B(\Lambda/I) \sim pt.$

Observe that Λ/I is equivalent to the poset of finite non empty subsets of S^1, except that [redacted] these subsets carry a topology. The classifying space should be a giant simplex with vertices the points of S^1. Thus the classifying space is the space of probability measures on S^1 with finite support, and probably the topology allows the support to move. In any case one has a linear path joining [redacted] any two probability measures, so this classifying space should be contractible.

If we divide **out** by the circle action on $B(\Lambda/I)$ we get the [redacted] classifying space of the fibred cat over Λ which associates to S the open simplex of possible ways of assigning positive lengths to the edges of S so that the total length is 1.

One thing about the $\underleftarrow{\text{holim}}$ picture for torsors on S is that the natural [redacted] maps $S' \to S$ to consider [redacted] are inclusions, i.e. where vertices are omitted rather than where edges are collapsed. ~~[illegible]~~

(May 25)

Ideas from scratch paper

Bigraded DGA: $T_A(E)$

$$A \leftarrow E \leftleftarrows E\otimes_A E \cdots$$

If degeneracy: $E \xrightarrow{\Delta} E\otimes_A E$ exists it gives contraction of b' column.

Construction of $T_A(E)$ analogous to constructing BG by an infinite process.

Join operation and Milnor's model for BG

$$\{P\to A\}\otimes_A\{Q\to A\} = \{P\otimes_A Q \longrightarrow P\oplus Q \longrightarrow A\}$$

$P * G$ $\overbrace{\text{unique for}}^{\text{G-space}}$ a ^{G}map from P together with a (not necessarily equivariant contraction to a point. Thus P universal (contractible) $\Rightarrow$ $P * G$ retracts equiv. onto P so we can expect

$$P \subset P*G \subset \cdots \subset P*G^{(\infty)}$$

to equiv. retract & maybe deform to P. However one doesn't seem to get an equiv. map $P*G^{(\infty)} \longrightarrow G^{(\infty)}$. At this point I decided the join approach is not self contained.

More on cyclic category. Graeme's circle bundle & ~~putting~~ lengths on edges.

Also simplicial complex maps with S^1 fibre leads to polygons & collapsing edges.

May 26, 1992:

HPT again. The application we have in mind is as follows. Let P be a (chain) complex of projective right A-modules and Q a complex of left A-modules which is acyclic. Then we know $P \otimes_A Q$ is acyclic, in fact it's enough for P to be flat.

Suppose we want to construct a contracting ~~homotopy~~ homotopy for $P \otimes_A Q$. We suppose given a contracting homotopy for Q as a complex of vector spaces, and also we suppose given connections on right modules P_n for all n. This gives a liftings of right A-modules

$$\begin{tikzcd} P \arrow[r, shift right, "s"'] & P \otimes A \arrow[l, shift right, "m"'] \end{tikzcd} \qquad ms = 1.$$

compatible with the grading but not the diffl in P. Then we get upon tensoring with Q maps.

$$\begin{tikzcd} P \otimes_A Q \arrow[r, shift right, "\bar{s}"'] & P \otimes Q \arrow[l, shift right, "\bar{m}"'] \end{tikzcd} \qquad \bar{m}\bar{s} = 1$$

These are bigraded, really bicomplexes with horizontal and vertical differentials $d \otimes 1$, $1 \otimes d$ respectively. Let h be a contracting homotopy for Q, i.e. $[d_Q, h] = 1$. Take $1 \otimes h$ on $P \otimes Q$ and compress it to an operator on $P \otimes_A Q$; this gives $\bar{m}(1 \otimes h)\bar{s}$. Since $\bar{m}, \bar{s}$ are compatible with $1 \otimes d$, this gives a contracting homotopy for $1 \otimes d$ on $P \otimes_A Q$.

So we have a double complex and a contracting homotopy for the columns.

We now treat the horizontal differential as a perturbation and apply HPT. Recall the formulas

$$\begin{tikzcd} E \arrow[d, shift right, "p"'] \\ F \arrow[u, shift right, "i"'] \end{tikzcd} \qquad \begin{aligned} [d,p] &= [d,i] = 0 \\ [d,h] &= 1 - ip \\ [d\theta] &= \theta^2 \end{aligned}$$

Then $\theta' = p\theta\frac{1}{1-h\theta}i \qquad h' = h\frac{1}{1-\theta h}$

$p' = p\frac{1}{1-\theta h} \qquad i' = \frac{1}{1-h\theta}i$

satisfy

$$(d-\theta')^2 = 0$$

$$(d-\theta')p' = p'(d-\theta)$$

$$(d-\theta)i' = i'(d-\theta')$$

$$[d-\theta, h'] = 1 - i'p'$$

An important special case is where θ descends to F via p. We assume

$$p\theta = \theta p \quad , \quad ph = 0, \quad pi = 1.$$

In this case $\theta' = \theta$, $p' = p$.

Now we are concerned with the case where $F = 0$. In this case we have simply

$$[d,h] = 1$$

$$d\theta = \theta^2$$

and we construct $h' = h\frac{1}{1-\theta h}$ satisfying

$$[d-\theta, h'] = 1.$$

Proof: $\left[d-\theta, h\frac{1}{1-\theta h}\right] = [d-\theta, h]\frac{1}{1-\theta h} - h\left[d^{-\theta}_{A}, \frac{1}{1-\theta h}\right]$

$$= (1-\theta h-h\theta)\frac{1}{1-\theta h} - h\frac{1}{1-\theta h}\underbrace{[\overset{-\theta}{d},\theta h]}\frac{1}{1-\theta h}$$

$$\theta^2 h - \theta - \cancel{\theta^2 h} + \theta h\theta$$

$$= \left\{1-\theta h-h\theta - h\frac{1}{1-\theta h}\overset{-(1-\theta h)\theta}{\text{[redacted]}}\right\}\frac{1}{1-\theta h}$$

$$= \{1-\theta h-\cancel{h\theta}+\cancel{h\theta}\}\frac{1}{1-\theta h} = 1.$$

Now suppose we start with a ~~situation~~

$$\begin{tikzcd} E \arrow[d, shift right, "p"'] \\ F \arrow[u, shift right, "i"'] \end{tikzcd}$$

$$[d,h] = 1-ip \ , \quad [d,p]=[d,i]=0$$
$$pi = 1.$$

and form the mapping cone so as to reduce to the acyclic case. Thus we consider

$$\Sigma E \oplus F \quad \text{with} \quad \tilde{d} = \begin{pmatrix} -d & 0 \\ p & d \end{pmatrix}, \quad \tilde{h} = \begin{pmatrix} -h & i \\ 0 & 0 \end{pmatrix}$$

and note

$$\tilde{d}^2 = \begin{pmatrix} -d & 0 \\ p & d \end{pmatrix}\begin{pmatrix} -d & 0 \\ p & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -pd+dp & 0 \end{pmatrix} = 0 \iff [d,p]=0$$

$$[\tilde{d},\tilde{h}] = \begin{pmatrix} dh+hd+ip & -di+id \\ -ph & pi \end{pmatrix} = I \iff \begin{matrix} [d,h]=1-ip \\ [d,i]=0 \\ ph=0 \\ pi=1. \end{matrix}$$

Now suppose given θ on $E, F \ni p\theta = \theta p$. and $[d,\theta]=\theta^2$. Then we put

$$\tilde{\theta} = \begin{pmatrix} -\theta & 0 \\ 0 & \theta \end{pmatrix}$$

and find

$$[\tilde{d},\tilde{\theta}] = \begin{pmatrix} -d & \\ p & d \end{pmatrix}\begin{pmatrix} -\theta & \\ & \theta \end{pmatrix} + \begin{pmatrix} -\theta & \\ & \theta \end{pmatrix}\begin{pmatrix} -d & \\ p & d \end{pmatrix} = \begin{pmatrix} [d,\theta] & 0 \\ -p\theta+\theta p & [d,\theta] \end{pmatrix}$$

so $[d\tilde\theta] = \tilde\theta^2$

We have

$$1-\tilde h\tilde\theta = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}-h & i\\ 0 & 0\end{pmatrix}\begin{pmatrix}-\theta & \\ & \theta\end{pmatrix}$$

$$= \begin{pmatrix}1-h\theta & -i\theta\\ 0 & 1\end{pmatrix}$$

$$(1-\tilde h\tilde\theta)^{-1} = \begin{pmatrix}\frac{1}{1-h\theta} & \frac{1}{1-h\theta}i\theta\\ & 1\end{pmatrix}$$

$$\cdot\begin{pmatrix}-h & i\\ 0 & 0\end{pmatrix}$$

so $\tilde h' =$ [redacted] $(1-\tilde h\tilde\theta)^{-1}\tilde h = \begin{pmatrix}-\frac{1}{1-h\theta}h & \frac{1}{1-h\theta}i\\ 0 & 0\end{pmatrix} = \begin{pmatrix}-h' & i'\\ 0 & 0\end{pmatrix}$

satisfies $[\tilde d-\tilde\theta, \tilde h'] = 1$ which [redacted] is equivalent to the identities

[redacted]

$$[d-\theta, p] = [d-\theta, i'] = 0$$
$$[d-\theta, h'] = 1 - ip$$
$$ph' = 0 \ , \quad pi' = 1$$

So the conclusion is that the SDR situation is essentially reduced to the acyclic case by the cone construction.

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May 27, 1992

Consider E a projective bimodule chain complex with surjective augmentation to A. We have a bigraded DGA

1) $$A \longleftarrow E \longleftarrow E\otimes_A E \Longleftarrow \cdots$$

which is acyclic. A horizontal contracting homotopy is given by left (or right) multiplication by an elt $\xi \in E_0$ lying over $1 \in A$. This means 1) has a canonical (up to the choice of ξ) left (or right) module contraction.

1) can also be obtained from a presimplicial object of complexes of bimodules

1)' $$A \longleftarrow E \underset{d_1 = 1\otimes\varepsilon}{\overset{d_0 = \varepsilon\otimes 1}{\leftleftarrows}} E\otimes_A E \Lleftarrow E\otimes_A E\otimes_A E \cdots$$

with augmentation to A. If we apply $?\otimes_A$ to this presimplicial object we obtain the b complex associated to the precyclic object $[E\otimes_A]^{(*+1)}$. If we tensor 1)' with E on one side we obtain

2)' $$E \xleftarrow{d_0} E\otimes_A E \underset{d_1}{\overset{d_0}{\leftleftarrows}} E\otimes_A E\otimes_A E \cdots$$

which gives rise after applying $?\otimes_A$ to the b' complex. (It looks like we want to tensor with E on the right.)

Point: If we have given a left connection on E then combining with the left contraction of 1) (given by right multiplication by ξ (with signs)) we obtain a contraction of 2)'. This means that up to choosing a left connection in E and a ξ we have a contraction for the b'-complex associated to the cyclic object $[E\otimes_A]^{(*+1)}$.

May 29, 1992

I propose to look at the fundamental lemma of homological algebra and give a constructive version using connections.

[redacted] This lemma says that a projective complex P has the lifting property wrt respect to surjective quasi-isomorphisms $X \longrightarrow Y$. It then should follow that $\mathrm{Hom}_A(P, X) \longrightarrow \mathrm{Hom}_A(P, Y)$ is a surjective quasi-isomorphism. (Here I would with complexes of A modules bounded below to be safe.)

I now want a constructive or concrete version. Assume A is an algebra over $\mathbb{C}$, and take as a concrete form of surjective quis a strong deformation retraction (in some sense [redacted] to be made precise) [redacted] where the A module structure is ignored. To keep things simple at the beginning suppose $Y = 0$. We than wish to show that X contractible $\Longrightarrow$ $\mathrm{Hom}_A(P, X)$ contractible, or rather to understand this on a concrete level where we have a contraction h for X and a connection in P.

Let's begin with a cocycle in $\mathrm{Hom}_A(P, X)$, i.e. a map of A module complexes $f: P \to X$. We have a standard way to construct a contracting homotopy for f:

$$\begin{tikzcd}
P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] \arrow[dl, "hf_0"'] & 0 \\
X_2 \arrow[r] & X_1 \arrow[r, shift right] & X_0 \arrow[r] \arrow[l, shift right, "h"'] & X_{-1}
\end{tikzcd}$$

The first step is :

$$\begin{tikzcd} & X_1 \arrow[d, "d"] \arrow[d, dashed, bend left, leftarrow, "h"'] \\ P_0 \arrow[r, "f_0"] \arrow[ur, dashed] & \mathrm{Ker}(X_0 \to X_{-1}) \end{tikzcd}$$

$dh = 1$

Here we have a surjection d relative to which we want to lift f_0. We have the section h of d, but h is not an A-module map. We need a flattening process to convert $hf_0 : P \longrightarrow X_1$ to an A-module map. Given a connection or rather an A-module lifting

$$\begin{tikzcd} A \otimes P_0 \arrow[d, "m"] \\ P_0 \arrow[u, dashed, bend left, "s"] \end{tikzcd}$$

$ms = 1$

we can then define $H_0 : P_0 \longrightarrow X_1$ to be

$$H_0 = \widetilde{hf_0}\, s : P_0 \xrightarrow{s} A \otimes P_0 \longrightarrow X_1$$

$$a \otimes p \longmapsto a(hf_0\, p)$$

Observe $dH_0 = d\widetilde{hf_0}\, s = \widetilde{dhf_0}\, s = \tilde{f_0} s = f_0$.

If I write $\Phi(hf_0)$ for $\widetilde{hf_0}\, s$, (Φ is the flattening process), then we have

$$H_0 = \Phi(hf_0)$$

$$\begin{tikzcd} {} \arrow[r] & P_1 \arrow[r, "d"] \arrow[d, "f_1"] & P_0 \arrow[d] \arrow[dl, "H_0"'] \\ {} \arrow[r] & X_1 \arrow[r] \arrow[l, bend right, "h"'] & X_0 \end{tikzcd}$$

Repeat for $f_1 - H_0 d$ and we get

$$H_1 = \Phi(h(f_1 - H_0 d)) = \Phi(hf_1) - \Phi(h\Phi(hf_0)d)$$

Similarly

$$H_2 = \Phi(h\{f_2 - H_2 d\})$$

$$\Phi(h\Phi(h\Phi(hf_0)d)d)$$

$$= \Phi(hf_2) - \Phi(h\Phi(hf_1)d) +$$ [redacted]

This is messy but if we use the double complex picture for $\mathrm{Hom}_A(P, X)$, then $-d$, d the differential on P, is the perturbation Θ. So the above iteration looks like

$$h + h\Theta h + h\Theta h\Theta h + \ldots$$

where h here stands for the operation $f \mapsto \Phi(hf)$ above. The conclusion is that the [redacted] standard construction should be equivalent to what we obtain using perturbation theory.

Review from May 26. Observe that given a deformation retraction situation

$$\begin{tikzcd} X \arrow[d, shift right, "p"'] \\ Y \arrow[u, shift right, "i"'] \end{tikzcd} \qquad [d, h] = 1 - ip \qquad [p, d] = 0$$

$$pi = 1 \qquad [i, d] = 0$$

it more natural to form $X \times_Y Y^I \times_Y pt$ the mapping fibre, rather than the mapping cone. We have

$$(X \times_Y Y^I \times_Y pt)_n = X_n \oplus Y_{n-1}$$

$$\tilde{d} = \begin{pmatrix} d & 0 \\ p & -d \end{pmatrix} \qquad \tilde{h} = \begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix}$$

$$[\tilde{d}, \tilde{h}] = \begin{pmatrix} dh + hd + ip & di - id \\ ph & pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ph & 1 \end{pmatrix}$$

Notice $ph = 0$ means that the homotopy h of X [redacted] is fibrewise (geometrically the homotopy $h : X \times I \longrightarrow X$ when composed with $p : X \to Y$ is constant). The condition

$$hi = 0$$

means that the section i is left fixed under the deformation.

We have $\tilde{h}^2 = \begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h & i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} h^2 & hi \\ 0 & 0 \end{pmatrix}$

so that $\tilde{h}^2 = 0 \iff h^2 = hi = 0$.

Recall that a deformation retraction can be viewed as an embedding ι and retraction p, or also as a projection p and section i. The former picture leads to

$$(\text{Cone } i)_n = \Big(X \underset{Y\times 0}{\cup} Y \times I \underset{Y \times 1}{\cup} pt\Big)_n = X_n \oplus Y_{n-1}$$

$$\tilde{d} = \begin{pmatrix} d & 0 \\ i & -d \end{pmatrix} \qquad \tilde{h} = \begin{pmatrix} h & 0 \\ p & 0 \end{pmatrix}$$

and then $[\tilde{d}, \tilde{h}] = \begin{pmatrix} dh + hd + ip & hi \\ +pd - dp & pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

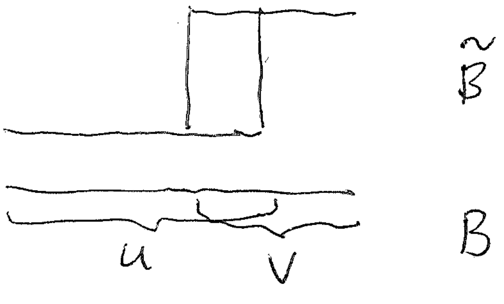
gives the extra condition $hi = 0$ rather than $ph = 0$.

Question: Is there a geometric meaning to $h^2 = 0$?

May 30, 1992

Recall that if P is a projective complex and $X \to Y$ is a quis of complexes of A modules, then $\mathrm{Hom}_A(P, X) \longrightarrow \mathrm{Hom}_A(P, Y)$ is a quis of complexes. This is a basic fact which I want to make constructive. I'm really interested in the case of $P \otimes_A X \longrightarrow P \otimes_A Y$ where P is a projective right module complex.

Consider a geometric version of this result. Let P be a principal G bundle, let $X \to Y$ be a G-map which is a heq when the G action is ignored. Then $P \times^G X \longrightarrow P \times^G Y$ is a heq. To prove this one might ~~use~~ an open covering of $B = P/G$ together with a partition of unity and trivializations of P over the open sets in the covering. Graeme's work shows this amounts to replacing B by the realization of a simplicial space



A simpler simplicial space we can take together with G-bundle over it is

$$\begin{tikzcd} \tilde{P}: & {} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & P\times G\times G \arrow[r, shift left] \arrow[r, shift right] & P\times G \arrow[r, "|" marking] & P \end{tikzcd}$$

$$\begin{tikzcd} \tilde{B}: & {} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & P\times G \arrow[r, shift left] \arrow[r, shift right] & P \arrow[r, "|" marking] & B \end{tikzcd}$$

Thus the map $\tilde{P} \times^G X \longrightarrow \tilde{P} \times^G Y$ is

$$\begin{tikzcd}
{} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & P\times G\times X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P\times X \arrow[r, "|" marking] \arrow[d] & P\times^G X \arrow[d] \\
{} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & P\times G\times Y \arrow[r, shift left] \arrow[r, shift right] & P\times Y \arrow[r, "|" marking] & P\times^G Y
\end{tikzcd}$$

One then applies the basic lemma that a map of simplicial spaces which is degree wise a h.eq gives rise upon realization to a h.eq. What this proof amounts to is

$$\begin{tikzcd}
\tilde{P}\times^G X \arrow[r] \arrow[d] & P\times^G X \arrow[d] \\
\tilde{P}\times^G Y \arrow[r] & P\times^G Y
\end{tikzcd}$$

where the horizontal arrows are SDR's because $\tilde{P}\longrightarrow P$ is an equivariant SDR, and the the left vertical arrow is a h.eq because of a suitable inductive construction using a "skeletal" filtration.

Given $X \xrightarrow{p} Y$ consider $Fib(p)$:

$$Fib(p)_n = X_n \oplus Y_{n+1} \qquad \tilde{d} = \begin{pmatrix} d & 0 \\ p & -d \end{pmatrix}$$

Suppose we have a homotopy inverse for p, namely $i : Y \longrightarrow X$, $[d,i] = 0$ together with homotopies in X, Y both denoted h such that

$$1 - ip = [d, h_X]$$
$$1 - pi = [d, h_Y]$$

Let

$$\tilde{h} = \begin{pmatrix} h & i \\ & -h \end{pmatrix} \quad : \quad \begin{tikzcd} X_n \arrow[r, "h"] \arrow[d, phantom, "\oplus"] & X_{n+1} \arrow[d, phantom, "\oplus"] \\ Y_{n+1} \arrow[r, "-h"'] \arrow[ur, "i"] & Y_{n+2} \end{tikzcd}$$

Then

$$[\tilde{d}, \tilde{h}] = \begin{pmatrix} d & 0 \\ p & -d \end{pmatrix}\begin{pmatrix} h & i \\ 0 & -h \end{pmatrix} + \begin{pmatrix} h & i \\ 0 & -h \end{pmatrix}\begin{pmatrix} d & 0 \\ p & -d \end{pmatrix}$$

$$= \begin{pmatrix} dh + hd + ip & di - id \\ ph - hp & pi + dh + hd \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ [p,h] & 1 \end{pmatrix}$$

This commutes with $\tilde{d}$, so we can get a contracting homotopy by multiplying $\tilde{h}$ with the inverse:

$$\begin{pmatrix} 1 & 0 \\ -[p,h] & 1 \end{pmatrix}\tilde{h} = \begin{pmatrix} h & i \\ -[p,h]h & -[p,h]i - h \end{pmatrix}$$

However this is not very canonical. The good thing seems to be to assume $[p,h] = 0$. This is a bit strong, but it generalizes the case considered before where we took $h = 0$ in Y and required $ph = 0$.

Thus we have found a generalization of our former version of HPT as follows.

Assumptions are:

$$\begin{tikzcd} X \arrow[r, shift right, "p"'] & Y \arrow[l, shift right, "i"'] \end{tikzcd} \qquad \begin{aligned} [d,i] &= [d,p] = 0 \\ [d,h] &= 1-pi \\ [d,h] &= 1-ip \\ [p,h] &= 0. \end{aligned}$$

Then we form $F(p)$ with

$$\tilde{d} = \begin{pmatrix} d & \\ p & -d \end{pmatrix} \qquad \tilde{h} = \begin{pmatrix} h & i \\ & -h \end{pmatrix}$$

and we have $[\tilde{d}, \tilde{h}] = 1$.

Next suppose we have a perturbed diffl $d-\theta$ in both X, Y. Assume $[p,\theta]=0$. Then setting $\tilde{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}$ we find

$$(\tilde{d}-\tilde{\theta})^2 = \begin{pmatrix} d-\theta & 0 \\ p & -d+\theta \end{pmatrix}\begin{pmatrix} d-\theta & 0 \\ p & d-\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \underbrace{p(d-\theta) - (d-\theta)p}_{0} & 0 \end{pmatrix}$$

HPT given the contracting homotopy wrt $\tilde{d}-\tilde{\theta}$:

$$\tilde{h}' = \tilde{h}\frac{1}{1-\tilde{\theta}\tilde{h}} = \begin{pmatrix} h & i \\ 0 & -h \end{pmatrix}\begin{pmatrix} 1-\theta h & -\theta i \\ & 1-\theta h \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} h & i \\ 0 & -h \end{pmatrix}\begin{pmatrix} \frac{1}{1-\theta h} & \frac{1}{1-\theta h}\theta i \frac{1}{1-\theta h} \\ & \frac{1}{1-\theta h} \end{pmatrix}$$

$$= \begin{pmatrix} h\frac{1}{1-\theta h} & h\frac{1}{1-\theta h}\theta i \frac{1}{1-\theta h} + i\frac{1}{1-\theta h} \\ & -h\frac{1}{1-\theta h} \end{pmatrix}$$

$$\begin{aligned} & h\frac{1}{1-\theta h}\theta + 1 \\ &= \frac{1}{1-h\theta}h\theta + \frac{1}{1-h\theta}(1-h\theta) \\ &= \frac{1}{1-h\theta} \end{aligned}$$

Thus

$$\tilde{h}' = \begin{pmatrix} h\frac{1}{1-\theta h} & \frac{1}{1-h\theta}\, i\, \frac{1}{1-\theta h} \\ & -h\frac{1}{1-\theta h} \end{pmatrix}$$

Summarize: Given $X \underset{p}{\overset{i}{\rightleftarrows}} Y$ with

$$[d,i] = [d,p] = 0\ , \quad [d,h] = \begin{cases} 1-ip \\ 1-pi \end{cases}, \quad [p,h]=0$$

and a perturbation θ of d on X,Y: $\left|\begin{array}{l}(d-\theta)^2 = 0 \\ [p,\theta] = 0\end{array}\right.$

then we have

$$[d-\theta, i'] = [d-\theta, p] = 0\ , \quad [d-\theta, h'] = \begin{cases} 1-i'p \\ 1-pi' \end{cases}, \quad [p,h']=0$$

where $h' = h\frac{1}{1-\theta h}$ and $i' = \frac{1}{1-h\theta}\, i\, \frac{1}{1-\theta h}$.

<u>More formulas:</u> Let $P(p) \longleftrightarrow X \times_Y Y^I$. Thus

$$P(p)_n = X_n \oplus Y_{n+1} \oplus Y_n$$

$$\tilde{d} = \begin{pmatrix} d & & \\ p & -d & 1 \\ & & d \end{pmatrix} \cdot \begin{pmatrix} d & & \\ p & -d & 1 \\ & & d \end{pmatrix} = \begin{pmatrix} d^2 & 0 & 0 \\ -[d,p] & d^2 & 0 \\ 0 & 0 & d^2 \end{pmatrix}$$

Observe $P(p) \underset{\begin{pmatrix}1\\0\\-p\end{pmatrix}}{\overset{(1\ 0\ 0)}{\rightleftarrows}} X$ $\qquad \begin{pmatrix} d & & \\ p & -d & 1 \\ & & d \end{pmatrix}\begin{pmatrix}1\\0\\-p\end{pmatrix} = \begin{pmatrix} d \\ 0 \\ -dp \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -p \end{pmatrix} d$

$$\left[\begin{pmatrix} d & & \\ p & -d & 1 \\ & & d \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ p & 0 & 1 \end{pmatrix} = I - \begin{pmatrix} 1 \\ 0 \\ -p \end{pmatrix}(1\ 0\ 0)$$

Also $\begin{pmatrix} & & \\ & & \\ & 1 & \end{pmatrix}^2 = 0$, $\begin{pmatrix} & & \\ & & \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -p \end{pmatrix} = 0$, $(1\ 0\ 0)\begin{pmatrix} & & \\ & & \\ & 1 & \end{pmatrix} = 0$

so $P(p)$ has X canonically as SDR.

Next suppose i, h as above and put $\tilde{h} = \begin{pmatrix} h & i & \\ & -h & \\ & & \end{pmatrix}$

Then

$$\begin{pmatrix} d & & \\ p & -d & 1 \\ & & d \end{pmatrix}\begin{pmatrix} h & i & \\ & -h & \\ & & \end{pmatrix} + \begin{pmatrix} h & i & \\ & -h & \\ & & \end{pmatrix}\begin{pmatrix} d & & \\ p & -d & 1 \\ & & d \end{pmatrix}$$

$$= \begin{pmatrix} dh+hd+ip & di-id & i \\ ph-hp & pi+dh+hd & -h \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -h \\ 0 & 0 & 0 \end{pmatrix}$$

$$= I - \begin{pmatrix} -i \\ h \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

I should have noted

$$\begin{tikzcd} P(p) \arrow[r, shift left, "(0\ 0\ 1)"] & Y \arrow[l, shift left, "\begin{pmatrix} -i \\ h \\ 1 \end{pmatrix}"] \end{tikzcd} \qquad \begin{pmatrix} d & & \\ p & -d & 1 \\ & & d \end{pmatrix}\begin{pmatrix} -i \\ h \\ 1 \end{pmatrix} = \begin{pmatrix} -di \\ -pi-dh+1 \\ d \end{pmatrix} = \begin{pmatrix} -i \\ h \\ 1 \end{pmatrix} d$$

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} h & i & \\ & -h & \\ & & \end{pmatrix} = 0$$

$$\begin{pmatrix} h & i & \\ & -h & \\ & & \end{pmatrix}\begin{pmatrix} -i \\ h \\ 1 \end{pmatrix} \text{[redacted]} = \begin{pmatrix} -hi+ih \\ -h^2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} h & i & \\ & -h & \\ & & \end{pmatrix}\begin{pmatrix} h & i & \\ & -h & \\ & & \end{pmatrix} = \begin{pmatrix} h^2 & hi-ih & \\ & h^2 & \\ & & \end{pmatrix}$$

Thus this will give a deformation retraction [redacted] such that the surjection kills the homotopy in general. It will give Y as an SDR of $P(p)$ when we have the additional conditions $[h, i] = 0$ and $h^2 = 0$.

May 31, 1992

There seems to be something subtle happening with homotopy equivalences. Consider

1)
$$\begin{tikzcd} X \arrow[r, shift right, "p"'] & Y \arrow[l, shift right, "i"'] \end{tikzcd}$$

$$[d,p] = [d,i] = 0$$
$$[d,h] = \begin{cases} 1-ip \\ 1-pi \end{cases}$$

where h really means h_X on X and h_Y on Y. This is the ~~first~~ approximation to the notion of h.eq. To see that it isn't enough suppose $pi = ip = 1$, i.e. p, i are inverse isomorphisms. Then $[h_X] \in R\mathrm{Hom}^1(X,X)$, $[h_Y] \in R\mathrm{Hom}^1(Y,Y)$ are arbitrary and should be zero.

An idea is to strengthen 1) by requiring in addition

2)
$$[p,h] = [i,h] = 0$$

I recall that the condition $[p,h]=0$ means that we get a simple contraction of $F(p)$, the homotopy fibre of p:

$$\left[\begin{pmatrix} d & 0 \\ p & -d \end{pmatrix}, \begin{pmatrix} h & i \\ 0 & -h \end{pmatrix}\right] = I$$

(Presumably $[i,h]=0$ means similarly for $\mathrm{Cone}(i)$. This is clear as the homotopy fibre and cone agree up to suspension.)

Question: Do h.eq.'s of the form 1), 2) compose? Suppose we have

$$\begin{tikzcd} X \arrow[r, shift right, "p"'] & Y \arrow[l, shift right, "i"'] \arrow[r, shift right, "q"'] & Z \arrow[l, shift right, "j"'] \end{tikzcd}$$

$$[d,p] = [d,i] = [d,q] = [d,j] = 0$$
$$[d,h] = \begin{cases} 1-ip \\ 1-pi \end{cases}, \quad [d,k] = \begin{cases} 1-jq \\ 1-qj \end{cases}$$
$$[p,h] = [i,h] = [q,k] = [j,k] = 0.$$

Then

$$\begin{aligned} 1 - ijgp &= 1 - i(1-[d,k])p \\ &= 1 - ip + [d, ikp] \\ &= [d, h+ikp] \\ 1 - gpij &= [d, k+ghj] \end{aligned}$$

$$gp(h+ikp) \overset{?}{=} (k+ghj)gp$$

$$\begin{array}{ll} \| & \| \\ gph + g(1-[d,h])kp & kgp + gh(1-[d,k])p \\ \| \quad (\text{use } [p,h]=0) & \| \quad (\text{use } [g,k]=0) \\ ghp + gkp - g[d,h]kp & gkp + ghp - gh[d,k]p \\ \| & \| \\ g(h+k-[d,h]k)p & g(h+k-h[d,k])p \end{array}$$

These are not the same but are homotopic since

$$[d,hk] = [d,h]k - h[d,k]$$

Similarly using $[i,h]=0$ and $[j,k]=0$ we have

$$\begin{aligned} (h+ikp)ij &= i(h+k-k[d,h])j \\ ij(h+ikp) &= i(h+k-[d,k]h)j \end{aligned}$$

Also

$$\begin{aligned} (h+ikp)^2 &= h^2 + i(kh+hk)p + i(k^2-k[d,h]k)p \\ (k+ghj)^2 &= k^2 + g\{hk+kh+h^2-h[d,k]h\}j \end{aligned}$$

so if we work with the additional conditions $h^2=k^2=0$, then if we assume $hk=kh=0$ the composition satisfies the same conditions.

June 1, 1992

Idea: Consider $T_A(A \otimes A)$:

$$A \xleftarrow{b'} A \otimes A \xleftarrow{b'} A \otimes A \otimes A \xleftarrow{b'}$$

This has contraction s = left multiplication by $1 \otimes 1$, that is $s(a_0, .., a_n) = (1, a_0, .., a_n)$.

Let us regard $b = b' + c$, c = crossover, as a perturbation of b' (here $\theta = -c$). If [redacted] $1 + cs$ were invertible, then we would have a contracting homotopy for b:

$$[b, s \frac{1}{1+cs}] = 1.$$

But this can't work, in fact

$$(cs)(a_0, .., a_n) = c(1, a_0, .., a_n)$$
$$= (-1)^{n+1}(a_n, a_0, .., a_{n-1})$$

so we have $\boxed{-cs = \lambda}$. We could also use s' = right multiplication by $1 \otimes 1$ with sign:

$$s'(a_0, .., a_n) = (-1)^n(a_0, .., a_n, 1)$$

$$cs'(a_0, .., a_n) = -(a_0, .., a_n)$$

so in this case $\boxed{-cs' = 1}$

Method to mimic: Consider principal G-bundles, and the result that for any principal G-bundle P and [redacted] contractible bundle PG, there is an equivariant map $P \longrightarrow PG$.

The geometric proof is to form Borel's mixing diagram

$$\begin{tikzcd} P & P \times PG \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\text{cart}"] & PG \arrow[d] \\ P/G \arrow[from=u] & P \times^G PG \arrow[l] & \end{tikzcd}$$

Then $P \times^G PG \longrightarrow P/G$ is a fibration (uses local triviality of action) with contractible fibre PG, so it has a section, and this section is equivalent to an equivariant map $P \to PG$. (Suffices PG is a G-space (not nec. equivariantly) contractible.)

Good method is to use the free G-biset resolution of G:

1)
$$\begin{tikzcd} G & G \underset{\wedge}{\times} G \arrow[l, "|" marking] & G \underset{\wedge}{\times} G \underset{\wedge}{\times} G \arrow[l, shift right, "m\times 1"'] \arrow[l, shift left, "1\times m"] & {} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

in order to couple a right G space P to a left G space X:

$$\begin{tikzcd} P \times^G X & P \underset{\wedge}{\times} X \arrow[l, "|" marking] & P \underset{\wedge}{\times} G \underset{\wedge}{\times} X \arrow[l, shift right] \arrow[l, shift left] & {} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

(The carets $\wedge$ denote the location of the (imaginary) vertices which are deleted by the face operators.)

Take 1) and couple with P on the right to get

2)
$$\begin{tikzcd} P & P \underset{\wedge}{\times} G \arrow[l, "|" marking] \arrow[d] & P \underset{\wedge}{\times} G \underset{\wedge}{\times} G \arrow[l, shift right] \arrow[l, shift left] \arrow[d] & {} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\ & {}_{\wedge}G & {}_{\wedge}G \underset{\wedge}{\times} G \arrow[l, shift right] \arrow[l, shift left] & {} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

The "base" is the standard simplicial PG. It's pretty clear that the "total space" is the product of P and PG, so this diagram 2) is

$$\begin{tikzcd}
P & P \times PG \arrow[l, "pr_1"'] \arrow[d, "pr_2"] \\
 & PG
\end{tikzcd}$$

Dividing by the G-action we get

3)
$$\begin{tikzcd}
P/G & P_{\wedge} \arrow[l, "|" marking] \arrow[d] & P \underset{\wedge}{\times} G_{\wedge} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & P \underset{\wedge}{\times} G \underset{\wedge}{\times} G_{\wedge} \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\
 & pt & {}_{\wedge}G_{\wedge} \arrow[l, shift left] \arrow[l, shift right] & {}_{\wedge}G \underset{\wedge}{\times} G_{\wedge} \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

which is just

$$\begin{tikzcd}
P/G & P \times^{G} PG \arrow[l] \arrow[d] \\
 & BG
\end{tikzcd}$$

(The reason for the preceding is to make me a little happier about using the model

$$\begin{tikzcd}
{}_{\wedge}G & {}_{\wedge}G \underset{\wedge}{\times} G \arrow[l, shift left] \arrow[l, shift right] & \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

for PG, where G acts by right multiplication on the right factor, instead of

$$\begin{tikzcd}
G & \underset{\wedge}{G} \times \underset{\wedge}{G} \arrow[l, shift left] \arrow[l, shift right] & \underset{\wedge}{G} \times \underset{\wedge}{G} \times \underset{\wedge}{G} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

where G acts diagonally on the right. The two models are of course isomorphic.)

Next [redacted] let's go to bimodules.

Let P be a projective bimodule complex. Suppose we have a map $P \to Y$ of bimodule complexes. [redacted] Consider the standard bimodule resolution $\mathcal{B}$:

$$4)\qquad \begin{tikzcd} A & A \hat{\otimes} A \arrow[l] & A \hat{\otimes} A \hat{\otimes} A \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

as an analogue of 1). Then we have the standard resolution $Y \otimes_A \mathcal{B}$ of Y, and we would like to map P to it (over Y). Form

$$\begin{tikzcd} P & P \otimes_A \mathcal{B} \arrow[l] \arrow[d] \\ Y \arrow[from=u] & Y \otimes_A \mathcal{B} \arrow[l] \end{tikzcd}$$

i.e.

$$\begin{tikzcd} P \arrow[d] & P \otimes A \arrow[l] \arrow[d] & P \otimes A \otimes A \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & \cdots \\ Y & Y \otimes A \arrow[l] & Y \otimes A \otimes A \arrow[l, shift left] \arrow[l, shift right] & \cdots \end{tikzcd}$$

The problem is to construct a lifting of P into $P \otimes_A \mathcal{B}$. This step is to be done by skeletal induction, and I think it reduces to a construction for a projective bimodule (i.e. P supported in degree 0) together with HPT.

If $[d,h]=1$ but $h^2 \neq 0$, then one has $[d, hdh] = 1$ and $(hdh)^2 = 0$. In effect $dh+hd=1$ implies dh, hd commute, and as $hd\,dh = 0$, we get hd, dh are orthogonal idempotents. Thus $hdhhdh = 0$ and $[d, hdh] = [d,h]dh \; \underline{h}\cancel{[d,d]}h + hd[d,h] = dh + hd = 1$.

Remark about HPT. Recall

$$[d, h] = 1$$
$$(d-\theta)^2 = 0$$

$$\left[d-\theta,\ h\frac{1}{1-\theta h}\right] = 1$$

Now assume the special condition $h^2 = 0$. Then we can write

$$\begin{aligned} h\frac{1}{1-\theta h} &= h + h\theta h + h\theta h\theta h + \dots \\ &= h + h[\theta, h] + h[\theta, h][\theta, h] \\ &= h\frac{1}{1-[\theta, h]} \end{aligned}$$

Another way to see this:

$$\begin{aligned} h\frac{1}{1-[\theta,h]} &= h\frac{1}{1-\theta h - h\theta} = h\frac{1}{(1-\theta h)(1-h\theta)} \\ &= h\frac{1}{(1-h\theta)}\frac{1}{(1-\theta h)} = h\frac{1}{1-\theta h} \end{aligned}$$

where we use $(AB)^{-1} = B^{-1}A^{-1}$.

Let P be a projective bimodule. We want to contract $\mathrm{Cone}(P\otimes_A B \longrightarrow P)$:

5) $$P \xleftarrow{m} P\underset{\wedge}{\otimes}A \Lleftarrow P\underset{\wedge}{\otimes}A\underset{\wedge}{\otimes}A \quad \cdots$$

The idea is to use a right connection in P together with the right contraction of $\mathrm{Cone}(B\to A)$ given by $s\omega = 1\otimes\omega$. (This is left mult. by $1\otimes 1$ wrt the DG algebra structure) Notice a contraction of 5) in particular gives a bimodule lifting $l: P \longrightarrow P\otimes A$, or equivalently a right connection.

Consider

$$\begin{tikzcd}
(P\otimes A)\otimes_A B = P\otimes B \arrow[d, "m\otimes 1"] \\
P\otimes_A B
\end{tikzcd}$$

i.e.

$$\begin{tikzcd}
P\otimes A \arrow[r, dotted, bend left, "\cdot\otimes 1_A\otimes\cdot\, =\, id_P\otimes s"] \arrow[d, "m"] & P\otimes A\underset{\wedge}{\otimes} A \arrow[l] \arrow[r, dotted, bend left, "id_P\otimes s"] \arrow[d, "m\otimes 1"] & P\otimes A\underset{\wedge}{\otimes} A\underset{\wedge}{\otimes} A \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \arrow[d, "m\otimes 1\otimes 1"] & \cdots \\
P \arrow[u, dotted, bend left, "l"] & P\underset{\wedge}{\otimes} A \arrow[l] \arrow[u, dotted, bend left, "l\otimes 1"] & P\underset{\wedge}{\otimes} A\underset{\wedge}{\otimes} A \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \arrow[u, dotted, bend left, "l\otimes 1"] & \cdots
\end{tikzcd}$$

Observe that $(m\otimes 1)(id_P\otimes s) = id$, so it seems that our contracting homotopy for $P\otimes_A B$ is just

$$P\otimes A^{\otimes n} \xrightarrow{l\otimes 1} (P\otimes A)\otimes A^{\otimes n+1} = P\otimes A^{\otimes n+1}$$

Another way to think of this is to use the simplicial ~~identity~~

$$d_i s_0 = \begin{cases} 1 & i = 0, 1 \\ s_0 d_{i-1} & i > 1. \end{cases}$$

This is why $P\otimes B$ is contractible; it is $P\otimes_A B$ with d_0 forgotten and a shift in indices. The s_0 for $P\otimes_A B$ becomes an s_{-1} on $P\otimes B$, which is a well-known contracting homotopy. It's clear that l extended to $P\otimes_A B$ in the way indicated is just an s_{-1}.

So now we know how a lifting $l: P\to P\otimes A$ contracts Cone$\{P\otimes_A B \to P\}$

June 2, 1992

Yesterday we learned how a lifting $l : P \longrightarrow P \otimes A$ leads to a contraction of $\mathrm{Cone}\{P \otimes_A B \longrightarrow P\}$:

$$P \longleftarrow P \underset{\wedge}{\otimes} A \leftleftarrows P \underset{\wedge}{\otimes} A \underset{\wedge}{\otimes} A \cdots$$

One can think of l as an extra degeneracy s_{-1}. Here B is the unnormalized ~~Resolution~~ standard resolution of the bimodule A.

Another idea is the desirability of a homotopy h such that $h^2 = 0$. For this reason one wants instead of B the normalized standard resolution $\Omega A \otimes A$:

$$A \longleftarrow\!\!|\!\!- A \otimes A \xleftarrow{b'} \Omega^1 A \otimes A \xleftarrow{b'} \cdots$$

where inserting 1 in the first place becomes $d \otimes 1$. Note that $\Omega A \otimes A$ is a ~~quotient~~ of B. This is even true for the DG algebras structures on the cones it seems.

The next point is to recall what happens for right connections. Recall formulas

$$-\xi da \xleftarrow{-p} \xi \otimes a, \quad \xi \otimes 1 \longleftarrow\!\!| \ \xi$$

$$0 \longrightarrow P \otimes_A \Omega^1 A \xrightarrow[j]{} P \otimes A \xrightarrow[m]{} P \longrightarrow 0$$

$$\boxed{\xi \otimes 1 - l(\xi) = j(\nabla \xi)}$$

$$\xi a \otimes 1 - l(\xi a) = j(\nabla(\xi a))$$

$$- \quad \xi \otimes a - l(\xi) a = j(\nabla \xi\, a)$$

$$j(\xi da) - [l(\xi a) - l(\xi) a] = j(\nabla(\xi a) - \nabla \xi\, a)$$

so we have $\nabla \xi = (-p) l(\xi)$ where $(-p)(\xi \otimes a) = \xi da$

Diagram

$$\begin{tikzcd}
P\otimes A \arrow[r, "1\otimes 1_A\otimes 1"] \arrow[d, shift left, "m"] & P\otimes A\otimes A \arrow[r, "1\otimes d\otimes 1"] \arrow[d, shift left, "m\otimes 1"] & P\otimes\Omega^1A\otimes A \arrow[r] \arrow[d, shift left, "\pi\otimes 1"] & {} \\
P \arrow[r, "\ell"'] \arrow[u, shift left, "\ell"] & P\otimes A \arrow[r, "\nabla\otimes 1"'] \arrow[u, shift left, "\ell\otimes 1"] & P\otimes_A\Omega^1A\otimes A \arrow[r, "\nabla\otimes 1"'] \arrow[u, shift left, "\ell\otimes 1"] & {}
\end{tikzcd}$$

The point is that ℓ makes P a direct factor of $P\otimes A$, hence it makes $P\otimes_A \mathrm{Cone}(\Omega A\otimes A\to A)$ a direct factor of $P\otimes\mathrm{Cone}(\Omega A\otimes A\to A)$. Ons this cone we have the contraction given by $1_A\otimes 1$ in degree 0 an $d\otimes 1$ in degrees >0. This leads to a contraction of $P\otimes_A \mathrm{Cone}(\Omega A\otimes A\to A)$ given by ℓ in degree 0 and $\nabla\otimes 1$ in degrees >0.

Now I would like to the case where $P = \Omega A\otimes A$. In this case

$$P\otimes_A \Omega A\otimes A = \Omega A\otimes\Omega A\otimes A$$

This is a double chain complex with vertical augmentation to $\Omega A\otimes A$, where the augmentation is

$$\Omega^pA\otimes\Omega^0A\otimes A\xrightarrow{1\otimes m}\Omega^pA\otimes A$$

We have an obvious connection in P which together with the contraction in the vertical $\Omega A\otimes A$ gives rise to a vertical homotopy $h = 1\hat{\otimes} d\otimes 1$ in $\Omega A\otimes\Omega A\otimes A$. At this point we have to make horizontal + vertical differentials anticommute, so we take $b'\otimes_A 1$ and $1\hat{\otimes}_A b'$ on $(\Omega A\otimes A)\otimes_A(\Omega A\otimes A)$

We treat $b' \otimes_A 1 = -\theta$ as perturbation of $1 \hat{\otimes}_A b'$. To compute

$$h \frac{1}{1-\theta h} = h \frac{1}{1-[\theta, h]} \text{ and}$$

$$\frac{1}{1-h\theta} i = \frac{1}{1-h\theta} \frac{1}{1-\theta h} i = \frac{1}{(1-\theta h)(1-h\theta)} i = \frac{1}{1-[\theta, h]} i$$

we need $[\theta, h]$. Notice θ and $b' \otimes_A 1$ do not involve $\otimes A$ at the right, so can calculate in $\Omega A \otimes \Omega A$

$$\begin{tikzcd}
\omega\, da \otimes \alpha \arrow[r, "h"] \arrow[dd, "b' \otimes_A 1"] & (-1)^{|\omega|+1} \omega\, da \otimes d\alpha \arrow[d, "b' \otimes_A 1"] \\
& -\omega a \otimes d\alpha + \omega \otimes a\, d\alpha \\
(-1)^{|\omega|}(\omega a \otimes \alpha - \omega \otimes a\alpha) \arrow[r, "h"] & \underline{\omega a \otimes d\alpha - \omega \otimes d(a\alpha)} \\
& -\omega \otimes da\, \alpha
\end{tikzcd}$$

Thus we have

$$\underbrace{-[b' \otimes_A 1, h]}_{[\theta, h]} (\omega\, da \otimes \alpha \otimes a') = \omega \otimes da\, \alpha \otimes a'$$

From this we ought to be able to get a SDR for $1 \otimes \varepsilon : P \otimes_A P \longrightarrow P$. In particular we should get a section $P \longrightarrow P \otimes_A P$ of $1 \otimes \varepsilon$ as $\frac{1}{1-[\theta, h]} i$. It is clear what this does is to send $a_0 da_1 \cdots da_n \otimes a'$ to

$$\sum_{j=0}^{n} a_0 da_1 \cdots da_j \otimes da_{j+1} \cdots da_n \otimes a'$$

This should be the diagonal for the cup product of normalized cochains.

June 4, 1992

Recall the identification

$$\{E \to A\} \longmapsto \{\mathrm{Cone}(E \to A)\}$$

chain complexes of bimodules equipped with augmentation	chain complexes of bimodules $= A$ in degree 0

The tensor product on the right: $K \otimes_A L$ corresponding to join on the left:

$$E * E' = \underset{(\text{or Cone})}{\mathrm{Total}}\left(E \oplus E' \longleftarrow E \otimes_A E' \right)$$

[redacted] A-Algebras on the right correspond to 'join algebras' on the left. The A-algebra generated by $K_E = \mathrm{Cone}(E \to A)$ is

$$R_A(K_E) = T_A(K_E)/(1 = 1_A) \simeq T_A(\Sigma E)$$

and this corresponds to the join algebra generated by E which can be described as

$$\mathrm{Total}\left(E \leftleftarrows E \otimes_A E \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; E \otimes_A E \otimes_A E \cdots \right)$$

or

$$\varinjlim_{[n]} E * \overset{n+1}{\cdots} * E$$

where $[n]$ ranges over Δ', the category of nonempty finite totally ordered sets and [redacted] injections

Suppose $E_0 \to A$ surjective, or equivalently K_E connected. Then we know $R_A(K_E)$ is acyclic with contracting homotopy left mult. by any $\xi \in E_0$ mapping onto [redacted] 1_A. The same argument applies to any chain algebra B with $B_0 = A$ and $H_0(B) = 0$. [redacted] Another way to see this is to use the fact that because B is an A-algebra, hense there is a canonical retraction

$$R(B) \longrightarrow B$$

Thus $R(B)$ acyclic implies B acyclic.

Our problem is to show that when K_E is acyclic (i.e. $E \to A$ is a quis) and E is projective, then the inclusion $K_E \subset R(K_E)$ is a h.eq of bimodule complexes, in fact an SDR exists. If K_E has a DG algebra structure, then we have a retraction $R(K_E) \longrightarrow K_E$ as we have seen, so a first step might be to construct the homotopy in this case. Writing B for K_E we can use the canonical isomorphism

$$R(B) = \bigoplus_{n \geq 0} \Omega_A^{2n}(B)$$

to reduce to producing A-bimodule contractions for $\Omega_A^{2n}(B)$, $n > 0$. But we have

$$\Omega_A^k(B) = B \otimes_A (B/A) \otimes_A \overset{k}{\cdots} \otimes_A (B/A)$$

and we know such a contraction exists for $k>0$ because B is acyclic and B/A is projective. (Strictly speaking we need a left A-module contraction for B which is OKAY as it is projective as left A-module since A and B/A are.)

Some nice things about this argument: [redacted] Recall the problem arising with [redacted] a constant presimplicial module

$$V \leftleftarrows V \substack{\leftarrow\\\leftarrow\\\leftarrow}$$

[redacted] The associated complex is

$$V \xleftarrow{0} V \xleftarrow{1} V \xleftarrow{0} V \xleftarrow{1}$$

so the skeletal filtration is not suited to proving acyclicity. Instead we should be using [redacted] the odd half of the [redacted] skeletal filtration:

$$F_1 \subset F_3 \subset F_5 \subset \cdots\cdots$$

Recall the [redacted] natural increasing + decreasing filtrations for $R = R_A(B)$, $I = I_A(B)$

$$A \subset B \subset B^2 \subset \cdots\cdots \subset R$$

$$R \supset I \supset I^2 \supset \cdots\cdots$$

and the basic fact $\boxed{B^{2n+1} \oplus I^{n+1} = R}$ so that

$I^n/I^{n+1} \simeq$ [redacted] $I^n \cap B^{2n+1} \simeq B^{2n+1}/B^{2n-1}$. The problem of making $B \subset R_A(B)$ an SDR ~~of~~ bimodule complexes reduces to contracting $\Omega_A^{2n}(B)$ for $n \geq 1$. Here I am assuming B [redacted] is a DG algebra, but one can pose similar questions for $K = K_E$.

Thus consider $K = K_E$ a bimodule chain [redacted] complex $= A$ in degree 0, assumed to be [redacted] acyclic and also projective in degrees > 0. We have

$$A \subset K \subset K^2 \subset \cdots\cdots \subset R = R(K)$$

and also $R = \varinjlim_{[n]}$ [redacted] $K \otimes_A \overset{n+1}{\cdots} \otimes_A K$

and $K^n/K^{n-1} = \Sigma E \otimes_A \overset{n}{\cdots} \otimes_A \Sigma E$

Observe $K^2/K^0 = (A \Leftarrow E \Leftarrow E \otimes_A E)/A$

$$= \Sigma(E \Leftarrow E \otimes_A E)$$

is not acyclic. It is thus not the same as $\Omega^1 B = B \otimes_A (B/A)$ in the DG alg case.

Let's go over the geometric picture where we consider the join on G-spaces, G a discrete group. Before bringing in the group let's discuss the realization of the presimplicial space

$$1) \qquad X \leftleftarrows X \times X \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; X \times X \times X \cdots\cdots$$

where the faces are projections (omitting a component). In general the realization of F. is an inductive limit

$$\varinjlim_{\Delta'/F} \Delta_n$$

In the case of 1 we have an n-simplex for each element of X^{n+1}, this means we have $X * \overset{n+1}{\cdots} * X$, and these have to be glued according to the face map. This gives

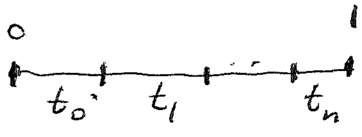
$$\varinjlim_{[n] \in \Delta'} X * \overset{n+1}{\cdots} * X$$

which should be the join monoid generated by X.

Take $X = pt$. Then we get

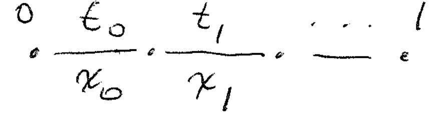
$$\varinjlim_{[n] \in \Delta'} \Delta_n$$

Regard Δ_n as $\left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$. We can also identify the interior with a triangulation of $[0, 1]$ with $n+2$ vertices



One fits these together for different n in the obvious way. I guess this is the obvious topology on finite subsets of $[0, 1]$ containing $\{0, 1\}$.

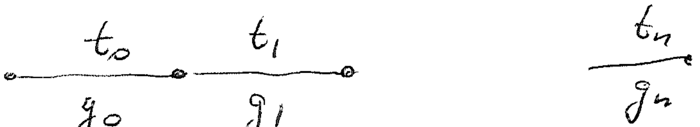
For a general X we obtain



triangulations with edges labelled by elements of X. If we take the model for PG:

$$G \leftleftarrows {}_{\wedge}G\underset{\wedge}{\times} G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; {}_{\wedge}G \underset{\wedge}{\times} G \underset{\wedge}{\times} G \cdots$$

then the realization consists of



so that if $t_0 \downarrow 0$, then g_0 disappears, but if $t_j \downarrow 0$ for $j>0$, then g_j multiplies with g_{j-1} on the right: $g_{j-1} \mapsto g_{j-1}g_j$.

It will be better to consider the model where G acts diagonally by right multiplication on $G\times\cdots\times G$. Let be the realization

$$P = \varinjlim G * \overset{n+1}{\cdots} * G$$

This is a different version of Milnor's model and it is the join monoid generated by G.

Can I see geometrically that the join monoid on P has an equivariant SDR to P?

Let $J(X) = \varinjlim_{[n]} X * \overset{n+1}{\cdots} * X$ be the join monoid generated by X. I want to show that the inclusion $J(X) \subset JJ(X)$ extends to an SDR. A point of $J(X)$ we recall is a subdivision of $[0,1]$ with edges labelled by elts of X

About affine spaces. Recall that an affine space is essentially identical to a surjection of vector spaces $V \twoheadrightarrow \mathbb{C}$, the affine space being the inverse image of $1 \in \mathbb{C}$. Given two affine spaces $V \twoheadrightarrow \mathbb{C}$, $W \twoheadrightarrow \mathbb{C}$ we get a third given by

$$V \oplus W \twoheadrightarrow \mathbb{C} \oplus \mathbb{C} \xrightarrow{+} \mathbb{C}$$

This is the natural join operation; the dimension is the sum of the dimensions plus 1. On the other hand if we choose an embedding $\mathbb{C} \hookrightarrow \mathbb{C} \oplus \mathbb{C}$, $1 \longmapsto (1-t, t)$, then the fibre product

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, hook] \\ V \oplus W \arrow[r, two heads] & \mathbb{C} \oplus \mathbb{C} \end{tikzcd}$$

isomorphic to the product of the affine spaces of V, W for $t \neq 0, 1$.

Next consider joins. Given a space X form $\mathbb{R}_{\geq 0} \times X$ and collapse $0 \times X$ to a point to obtain the infinite cone on X.

If $X \subset S(V)$, where V is a real (say) finite dimensional v.s. and $S(V)$ is the unit sphere for some norm, then this cone can be identified with the cone on X in V. Let's denote this cone by R^+X and note ~~that~~ there is a canonical map $R^+X \longrightarrow R^+$ $(= \mathbb{R}_{\geq 0})$.

Next take two spaces X, Y and form the product $R^+X \times R^+Y$. This maps by sum to $R^+ \times R^+$ and we can compose with the sum $\mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. The inverse image of 1 is the join $X * Y$ and we have a natural identification

$$\mathbb{R}^+ X \times \mathbb{R}^+ Y \simeq \mathbb{R}^+(X * Y)$$

June 6th, 1992

Let B be a chain algebra with $B_0 = A$, $H(B) = 0$, and $B/A = \Sigma E$ a projective A-bimodule. Recall that

$$R_A(B) = \text{tot}(A \leftarrow E \leftleftarrows E\otimes_A E \cdots\cdots) = T_A(\Sigma E)$$

Because of the algebra structure on B we have a canonical isomorphism of A-bimodules

$$R_A(B) = \Omega_A^{ev}(B) = B \oplus \Omega_A^2(B) \oplus \cdots$$

We have seen this makes B a SDR of $R_A(B)$. In effect it suffices to produce bimodule contractions of

$$\Omega^n_A(B) = B\otimes_A (B/A)\otimes_A \overset{n}{\cdots}\otimes_A (B/A)$$

for all $n \geq 1$. For this it suffices to do the case $n=1$, and we have seen that a contraction of $B\otimes_A(B/A)$ can be obtained from a left A-module contraction of B (say given by right multiplication by a $\xi \in B_0$ lying over $1_A \in A$), [redacted] together with a left connection on B/A.

Observe that

$$\Omega^1_A(B) = B\otimes_A(B/A) = \Sigma\,\text{tot}\left\{E \xleftarrow{\varepsilon\otimes 1} E\otimes_A E\right\}$$

$$\Omega^2_A(B) = B\otimes_A(B/A)\otimes_A(B/A) = \Sigma^2\,\text{tot}\left\{E\otimes_A E \xleftarrow{\varepsilon\otimes 1\otimes 1} E\otimes_A E\otimes_A E\right\}$$

Now this is curious because we know that

$$\Omega^2_A(B) = I_A(B)/I_A(B)^2 \simeq F_3 R_A(B)/F_1 R_A(B)$$

$$= \Sigma^2\,\text{tot}\left\{E\otimes_A E \xrightarrow{\partial} E\otimes_A E\otimes_A E\right\}$$

where $\partial = \varepsilon\otimes 1\otimes 1 - 1\otimes\varepsilon\otimes 1 + 1\otimes 1\otimes\varepsilon$

Somehow the product on B has been used to transform ∂ to $\varepsilon \otimes 1 \otimes 1$.

June 7, 1992

Let us return to the idea that a projective bimodule resolution P of A yields a functor $X \mapsto P \otimes_A X$ from complexes of A-modules to projective complexes of A-modules, and that we thus obtain a functorial projective resolution $P \otimes_A X \longrightarrow X$. We can thus try to bring the known facts about construction of projective resolutions to bear on the properties of P which we want to use.

The first property is that $P \otimes_A X \longrightarrow X$ is a quis of A-modules. A concrete form of this is that this map is a homotopy equivalence when A-module structures are ignored, and this translates more or less into a homotopy equivalence $P \longrightarrow A$ compatible with right multiplication.

Another property is that if $X \longrightarrow Y$ is a quis of A-module complexes, then $P \otimes_A X \longrightarrow P \otimes_A Y$ is a heq of A-module complexes. A concrete version of this is provided by a right connection in P, i.e. a lifting $P \hookrightarrow P \otimes A$ wrt right mult. not necessarily compatible with the differentials of P.

Next we want the uniqueness of projective resolutions up to homotopy. Thus if we have $E \longrightarrow X$ a projective resolution we want a h.eq of E and $P \otimes_A X$. Consider

$$\begin{tikzcd}
P \otimes_A E \arrow[r, "h.eq."] \arrow[d] & P \otimes_A X \arrow[d] \\
E \arrow[r] & X
\end{tikzcd}$$

Thus we need to know that

$$P\otimes_A E \longrightarrow E$$

is [redacted] h eq when E is projective.

Constructively [redacted] we need a left A-module h eq $P \longrightarrow A$ and a left connection in E.

So far we want

a h eq $P \longrightarrow A$ comp. with right mult.

a h eq $P \longrightarrow A$ ——— left mult.

a right connection in P

If we consider projective bimodule resolutions $P \longrightarrow A$, then to prove two are homotopy equivalent we might try

$$P \xleftarrow{(1)} P\otimes_A P' \xrightarrow{(2)} P'$$

For (1) [redacted] to be a h eq you need [redacted]

right conn. on P

right module h eq $P' \longrightarrow A$

For (2) to be a h eq you need

left conn. on P'

left module h eq $P \longrightarrow A$

June 8, 1992

Motivation: For P a proj. bimodule resolution of A it seems we want to [redacted] product maps

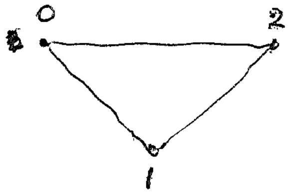
$$\begin{tikzcd} P \arrow[r, "\Delta"] & P \otimes_A P \arrow[r, shift left=2, "\varepsilon\otimes 1"] \arrow[r, shift right=2, "1\otimes\varepsilon"'] \arrow[r, phantom, "\Downarrow h"] & P \end{tikzcd}$$

and maybe something else in order to construct the perturbed differential on $P \otimes_A$ giving rise to the cyclic theory of A. The above suggests the maps

$$\begin{tikzcd} X \arrow[r] & X^I \arrow[r, shift left] \arrow[r, shift right] & X \end{tikzcd}$$

one encounters with the path space. Maybe it's better to think of P and X^I as analogous, so that $P \otimes_A$ corresponds to the free loop space X^{S^1}.

Thus to a 1 simplex we want to associate a path space and to a cyclic graph the space of maps from the realization to X. Unfortunately we run into problems with the morphisms, e.g. if we take a face, [redacted] say deleting the vertex 1 in



then we need a map from maps from  to X to maps  to X. It seems we can take care of this problem by [redacted] introducing lengths for the edges.

Let's return to Graeme's picture of Δ', which we take to have the objects $[n] = \{0, 1, \dots, n\}$ and injective maps preserving the cyclic ordering. Let

$$I([n]) = \text{space of embeddings } [n] \overset{\alpha}{\hookrightarrow} S^1$$
compatible with the cyclic order

Then 1) $I : (\Lambda')^{op} \longrightarrow \text{Spaces}$ is a contravariant functor. 2) The map $I([n]) \longrightarrow S^1$, $\alpha \mapsto \alpha(0)$ is a fibring with fibre over z ~~the~~ open n-simplex of embeddings $\{1, \dots, n\} \hookrightarrow S^1 - \{z\}$ compatible with the linear order. In particular $I([n]) \sim S^1$.

3) S^1 acts freely on $I([n])$ by translation and the quotient is the ~~open~~ n-simplex of ways of assigning positive lengths to edges of ~~the~~ cyclic graph associated to $[n]$ so the total length is one.

4) The [redacted] fibred category over Λ' defined by I should have as its realization the space of probability measures on the circle of finite support. (?) This is convex hence contractible.

[redacted] (We have to be more precise. The fibred category Λ'/I is equivalent to the poset of non empty finite subsets of S^1 under inclusion. The realization of this poset is the barycentric subdivision of the infinite simplex with points of S^1 as vertices, i.e. the probability measures on S^1 with finite support. Note the significance of the equivalence - S^1 acts freely on the fibres [redacted] $I([n])$ but not freely on the space of probability measures.)

Now given X we can form the cyclic space

$$[n] \longmapsto \underbrace{I([n]) \times_{S^1}}\left(X^{S^1}\right)$$

A point of is a cyclic graph with

parametrization together with a map of this graph into X. One can think of a point of $I([n]))/S^1$ as a parametrized circle together with an embedding of $[n]$ into it compatible with cyclic order.

This trick of introducing lengths is probably related to the Moore loop space construction.

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We have seen how to thicken the category of cyclic graphs and embeddings by introducing parametrizations. Over the latter category we have a principal S^1-bundle and we get a functor assigning to a parametrized graph the space of maps to X. Thus we have a fibre bundle with fibre X^{S^1}.

I would like to do the same sort of thing for X^I. I tend to think of P as analogous to X^I.

Instead of cyclic graphs consider linear graphs and embedding, i.e. the category Δ of non empty linearly ordered finite sets and order-compatible **injections**.